Contributor Profiles: Peter Y. Woo

Peter was raised in Hong Kong under the British system. He became interested in mathematics at an early age because his father and grandfather were good at it. In high school, he was fortunate to have had a kind and loving teacher, Terry Chamberlain. From him, Peter learned about Steiner's inversions of the plane across a circle and how to decompose some simple rational functions into partial fractions by inspection.

At university, most of his classmates wanted to become the next Einstein or Madame Curie. As a result, he also worked very hard at physics. However, most of his teachers in physics and mathematics at the University of Hong Kong (HKU) were not very encouraging.

After receiving a B.Sc. from HKU, Peter got a scholarship to do graduate studies at the University of Southern California (USC) in mathematics. There he met Herbert Buseman, who one day asked him to become a disciple. Under Buseman, he learned about convex functions, convex bodies, and all manners of geometry in two and three dimensions. He received his M.A. and Ph.D. at USC. His dissertation was in geometry, of course.

Peter then worked in the computer software industry for 20 years, occasionally teaching college mathematics courses. In 1988 he left industry to become an associate professor in mathematics and computer science at Biola University, where he remained until he fully retired 2 years ago. He still teaches one or two courses on a part-time basis.

Peter's interest in problem-solving can be traced to a 1974 article in Mathematics Magazine on the arbelos by Leon Bankoff. He subsequently got to know Bankoff and, together with Paul Yiu, C.W. Dodge, and T. Schoch, they published a follow-up paper in Mathematics Magazine. This experience piqued his interest in solving problems, and he has been solving ***CRUX with MAYHEM*** problems since 1998. Most of our readers will recognize Peter's name as a regular and prolific contributor of solutions.

Now that he has retired, Peter currently spends a lot of time in China. During his travels, he became aware of the lack of educational opportunities available to many children there. Peter himself has identified a number of orphans and poor children who are eager to have a chance to go to high school, yet who cannot afford to go. On returning to America, he has been active in encouraging others to provide support for such children.

SKOLIAD No. 107

Robert Bilinski

Please send your solutions to the problems in this edition by ***July 1, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 1** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Erratum: the contest in the December 2007 Skoliad was the 2007 Maritime Contest, not the 2006 Maritime Contest.

Our problems this month come from the Collège Montmorency Contest, 2005–06. Our thanks go to André Labelle, Collège Montmorency, who looks after this contest designed for secondary students in the Laval region.

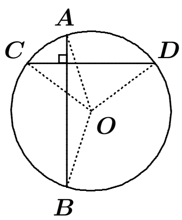
Montmorency Contest 2005–06
Sec V, November 2005

1. Evaluate the following, giving the answer as a fraction:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{2005^2}\right).$$

2. In a circle having centre O, the chords AB and CD are perpendicular to each other and neither chord passes through the centre. Show that

$$\angle AOD+\angle BOC = 180^\circ .$$

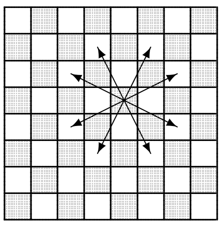


3. Show that for real x and y where $x \neq 0$ and $y \neq 0$, there is no solution for the equation

$$(x+y)^4 = x^4+y^4 .$$

4. (a) How many positions on average can a knight reach in one move on a 8×8 chessboard?

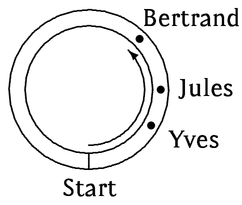
(b) What result do we get for an $n \times n$ chessboard?



The movement of a knight has the shape of an "L": a shift of two squares horizontally followed by one vertically, or vice versa.

5. Show that by placing 5 points inside a right-triangle with sides 6 cm, 8 cm, and 10 cm, at least two of them have a distance smaller than 5 cm.

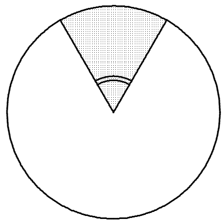
6. Three runners, Yves, Jules, and Bertrand meet one day at a circular track. Knowing that they start running at the same moment, that Bertrand, the fastest of the three, does a full revolution of the track in 2 minutes and that Yves does it in 8 minutes, how long does it take Jules to do a full revolution, if we further note that all three runners meet at the same place on the track before Yves, the slowest of the three runners, has done a full revolution?



7. How many numbers are there smaller than 10,000 that contain the digit 7 at least once?

8. We build a cone from a circular piece of cardboard having a radius of 10 cm by cutting out a sector from it. Determine the volume of the cone we obtain as a function of the angle, in radians, at the center of the sector.

Recall that Volume $= \frac{1}{3}\pi r^2 h$, where r is the radius of the base of the cone, h is the altitude of the cone, and 2π radians $= 360°$.



Concours Montmorency 2005–06

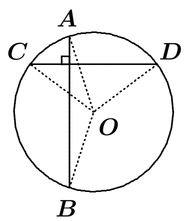
Sec V, novembre 2005

1. Effectuer et donner sous la forme fractionnaire :

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{2005^2}\right).$$

2. Dans un cercle de centre O, les cordes AB et CD sont perpendiculaires entre-elles et ne passent pas (ni l'une ni l'autre) par le centre. Montrez que

$$\angle AOD + \angle BOC \;=\; 180° \,.$$

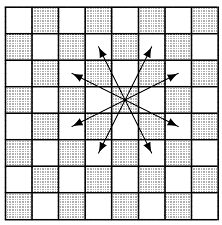


3. Montrez que pour x et y réels où $x \neq 0$ et $y \neq 0$, il n'existe pas de solution pour l'équation

$$(x+y)^4 \;=\; x^4 + y^4 \,.$$

4. (a) Combien de cases un cavalier peut-il atteindre en moyenne en un coup sur un échiquier 8×8 ?

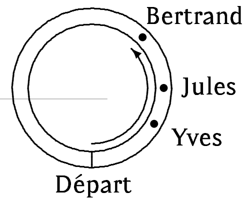
(b) Qu'en est-il sur un échiquier $n \times n$?



Le mouvement du cavalier est en forme de "L" : un déplacement de deux cases horizontalement suivi d'une case verticalement ou inversement.

5. Montrez qu'en plaçant cinq points à l'intérieur d'un triangle rectangle de côté 6 cm, 8 cm et 10 cm, il y en a au moins deux distant de moins de 5 cm.

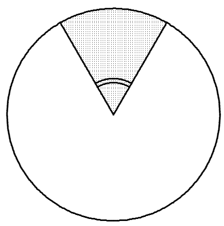
6. Trois coureurs : Yves, Jules et Bertrand arrivent un jour à une piste d'athlétisme circulaire. Sachant qu'ils partent tous ensemble, que Bertrand, le plus rapide des trois, fait le tour de la piste en 2 minutes et qu'Yves l'effectue en 8 minutes. En combien de temps Jules fait-il le tour de cette piste, si on s'aperçoit que les trois amis se rencontrent à un certain endroit de la piste avant même qu'Yves, le coureur le plus lent, en ait effectué un tour complet ?



7. Combien existe t'il de nombres entiers inférieurs à 10 000 qui contiennent au moins une fois le chiffre 7 ?

8. On fabrique un cône à partir d'un morceau de carton circulaire de 10 cm de rayon en lui coupant un secteur. Déterminer le volume du cône obtenu en fonction de l'angle au centre en radians du secteur.

On rappel que Volume $= \frac{1}{3}\pi r^2 h$ où r est le rayon de las base du cône, h est la hauteur du cône et 2π radians $= 360°$.



Next we give the solutions to the 2006 British Columbia Colleges Senior High School Mathematics Contest [2007 : 193–195].

1. Determine the number of sequences of consecutive integers whose sum is 100.

Official solution.

Let k be the first integer in the sequence and n be the number of integers in the sequence. Then the sum of the consecutive integers is

$$\begin{aligned} k + (k+1) + (k+2) + \cdots + (k+n-1) &= nk + 1 + 2 + \cdots + n - 1 \\ &= nk + \tfrac{1}{2}n(n-1) \;=\; 100\,; \end{aligned}$$

that is, $2nk + n(n-1) = n(2k+n-1) = 200$, or

$$2k + n - 1 \;=\; \frac{200}{n}\,.$$

Thus, n must be a divisor of 200 with $n > 1$; that is,

$$n \in \{2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200\}.$$

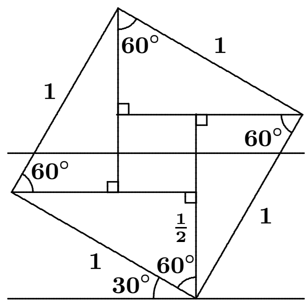
Consider the following table:

n	Equation for k	k
2	$2k+1=100$	no integer solution
4	$2k+3=50$	no integer solution
5	$2k+4=40$	$k=18$
8	$2k+7=25$	$k=9$
10	$2k+9=20$	no integer solution
20	$2k+19=10$	no integer solution
25	$2k+24=8$	$k=-8$
40	$2k+39=5$	$k=-17$
50	$2k+49=4$	no integer solution
100	$2k+99=2$	no integer solution
200	$2k+199=1$	$k=-99$

Thus, there are five sequences of consecutive integers whose sum is 100. Note that the sequence for $n=25$ is the sequence for $n=8$ with the 17 consecutive integers from -8 to 8, which add up to zero, included. In the same way, the sequence for $n=40$ is the sequence for $n=5$ with the 35 consecutive integers from -17 to 17, which add up to zero, included. There are five sequences of consecutive integers whose sum is 100.

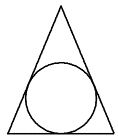
2. A cubical glass tank with sides of length one metre is placed on a horizontal table and half filled with water. Thus, the depth of the water in the tank (the distance of the surface of the water from the surface of the table) is one half metre. The tank is rotated about one of the edges that is on the table so that one face of the tank makes a 30° angle with the table. Find the depth of the water in the tank after the rotation.

Solution by Jaclyn Chang, student, John Ware Junior High School, Calgary, AB, modified by the editor.



Let us represent a cross section of the cube by the square in the diagram. Any straight line that divides a square's area into two equal parts has to pass through the centre by symmetry. The water level in the square is parallel to the table surface, and must pass through the centre of the square. Let us draw four 30°–60°–90° triangles inside the square, as shown. Then by symmetry, the inner square is also cut in two by the water line, and two of the inner square's edges are parallel to the table surface because of the angles in the situation. Since the hypotenuse of each of the four triangles is 1, the other sides are $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$. This means that the sides of the inner square have length $\frac{\sqrt{3}-1}{2}$. The depth of the water is therefore $\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}-1}{2}\right)=\frac{1+\sqrt{3}}{4}$.

3. The lengths of the sides of a triangle are 13, 13, and 10. The *inscribed circle* of this triangle is the circle with centre inside the triangle that is tangent to each of the three sides of the triangle. (See the diagram.) Find the radius of the inscribed circle.



Solution by Jaclyn Chang, student, John Ware Junior High School, Calgary, AB, modified by the editor.

Connecting the centre of the inscribed circle to the vertices of the triangle, one can split the triangle into three triangles: two with a base of 13 and one with a base of 10. Each of these three new triangles will have the same height (which is the radius of the inscribed circle). Since the given triangle is isosceles, we can apply the Pythagorean Theorem to either right triangle obtained by bisecting the apex to see that the height of the given triangle is 12, implying that the area is 60. Computing and summing the areas of each of the three newly created triangles, we get

$$\begin{aligned} \tfrac{1}{2}\cdot 13r + \tfrac{1}{2}\cdot 13r + \tfrac{1}{2}\cdot 10r &= 60\,, \\ 13r + 5r &= 60\,, \\ 18r &= 60\,, \\ r &= \tfrac{10}{3}\,. \end{aligned}$$

The radius of the inscribed circle is $\frac{10}{3}$.

4. Five positive integers a, b, c, d, and e greater than one make the following conditions true:

$$\begin{aligned} a(b+c+d+e) &= 128\,, \\ b(a+c+d+e) &= 155\,, \\ c(a+b+d+e) &= 203\,, \\ d(a+b+c+e) &= 243\,, \\ e(a+b+c+d) &= 275\,. \end{aligned}$$

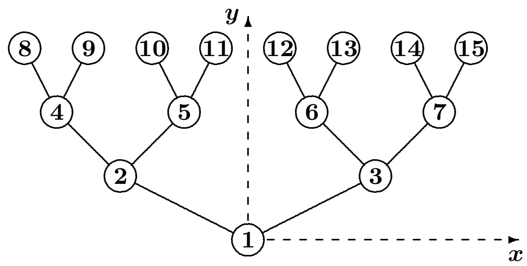
Find the five integers.

Official solution.

First note that since a, b, c, d, and e are integers greater than 1; that is, a, b, c, d, $e \geq 2$, the sum of any four of them is at least 8. Then, since $b(a+c+d+e) = 155 = 5\cdot 31$, where 5 and 31 are prime, we must have $b = 5$ and $a+c+d+e = 31$, since $a+c+d+e \geq 8$. In the same way, $c(a+b+d+e) = 203$ gives $c = 7$ and $a+b+d+e = 29$. Either of these results yields $a+d+e = 24$, which implies that $a+b+c+d+e = 36$. Then the equation $a(b+c+d+e) = 128$ becomes $a(36-a) = 128$, or $a^2 - 36a + 128 = (a-32)(a-4) = 0$. The only possible solution is $a = 4$, since $a = 32$ makes $a+b+c+d+e \geq 40$. Therefore, $a+b+c = 16$ and the equation $e(a+b+c+d) = 275$ becomes $e(16+d) = 275$ where

$d + e = 36 - a - b - c = 20$. Since $275 = 11 \cdot 25$ and $16 + d \geq 18$, we must have $e = 11$ and $d = 25 - 16 = 9$. Note that the other factoring of $275 = 5 \cdot 55$ yields $d = 39$, which is too large. The five numbers are $a = 4$, $b = 5$, $c = 7$, $d = 9$, and $e = 11$.

5. A full binary tree consists of a root **node** which has two **children**, a right child node and a left child node, and each child node has two children, until the top of the tree is reached, where each node has no children. In a certain full binary tree each node is numbered, starting with 1 at the root, numbering from left to right across each level. The diagram shows the first four levels of such a tree. The root of the tree is placed at the origin of an xy–coordinate system, with the x–axis horizontal and the y–axis vertical, as shown. If the spacing between the levels of the tree is 2 units in the y–direction and the spacing between the nodes at the top level that contains the node numbered 2006 is 2 units in the x–direction, find the coordinates of the node numbered 2006.



Solution by Jaclyn Chang, student, John Ware Junior High School, Calgary, AB, modified by the editor.

The left-most number on each level is a power of two. Specifically, the left-most number on level n is 2^{n-1}. Since 2006 is between $1024 = 2^{10}$ and $2047 = 2^{11} - 1$, it appears on level 11; thus, the y-coordinate is $2 \cdot 10 = 20$. Since $1536 = 1024 + 512 = 2048 - 512$, the number 1536 is 1 unit to the right of the y-axis on level 11. Since 2006 is 470 greater than 1536, the x-coordinate is $1 + 2 \cdot 470 = 941$. Therefore, the coordinates of node 2006 are $(941, 20)$.

That brings us to the end of another issue. This month's winner of a past volume of Mayhem is Jaclyn Chang. Congratulations, Jaclyn! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Please send your solutions to the problems in this edition by **1 May 2008**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M326. *Proposed by the Mayhem Staff.*

The notation $\underline{a}89\underline{b}$ means the four-digit (base 10) integer whose thousands digit is a, whose hundreds digit is 8, whose tens digit is 9, and whose units digit is b. Determine all pairs of non-zero digits a and b such that $\underline{a}89\underline{b} - 5904 = \underline{b}98\underline{a}$.

M327. *Proposed by Lino Demasi, student, University of Waterloo, Waterloo, ON.*

Kaitlyn bought a new eraser. Her new eraser is in the shape of a rectangular prism. She calculates the lengths of the diagonals of the faces to be 10, $\sqrt{61}$, and $\sqrt{89}$. What is the volume of Kaitlyn's eraser?

M328. *Proposed by Hugo Cuéllar, Columbia Aprendiendo, Zipaquirá, Colombia.*

Prove that, if from any positive integer we subtract the sum of each of its digits raised to any odd power (not necessarily the same), then the result is always a multiple of 3.

M329. *Proposed by the Mayhem Staff.*

Determine the value of

$$\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 89^\circ + \cos^2 90^\circ .$$

M330. *Proposed by the Mayhem Staff.*

If n is a positive integer, the n^{th} triangular number is defined as $T_n = 1 + 2 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)$. Determine all pairs of triangular numbers whose difference is 2008.

M331. *Proposed by the Mayhem Staff.*

In trapezoid $ABCD$, side AB is parallel to DC, and diagonals AC and BD intersect at P.

(a) If the area of $\triangle APB$ is 4 and the area of $\triangle DPC$ is 9,

(i) prove that $AP : PC = 2 : 3$,

(ii) explain why the ratio of the area of $\triangle BPC$ to the area of $\triangle BPA$ equals $3 : 2$, and

(iii) determine the area of trapezoid $ABCD$.

(b) If the area of $\triangle APB$ is x and the area of $\triangle DPC$ is y, determine the area of trapezoid $ABCD$ in terms of x and y.

. . . .

M326. *Proposé par l'Équipe de Mayhem.*

L'expression $\underline{a}89\underline{b}$ dénote un entier de quatre chiffres (écrit en base 10) dont le chiffre des milliers est a, celui des centaines est 8, celui des dizaines est 9, et celui des unités est b. Déterminer toutes les paires de chiffres non nuls a et b tels que $\underline{a}89\underline{b} - 5904 = \underline{b}98\underline{a}$.

M327. *Proposé par Lino Demasi, étudiant, Université de Waterloo, Waterloo, ON.*

Catherine s'est acheté une nouvelle gomme à effacer, en forme de prisme rectangulaire, dont les diagonales des faces mesurent 10, $\sqrt{61}$, et $\sqrt{89}$. Quel est le volume de la gomme de Catherine ?

M328. *Proposé par Hugo Cuéllar, Columbia Aprendiendo, Zipaquirá, Colombie.*

Montrer que si l'on soustrait d'un entier positif arbitraire, la somme de chacun de ses chiffres élevé à une puissance impaire quelconque (pas nécessairement la même),on obtient toujours un multiple de 3.

M329. *Proposé par l'Équipe de Mayhem.*

Déterminer la valeur de

$$\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 89^\circ + \cos^2 90^\circ .$$

M330. *Proposé par l'Équipe de Mayhem.*

Soit n un entier positif. On définit le $n^{\text{ième}}$ nombre triangulaire T_n comme la somme $T_n = 1 + 2 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)$. Déterminer toutes les paires de nombres triangulaires dont la différence est 2008.

