

## SKOLIAD No. 98

Robert Bilinski

The year 2006 is now at an end, and so is my second year as Skoliad Editor. A lot of things have happened in my life, not all of them happy. Thankfully, *Crux Mathematicorum* and my little part in it have brought me a joyful reprieve. Hence, I would like to thank you all.

In particular, I would like to thank Jim Totten, for his patience with my testing of the limits of the editing schedule, and the other editors of *Crux Mathematicorum* for their comments.

I also want to thank the contest providers for helping me have a “regular”, but mostly an interesting and diversified, rotation of contests. Namely, my heartfelt thanks go out to: David Horrocks, Véronique Hussin, André Labelle, Clint Lee, John Grant McLoughlin, Warren Palmer, Ron Persky, and Don Rideout. My thanks go out not only for the help they bring to Skoliad, but also for the investment of time and energy they give to the promotion of mathematics in their neck-of-the-woods.

Lastly, I also thank our problem solvers (young and old, regular and new contributors) for solutions sent in during the past year. Thanks to Jean-François Désilets, Jean-David Houle, Vedula N. Murty, Khartik Natarajan, Carl O'Connor, Alex Remorov, Jia-Xi Sun, and Edward T.H. Wang.

I hope to see all of you in the coming year. Happy New Year!

Please send your solutions to the problems in this edition by **1 June, 2007**. A copy of **MATHEMATICAL MAYHEM Vol. 8** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our questions this time come from the 2006 Maritime Mathematics Contest. Thanks go to David Horrocks from the University of Prince Edward Island and John Grant McLoughlin from the University of New Brunswick.

### Concours de Mathématiques des Maritimes 2006

**1.** Après neuf heures, quelle est la prochaine heure à laquelle les aiguilles d'une horloge forment un angle droit ?

**2.** Pour un nombre positif comme 3, 14, on appelle 3 la *partie entière* et 0, 14 la *partie décimale*. Trouver un nombre positif tel que sa partie décimale, sa partie entière et le nombre lui-même sont trois termes consécutifs

(a) d'une suite arithmétique ;                      (b) d'une suite géométrique.

(Une suite  $a_1, a_2, a_3, a_4, \dots$  est dite *arithmétique* si pour un certain nombre  $d$ , on a  $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \dots = d$ ; elle est dite *géométrique* si pour un certain nombre  $r \neq 0$ , on a  $a_2/a_1 = a_3/a_2 = a_4/a_3 = \dots = r$ .)

**3.** Une cuve rectangulaire de longueur 60 cm, de largeur 60 cm et de hauteur 40 cm est remplie d'eau jusqu'à une profondeur de 15 cm et repose sur une table horizontale. Soient  $A$ ,  $B$ ,  $C$  et  $D$  en ordre cyclique les quatre coins de la base de la cuve. On vide partiellement la cuve en soulevant lentement l'arête  $BC$  de sorte que la cuve pivote autour de l'arête  $AD$ . Quand l'angle que fait l'arête  $AB$  avec le plan de la table atteint  $60^\circ$ , on retourne la cuve à sa position originale. Quelle est maintenant la profondeur de l'eau dans la cuve?

**4.** Écrivons les entiers positifs en spirale à droite. Où se situe le nombre 2006 relatif au nombre 1? (Par exemple, relatif à 1, le nombre 10 est situé une rangée plus haut et deux colonnes à la droite.)

7	8	9	10
6	1	2	11
5	4	3	12
...	14	13	

**5.** Il se peut pour une paire  $(x, y)$  d'entiers positifs que  $x + y$  et  $xy$  soient tous les deux des carrés parfaits. Par exemple,  $(5, 20)$  est une telle paire puisque  $5 + 20 = 25$  et  $5 \times 20 = 100$  sont des carrés parfaits. Montrer que le nombre 3 n'appartient à aucune telle paire.

**6.** Trouver toutes les valeurs réelles de  $x$  et  $y$  qui satisfont au système d'équations ci-dessous.

$$\begin{aligned} 2(x + y - 2) &= y(x - y + 2), \\ x^2(y - 1) + y^2(x - 1) &= xy - 1. \end{aligned}$$

## 2006 Maritime Mathematics Contest

**1.** At 9 o'clock, the hour and minute hands on a clock form a right angle. After 9 o'clock, what is the next time at which the clock hands form a right angle?

**2.** For a positive number such as 3.14, we call 3 the *integer part* and 0.14 the *fractional part*. Find a positive number such that the fractional part, the integer part, and the number itself are three consecutive terms

(a) in an arithmetic sequence;                      (b) in a geometric sequence.

(The sequence  $a_1, a_2, a_3, a_4, \dots$  is called *arithmetic* if there is a number  $d$  such that  $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \dots = d$ ; it is called *geometric* if there is a number  $r \neq 0$  such that  $a_2/a_1 = a_3/a_2 = a_4/a_3 = \dots = r$ .)

**3.** A rectangular tank having length 60 cm, width 60 cm, and height 40 cm is filled with water to a depth of 15 cm and rests on a horizontal table. Let  $A$ ,  $B$ ,  $C$ , and  $D$  in cyclic order be the four bottom corners of the tank. Suppose that the edge  $BC$  is slowly raised so that the edge  $AD$  remains on the table. As water flows out, the tank is raised until the edge  $AB$  makes an angle of  $60^\circ$  with the table. The edge  $BC$  is then lowered until the tank once again rests on the table. At this point, what is the depth of water in the tank?

4. Suppose that the positive integers are written in a spiral as shown. Relative to the number 1, where does the number 2006 appear? (For example, 10 appears one unit up and two units to the right of 1.)

7	8	9	10
6	1	2	11
5	4	3	12
...	14	13	

5. A *square pair* is a pair  $(x, y)$  of positive integers such that  $x + y$  and  $xy$  are both perfect squares. For example,  $(5, 20)$  is a square pair since  $5 + 20 = 25$  and  $5 \times 20 = 100$  are both perfect squares. Show that no square pair exists in which one of the numbers is 3.

6. Find all solutions in real numbers  $x$  and  $y$  for the system of equations:

$$\begin{aligned} 2(x + y - 2) &= y(x - y + 2), \\ x^2(y - 1) + y^2(x - 1) &= xy - 1. \end{aligned}$$

Next we give solutions to the Concours de l'Association Mathématique du Québec 2004 (niveau secondaire) [2006 : 130–133].

1. (Les vases d'eau salée.) Deux vases,  $A$  et  $B$ , d'une capacité de six litres chacun, contiennent chacun quatre litres d'eau salée, selon les concentrations suivantes :  $A$  contient 5% de sel et  $B$  contient 10% de sel. On vide un litre d'eau salée du vase  $A$  dans le vase  $B$  puis on mélange. On vide ensuite un litre du vase  $B$  dans le vase  $A$ , puis on mélange à nouveau. Quelle concentration de sel (en pourcentage) chacun des vases  $A$  et  $B$  contiennent-ils maintenant ?

[Ed: The English version of this was incorrectly stated. The first sentence should have read: "Two vases,  $A$  and  $B$ , each having a capacity of 6 litres, each contain 4 litres of salt-water solutions in the following concentrations: ..."]

*Solution par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC.*

En versant 1 litre du vase  $A$  dans le vase  $B$ , on obtient 3 litres d'eau salée à 5% dans le vase  $A$ . Pour le vase  $B$ , on a

$$\begin{aligned} c_1 v_1 + c_2 v_2 &= c_f v_f, \\ (5\%)(1L) + (10\%)(4L) &= c_f(5L), \\ c_f &= \frac{1}{5}(5\% + 40\%) = 9\%. \end{aligned}$$

Donc il y a 5 litres d'eau salée à 9% dans le vase  $B$ . Ensuite, on verse 1 litre du vase  $B$  dans le vase  $A$  pour obtenir 4 litres d'eau salée à 9% dans le vase  $B$ . Pour le vase  $A$ , on a

$$\begin{aligned} c_1 v_1 + c_2 v_2 &= c_f v_f, \\ (9\%)(1L) + (5\%)(3L) &= c_f(4L), \\ c_f &= \frac{1}{4}(9\% + 15\%) = 6\%. \end{aligned}$$

Le résultat final est donc 6% de sel dans le vase  $A$  et 9% de sel dans le vase  $B$ .

Aussi solutionné par Alexander Remorov, étudiant, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

Commentaire par Alexandre Remorov : On a  $5\%(4) + 10\%(4) = 0,6$  litres de sel au début ; à la fin, on a  $6\%(4) + 9\%(4) = 0,6$  litres, la même qu'au début. On a donc une bonne manière de vérifier que nos calculs ont une bonne chance d'être bons.

**2.** (La multiplication de Koallo.) Koallo habite le joli village d'Oloko, au Nigéria. Comme il aime les mathématiques, il a remarqué récemment, qu'avec une correspondance appropriée entre les chiffres et les lettres et en multipliant par 11 le nom de son village, il obtenait son nom ! Êtes-vous capable de faire comme lui ? Plus précisément, pouvez-vous trouver les chiffres différents que doivent représenter les lettres  $O$ ,  $L$ ,  $K$  et  $A$  pour que l'équation  $OLOKO \times 11 = KOALLO$  soit vraie. Attention,  $OLOKO$  doit être vu comme un nombre de cinq chiffres et non comme le produit  $O \times L \times O \times K \times O$ . Il en va de même pour  $KOALLO$ .

*Solution officielle.*

En effectuant la multiplication de la façon usuelle, on obtient la somme illustrée à droite. On désignera les rangs des colonnes en commençant par la droite. La 6<sup>e</sup> colonne nous apprend que  $K = O + 1$  et que la 5<sup>e</sup> colonne génère une retenue. Sachant cela et examinant la 5<sup>e</sup> colonne, on déduit que  $L = 9$  et que la 4<sup>e</sup> colonne génère une retenue et donc que  $O = A + 1$ . On a donc que les nombres  $A$ ,  $O$  et  $K$  sont consécutifs.

Puisque  $L = 9$ , les 2<sup>e</sup> et 3<sup>e</sup> colonnes ne génèrent pas de retenues et on a  $K + O = 9$ . Cela nous donne  $K = 5$ ,  $O = 4$  et alors  $A = 3$ .

En somme,  $(O, L, K, A) = (4, 9, 5, 3)$ .

Aussi solutionné par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC ; et Alexander Remorov, étudiant, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

**3.** (Les nombres de Fibonacci dans des triangles de Pythagore.) La suite : 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... est la célèbre suite de Fibonacci. On voit que si on commence avec 1 et 2, les termes qui suivent sont toujours obtenus comme la somme des deux nombres précédents de la suite. Ainsi, par exemple, on a  $3 = 2 + 1$ ,  $5 = 3 + 2$ ,  $8 = 5 + 3$ . Un triangle de Pythagore est, quant à lui, un triangle rectangle dont la longueur de chacun des côtés est un nombre entier.

On remarque alors que si nous prenons quatre termes consécutifs de la suite de Fibonacci, quelques opérations simples nous permettent de former des triangles de Pythagore. Par exemple, soit les quatre nombres 3, 5, 8 et 13, alors un premier côté  $x$  du triangle est obtenu en prenant deux fois le produit des deux nombres du milieu ( $x = 2 \times 5 \times 8 = 80$ ), le deuxième côté  $y$  est obtenu en multipliant le premier et le dernier des quatre nombres ( $y = 3 \times 13 = 39$ ) et le dernier côté  $z$  est égal à la somme des carrés des deux nombres du milieu ( $z = 5^2 + 8^2 = 89$ ). Ainsi, on a bien obtenu un triangle de Pythagore car on a  $80^2 + 39^2 = 89^2$ .

- (a) Vérifiez que cela marche aussi si on prend les nombres 2, 3, 5 et 8.  
 (b) Pouvez-vous montrer que cela marche tout le temps? Plus précisément, si  $a, b, c$  et  $d$  désignent quatre nombres consécutifs de la suite de Fibonacci et que l'on pose  $x = 2bc$ ,  $y = ad$  et  $z = b^2 + c^2$ , montrer que  $x, y$  et  $z$  forment les côtés entiers d'un triangle rectangle.

*Solution identique par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC; et Alexander Remorov, étudiant, William Lyon Mackenzie Collegiate Institute, Toronto, ON.*

(a) Avec les nombres 2, 3, 5 et 8, on obtient  $x = 2 \times 3 \times 5 = 30$ ,  $y = 2 \times 8 = 16$  et  $z = 3^2 + 5^2 = 34$ . Et on obtient bien un triangle de Pythagore car  $30^2 + 16^2 = 34^2$ .

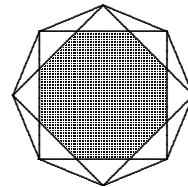
(b) Puisque  $a, b, c$  et  $d$  font parties de la suite de Fibonacci, on peut écrire  $a = c - b$  et  $d = b + c$ . Ainsi, on a  $x = 2bc$ ,  $y = ad = c^2 - b^2$  et  $z = b^2 + c^2$ . Mais  $x^2 + y^2 = (2bc)^2 + (c^2 - b^2)^2 = (c^2 + b^2)^2 = z^2$ , ce qui prouve que  $x, y$  et  $z$  forment les côtés entiers d'un triangle rectangle.

**4.** (That figures!) Find the number of digits and the sum of the digits for the integer  $16^8 \times 5^{30}$ .

*Identical solutions by Jean-David Houle, student, Cégep de Drummondville, Drummondville, QC and Alexander Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.*

We observe that  $16^8 \times 5^{30} = 2^{32} \times 5^{30} = 4 \times 10^{30}$ . Hence, the product has 31 digits, namely a 4 and thirty 0s, which means they sum to 4.

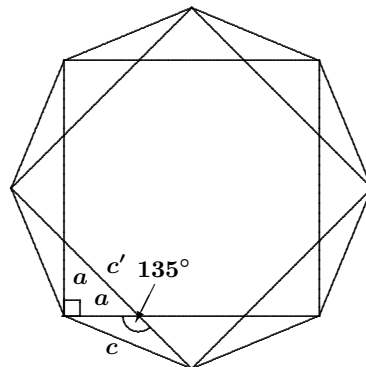
**5.** (L'octogone.) Si on relie entre eux les sommets d'un octogone régulier qui ont un sommet voisin en commun, on obtient au centre de la figure un nouvel octogone régulier, décalé et plus petit que le premier (en gris sur le dessin). Si l'aire de l'octogone initial est 1, quelle est l'aire du nouvel octogone?



*Indice :* lorsque deux figures sont semblables, le rapport de leurs aires est égal au carré du rapport de leurs côtés homologues.

*Solution par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC.*

Puisque le grand octogone est régulier, alors chacun des angles intérieurs vaut  $180^\circ \times (8 - 2)/8 = 135^\circ$ . On note  $c$  la longueur d'un côté du grand octogone,  $c'$  le côté du petit octogone et  $a$  la longueur illustrée sur le dessin. Par Pythagore, on a  $(c')^2 = a^2 + a^2 = 2a^2$ , ou bien  $c' = a\sqrt{2}$ .



Soit  $A$  l'aire du petit octogone. Ainsi, puisque le petit et le grand octogone sont semblables, nous avons :

$$A = \frac{A}{1} = \left(\frac{c'}{c}\right)^2 = \left(\frac{a\sqrt{2}}{c}\right)^2 = 2\left(\frac{a}{c}\right)^2.$$

Par la loi des sinus, on trouve :

$$\frac{\sin(45^\circ/2)}{a} = \frac{\sin(135^\circ)}{c}, \quad \text{ou bien} \quad \frac{a}{c} = \frac{\sin(45^\circ/2)}{\sin(135^\circ)}.$$

Ainsi on a

$$A = \frac{2 \sin^2(45^\circ/2)}{\sin^2 135^\circ} = \frac{1 - \cos 45^\circ}{\sin^2 135^\circ} = \frac{1 - (1/\sqrt{2})}{(1/\sqrt{2})^2} = 2 - \sqrt{2}.$$

*Aussi solutionné par Alexander Remorov, étudiant, William Lyon Mackenzie Collegiate Institute, Toronto, ON.*

**6.** (Presto . . . without calculators!) Explain why the following equality holds.

$$\begin{aligned} \frac{2004^2}{2003 \times 2005} + \frac{2005^2}{2004 \times 2006} + \cdots + \frac{3004^2}{3003 \times 3005} \\ = 1001 + \frac{1}{2} \left( \frac{1}{2003} + \frac{1}{2004} - \frac{1}{3004} - \frac{1}{3005} \right). \end{aligned}$$

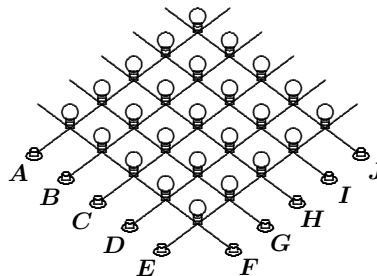
*Hint:* Decompose each of the terms  $\frac{2004^2}{2003 \times 2005}$ ,  $\frac{2005^2}{2004 \times 2006}$ , etc. appropriately, then add them all.

*Identical solutions by Jean-David Houle, student, Cégep de Drummondville, Drummondville, QC; and Alexander Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.*

We have  $\frac{(n+1)^2}{n(n+2)} = 1 + \frac{1}{n(n+2)} = 1 + \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right)$ . Hence,

$$\begin{aligned} \sum_{n=2003}^{3003} \frac{(n+1)^2}{n(n+2)} &= 1001 + \frac{1}{2} \sum_{n=2003}^{3003} \left( \frac{1}{n} - \frac{1}{n+2} \right) \\ &= 1001 + \frac{1}{2} \left( \frac{1}{2003} - \frac{1}{2005} + \frac{1}{2004} - \frac{1}{2006} + \cdots + \frac{1}{3003} - \frac{1}{3005} \right) \\ &= 1001 + \frac{1}{2} \left( \frac{1}{2003} + \frac{1}{2004} - \frac{1}{3004} - \frac{1}{3005} \right). \end{aligned}$$

**7.** (Les ampoules de Raoul.) Raoul se confectionne un circuit électrique formé de vingt-cinq ampoules disposées en carrés et de dix interrupteurs, notés de  $A$  à  $J$ , comme sur le dessin. S'il appuie sur un interrupteur, alors les cinq ampoules situées sur la ligne de cet interrupteur voient leur état inversé : celles qui sont allumées s'éteignent tandis que celles qui sont éteintes s'allument.



- (a) Montrer que, quelque soit l'état initial des ampoules (certaines ampoules peuvent être allumées tandis que d'autres, non), il est toujours possible de manipuler les interrupteurs de telle sorte que, dans chacune des dix rangées correspondantes, il y ait toujours plus d'ampoules allumées qu'éteintes.
- (b) Est-il toujours possible d'allumer toutes les ampoules en même temps ? Que votre réponse soit oui ou non, il faut donner la preuve de ce que vous avancez.

*Solution par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC.*

(a) Si une ligne ne contient pas plus d'ampoules allumées que d'ampoules éteintes, et le nombre d'ampoules étant impair, alors appuyer sur l'interrupteur correspondant à cette ligne permettra d'obtenir plus d'ampoules allumées que d'ampoules éteintes. Ainsi, si on appuie sur l'interrupteur de chacune des lignes non-conformes, alors le nombre fini d'ampoules éteintes diminuera.

Puisque les interrupteurs sont indépendants, il n'y aura pas de situations en boucle, en ce sens que, si on répète le processus ci-dessus, alors on arrivera nécessairement à un instant où toutes les lignes contiendront plus d'ampoules allumées que d'ampoules éteintes.