M331. *Proposé par l'Équipe de Mayhem.*

Soit un trapèze $ABCD$ dont les côtés AB et DC sont parallèles et les diagonales AC et BD se coupent en P.

(a) Si l'aire du triangle APB est 4 et celle du triangle DPC est 9,

(i) montrer que $AP : PC = 2 : 3$,

(ii) expliquer pourquoi le rapport des surfaces des triangles BPC et BPA est égal à $3 : 2$, et

(iii) déterminer l'aire du trapèze $ABCD$.

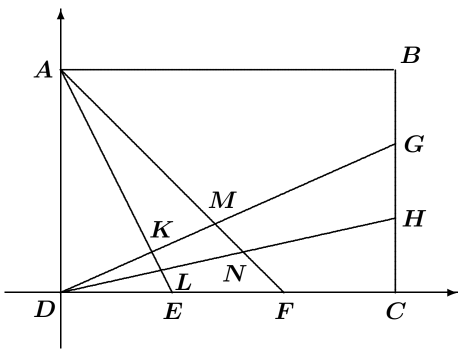
(b) Si l'aire du triangle APB est x et celle du triangle DPC est y, déterminer l'aire du trapèze $ABCD$ en fonction de x et y.

Mayhem Solutions

M276. *Proposed by Babis Stergiou, Chalkida, Greece.*

In rectangle $ABCD$, points E and F divide side DC into three equal parts $DE = EF = FC$, and points G and H divide side BC into three equal parts $BG = GH = HC$. The line AE cuts the lines DG and DH at points K and L, respectively. Similarly, the line AF cuts the lines DG and DH at points M and N, respectively. Show that $KN \parallel CD$.

Combined solution from George Apostolopoulos, Mesologi, Greece; Taichi Maekawa, Takatsuki City, Osaka, Japan; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Justin Yang, student, Lord Byng Secondary School, Vancouver, BC.



Rectangle $ABCD$ is placed into the Cartesian plane such that its vertices are as follows: $A(0,3b)$, $B(3a,3b)$, $C(3a,0)$, and $D(0,0)$. The coordinates of the remaining points are then $G(3a,2b)$, $H(3a,b)$, $F(2a,0)$, and $E(a,0)$. The equations of lines DH and AF are obtained as $y = \frac{b}{3a}\,x$ and $y = -\frac{3b}{2a}\,x + 3b$, respectively. Solving these, we get the y-coordinate of their point of intersection, N, as $y = \frac{6b}{11}$. Similarly, the equations of lines DG and AE are found to be $y = \frac{2b}{3a}\,x$ and $y = -\frac{3b}{a}\,x + 3b$, respectively. This yields the y-coordinate of their point of intersection, K, as $y = \frac{6b}{11}$.

Hence, since points K and N are equidistant from side DC, line segments KN and DC must be parallel.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comănești, Romania.

M277. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most two candies. For example, if $n = 3$, then $f(3,7) = 0$, $f(3,6) = 1$, and $f(3,4) = 6$. Determine the value of

$$f(2007,1) + f(2007,4) + f(2007,7) + \cdots + f(2007,4012)\,.$$

Solution by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

First note that $f(n,1) = n$ and $f(n,2) = n + C_2^n$ for integers $n \geq 1$. Define $f(n,k) = 0$ for integers $n \geq 1$ and $k < 0$. Then we can arrange the $f(n,k)$s as:

$$\begin{array}{ccccccc} \cdots & f(1,-1) & f(1,0) & f(1,1) & f(1,2) & f(1,3) & \cdots \\ \cdots & f(2,0) & f(2,1) & f(2,2) & f(2,3) & f(2,4) & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

or equivalently as:

$$\begin{array}{ccccccc} \cdots & 0 & 1 & 1 & 1 & 0 & \cdots \\ \cdots & 1 & 2 & 3 & 2 & 1 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The number of ways of distributing k candies to n children is the same as the number of ways of distributing 0 candies to a certain child while distributing k candies to the $n-1$ others, plus the number of ways of distributing 1 candy to the same child while distributing $k-1$ candies to the others, plus the number of ways of distributing 2 candies to the same child while distributing $k-2$ candies to the others; that is,

$$f(n,k) \;=\; f(n-1,k) + f(n-1,k-1) + f(n-1,k-2)\,.$$

Let R_n be the sum of the numbers in the n^{th} row of the above table for integers $n \geq 1$. Then $R_{n+1} = 3R_n$ for integers $n \geq 1$, and since $R_1 = 3$, we get $R_n = 3^n$ for integers $n \geq 1$. Finally, we find that $f(2006,k) = 0$ for integers $k < 0$ or $k > 4012$. Therefore,

$$\begin{aligned} f(2007,1) &+ f(2007,4) + f(2007,7) + \cdots + f(2007,4012) \\ &= \sum_{k=0}^{4012} f(2006,k) \;=\; R_{2006} \;=\; 3^{2006}\,. \end{aligned}$$

Also solved by DIVYANSHU RANJAN, high school student, Delhi, India; and JUSTIN YANG, student, Lord Byng Secondary School, Vancouver, BC. One incomplete solution was also submitted.

M278. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Find sixteen 16–digit palindromes, in each of which the product of the non-zero digits and the sum of the digits are both equal to 16. How many such numbers are there?

A composite of solutions from Daniel Tsai, student, Taipei American School, Taipei, Taiwan; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Justin Yang, student, Lord Byng Secondary School, Vancouver, BC.

A 16–digit palindrome must have the form $d_1d_2\ldots d_8d_8\cdots d_2d_1$, with $d_i \in \{0, 1, 2, \ldots, 9\}$ for $1 \le i \le 8$, and $d_1 \neq 0$. For the sum of its digits to be 16, the sum $d_1 + d_2 + \cdots + d_8$ must be 8. Simultaneously, for the product of the non-zero digits of the palindrome to be 16, the product of the non-zero digits among its first eight digits must be 4. It can easily be seen that the first eight digits of the required palindromes must be arrangements, either of digits 2, 2, 1, 1, 1, 1, 0, 0 or of digits 4, 1, 1, 1, 1, 0, 0, 0, with 0 excluded as first digit. The following are 16 such palindromes:

4111100000011114 ,	1411100000011141 ,
1141100000011411 ,	1114001001004111 ,
1100141001410011 ,	1010410110140101 ,
1100140110410011 ,	1004111001114001 ,
1041110000111401 ,	4000111111110004 ,
2211110000111122 ,	2121110000111212 ,
2112110000112112 ,	2111210000121112 ,
2111120000211112 ,	2011121001211102 .

The total number of such palindromes, with 0 excluded as first digit, is obtained as follows:

If the first 8 digits of the palindrome contain two 2s, four 1s, and two 0s, then the number of arrangements of these 8 digits is

$$\frac{6(7!)}{2!2!4!} = 315\,.$$

If the first 8 digits of the palindrome contain one 4, four 1s, and three 0s, then the number of arrangements is

$$\frac{5(7!)}{4!3!} = 175\,.$$

Thus, there are $315 + 175 = 490$ such numbers.

Also solved by ELIAS C. BUISSANT DES AMORIE, CJ Castricum, the Netherlands; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA. There were also two incorrect solutions submitted.

M279. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine an infinite set of rational number solutions (α, β) to the equation $\alpha^2 + \beta^2 = \alpha^3 + \beta^3$.

Composite solution from Hasan Denker, Istanbul, Turkey; and Geoffrey A. Kandall, Hamden, CT, USA.

The given equation is equivalent to

$$\begin{aligned}(\alpha+\beta)^2 - 2\alpha\beta &= \alpha^2+\beta^2 = \alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2) \\ &= (\alpha+\beta)\big[(\alpha+\beta)^2 - 3\alpha\beta\big] .\end{aligned}$$

Setting $\alpha + \beta = p$ and $\alpha\beta = q$, we obtain $p^2 - 2q = p(p^2 - 3q)$. Solving this equation for q yields

$$q = \frac{p^2(p-1)}{3p-2} . \qquad (1)$$

Note that α and β are roots of the quadratic equation $x^2 - px + q = 0$. The discriminant of this equation is $D = p^2 - 4q$. Using (1), we obtain

$$D = \frac{p^2(2-p)}{3p-2} .$$

Since we want α and β to be rational numbers, we must have $\sqrt{D}$ rational. Let $D = p^2k^2$, where $k \in \mathbb{Q}$ and $k^2 = \dfrac{2-p}{3p-2}$. Solving this equation for p yields $p = \dfrac{2(1+k^2)}{1+3k^2}$.

Hence, an infinite set of solutions to the equation $\alpha^2 + \beta^2 = \alpha^3 + \beta^3$ is given by the set

$$\begin{aligned}&\Big\{(\alpha,\beta) = \Big(\tfrac{p \pm pk}{2}, \tfrac{p \mp pk}{2}\Big) : k \in \mathbb{Q}\Big\} \\ &\quad = \left\{(\alpha,\beta) = \left(\frac{(1+k^2)(1\pm k)}{1+3k^2}, \frac{(1+k^2)(1\mp k)}{1+3k^2}\right) : k \in \mathbb{Q}\right\} .\end{aligned}$$

Also solved by MIHÁLY BENCZE, Brasov, Romania; ELIAS C. BUISSANT DES AMORIE, CJ Castricum, the Netherlands; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VEDULA N. MURTY, Dover, PA, USA; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan. There was one incorrect solution submitted.

M280. *Proposed by the Mayhem Staff.*

An equilateral triangle lies in the plane with two of its vertices at points $(0, 0)$ and $(0, n)$. Determine the number of points (x, y) with integer coordinates which lie in the interior of the triangle.

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

We will consider the triangle with the third vertex in the first quadrant. If $n = 1$, it is clear that there are no points with integer coordinates in the interior of the triangle; if $n = 2$, it is also clear that there is exactly one such point in the interior of the triangle, namely $(1, 1)$. Let us now assume that $n \geq 3$.

Two of the lines which determine the triangle are $y = \left(\tan \frac{\pi}{6}\right)x = \frac{\sqrt{3}}{3}\,x$ and $y = n - \left(\tan \frac{\pi}{6}\right)x = n - \frac{\sqrt{3}}{3}\,x$; thus, the third vertex has coordinates $\left(\frac{\sqrt{3}}{2}\,n, \frac{1}{2}\,n\right)$. For each positive integer i, the number $\sqrt{3}i$ is not an integer; hence, there are $\lfloor\sqrt{3}i\rfloor$ lattice points inside the triangle and on the line $y = i$. By symmetry with respect to $y = n/2$, we see that, for odd n with $n \geq 3$, the number of lattice points is

$$2\sum_{i=1}^{(n-1)/2} \left\lfloor\sqrt{3}i\right\rfloor .$$

In the same way, for even n with $n \geq 4$, the number of lattice points is

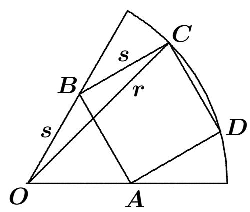
$$\left(2\sum_{i=1}^{(n/2)-1} \left\lfloor\sqrt{3}i\right\rfloor\right) + \left\lfloor\sqrt{3}n/2\right\rfloor .$$

HASAN DENKER, Istanbul, Turkey provided a very good approximation using Pick's Formula. Two incorrect solutions were also submitted.

M281. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A square of side length s is inscribed symmetrically inside a sector of a circle with radius of length r and central angle of $60°$, such that two vertices lie on the straight sides of the sector and two vertices lie on the circular arc of the sector. Determine the exact value of s/r.

Solution by Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA.



By symmetry, we see that $OA = OB$. Since $\angle AOB = 60°$, we see that $\triangle AOB$ is equilateral. Thus, $OB = AB = BC = s$. We can now use the Law of Cosines on $\angle OBC$ to successively obtain:

$$\begin{aligned} r^2 &= s^2 + s^2 - 2s^2 \cos 150^\circ = s^2 \left(2 + 2 \cdot \frac{\sqrt{3}}{2}\right), \\ \left(\frac{r}{s}\right)^2 &= 2 + \sqrt{3}, \\ \frac{s}{r} &= \frac{1}{\sqrt{2+\sqrt{3}}} = \sqrt{2 - \sqrt{3}}. \end{aligned}$$

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; JUSTIN YANG, student, Lord Byng Secondary School, Vancouver, BC; and TITU ZVONARU, Comănești, Romania. Two incorrect solutions were also submitted.

Problem of the Month

Ian VanderBurgh

After a long winter's break, it's time to stop playing games and, well, . . . start playing games!

Problem (2006 Cayley Contest)
Anne and Brenda play a game which begins with a pile of n toothpicks. They alternate turns with Anne going first. On each player's turn, she must remove 1, 3, or 4 toothpicks from the pile. The player who removes the last toothpick wins the game. For which of the values of n from 31 to 35 inclusive does Brenda have a winning strategy?

With this game problem, as with any such problem, the best thing to do initially is to get out a pencil and a piece of paper (or better yet, a pile of toothpicks) and give it a try, starting with some small values for n.

Even before we do this, though, it's probably worth remembering what it means to have a "winning strategy" in such a game. A player has a winning strategy if, regardless of what the other player does, there are moves that she can make that *guarantee* that she will win. (In fact, it's worth checking out some back issues of *Mayhem* at this point—my trusty colleague, John Grant McLoughlin, wrote a couple of *Pólya's Paragon* columns in 2006 about mathematical games [2006 : 275–276, 369–371].)

Let's try some small values of n to see if we can get a feel for this game. We'll abbreviate the players' names (conveniently) as A and B. I would suggest that you get out a pencil and a piece of paper and try the cases $n = 1$ to $n = 6$ before we go through this together.

If $n = 1$, A wins by immediately taking 1 (leaving 0).

If $n = 2$, A cannot take 3 or 4; thus, A must take 1, leaving 1, and B wins by taking 1 (leaving 0).

If $n = 3$ or $n = 4$, A wins by immediately taking 3 or 4, respectively.

The value $n = 5$ is where things start to get more interesting. If A removes 1, B receives a pile of 4, and wins by taking 4 (leaving 0). So A doesn't want to remove 1. If A removes 4, B receives a pile of 1, and wins by taking 1. So A doesn't want to remove 4 either. If A removes 3, B receives a pile of 2. Here, B can only remove 1 (not 3 or 4), and A then receives a pile of 1, and so wins by removing 1.

What does this case tell us? Who has the winning strategy? If A chooses 1 or 4, she loses, but if she chooses 3, she wins. Thus, A has a winning strategy, as she controls her own fate by choosing first (and hopefully choosing 3).

When $n = 5$, A's winning strategy was to remove 3, leaving B with 2. We showed by looking at what B and A can remove that A must win. But is there a way of looking at this that might be easier to generalize? This could be really useful. Think about this as we're looking at the case of $n = 6$.

If $n = 6$, A can remove 1, 3, or 4, leaving B with 5, 3, or 2, respectively. If A removes 1 leaving B with 5, then B could be clever and remove 3, leaving A with 2. But we saw above that A loses when choosing from a pile of size 2. Thus, if A removes 1, then B can force A to lose. If A removes 3 leaving B with 3, then B can remove 3 and win. That is, if A removes 3, then B can force A to lose. If A removes 4 leaving B with 2, then B will lose as the first person choosing from a pile of size 2 will lose.

Wait! That's the key right there. Starting with $n = 6$, A can reduce the pile to 2, 3, or 5. But we've already looked at these cases. The first player to choose should win starting with a pile of 3 or 5, but should be forced to lose starting with a pile of 2. After A has chosen and passed the pile to B, then B will be the first one to choose (the tables have been turned). So if A chooses 4, then B will lose.

Can we generalize this now? Starting with a pile of size n, if A can choose in such a way as to reduce the pile to one where the first player to choose (now B) does not have a winning strategy, then A will win. If all three of the positions to which A can reduce the pile are positions where the first player to choose (now B) has a winning strategy, then A will lose (as B can follow a winning strategy no matter how A chooses initially).

If $n = 7$, then A can remove 1, 3, or 4, leaving B with 6, 4, or 3, respectively, and all three of these possibilities have a winning strategy for the player who chooses first (here, B). Hence, B can force A to lose, so A does not have a winning strategy for $n = 7$.

We're now ready to write down a solution to the original problem. You will see that our solution will be quite short because of all of the work we've done in advance. (This is a great technique—do the legwork beforehand so that things are simpler later on.) If you don't feel like you've got a grip on the problem, then try doing some larger cases, such as $n = 8$ to $n = 12$. Then read on.

Solution: From above, A has a winning strategy if $n = 1, 3, 4, 5, 6$, but does not have a winning strategy if $n = 2$ or $n = 7$.

Starting with a pile of size n, A must reduce the pile to one of size $n-1$, $n-3$, or $n-4$ and pass it to B. If the first person to choose (now B) has a winning strategy starting with a pile of each of these sizes, then A will lose. In other words, if A has a winning strategy starting with piles of size $n-1$, $n-3$, and $n-4$, then A will lose starting with a pile of size n, since B can implement A's strategy for the smaller pile and win, no matter what A does.

If one or more of these pile sizes are such that the first person does not have a winning strategy, then A should reduce to this size, which prevents B from being able to win. Thus, A herself will win.

From $n = 8$, A can reduce to 7, 5, or 4. Since the first player does not win when starting with 7, then A wins for $n = 8$ by taking 1 toothpick and reducing the pile to 7.

For $n = 9$, A can reduce the pile to 8, 6, or 5. Since the first player has a winning strategy for each of these sizes, we see that A loses when $n = 9$.

For $n = 10$ and $n = 11$, A can reduce the pile to 7 by removing 3 and 4, respectively. Since the first player does not have a winning strategy starting with 7, then A wins starting with $n = 10$ and $n = 11$.

For $n = 12$ and $n = 13$, A can reduce the pile to 9 by removing 3 and 4, respectively. Since the first player does not have a winning strategy starting with 9, then A wins starting with $n = 12$ and $n = 13$.

Continuing to examine cases in this way, we can list the winning and losing starting positions for A:

Winning: 15, 17, 18, 19, 20, 22, 24, 25, 26, 27, 29, 31, 32, 33, 34, ...

Losing: 14, 16, 21, 23, 28, 30, 35, ...

Therefore (to answer the question that was asked!), Anne has a winning strategy when n is 31, 32, 33, and 34, and does not when n is 35.

This is an appealing problem in many ways. There are several interesting things to think about—what a winning strategy means, how winning strategies for some positions correspond to winning strategies at other positions, and so on.

There are also some interesting extensions here for those of you who like to look a little bit beyond. Can you figure out who has a winning strategy if $n = 100$? Can you determine a complete list of winning positions for the two players? What happens if the players can remove 1, 2, or 4 instead of 1, 3, or 4? How about 1, 3, or 6? There is always more to think about!