(b) Si on appuie un nombre pair de fois sur un interrupteur, alors la situation de chacune des ampoules de cette ligne adopte son état initial. Si on appuie un nombre impair de fois sur un interrupteur, alors la situation de chacune des ampoules de cette ligne voit son état inversé.

De plus, l'ordre dans lequel on appuie sur les interrupteurs n'a aucune importance, car les interrupteurs sont indépendants les uns des autres. On en conclut donc que toutes les configurations d'ampoules possibles avec un tel système peuvent s'obtenir en appuyant au plus une fois sur chaque interrupteur.

Le nombre de configurations atteignables avec les interrupteurs est donc de  $2^{10}$ . Par contre, le nombre de configurations possibles pour les ampoules est de  $2^{25}$  (car il y a 25 ampoules).

On voit donc que, grâce aux interrupteurs, on ne peut pas accéder à tous les états possibles. Il n'est donc pas toujours possible d'allumer toutes les ampoules à partir de toutes les configurations initiales.

*La partie (b) a aussi été solutionnée par Alexander Remorov, étudiant, William Lyon Mackenzie Collegiate Institute, Toronto, ON.*

That brings us to the end of another issue. This month's winners of a past Volume of Mayhem are Jean-David Houle, student, Cégep de Drummondville, Drummondville, QC and Alexander Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON. Congratulations Jean-David and Alexander ! Continue sending in your contests and solutions.

## MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is Jeff Hooper (Acadia University). The other staff members are John Grant McLoughlin (University of New Brunswick), Ian VanderBurgh (University of Waterloo), Larry Rice (University of Waterloo), Eric Robert (Leo Hayes High School, Fredericton), Monika Khbeis (Father Michael Goetz Secondary School, Mississauga), Mark Bredin (St. John's-Ravenscourt School, Winnipeg), and Ron Lancaster (University of Toronto).

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### Mayhem Problems

*Veillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier avril 2007**. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

*Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.*

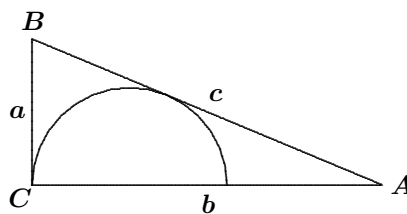
*La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.*

**M269.** *Proposé par Bruce Sawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans le carré  $ABCD$ , soit  $E$  le point milieu du côté  $AD$ , soit  $F$  le point sur  $EB$  tel que  $CF$  soit perpendiculaire à  $EB$ , et soit  $G$  le point sur  $EB$  tel que  $AG$  soit perpendiculaire à  $EB$ . Montrer que  $DF = CG$ .

**M270.** *Proposé par Bruce Sawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Les côtés d'un triangle rectangle sont de longueur  $a$  et  $b$ , son hypoténuse est de longueur  $c$ . Un demi-cercle, ayant comme diamètre le côté de longueur  $b$ , est tangent aux deux autres côtés. Déterminer le rayon du demi-cercle en fonction de  $a$ ,  $b$  et  $c$ .





**M271.** *Proposé par Yakub N. Aliyev, Université d'Etat de Bakou, Bakou, Azerbaïdjan.*

Sachant que dans un hexagone convexe  $ABCDEF$ , les côtés  $BC$ ,  $DE$  et  $FA$  sont respectivement parallèles aux diagonales  $AD$ ,  $CF$  et  $EB$ , on désigne respectivement par  $K$ ,  $L$  et  $M$  les intersections des droites  $AB$  avec  $CD$ ,  $CD$  avec  $EF$ , et  $EF$  avec  $AB$ ; on désigne enfin par  $P$ ,  $Q$  et  $R$  les intersections respectives de  $CF$  avec  $BE$ , de  $BE$  avec  $AD$ , et de  $AD$  avec  $CF$ . Montrer que  $KP$ ,  $MR$  et  $LQ$  se coupent en un même point.

**M272.** *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Soit  $P$  un point situé sur le côté  $AB$  d'un parallélogramme  $ABCD$ . Sachant que le rapport de l'aire du triangle  $ABC$  et celle du quadrilatère  $APCD$  est  $m/n$ , déterminer le rapport de  $AP$  et  $PB$ .

**M273.** *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Soit  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ ,  $G$  et  $H$  des lettres représentant des chiffres de 0 à 9, distincts. Déterminer leur valeur, sachant que les deux produits ci-dessous sont justes. (Noter que le premier chiffre d'un nombre doit être non nul.)

$$\begin{array}{r} ABCD \\ \times E \\ \hline DCBA \end{array} \qquad \begin{array}{r} BFDG \\ \times G \\ \hline GDFB \end{array}$$

**M274.** *Proposé par Neven Jurič, Zagreb, Croatie.*

Déterminer l'aire du polygone dont tous les sommets sont sur le cercle d'équation  $x^2 + y^2 = 100$ , leurs coordonnées étant toutes des entiers.

**M275.** *Proposé par K. R. S. Sastry, Bangalore, Inde.*

Un triangle pythagorique primitif (TPP en bref) est un triangle rectangle avec, comme longueurs des trois côtés, des entiers dont le plus grand commun diviseur est 1. Parmi les paires de TPPs non congruents possédant des cercles inscrits congruents à rayon entier, trouver une paire de rayon minimal.

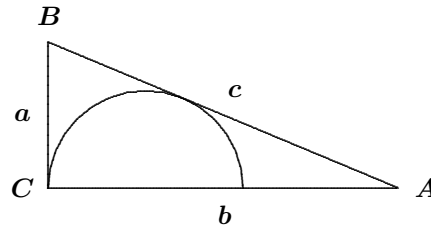
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**M269.** *Proposed by Bruce Sawyer, Memorial University of Newfoundland, St. John's, NL.*

Let  $ABCD$  be a square. Let  $E$  be the mid-point of the side  $AD$ , let  $F$  be the point on  $EB$  such that  $CF$  is perpendicular to  $EB$ , and let  $G$  be the point on  $EB$  such that  $AG$  is perpendicular to  $EB$ . Show that  $DF = CG$ .

**M270.** Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

A right triangle has legs of lengths  $a$  and  $b$  and a hypotenuse of length  $c$ . A semicircle has its diameter on the side of length  $b$  and is tangent to the other two sides. Determine the radius of the semicircle in terms of  $a$ ,  $b$ , and  $c$ .



**M271.** Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.

For the convex hexagon  $ABCDEF$ , it is known that the sides  $BC$ ,  $DE$ , and  $FA$  are parallel to the diagonals  $AD$ ,  $CF$ , and  $EB$ , respectively. We denote by  $K$ ,  $L$ , and  $M$  the respective intersections of the lines  $AB$  with  $CD$ ,  $CD$  with  $EF$ , and  $EF$  with  $AB$ ; we further denote by  $P$ ,  $Q$ , and  $R$  the respective intersections of  $CF$  with  $BE$ ,  $BE$  with  $AD$ , and  $AD$  with  $CF$ . Prove that  $KP$ ,  $MR$ , and  $LQ$  intersect at the same point.

**M272.** Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.

Let  $ABCD$  be a parallelogram, and let  $P$  be a point situated on  $AB$ . If the ratio of the area of triangle  $ABC$  to that of quadrilateral  $APCD$  is  $m/n$ , determine the ratio of  $AP$  to  $PB$ .

**M273.** Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.

The letters  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ ,  $G$ , and  $H$  represent distinct digits. Determine their values given that the two products shown are true. (Note that the first digit of a number must be non-zero)

$$\begin{array}{r} ABCD \\ \times E \\ \hline DCBA \end{array} \qquad \begin{array}{r} BFDG \\ \times G \\ \hline GDFB \end{array}$$

**M274.** Proposed by Neven Jurić, Zagreb, Croatia.

Determine the area of the polygon whose vertices are all the points on the circle  $x^2 + y^2 = 100$  where both coordinates are integers.

**M275.** Proposed by K.R.S. Sastry, Bangalore, India.

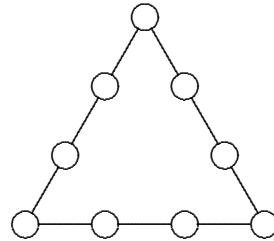
A primitive Pythagorean triangle (PPT) is a right triangle whose sides have lengths which are integers with a greatest common divisor of 1. Among all pairs of non-congruent PPTs which have congruent incircles with an integer radius, find a pair for which this radius is minimized.

## Mayhem Solutions

**M219.** Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.

Place each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in exactly one of the circles in such a way that:

1. the sums of the four numbers on each side of the triangle are equal; and
2. the sums of the squares of the four numbers on each side of the triangle are equal.



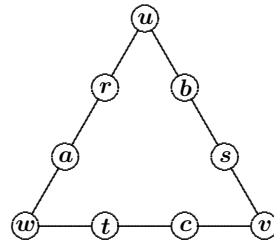
*Solution by Messiah College Problem Solving Group, Messiah College, Grantham, PA, USA.*

Assign variables to the nine positions in the triangle as shown. Since the sum of all nine numbers is 45 and the sums on each side are equal, we have the following equations:

$$a + b + c + r + s + t + u + v + w = 45, \quad (1)$$

$$a + r + w = b + s + v, \quad (2)$$

$$a + r + u = t + c + v. \quad (3)$$



Adding these equations and dividing by 3 gives

$$a = 15 - r + \frac{1}{3}(-2u + v - 2w).$$

Since  $a$  is an integer, we must have  $-2u + v - 2w \equiv 0 \pmod{3}$ , which is equivalent to

$$u + v + w \equiv 0 \pmod{3}. \quad (4)$$

We now consider condition 2 of the problem statement modulo 3. Among the squares of the nine numbers 1, 2, ..., 9, the squares of 3, 6, and 9 are congruent to 0 modulo 3 and the others are congruent to 1. Each of the squares of the variables  $u$ ,  $v$ , and  $w$  at the corners of the triangle may be congruent to 0 or 1 (modulo 3). But a check of the various cases shows that the three sums of squares cannot be equal, modulo 3, unless either

$$u^2 \equiv v^2 \equiv w^2 \equiv 0 \pmod{3} \quad \text{or} \quad u^2 \equiv v^2 \equiv w^2 \equiv 1 \pmod{3}.$$

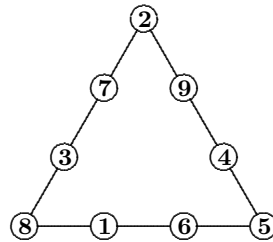
These additional constraints, along with (4), leave only three possibilities for  $(u, v, w)$ , namely (1, 4, 7), (2, 5, 8), and (3, 6, 9).

Setting  $(u, v, w) = (1, 4, 7)$  in (1), (2), and (3) and simplifying yields  $a + r = 11$ . The possibilities for  $\{a, r\}$  are  $\{2, 9\}$ ,  $\{3, 8\}$ , and  $\{5, 6\}$ . None of these yields a solution to the problem.

Setting  $(u, v, w) = (3, 6, 9)$  in (1), (2), and (3) and simplifying yields  $a + r = 9$ . The possibilities for  $\{a, r\}$  are  $\{1, 8\}$ ,  $\{2, 7\}$ , and  $\{4, 5\}$ . Again, none of these yields a solution to the problem.

Setting  $(u, v, w) = (2, 5, 8)$  in (1), (2), and (3) and simplifying yields  $a + r = 10$ . The possibilities for  $\{a, r\}$  are  $\{1, 9\}$ ,  $\{3, 7\}$ , and  $\{4, 6\}$ . The set  $\{3, 7\}$  gives us a solution to the problem, as shown.

Any triangle with 2, 5, and 8 on the corners, and with side numbers  $\{2, 3, 7, 8\}$ ,  $\{2, 4, 5, 9\}$ , and  $\{1, 5, 6, 8\}$  is a solution.



Also solved by MIGUEL MARANÓN GRANDES, Grade 12 student, I.E.S. Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JOSH TREJO and MANDY RODGERS, Angelo State University, San Angelo, TX, USA.

**M220.** Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.

Show how to number the faces of an octahedral die using the numbers 1 through 8 in such a way that the sum of the numbers on the four faces joining at each vertex is always the same.

*Solution by Messiah College Problem Solving Group, Messiah College, Grantham, PA, USA.*

Since the numbers 1 through 8 sum to 36 and each is counted in a sum three times (once at each vertex of its triangular face), the total sum over all 6 vertices is 108, and hence the sum at each vertex must be 18.

When two faces share an edge, we shall refer to them as *neighbouring faces*. The numbers on a given pair of neighbouring faces are together in two of the vertex sums. Each of these two vertex sums involves two more numbers, which we will refer to as an *adjoining pair* for the given pair of neighbouring faces.

We examine the possibilities for neighbouring faces and their adjoining pairs, where one of the neighbouring faces is the one labelled 1.

Neighbouring Faces	Sum	Sum of Adjoining Pairs	Possible Adjoining Pairs
(1, 2)	3	15	(7, 8)
(1, 3)	4	14	(6, 8)
(1, 4)	5	13	(5, 8), (6, 7)
(1, 5)	6	12	(4, 8)
(1, 6)	7	11	(3, 8), (4, 7)
(1, 7)	8	10	(2, 8), (4, 6)
(1, 8)	9	9	(2, 7), (3, 6), (4, 5)

In a properly numbered die, it is clear that a pair of neighbouring vertices must have two distinct adjoining pairs. Thus, none of the pairs (1, 2), (1, 3), or (1, 5) can occur as a neighbouring pair. This leaves only four numbers which can be neighbours of 1, namely 4, 6, 7, or 8. Any solution must use

three of these four. If 6 is a neighbour of 1, then 4 and 7 cannot both be neighbours of 1, since (4, 7) and (3, 8) must both be adjoining pairs for (1, 6). This leaves only 3 possibilities to check, namely {4, 6, 8}, {4, 7, 8}, and {6, 7, 8}. Each possibility yields a solution, as shown.



These are the only possibilities up to symmetry.

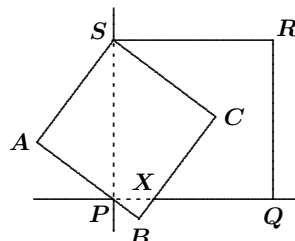
Also solved by SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College Saratoga Springs, NY, USA; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

**M221.** Proposed by Bill Sands, University of Calgary, Calgary, AB.

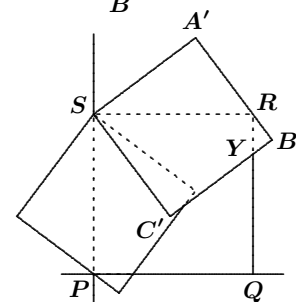
Prove that a  $5 \times 5$  square can be covered by three  $4 \times 4$  squares.

Solution by Skidmore College Problem Solving Group, Skidmore College Saratoga Springs, NY, USA.

Place the  $5 \times 5$  square  $PQRS$  in the first quadrant with vertices  $P(0, 0)$ ,  $Q(5, 0)$ ,  $R(5, 5)$ , and  $S(0, 5)$ . Place a  $4 \times 4$  square  $\mathcal{S}_1 = ABCS$  so that the side  $AB$  passes through the origin  $P$ , with  $PA = 3$  and  $PB = 1$ . By similar triangles, the side  $BC$  will intersect the  $x$ -axis at the point  $X(\frac{5}{4}, 0)$ .

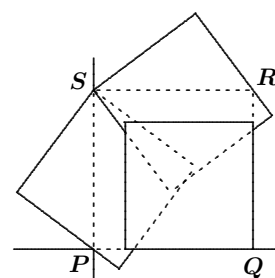


Place a second  $4 \times 4$  square  $\mathcal{S}_2 = A'B'C'S$  over the  $5 \times 5$  square  $PQRS$  so that the side  $A'B'$  passes through the vertex  $R$ , with  $RA' = 3$  and  $RB' = 1$ . By symmetry and similar triangles, the side  $B'C'$  will intersect  $RQ$  at the point  $Y(5, \frac{15}{4})$ .



The slopes of the line segments  $SC$  and  $SC'$  are  $-\frac{3}{4}$  and  $-\frac{4}{3}$ , respectively. Therefore, the two squares  $\mathcal{S}_1$  and  $\mathcal{S}_2$  overlap each other in the interior of  $PQRS$ .

Place a third  $4 \times 4$  square  $\mathcal{S}_3$  at the bottom right corner of  $PQRS$  so that its vertices are  $(1, 0)$ ,  $(5, 0)$ ,  $(5, 4)$ , and  $(1, 4)$ . The vertices  $(1, 0)$  and  $(5, 4)$  of this third square are covered by the squares  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , respectively, because the intersection point  $X(\frac{5}{4}, 0)$  lies to the right of  $(1, 0)$  and the intersection point  $Y(5, \frac{15}{4})$  lies below  $(5, 4)$ . The point  $(1, 4)$  is covered by both squares  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Therefore, every point in the  $L$ -shaped region inside  $PQRS$  and outside  $\mathcal{S}_3$  is covered by at least one of the squares  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .



Also solved by MESSIAH COLLEGE PROBLEM SOLVING GROUP, Messiah College, Grantham, PA, USA; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

**M222.** Proposed by Bill Sands, University of Calgary, Calgary, AB.

Suppose that  $30a + 40b$  and  $40a + 30b$  are the sides of a right triangle and that  $50a + kb$  is the hypotenuse, where  $a$ ,  $b$ , and  $k$  are positive integers. Find the smallest possible values of  $a$ ,  $b$ , and  $k$ .

*Essentially the same solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain; Richard I. Hess, Rancho Palos Verdes, CA, USA; and Messiah College Problem Solving Group.*

By the Pythagorean Theorem, we see that

$$(30a + 40b)^2 + (40a + 30b)^2 = (50a + kb)^2.$$

Expanding and simplifying, we get  $4800ab + 2500b^2 = 100abk + k^2b^2$ . Dividing by  $b$  (since  $b > 0$ ) and rearranging gives

$$b(50 - k)(50 + k) = 100a(k - 48).$$

Since  $a$ ,  $b$ , and  $k$  are all positive, the factors  $50 - k$  and  $k - 48$  must have the same sign. This happens only when  $k = 49$ . Setting  $k = 49$ , we obtain  $99b = 100a$ . Since 99 and 100 are relatively prime and we are seeking the smallest values of  $a$  and  $b$ , we conclude that  $a = 99$  and  $b = 100$ . Thus, the minimum solution is  $(a, b, k) = (99, 100, 49)$ .

**M223.** Proposed by Larry Rice, University of Waterloo, Waterloo, ON.

The fraction  $\frac{3}{10}$  can be written as the sum of two positive rational numbers with numerator 1 in exactly two ways, namely as  $\frac{1}{10} + \frac{1}{5}$  and  $\frac{1}{20} + \frac{1}{4}$ .

Determine the number of ways that  $\frac{3}{2006}$  can be expressed as the sum of two positive rational numbers with numerator 1.

*Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Let  $p$  and  $q$  be positive integers such that  $\frac{3}{2006} = \frac{1}{p} + \frac{1}{q}$ . Then

$$3pq = 2006(p + q) = 2 \cdot 17 \cdot 59(p + q). \quad (1)$$

To find values of  $p$  and  $q$ , we look at 4 cases.

Case 1:  $2 \cdot 17 \cdot 59$  divides  $p$ .