THE OLYMPIAD CORNER

No. 267

R.E. Woodrow

How the time seems to fly by! Here it is another year past and the start of a new volume of ***CRUX with MAYHEM***, with a new in-coming Editor-in-Chief—Václav (Vazz) Linek—coming on board to take over leadership. I would like to take this opportunity to express my particular thanks to Jim Totten, the out-going Editor-in-Chief and to his Associate Editor, Bruce Crofoot, for the careful proof-reading and correction of the typos and errors I let slip through. It is also appropriate, I think, to thank those who have contributed problem sets, comments, solutions and generalizations for our use in the *Corner*.

Houda Anoun
Miguel Amengual Covas
Mohammed Aassila
Michel Bataille
Robert Bilinski
David Bradley
Pierre Bornsztein
Ricardo Barroso Campos
Bruce Crofoot
José Luis Díaz-Barrero
J. Chris Fisher
Ovidiu Furdui
Joan P. Hutchison
Geoffrey A. Kandall
Ioannis Katsikis
Matti Lehtinen
Andy Liu
Pavlos Maragoudakis
Vedula N. Murty
Henry Ricard
D.J. Smeenk
Christopher Small
Jim Totten
Edward T.H. Wang
Li Zhou

And a special thanks to Joanne Canape, whose skill at turning my scribbles, notes, requests, and pleas into a clean well-presented LaTeX file, usually on short notice, continues to amaze me.

As a first problem set this number, we give the problems of the Italian Team Selection Test given at Pisa in May 2005. My thanks go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

ITALIAN TEAM SELECTION TEST
20-21 May 2005, Pisa

1. A stage course is attended by n students ($n \geq 4$). The day before the final test, each group of three students conspires against another student, to throw him or her out of the competition. Prove that there is a student against whom there are at least $\sqrt[3]{(n-1)(n-2)}$ conspirators. [From Slovenia 2004.]

2. (a) Prove that in a triangle the sum of the distances from the centroid to the three sides is greater than or equal to three times the radius of the incircle. Determine when exact equality holds.

(b) Determine the points in a triangle such that the sum of the distances from the sides is minimal.

3. For a positive integer n, let $\psi(n) = \sum_{k=1}^{n} \gcd(k, n)$.

(a) Prove that $\psi(mn) = \psi(m)\psi(n)$ for m and n relatively prime positive integers.

(b) Prove that, for every positive integer a, the equation $\psi(x) = ax$ has at least one solution.

[From IMO Short List 2004.]

4. Let $S_n = \{1, 2, \ldots, n\}$. Let $f : S_{1600} \to S_{1600}$ be such that

$$f^{(2005)}(x) = x\,, \quad \text{for } x = 1, 2, \ldots, 1600 \text{ and } f(1) = 1. \qquad (1)$$

(a) Prove that f has at least another fixed point.

(b) Determine those $n > 1600$ such that any $f : S_n \to S_n$ satisfying (1) has at least two fixed points.

5. A circle γ and a line ℓ have no points in common. Let AB be the diameter of γ perpendicular to ℓ, with B closer to ℓ than A. Let C be a point on γ different from A and B. The line AC intersects ℓ at D. The line DE is tangent to γ at E, with B and E on the same side of AC. The line BE intersects ℓ at F, and G is the other intersection of the line AF with γ. Let H be symmetric to G with respect to AB. Prove that F, C, and H are on the same line. [From IMO Short List 2004.]

6. Let N be a positive integer. Alberto and Barbara write a number on the blackboard taking turns, according to the following rules: Alberto starts by writing 1 on the blackboard; subsequently, if a player wrote a number n on the blackboard, then, on the next move, the other player chooses to write either $n + 1$ or $2n$ as long as the number is not greater than N. The player who writes N on the blackboard wins.

(a) Determine which player has a winning strategy for $N = 2005$.

(b) Determine which player has a winning strategy for $N = 2004$.

(c) Determine for how many integers N, $1 \le N \le 2005$, Barbara has a winning strategy. [From IMO Short List 2004.]

Next, we give the 11th Form of the Final Round of the XXXI Russian Mathematical Olympiad for 2004–2005. My thanks again go to Felix Recio, Canadian Team Leader to the IMO in Mexico, for collecting them for our use.

XXXI RUSSIAN MATHEMATICAL OLYMPIAD
Final Round - 11th Form

1. *(I. Rubanov)* Let $\{a_1, a_2, \ldots, a_{50}, b_1, b_2, \ldots, b_{50}\}$ be a set of of 100 real numbers. Suppose that the equation

$$|x - a_1| + \cdots + |x - a_{50}| \;=\; |x - b_1| + \cdots + |x - b_{50}|$$

has N solutions (N is finite). Find the maximal value of N.

2. *(I. Bogdanov)* Different numbers are written on the reverse sides of 2005 cards (one number on each card). In one step, one may select three cards and be told the set of three numbers written on them. Find the least number of steps necessary to determine with certainty which number is written on each card.

3. *(L. Emelyanov)* The excircles of $\triangle ABC$ touch the corresponding sides at A', B', and C'. The circumcircles of triangles $A'B'C$, $AB'C'$, and $A'BC'$ meet the circumcircle of $\triangle ABC$ for the second time at C_1, A_1, and B_1, respectively. Prove that $\triangle A_1B_1C_1$ is similar to the triangle whose vertices are the tangent points of the incircle of $\triangle ABC$ with its sides.

4. *(V. Senderov)* Positive integers x, y, and z (where $x > 2$, $y > 1$) satisfy $x^y + 1 = z^2$. Let p denote the number of different prime divisors of x, and let q denote the number of different prime divisors of y. Prove that $p \geq q + 2$.

5. *(N. Agakhanov)* Does there exist a bounded function $f : \mathbb{R} \to \mathbb{R}$ with $f(1) > 0$ such that

$$f^2(x + y) \;\geq\; f^2(x) + 2f(xy) + f^2(y)$$

for all x, $y \in \mathbb{R}$?

6. *(A. Akopyan)* The edges of 12 rectangular parallelepipeds $P_1, P_2, \ldots, P_{12}$ are parallel to the x-, y-, and z-axes. Is it possible that P_2 intersects (that is, has a point in common with) each of the other 11 parallelepipeds except for P_1 and P_3; P_3 intersects each of the others except for P_2 and P_4; $\ldots$; P_{12} intersects each of the others except for P_{11} and P_1; and P_1 intersects each of the others except for P_{12} and P_2? (The surface of a parallelepiped belongs to the parallelepiped.)

7. *(A. Zaslavsky, M. Isaev, and D. Tsvetov)* Let $ABCD$ be a quadrilateral having an incircle and having no two sides parallel. Let O be its incentre. Prove that O lies on two lines joining the mid-points of the opposite sides of $ABCD$ if and only if $OA \cdot OC = OB \cdot OD$.

8. *(S. Berlov)* Seated at a round table are 100 representatives of 25 countries (four persons from each country). Prove that they can be partitioned into four groups satisfying the following conditions: (i) each group contains a representative of each country, (ii) each person has no neighbours among the members of his group.

Our last set of problems comes from the Selected Problems from the Taiwan Mathematical Olympiad of July 2005. Again thanks go to Felix Recio for collecting them for us.

TAIWAN MATHEMATICAL OLYMPIAD
Selected Problems
July 6, 2005

1. A $\triangle ABC$ is given with side lengths a, b, and c. A point P lies inside $\triangle ABC$, and the distances from P to three sides are p, q, and r, respectively. Prove that

$$R \leq \frac{a^2+b^2+c^2}{18\sqrt[3]{pqr}},$$

where R is the circumradius of $\triangle ABC$. When does equality hold?

2. Suppose that G is a graph of order n which does not contain a complete graph of order k as a subgraph. Prove that G contains at most

$$\frac{k-2}{k-1}\cdot\frac{n^2-r^2}{2}+\binom{r}{2}$$

edges, where $n \equiv r \pmod{(k-1)}$ and $0 \leq r \leq k-2$.

3. The IMO will take place on the 13th and 14th of July 2005. Prove that the sum

$$\sum_{\substack{1\leq i,j,k\leq 3\\ i\neq j\neq k}} \csc^{13}\left(\frac{2^i\pi}{7}\right)\csc^{14}\left(\frac{2^j\pi}{7}\right)\csc^{2005}\left(\frac{2^k\pi}{7}\right)$$

is a rational number.

4. Let $a \bmod b$ denote $a - b\left\lfloor\frac{a}{b}\right\rfloor$; that is, the remainder when a is divided by b. Find all solutions in positive integers (x,y,z) to the equations

$$xy \bmod z \;=\; yz \bmod x \;=\; zx \bmod y \;=\; 2\,.$$

5. Given $\triangle ABC$, a circle Γ with centre O passes through B and C and intersects sides AC and AB at points D and E, respectively. Let F be the intersection of BD and CE. The line OF intersects the circumcircle of $\triangle ABC$ at P. Prove that the incentre of $\triangle PBD$ coincides with the incentre of $\triangle PCE$.

6. Find all positive integers $n \geq 3$ with the following property: there exists a positive integer M_n such that, for any given n positive real numbers $a_1, a_2, \ldots, a_n$, the following equality holds:

$$\frac{a_1 + a_2 + \cdots + a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}} \leq M_n \left(\frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_{n-1}} + \frac{a_1}{a_n} \right) .$$

We have had some readers challenge the value 47 as the least possible answer to problem 4 of the Category B Belarus Mathematical Olympiad [2006 : 438; 2007 : 421].

4. (*I. Voronovich*) Pairwise distinct positive integers a, b, c, d, e, f, g, h, and n satisfy the equalities $n = ab + cd = ef + gh$.

Find the smallest possible value of n.

Comments and solutions by Stan Wagon, Macalester College, St. Paul, MN, USA; and John P. Robertson, National Council on Compensation Insurance, Inc., Boca Raton, FL, USA.

Using a computer search for other solutions to $n = ab + cd = ef + gh$, Wagon found that $18 = 2 \cdot 9 = 3 \cdot 6$ and $20 = 1 \cdot 20 = 4 \cdot 5$, which implies that $38 = 2 \cdot 9 + 1 \cdot 20 = 3 \cdot 6 + 4 \cdot 5$, improving considerably on 47. Robertson did a more general search finding $31 = 1 \cdot 7 + 4 \cdot 6 = 2 \cdot 8 + 3 \cdot 5$. Robertson notes this uses the smallest possible set of a, b, $\ldots$, h (smallest in the sense of lexicographic ordering of possible sorted a, b, $\ldots$, h. He generated all $n = ab + cd$ for $a = 4$ to 50, with b, c, d smaller than a and distinct, and looked for an n with two representations using 8 distinct elements. Since $50 > 31$ and $n > a$, there cannot be any solution smaller than 31. The checking involved about 700,000 cases, taking a minute or two on a personal computer.

Next we give the solution by Przemyslaw Mazur, Jan Sobieski High School, Krakow, Poland.

The answer is $31 = 8 \cdot 2 + 3 \cdot 5 = 6 \cdot 4 + 1 \cdot 7$.

To prove minimality we will show that:

(i) The minimum possible value of $S = ab + cd + ef + gh$ is 60.

(ii) In the case where $S = 60$, the terms in the sum are 8, 14, 18, and 20. Therefore, there is no way to rearrange them into pairs giving the same sums.

To prove both claims, let us assume that a, b, c, d, e, f, g, and h minimize the sum S. If any of the eight numbers—let us call it x—is greater than 8, then one of the numbers from $\{1, 2, 3, 4, 5, 6, 7, 8\}$ must remain unused—let us call it y. Replacing x with y decreases S, a contradiction.

This means that $\{a, b, c, d, e, f, g, h\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ which are grouped into four pairs that give the terms. Assume that 8 is paired with something other than 1, say 8 is paired with w. Naturally, 1 is paired with yet another number, say z. Changing pairs $(8, w)$ and $(1, z)$ into $(8, 1)$ and (w, z) decreases S (because $w > 1$ and $z < 8$), a contradiction.

Repeating the above argument, we prove that the pairs are $(8, 1)$, $(7, 2)$, $(6, 3)$, and $(5, 4)$. This completes the proof.

Next we turn to readers' solutions to problems from the Hungarian National Olympiad 2003–2004, Grades 11–12, Round 2, given in [2007 : 83].

1. Let n be an integer, $n > 1$. Define

$$\begin{aligned} A &= \frac{\sqrt{n+1}}{n} + \frac{\sqrt{n+4}}{n+3} + \frac{\sqrt{n+7}}{n+6} + \frac{\sqrt{n+10}}{n+9} + \frac{\sqrt{n+13}}{n+12} \\ \text{and} \quad B &= \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n+2}} + \frac{1}{\sqrt{n+5}} + \frac{1}{\sqrt{n+8}} + \frac{1}{\sqrt{n+11}}\,. \end{aligned}$$

Determine which of the following relations holds (depending on n): $A > B$, $A = B$, or $A < B$.

Solved by Houda Anoun, Bordeaux, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Ioannis Katsikis, Athens, Greece; Andrea Munaro, student, University of Trento, Trento, Italy; Vedula N. Murty, Dover, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Anoun's write-up.

Let p be a positive integer such that $p > 1$. We can easily prove that

$$\frac{\sqrt{p+1}}{p} < \frac{1}{\sqrt{p-1}}\,. \tag{1}$$

In fact, this is equivalent to $(p-1)(p+1) = p^2 - 1 < p^2$, which is true.

By applying (1), we have for each $i \in \{0, \ldots, 4\}$,

$$\frac{\sqrt{n+3i+1}}{n+3i} < \frac{1}{\sqrt{n+3i-1}}\,;$$

hence, we can straightforwardly deduce that $A < B$. This result can be generalized as follows:

$$\sum_{i=0}^{k} \frac{\sqrt{n+3i+1}}{n+3i} < \sum_{i=1}^{k} \frac{1}{\sqrt{n+3i-1}}\,.$$

2. Let a, b, and c denote the sides of a triangle opposite the angles A, B, and C, respectively. Let r be the inradius and R the circumradius of the triangle. If $\angle A \geq 90^\circ$, prove that

$$\frac{r}{R} \leq \frac{a \sin A}{a+b+c}\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Ioannis Katsikis, Athens, Greece; Andrea Munaro, student, University of Trento, Trento, Italy; Vedula N. Murty, Dover, PA, USA; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; and Titu Zvonaru, Comăneşti, Romania. We first give Bataille's solution.

If we let F be the area of $\triangle ABC$ and let h_a be the altitude from A, the proposed inequality is successively equivalent to

$$\begin{aligned} r(a+b+c) &\leq aR\sin A\,, \\ 2F &\leq \tfrac{1}{2}a\cdot 2R\sin A\,, \\ ah_a &\leq \tfrac{1}{2}a^2\,, \\ h_a &\leq \tfrac{1}{2}a\,. \end{aligned} \tag{2}$$

Now, the median m_a from A satisfies

$$\begin{aligned} m_a^2 &= \tfrac{1}{2}(b^2+c^2) - \tfrac{1}{4}a^2 \\ &= \tfrac{1}{2}(a^2+2bc\cos A) - \tfrac{1}{4}a^2 \\ &= \tfrac{1}{4}a^2 + bc\cos A \;\leq\; \tfrac{1}{4}a^2\,, \end{aligned}$$

where the final inequality follows from $\cos A \leq 0$ (since $\angle A \geq 90^\circ$).

Thus, $m_a \leq \frac{1}{2}a$, and (2) follows from the obvious inequality $h_a \leq m_a$.

Next we give the version of Bornsztein.

Soit S l'aire du triangle. Il est bien connu que

$$S = \tfrac{1}{2}(a+b+c)r = \tfrac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}\,.$$

D'autre part, d'après la loi des sinus, on a $\dfrac{\sin(A)}{a} = \dfrac{1}{2R}$. L'inégalité désirée est donc équivalente à

$$a^4 \geq (a+b+c)(a+b-c)(a-b+c)(-a+b+c)\,. \tag{1}$$

Or, comme $\angle A \geq 90^\circ$, on a $a^2 = b^2+c^2-2bc\cos(A) \geq b^2+c^2$. Par la suite, d'après l'inégalité arithmético–géométrique,

$$(b+c)^2 = b^2+c^2+2bc \leq 2(b^2+c^2) \leq 2a^2\,.$$

Et ainsi

$$\begin{aligned} &(a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\ &\qquad = (a^2-(b-c)^2)((b+c)^2-a^2) \;\leq\; a^2\cdot a^2 \end{aligned}$$

ce qui prouve que (1) est vraie. On peut noter qu'il y a égalité si et seulement si $b=c$ et $a^2=b^2+c^2$; c'est à dire, le triangle est rectangle et isocèle en A.

3. Prove that the equation $x^3+2px^2+2p^2x+p=0$ cannot have three distinct real roots, for any real number p.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Vedula N. Murty, Dover, PA, USA; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; and Titu Zvonaru, Comăneşti, Romania. We first give the solution by Tsai.

We prove the slightly stronger result that $x^3 + 2px^2 + 2p^2x + p = 0$ cannot have at least two distinct real roots, for any real number p. Assume to the contrary that the solutions (counting multiplicity) α, β, and γ to $x^3 + 2px^2 + 2p^2x + p = 0$ are all real such that there are at least two distinct ones. Then

$$\begin{aligned} & x^3 + 2px^2 + 2p^2x + p \\ & \qquad = \; x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma \end{aligned}$$

for any real number x, and equating coefficients gives

$$\begin{aligned} 2p &= -(\alpha+\beta+\gamma)\,, && (1)\\ 2p^2 &= \alpha\beta+\beta\gamma+\gamma\alpha\,, && (2)\\ \text{and} \qquad p &= -\alpha\beta\gamma\,. && (3) \end{aligned}$$

Squaring (1) yields $4p^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ and by (2), we get $\alpha^2 + \beta^2 + \gamma^2 = 0$, which cannot be.

Next we give the approach of Kandall.

Suppose the polynomial $x^3 + 2px^2 + 2p^2x + p$ has three distinct real roots $x_1 < x_2 < x_3$. Then, by Rolle's Theorem, its derivative $3x^2+4px+2p^2$ has two distinct real roots: one between x_1 and x_2, the other between x_2 and x_3. Therefore,

$$(4p)^2 - 4(3)(2p^2) \;=\; -8p^2 \;>\; 0\,,$$

which is impossible.

Note that, if p and q are any real numbers, then the same argument applies to a polynomial $x^3 + 2px^2 + 2p^2x + q$.

4. Let $ABCD$ be a cyclic quadrilateral with $AB = 2AD$ and $BC = 2CD$. Let $d = AC$ and $\alpha = \angle BAD$ be given. Express the area of $ABCD$ in terms of d and α.

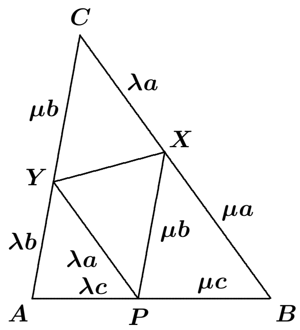
Comments and solutions by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Andrea Munaro, student, University of Trento, Trento, Italy; Vedula N. Murty, Dover, PA, USA; and Titu Zvonaru, Comăneşti, Romania.

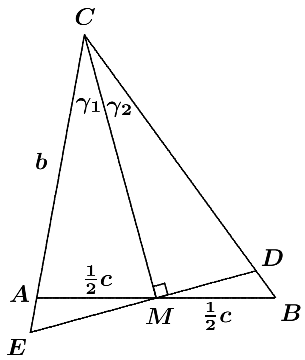
Bataille comments that this is problem 7 of the Belarus Olympiad 2002, discussed in the November 2007 *Corner* [2006 : 439; 2007 : 424–425].

Next we turn to the Hungarian National Olympiad 2003–2004, Grades 11–12, Final Round given in [2007 : 83–84].

1. Let ABC be an acute triangle, and let P be a point on side AB. Draw lines through P parallel to AC and BC, and let them cut BC and AC at X and Y, respectively. Construct (with straightedge and compass) the point P which gives the shortest length XY. Prove that the shortest XY is perpendicular to the median of ABC through C.

Solved by Andrea Munaro, student, University of Trento, Trento, Italy; and D.J. Smeenk, Zaltbommel, the Netherlands. We present Smeenk's solution.





Let λ and μ be the positive real numbers such that $AP = \lambda AB = \lambda c$ and $BP = \mu AB = \mu c$. Then $\lambda + \mu = 1$, and we also have

$$PY = \lambda a\,, \quad AY = \lambda b\,, \quad BX = \mu a\,, \quad XC = \lambda a\,, \quad YC = \mu b\,.$$

Let $f(\lambda, \mu)$ be the square of the distance from X to Y. If $\gamma = \angle C = \angle XPY$, then, by the Law of Cosines in $\triangle PXY$ and $\triangle ABC$, we have

$$\begin{aligned} f(\lambda,\mu) = XY^2 &= \lambda^2 a^2 + \mu^2 b^2 - 2\lambda\mu ab\cos\gamma \\ &= \lambda^2 a^2 + \mu^2 b^2 - \lambda\mu(a^2+b^2-c^2) \\ &= \lambda^2 a^2 + (\lambda^2 - 2\lambda + 1)b^2 + \lambda(\lambda-1)(a^2+b^2-c^2) \\ &= \lambda^2(2a^2+2b^2-c^2) - \lambda(a^2+3b^2-c^2) + b^2 \\ &= \lambda^2 \cdot 4m_c^2 - \lambda(a^2+3b^2-c^2) + b^2\,. \end{aligned}$$

Setting $\dfrac{df}{d\lambda} = 0$, we get

$$\lambda = \frac{a^2+3b^2-c^2}{8m_c^2} \qquad \left(\text{and} \quad \mu = 1 - \lambda = \frac{3a^2+b^2-c^2}{8m_c^2}\right)\,.$$

Therefore,

$$CX = \lambda a = a\frac{a^2+3b^2-c^2}{8m_c^2} \quad \text{and} \quad CY = \mu b = b\frac{3a^2+b^2-c^2}{8m_c^2}\,. \tag{1}$$

Let M be the mid-point of AB and set $\gamma_1 = \angle ACM$ and $\gamma_2 = \angle BCM$. By the Law of Cosines in $\triangle ACM$,

$$\begin{aligned} AM^2 &= CA^2 + CM^2 - 2CA \cdot CM \cdot \cos\gamma_1 \,, \\ \text{or} \qquad c^2 &= 4b^2 + 4m_c^2 - 8bm_c \cos\gamma_1 \,. \end{aligned}$$

Similarly, we get $c^2 = 4a^2 + 4m_c^2 - 8am_c \cos\gamma_2$. Since $4m_c^2 = 2a^2 + 2b^2 - c^2$, we have

$$\cos\gamma_1 \;=\; \frac{a^2 + 3b^2 - c^2}{4bm_c} \qquad \text{and} \qquad \cos\gamma_2 \;=\; \frac{3a^2 + b^2 - c^2}{4am_c} \,,$$

from which we have

$$\frac{\cos\gamma_1}{\cos\gamma_2} \;=\; \frac{a(a^2 + 3b^2 - c^2)}{b(3a^2 + b^2 - c^2)} \,.$$

Let the line through M perpendicular to CM intersect BC at D and AC at E. Applying the Law of Sines to $\triangle CDE$, we obtain

$$\frac{CD}{CE} \;=\; \frac{\sin\angle CED}{\sin\angle CDE} \;=\; \frac{\cos\gamma_1}{\cos\gamma_2} \;=\; \frac{a(a^2 + 3b^2 - c^2)}{b(3a^2 + b^2 - c^2)} \,. \tag{2}$$

By (1) and (2), we have

$$\frac{CX}{CY} \;=\; \frac{a(a^2 + 3b^2 - c^2)}{b(3a^2 + b^2 - c^2)} \;=\; \frac{CD}{CE} \,.$$

Since $\triangle CXY$ and $\triangle CDE$ also have the angle at C in common, we see that they must be similar. Since $CM \perp ED$, it follows that $CM \perp XY$.

Now we turn to readers' solutions to problems of the Hungarian National Olympiad 2003–2004 (Specialized Mathematics Classes), First Round, Grades 11–12 which appeared in [2007 : 84].

1. Let n be a positive integer, and let a and b be positive real numbers. Prove that

$$\begin{aligned} \log(a^n) + \binom{n}{1}\log(a^{n-1}b) + \binom{n}{2}\log(a^{n-2}b^2) + \cdots + \log(b^n) \\ = \; \log\left((ab)^{n2^{n-1}}\right) \,. \end{aligned}$$

Solved by Houda Anoun, Bordeaux, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Ioannis Katsikis, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

Let S denote the left side of the identity to be proved. Then

$$\begin{aligned} S &= \sum_{k=0}^{n} \binom{n}{k} \log(a^{n-k}b^k) \;=\; \log\left(\prod_{k=0}^{n} (a^{n-k}b^k)^{\binom{n}{k}}\right) \\ &= \log\left(a^{\sum (n-k)\binom{n}{k}} b^{\sum k\binom{n}{k}}\right) \,, \end{aligned} \tag{1}$$

where both summations in the last line are from $k = 0$ to $k = n$.

Using the standard well-known method, we differentiate the binomial expansion

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

to obtain

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}.$$

Multiplying by x, we have $nx(1+x)^{n-1} = \sum\limits_{k=1}^{n} k\binom{n}{k}x^k$.

Setting $x = 1$, we then obtain

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}. \qquad (2)$$

Hence,

$$\begin{aligned}\sum_{k=0}^{n}(n-k)\binom{n}{k} &= n\sum_{k=0}^{n}\binom{n}{k} - \sum_{k=0}^{n} k\binom{n}{k} \\ &= n2^n - n2^{n-1} = n2^{n-1}. \qquad (3)\end{aligned}$$

Substituting (2) and (3) into (1) yields the result immediately.

2. Let H be a finite set of positive integers none of which has a prime factor greater than 3. Show that the sum of the reciprocals of the elements of H is smaller than 3.

Solved by Houda Anoun, Bordeaux, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Anoun's write-up.

Let $H = \{n_1, \ldots, n_k\}$, where each n_i is a positive integer such that $n_i = 2^{a_i}3^{b_i}$ (a_i and b_i are positive integers).

Let $S(k) = \sum\limits_{i=1}^{k} \dfrac{1}{n_i}$. We have

$$S(k) = \sum_{i=1}^{k} \frac{1}{2^{a_i}3^{b_i}}.$$

Hence,

$$S(k) < \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{2^i 3^j} = \left(\sum_{i=0}^{\infty}\frac{1}{2^i}\right)\cdot\left(\sum_{j=0}^{\infty}\frac{1}{3^j}\right).$$

Since

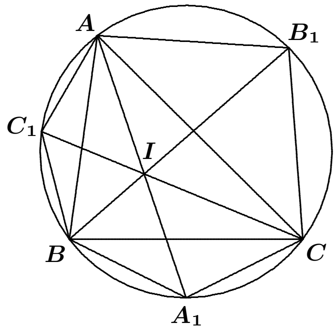
$$\sum_{j=0}^{\infty}\frac{1}{3^j} = \frac{3}{2} \qquad \text{and} \qquad \sum_{i=0}^{\infty}\frac{1}{2^i} = 2,$$

we deduce that $S(k) < 3$.

3. Consider the three disjoint arcs of a circle determined by three points on the circle. For each of these arcs, draw a circle centred at the mid-point of the arc and passing through the end-points of the arc. Prove that the three circles have a common point.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

The following theorem is well-known: Let I be the incentre of $\triangle ABC$. Then the lines AI, BI, and CI intersect the circumcircle for the second time in the mid-points A_1, B_1, and C_1 of the arcs BC, CA, and AB, respectively. Furthermore,



$$\begin{aligned} A_1B &= A_1I = A_1C\,, \\ B_1C &= B_1I = B_1A\,, \\ \text{and} \quad C_1A &= C_1I = C_1B\,. \end{aligned}$$

The common point of the three circles is thus the incentre of $\triangle ABC$.

4. A palace which has a square shape is divided into 2003×2003 square rooms of the same size which form a square grid. There might be a door between two rooms if they have a common side. The main gate leads to the room at the northwest corner. Someone has entered the palace, walked around for a while and upon returning to the room at the northwest corner for the first time, immediately left the palace. It turned out that this person visited each of the other rooms 100 times, except the room at the southeast corner. How many times did this person visit the room at the southeast corner?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

La pièce a été visitée 99 fois.

Plus généralement, on considère un palace de $n \times n$ pièces, chacune d'entre elles ayant été visitée exactement $p \geq 1$ fois, sauf celle située au coin nord-ouest qui a été visitée exactement deux fois, et celle située au coin sud-est, qui a été visitée exactement x fois. On va prouver qu'alors $x = p - 1$ si n est impair, et $x = 2p - 1$ si n est pair.

On commence par colorier les pièces alternativement en noir et blanc, comme sur un échiquier, de sorte que les deux cases situées au coins nord-ouest et sud-est soient toutes les deux noires. En particulier, on passe toujours d'une pièce blanche à une pièce noire, et réciproquement.

Cas 1. n est impair.

En tout, il y a alors $\frac{1}{2}(n^2+1)$ pièces noires et $\frac{1}{2}(n^2-1)$ pièces blanches.

Puisque chaque pièce blanche a été visitée exactement p fois et que la personne est sortie d'une pièce blanche pour entrer dans une pièce noire, le nombre total de passages d'une pièce blanche à une pièce noire réalisés pendant la promenade est $\frac{1}{2}(n^2-1)p$.

Mais, si l'on élimine la première entrée dans la pièce du coin nord-ouest (puisque l'on vient de l'extérieur), cela correspond au nombre total de visites des pièces noires au cours de la promenade. Or, la pièce du coin nord-ouest a été visitée une seule fois (puisque la promenade s'arrête après le premier retour dans cette pièce) et celle du coin sud-est a été visitée x fois, alors que les $\frac{1}{2}(n^2+1)-2$ autres pièces noires ont été visitées chacune exactement p fois. D'où

$$1+x+\left(\tfrac{1}{2}(n^2+1)-2\right)p \;=\; \tfrac{1}{2}(n^2-1)p\,.$$

On en déduit facilement que $x=p-1$.

Cas 2. n est pair.

En tout, il y a alors $\frac{1}{2}n^2$ pièces noires et $\frac{1}{2}n^2$ pièces blanches. Le reste du raisonnement est le même pour arriver à

$$1+x+\left(\tfrac{1}{2}n^2-2\right)p \;=\; \tfrac{1}{2}n^2p\,.$$

On en déduit facilement que $x=2p-1$.

Now we turn to problems of the Finnish High School Math Contest 2004, Final Round, given at [2007 : 85].

1. The equations

$$x^2+2ax+b^2 \;=\; 0 \qquad\text{and}\qquad x^2+2bx+c^2 \;=\; 0$$

both have two different real roots. Determine the number of real roots of the equation

$$x^2+2cx+a^2 \;=\; 0\,.$$

Solved by Houda Anoun, Bordeaux, France; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Zvonaru.

Since $x^2+2ax+b^2=0$ and $x^2+2bx+c^2=0$ each have different real roots, we have

$$4(a^2-b^2) \;>\; 0 \qquad\text{and}\qquad 4(b^2-c^2) \;>\; 0\,.$$

Summing these inequalities, we deduce that $a^2-c^2>0$, or $c^2-a^2<0$, which implies that the equation $x^2+2cx+a^2=0$ has no real roots.

2. Let a, b, and c be positive integers such that

$$\frac{a\sqrt{3}+b}{b\sqrt{3}+c}$$

is a rational number. Show that

$$\frac{a^2+b^2+c^2}{a+b+c}$$

is an integer.

Solved by Houda Anoun, Bordeaux, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; Daniel Tsai, student, Taipei American School, Taipei, Taiwan; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Kandall's write-up.

We first observe that

$$\frac{a\sqrt{3}+b}{b\sqrt{3}+c} = \frac{a\sqrt{3}+b}{b\sqrt{3}+c}\cdot\frac{b\sqrt{3}-c}{b\sqrt{3}-c} = \frac{(3ab-bc)+(b^2-ac)\sqrt{3}}{3b^2-c^2}\,.$$

Since this number is rational, we must have $b^2 = ac$. Then

$$\begin{aligned} a^2+b^2+c^2 &= a^2+ac+c^2 = (a+c)^2-ac \\ &= (a+c)^2-b^2 = (a+c+b)(a+c-b)\,. \end{aligned}$$

Consequently,

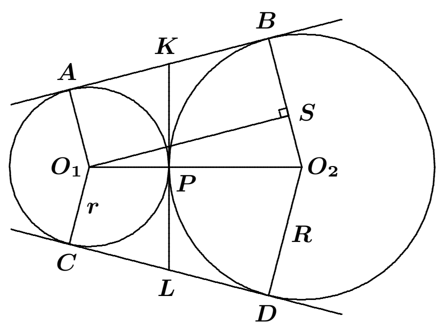
$$\frac{a^2+b^2+c^2}{a+b+c} = a+c-b\,,$$

an integer.

3. Two circles with radii r and R are externally tangent at a point P. Determine the length of the segment cut from the common tangent through P by the other common tangents.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; Ioannis Katsikis, Athens, Greece; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Katsikis.

Without loss of generality, we may assume that $r \le R$. Let the circle with radius r have centre O_1 and the circle with radius R have centre O_2. Let P be their point of tangency. Let the common external tangents meet the circles at A, B, C, and D, as in the diagram. Let the internal common tangent meet the external common tangents at K and L.



Let S be the point on O_2B such that $O_1S \perp O_2B$. Then $O_1S = AB$ and $O_2S = R - r$. Also

$$O_1S \;=\; \sqrt{(O_1O_2)^2 - (O_2S)^2} \;=\; \sqrt{(R+r)^2 - (R-r)^2} \;=\; 2\sqrt{Rr}\,.$$

Thus, $KP = \frac{1}{2}AB = \sqrt{Rr}$. Similarly, since $CD = AB = 2\sqrt{Rr}$, we have $PL = \sqrt{Rr}$, which implies that $KL \;=\; 2\sqrt{Rr}$.

4. The numbers $2005! + 2$, $2005! + 3$, $\ldots$, $2005! + 2005$ form a sequence of 2004 consecutive integers, none of which is a prime number. Does there exist a sequence of 2004 consecutive integers containing exactly 12 prime numbers?

Solution byPierre Bornsztein, Maisons-Laffitte, France.

Si l'on n'impose pas qu'il s'agisse d'entiers positifs, la conclusion est triviale puisqu'il suffit de prendre les entiers positifs jusqu'aux 12 premiers nombres premiers et de compléter la suite par des nombres négatifs.

Soient $n > 0$ un entier et $\pi(n)$ le nombre de nombres premiers inférieurs ou égaux à n.

Soit enfin $f(n) = \pi(n + 2004) - \pi(n)$. Alors, $f(n)$ est le nombre de nombres premiers dans $\{n + 1, n + 2, \ldots, n + 2004\}$, ensemble formé de 2004 entiers consécutifs. Il s'agit donc de prouver qu'il existe $n > 0$ tel que $f(n) = 12$.

On commence par noter que $f(1) = \pi(2005) > 12$ et que, d'après le rappel de l'énoncé, $f(2005! + 1) = 0$.

Soit $n > 0$ un entier. On a

$$\begin{aligned} & f(n+1) - f(n) \\ &\qquad = \pi(n + 2005) - \pi(n + 2004) + \pi(n) - \pi(n + 1) \\ &\qquad = \begin{cases} 0 & \text{si } n+1 \text{ et } n+2005 \text{ sont tous les deux premiers} \\ & \qquad\qquad \text{ou tous les deux composés,} \\ 1 & \text{si } n+2005 \text{ est premier et } n+1 \text{ est composé,} \\ -1 & \text{si } n+1 \text{ est premier et } n+2005 \text{ est composé.} \end{cases} \end{aligned}$$

On en déduit que la fonction f passe de la valeur $f(1)$ à la valeur $f(2005!+1)$ en ne sautant aucun entier strictement positif. En particulier, elle prend au moins une fois la valeur 12, ce qui permet d'affirmer qu'il existe une suite de 2004 entiers strictement positifs consécutifs qui contienne exactement 12 nombres premiers.

5. Finland is going to change its monetary system again and replace the Euro by the Finnish Mark. The Mark is divided into 100 pennies. There shall be coins of three denominations only, and the number of coins a person has to carry in order to be able to pay for any purchase less than one Mark should be minimal. Determine the coin denominations.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Si l'on veut pouvoir payer une somme de 1 penny il faut que l'une des pièces soit de ce montant. On note a et b les autres montants, avec $a < b$.

Une personne qui possède x pièces de 1 penny, y pièces de a pennies et z pièces de b pennies pourra payer tout prix inférieur ou égal à 99 pennies seulement si avec ces pièces il peut former au moins 100 valeurs différentes (on compte le cas d'un prix nul).

Il faut donc que $(x+1)(y+1)(z+1) \geq 100$.

Or, d'aprés l'inégalité arithmético–géométrique, cela implique que

$$x + y + z \;\geq\; 3\sqrt[3]{100} - 3 \;=\; 10{,}92\ldots$$

et donc que $x + y + z \geq 11$.

Ainsi, quels que soient a et b, il faudra toujours au moins 11 pièces pour pouvoir payer toute somme inférieure ou égale à 99 pennies.