Then there exists a positive integer  $r$  such that  $p = 2 \cdot 17 \cdot 59r$ . In this case, (1) takes the form  $3qr = 2 \cdot 17 \cdot 59r + q$ , or  $(3r - 1)q = 2 \cdot 17 \cdot 59r$ . Therefore,  $q = 2 \cdot 17 \cdot 59r / (3r - 1)$ . This last equation has positive integer solutions for  $q$  when  $3r - 1$  takes the values 2, 17, 59, or  $2 \cdot 17 \cdot 59$ . [Ed.: the values  $2 \cdot 17$ ,  $2 \cdot 59$ , and  $17 \cdot 59$  cannot be expressed in the form  $3r - 1$ .] The corresponding values of  $r$  are 1, 6, 20, and 669, which generate the following  $(p, q)$  pairs: (2006, 1003), (12036, 708), (40120, 680) and (1342014, 669).

Case 2:  $2 \cdot 17$  divides  $p$ , but 59 does not.

Then there exists a positive integer  $r$ , not a multiple of 59, such that  $p = 2 \cdot 17r$  and  $q = 59s$  for some positive integer  $s$ . In this case, (1) takes the form  $(3r - 59)s = 2 \cdot 17r$ , and then  $s = 2 \cdot 17r / (3r - 59)$ . The values for  $r$ , not multiples of 59, which give positive integer solutions for  $s$  are 20 and 31 (corresponding to the equations  $3r - 59 = 1$  and  $3r - 59 = 2 \cdot 17$ ). Only the second value generates a new  $(p, q)$  pair: (1054, 1829).

Case 3:  $2 \cdot 59$  divides  $p$ , but 17 does not.

Then there exists a positive integer  $r$ , not a multiple of 17, such that  $p = 2 \cdot 59r$  and  $q = 17s$  for some positive integer  $s$ . In this case, (1) takes the form  $(3r - 17)s = 2 \cdot 59r$ , and then  $s = 2 \cdot 59r / (3r - 17)$ . The values for  $r$ , not multiples of 17, which give positive integer solutions for  $s$  are 6 and 45 (corresponding to the equations  $3r - 17 = 1$  and  $3r - 17 = 2 \cdot 59$ ). Only the second value generates a new  $(p, q)$  pair: (5310, 765).

Case 4:  $17 \cdot 59$  divides  $p$ , but 2 does not.

Then there exists a positive integer  $r$ , not a multiple of 2, such that  $p = 17 \cdot 59r$  and  $q = 2s$  for some positive integer  $s$ . In this case, (1) takes the form  $(3r - 2)s = 17 \cdot 59r$ , and then  $s = 17 \cdot 59r / (3r - 2)$ . The odd values for  $r$ , which give positive integer solutions for  $s$  are 1 and 335 (corresponding to the equations  $3r - 2 = 1$  and  $3r - 2 = 17 \cdot 59$ ). Only the second value generates a new  $(p, q)$  pair: (336005, 670).

[Ed: The remaining possible cases, where one of the primes 2, 17, and 59 divides  $p$  and the other two do not, simply exchange the roles of  $p$  and  $q$ , and thus do not lead to any new pairs.]

In conclusion, there are exactly seven ways in which  $3/2006$  can be expressed as the sum of two positive rational numbers with numerator 1:

$$\frac{1}{669} + \frac{1}{1342014}, \quad \frac{1}{670} + \frac{1}{336005}, \quad \frac{1}{680} + \frac{1}{40120}, \quad \frac{1}{708} + \frac{1}{12036}, \\ \frac{1}{765} + \frac{1}{5310}, \quad \frac{1}{1003} + \frac{1}{2006}, \quad \frac{1}{1054} + \frac{1}{1829}.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and Messiah College Problem Solving Group. One incomplete solution was submitted.

**M224.** Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.

Let  $A(-1, 1)$  and  $B(3, 9)$  be two points on the parabola  $y = x^2$ . Take another point  $M(m, m^2)$  on the parabola lying between  $A$  and  $B$ . Let  $H$  be the point on the line segment joining  $A$  to  $B$  that has the same  $x$ -coordinate as  $M$ .

Show that if the length of  $MH$  is  $k$  units, then triangle  $AMB$  has area  $2k$  square units. Does this relationship still hold if  $M$  is not between  $A$  and  $B$ ?

*Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.*

The equation for the line through points  $A(-1, 1)$  and  $B(3, 9)$  can easily be found to be  $y = 2x + 3$ . Let  $M(m, m^2)$  be any point on the parabola (not necessarily between  $A$  and  $B$ ). The length of the line segment joining points  $M(m, m^2)$  and  $H(m, 2m + 3)$  is  $k = |m^2 - 2m - 3|$ , and the area of triangle  $ABM$  can be calculated as follows:

$$\begin{aligned} [ABC] &= \frac{1}{2} \left\| \overrightarrow{AM} \times \overrightarrow{AB} \right\| = \frac{1}{2} \left\| \langle m+1, m^2-1, 0 \rangle \times \langle 4, 8, 0 \rangle \right\| \\ &= \frac{1}{2} \left\| \langle 0, 0, -4m^2 + 8m + 12 \rangle \right\| = 2|m^2 - 2m - 3| = 2k. \end{aligned}$$

Thus, the given relationship holds no matter whether  $M$  is between  $A$  and  $B$  or not.

*Also solved by ESTHER MARÍA GARCÍA-CABALLERO, Universidad de Jaén, Jaén, Spain.*

**M225.** *Proposed by Zun Shan, Normal University, China; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Define a sequence  $\{x_n\}$  by  $x_1 = 1/2005$  and  $x_{n+1} = x_n + x_n^2$  for  $n \geq 1$ . Set

$$S = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \dots + \frac{x_{2005}}{x_{2006}}.$$

Determine  $\lfloor S \rfloor$ , the greatest integer not exceeding  $S$ .

*Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.*

It is easy to prove, by induction, that

$$x_n = x_1(1 + x_1)(1 + x_1 + x_1^2) \cdots (1 + x_1 + x_1^2 + \cdots + x_1^{n-1}).$$

Then,

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_{n+1}} = \frac{1}{1 + x_1} + \frac{1}{1 + x_1 + x_1^2} + \cdots + \frac{1}{\sum_{i=1}^{n+1} x_1^{i-1}}.$$

Setting  $x_1 = 1/2005$  and  $n = 2005$ , we see that

$$\begin{aligned} S &= \frac{1}{1 + \frac{1}{2005}} + \frac{1}{1 + \frac{1}{2005} + \left(\frac{1}{2005}\right)^2} + \cdots \\ &\quad + \frac{1}{1 + \frac{1}{2005} + \left(\frac{1}{2005}\right)^2 + \cdots + \left(\frac{1}{2005}\right)^{2005}} = \sum_{n=1}^{2005} \frac{1}{T_n}, \end{aligned}$$

where  $T_n = \sum_{i=0}^n \left(\frac{1}{2005}\right)^i$ .

Since  $T_n < \sum_{i=0}^{\infty} \left(\frac{1}{2005}\right)^i = \frac{2005}{2004}$ , we have  $\frac{1}{T_n} > \frac{2004}{2005}$ , and therefore,

$S > \sum_{n=1}^{2005} \frac{2004}{2005} = 2004$ . But we also have  $\frac{1}{T_n} \leq \frac{1}{1 + \frac{1}{2005}} = \frac{2005}{2006}$  for each  $n$ .



Therefore,

$$S < \sum_{n=1}^{2005} \frac{2005}{2006} = \frac{2005^2}{2006} < 2005.$$

Now, since  $2004 < S < 2005$ , we see that  $\lfloor S \rfloor = 2004$ .

*One incorrect solution was received.*

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## Problem of the Month

Ian VanderBurgh

This month's problem combines some geometry, some number theory, and some algebra:

**Problem.** A *Pythagorean triple* is a set of three positive integers  $(a, b, c)$  such that  $a < b < c$  and  $a^2 + b^2 = c^2$ . Determine all primitive Pythagorean triples that satisfy the equation  $a + b - c = 20$ .

(A variation of this problem was used at the annual Canadian Mathematics Competitions Mathematics Contests Seminar in June 2006.)

No, "primitive" does not mean that the Pythagorean triples live in caves! A Pythagorean triple  $(a, b, c)$  is called *primitive* if the integers  $a$ ,  $b$ , and  $c$  do not all share a common factor greater than 1.

First let's solve this problem in a way that does not assume any prior knowledge on the subject of Pythagorean triples.

*Solution 1:* From the given equation  $a + b - c = 20$ , we get

$$\begin{aligned} a + b - 20 &= c, \\ (a + b - 20)^2 &= c^2, \\ a^2 + b^2 + 400 + 2ab - 40a - 40b &= c^2, \\ 400 + 2ab - 40a - 40b &= 0 \quad (\text{since } a^2 + b^2 = c^2), \\ ab - 20a - 20b &= -200, \\ ab - 20a - 20b + 400 &= 200, \\ (a - 20)(b - 20) &= 200. \end{aligned}$$

The factors  $a - 20$  and  $b - 20$  on the left side of this equation must be integers because  $a$  and  $b$  are integers. Since  $a + b - c = 20$  and  $a < b < c$ , we have  $b > a = 20 + (c - b) > 20$ . Thus, the factors  $a - 20$  and  $b - 20$  are both positive, and  $b - 20 > a - 20$ .

The easiest thing to do at this stage is to make a table of all possible pairs  $a - 20$ ,  $b - 20$ .

$a - 20$	$b - 20$	$a$	$b$	$c = a + b - 20$	Primitive
1	200	21	220	221	Yes
2	100	22	120	122	No
4	50	24	70	74	No
5	40	25	60	65	No
8	25	28	45	53	Yes
10	20	30	40	50	No

Thus, the two primitive Pythagorean triples that work are (21, 220, 221) and (28, 45, 53). (Notice that each triple that we rejected above had a common factor of 2 among the three integers.)

That was a good way to solve this problem. It did not require any advanced machinery.

However, there is some structure to the set of all primitive Pythagorean triples which we could have used. Every primitive Pythagorean triple  $(a, b, c)$  with  $a < b < c$  can be written in one of the forms  $(2mn, m^2 - n^2, m^2 + n^2)$  or  $(m^2 - n^2, 2mn, m^2 + n^2)$ , where  $m$  and  $n$  are positive integers of opposite parity (that is, one is even and one is odd) with no common factors greater than 1. We will take this on faith for a little while and come back to the reasoning later.

*Solution 2:* Using the above forms for primitive Pythagorean triples, we rewrite the equation  $a + b - c = 20$  as  $m^2 - n^2 + 2mn - (m^2 + n^2) = 20$ . (Notice that we get the same equation from each of the two possible forms.) Thus,  $2mn - 2n^2 = 20$ , or  $n(m - n) = 10$ . Here  $m$  and  $n$  are positive integers, and therefore  $n$  and  $m - n$  are positive factors of 10.

Now we make a table showing the possibilities for  $n$  and  $m - n$ .

$n$	$m - n$	$m$	$2mn$	$m^2 - n^2$	$m^2 + n^2$	$(a, b, c)$	Primitive
1	10	11	22	120	122	(22, 120, 122)	No
2	5	7	28	45	53	(28, 45, 53)	Yes
5	2	7	70	24	74	(24, 70, 74)	No
10	1	11	220	21	221	(21, 220, 221)	Yes

Therefore, the primitive Pythagorean triples satisfying  $a + b - c = 20$  are (28, 45, 53) and (21, 220, 221).

Wonderful! This solution was a fair bit less complicated than Solution 1. Interestingly, the “endgame” was remarkably similar to that in Solution 1.

The problem can be interpreted geometrically. Each Pythagorean triple  $(a, b, c)$  corresponds to a right triangle whose sides have integer lengths  $a$ ,  $b$ , and  $c$ . The equation  $a^2 + b^2 = c^2$  is the Pythagorean Theorem (which, of course, is why we call  $(a, b, c)$  a Pythagorean triple). The diameter of the inscribed circle of such a triangle is  $a + b - c$ . (Can you show this?) A primitive Pythagorean triple corresponds to a right triangle which is not similar to any smaller right triangle with integer side lengths. The problem asks us to find all such triangles which have an inscribed circle of diameter 20.

### Postscript.

Before wrapping up this article, we should look at the formulae for generating primitive Pythagorean triples, which we will abbreviate as PPTs.

First, we check that any triple of the form  $(2mn, m^2 - n^2, m^2 + n^2)$  or  $(m^2 - n^2, 2mn, m^2 + n^2)$  is, in fact, a Pythagorean triple. This requires squaring each of the three terms and checking that the sum of the first two squares equals the third. I'll let you think about why such a triple must be primitive, given that  $m$  and  $n$  are positive, have opposite parity, and have no common factors greater than 1.

Next, we check that every PPT is of one of these two forms. We will use one of the well-known methods to do this and break the procedure into a number of steps. (There are certainly other methods that can be used to derive these formulae.)

*Step 0: Odd and even perfect squares.*

If  $a$  is even, then  $a = 2k$  for some integer  $k$ ; thus,  $a^2 = 4k^2$  which gives a remainder of 0 when divided by 4.

If  $a$  is odd, then  $a = 2k + 1$  for some integer  $k$ ; thus,  $a^2 = 4k^2 + 4k + 1$  which gives a remainder of 1 when divided by 4.

This doesn't seem relevant immediately, but hang on!

*Step 1: Parities of  $a, b, c$ .*

Suppose  $(a, b, c)$  is a PPT. If  $a$  and  $b$  were both even, then  $c$  would be even, since  $c^2 = a^2 + b^2$ . Then  $a, b,$  and  $c$  would all be divisible by 2, contradicting our assumption that the Pythagorean triple  $(a, b, c)$  is primitive.

If  $a$  and  $b$  were both odd, then  $c$  would be even. In this case, the remainder upon dividing  $c^2$  by 4 would be 0 and the remainder upon dividing  $a^2 + b^2$  by 4 would be 2, leading to a contradiction.

Therefore, having eliminated all of the other cases, we are left with the case where one of  $a$  or  $b$  is even and the other is odd (implying that  $c$  is odd). Assume that  $a$  is odd and  $b$  is even.

*Step 2: Some algebra.*

Since  $a^2 + b^2 = c^2$ , then  $b^2 = c^2 - a^2 = (c - a)(c + a)$ . Dividing both sides by 4, we get  $(\frac{1}{2}b)^2 = (\frac{1}{2}(c - a))(\frac{1}{2}(c + a))$ . Note that  $\frac{1}{2}b$  is an integer, because  $b$  is even, and  $\frac{1}{2}(c - a)$  and  $\frac{1}{2}(c + a)$  are integers, because  $a$  and  $c$  are both odd (making  $c - a$  and  $c + a$  even).

*Step 3: Analysis of the factors  $\frac{1}{2}(c - a)$  and  $\frac{1}{2}(c + a)$ .*

If  $\frac{1}{2}(c - a)$  and  $\frac{1}{2}(c + a)$  had a common factor larger than 1, then this factor would also be a factor of their sum (which equals  $c$ ) and of their difference (which equals  $a$ ). Then  $b$  would also share this common factor, since  $b^2 = c^2 - a^2$ . This can't happen, because  $a, b,$  and  $c$  have no common factor. Also,  $\frac{1}{2}(c - a)$  and  $\frac{1}{2}(c + a)$  cannot both be odd. If they were odd, then their sum  $c$  and their difference  $a$  would be even (and we know that they are odd).

So  $\frac{1}{2}(c - a)$  and  $\frac{1}{2}(c + a)$  have no common factors, one is even and one is odd, and their product is a perfect square.

*Step 4: Introduction of  $m$  and  $n$ .*

It must be the case, therefore, that each of  $\frac{1}{2}(c - a)$  and  $\frac{1}{2}(c + a)$  is a perfect square itself. (Think about this.) We can write  $\frac{1}{2}(c - a) = n^2$  and  $\frac{1}{2}(c + a) = m^2$ , with  $m$  and  $n$  positive integers.

Then  $m$  and  $n$  cannot have a common factor larger than 1 (since  $m^2$  and  $n^2$  don't), and  $m$  and  $n$  must have opposite parity (since  $m^2$  and  $n^2$  do).

*Step 5: The big finish.*

Since  $\frac{1}{2}(c - a) = n^2$  and  $\frac{1}{2}(c + a) = m^2$ , then  $c = m^2 + n^2$  (adding) and  $a = m^2 - n^2$  (subtracting). Also,  $(\frac{1}{2}b)^2 = m^2n^2$ ; that is,  $b^2 = 4m^2n^2$ . Then  $b = 2mn$ , since  $b$ ,  $m$ , and  $n$  are positive.

Thus, each PPT has exactly the form that we had hoped.

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## Mayhem Year End Wrap Up

Shawn Godin

Another year has come and gone. For me, on this cold, windy Sunday morning in October, this is one of my last tasks as Editor of the Mayhem section. This task is bitter-sweet for me. For the last six years I have worked with some great people and reshaped Mayhem to better fit inside its mother journal, ***Crux Mathematicorum***. The job has consumed a big chunk of my meager free time, so the freedom will be a welcome change. Having said that, I must add that I will miss the wonderful problems, solutions, and articles sent in by our readers and staff members. It has been an honour to be associated with such a well-respected and world-class journal.

At this point, I need to thank a couple of members of the staff, without whom Mayhem would not be. First, I must thank Mayhem Assistant Editor and future Editor, JEFF HOOPER. Jeff's thoughtful suggestions always help deliver a better issue. Secondly, I must thank IAN VANDERBURGH. Ian continues to present our readers with great material in his regular column, The Problem of the Month.

I also need to thank those people behind the scenes: ED BARBEAU, ROBERT BILINSKI, MARK BREDIN, RICHARD HOSHINO, MONIKA KHEBEIS, RON LANCASTER, JOHN GRANT McLOUGHLIN, PAUL OTTAWAY, LARRY RICE, ERIC ROBERT, BRUCE SHAWYER, and GRAHAM WRIGHT. They were there when I needed them and their contributions were always appreciated, although not always acknowledged. Thanks everyone!

All the best of the season to all our readers and contributors! I hope you have a great year in 2007. Thank you for making the last six years so enjoyable. Happy problem solving!

# THE OLYMPIAD CORNER

No. 258

R.E. Woodrow

For our problems in this issue we give a first installment of the short-listed problems from the 44<sup>th</sup> IMO in Japan. My thanks go to Andy Liu, Canadian Team Leader to the IMO, for collecting them for our use.

## 44<sup>th</sup> INTERNATIONAL MATHEMATICAL OLYMPIAD Short-listed Problems

### Algebra

**A1.** Let  $a_{ij}$  ( $i = 1, 2, 3; j = 1, 2, 3$ ) be real numbers such that  $a_{ij}$  is positive for  $i = j$  and negative for  $i \neq j$ . Prove that there exist positive real numbers  $c_1, c_2, c_3$  such that the numbers  $a_{11}c_1 + a_{12}c_2 + a_{13}c_3$ ,  $a_{21}c_1 + a_{22}c_2 + a_{23}c_3$ , and  $a_{31}c_1 + a_{32}c_2 + a_{33}c_3$  are all negative, all positive, or all zero.

**A2.** Find all non-decreasing functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

(a)  $f(0) = 0, f(1) = 1;$

(b)  $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$  for all real numbers  $a$  and  $b$  such that  $a < 1 < b$ .