Réciproquement : Tout entier $p \in \{1, \ldots, 99\}$ se décompose en base 5 sous la forme $p = x + 5y + 25z$, avec $x, y, z \leq 4$. Mais, puisque $p < 100$, on a même $z \leq 3$. Par conséquent, pour $a = 5$ et $b = 25$, on pourra payer tout achat de moins d'un Mark en partant avec 4 pièces de 1 penny, 4 pièces de 5 pennies et 3 pièces de 25 pennies, soit donc un total de 11 pièces et l'on a vu que l'on ne pourra pas faire mieux.

Donc, avec des pièces de 1 penny, 5 pennies et 25 pennies, l'objectif sera atteint.

Cela étant, il n'y a pas unicité car l'objectif sera atteint également avec 4 piéces de 1 penny, 3 pièces de 5 pennies et 4 pièces de 20 pennies (soit donc à nouveau un total de 11 pièces). En effet, si $p \in \{1, \ldots, 99\}$ et $\lfloor \cdot \rfloor$ désigne la partie entière alors, en posant $z = \left\lfloor \frac{p}{20} \right\rfloor$ on a $z \leq 4$ et $p = 20z + q$ avec $q \in \{0, \ldots, 19\}$. En particulier, $q = 5y + x$ avec $y \leq 3$ et $x \leq 4$. Ainsi, $p = x + 5y + 20z$ et on pourra donc payer un prix de p pennies avec les pièces que l'on a en poche.

That completes the material for this issue of the *Corner*. Send me your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

How Euler Did It
By C. Edward Sandifer, Mathematical Association of America, 2007
ISBN:978-0-88385-563-8, hardcover, 237+xiv pages, US$51.95
Reviewed by ***J. Chris Fisher****, University of Regina, Regina, SK*

How Euler Did It brings together the 40 essays that appeared as monthly columns under this title from November 2003 to February 2007 on *MAA Online* (http://www.maa.org/news/howeulerdidit.html). The essays average about 5 pages in length. Most begin with a short introduction that provides the historical and mathematical setting and explains why Sandifer finds the month's topic interesting. Then comes an account of some aspect of Euler's work, usually using Euler's notation and sometimes even his own words in translation. The author gently guides the reader through arguments that might involve little more than high-school tools, but display a brilliance that demands some effort from the reader. The exposition maintains a light-hearted tone and would appeal to anybody with a mathematical bent, while the details are kept to a level that is suitable for any reader of ***CRUX with MAYHEM***. Each essay concludes with a list of references that always includes the original source, which can be found on-line in the *Euler Archive* (http://www.eulerarchive.org).

A brief biographical note informs us that Sandifer can read the works of Euler in their original Latin, French, and German. Evidently he is familiar with many of those works, having also written *The Early Mathematics of Leonhard Euler* (also published by the MAA) and helped edit *Leonhard Euler: Life, Work and Legacy*. The author does an admirable job of providing background information about 18$^{\text{th}}$ century mathematics and mathematicians; he reminds us more than once that Euler used 18$^{\text{th}}$ century techniques and wrote for an 18$^{\text{th}}$ century audience. This gives him an opportunity to contrast Euler's methods with today's and to describe the Euler impact.

The first essay presents an overview of Euler's work, while the last is a report on a meeting of the Euler Society. The other 38 have been grouped into four parts: 6 on geometry, another 6 on number theory, 5 on combinatorics, and 21 on analysis. Three of the essays that were placed under the analysis heading—probably by a publisher's error—deal with scientific applications which, we learn, occupied much of Euler's time and effort.

Many of the essays were inspired by questions raised by members of the Euler Society. For example, why do some sources credit Lambert with the first proof that e is irrational while others credit Euler? The answer was not easily found: there are 866 items in the authoritative Eneström list of Euler's publications, many of which are rather large books, and there are also 1000 of his letters extant. It turned out that Euler really did prove that e is irrational, although the details are somewhat buried in a long paper on continued fractions from 1737.

Another pair of essays inspired by the Euler Society involve the "Euler formula" $V - E + F = 2$. Some half-truths have arisen concerning what Euler did and what he proved. To uncover the whole truth, Sandifer had to study two papers of Euler together with a paper of Descartes from 1649, a century before Euler's work, which established a formula that is equivalent to Euler's.

Some of Sandifer's stories are based on items he just stumbled across while looking for something else. One example of this is a 1770 paper in which Euler anticipated orthogonal matrices some 80 years before the concept came into mathematics. Euler failed to say what he was thinking when he formulated the problem that led him to these results.

I learned something from every essay, even those on topics about which I already knew something. I learned about the importance of letters for the exchange of ideas in the 18^{th} century. I learned that number theory attracted little interest at the time, so that the impact of Euler's substantial work in the area was delayed for a generation or two. ***CRUX with MAYHEM*** readers might recall G.C. Shephard's article on Euler's triangle theorem [1999 : 148–153]: *If O is any point in the plane of triangle ABC that does not lie on a side, and D, E, and F are the feet of the cevians AO, BO, and CO, respectively, then*

$$\frac{AO}{OD}\cdot\frac{BO}{OE}\cdot\frac{CO}{OF} \;=\; \frac{AO}{OD}+\frac{BO}{OE}+\frac{CO}{OF}+2\,,$$

where the quotients are positive when O is between the vertex and foot, and negative otherwise. Euler provided his first proof of the theorem in 1780, just three years before he died. (In his version of the story, Sandifer raises the question of how a person who has been blind for some 15 years could possibly discover such a theorem! He can explain only that Euler had assistants who wrote down his proofs.) Both Shephard and Sandifer complain of the awkwardness of that first proof. Sandifer, however, provides Euler's beautiful alternative proof that Euler seems to have appended to his paper at a later date. He speculates that Euler lacked sufficient time before his death to revise and polish this paper; thus, it gives us a glimpse of how Euler discovered things as he wrote a paper and how he came back later to improve his solutions.

All forty columns are pleasant and informative. The book can be enjoyed whether you read it from cover to cover or browse through random essays. But should you buy the book? After all, the whole thing (and more recent material) is readily available on-line. On the one hand, it is surely a good idea to support the MAA (Mathematical Association of America), which published the book. Besides, the book would be an excellent resource for teachers, either for directed reading or for term projects in any number of courses. It would make a great gift for anybody who loves math. On the other hand, the MAA could have and should have done a much better job of editing the manuscript. Although Sandifer thanks the members of the editorial board for their "conscientious and rigorous editing," it looked to me as if they did little more than download the computer files and add a table of contents and index. There are just too many typos—nothing particularly serious, but just what one would expect from an on-line document, not a published text. More serious was the clumsy cross-referencing. Sandifer often refers to other columns by date and title, which is great when one has handy computer links, but inconvenient in a 200-page book. Page numbers, or even chapter numbers, should have been provided for each cross-reference. Moreover, the introductory remarks from many of the columns need revising. In essay 3 he repeats what he said in the previous month's column; in the book that means that consecutive pages contain identical paragraphs. It would have been easy for an editor to deal with these minor problems. We should expect more for our money.

Nonplussed! Mathematical Proof of Implausible Ideas
By Julian Havil, published by Princeton University Press, 2007
ISBN 0-691-12056-0, hardcover, 196+xiii pages, US$24.95
Reviewed by **Robert D. Poodiack**, *Norwich University, Northfield, VT, USA*

Among my extracurricular activities, I travel to Vermont high schools to give talks about paradoxes in probability. Thanks to Julian Havil's wonderful book, I've just found several new ways to expand my lectures.

The 14 chapters of *Nonplussed!* deal primarily with paradoxes in probability and combinatorics, but other examples are pulled in from calculus, mechanics, number theory, and game theory. Many will be familiar to experienced mathematics readers. The Birthday Problem, derangements (permutations in which there is no fixed point), Buffon's needle, and Torricelli's Trumpet (also known as Gabriel's horn—the surface of revolution for $y = 1/x$ about the x-axis for $x \geq 1$) all make appearances.

The chapters on the slightly less-known topics were thoroughly enjoyable. The mathematical work is pitched at the level of an undergraduate mathematics journal, a decision that shows why Havil is a master teacher. Havil's conversational tone entices the reader into areas which may be entertaining but, a few chapters in, quite confounding. The first chapter on "Three Tennis Paradoxes" is thick with algebra and the level of mathematics takes off from there. Havil notes in his introduction that the difficulty level generally rises from chapter to chapter and the reader moves from algebra and basic probability through trigonometry, differential and integral calculus, sequences and series, combinatorics, generating functions, differential equations, and modular arithmetic. Havil optimistically writes that "none of it is beyond a committed senior high school student," but university-level students with some mathematical experience might better appreciate the articles. (I will have to wave my hands over some areas for my high school audiences.) However, Havil never forgets that each problem stems from a great story and even novices will be attracted by the histories of the 14 problems, one associated with each chapter.

In the best chapters, Havil takes a topic with which many readers might be familiar and summarizes or describes it from a new angle. The most accessible may be Chapter 3 on the Birthday Problem.

Many of us learned about this problem as either an amazing fact (it takes a gathering of only 23 people to have a better than 50% chance of having two people with the same birthday) or a lesson in computing probabilities via complements. Havil expands the problem in two interesting directions. First, he dispenses with a misstatement of the problem—how many people are needed to have a better than 50% chance of someone having the same birthday as you? (The answer: a lot more than 23.) After proving the usual result, Havil generalizes the Birthday Problem in various directions. How many people are needed in a group to have three, four, or more with the same birthday? How many people are needed to have more than two birthdays separated by a certain number of days? The answer, via Paul Halmos, brings in the Arithmetic Mean–Geometric Mean Inequality.

Havil concludes the chapter with an application of the Birthday Problem to identifying web browsers and computers. This is just one example of how Havil connects historical problems to modern applications. In the chapter on Parrondo Games—winning games that are constructed from two losing games—Havil mentions a demonstration in a New York Times article from 2000 of how two losing stock portfolios can be combined into a winning one.

As with the best classroom teachers, Havil's enthusiasm for the material carries the reader even through the tough spots. The chapter on hyperdimensions is quite stunning in its rigour, but Havil continues to pull ever more amazing facts and identities from his bag of tricks. I questioned some of the techniques Havil introduced here for their lack of physical meaning, but I became a believer when I saw Gelfond's constant (e^{π}), the Gamma function, and the double factorial being introduced to a new audience.

I'm still not sure I understand the closing chapter on John Conway's Fractran programming language, after several attempts to do so and even after reading Conway's own exposition in his *Book of Numbers*, written with Richard Guy. Havil so obviously enjoys the material, though, that I want to keep trying! (The chapter on Conway's checkerboard game is much easier to digest.)

Havil's presentation on Fractran follows Guy's 1983 article from *Mathematics Magazine*. Most of the chapters explain, interpret, and synthesize previously published articles from other authors. Havil cites all his sources throughout the text, but my one wish would have been for a united bibliography somewhere in the book.

Nonplussed! is a wonderful collection of astounding paradoxes and stunning turnarounds. Although the mathematics can get fairly dense, the brimming delight Havil has for his material repays the work readers must invest. Fans of puzzle, paradox, or game books should find much to enjoy. I look forward to sharing a lot of this with Vermont high school students in the next few years.

A Useful Inequality

Roy Barbara

We present and prove a new inequality that further turns out to offer an alternative approach to solving a large class of inequalities. When applicable, the method allows reducing a symmetric inequality with three variables to an inequality in just **one** variable.

The main result is

Theorem 1 Let a, b, and c be non-negative real numbers. Then we have the following inequality

$$\big((a+b+c)(2a^2+2b^2+2c^2-5ab-5bc-5ca)+27abc\big)^2 \\ \leq\; 4(a^2+b^2+c^2-ab-bc-ca)^3\,.$$

We shall put this symmetric inequality into an equivalent form, which we will then prove and use. Setting $S=a+b+c$, $P=abc$, and $Q=ab+bc+ca$, we start by writing the above inequality as

$$(9SQ-2S^3-27P)^2 \;\leq\; 4(S^2-3Q)^3\,.$$

If we then take square roots, note that $S^2-3Q\geq 0$, and eliminate the absolute value $|9SQ-2S^3-27P|$, we obtain the equivalent result.

Theorem 2 Let a, b, c be non-negative real numbers. Set $S=a+b+c$, $P=abc$, and $Q=ab+bc+ca$. Then we have the double inequality

$$9SQ-2S^3-2(S^2-3Q)^{3/2} \;\leq\; 27P \;\leq\; 9SQ-2S^3+2(S^2-3Q)^{3/2}\,.$$

Lemma Let $0\leq k<1$. Set $x_1=1-2\sqrt{1-k}$ and $x_4=1+2\sqrt{1-k}$. Let $x\in[x_1,x_4]$. Then

$$3k-2-2(1-k)^{3/2} \;\leq\; x^3-3x^2+3kx \;\leq\; 3k-2+2(1-k)^{3/2}\,.$$

Proof of the Lemma: Set $x_2=1-\sqrt{1-k}$ and $x_3=1+\sqrt{1-k}$. Clearly, $x_1<x_2<x_3<x_4$. Define $f(x)=x^3-3x^2+3kx$ on $[x_1,x_4]$. Its derivative $f'(x)=3(x^2-2x+k)$ has zeroes at x_2 and x_3. The sign of f' shows, as x varies from x_1 to x_2, then from x_2 to x_3, and finally from x_3 to x_4, that $f(x)$ increases, then decreases, and then increases again.

Hence, for any $x\in[x_1,x_4]$, one has

$$\min\{f(x_1),f(x_3)\} \;\leq\; f(x) \;\leq\; \max\{f(x_2),f(x_4)\}\,.$$

We claim that $f(x_2) = f(x_4)$ and $f(x_1) = f(x_3)$. In order to evaluate $f(x_1)$ and $f(x_4)$, note that x_1 and x_4 are the zeroes of $x^2 - 2x + (4k-3) = 0$. The Division Algorithm provides

$$\begin{aligned} f(x) &= x^3 - 3x^2 + 3kx \\ &= \big(x^2 - 2x + (4k-3)\big)(x-1) + (1-k)x + (4k-3)\,. \end{aligned}$$

Hence, $f(x_1) = (1-k)x_1 + (4k-3) = 3k - 2 - 2(1-k)^{3/2}$. Similarly, one finds that $f(x_4) = 3k - 2 + 2(1-k)^{3/2}$.

To evaluate $f(x_2)$ and $f(x_3)$, we note first that x_2 and x_3 are the roots of $x^2 - 2x + k = 0$ and that

$$f(x) \;=\; x^3 - 3x^2 + 3kx \;=\; (x^2 - 2x + k)(x-1) - 2(1-k)x + k\,.$$

A careful check verifies the claim and then it follows that, for $x \in [x_1, x_4]$, one has $f(x_1) \le f(x) \le f(x_4)$, as desired.

Proof of the Theorem: The result being obvious for $S = 0$, we assume $S > 0$. Further, we note that the inequality is homogeneous. Thus, it suffices (by rescaling) to prove it when $S = 3$. We also set $Q = 3k$, for some non-negative real number k. Since $0 \le Q \le S^2/3$, we get $0 \le k \le 1$. The case $k = 1$ (where $Q = S^2/3$ is the maximum) corresponds to $a = b = c$ and the result is obvious. Hence, we may assume $0 \le k < 1$. Dividing by 27, we have to prove that

$$3k - 2 - 2(1-k)^{3/2} \;\le\; P \;\le\; 3k - 2 + 2(1-k)^{3/2}\,.$$

Now set $A = a + b$ and $B = ab$. Then $A = 3 - c$ and

$$B \;=\; Q - cA \;=\; 3k - c(3-c) \;=\; c^2 - 3c + 3k\,.$$

Substituting $A = 3 - c$ and $B = c^2 - 3c + 3k$ into $A^2 - 4B \ge 0$ yields

$$c^2 - 2c + (4k-3) \;\le\; 0\,. \tag{1}$$

Clearly, c must lie between the two distinct roots of the quadratic in (1); that is, $1 - 2\sqrt{1-k} \le c \le 1 + 2\sqrt{1-k}$. Since $P = abc = Bc = c^3 - 3c^2 + 3kc$, the result follows from the Lemma.

Applications of Theorem 2

We will use the transformation $u = S^2 - 3Q$. Clearly, $0 \le u \le S^2$. If a, b, and c are also the sides of a triangle, we leave it as an exercise to check that $S^2/4 \le Q \le S^2/3$; thus, the range of u reduces to $0 \le u \le S^2/4$.

Example 1: ([2]) Suppose that a, b, and c are positive real numbers. Prove that

$$\frac{2(a^3+b^3+c^3)}{abc} + \frac{9(a+b+c)^2}{a^2+b^2+c^2} \;\ge\; 33\,. \tag{2}$$

Inequality (2) is homogeneous; so we may take $S = 1$. Using

$$\sum_{\text{cyclic}} a^2 \;=\; 1 - 2Q\,, \qquad \sum_{\text{cyclic}} a^3 \;=\; 1 + 3P - 3Q\,,$$

and a little algebra, equation (2) reduces to $9P(1-3Q) \leq (1-2Q)(1-3Q)$. Now $1 - 3Q = S^2 - 3Q \geq 0$. If $1 - 3Q = 0$, then we are done. If $1 - 3Q > 0$, this latter simplifies to $9P \leq 1 - 2Q$, or $27P \leq 3 - 6Q$. By Theorem 2, it suffices to prove that $9Q - 2 + 2(1-3Q)^{3/2} \leq 3 - 6Q$; that is, $2(1-3Q)^{3/2} \leq 5(1-3Q)$. Setting $u = 1 - 3Q$, we see that this becomes $2u^{3/2} \leq 5u$, or $u(5 - 2\sqrt{u}) \geq 0$ for $0 \leq u \leq 1$. This, however, is trivial.

Example 2: ([1]) Let x, y, and z be non-negative real numbers. Prove that

$$\big((x+y)(y+z)(z+x)\big)^2 \;\geq\; xyz(2x+y+z)(2y+z+x)(2z+x+y)\,. \quad (3)$$

Inequality (3) is also homogeneous, and we may again take $S = 1$. Now we rewrite (3) successively as

$$\begin{aligned}
\big((1-z)(1-x)(1-y)\big)^2 &\;\geq\; P(1+x)(1+y)(1+z)\,,\\
(Q-P)^2 &\;\geq\; P(2+Q+P)\,,\\
Q^2 &\;\geq\; P(2+3Q)\,,\\
27Q^2 &\;\geq\; 27P(2+3Q)\,.
\end{aligned}$$

By Theorem 2, it suffices to prove $(2+3Q)(9Q-2+2(1-3Q)^{3/2}) \leq 27Q^2$; that is, $2(3Q)+2(1-3Q)^{3/2}+(3Q)(1-3Q)^{3/2} \leq 2$. Using $u = 1-3Q$, we see that this simplifies to $2u + u^{5/2} \geq 3u^{3/2}$, or $u(1-\sqrt{u})(2-u-\sqrt{u}) \geq 0$ for $0 \leq u \leq 1$, which is true.

Example 3: Let a, b, and c be positive real numbers. Then we have the following non-homogeneous inequality:

$$\begin{aligned}
&\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\big((a+b+c)(2a+2b+2c+3)+1\big)\\
&\qquad\leq\; 9 + \frac{(a+b+c)^2(a+b+c+1)^2}{3abc}\\
&\qquad\qquad + (ab+bc+ca)\left(\frac{3}{a}+\frac{3}{b}+\frac{3}{c}\right)\,. \qquad (4)
\end{aligned}$$

In terms of P, Q, and S, this is $\dfrac{Q}{P}\big(S(2S+3)+1\big) \leq 9 + \dfrac{S^2(S+1)^2}{3P} + Q\dfrac{3Q}{P}$, which can be rearranged as $27P \geq 9SQ - 2S^3 - (S^2-3Q) - (S^2-3Q)^2$. By Theorem 2, it suffices to prove that

$$9SQ - 2S^3 - 2(S^2-3Q)^{3/2} \;\geq\; 9SQ - 2S^3 - (S^2-3Q) - (S^2-3Q)^2\,.$$

Using $u = S^2 - 3Q$, we have $u + u^2 \geq 2u^{3/2}$ with $0 \leq u \leq S^2$, and this holds by the AM–GM Inequality.

Example 4: Let a, b, and c be positive real numbers. Prove the following inequality

$$\frac{(a+b)^3}{c} + \frac{(b+c)^3}{a} + \frac{(c+a)^3}{b} \;\geq\; \frac{8}{3}(a+b+c)^2 \,. \tag{5}$$

Since inequality (5) is homogeneous, we take $S = 1$. The left side of (5) is

$$\begin{aligned}\sum_{\text{cyclic}} \frac{(1-a)^3}{a} &= \sum_{\text{cyclic}} \left(\frac{1}{a} - 3 + 3a - a^2\right) \\ &= \frac{Q}{P} - 9 + 3 - (1-2Q) \;=\; \frac{Q}{P} + 2Q - 7 \,.\end{aligned}$$

Hence, inequality (5) becomes $(Q/P) + 2Q \geq 29/3$, or equivalently $27P(29 - 6Q) \leq 81Q$. Note that $29 - 6Q > 0$, since $Q \leq \frac{1}{3}S^2 = \frac{1}{3}$. By Theorem 2, it suffices to prove that

$$(29 - 6Q)\bigl(9Q - 2 + 2(1-3Q)^{3/2}\bigr) \;\leq\; 81Q \,.$$

With $u = 1 - 3Q$, this becomes

$$\bigl(29 - 2(1-u)\bigr)\bigl(3(1-u) - 2 + 2u^{3/2}\bigr) \;\leq\; 27(1-u) \,,$$

which simplifies to $2u^{5/2} + 27u^{3/2} \leq 3u^2 + 26u$, or

$$u(1 - \sqrt{u}\,)(26 + 2u - \sqrt{u}\,) \;\geq\; 0$$

for $0 \leq u \leq 1$, which is true.

Here is a list of inequalities that can be treated similarly by using Theorem 2.

- Let a, b, and c be non-negative real numbers. Prove that

$$a^3(b+c) + b^3(c+a) + c^3(a+b) \;\geq\; 2abc(a+b+c) \,.$$

- Let a, b, and c be non-negative real numbers. Prove that

$$a^2(b+c)^2 + b^2(c+a)^2 + c^2(a+b)^2 \;\leq\; \tfrac{4}{3}(a^3+b^3+c^3)(a+b+c) \,.$$

- Let a, b, and c be non-negative real numbers. Prove that

$$a^4 + b^4 + c^4 + 3a^2b^2 + 3b^2c^2 + 3c^2a^2 \;\geq\; \tfrac{4}{27}(a+b+c)^4 \,.$$

- Let a, b, and c be non-negative real numbers with $a+b+c = 1$. Prove that

$$\frac{4}{1+a} + \frac{4}{1+b} + \frac{4}{1+c} \;\leq\; \frac{3}{ab+bc+ca} \,.$$

- Let a, b, and c be positive real numbers. Prove that

$$\frac{9}{a+b+c}+\frac{4(a+b+c)^2}{3abc} \;\geq\; \frac{5}{a}+\frac{5}{b}+\frac{5}{c}\,.$$

- ([4]) Let a, b, and c be positive real numbers. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \;\leq\; \frac{(a+b+c)^2}{6abc}\,.$$

- ([3]) Find the smallest value of α for which

$$\tfrac{1}{27}-xyz \;\leq\; \alpha\left[\tfrac{1}{3}-(xy+yz+zx)\right]$$

  holds for all non-negative x, y, and z which satisfy $x+y+z=1$.

- ([6]) Let a, b, and c be non-negative real numbers with $a+b+c=1$. Prove that

$$\frac{(1-a)(1-b)(1-c)}{(1-a^2)^2+(1-b^2)^2+(1-c^2)^2} \;\leq\; \frac{1}{8}\,.$$

- Let a, b, and c be non-negative real numbers satisfying the consraint $a^2+b^2+c^2-1=ab+bc+ca$. Prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \;\geq\; 9+\frac{2(a+b+c-1)}{3abc}\,.$$

  Hint: Write the constraint as $S^2-3Q=1$.

- ([5]) Let a, b, and c be the sides of a triangle. Show that

$$\frac{abc(a+b+c)^2}{a^2+b^2+c^2} \;\geq\; 2abc+\prod_{\text{cyclic}}(b+c-a)\,.$$

- Let x, y, and z be positive real numbers with $1/x+1/y+1/z=1$. Prove that

$$\frac{15+x+y+z}{xyz}+\frac{12(x^2+y^2+z^2)}{x^2y^2z^2} \;\leq\; \frac{4}{3}\,.$$

  Hint: Set $a=1/x$, $b=1/y$, $c=1/z$.

- In any triangle with sides a, b, and c, we have

$$\frac{(a^2+b^2+c^2)(a+b+c)+4abc}{(a+b+c)^3} \;\geq\; \frac{13}{27}\,.$$

- In any triangle with sides a, b, and c, and inradius r, we have

$$12r^2(a+b+c)^2 \;\le\; (a^2+b^2+c^2)^2\,.$$

Hint: Take $S = 2$, which means $s = 1$ for the semi-perimeter s. Use Heron's Formula to get $r^2 = Q - P - 1$.

- ([7]) If a, b, and c are positive real numbers, prove that

$$\frac{a^2(b+c-a)}{b+c} + \frac{b^2(c+a-b)}{c+a} + \frac{c^2(a+b-c)}{a+b} \;\le\; \frac{ab+bc+ca}{2}\,.$$

- Let a, b, and c be positive real numbers summing to 3. Prove that

$$\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} + \frac{135}{abc} + \frac{9ab}{c} + \frac{9bc}{a} + \frac{9ac}{b} + 81 \;\ge\; 82\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\,.$$

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 August 2008**. *An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3281. Correction. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let a_1, a_2, $\dots$, a_n be positive real numbers. Prove that

$$\left(\sum_{k=1}^{n} a_k^{\frac{n+1}{2}}\right)^n \leq \prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^k\right).$$

3301. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Prove that

$$\sum_{n=1}^{\infty} \frac{\ln 2 - \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)}{n} = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{(2n+1)(2n+2)}.$$

What is this common value?

3302. *Proposed by Mihály Bencze, Brasov, Romania.*

Let s, r, and R denote the semiperimeter, the inradius, and the circumradius of a triangle ABC, respectively. Show that

$$(s^2 + r^2 + 4Rr)(s^2 + r^2 + 2Rr) \geq 4Rr(5s^2 + r^2 + 4Rr),$$

and determine when equality holds.

3303. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$\prod_{\text{cyclic}} \left(2(a+b)^3\right) \geq \prod_{\text{cyclic}} \left((a+s_1)(bc+s_2)\right),$$

where $s_1 = a + b + c$ and $s_2 = ab + bc + ca$.

3304. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers and identify a_{n+1} with a_1. Prove that

$$\sum_{k=1}^{n} a_k^3 \;\geq\; \sum_{k=1}^{n} a_k a_{k+1}^2 \,.$$

3305. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Prove that

$$\begin{aligned} \tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} &= \tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13} \\ &= \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13} \;=\; \sqrt{13+2\sqrt{13}} \,. \end{aligned}$$

3306. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Find a real number t and polynomials $f(x)$, $g(x)$, and $h(x)$ with integer coefficients, such that

$$f(t) = \sqrt{2}\,, \qquad g(t) = \sqrt{3}\,, \qquad \text{and} \qquad h(t) = \sqrt{7}\,.$$

3307. *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Eliminate θ from the system

$$\begin{aligned} \lambda\cos(2\theta) &= \cos(\theta+\alpha)\,, \\ \lambda\sin(2\theta) &= 2\sin(\theta+\alpha)\,. \end{aligned}$$

3308. *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Given $\triangle ABC$, let AD be the altitude to BC. If $AB : AC = 1 : \sqrt{3}$, prove that $AD \leq \frac{\sqrt{3}}{2} BC$. When does equality hold?

3309. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let α, β, and γ be fixed non-zero real numbers. Show that the system

$$\begin{aligned} \alpha x + \beta y + \gamma z &= 1\,, \\ xy + yz + zx &= 1\,, \end{aligned}$$

has a unique solution for (x, y, z) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 + 1 \;=\; 2(\alpha\beta + \beta\gamma + \gamma\alpha)\,,$$

and, in this case find that unique solution.

3310. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b, and c denote, as usual, the lengths of the sides BC, CA, and AB, respectively, in $\triangle ABC$. Let s be the semiperimeter of $\triangle ABC$, r the inradius, h_a the altitude to side BC, and r_a, r_b, and r_c the exradii to A, B, and C, respectively.

(a) Show that for $x > 0$, we have $h_a = \dfrac{2s(s-a)x}{x^2 + s(s-a)}$ if and only if $x = r_b$ or $x = r_c$.

(b) Show that for $x > 0$, we have $h_a = \dfrac{2(s-b)(s-c)x}{|x^2 - (s-b)(s-c)|}$ if and only if $x = r$ or $x = r_a$.

3311. *Proposed by Michel Bataille, Rouen, France.*

Let n be an integer with $n \geq 2$. Suppose that for $k = 0, 1, \ldots, n-2$ we have

$$\binom{n-2}{k} \equiv (-1)^k(k+1) \pmod{n} .$$

Show that n is a prime.

3312. *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer congruent to 1 modulo 6. Show that $3/n$ can be expressed as

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$$

for some distinct positive integers $a_1, a_2, \ldots, a_k$, and find the minimal value of k.

. . . .

3281. Correction. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne et Pantelimon George Popescu, Bucarest, Roumanie.*

Soit $a_1, a_2, \ldots, a_n$ des nombres réels positifs. Montrer que

$$\left(\sum_{k=1}^{n} a_k^{\frac{n+1}{2}}\right)^n \leq \prod_{k=1}^{n}\left(\sum_{j=1}^{n} a_j^k\right) .$$

3301. *Proposé par Ovidiu Furdui, Université de Toledo, Toledo, OH, É-U.*

Montrer que

$$\sum_{n=1}^{\infty} \frac{\ln 2 - \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)}{n} = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{(2n+1)(2n+2)} .$$

Quelle est cette valeur commune ?

3302. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit respectivement s, r et R le demi-périmètre, le rayon du cercle inscrit et le rayon du cercle circonscrit du triangle ABC. Montrer que

$$(s^2+r^2+4Rr)(s^2+r^2+2Rr) \;\geq\; 4Rr(5s^2+r^2+4Rr)\,,$$

et déterminer quand il y a égalité.

3303. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\prod_{\text{cyclique}} \left(2(a+b)^3\right) \;\geq\; \prod_{\text{cyclique}} \left((a+s_1)(bc+s_2)\right)\,,$$

où $s_1 = a+b+c$ et $s_2 = ab+bc+ca$.

3304. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a_1, a_2, $\ldots$, a_n des nombres réels positifs, et identifier a_{n+1} avec a_1. Montrer que

$$\sum_{k=1}^{n} a_k^3 \;\geq\; \sum_{k=1}^{n} a_k a_{k+1}^2\,.$$

3305. *Proposé par Stanley Rabinowitz, MathPro Press, Chelmsford, MA, É-U.*

Montrer que

$$\begin{aligned}\tan\frac{2\pi}{13}+4\sin\frac{6\pi}{13} &= \tan\frac{4\pi}{13}+4\sin\frac{\pi}{13}\\ &= \tan\frac{5\pi}{13}+4\sin\frac{2\pi}{13} \;=\; \sqrt{13+2\sqrt{13}}\,.\end{aligned}$$

3306. *Proposé par Stanley Rabinowitz, MathPro Press, Chelmsford, MA, É-U.*

Trouver un nombre réel t et des polynômes $f(x)$, $g(x)$ et $h(x)$ à coefficients entiers, de sorte que

$$f(t)=\sqrt{2}\,,\qquad g(t)=\sqrt{3}\,,\qquad \text{et}\qquad h(t)=\sqrt{7}\,.$$

3307. *Proposé par D.E. Prithwijit, University College Cork, République d'Irlande.*

Eliminer θ du système

$$\begin{aligned}\lambda\cos(2\theta) &= \cos(\theta+\alpha)\,,\\ \lambda\sin(2\theta) &= 2\sin(\theta+\alpha)\,.\end{aligned}$$

3308. *Proposé par D.E. Prithwijit, University College Cork, République d'Irlande.*

Dans un triangle $\triangle ABC$, soit AD la hauteur abaissée sur le côté BC. Si $AB : AC = 1 : \sqrt{3}$, montrer que $AD \leq \frac{\sqrt{3}}{2} BC$. Quand y a-t-il égalité ?

3309. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit α, β et γ trois nombres réels non nuls donnés. Montrer que le système

$$\begin{aligned} \alpha x + \beta y + \gamma z &= 1 , \\ xy + yz + zx &= 1 , \end{aligned}$$

possède une seule solution (x, y, z) si et seulement si

$$\alpha^2 + \beta^2 + \gamma^2 + 1 = 2(\alpha\beta + \beta\gamma + \gamma\alpha) ,$$

et dans ce cas, trouver cette solution.

3310. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit respectivement a, b et c les longueurs des côtés BC, CA et AB du triangle ABC. Soit s le demi-périmètre du triangle ABC, r le rayon de son cercle inscrit, h_a la hauteur abaissée du sommet A sur le côté BC, ainsi que r_a, r_b et r_c les rayons des cercles exinscrits correspondants respectivement aux sommets A, B et C.

(a) Montrer que pour $x > 0$, on a $h_a = \dfrac{2s(s-a)x}{x^2 + s(s-a)}$ si et seulement si $x = r_b$ ou $x = r_c$.

(b) Montrer que pour $x > 0$, on a $h_a = \dfrac{2(s-b)(s-c)x}{|x^2 - (s-b)(s-c)|}$ si et seulement si $x = r$ ou $x = r_a$.

3311. *Proposé par Michel Bataille, Rouen, France.*

Soit n un entier avec $n \geq 2$. On suppose que pour $k = 0, 1, \ldots, n-2$ on a

$$\binom{n-2}{k} \equiv (-1)^k(k+1) \pmod{n} .$$

Montrer que n est un nombre premier.

3312. *Proposé par Michel Bataille, Rouen, France.*

Soit n un entier positif congruent à 1 modulo 6. Montrer que $3/n$ peut être exprimé comme

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$$

pour certains entiers positifs distincts a_1, a_2, $\ldots$, a_k, et trouver la valeur minimale de k.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3201. [2007 : 41, 44] *Proposed by G.P. Henderson, Garden Hill, Campbellcroft, ON, in memory of Murray S. Klamkin.*

Given positive integers m and n, consider the real monic polynomials $P(x) = \sum_{i=0}^{m} a_i x^i$ and $Q(x) = \sum_{j=0}^{n} b_j x^j$ with non-negative coefficients. We are interested in whether P and Q satisfy the condition

$$P(x)Q(x) = \sum_{k=0}^{m+n} x^k .$$

(a) Prove that if m and n are both odd, there are no such polynomials.

(b) Prove that if $m = n$, there are no such polynomials.

(c) Show that for each m there is an infinite set of values of n for which there do exist such polynomials.

(d) Prove that the coefficients in every such pair of polynomials are either 0 or 1.

(Compare problem 266 in Edward J. Barbeau, Murray S. Klamkin, and William O.J. Moser, *Five Hundred Mathematical Challenges*, where $m = n = 5$.)

Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela.

For s a positive integer, let $R_s(x) = \sum_{k=0}^{s} x^k$. Since

$$(x-1)R_s(x) = x^{s+1} - 1 ,$$

it follows that the roots of R_s are roots of unity. If s is odd, then R_s has just one real root, namely -1; if s is even, then all the roots of R_s are complex. If P is a real monic polynomial that divides R_s, then each root of P is necessarily a root of unity. If e^{ai} is a complex root of P, then its conjugate e^{-ai} is also a root, and $P(x)$ is divisible by $(x - e^{ai})(x - e^{-ai}) = x^2 - 2(\cos a)x + 1$. Hence, $P(x)$ factors into a product of these quadratic polynomials together with, if it has odd degree, the linear factor $x + 1$. Since all these factors are symmetric, so is P. In other words, if $P(x) = \sum_{j=0}^{m} a_j x^j$, then $a_j = a_{m-j}$

for $j = 0, \ldots, m$. Of course, the same discussion applies to the other given polynomial $Q(x)$; in particular, since P and Q are monic,

$$a_0 = a_m = 1 = b_n = b_0 .$$

We are now ready for the answers.

(a) If m and n are both odd, then P and Q should have one real root each; but $m + n$ is even so that their supposed product R_{m+n} would have no real roots, a contradiction.

(b) If $m = n$, then the coefficient of x^m in the product $PQ = R_{m+n}$ would be

$$1 = \sum_{j=0}^{m+n} a_j b_{m-j} \geq a_0 b_m + a_m b_0 = 2 ,$$

a contradiction.

(c) Set $n = k(m + 1)$, where k is an arbitrary positive integer. If $P(x) = \sum_{j=0}^{m} x^j$ and $Q(x) = \sum_{j=0}^{k} x^{j(m+1)}$, then

$$P(x)Q(x) = \sum_{j=0}^{k(m+1)+m} x^j = R_{m+n}(x) ,$$

as desired.