**A3.** Consider pairs of sequences of positive real numbers

$$a_1 \geq a_2 \geq a_3 \geq \dots \quad \text{and} \quad b_1 \geq b_2 \geq b_3 \geq \dots$$

For any such pair, define  $c_i = \min\{a_i, b_i\}$  for  $i = 1, 2, 3, \dots$ . For  $n = 1, 2, 3, \dots$ , define the sums

$$A_n = a_1 + \dots + a_n, \quad B_n = b_1 + \dots + b_n, \quad C_n = c_1 + \dots + c_n.$$

(a) Does there exist a pair  $(a_i)_{i \geq 1}, (b_i)_{i \geq 1}$  such that the sequences  $(A_n)_{n \geq 1}$  and  $(B_n)_{n \geq 1}$  are unbounded while the sequence  $(C_n)_{n \geq 1}$  is bounded?

(b) Does the answer to question (a) change if the additional assumption is made that  $b_i = 1/i$ , for  $i = 1, 2, \dots$ ?

Justify your answer.

### Combinatorics

**C1.** Let  $A$  be a 101-element subset of the set  $S = \{1, 2, \dots, 1000000\}$ . Prove that there exist numbers  $t_1, t_2, \dots, t_{100}$  in  $S$  such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \quad j = 1, 2, \dots, 100$$

are pairwise disjoint.

**C2.** Let  $D_1, \dots, D_n$  be closed discs in the plane. (A closed disc is the region bounded by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs  $D_i$ . Prove that there exists a disc  $D_k$  which intersects at most  $7 \cdot 2003 - 1$  other discs  $D_i$ .

**C3.** Let  $n \geq 5$  be a given integer. Determine the greatest integer  $k$  for which there exists a polygon with  $n$  vertices (convex or not, with a boundary which is not self-intersecting) having  $k$  internal right angles.

### Geometry

**G1.** Let  $ABCD$  be a cyclic quadrilateral. Let  $P, Q, R$  be the feet of the perpendiculars from  $D$  to the lines  $BC, CA, AB$ , respectively. Show that  $PQ = QR$  if and only if the bisectors of  $\angle ABC$  and  $\angle ADC$  are concurrent with  $AC$ .

**G2.** Three distinct points  $A, B, C$  are fixed on a line in this order. Let  $\Gamma$  be a circle passing through  $A$  and  $C$  whose centre does not lie on the line  $AC$ . Denote by  $P$  the intersection of the tangents to  $\Gamma$  at  $A$  and  $C$ . Suppose  $\Gamma$  meets the segment  $PB$  at  $Q$ . Prove that the intersection of the bisector of  $\angle AQC$  and the line  $AC$  does not depend on the choice of  $\Gamma$ .

**G3.** Let  $ABC$  be a triangle, and let  $P$  be a point in its interior. Denote by  $D, E, F$  the feet of the perpendiculars from  $P$  to the lines  $BC, CA, AB$ , respectively. Suppose that  $AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2$ . Denote by  $I_A, I_B, I_C$  the excentres of the triangle  $ABC$ . Prove that  $P$  is the circumcentre of the triangle  $I_A I_B I_C$ .

**G4.** Let  $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$  be distinct circles such that  $\Gamma_1$  and  $\Gamma_3$  are externally tangent at  $P$ , and  $\Gamma_2$  and  $\Gamma_4$  are externally tangent at the same point  $P$ . Suppose that  $\Gamma_1$  and  $\Gamma_2$  meet at  $A$ ,  $\Gamma_2$  and  $\Gamma_3$  meet at  $B$ ,  $\Gamma_3$  and  $\Gamma_4$  meet at  $C$ , and  $\Gamma_4$  and  $\Gamma_1$  meet at  $D$ , where the points  $A, B, C, D$  are different from  $P$ . Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

### Number Theory

**N1.** Let  $m$  be a fixed integer greater than 1. The sequence  $x_0, x_1, x_2, \dots$  is defined as follows:

$$x_i = \begin{cases} 2^i & \text{if } 0 \leq i \leq m-1, \\ \sum_{j=1}^m x_{i-j} & \text{if } i \geq m. \end{cases}$$

Find the greatest  $k$  for which the sequence contains  $k$  consecutive terms divisible by  $m$ .

**N2.** Each positive integer  $a$  undergoes the following procedure in order to obtain the number  $d = d(a)$ :

- (i) Move the last digit of  $a$  to the first position to obtain the number  $b$ ;
- (ii) Square  $b$  to obtain the number  $c$ ;
- (iii) Move the first digit of  $c$  to the end to obtain the number  $d$ .

(All numbers in the problem are considered to be represented in base 10.) For example, for  $a = 2003$ , we obtain  $b = 3200$ ,  $c = 10240000$ , and  $d = 02400001 = 2400001 = d(2003)$ .

Find all numbers  $a$  for which  $d(a) = a^2$ .

**N3.** Determine all pairs of positive integers  $(a, b)$  such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

**N4.** Let  $b$  be an integer greater than 5. For each positive integer  $n$ , consider the number

$$x_n = \underbrace{11 \cdots 1}_{n-1} \underbrace{22 \cdots 2}_n 5,$$

written in base  $b$ .

Prove that the following condition holds if and only if  $b = 10$ : there exists a positive integer  $M$  such that for any integer  $n$  greater than  $M$ , the number  $x_n$  is a perfect square.

A package arrived recently from Ioannis Katsikis, Athens, Greece containing a number of solutions to problems whose solutions have already been given in the *Corner*. In the package were solutions to the following:

- Singapore Mathematical Olympiad 2002, Open Section Part A [2005 : 215], [2006 : 378–383], Problems 1, 3, 4, 5, 6, 7, 9, and 10; and Part B [2005 : 216], [2006 : 384–386], Problems 3 and 4;
- XVIII Italian Mathematical Olympiad [2005 : 217], [2006 : 386–388], Problems 1, 3, and 4;
- 2001–2002 British Mathematical Olympiad, Round 1 [2005 : 287], [2006 : 423–426], Problems 1, 2, 3, 4, and 5; and Round 2 [2005 : 288], [2006 : 426–428], Problems 1, 2, and 3;
- the 15<sup>th</sup> Korean Mathematical Olympiad [2005 : 288–289], [2006 : 429–432], Problems 2 and 5.

Since we have not previously published a solution to Problem 5 of the 15<sup>th</sup> Korean Mathematical Olympiad, we now give the solution of Katsikis.

**5.** Let  $ABC$  be an acute triangle, and let  $O$  be its circumcircle. Let the perpendicular line from  $A$  to  $BC$  meet  $O$  at  $D$ . Let  $P$  be a point on  $O$ , and let  $Q$  be the foot of the perpendicular line from  $P$  to the line  $AB$ . Prove that if  $Q$  is on the outside of  $O$  and  $2\angle QPB = \angle PBC$ , then  $D, P, Q$  are collinear.

*Solution by Ioannis Katsikis, Athens, Greece, modified by the editor.*

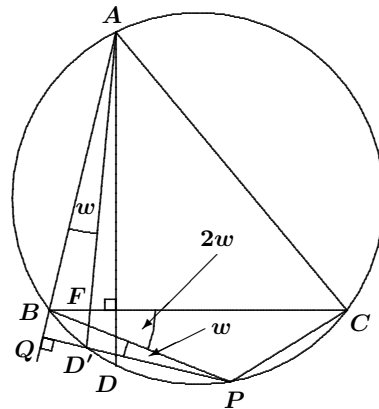
Let  $w = \angle QPB$ . Then  $\angle PBC = 2w$  and  $\angle QBP = 90^\circ - w = \angle ABC$ .

**Case 1.** The line  $PQ$  intersects the circle  $O$  at a point  $D'$  between  $P$  and  $Q$ .

Let  $F$  be the intersection of  $AD'$  and  $BC$ . We have  $\angle BAD' = \angle BPD'$  (since these angles subtend the same arc), and hence  $\angle BAD' = \angle BPQ = w$ . Then

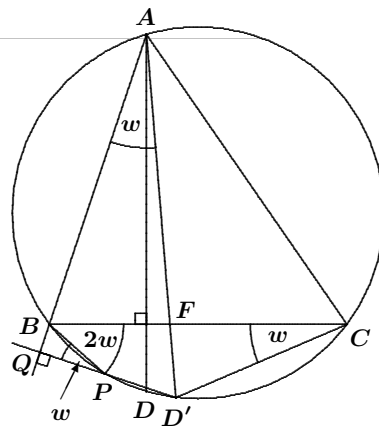
$$\begin{aligned}\angle AFB &= 180^\circ - \angle BAD' - \angle ABC \\ &= 180^\circ - w - (90^\circ - w) \\ &= 90^\circ.\end{aligned}$$

Then  $D' = D$ . Thus,  $D, P, Q$  are collinear.



**Case 2.** The line  $PQ$  intersects the circle  $O$  at a point  $D'$  such that  $P$  is between  $D'$  and  $Q$ .

Let  $F$  be the intersection of  $AD'$  and  $BC$ . Since quadrilateral  $BPD'C$  is inscribed in the circle  $O$ , we see that  $\angle BCD' = 180^\circ - \angle BPD' = w$ . Hence,  $\angle BAD' = w$ . In the triangle  $ABF$ , we have  $\angle BAF = \angle BAD' = w$  and  $\angle ABF = \angle ABC = 90^\circ - w$ . Therefore,  $\angle AFB = 90^\circ$ . Then  $D' = D$ . Thus,  $D, P, Q$  are collinear.



**Case 3.** The line  $PQ$  is tangent to the circle  $O$  at  $P$ .

The proof is similar to Case 2. Quadrilateral  $BPD'C$  now degenerates into triangle  $BPC$ . Then  $\angle BCP = w$  by the Tangent-Chord Theorem, and the rest of Case 2 follows.



Next we consider readers' solutions to the Yugoslav Qualification for IMO 2002 First Round and Second Round given in [2005 : 373–374].

### First Round

**1.** A man standing at the point  $(1, 1)$  in the coordinate plane wants to find an object that lies at some point  $(\alpha, \beta)$ , where  $\alpha \in \{1, 2, \dots, m\}$ , and  $\beta \in \{1, 2, \dots, n\}$ . After finding the object, he will return to the starting point. Find the minimal worst case time needed for doing this job, if he does not know exactly at which point the object lies, and if he can move in any direction with velocity not greater than one.

*Solution by Pierre Bornsstein, Maisons-Laffitte, France.*

The minimal worst case time is equal to the minimal worst case length of a path needed for doing the job.

Any chosen closed path has to go through each of the points  $(\alpha, \beta)$ , where  $\alpha \in \{1, 2, \dots, m\}$  and  $\beta \in \{1, 2, \dots, n\}$ ; otherwise, the man cannot be sure of finding the object. Therefore, any chosen closed path must have length at least  $mn$ .

If  $m = 1$ , it is easy to see that the minimal length of a closed path going through each point is  $2n$ . In the same way, if  $n = 1$ , the minimal length is  $2m$ .

If  $m$  and  $n$  are both greater than 1, then any closed path must have even length because the number of left moves is equal to the number of right moves, and the number of up moves is equal to the number of down moves.

**Case 1.**  $mn$  is even, say  $m$  even.

There exists a closed path with length  $mn$ , going through each point and back to  $(1, 1)$ . We may use:

$$\begin{aligned} & (1, 1) \rightarrow (2, 1) \rightarrow \dots \rightarrow (m, 1) \rightarrow \\ & \rightarrow (m, 2) \rightarrow (m, 3) \rightarrow \dots \rightarrow (m, n) \rightarrow \\ & \rightarrow (m-1, n) \rightarrow (m-1, n-1) \rightarrow \dots \rightarrow (m-1, 2) \rightarrow \\ & \rightarrow (m-2, 2) \rightarrow (m-2, 3) \rightarrow \dots \rightarrow (m-2, n) \rightarrow \\ & \quad \vdots \\ & \rightarrow (1, n) \rightarrow (1, n-1) \rightarrow \dots \rightarrow (1, 1). \end{aligned}$$

**Case 2.**  $mn$  is odd.

Then the minimal length is at least  $mn + 1$ . Conversely, there exists a closed path with length  $mn + 1$ , going through each point and returning back to  $(1, 1)$ . We may use the same path as above until we reach  $(3, n)$  and then use:

$$\begin{aligned} & (3, n) \rightarrow (2, n) \rightarrow (1, n) \rightarrow (1, n-1) \rightarrow (2, n-1) \rightarrow \\ & \rightarrow (2, n-2) \rightarrow (1, n-2) \rightarrow \dots \rightarrow (2, 2) \rightarrow (2, 1) \rightarrow (1, 1). \end{aligned}$$

**2.** Let  $p$  be the semiperimeter of the triangle  $ABC$ . Let the points  $E$  and  $F$  lie on the line  $AB$  such that  $CE = CF = p$ . Prove that the circumcircle of the triangle  $EFC$  and the circle that touches the side  $AB$  and the extension of the sides  $AC$  and  $BC$  of the triangle  $ABC$  meet in one point.

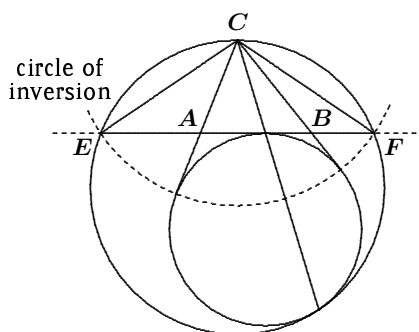
*Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Tangency invites inversion; thus, we invert in the circle with centre  $C$  and radius  $p = CE = CF$ .

The circumcircle of  $\triangle EFC$  becomes the straight line through  $E$  and  $F$ .

Since the tangents from  $C$  (or any other vertex) to the excircle beyond the opposite side are of length  $p$ , the excircle that touches the side  $AB$  is invariant under this inversion.

Tangency is preserved under inversion; therefore, the circumcircle of  $\triangle EFC$  and the excircle beyond  $AB$  meet in one point.



**3.** Let  $\{x_n\}_{n \geq 2}$ , be a sequence such that  $x_2 = 1$ ,  $x_3 = 1$ , and, for  $n \geq 3$ ,

$$(n+1)(n-2)x_{n+1} = n(n^2 - n - 1)x_n - (n-1)^3 x_{n-1}.$$

Prove that  $x_n$  is an integer if and only if  $n$  is a prime.

*Solved by Pierre Bornshtein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornshtein's write-up.*

Let  $y_n = nx_n - 1$ . Then  $y_2 = 1$ ,  $y_3 = 2$  and, for  $n \geq 3$ ,

$$(n-2)y_{n+1} = (n^2 - n - 1)y_n - (n-1)^2 y_{n-1};$$

thus,

$$(n-2)(y_{n+1} - y_n) = (n-1)^2 (y_n - y_{n-1}).$$

Let  $z_n = y_n - y_{n-1}$ . Then  $z_3 = 1$  and  $z_{n+1} = \frac{(n-1)^2}{n-2} z_n$  for  $n \geq 3$ .

It follows easily that  $z_n = (n-2) \cdot (n-2)!$  for  $n \geq 3$ . Then

$$\begin{aligned} y_n &= (y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \cdots + (y_3 - y_2) + y_2 \\ &= z_n + z_{n-1} + \cdots + z_3 + y_2 = 1 + \sum_{k=1}^{n-2} k \cdot k! = (n-1)! \end{aligned}$$

(This last equality follows by an easy induction.) Therefore, for  $n \geq 2$ , we have  $x_n = \frac{(n-1)! + 1}{n}$ .

Thus,  $x_n$  is an integer if and only if  $n$  divides  $(n-1)! + 1$ , which is equivalent to  $n$  being prime, according to Wilson's Theorem.

### Second Round

**1.** What is the maximal value of the expression  $a + b + c + abc$ , if  $a, b, c$  are non-negative numbers such that  $a^2 + b^2 + c^2 + abc \leq 4$ ?

*Solved by Pierre Bornshtein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornshtein's write-up.*

Let  $f(a, b, c) = a + b + c + abc$  and  $g(a, b, c) = a^2 + b^2 + c^2 + abc$ . Let  $S$  be the set of the triples  $(a, b, c)$  of non-negative numbers such that  $g(a, b, c) \leq 4$ .

First, it is clear that  $S \subset [0, 2]^3$  and that  $S$  is a closed set. Since  $f$  is continuous, the desired maximum exists. Let  $M$  denote this maximum. Since  $f(1, 1, 1) = 4$ , we must have

$$M \geq 4. \quad (1)$$

Let  $(x, y, z) \in S$  such that  $f(x, y, z) = M$ . If one of  $x, y, z$  is 2, say  $x = 2$ , then  $y = z = 0$  and  $f(x, y, z) = 2$ , which is a contradiction since  $f(x, y, z) = M \geq 4$ . If one of  $x, y, z$  is 0, say  $x = 0$ , then  $y^2 + z^2 \leq 4$  and, using the inequality between the arithmetic and quadratic means, we have

$$f(x, y, z) = y + z \leq 2\sqrt{\frac{y^2 + z^2}{2}} \leq 2\sqrt{2} < M,$$

a contradiction. Thus,

$$0 < x, y, z < 2. \quad (2)$$

Now we will prove that, if  $x \neq y$ , then  $(\frac{1}{2}(x + y), \frac{1}{2}(x + y), z) \in S$  and  $f(\frac{1}{2}(x + y), \frac{1}{2}(x + y), z) > f(x, y, z)$ .

We have

$$\begin{aligned} g\left(\frac{1}{2}(x + y), \frac{1}{2}(x + y), z\right) &= \left(\frac{1}{2}(x + y)\right)^2 + \left(\frac{1}{2}(x + y)\right)^2 + z^2 + \left(\frac{1}{2}(x + y)\right)^2 z \\ &= x^2 + y^2 + z^2 + xyz - \frac{1}{4}(x - y)^2(2 - z) \\ &= g(x, y, z) - \frac{1}{4}(x - y)^2(2 - z) \leq g(x, y, z), \end{aligned}$$

since  $z < 2$ . This proves that  $(\frac{1}{2}(x + y), \frac{1}{2}(x + y), z) \in S$ . Moreover,

$$f\left(\frac{1}{2}(x + y), \frac{1}{2}(x + y), z\right) - f(x, y, z) = \left[\left(\frac{1}{2}(x + y)\right)^2 - xy\right]z > 0,$$

by the AM–GM Inequality and (2). Thus,

$$f\left(\frac{1}{2}(x + y), \frac{1}{2}(x + y), z\right) > f(x, y, z),$$

as claimed.

Since  $f(x, y, z) = M$  is the maximal value of  $f$ , this proves that  $x = y$ . We can prove in the same way that  $x = z$ , so that  $x = y = z$ . Let  $t$  be this common value. Then  $3t^2 + t^3 \leq 4$ , which forces  $t \leq 1$ . On the other hand, for  $t \leq 1$ , we clearly have  $f(t, t, t) = 3t + t^3 \leq 4$ . Then  $M \leq 4$ .

In view of (1), it follows that  $M = 4$ .

Next we look at readers' solutions to problems of the 27<sup>ième</sup> Olympiade Mathématique Belge, Midi Finale, given in [2005 : 374–375].