(d) Without loss of generality, assume that $m < n$. By symmetry, $\sum_{j=0}^{m} a_j b_j = \sum_{j=0}^{m} a_{m-j} b_j = 1$ (the coefficient of x^m in PQ). Since $a_0 = b_0 = 1$, we get $\sum_{j=1}^{m} a_j b_j = 0$; moreover, since the coefficients of P and Q are non-negative for $j = 1, \ldots, m$, at least one member of the pair (a_j, b_j) is 0 for all j, $1 \leq j \leq m$. Now let k be an integer, $1 \leq k \leq m$, and suppose that

$$\text{each } a_j \text{ and each } b_j \text{ with } 0 \leq i,\ j < k \text{ is either 0 or 1.} \qquad (1)$$

Then the coefficient of x^k in PQ is

$$1 = \sum_{j=0}^{k} a_j b_{k-j} = a_k + b_k + \sum_{j=1}^{k-1} a_j b_{k-j} .$$

By our assumption (1), each product $a_j b_{k-j}$ in the sum is either 1 or 0 for each j, $1 \leq j \leq k - 1$. If one of them is 1, it follows that all other terms in the sum are 0; in particular, we would have $a_k = b_k = 0$. Otherwise (if each $a_j b_{k-j}$ is 0), we get $a_k + b_k = 1$ and, since $a_k b_k = 0$, one of a_k or b_k is 0 and the other is 1. Thus, by induction we see that each coefficient of P is either 1 or 0. Dividing R_{m+n} by P, we see that Q must have integer coefficients. But no b_j can exceed 1 (it is bounded by the coefficient, namely 1, of x^j in PQ); whence, each b_j is either 1 or 0.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3202. [2007 : 41, 44] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let Γ be a circle with radius r, let A be any point on Γ, and let t be the tangent line to Γ at A. Let B and C be points of t on opposite sides of A such that $AB = mr$ and $AC = nr$ for some positive real numbers m and n. Let P be any point of Γ different from A. Show that $\cot\angle APB + \cot\angle APC$ is a constant for all such points P, and determine this constant value in terms of m and n.

I. *Similar solutions by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela; and the proposer.*

Set $\alpha = \angle CAP$, $\beta = \angle APB$, and $\gamma = \angle APC$. Since the angle between a chord and tangent equals half the angle at the centre subtended by the chord, we see that $AP = 2r\sin\alpha$; moreover, $\angle ABP = \alpha - \beta$. The Sine Law applied to $\triangle APB$ gives

$$\frac{\sin\beta}{mr} = \frac{\sin(\alpha-\beta)}{2r\sin\alpha} = \frac{\sin\alpha\cos\beta - \sin\beta\cos\alpha}{2r\sin\alpha};$$

whence,

$$\frac{2}{m} = \cot\beta - \cot\alpha. \qquad (1)$$

Similarly, since $\angle ACP = 180^\circ - \alpha - \gamma$, the Sine Law applied to $\triangle APC$ yields

$$\frac{\sin\gamma}{nr} = \frac{\sin(180^\circ-\alpha-\gamma)}{2r\sin\alpha} = \frac{\sin(\alpha+\gamma)}{2r\sin\alpha} = \frac{\sin\alpha\cos\gamma + \sin\gamma\cos\alpha}{2r\sin\alpha};$$

whence,

$$\frac{2}{n} = \cot\gamma + \cot\alpha. \qquad (2)$$

Finally, add (1) and (2) to obtain $\cot\beta + \cot\gamma = \frac{2}{m} + \frac{2}{n}$, which is a constant, as claimed.

II. *Similar solutions by Václav Konečný, Big Rapids, MI, USA; and Edmund Swylan, Riga, Latvia.*

We shall let $\alpha = \angle CAP$ (as in solution I) and use directed distances. Let B' and C' be the feet of the perpendiculars to PA from B and C, respectively. Then

$$\begin{aligned} PA &= 2r\sin\alpha\,, & & \\ PB' = PA + AB' &= 2r\sin\alpha + mr\cos\alpha\,, & BB' &= mr\sin\alpha\,, \\ PC' = PA - AC' &= 2r\sin\alpha - nr\cos\alpha\,, & CC' &= nr\sin\alpha\,. \end{aligned}$$

Thus,

$$\begin{aligned}\cot\angle APB + \cot\angle APC &= \frac{PB'}{BB'} + \frac{PC'}{CC'} \\ &= \frac{2r\sin\alpha + mr\cos\alpha}{mr\sin\alpha} + \frac{2r\sin\alpha - nr\cos\alpha}{nr\sin\alpha} \\ &= 2\,\frac{m+n}{mn}\,.\end{aligned}$$

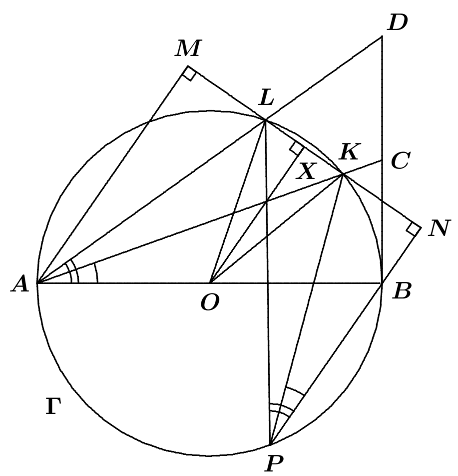
Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; GEOFFREY A. KANDALL, Hamden, CT, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania. There was one incorrect solution submitted.

3203. [2007 : 41, 45] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let AB be the diameter of a semicircle Γ. Let D be any point on the tangent to Γ at B and lying on the same side of AB as Γ, and let C be the mid-point of BD. The segments AC and AD intersect Γ for the second time at the points K and L, respectively. If M and N are the projections onto KL of A and B, respectively, show that $ML = LK = KN$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let P be the second point of intersection of the line NB and the circle Γ. Then $\angle BPL = \angle BAL$ and $\angle BPK = \angle BAK$, so that the triangles LPN and DAB are similar. Hence PK is the median of $\triangle LPN$, because AC is the median of $\triangle DAB$. Thus, $LK = KN$.



Let X be the mid-point of LK. Since $OL = OK$, then $OX \perp LK$. On the other hand, O is the mid-point of AB and $ABNM$ is a trapezoid with right angles at M and N; whence, X is also the mid-point of MN. It follows that $ML = KN$, which completes the proof.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, *MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; EDMUND SWYLAN, Riga, Latvia; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; TITU ZVONARU, Comănești, Romania; and the proposer.*

3204. [2007 : 42, 45] *Proposed by Mihály Bencze, Brasov, Romania.*

Let A, $J \in M_{n\times n}(\mathbb{R})$, where J is the matrix all of whose entries are 1, and let $b \in \mathbb{R}$. Set $B = bJ$, and for $k = 1, 2, \ldots, n$, denote by A_k the matrix obtained from A by replacing each element in row k with the value b. Prove that

$$\det(A+B)\det(A-B) \;=\; (\det A)^2 - \left(\sum_{k=1}^{n} \det A_k\right)^2 .$$

Similar solutions by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA.

For any $K \in M_{n\times n}(\mathbb{R})$, let $K^{(i)}$ denote the i^{th} row of K and let $U = (b\; b\; \ldots\; b) = B^{(i)}$, $i = 1, 2, \ldots, n$. We have

$$\begin{aligned}\det(A+B) &= \det\bigl(A^{(1)}+B^{(1)}, A^{(2)}+B^{(2)}, \ldots, A^{(n)}+B^{(n)}\bigr)\\ &= \det\bigl(A^{(1)}+U, A^{(2)}+U, \ldots, A^{(n)}+U\bigr) .\end{aligned}$$

Since a size n determinant is an n-linear function of its rows, we obtain

$$\begin{aligned}\det(A+B) &= \det\bigl(A^{(1)}, A^{(2)}, \ldots, A^{(n)}\bigr) + \det\bigl(U, A^{(2)}, \ldots, A^{(n)}\bigr)\\ &\quad + \det\bigl(A^{(1)}, U, \ldots, A^{(n)}\bigr) + \cdots + \det\bigl(A^{(1)}, A^{(2)}, \ldots, U\bigr) ,\end{aligned}$$

where a number of terms, being determinants with two rows equal to U, have vanished. Thus,

$$\det(A+B) \;=\; \det(A) + \sum_{k=1}^{n} \det A_k .$$

In the same way, with $-U$ instead of U, we obtain

$$\det(A-B) \;=\; \det(A) - \sum_{k=1}^{n} \det A_k .$$

Therefore,

$$\det(A+B)\det(A-B) \;=\; (\det A)^2 - \left(\sum_{k=1}^{n} \det A_k\right)^2 .$$

Also solved by JOE HOWARD, Portales, NM, USA; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA (second solution); JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

3205. [2007 : 42, 45] *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

Let $A_1A_2\cdots A_n$ be a convex polygon, and let P be any interior point of the polygon. For $k=1, 2, \dots, n$, let G_k be the centroid of the polygon $A_1A_2\cdots A_{k-1}A_{k+1}\cdots A_n$ (the polygon obtained by removing vertex A_k from $A_1A_2\cdots A_n$). If B_k is the reflection of G_k through the point P, prove that the lines A_iB_i are concurrent for $i=1, 2, \dots, n$.

I. *Solution by Michel Bataille, Rouen, France.*

For $k=1, 2, \dots, n$, we have

$$(n-1)\overrightarrow{G_k} \;=\; \sum_{j\neq k}\overrightarrow{A_j} \qquad \text{and} \qquad 2\overrightarrow{P} \;=\; \overrightarrow{G_k}+\overrightarrow{B_k}\,;$$

hence,

$$(n-1)\overrightarrow{B_k}+\sum_{j\neq k}\overrightarrow{A_j} \;=\; 2(n-1)\overrightarrow{P}\,.$$

Therefore,

$$(n-1)\overrightarrow{B_k}-\overrightarrow{A_k} \;=\; 2(n-1)\overrightarrow{P}-\sum_{j=1}^{n}\overrightarrow{A_j}\,,$$

a value independent of k. Now define U by $\overrightarrow{A_1U}=\frac{n-1}{n-2}\overrightarrow{A_1B_1}$. Then

$$(n-2)\overrightarrow{U} \;=\; (n-1)\overrightarrow{B_1}-\overrightarrow{A_1} \;=\; (n-1)\overrightarrow{B_k}-\overrightarrow{A_k}$$

for all k, $1\le k\le n$. Thus, the point U lies on each line A_kB_k, and the result follows.

Remarks.

1. There is no need for P to be interior to the polygon, nor for the polygon to be convex.
2. If G denotes the centroid of the polygon $A_1A_2\cdots A_n$, then

$$(n-2)\overrightarrow{U} \;=\; 2(n-1)\overrightarrow{P}-\sum_{j=1}^{n}\overrightarrow{A_j} \;=\; 2(n-1)\overrightarrow{P}-n\overrightarrow{G}\,.$$

This shows that U also lies on the line PG.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

We will generalize the result to $Z=A_1A_2\cdots A_n$, a closed polygonal path in n-dimensional space. Let G be the centroid of all n points. Then G lies on the line segment A_iG_i such that $GG_i : GA_i = 1 : n-1$.

Therefore, the polygonal path $Z'=G_1G_2\cdots G_n$ is similar to Z with each side G_iG_{i+1} being parallel to the side A_iA_{i+1} for all i, $1\le i\le n$.

After reflection through the point P, we similarly have the polygonal path $Z'' = B_1B_2\cdots B_n$ congruent to Z' with each side G_iG_{i+1} being parallel to the side B_iB_{i+1} for all i, $1 \le i \le n$.

Hence, Z' and Z'' are similar with each side B_iB_{i+1} being parallel to A_iA_{i+1} and having length $1/(n-1)$ that of the side A_iA_{i+1}.

Therefore, there is a centre of similitude between the similar polygonal paths Z and Z''.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece (2 solutions); JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposers.

Woo pointed out that the term "centroid of a polygon" is ambiguous; it may mean (i) the centroid of the vertices with equal weights, (ii) the centroid of the polygonal lamina, of (iii) the centroid of the set of n sides, like metal wires.

3206. [2007 : 42, 45] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer and x a real number. Prove that

$$\lfloor x\rfloor^2 + \left\lfloor x+\frac{1}{n}\right\rfloor^2 + \cdots + \left\lfloor x+\frac{n-1}{n}\right\rfloor^2 \;=\; \lfloor nx\rfloor^2$$

if and only if $\lfloor nx\rfloor = \lfloor x\rfloor$, where $\lfloor\cdot\rfloor$ denotes the greatest integer function.

Solution by Edmund Swylan, Riga, Latvia, modified by the editor.

The problem, as stated, is incorrect. An easy counterexample is $x = \frac{1}{2}$, $n = 2$. First we note that both

$$\lfloor x\rfloor^2 + \left\lfloor x+\frac{1}{n}\right\rfloor^2 + \cdots + \left\lfloor x+\frac{n-1}{n}\right\rfloor^2 \;=\; \lfloor nx\rfloor^2 \tag{1}$$

and

$$\lfloor nx\rfloor \;=\; \lfloor x\rfloor \tag{2}$$

are equations in the variables n and x, where $n \in \mathbb{N}^+$ and $x \in \mathbb{R}$. We solve these two equations and compare the sets of solutions. In the process, we establish the conditions under which the two equations are equivalent.

Let $x' = x - \lfloor x\rfloor$. Then $0 \le x' < 1$, so that $0 \le nx' < n$, and therefore,

$$\lfloor nx'\rfloor \in \{0, 1, \ldots, n-1\}\,.$$

In particular, $\lfloor nx'\rfloor = m$ if and only if $m \le nx' < m+1$ if and only if $m/n \le x' < (m+1)/n$.

Now, if $k \in \{0, 1, \ldots, n - \lfloor nx'\rfloor - 1\}$, then

$$\begin{aligned} 0 \;\le\; \frac{k}{n} \;\le\; x' + \frac{k}{n} \;&\le\; x' + \frac{n - \lfloor nx'\rfloor - 1}{n} \\ &=\; \frac{nx' - \lfloor nx'\rfloor + n - 1}{n} \;<\; \frac{1 + n - 1}{n} \;=\; 1\,, \end{aligned}$$

so that $\left\lfloor x' + \dfrac{k}{n}\right\rfloor = 0$.

Similarly, if $k \in \{n - \lfloor nx' \rfloor, n - \lfloor nx' \rfloor + 1, \ldots, n-1\}$, then

$$1 \le \frac{nx' + n - \lfloor nx' \rfloor}{n} \le x' + \frac{k}{n} \le x' + \frac{n-1}{n} < 2,$$

so that $\left\lfloor x' + \frac{k}{n} \right\rfloor = 1$.

Thus,

$$\begin{aligned} \left\lfloor x + \frac{k}{n} \right\rfloor &= \left\lfloor \lfloor x \rfloor + x' + \frac{k}{n} \right\rfloor = \lfloor x \rfloor + \left\lfloor x' + \frac{k}{n} \right\rfloor \\ &= \begin{cases} \lfloor x \rfloor, & \text{if } k \in \{0, 1, \ldots, n - \lfloor nx' \rfloor - 1\}, \\ \lfloor x \rfloor + 1, & \text{if } k \in \{n - \lfloor nx' \rfloor, n - \lfloor nx' \rfloor + 1, \ldots, n-1\}. \end{cases} \end{aligned}$$

Hence, the left side of equation (1) is

$$\big(n - \lfloor nx' \rfloor\big)\lfloor x \rfloor^2 + \lfloor nx' \rfloor\big(\lfloor x \rfloor + 1\big)^2 = n\lfloor x \rfloor^2 + 2\lfloor nx' \rfloor\lfloor x \rfloor + \lfloor nx' \rfloor .$$

Since $\lfloor nx \rfloor = \lfloor n(\lfloor x \rfloor + x') \rfloor = n\lfloor x \rfloor + \lfloor nx' \rfloor$, its right side is

$$\lfloor nx \rfloor^2 = n^2\lfloor x \rfloor^2 + 2n\lfloor x \rfloor\lfloor nx' \rfloor + \lfloor nx' \rfloor^2 .$$

Thus, equation (1) can be written as

$$(n-1)n\lfloor x \rfloor^2 + 2(n-1)\lfloor nx' \rfloor\lfloor x \rfloor + \lfloor nx' \rfloor^2 - \lfloor nx' \rfloor = 0 .$$

This is true if $n = 1$. Suppose $n \ge 2$. Since $\lfloor nx' \rfloor^2 - \lfloor nx' \rfloor \ge 0$, then

$$(n-1)\lfloor x \rfloor\big(n\lfloor x \rfloor + 2\lfloor nx' \rfloor\big) \le 0 .$$

If $\lfloor x \rfloor \le -2$, then $n\lfloor x \rfloor \le -2n$ and $2\lfloor nx' \rfloor \le 2n - 2$, so that

$$(n-1)\lfloor x \rfloor\big(n\lfloor x \rfloor + 2\lfloor nx' \rfloor\big) > 0 .$$

Similarly, if $\lfloor x \rfloor \ge 1$, then

$$(n-1)\lfloor x \rfloor\big(n\lfloor x \rfloor + 2\lfloor nx' \rfloor\big) > 0 .$$

If $\lfloor x \rfloor = -1$, then equation (1) becomes

$$(n-1)n - (2n-1)\lfloor nx' \rfloor + \lfloor nx' \rfloor^2 = 0 ,$$

or

$$\big((n-1) - \lfloor nx' \rfloor\big)\big(n - \lfloor nx' \rfloor\big) = 0 .$$

Since $n - \lfloor nx' \rfloor > 0$, then $\lfloor nx' \rfloor = n - 1$, which gives $(n-1)/n \le x' < 1$, or,

$$x = \lfloor x \rfloor + x' \in \big[-\tfrac{1}{n}, 0\big) .$$

If $\lfloor x \rfloor = 0$, then equation (1) becomes

$$\lfloor nx' \rfloor^2 - \lfloor nx' \rfloor = 0 ,$$

which happens if $\lfloor nx' \rfloor = 0$ or $\lfloor nx' \rfloor = 1$. This gives $0 \leq x' < 1/n$ or $1/n \leq x' < 2/n$; whence,

$$x = \lfloor x \rfloor + x' \in \left[0, \tfrac{2}{n}\right).$$

In summary, equation (1) has solutions

$$\{(1, x) \mid x \in \mathbb{R}\} \qquad \text{and} \qquad \left\{(n, x) \mid n \in \mathbb{N}^+ \setminus \{1\}, x \in \left[-\tfrac{1}{n}, \tfrac{2}{n}\right)\right\}.$$

Since $\lfloor nx \rfloor = \lfloor n(\lfloor x \rfloor + x') \rfloor = n\lfloor x \rfloor + \lfloor nx' \rfloor$, equation (2) can be written as

$$n\lfloor x \rfloor + \lfloor nx' \rfloor = \lfloor x \rfloor$$

or

$$\lfloor nx' \rfloor = (1 - n)\lfloor x \rfloor.$$

This is clearly true for $n = 1$. Suppose $n \geq 2$. Since $\lfloor nx' \rfloor \geq 0$, then $(1 - n)\lfloor x \rfloor \geq 0$, while $1 - n < 0$, so that $\lfloor x \rfloor \leq 0$. If $\lfloor x \rfloor \leq -2$, then $(1 - n)\lfloor x \rfloor \geq 2(n - 1)$, while $\lfloor nx' \rfloor \leq (n - 1)$. Therefore, $\lfloor x \rfloor$ can only be 0 or -1. Now, $\lfloor x \rfloor = 0$ implies that $\lfloor nx' \rfloor = 0$, which gives $x = \lfloor x \rfloor + x' \in \left[0, \frac{1}{n}\right)$. Likewise, $\lfloor x \rfloor = -1$ implies that $\lfloor nx' \rfloor = n - 1$, which gives $x = \lfloor x \rfloor + x' \in \left[-\frac{1}{n}, 0\right)$. In summary, equation (2) has solutions

$$\{(1, x) \mid x \in R\} \qquad \text{and} \qquad \left\{(n, x) \mid n \in \mathbb{N}^+ \setminus \{1\}, x \in \left[-\tfrac{1}{n}, \tfrac{1}{n}\right)\right\}.$$

Consequently, equations (1) and (2) are not equivalent, in general. They are equivalent under the additional assumption $x \notin \left[\frac{1}{n}, \frac{2}{n}\right)$ for $n \geq 2$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA. There were three incomplete solutions submitted.

3207. [2007 : 42, 45] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let a convex quadrilateral $APQC$ have its sides AP, PQ, and QC tangent to a minor circular arc ABC at the points A, B, and C, respectively. Let E be the projection of B onto AC. Let a semicircle with PQ as diameter cut AC at H and K, with H between A and K.

Without using trigonometry, prove that BE bisects $\angle PEQ$ and that PH bisects $\angle APB$.

Solution by Edmund Swylan, Riga, Latvia.

We show that $\angle BEP = \angle BEQ$. Let P' and Q' be the feet of the perpendiculars from P and Q to AC. Since the chord AC makes equal angles with the tangents PA and QC at its end-points, the right triangles PAP' and QCQ' are similar, so that

$$\frac{PP'}{QQ'} = \frac{PA}{QC}\,.$$

But tangents from an external point have equal lengths, namely $PA = PB$ and $QC = QB$; consequently,

$$\frac{PP'}{QQ'} = \frac{PB}{QB}\,. \tag{1}$$

Because the lines PP', BE, and QQ' are perpendicular to AC, they are parallel to one another and cut the transversals AC and PQ proportionally; specifically

$$\frac{PB}{QB} = \frac{P'E}{Q'E}\,. \tag{2}$$

Equations (1) and (2) imply that the right triangles PEP' and QEQ' are similar; whence, $\angle PEP' = \angle QEQ'$, and their complements, $\angle BEP$ and $\angle BEQ$ are equal, as desired.

To show that PH bisects $\angle APB$, we let M be the mid-point of PQ and H' be the point where the line through M parallel to PA meets AC. Then $MH' = \frac{1}{2}(PA + QC)$. [*Ed:* This claim was made without any justification, but this editor fails to see why it might be either obvious or well known; however, it is easily verified: Let C' be the point where the line through Q parallel to PA meets AC. Then $APQC'$ is a trapezoid whose mid-line MH' is the average of the two parallel sides; that is, $MH' = \frac{1}{2}(PA + QC')$. Moreover, $QC = QC'$ because triangle QCC' has equal angles at C and C' (since $\angle QCC' = \angle PAC$ as before, while the latter equals $\angle CC'Q$ because they are corresponding angles formed by the transversal AC'.)] Again using the equal tangents ($PA = PB$ and $QC = QB$), we deduce that $MH' = \frac{1}{2}(PB + BQ) = MP$. Thus, $H' = H$ because it is the point closest to A where the circle on diameter PQ (with radius MP) intersects AC. We therefore have

$$\begin{aligned}
\angle HPB = \angle HPM &= \frac{\pi - \angle PMH}{2} = \frac{\pi - \angle PQC'}{2}\\
&= \frac{\pi - (\angle PQQ' - \angle C'QQ')}{2}\\
&= \frac{(\pi - \angle PQQ') + \angle APP'}{2}\\
&= \frac{\angle P'PM + \angle APP'}{2} = \frac{\angle APB}{2}\,.
\end{aligned}$$

Thus, PH bisects $\angle APB$, as claimed.

Also solved by MICHEL BATAILLE, Rouen, France; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer. There was one incomplete solution.

3208. [2007 : 42, 45] *Proposed by Shaun White, student, Vincent Massey Secondary School, Windsor, ON.*

Find the largest integer k such that for all positive real numbers a, b, c, we have

$$(a^3+3)(b^3+6)(c^3+12) \;>\; k(a+b+c)^3\,.$$

Composite of nearly identical solutions by Apostolis Vergos, student, University of Patras, Patras, Greece; and the proposer.

Applying the generalized Hölder Inequality, we have

$$\begin{aligned}(a^3+3)^{1/3}&(b^3+6)^{1/3}(c^3+12)^{1/3}\\ &= (a^3+1+2)^{1/3}(2+b^3+4)^{1/3}(4+8+c^3)^{1/3}\\ &\ge a\sqrt[3]{2}\sqrt[3]{4}+b\sqrt[3]{8}+c\sqrt[3]{2}\sqrt[3]{4} \;=\; 2(a+b+c)\,.\end{aligned}$$

Hence, $(a^3+3)(b^3+6)(c^3+12) \ge 8(a+b+c)^3$. For equality to hold, we must have $\dfrac{a^3}{1}=\dfrac{2}{b^3}=\dfrac{4}{8}$ and $\dfrac{a^3}{2}=\dfrac{2}{4}=\dfrac{4}{c^3}$. In particular, $a^3=\dfrac{1}{2}$ and $a^3=1$, which is impossible. On the other hand, if $a=b=1$ and $c=2$, then

$$\begin{aligned}(a^3+3)(b^3+6)(c^3+12) &= 4\times 7\times 20 \;=\; 560 \;<\; 576 \;=\; 9\times 4^3\\ &= 9(a+b+c)^3\,.\end{aligned}$$

Therefore, $k=8$.

Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and STAN WAGON, Macalester College, St. Paul, MN, USA. There was also an incomplete solution.

Lau proved that the largest real value λ such that

$$(a^3+3)(b^3+6)(c^3+12) \;\ge\; \lambda(a+b+c)^3$$

is $\lambda=\tau^2$ where $\tau=7\cos(\frac{1}{3}\cos^{-1}\frac{89}{343})-\frac{7}{2}$ is the unique positive root of the cubic equation $2x^3+21x^2-216=0$. He computed $\lambda=8.093$, to 3 decimal places, while Hess and Wagon calculated λ to 8 and 15 places, respectively. Wagon's solution is based on computer algebra.

3209. [2007 : 42, 46] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let f be a convex function on an interval I. For $i=1, 2, \ldots, n$, let $a_i\in I$. Define $a=\dfrac{1}{n}\sum\limits_{i=1}^{n}a_i$. Prove that

$$\frac{n(n-2)}{2}f(a)+\sum_{i=1}^{n}f(a_i) \;\ge\; \frac{n}{2(n-1)}\sum_{i\ne j}f\left(a+\frac{a_i-a_j}{n}\right)\,.$$

Solution by Michel Bataille, Rouen, France.

For $i \neq j$ let $b_j = \frac{1}{n-1}\sum\limits_{k\neq j} a_k$; hence, by Jensen's Inequality,

$$f\left(a + \frac{a_i - a_j}{n}\right) = f\left(\frac{a_i + (n-1)b_j}{n}\right) \leq \frac{1}{n}f(a_i) + \frac{n-1}{n}f(b_j)\,.$$

Thus,

$$\begin{aligned}
&\frac{n}{2(n-1)}\sum_{i\neq j} f\left(a + \frac{a_i - a_j}{n}\right) \\
&\quad\leq \frac{n}{2(n-1)}\sum_{i\neq j}\left(\frac{1}{n}f(a_i) + \frac{n-1}{n}f(b_j)\right) \\
&\quad= \frac{n}{2(n-1)}\left(\frac{1}{n}(n-1)\sum_{i=1}^{n} f(a_i) + \frac{n-1}{n}(n-1)\sum_{j=1}^{n} f(b_j)\right) \\
&\quad= \frac{1}{2}\sum_{i=1}^{n} f(a_i) + \frac{n-1}{2}\sum_{j=1}^{n} f(b_j)\,.
\end{aligned}$$

From a generalization of Popoviciu's Inequality proved by the proposer of this problem and presented in this journal ([2005 : 313–318]), we have

$$(n-1)\sum_{j=1}^{n} f(b_j) \leq \sum_{i=1}^{n} f(a_i) + n(n-2)f(a)\,,$$

and therefore

$$\begin{aligned}
&\frac{n}{2(n-1)}\sum_{i\neq j} f\left(a + \frac{a_i - a_j}{n}\right) \\
&\qquad\qquad \leq \frac{1}{2}\sum_{i=1}^{n} f(a_i) + \frac{1}{2}\left(\sum_{i=1}^{n} f(a_i) + n(n-2)f(a)\right).
\end{aligned}$$

Also solved by ALEXANDROS SYGELAKIS, student, University of Crete, Heraklion, Greece; and the proposer.

3210. [2007 : 43, 46] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Prove that, for all real numbers $a_1, a_2, \ldots, a_n \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, we have

$$\sum_{i=1}^{n} \frac{3}{a_i + 2a_{i+1}} \geq \sum_{i=1}^{n} \frac{2}{a_i + a_{i+1}}\,,$$

where the subscripts are taken modulo n.

Solution by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

Rewrite the inequality as

$$\sum_{i=1}^{n} \frac{a_i - a_{i+1}}{(a_i + 2a_{i+1})(a_i + a_{i+1})} \;\geq\; 0\,.$$

Denote the left side of this inequality by L. Then, since $\sum\limits_{i=1}^{n} \dfrac{a_i - a_{i+1}}{a_i a_{i+1}} = 0$, we have

$$\begin{aligned} 6L &= \sum_{i=1}^{n} \frac{6(a_i - a_{i+1})}{(a_i + 2a_{i+1})(a_i + a_{i+1})} - \sum_{i=1}^{n} \frac{a_i - a_{i+1}}{a_i a_{i+1}} \\ &= \sum_{i=1}^{n} \frac{(a_i - a_{i+1})^2(2a_{i+1} - a_i)}{a_i a_{i+1}(a_i + 2a_{i+1})(a_i + a_{i+1})}\,. \end{aligned}$$

Since $\frac{1}{\sqrt{2}} \leq a_i \leq \sqrt{2}$, we have $2a_{i+1} - a_i \geq 0$ for all $i = 1, 2, \ldots, n$. Hence, $L \geq 0$ and equality holds if and only if all the a_is are equal.

Also solved by the proposer with essentially the same solution.

3211. [2007 : 43, 46] *Proposed by an anonymous proposer.*

Let $ABCD$ be a quadrilateral which is inscribed in a circle Γ. Further suppose that $ABCD$ itself has an incircle. Let EF be the diameter of Γ which is perpendicular to BD, with E lying on the same side of BD as A. Let BD intersect EF at M and AC at S.

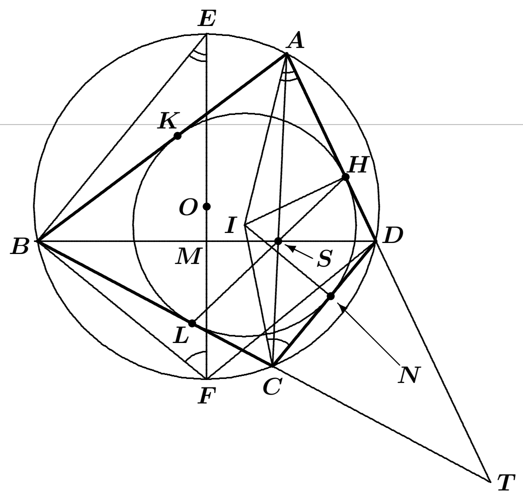
Prove that $AS : SC = EM : MF$.

[*Ed:* This problem came into the University of Regina's Math Central website, but the name of the proposer has subsequently been lost.]

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.

Let I be the centre of the inscribed circle and let K, L, N, and H be the tangency points of the sides AB, BC, CD, and DA, respectively. Finally, define T to be the intersection point of lines AD and BC. The inscribed angles $\angle BCD$ and $\angle BFD$ are equal; because $\angle ICN = \angle ICD = \frac{1}{2}\angle BCD$ and $\angle BFM = \angle BFE = \frac{1}{2}\angle BFD$, it follows that $\angle ICN = \angle BFM$. Consequently, the right triangles ICN and BFM are similar; whence, $\dfrac{FM}{CN} = \dfrac{BM}{IN}$. Similarly, triangles IAH and BEM are similar, and we get $\dfrac{EM}{AH} = \dfrac{BM}{IH}$. Since the inradii $IN = IH$, we have $\dfrac{FM}{CN} = \dfrac{EM}{AH}$, or

$$\frac{EM}{FM} = \frac{AH}{CN}\,. \tag{1}$$



For the other ratio we observe that Brianchon's Theorem applied to $ABCD$ implies that S is on HL; that is, if a conic (here the given incircle) is inscribed in a quadrangle, the chords joining the points of tangency of opposite sides (namely HL and KN) go through the intersection point of the two diagonals (namely $S = AC \cap BD$). By Menelaus' Theorem applied to the transversal HSL of $\triangle TAC$,

$$\frac{TH}{HA} \cdot \frac{AS}{SC} \cdot \frac{CL}{LT} = -1\,;$$

or, since $TH = TL$ and $CL = CN$,

$$\frac{AS}{SC} = \frac{AH}{CN}\,. \tag{2}$$

The result follows from equations (1) and (2).

Also solved by MICHEL BATAILLE, Rouen, France; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the UNIVERSITY OF REGINA MATH CENTRAL CONSULTANTS. There was one incomplete solution submitted.

Malikić informed us that the problem has previously appeared as problem 2027 [1995 : 90; 1996 : 94-95], *proposed by D.J. Smeenk. Since the notation here is identical to that used twelve years ago, it seems clear that Smeenk was the original source of the problem. Several of the solvers this time around devised an elegant area argument much like that of the featured solution from 1996.*

3212. [2007 : 43, 46] *Proposed by José Luis Díaz-Barrero and Francisco Palacios Quiñonero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a_1, a_2, ..., a_n be real numbers such that $a_k \geq 1$ for $1 \leq k \leq n$. Prove that

$$\prod_{k=1}^{n} a_k^{\left(\frac{2k}{n(n+1)}\right)^{1/2}} \leq \exp\left(\sqrt{\sum_{k=1}^{n} \ln^2 a_k}\right)\,.$$

Solution by Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain.

Setting $x_k = \ln a_k$, the inequality becomes

$$\prod_{k=1}^{n} \exp\left(\sqrt{\frac{2k}{n(n+1)}}\, x_k\right) \;\le\; \exp\left(\sqrt{\sum_{k=1}^{n} x_k^2}\right),$$

or equivalently,

$$\sum_{k=1}^{n} \sqrt{k}\, x_k \;\le\; \sqrt{\frac{n(n+1)}{2} \sum_{k=1}^{n} x_k^2},$$

which follows from the Cauchy-Schwarz Inequality since

$$\left(\sum_{k=1}^{n} \sqrt{k}\, x_k\right)^2 \;\le\; \left(\sum_{k=1}^{n} k\right)\left(\sum_{k=1}^{n} x_k^2\right) \;=\; \frac{n(n+1)}{2} \sum_{k=1}^{n} x_k^2 .$$

Equality holds if and only if $\frac{x_1}{\sqrt{1}} = \frac{x_2}{\sqrt{2}} = \cdots = \frac{x_n}{\sqrt{n}}$.

Also solved, using essentially the same method, by DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; JOE HOWARD, Portales, NM, USA; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; ALEXANDROS SYGELAKIS, student, University of Crete, Heraklion, Greece; APOSTOLIS VERGOS, student, University of Patras, Patras, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3213. [2007 : 109, 113] *Proposed by Mihály Bencze, Brasov, Romania.*

(a) Let a and b be positive real numbers with $a < b$ and let $f : [a,b] \to \mathbb{R}$ be a continuously differentiable and strictly monotone function. Show that there is a real number $c \in (a,b)$ such that

$$(a+b)f(c) \;=\; af(a) + bf(b) .$$

(b)★ Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $a_1 < a_2 < \cdots < a_n$ and let $f : [a_1, a_n] \to \mathbb{R}$ be a continuously differentiable and strictly monotone function. Show that there is a real number $c \in (a_1, a_n)$ such that

$$\left(\sum_{k=1}^{n} a_k\right) f(c) \;=\; \sum_{k=1}^{n} a_k f(a_k) .$$

Solution by Michel Bataille, Rouen, France.

For part (a) we assume that f is continuous and that $f(a) \neq f(b)$. If $m = \dfrac{af(a)+bf(b)}{a+b}$, then

$$m - f(a) \;=\; \frac{b(f(b)-f(a))}{a+b} \qquad \text{and} \qquad m - f(b) \;=\; \frac{a(f(a)-f(b))}{a+b}\,.$$

Since $f(a) \neq f(b)$, we see that $m - f(a)$ and $m - f(b)$ have opposite signs and that m is between $f(a)$ and $f(b)$. Since f is continuous, it follows by the Intermediate Value Theorem that $m = f(c)$ for some $c \in (a, b)$, as required.

For part (b) it is enough to assume that f is continuous and that either $f(a_1) < f(a_k) < f(a_n)$ or $f(a_1) > f(a_k) > f(a_n)$ for $k = 2, \ldots, n-1$. Letting $s = \sum\limits_{k=1}^{n} a_k$, it follows from either of the above inequalities that $\frac{1}{s}\sum\limits_{k=1}^{n} a_k f(a_k)$ is between $f(a_1)$ and $f(a_n)$. Note that $\frac{a_k}{s} > 0$ and $\sum\limits_{k=1}^{n} \frac{a_k}{s} = 1$. Therefore, $\sum\limits_{k=1}^{n} a_k f(a_k)$ is a convex combination of $f(a_k)$ for $k = 1, 2, \ldots, n$. Again, by the Intermediate Value Theorem, for some $c \in (a_1, a_n)$, we have

$$\frac{1}{s}\sum_{k=1}^{n} a_k f(a_k) \;=\; f(c)\,.$$

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. All solvers indicated that the differentiability of the function is not necessary.