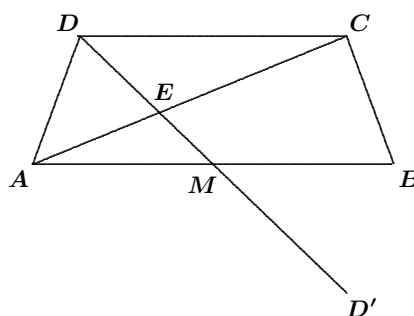
**1.** Soit  $M$  le milieu de la base  $[AB]$  d'un trapèze isocèle  $ABCD$  et  $E$  le point d'intersection de  $MD$  et de  $AC$ . Si  $M$  est le centre du cercle circonscrit au trapèze et si  $[AD]$  et  $[DE]$  ont la même longueur, déterminer l'amplitude de l'angle  $\angle DAB$ .

*Solution by Pavlos Maragoudakis, Pireas, Greece.*

Let  $\Gamma$  denote the circumcircle of  $ABCD$ . We assume that  $\Gamma$  has radius 1 unit. We will use the notation  $\widehat{XY}$  for the length of the minor arc between points  $X$  and  $Y$  on  $\Gamma$ .

Since  $AD = DE$ , we have

$$\begin{aligned}\angle DAC &= \angle DEA \\ &= \angle DCE + \angle CDE \\ &= \angle DCA + \angle CDD',\end{aligned}$$



where  $DD'$  is a diameter of  $\Gamma$ . Note that  $\angle DAC = \frac{1}{2}\widehat{DC}$ ,  $\angle DCA = \frac{1}{2}\widehat{AD}$ , and  $\angle CDD' = \frac{1}{2}\widehat{CD}'$ . Thus,

$$\widehat{DC} = \widehat{AD} + \widehat{CD}' = \widehat{AD} + 180^\circ - \widehat{DC},$$

or

$$2\widehat{DC} = \widehat{AD} + 180^\circ. \quad (1)$$

We also have  $\widehat{AD} + \widehat{DC} + \widehat{CB} = 180^\circ$ . Since  $\widehat{AD} = \widehat{CB}$ , this gives

$$2\widehat{AD} + \widehat{DC} = 180^\circ. \quad (2)$$

Solving (1) and (2), we get  $\widehat{AD} = 36^\circ$  and  $\widehat{DC} = 108^\circ$ . Finally,

$$\angle DAB = \frac{1}{2}\widehat{DB} = \frac{1}{2}(\widehat{DC} + \widehat{CB}) = \frac{1}{2}(108^\circ + 36^\circ) = 72^\circ.$$

**2.** La somme de quatre nombres réels est nulle ; la somme de leurs cubes est également nulle. Est-il vrai qu'alors deux de ces quatre nombres sont nécessairement opposés ?

*Solved by Pierre Bornshtein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornshtein's write-up.*

Oui. Soient  $x, y, z, t$  ces quatre réels.

Par l'absurde : supposons que la somme de deux quelconques d'entre eux n'est jamais nulle. Puisque  $x + y = -(z + t)$ , on a

$$\begin{aligned}x^3 + y^3 + z^3 + t^3 &= (x + y)(x^2 - xy + y^2) + (z + t)(z^2 - zt + t^2) \\ &= (x + y)[(x^2 - xy + y^2) - (z^2 - zt + t^2)],\end{aligned}$$

d'où  $x^2 - xy + y^2 = z^2 - zt + t^2$ .

De même,  $x^2 - xt + t^2 = z^2 - zy + y^2$ . En sommant ces deux égalités, il vient  $2x^2 - x(y+t) = 2z^2 - z(y+t)$ ; c'est à dire,  $(x-z)(2x+2z-y-t) = 0$  ou encore  $3(x-z)(x+z) = 0$  et ainsi  $x = z$ .

En raisonnant de la même façon, on prouve que  $x = y = z = t$ . Mais alors, puisque leur somme est nulle, c'est qu'ils sont tous égaux à 0, ce qui contredit notre point de départ.

**3.** (a) Existe-t-il quatre nombres naturels distincts non nuls tels que la somme de trois quelconques d'entre eux soit toujours un nombre premier ?

(b) Existe-t-il cinq nombres naturels distincts non nuls tels que la somme de trois quelconques d'entre eux soit toujours un nombre premier ?

*Solved by Pierre Bornshtein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornshtein's write-up.*

(a) Oui, par exemple 7, 11, 13 et 23.

(b) Non. Soient  $a, b, c, d, e$  cinq nombres naturels distincts et non nuls. Si trois de ces nombres ont le même reste modulo 3, leur somme est un nombre divisible par 3 et strictement supérieur à 3. Ce ne peut donc pas être un nombre premier. Donc, au plus deux des nombres sont dans une classe donnée modulo 3. Comme il y a 5 nombres à répartir et trois classes possibles, c'est donc que chaque classe contient au moins un nombre. Mais alors, en choisissant un nombre dans chacune des classes, on obtient la même impossibilité. D'où la conclusion.

**4.** Soit un rectangle  $ABCD$ ,  $P$  un point situé sur un des côtés de ce rectangle,  $E$  et  $F$  les pieds des hauteurs abaissées de  $P$  sur les diagonales du rectangle. Démontrer que la somme  $|PE| + |PF|$  reste constante lorsque  $P$  parcourt le périmètre de  $ABCD$ .

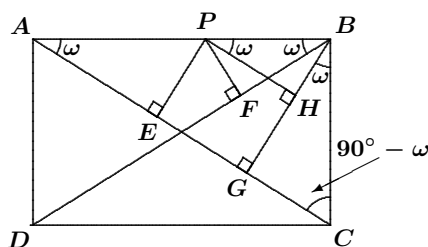
*Solution by Pavlos Maragoudakis, Pireas, Greece.*

Let  $G$  be the projection of  $B$  on  $AC$  and  $H$  the projection of  $P$  on  $BG$ .

If  $\angle BPH = \omega$ , then we have  $\angle GBC = \omega$ ,  $\angle BCA = 90^\circ - \omega$ ,  $\angle CAB = \omega$ , and  $\angle ABD = \omega$ . Now  $\triangle PFB \cong \triangle PHB$  (AAS), since  $PB$  is common,  $\angle PFB = \angle PHB = 90^\circ$ , and  $\angle HPB = \angle FBP = \omega$ . Thus,  $PF = BH$ .

Also,  $PE = HG$  from the rectangle  $PEGH$ .

Finally  $PE + PF = HG + BH = BG$ , which is constant.



Next we look at readers' solutions to problems of the 27<sup>ième</sup> Olympiad Mathématique Belge, Maxi Finale, given in [2005 : 375].

**1.** Soit la suite  $(a_n)_{n \in \mathbb{N}}$  telle que  $a_n = n + \lfloor \sqrt{n} \rfloor$  pour tout  $n \in \mathbb{N}$ . Déterminer le plus petit entier naturel  $k$  pour lequel  $a_k, a_{k+1}, \dots, a_{k+2001}$  constituent une suite de 2002 entiers consécutifs. (Note :  $\lfloor x \rfloor$  désigne le plus grand entier plus petit ou égal à  $x$ .)

*Solved by Pierre Bornshtein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornshtein's write-up.*

Let  $n \in \mathbb{N}$ . There exists a unique non-negative integer  $k$  such that  $k^2 \leq n < (k+1)^2$ . Then  $k = \lfloor \sqrt{n} \rfloor$  and  $a_n = n + k$ .

It follows that, for  $q = 0, 1, \dots, 2k$ , the numbers  $a_{k^2+q}$  are  $2k+1$  consecutive integers. Moreover, we have  $a_{(k+1)^2-1} = (k+1)^2 - 1 + k$  and  $a_{(k+1)^2} = (k+1)^2 + (k+1) = a_{(k+1)^2-1} + 2$ .

Hence, the sequence  $\{a_n\}$  is formed by groups of  $1, 3, \dots, 2n+1, \dots$  consecutive integers, with a gap of 2 between consecutive groups. For  $n \geq 1$ , the first term of the  $n^{\text{th}}$  group (the one with length  $2n-1$ ) is  $a_{(n-1)^2}$ . Thus, the first group of at least 2002 consecutive integers is the one with length  $2003 = 2 \times 1002 - 1$  and first term  $a_{1001^2}$ .

Therefore, the least integer  $k$  such that  $a_k, a_{k+1}, \dots, a_{k+2001}$  are 2002 consecutive integers is  $k = 1001^2$ .

**2.** (a) Dans le plan, soient  $AB_1C_1D_1$  et  $AB_2C_2D_2$  deux carrés ayant un sommet commun (les sommets sont cités dans le même sens). Si  $B, C$  et  $D$  sont respectivement les milieux des segments  $[B_1B_2], [C_1C_2]$  et  $[D_1D_2]$ , le quadrilatère  $ABCD$  est-il aussi un carré ?

(b) Qu'en est-il si les sommets des carrés  $AB_1C_1D_1$  et  $AB_2C_2D_2$  sont cités en sens opposés ?

*Solved by Bruce Crofoot, Thompson Rivers University, Kamloops, BC; and Pavlos Maragoudakis, Pireas, Greece. We present Crofoot's solution.*

Place the squares in the complex plane  $\mathbb{C}$  with  $A$  at the origin. There is some non-zero complex number  $\zeta$  such that the square  $AB_2C_2D_2$  is the image of  $AB_1C_1D_1$  under the transformation  $z \mapsto \zeta z$  ( $z \in \mathbb{C}$ ). Without loss of generality, assume that the square  $AB_1C_1D_1$  has sides of length 1. Place this square so that  $B_1 = 1, C_1 = 1 + i$ , and  $D_1 = i$ .

Note that  $B = \frac{1}{2}(B_1 + B_2)$ , since  $B$  is the mid-point of the segment  $B_1B_2$ . Similarly,  $C = \frac{1}{2}(C_1 + C_2)$  and  $D = \frac{1}{2}(D_1 + D_2)$ .

(a) Suppose the vertices  $A, B_2, C_2, D_2$  are in counterclockwise order (the same as the order of  $A, B_1, C_1, D_1$ ). Then  $B_2 = \zeta B_1 = \zeta$ ,  $C_2 = \zeta C_1 = \zeta(1 + i)$ , and  $D_2 = \zeta D_1 = \zeta i$ . Hence,

$$B = \frac{1}{2}(1 + \zeta), \quad C = \frac{1}{2}(1 + \zeta)(1 + i), \quad D = \frac{1}{2}(1 + \zeta)i.$$

Therefore, the quadrilateral  $ABCD$  is the image of  $AB_1C_1D_1$  under the

transformation  $z \mapsto \frac{1}{2}(1 + \zeta)z$  ( $z \in \mathbb{C}$ ). It follows that  $ABCD$  is a square unless  $\zeta = -1$ .

When  $\zeta = -1$ , the squares  $AB_1C_1D_1$  and  $AB_2C_2D_2$  are the same size and the angle of rotation from one to the other is  $180^\circ$ . In this case, the points  $A, B, C, D$  coincide and there is no quadrilateral  $ABCD$ .

(b) Suppose the vertices  $A, B_2, C_2, D_2$  are in clockwise order (the opposite of the order of  $A, B_1, C_1, D_1$ ). Then  $B_2 = \zeta D_1 = \zeta i$ ,  $C_2 = \zeta C_1 = \zeta(1 + i)$ , and  $D_2 = \zeta B_1 = \zeta$ . Hence,

$$B = \frac{1}{2}(1 + \zeta i), \quad C = \frac{1}{2}(1 + \zeta)(1 + i), \quad D = \frac{1}{2}(i + \zeta).$$

Thus,  $\overrightarrow{AB} = \frac{1}{2}(1 + \zeta i)$  and  $\overrightarrow{AD} = \frac{1}{2}(i + \zeta)$ .

We have  $|\overrightarrow{AB}| = |\overrightarrow{AD}|$  if and only if  $|1 + \zeta i|^2 = |i + \zeta|^2$ ; that is,

$$(1 + \zeta i)(1 - \bar{\zeta} i) = (i + \zeta)(-i + \bar{\zeta}),$$

which simplifies to  $\zeta = \bar{\zeta}$ . Thus,  $|\overrightarrow{AB}| = |\overrightarrow{AD}|$  if and only if  $\Im \zeta = 0$ .

We have  $\overrightarrow{AB} \perp \overrightarrow{AD}$  if and only if  $\Re\{(1 + \zeta i)(\bar{i} + \bar{\zeta})\} = 0$ . Since

$$(1 + \zeta i)(\bar{i} + \bar{\zeta}) = (1 + \zeta i)(-i + \bar{\zeta}) = 2\Re\zeta + i(|\zeta|^2 - 1),$$

we conclude that  $\overrightarrow{AB} \perp \overrightarrow{AD}$  if and only if  $\Re\zeta = 0$ .

From these calculations, we see that there is no case where both  $|\overrightarrow{AB}| = |\overrightarrow{AD}|$  and  $\overrightarrow{AB} \perp \overrightarrow{AD}$ . Thus,  $ABCD$  is never a square. (However,  $ABCD$  is always a parallelogram when it is non-degenerate, as can be easily checked.)

**3.** Voici une vue partielle d'une table de multiplication dans laquelle un tableau rectangulaire a été sélectionné.

1	2	3	4	5	6	...
2	4	6	8	10	12	...
3	6	9	12	15	18	...
4	8	12	16	20	24	...
5	10	15	20	25	30	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

Pour chaque tableau dont l'élément du coin supérieur gauche et celui du coin inférieur droit sont respectivement 1 et 2002, on calcule la somme de tous ses éléments. Quelle est la plus petite des sommes ainsi obtenues ?

*Solved by Pierre Bornsztejn, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztejn's write-up.*

Soient  $a$  et  $b$  les nombres situés respectivement dans le coin inférieur gauche et dans le coin supérieur droit du tableau sélectionné. Ainsi on a  $ab = 2002$ . La somme  $S(a, b)$  de tous les nombres du tableau est

$$\begin{aligned} S(a, b) &= \sum_{k=1}^a (k + 2k + \dots + bk) = \left( \sum_{k=1}^a k \right) \left( \sum_{k=1}^b k \right) \\ &= \frac{a(a+1)b(b+1)}{4} = \frac{2002(2003+a+b)}{4}, \end{aligned}$$

comme  $ab = 2002$ .

Il s'agit donc de minimiser  $a + b$  sous la contrainte  $ab = 2002$ . Comme  $2002 = 2 \times 7 \times 11 \times 13$ , le nombre 2002 a 16 diviseurs positifs. Par symétrie des rôles, il suffit d'étudier les cas  $a \in \{1, 2, 7, 11, 13, 14, 22, 26\}$ .

On vérifie à la main que la valeur minimale de  $a + b$  est alors  $103 = 26 + 77$ . La somme minimale cherchée est donc

$$S(26, 77) = \frac{2002(2003 + 103)}{4} = 1054053.$$

**4.** Trouver tous les nombres premiers  $a$  et  $b$  tels que  $a^{a+1} + b^{b+1}$  est aussi un nombre premier.

*Solved by Pierre Bornsztejn, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztejn's write-up.*

Soient  $a$  et  $b$  deux nombres premiers tel que  $a^{a+1} + b^{b+1} = p$  est premier. Comme  $a, b \geq 2$ , on a  $p > 2$  et donc  $p$  est impair. Par suite, parmi  $a$  et  $b$  l'un est pair et l'autre est impair, disons  $a$  pair. Comme  $a$  est premier, c'est donc que  $a = 2$ .

L'équation se réécrit alors  $8 + b^{b+1} = p$ . Si  $b \geq 5$  alors, puisque  $b + 1$  est pair, on a  $b^{b+1} \equiv 1 \pmod{3}$  et donc  $p \equiv 0 \pmod{3}$ . Or,  $p > 8$ , donc  $p$  ne peut être premier. Par suite  $b = 3$ . On a alors  $p = 89$ , qui est bien premier.

Finalement, à l'ordre près, la seule solution est  $(a, b) = (2, 3)$ .

That completes the *Corner* for this issue. Send me your nice solutions, and soon for problems that have appeared in 2006 numbers of the *Corner*.



## BOOK REVIEW

John Grant McLoughlin

*Combinatorial Explorations*

By Richard Hoshino and John Grant McLoughlin, published by the Canadian Mathematical Society, 2005.

ISBN 0-919558-14-3, coilbound, 60+iv pages, CDN\$12.00.

Reviewed by **Jim Totten**, Thompson Rivers University Kamloops, BC.

*Combinatorial Explorations* is Volume V in a series of booklets called “A Taste of Mathematics” (ATOM), published by the Canadian Mathematical Society (CMS). According to the CMS website, the booklets in the series “are designed as enrichment materials for high school students with an interest in and aptitude for mathematics.” Some of the booklets “also cover the materials useful for mathematical competitions at national and international levels”.

The booklet discusses three core problems and some extensions to those problems. The first core problem is the Handshake Problem (how many handshakes take place among  $n$  people if everyone shakes everyone else’s hand?); the second core problem is the Route Problem (in how many ways can one travel from point  $A$  to point  $B$ , located at diagonally opposite corners of a rectangle, by proceeding only along a given finite number of grid lines parallel to the edges of the rectangle, and ensuring that the minimum distance is travelled?); and the third core problem is the Checkerboard Problem (how many squares of all sizes appear on an  $8 \times 8$  checkerboard?).

In the process of extending these core problems, the reader comes across many other interesting combinatorial ideas, including (but not limited to) Pascal’s Triangle, the Binomial Theorem, Fibonacci Numbers, sums of cubes, and telescoping series. Each of these core problems leads quite naturally to an investigation, where the ideas developed in that section need to be extended and applied to new problems. For example, the discussion of the Handshake Problem leads to a guided investigation of Ramsey Theory.

Since there is both a glossary at the end of the book and an introduction which defines all the terms needed to understand the material, even a novice to combinatorics is well-served by this book.

The book is very well written and is at a level that makes it understandable for most high school students. I would recommend this book for any student seeking some enrichment in mathematics, or for a teacher who is looking for ideas to enrich the mathematics course(s) she is teaching.

## PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le 1er juin 2007. Une étoile (\*) après le numéro indique que le problème a été soumis sans solution.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

**3135.** Correction. *Proposé par Marian Marinescu, Monbonnot, France.*

Soit  $\mathbb{R}^+$  l'ensemble des nombres réels non négatifs. Pour tout  $a, b$  et  $c \in \mathbb{R}^+$ , soit  $H(a, b, c)$  l'ensemble de toutes les fonctions  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  telles que

$$h(x) \geq h(h(ax)) + h(bx) + cx$$

pour tout  $x \in \mathbb{R}^+$ . Montrer que  $H(a, b, c)$  est non vide si et seulement si  $b \leq 1$  et  $4ac \leq (1 - b)^2$ .

**3188.** *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit  $x, y$  et  $z$  des nombres réels positifs. Montrer que

$$\left(\frac{x}{y} + \frac{z}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{y}{z} + \frac{x}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{z}{x} + \frac{y}{\sqrt[3]{xyz}}\right)^2 \geq 12.$$

**3189.** *Proposé par K. R. S. Sastry, Bangalore, Inde.*

Soit  $A$  le plus grand des trois angles du triangle  $ABC$ . Soit  $\alpha$  la mesure de l'angle  $A$ , et soit respectivement  $h, w$  et  $m$  les longueurs de la hauteur, de la bissectrice intérieure et de la médiane, toutes mesurées de  $A$  jusqu'au côté  $BC$ .

- (a) Déterminer l'aire du triangle  $ABC$  en fonction de  $\alpha, h$  et  $w$ .
- (b) Déterminer l'aire du triangle  $ABC$  en fonction de  $\alpha, m$  et  $w$ .

[Ed : Le lecteur pourra consulter le problème M63 de Mayhem et sa solution [2003 : 427–428].]

**3190.** *Proposé par D. J. Smeenk, Zaltbommel, Pays-Bas.*

Soit  $A$  un point sur le cercle  $\Gamma$ , et soit  $P$  un point en dehors de  $\Gamma$ . Construire une droite  $\ell$  par  $P$  coupant  $\Gamma$  en  $B$  et  $C$  de telle sorte que

$$2(BC) = AB + AC.$$

**3191.** *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit  $\Gamma$  le cercle circonscrit au triangle  $ABC$ , soit  $AD$  la bissectrice interne de l'angle  $BAC$  avec  $D$  sur  $BC$ , et soit  $E$  le point où  $AD$  coupe  $\Gamma$  pour la seconde fois. Soit  $\Gamma'$  le cercle de diamètre  $AE$ , et soit  $F$  un point de  $\Gamma'$  tel que  $DF$  soit perpendiculaire à  $AE$ . Montrer que  $EF = EC$ .

**3192.** *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit  $k \in (0, 1)$ , et soit la suite  $\{B_n\}_{n=0}^\infty$  définie par  $B_0 = k$ ,  $B_1 = k^2$  et  $B_{n+2} = kB_{n+1} + k^2B_n$  pour les  $n$  entiers non négatifs. Trouver  $\sum_{n=0}^{\infty} \frac{B_n}{n+1}$ .

**3193.** *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit le triangle  $ABC$  et  $A_1, B_1$  et  $C_1$  trois points sur les côtés respectifs  $BC, CA$  et  $AB$ , de sorte que

$$\frac{AC_1}{C_1B} = \frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = k,$$

où  $k$  est une constante positive. Soit  $H$  et  $H_1$  les orthocentres respectifs des triangles  $ABC$  et  $A_1B_1C_1$  avec  $O$  et  $O_1$  les centres de leur cercle circonscrit respectif. Montrer que  $OO_1$  est parallèle à  $HH_1$ .

**3194.** *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit  $n$  un entier positif quelconque, et soit  $x_k, y_k \in \mathbb{R}$  pour  $k = 1, 2, \dots, n$ . Montrer que

$$\min \left\{ \sum_{k=1}^n x_k^2, \sum_{k=1}^n y_k^2 \right\} \cdot \sum_{k=1}^n (x_k - y_k)^2 \geq \sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2.$$

**3195.** *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

(a) Soit  $n \geq 3$  un nombre naturel. Montrer qu'il existe un nombre réel  $q_n > 1$  tel que pour tous nombres réels  $a_1, a_2, \dots, a_n \in [1/q_n, q_n]$ ,

$$\frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + \dots + \frac{a_n}{a_n + a_1} \geq \frac{n}{2}.$$

(b)★ Existe-t-il un nombre réel  $q > 1$  tel que l'inégalité ci-dessus soit valide pour tout nombre naturel  $n \geq 3$  et pour tous nombres réels  $a_1, a_2, \dots, a_n \in [1/q, q]$ ?

**3196.** *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit  $x_1, x_2, \dots, x_n$   $n$  nombres réels positifs. Montrer que

$$\begin{aligned} & x_1^n + x_2^n + \dots + x_n^n + n(n-1)x_1x_2 \cdots x_n \\ & \geq x_1x_2 \cdots x_n(x_1 + x_2 + \dots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right). \end{aligned}$$

**3197.** *Proposé par Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.*

Soit  $AB$  un segment de droite fixé. Parmi les triangles qui satisfont la condition  $\angle AIO = \pi/2$ , où  $I$  est le centre du cercle inscrit du triangle  $ABC$  et  $O$  celui de son cercle circonscrit, trouver le triangle d'aire maximale. Quelle est la valeur de celle-ci ?

**3198.** *Proposé par Michel Bataille, Rouen, France.*

Soit  $ABCD$  un quadrilatère plan qui ne soit pas un parallélogramme. Soit  $C'$  et  $D'$  les projections orthogonales respectives des points  $C$  et  $D$  sur la droite  $AB$ . Les perpendiculaires de  $C$  sur  $AD$  et de  $D$  sur  $BC$  se coupent en  $P$ ; les perpendiculaires de  $C'$  sur  $AD$  et de  $D'$  sur  $BC$  se coupent en  $Q$ . Montrer que  $PQ$  est perpendiculaire à la droite passant par les milieux de  $AC$  et  $BD$ .

**3199.** *Proposé par Michel Bataille, Rouen, France.*

Trouver toutes les fonctions  $f : \mathbb{R} \rightarrow \mathbb{R}$  telles que, pour tous les réels  $x$  et  $y$ , on a  $f(xy) = f(f(x) + f(y))$ .

**3200.** *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit  $ABC$  un triangle avec l'angle en  $B$  plus grand que l'angle en  $C$ , et soit  $E$  le centre du cercle exinscrit opposé à  $A$ . Soit respectivement  $M$  et  $N$  les points sur  $AB$  et  $AC$ , tels que  $EM$  soit la bissectrice intérieure de l'angle  $AEB$  et  $EN$  celle de l'angle  $AEC$ . Si l'on prolonge  $MN$  jusqu'à son intersection  $L$  avec  $BC$ , montrer qu'alors la somme des angles  $BEL$  et  $CEL$  vaut  $180^\circ$ .

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**3135.** *Correction. Proposed by Marian Marinescu, Monbonnot, France.*

Let  $\mathbb{R}^+$  be the set of non-negative real numbers. For all  $a, b, c \in \mathbb{R}^+$ , let  $H(a, b, c)$  be the set of all functions  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that

$$h(x) \geq h(h(ax)) + h(bx) + cx$$

for all  $x \in \mathbb{R}^+$ . Prove that  $H(a, b, c)$  is non-empty if and only if  $b \leq 1$  and  $4ac \leq (1 - b)^2$ .

**3188.** *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let  $x, y, z$  be positive real numbers. Prove that

$$\left(\frac{x}{y} + \frac{z}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{y}{z} + \frac{x}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{z}{x} + \frac{y}{\sqrt[3]{xyz}}\right)^2 \geq 12.$$

**3189.** *Proposed by K.R.S. Sastry, Bangalore, India.*

In  $\triangle ABC$ , let  $A$  be the largest of the three angles. Let  $\alpha$  denote the measure of angle  $A$ , and let  $h$ ,  $w$ , and  $m$  denote the lengths of the altitude, the internal angle bisector, and the median, all measured from  $A$  to the side  $BC$ .

(a) Determine the area of  $\triangle ABC$  in terms of  $\alpha$ ,  $h$ , and  $w$ .

(b) Determine the area of  $\triangle ABC$  in terms of  $\alpha$ ,  $m$ , and  $w$ .

[*Ed:* The reader may wish to look at Mayhem problem M63 and its solution [2003 : 427–428].]

**3190.** *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let  $A$  be a point on the circle  $\Gamma$ , and let  $P$  be a point outside  $\Gamma$ . Construct a line  $\ell$  through  $P$  which intersects  $\Gamma$  at  $B$  and  $C$  such that

$$2(BC) = AB + AC.$$

**3191.** *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let  $\Gamma$  be the circumcircle of  $\triangle ABC$ , let  $AD$  be the internal angle bisector of  $\angle BAC$  with  $D$  on  $BC$ , and let  $E$  be the point where  $AD$  meets  $\Gamma$  for the second time. Let  $\Gamma'$  be the circle with  $AE$  as diameter, and let  $F$  be a point of  $\Gamma'$  such that  $DF \perp AE$ . Prove that  $EF = EC$ .

**3192.** *Proposed by Mihály Bencze, Brasov, Romania.*

Let  $k \in (0, 1)$ , and let the sequence  $\{B_n\}_{n=0}^{\infty}$  be defined by  $B_0 = k$ ,  $B_1 = k^2$ , and  $B_{n+2} = kB_{n+1} + k^2B_n$  for integers  $n \geq 0$ . Find  $\sum_{n=0}^{\infty} \frac{B_n}{n+1}$ .

**3193.** *Proposed by Mihály Bencze, Brasov, Romania.*

Let  $ABC$  be a triangle, and let  $A_1$ ,  $B_1$ ,  $C_1$  be on sides  $BC$ ,  $CA$ ,  $AB$ , respectively, such that

$$\frac{AC_1}{C_1B} = \frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = k,$$

where  $k$  is a positive constant. Let  $H$  and  $H_1$  be the orthocentres of  $\triangle ABC$  and  $\triangle A_1B_1C_1$ , respectively, and let  $O$  and  $O_1$  be their respective circumcentres. Prove that  $OO_1 \parallel HH_1$ .

**3194.** *Proposed by Mihály Bencze, Brasov, Romania.*

Let  $n$  be any positive integer, and let  $x_k, y_k \in \mathbb{R}$  for  $k = 1, 2, \dots, n$ . Prove that

$$\min \left\{ \sum_{k=1}^n x_k^2, \sum_{k=1}^n y_k^2 \right\} \cdot \sum_{k=1}^n (x_k - y_k)^2 \geq \sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i)^2.$$

**3195.** Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.

- (a) Let  $n$  be a natural number,  $n \geq 3$ . Prove that there is a real number  $q_n > 1$  such that for any real numbers  $a_1, a_2, \dots, a_n \in [1/q_n, q_n]$ ,

$$\frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + \dots + \frac{a_n}{a_n + a_1} \geq \frac{n}{2}.$$

- (b)★ Does there exist a real number  $q > 1$  such that the inequality in (a) holds for any natural number  $n \geq 3$  and for any real numbers  $a_1, a_2, \dots, a_n \in [1/q, q]$ ?

**3196.** Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.

Let  $x_1, x_2, \dots, x_n$  be positive real numbers. Prove that

$$\begin{aligned} & x_1^n + x_2^n + \dots + x_n^n + n(n-1)x_1x_2 \cdots x_n \\ & \geq x_1x_2 \cdots x_n(x_1 + x_2 + \dots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right). \end{aligned}$$

**3197.** Proposed by Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.

If  $AB$  is a fixed line segment, find the triangle  $ABC$  which has maximum area among those which satisfy  $\angle AIO = \pi/2$ , where  $I$  is the incentre of  $\triangle ABC$  and  $O$  is its circumcentre. What is this maximum area?

**3198.** Proposed by Michel Bataille, Rouen, France.

Let  $ABCD$  be a planar quadrilateral which is not a parallelogram. Let  $C'$  and  $D'$  be the orthogonal projections onto the line  $AB$  of the points  $C$  and  $D$ , respectively. The perpendiculars from  $C$  to  $AD$  and from  $D$  to  $BC$  meet at  $P$ ; the perpendiculars from  $C'$  to  $AD$  and from  $D'$  to  $BC$  meet at  $Q$ . Show that  $PQ$  is perpendicular to the line through the mid-points of  $AC$  and  $BD$ .

**3199.** Proposed by Michel Bataille, Rouen, France.

Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(xy) = f(f(x) + f(y))$  for all real numbers  $x$  and  $y$ .

**3200.** Proposed by Christopher J. Bradley, Bristol, UK.

Let  $ABC$  be a triangle with  $\angle B > \angle C$ , and let  $E$  be the centre of the excircle opposite angle  $A$ . Let  $M$  and  $N$  be points on  $AB$  and  $AC$ , respectively, such that  $EM$  is the internal bisector of  $\angle AEB$  and  $EN$  is the internal bisector of  $\angle AEC$ . If  $MN$  is extended to meet  $BC$  at  $L$ , prove that  $\angle BEL + \angle CEL = 180^\circ$ .

## SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

We apologize for omitting the name of Kee-Wai Lau, Hong Kong, China from the list of solvers of 3073.

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**3085.** [2005 : 459, 462] *Proposed by Neven Jurič, Zagreb, Croatia.*

A *magic square of order  $n$*  is an  $n \times n$  array containing the integers from 1 to  $n^2$  such that the sum of the elements in each row, in each column, and on each of the two diagonals is the same.

Let  $\mathfrak{M}$  be a magic square of odd order  $n \geq 3$ . Increase the values of all the entries in  $\mathfrak{M}$  by  $2n + 2$  to get a new  $n \times n$  array, say  $M_1$ . Place  $M_1$  in the interior of an  $(n + 2) \times (n + 2)$  array  $M'$ . Show that the border rows and columns of this can be filled in with the unused integers between 1 and  $(n + 2)^2$  to create a new magic square  $\mathfrak{M}'$  of order  $n + 2$ .

*Solution by Michel Bataille, Rouen, France.*

We will describe a process given in René Descombes, *Les carrés magiques*, Vuibert, 2000.

If  $\mathfrak{M}$  is a magic square, we denote by  $S(\mathfrak{M})$  the sum of the elements in each row, column, or diagonal. For a magic square of order  $n$ , we have

$$nS(\mathfrak{M}) = 1 + 2 + \cdots + n^2 = \frac{1}{2}n^2(n^2 + 1);$$

hence,  $S(\mathfrak{M}) = \frac{1}{2}n(n^2 + 1)$ . For the desired magic square  $\mathfrak{M}'$  of order  $n + 2$ , we must have  $S(\mathfrak{M}') = \frac{1}{2}(n + 2)(n^2 + 4n + 5)$ . In  $M_1$ , the sum of the elements in each row, column, or diagonal, is

$$S(\mathfrak{M}) + n(2n + 2) = \frac{1}{2}n(n^2 + 4n + 5).$$

Thus, in each row, column, or diagonal of  $M'$ , we must add two elements whose sum is

$$k = \frac{1}{2}(n+2)(n^2+4n+5) - \frac{1}{2}n(n^2+4n+5) = n^2+4n+5 = (n+2)^2+1.$$

These two elements are to be taken from the missing numbers

$$1, 2, \dots, 2n + 2, k - 1, k - 2, \dots, k - (2n + 2).$$

Clearly, it is sufficient to indicate the locations of  $1, 2, \dots, 2n + 2$  in the border rows and columns.

Now, set  $n = 2p - 1$  and label the blank squares of  $M'$  as follows:

$$\begin{array}{cccc}
 (1, 1) & (1, 2) & \cdots & (1, 2p + 1) \\
 (2, 1) & & & (2, 2p + 1) \\
 \vdots & & & \vdots \\
 (2p, 1) & & & (2p, 2p + 1) \\
 (2p + 1, 1) & (2p + 1, 2) & \cdots & (2p + 1, 2p + 1)
 \end{array}$$

In these blank squares, we distribute the numbers  $1, 2, \dots, 2n+2$  as follows:

$$\begin{array}{ll}
 4p \longrightarrow (1, p) & 4p - 1 \longrightarrow (p, 1) \\
 4p - 2 \longrightarrow (1, p - 1) & 4p - 3 \longrightarrow (p - 1, 1) \\
 \vdots & \vdots \\
 2p + 2 \longrightarrow (1, 1) & 2p + 3 \longrightarrow (2, 1) \\
 2p \longrightarrow (1, 2p + 1) & 2p + 1 \longrightarrow (p + 1, 1) \\
 2p - 2 \longrightarrow (2p + 1, p + 2) & 2p - 1 \longrightarrow (2p + 1, p + 1) \\
 2p - 4 \longrightarrow (2p + 1, p + 3) & 2p - 3 \longrightarrow (p + 2, 2p + 1) \\
 \vdots & 2p - 5 \longrightarrow (p + 3, 2p + 1) \\
 2 \longrightarrow (2p + 1, 2p) & \vdots \\
 & 1 \longrightarrow (2p, 2p + 1)
 \end{array}$$

In this way, each row, column, and diagonal of  $M_1$  receives exactly one extra number, and we can complete the new magic square.

For example, starting with the magic square of order 3:

4	9	2
3	5	7
8	1	6

we first obtain

6	8			4
7	12	17	10	
5	11	13	15	
	16	9	14	1
		3	2	

and finally the magic square of order 5:

6	8	23	24	4
7	12	17	10	19
5	11	13	15	21
25	16	9	14	1
22	18	3	2	20

With the same method, the latter yields the following magic square of



order 7:

8	10	12	45	46	48	6
9	18	20	35	36	16	41
11	19	24	29	22	31	39
7	17	23	25	27	33	43
47	37	28	21	26	13	3
49	34	30	15	14	32	1
44	40	38	5	4	2	42

Also solved by the proposer.

**3087.** [2005 : 459, 462] Proposed by Mihály Bencze, Brasov, Romania.

Let  $ABC$  be a triangle with sides  $a, b, c$  opposite the angles  $A, B, C$ , respectively. If  $R$  is the circumradius and  $r$  the inradius of  $\triangle ABC$ , prove that:

- (a)  $\frac{3R}{r} \geq \frac{a+c}{b} + \frac{b+a}{c} + \frac{c+b}{a} \geq 6$ ;
- (b)  $\left(\frac{R}{r}\right)^3 \geq \left(\frac{a}{b} + \frac{b}{a}\right) \left(\frac{b}{c} + \frac{c}{b}\right) \left(\frac{a}{c} + \frac{c}{a}\right) \geq 8$ .

(Both (a) and (b) are refinements of Euler's Inequality,  $R \geq 2r$ .)

Similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

(a) The right inequality follows by adding the easy to prove inequalities  $\frac{a}{b} + \frac{b}{a} \geq 2$ ,  $\frac{b}{c} + \frac{c}{b} \geq 2$ , and  $\frac{c}{a} + \frac{a}{c} \geq 2$ . The left inequality follows in the same way from the known inequality  $\frac{R}{r} \geq \frac{b}{c} + \frac{c}{b}$  (item 5.30 in [1]) and the cyclic versions of it, namely  $\frac{R}{r} \geq \frac{c}{a} + \frac{a}{c}$  and  $\frac{R}{r} \geq \frac{a}{b} + \frac{b}{a}$ .

(b) The same argument works for the second part of the problem upon substituting "adding" by "multiplying".

#### References

- [1] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (second solution); MICHEL BATAILLE, Rouen, France; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; TITU ZVONARU, Comănești, Romania; and the proposer.

**3088.** [2005 : 543, 546] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let  $ABC$  be a triangle and  $P$  a point in the plane of this triangle, not lying on any of the three lines determined by its sides. Let  $AD$ ,  $BE$ , and  $CF$  be the Cevians through the point  $P$ . The lines through  $A$  parallel to  $BE$  and  $CF$  meet the line  $BC$  at  $L$  and  $L'$ , respectively. Points  $M$ ,  $M'$ ,  $N$ , and  $N'$  are similarly defined. Prove that  $L$ ,  $L'$ ,  $M$ ,  $M'$ ,  $N$ ,  $N'$  all lie on a conic.

*Solution by John G. Heuver, Grande Prairie, AB.*

We will apply the converse of Carnot's Theorem to the triangle  $ABC$ : if, on the lines  $BC$ ,  $CA$ ,  $AB$ , pairs of points  $L$  and  $L'$ ,  $M$  and  $M'$ ,  $N$  and  $N'$ , respectively, are taken such that

$$\frac{AN}{BN} \cdot \frac{AN'}{BN'} \cdot \frac{BL}{CL} \cdot \frac{BL'}{CL'} \cdot \frac{CM}{AM} \cdot \frac{CM'}{AM'} = 1,$$

and if no three of the points  $L$ ,  $L'$ ,  $M$ ,  $M'$ ,  $N$ ,  $N'$  are collinear, then these six points lie on a non-degenerate conic. [A proof is given below; for further details see Howard Eves, *A Survey of Geometry*, Revised Edition (Allyn and Bacon, 1972), pages 256, 262, and 414.] The given parallelisms imply the following six equalities:

$$\begin{aligned} \frac{AN}{BN} &= \frac{DC}{BC}, & \frac{AN'}{BN'} &= \frac{AC}{EC}, & \frac{BL}{CL} &= \frac{EA}{CA}, \\ \frac{BL'}{CL'} &= \frac{BA}{FA}, & \frac{CM}{AM} &= \frac{FB}{AB}, & \frac{CM'}{AM'} &= \frac{CB}{DB}. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{AN}{BN} \cdot \frac{AN'}{BN'} \cdot \frac{BL}{CL} \cdot \frac{BL'}{CL'} \cdot \frac{CM}{AM} \cdot \frac{CM'}{AM'} \\ &= \frac{DC}{BC} \cdot \frac{AC}{EC} \cdot \frac{EA}{CA} \cdot \frac{BA}{FA} \cdot \frac{FB}{AB} \cdot \frac{CB}{DB} \\ &= \frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = 1, \end{aligned}$$

where the last equality holds by Ceva's Theorem. The converse of Carnot's Theorem then gives the desired result. [Ed: Should three of the given points be collinear, then the conic would degenerate into a pair of lines.]

*Also solved by MICHEL BATAILLE, Rouen, France; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

*Bataille's approach was essentially the same as that of our featured solution. As part of his solution he proved the converse of Carnot's Theorem. He used areal coordinates relative to triangle  $ABC$  to show that the points  $(0, 1, \lambda)$ ,  $(0, 1, \lambda')$ ,  $(\mu, 0, 1)$ ,  $(\mu', 0, 1)$ ,  $(1, \nu, 0)$ ,  $(1, \nu', 0)$  on the sides of the reference triangle all lie on a conic (or more precisely, their coordinates satisfy a second degree equation), if  $\lambda\lambda'\mu\mu'\nu\nu' = 1$ . His proof: Let  $r, s, t, u, v, w$  be such that  $\lambda = r/s$ ,  $\lambda' = s/t$ ,  $\mu = t/u$ ,  $\mu' = u/v$ ,  $\nu = v/w$ ,  $\nu' = w/r$ . Then one checks easily that the coordinates of all six points satisfy the equation*

$$vx^2 + ry^2 + tz^2 - \frac{s^2 + rt}{s}yz - \frac{u^2 + tv}{u}zx - \frac{w^2 + rv}{w}xy = 0.$$

(Remark. Carnot's Theorem says that  $\lambda\lambda'\mu\mu'\nu\nu' = 1$  is a necessary condition for the six points to lie on the same conic.)

**3089.** [2005 : 543, 546] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let  $ABC$  be a triangle and  $P$  a point in the plane of this triangle, not lying on any of the three lines determined by its sides. Let  $AD$ ,  $BE$ , and  $CF$  be the Cevians through the point  $P$ . The lines through  $E$  and  $F$  parallel to  $AD$  meet the line  $BC$  at  $L$  and  $L'$ , respectively. Points  $M$ ,  $M'$ ,  $N$ , and  $N'$  are similarly defined. Prove that  $L$ ,  $L'$ ,  $M$ ,  $M'$ ,  $N$ ,  $N'$  all lie on a conic.

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Using

$$\frac{BL'}{BD} = \frac{BF}{BA} \quad \text{and} \quad \frac{BD}{BC} = \frac{BN}{BF},$$

we see that

$$\frac{BL'}{BC} = \frac{BL'}{BD} \cdot \frac{BD}{BC} = \frac{BF}{BA} \cdot \frac{BN}{BF} = \frac{BN}{BA}.$$

It follows that  $NL' \parallel AC$ . Likewise,  $LM' \parallel BA$  and  $MN' \parallel CB$ . Therefore, opposite sides of the hexagon  $L'LM'MN'N$  are parallel and thus, its vertices lie on a conic by the converse of Pascal's Theorem.

*Also solved by MICHEL BATAILLE, Rouen, France; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.*

**3090.** [2005 : 543, 546] *Proposed by Arkady Alt, San Jose, CA, USA.*

Find all non-negative real solutions  $(x, y, z)$  to the following system of inequalities:

$$\begin{aligned} 2x(3 - 4y) &\geq z^2 + 1, \\ 2y(3 - 4z) &\geq x^2 + 1, \\ 2z(3 - 4x) &\geq y^2 + 1. \end{aligned}$$

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Without loss of generality, we may assume that  $x \leq \min\{y, z\}$ . From the first equation, we have

$$(3x - 1)^2 + 8x(y - x) + (z^2 - x^2) = z^2 + 1 - 2x(3 - 4y) \leq 0.$$

Each term of the sum on the left is non-negative; hence,  $x = y = z = \frac{1}{3}$ .

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer.*

**3091.** [2005 : 543, 546] *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

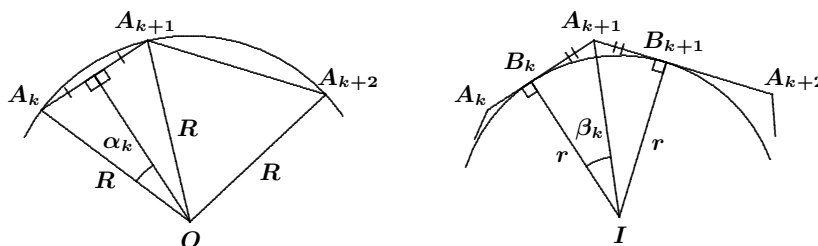
Let  $A_1A_2 \cdots A_n$  be a convex polygon which has both an inscribed circle and a circumscribed circle. Let  $B_1, B_2, \dots, B_n$  denote the points of tangency of the incircle with sides  $A_1A_2, A_2A_3, \dots, A_nA_1$ , respectively. Prove that

$$\frac{2sr}{R} \leq \sum_{k=1}^n B_k B_{k+1} \leq 2s \cos\left(\frac{\pi}{n}\right),$$

where  $R$  is the radius of the circumscribed circle,  $r$  is the radius of the inscribed circle,  $s$  is the semiperimeter of the polygon  $A_1A_2 \cdots A_n$ , and  $B_{n+1} = B_1$ .

*Solution by Joel Schlosberg, Bayside, NY, USA.*

Take all subscripts modulo  $n$ . Let  $O$  and  $I$  be the circumcentre and incentre, respectively, of  $A_1A_2 \cdots A_n$ .



Let  $\alpha_k = \frac{1}{2}\angle A_k O A_{k+1}$  and  $\beta_k = \frac{1}{2}\angle B_k I B_{k+1}$ . Note that  $0 < \alpha_k < \pi$  since  $0 < \angle A_k O A_{k+1} < 2\pi$ . Note also that

$$\sum_{k=1}^n \beta_k = \frac{1}{2} \sum_{k=1}^n \angle B_k I B_{k+1} = \pi,$$

and since the angles of quadrilateral  $B_k I B_{k+1} A_{k+1}$  sum to  $2\pi$ ,

$$\begin{aligned} 2\pi &= \angle B_k I B_{k+1} + \angle B_k A_{k+1} B_{k+1} + \angle I B_k A_{k+1} + \angle I B_{k+1} A_{k+1} \\ &= 2\beta_k + \angle A_k A_{k+1} A_{k+2} + (\pi/2) + (\pi/2), \end{aligned}$$

so that  $\beta_k = \frac{1}{2}(\pi - \angle A_k A_{k+1} A_{k+2})$ . Clearly,  $0 < \beta_k < \pi/2$ .

Triangle  $A_k O A_{k+1}$  gives  $A_k A_{k+1} = 2R \sin \alpha_k$ , so that

$$2s = \sum_{k=1}^n A_k A_{k+1} = 2R \sum_{k=1}^n \sin \alpha_k. \quad (1)$$

Similarly, triangle  $B_k I B_{k+1}$  gives  $B_k B_{k+1} = 2r \sin \beta_k$ , and therefore,

$$\sum_{k=1}^n B_k B_{k+1} = 2r \sum_{k=1}^n \sin \beta_k. \quad (2)$$

Finally, triangles  $B_k I A_{k+1}$  and  $A_{k+1} I B_{k+1}$  give

$$B_k A_{k+1} + A_{k+1} B_{k+1} = 2r \tan \beta_k,$$

so that

$$2s = \sum_{k=1}^n A_k A_{k+1} = \sum_{k=1}^n (B_k A_{k+1} + A_{k+1} B_{k+1}) = 2r \sum_{k=1}^n \tan \beta_k. \quad (3)$$

If  $A_{k+1}$  and  $O$  are on the same side of  $A_k A_{k+2}$ , then

$$\begin{aligned} \angle A_k A_{k+1} A_{k+2} &= \frac{1}{2} \angle A_k O A_{k+2} = \frac{1}{2} (2\pi - 2\alpha_k - 2\alpha_{k+1}) \\ &= \pi - \alpha_k - \alpha_{k+1}. \end{aligned}$$

If  $A_{k+1}$  and  $O$  are on opposite sides of  $A_k A_{k+2}$ , then

$$\begin{aligned} \angle A_k A_{k+1} A_{k+2} &= \pi - \frac{1}{2} \angle A_k O A_{k+2} = \pi - \frac{1}{2} (2\alpha_k + 2\alpha_{k+1}) \\ &= \pi - \alpha_k - \alpha_{k+1}. \end{aligned}$$

Hence,  $\beta_k = \frac{1}{2}(\pi - \angle A_k A_{k+1} A_{k+2}) = \frac{1}{2}(\alpha_k + \alpha_{k+1})$ .

Since the function  $\sin x$  is concave for  $x \in (0, \pi)$ , we have

$$\frac{1}{2}(\sin \alpha_k + \sin \alpha_{k+1}) \leq \sin \frac{1}{2}(\alpha_k + \alpha_{k+1}) = \sin \beta_k.$$

Then

$$\sum_{k=1}^n \sin \alpha_k = \sum_{k=1}^n \frac{1}{2}(\sin \alpha_k + \sin \alpha_{k+1}) \leq \sum_{k=1}^n \sin \beta_k.$$

Therefore, using (1) and (2), we see that

$$2s \frac{r}{R} = \frac{r}{R} 2R \sum_{k=1}^n \sin \alpha_k \leq 2r \sum_{k=1}^n \sin \beta_k = \sum_{k=1}^n B_k B_{k+1},$$

which completes the proof of the left inequality.

Since the function  $\sin x$  is concave for  $x \in (0, \pi/2)$ , we have

$$\sum_{k=1}^n \sin \beta_k \leq n \sin \frac{\beta_1 + \cdots + \beta_n}{n} = n \sin \frac{\pi}{n},$$

and since the function  $\tan x$  is convex for  $x \in (0, \pi/2)$ , we get

$$\sum_{k=1}^n \tan \beta_k \geq n \tan \frac{\beta_1 + \cdots + \beta_n}{n} = n \tan \frac{\pi}{n},$$

so that

$$\sum_{k=1}^n \sin \beta_k \leq n \sin \frac{\pi}{n} = n \tan \frac{\pi}{n} \cos \frac{\pi}{n} \leq \cos \frac{\pi}{n} \sum_{k=1}^n \tan \beta_k. \quad (4)$$

Applying (2), (3) and (4), we have

$$\sum_{k=1}^n B_k B_{k+1} = 2r \sum_{k=1}^n \sin \beta_k \leq 2r \cos \frac{\pi}{n} \sum_{k=1}^n \tan \beta_k = 2s \cos \frac{\pi}{n},$$

which completes the proof of the right inequality.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

**3092.** [2005 : 544, 546] *Proposed by Vedula N. Murty, Dover, PA, USA.*

(a) Let  $a$ ,  $b$ , and  $c$  be positive real numbers such that  $a + b + c = abc$ . Find the minimum value of  $\sqrt{1 + a^2} + \sqrt{1 + b^2} + \sqrt{1 + c^2}$ .

[Compare with **CRUX with MAYHEM** problem 2814 [2003 : 110; 2004 : 112].]

(b) Let  $a$ ,  $b$ , and  $c$  be positive real numbers such that  $a + b + c = 1$ . Find the minimum value of

$$\frac{1}{\sqrt{abc}} + \sum_{\text{cyclic}} \sqrt{\frac{bc}{a}}.$$

*Solution by Kee-Wai Lau, Hong Kong, China.*

(a) By the AM–GM Inequality, we have

$$\sqrt[3]{a + b + c} = \sqrt[3]{abc} \leq \frac{a + b + c}{3},$$

which implies that  $a + b + c \geq 3\sqrt{3}$ .

For  $x > 0$ , let  $f(x) = \sqrt{1 + x^2}$ . Then  $f'(x) = \frac{x}{\sqrt{1 + x^2}} > 0$  and  $f''(x) = \frac{1}{(1 + x^2)^{3/2}} > 0$ . Hence,

$$f(a) + f(b) + f(c) \geq 3f\left(\frac{a + b + c}{3}\right) \geq 3f(\sqrt{3}) = 6,$$

and equality holds when  $a = b = c = \sqrt{3}$ . This shows that the required minimum is 6.

(b) Since  $a + b + c = 1$ , we obtain

$$\begin{aligned} \frac{1}{\sqrt{abc}} + \sum_{\text{cyclic}} \sqrt{\frac{bc}{a}} &= \frac{1 + ab + bc + ca}{\sqrt{abc}} \\ &= \frac{1}{9} \left( \frac{3a + 1}{\sqrt{a}} \right) \left( \frac{3b + 1}{\sqrt{b}} \right) \left( \frac{3c + 1}{\sqrt{c}} \right) + \frac{5 - 27abc}{9\sqrt{abc}}. \end{aligned}$$

Since

$$\begin{aligned} abc &= \left( \sqrt[3]{abc} \right)^3 \leq \left( \frac{a + b + c}{3} \right)^3 = \frac{1}{27}, \\ \frac{3a + 1}{\sqrt{a}} &= \left( \sqrt{3} \sqrt[4]{a} - \frac{1}{\sqrt[4]{a}} \right)^2 + 2\sqrt{3} \geq 2\sqrt{3}, \end{aligned}$$

and similarly,  $\frac{3b + 1}{\sqrt{b}} \geq 2\sqrt{3}$  and  $\frac{3c + 1}{\sqrt{c}} \geq 2\sqrt{3}$ , we see that

$$\frac{1}{9} \left( \frac{3a + 1}{\sqrt{a}} \right) \left( \frac{3b + 1}{\sqrt{b}} \right) \left( \frac{3c + 1}{\sqrt{c}} \right) + \frac{5 - 27abc}{9\sqrt{abc}} \geq 4\sqrt{3},$$

and equality holds for  $a = b = c = \frac{1}{3}$ . This shows that the required minimum is  $4\sqrt{3}$ .

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; JOHN G. HEUVER, Grande Prairie, AB; JOEL SCHLOSBERG, Bayside, NY, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

There was quite a wide variety of methods used by our solvers for this problem. That is what made it a good problem.

**3093.** [2005 : 544, 547] Proposed by Mihály Bencze, Brasov, Romania.

Let  $p_k$  be the  $k^{\text{th}}$  prime. Show that the following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n(p_{n+1} - p_n)}.$$

I. Solution by Mohammed Aassila, Strasbourg, France, modified slightly by the editor.

A well-known consequence of the Prime Number Theorem states that  $p_n \sim n \ln n$ , which implies that  $p_{2N} < 3N \ln(2N)$  for sufficiently large  $N$ . Thus,  $p_{2N} < 3N(\ln 2 + \ln N) < 6N \ln N$ .

Using the AM–HM Inequality, we then have

$$\begin{aligned} \sum_{n=N}^{2N-1} \frac{1}{n(p_{n+1} - p_n)} &\geq \frac{1}{2N} \sum_{n=N}^{2N-1} \frac{1}{p_{n+1} - p_n} \\ &\geq \frac{1}{2N} \frac{N^2}{\sum_{n=N}^{2N-1} (p_{n+1} - p_n)} \\ &= \frac{N}{2(p_{2N} - p_N)} > \frac{N}{2p_{2N}} > \frac{1}{12 \ln N}. \end{aligned}$$

Summing over all  $N$  of the form  $2^k$  for sufficiently large  $k$ , we get

$$\sum_{n=1}^{\infty} \frac{1}{n(p_{n+1} - p_n)} > \sum_k \frac{1}{12 \ln(2^k)} = \frac{1}{12(\ln 2)} \sum_k \frac{1}{k}.$$

Since the harmonic series  $\sum \frac{1}{k}$  is divergent, the conclusion follows.

II. Essentially the same solution by Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

It is known (see, for example, Problem 3.2.71 on p. 84 of *Problems in Mathematical Analysis I*, AMS (1996) by W.J. Kaczor and M.T. Nowak) that if  $\{a_n\}$  is a monotonically increasing sequence of positive numbers such that

$\sum_{n=1}^{\infty} \frac{1}{a_n}$  is divergent, then  $\sum_{n=1}^{\infty} \frac{1}{(n+1)a_{n+1} - na_n}$  also diverges. Since

$$\frac{1}{n(p_{n+1} - p_n)} > \frac{1}{(n+1)p_{n+1} - np_n},$$

and since  $\sum_{n=1}^{\infty} \frac{1}{p_n}$  is well known to be divergent, the result follows immediately by letting  $a_n = p_n$ .

Also solved by OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

Janous pointed out that the result has been known for a long time and gave the following reference: L. Panaitopol, *Sur la suite des differences des nombres premiers consecutives*, (Romanian) Gaz. Mat., Bucur., Ser. A 79 (1974), 238–242.

Leong actually gave a proof for the result quoted in solution II above.

**3094.** [2005 : 544, 547] Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.

Let  $x_1, x_2, \dots, x_n$  be non-negative real numbers, where  $n \geq 3$ . Let  $S = \sum_{k=1}^n x_k$  and  $P = \prod_{k=1}^n (1 + x_k^2)$ . Prove that

$$(a) \quad P \leq \max_{1 \leq k \leq n} \left\{ \left( 1 + \frac{S^2}{k^2} \right)^k \right\};$$

$$(b) \quad P \leq \left( 1 + \frac{S^2}{n^2} \right)^n \text{ if } S > 2\sqrt{2}(n-1);$$

$$(c) \quad P \leq 1 + S^2 \text{ if } S \leq 2\sqrt{2}.$$

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

In fact, all three parts are valid for  $n \geq 1$ , and the bound on  $S$  in (b) can be improved. For  $n = 1$ , the inequalities are all trivial. We assume  $n \geq 2$  in our proofs.

(a) First consider the case  $n = 2$ . If  $P \leq 1 + S^2$ , then we are done. Otherwise, we have  $(1 + x_1^2)(1 + x_2^2) = P > 1 + S^2 = 1 + (x_1 + x_2)^2$ , which gives  $x_1 x_2 > 2$ ; then

$$\left( 1 + \frac{S^2}{2^2} \right)^2 - P = \frac{(x_1 - x_2)^2(x_1^2 + 6x_1x_2 + x_2^2 - 8)}{16} \geq 0.$$

Hence, the inequality is true for  $n = 2$ .

Assume, as an induction hypothesis, that the inequality is true for  $n - 1$  variables, for some  $n \geq 3$ . Consider the product

$$P(x_1, x_2, \dots, x_n) = \prod_{k=1}^n (1 + x_k^2).$$



For  $1 \leq i \leq n$ , let  $x_{i,0} = x_i$  and  $x_{i,m} = \frac{1}{2}(x_{i,m-1} + x_{i+1,m-1})$  for  $m \geq 1$ , where the first subscripts are taken modulo  $n$ . Note that  $\sum_{i=1}^n x_{i,m} = S$  for all  $m \geq 1$ . If, for some  $i$ , we have  $(1 + x_{i,0}^2)(1 + x_{i+1,0}^2) \leq 1 + (x_{i,0} + x_{i+1,0})^2$ , then the number of variables may be reduced to  $n - 1$ , and we are done. On the other hand, if  $(1 + x_{i,0}^2)(1 + x_{i+1,0}^2) > 1 + (x_{i,0} + x_{i+1,0})^2$  for all  $i$ , then, by the proven case  $n = 2$ , we have

$$(1 + x_{i,0}^2)(1 + x_{i+1,0}^2) \leq \left[ 1 + \left( \frac{x_{i,0} + x_{i+1,0}}{2} \right)^2 \right]^2$$

for all  $i$ , and thus,  $P(x_{1,0}, x_{2,0}, \dots, x_{n,0}) \leq P(x_{1,1}, x_{2,1}, \dots, x_{n,1})$ .

Now we iterate. If  $(1 + x_{i,m}^2)(1 + x_{i+1,m}^2) \leq 1 + (x_{i,m} + x_{i+1,m})^2$  for some  $i$  at some stage  $m$ , then we can reduce to the case of  $n - 1$  variables. Otherwise, we obtain an infinite sequence  $\{(x_{1,m}, x_{2,m}, \dots, x_{n,m})\}_{m=0}^{\infty}$  such that  $P(x_{1,m-1}, x_{2,m-1}, \dots, x_{n,m-1}) \leq P(x_{1,m}, x_{2,m}, \dots, x_{n,m})$  for all  $m \geq 1$ . This sequence converges to  $(S/n, S/n, \dots, S/n)$  (see [1]). The continuity of  $P$  implies that  $P(x_1, x_2, \dots, x_n) \leq P(S/n, S/n, \dots, S/n)$ , completing the proof.

(b) In the case  $n = 2$ , if  $S \geq 2\sqrt{2}$ , then

$$\left( 1 + \frac{S^2}{2^2} \right)^2 - P = \frac{(x_1 - x_2)^2(4x_1x_2 + S^2 - 8)}{16} \geq 0.$$

Now consider the case  $n = 3$ . By solving polynomial equations, we find that  $1 + S^2 \leq (1 + \frac{1}{9}S^2)^3$  for  $S \geq \frac{3}{2}\sqrt{2(\sqrt{33} - 3)} \approx 3.514$  and  $(1 + \frac{1}{4}S^2)^2 \leq (1 + \frac{1}{9}S^2)^3$  for  $S \geq a = \frac{3}{2}\sqrt{2(5\sqrt{105} + 33)} \approx 4.867$ . By part (a), we have  $P \leq (1 + \frac{1}{9}S^2)^3$  for  $S \geq a$ . This is an improvement of the given bound, since  $4\sqrt{2} > a$ .

Finally, suppose that  $n \geq 4$ . Let  $f(x) = \ln(1 + x^2)$  for  $x \geq 0$ . Then

$$f'(x) = \frac{2x}{1 + x^2} \quad \text{and} \quad f''(x) = \frac{2(1 - x^2)}{(1 + x^2)^2}.$$

Note that the equation  $f'(x) = f(x)/x$  has a unique positive root  $r \approx 1.98$ . The tangent line to the graph of  $f$  at the point where  $x = r$  passes through the origin. Let  $y = T(x)$  be the equation of this tangent line. Define

$$g(x) = \begin{cases} T(x) & \text{if } 0 \leq x < r, \\ f(x) & \text{if } r \leq x. \end{cases}$$

Then  $g$  is concave, and  $g(x) \geq f(x)$  for all  $x \geq 0$ . Using Jensen's Inequality, we get

$$\ln P = \sum_{k=1}^n f(x_k) \leq \sum_{k=1}^n g(x_k) \leq ng(S/n) = nf(S/n)$$

if  $S \geq rn$ . Thus,  $\ln P \leq n \ln(1 + (S/n)^2)$  for  $S \geq rn$ , and therefore the inequality in (b) holds for  $S \geq rn$ . The bound on  $S$  here is an improvement of the given bound, since  $2\sqrt{2}(n-1)/n \geq 3\sqrt{2}/2 > 2 > r$  for  $n \geq 4$ .

(c) By the AM-GM Inequality,  $x_1 x_2 \leq \left(\frac{x_1 + x_2}{2}\right)^2 \leq \left(\frac{S}{2}\right)^2 \leq 2$ . Hence,

$$(1 + x_1^2)(1 + x_2^2) - [1 + (x_1 + x_2)^2] = x_1 x_2 (x_1 x_2 - 2) \leq 0.$$

An easy induction completes the proof.

**Reference:**

- [1] G.-Z. Chang and T.W. Sederberg, "Over and Over Again", MAA, 1997, 29-31.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (c) only); and the proposer.

**3095.** [2005 : 544, 547] Proposed by Arkady Alt, San Jose, CA, USA.

Let  $a, b, c, p$ , and  $q$  be natural numbers. Using  $[x]$  to denote the integer part of  $x$ , prove that

$$\min \left\{ a, \left\lfloor \frac{c + pb}{q} \right\rfloor \right\} \leq \left\lfloor \frac{c + p(a + b)}{p + q} \right\rfloor.$$

*Solution by Joel Schlosberg, Bayside, NY, USA.*

We have

$$\begin{aligned} \frac{c + p(a + b)}{p + q} &= \frac{pa}{p + q} + \frac{c + pb}{p + q} = \frac{p}{p + q}a + \frac{q}{p + q} \frac{c + pb}{q} \\ &\geq \frac{p}{p + q}a + \frac{q}{p + q} \left\lfloor \frac{c + pb}{q} \right\rfloor \\ &\geq \frac{p}{p + q} \min \left\{ a, \left\lfloor \frac{c + pb}{q} \right\rfloor \right\} + \frac{q}{p + q} \min \left\{ a, \left\lfloor \frac{c + pb}{q} \right\rfloor \right\} \\ &= \min \left\{ a, \left\lfloor \frac{c + pb}{q} \right\rfloor \right\}. \end{aligned}$$

Since  $\min \left\{ a, \left\lfloor \frac{c + pb}{q} \right\rfloor \right\}$  is an integer,

$$\left\lfloor \frac{c + p(a + b)}{p + q} \right\rfloor \geq \min \left\{ a, \left\lfloor \frac{c + pb}{q} \right\rfloor \right\}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

**3096.** [2005 : 544, 547] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let  $ABC$  be a triangle with sides  $a, b, c$  opposite the angles  $A, B, C$ , respectively. Prove that

$$\sum_{\text{cyclic}} \frac{bc}{b+c} \sin^2 \left( \frac{A}{2} \right) \leq \frac{a+b+c}{8}.$$

*Similar solutions by Vedula N. Murty, Dover, PA, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Since  $\frac{bc}{b+c} \leq \frac{b+c}{4}$ ,  $\sin^2 \frac{A}{2} = \frac{1-\cos A}{2}$ , and  $a = b \cos C + c \cos B$ , we obtain

$$\begin{aligned} 8 \sum_{\text{cyclic}} \frac{bc}{b+c} \sin^2 \frac{A}{2} - \sum_{\text{cyclic}} a &\leq \sum_{\text{cyclic}} [(b+c)(1-\cos A)] - \sum_{\text{cyclic}} a \\ &= \sum_{\text{cyclic}} a - \sum_{\text{cyclic}} (b \cos C + c \cos B) = 0, \end{aligned}$$

which yields the desired inequality.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer.*

**3097.** [2005 : 545, 547] *Proposed by Mihály Bencze, Brasov, Romania.*

Let  $a$  and  $b$  be two positive real numbers such that  $a < b$ . Define  $A(a, b) = \frac{a+b}{2}$  and  $L(a, b) = \frac{b-a}{\ln b - \ln a}$ . Prove that

$$L(a, b) < L\left(\frac{a+b}{2}, \sqrt{ab}\right) < \left(A(\sqrt{a}, \sqrt{b})\right)^2 < A(a, b).$$

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

The inequality  $\left(A(\sqrt{a}, \sqrt{b})\right)^2 < A(a, b)$  is simply the Power-Mean Inequality. Applying the Hadamard's Inequality to the convex function  $f(x) = 1/x$ , we get

$$\begin{aligned} \frac{1}{L\left(\frac{1}{2}(a+b), \sqrt{ab}\right)} &= \frac{1}{\frac{1}{2}(a+b) - \sqrt{ab}} \int_{\sqrt{ab}}^{\frac{1}{2}(a+b)} f(x) dx \\ &> f\left(\frac{\frac{1}{2}(a+b) + \sqrt{ab}}{2}\right) = \frac{1}{\left(A(\sqrt{a}, \sqrt{b})\right)^2}. \end{aligned}$$

This gives the inequality  $L\left(\frac{1}{2}(a+b), \sqrt{ab}\right) < \left(A(\sqrt{a}, \sqrt{b})\right)^2$ .

The inequality  $L(a, b) < L\left(\frac{1}{2}(a+b), \sqrt{ab}\right)$  transforms successively into

$$\begin{aligned} (b-a) \ln\left(\frac{1}{2}(a+b)\right) &< (\sqrt{ab}-a) \ln a + (b-\sqrt{ab}) \ln b, \\ \left(\frac{1}{2}(a+b)\right)^{\sqrt{a}+\sqrt{b}} &< a^{\sqrt{a}} b^{\sqrt{b}}, \\ \left(\frac{1}{2}(1+b/a)\right) \left(\frac{1}{2}(1+a/b)\right) \sqrt{b/a} &< 1, \\ \left(\frac{1}{2}(1+x^2)\right) \left(\frac{x^2+1}{2x^2}\right)^x &< 1, \end{aligned}$$

with  $x = \sqrt{b/a} > 1$ . Now, for  $x > 1$ , let

$$f(x) = \ln\left(\frac{1}{2}(x^2+1)\right) + x \ln\left(\frac{x^2+1}{2x^2}\right).$$

Then

$$\begin{aligned} f'(x) &= \frac{2(x-1)}{x^2+1} + \ln\left(\frac{x^2+1}{2x^2}\right) \\ \text{and } f''(x) &= \frac{-2(x-1)^2(x+1)}{x(x^2+1)^2} < 0. \end{aligned}$$

Hence,  $f$  is strictly concave. Since  $f(1) = 0$  and  $f'(1) = 0$ , we conclude that  $f(x) < 0$ .

*Also solved by* ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

**3098★.** [2005 : 545, 548] Proposed by D.Z. Djokovic, University of Waterloo, Waterloo, ON; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON.

Let  $n$  and  $k$  be any positive integers such that  $k \leq n$ . Let  $S$  denote the sequence of length  $2n$  obtained by interlacing the two sequences  $n, n-1, \dots, 2, 1$  and  $-1, -2, \dots, -(n-1), -n$ , and let  $\mathcal{F}$  be the set of all  $\binom{2n-k+1}{k}$  subsequences  $K$  of  $S$  which have length  $k$  and do not contain any pair of consecutive terms of  $S$ . Prove that

$$\sum_{K \in \mathcal{F}} P(K) = 0,$$

where  $P(K)$  is the product of all  $k$  terms of the sequence  $K$ .

For example, if  $n = 3$  and  $k = 2$ , then  $S = 3, -1, 2, -2, 1, -3$ , and

$$\mathcal{F} = \{\{3, 2\}, \{3, -2\}, \{3, 1\}, \{3, -3\}, \{-1, -2\}, \{-1, 1\}, \\ \{-1, -3\}, \{2, 1\}, \{2, -3\}, \{-2, -3\}\};$$

hence,

$$\sum_{K \in \mathcal{F}} P(K) = 6 - 6 + 3 - 9 + 2 - 1 + 3 + 2 - 6 + 6 = 0.$$

[This result was obtained as a by-product of some research in Lie Algebra. The proposers have a proof for odd  $k$ . They hope that an elementary proof can be found.]

*Solution by Tom Leong, Brooklyn, NY, USA.*

Let  $S$  be the sequence  $a_1, b_1, a_2, b_2, \dots, a_n, b_n$ , where  $a_m = n - m + 1$  and  $b_m = -m$  for  $m = 1, 2, \dots, n$ . Let  $s_k(x) = \sum P(K)$ , where the sum is over all subsequences  $K \in \mathcal{F}$  whose first term is  $x$ . For instance, in the example given in the statement of the problem,

$$s_2(-1) = (-1)(-2) + (-1)(1) + (-1)(-3) = 4.$$

Note first that if  $K \in \mathcal{F}$  begins with  $a_m$  or  $b_m$ , then we must have  $k - 1 \leq n - m$  or  $m \leq n - k + 1$ . Thus, there are no subsequences in  $\mathcal{F}$  beginning with either  $a_m$  or  $b_m$  if  $m > n - k + 1$ .

We now prove by induction on  $k$  that

$$s_k(a_m) = (-1)^{k+1} k! \binom{n-m+1}{k} \binom{m+k-2}{k-1} \quad (1)$$

$$\text{and } s_k(b_m) = (-1)^k k! \binom{m+k-1}{k} \binom{n-m}{k-1} \quad (2)$$

for  $m = 1, 2, \dots, n - k + 1$ .

If  $k = 1$ , then equations (1) and (2) reduce to  $s_1(a_m) = n - m + 1$  and  $s_1(b_m) = -m$ , which are obviously true. Applying the inductive hypothesis, we have

$$\begin{aligned} & \sum_{i=m+1}^{n-k+1} (s_k(a_i) + s_k(b_i)) \\ &= (-1)^k k! \sum_{i=m+1}^{n-k+1} \left[ -\binom{n-i+1}{k} \binom{i+k-2}{k-1} + \binom{i+k-1}{k} \binom{n-i}{k-1} \right] \\ &= (-1)^k k! \binom{n-m}{k} \binom{m+k-1}{k} \end{aligned} \quad (3)$$

where (3) follows from (4) given below which we will prove later. Assuming

(3) holds for the moment, we have

$$\begin{aligned}
 s_{k+1}(a_m) &= a_m \sum_{i=m+1}^{n-k+1} (s_k(a_i) + s_k(b_i)) \\
 &= (n-m+1)(-1)^k k! \binom{n-m}{k} \binom{m+k-1}{k} \\
 &= (-1)^k (k+1)! \binom{n-m+1}{k+1} \binom{m+k-1}{k}.
 \end{aligned}$$

Similarly, using (3) and the inductive hypothesis again, we have

$$\begin{aligned}
 s_{k+1}(b_m) &= b_m \left( \sum_{i=m+2}^{n-k+1} s_k(a_i) + \sum_{i=m+1}^{n-k+1} s_k(b_i) \right) \\
 &= -m \left( \sum_{i=m+1}^{n-k+1} (s_k(a_i) + s_k(b_i)) - s_k(a_{m+1}) \right) \\
 &= -m(-1)^k k! \left[ \binom{n-m}{k} \binom{m+k-1}{k} \right. \\
 &\quad \left. + \binom{n-m}{k} \binom{m+k-1}{k-1} \right] \\
 &= -m(-1)^k k! \binom{n-m}{k} \left[ \binom{m+k-1}{k} + \binom{m+k-1}{k-1} \right] \\
 &= -m(-1)^k k! \binom{n-m}{k} \binom{m+k}{k} \\
 &= (-1)^{k+1} (k+1)! \binom{n-m}{k} \binom{m+k}{k+1}.
 \end{aligned}$$

This completes the induction.

To complete the solution, we prove the identity required for (3), namely

$$\begin{aligned}
 \sum_{i=m+1}^{n-k+1} \binom{i+k-1}{k} \binom{n-i}{k-1} - \sum_{i=m+1}^{n-k+1} \binom{n-i+1}{k} \binom{i+k-2}{k-1} \\
 = \binom{n-m}{k} \binom{m+k-1}{k}. \quad (4)
 \end{aligned}$$

We give a combinatorial argument. Let  $T = \{1, 2, \dots, n+k\}$ . The term  $\binom{i+k-1}{k} \binom{n-i}{k-1}$  counts the number of  $2k$ -element subsets of  $T$  whose  $(k+1)^{\text{st}}$  largest element is  $i+k$ . Thus, the first sum on the left side counts the number of  $2k$ -element subsets of  $T$  whose  $(k+1)^{\text{st}}$  largest element is greater than  $m+k$ . Similarly, the second sum on the left side counts the number of  $2k$ -element subsets of  $T$  whose  $k^{\text{th}}$  largest element is greater than  $m+k-1$ . Their difference counts the number of  $2k$ -element subsets of  $T$  whose  $k$  smallest elements are all less than  $m+k$  and whose  $k$  largest elements are greater than  $m+k$ . Clearly, the right side also counts this number.

There was also one incorrect submission.

For the truth of the result when  $k$  is odd, Leong gave the following simple argument: Let  $S$  be the sequence  $x_1, x_2, \dots, x_{2n}$ . For a subsequence  $K = x_{i_1}, x_{i_2}, \dots, x_{i_k}$ , let  $K' = x_{2n-i_k+1}, \dots, x_{2n-i_2+1}, x_{2n-i_1+1}$ . Note that  $K \in \mathcal{F}$  if and only if  $K' \in \mathcal{F}$ . Since  $x_i = -x_{2n-i+1}$ , we have  $P(K) = -P(K')$ , and consequently,  $\sum_{K \in \mathcal{F}} P(K) = 0$ .

**3099.** [2005 : 545, 548] Proposed by Mihály Bencze, Brasov, Romania.

Let  $a_1, a_2, \dots, a_n$  be positive real numbers. Prove that

$$\prod_{k=1}^n \ln(1 + a_k) \leq \left( \ln \left( 1 + \sqrt[n]{\prod_{k=1}^n a_k} \right) \right)^n.$$

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let  $f(x) = \ln(\ln(1 + e^x))$ . Then, by straightforward computations, we find that

$$\begin{aligned} f'(x) &= \frac{e^x}{(1 + e^x) \ln(1 + e^x)} \\ \text{and } f''(x) &= \frac{e^x(1 + e^x) \ln(1 + e^x) - e^x(e^x \ln(1 + e^x) + e^x)}{(1 + e^x) \ln(1 + e^x)^2} \\ &= \frac{e^x g(x)}{((1 + e^x) \ln(1 + e^x))^2}, \end{aligned}$$

where  $g(x) = \ln(1 + e^x) - e^x$ .

Since  $g'(x) = \frac{e^x}{1 + e^x} - e^x = \frac{-e^{2x}}{1 + e^x} < 0$  and  $\lim_{x \rightarrow -\infty} g(x) = 0$ , we deduce that  $g(x) < 0$  and  $f''(x) < 0$  for all real  $x$ .

Hence,  $f$  is strictly concave and, by Jensen's Inequality, we have

$$\exp \left( \frac{1}{n} \sum_{k=1}^n f(\ln a_k) \right) \leq \exp \left( f \left( \frac{1}{n} \sum_{k=1}^n \ln a_k \right) \right). \quad (1)$$

Note that

$$\frac{1}{n} \sum_{k=1}^n f(\ln a_k) = \frac{1}{n} \sum_{k=1}^n \ln(\ln(1 + a_k)) = \sum_{k=1}^n \ln \left( (\ln(1 + a_k))^{1/n} \right). \quad (2)$$

and

$$\begin{aligned} f \left( \frac{1}{n} \sum_{k=1}^n \ln a_k \right) &= \ln \left( \ln \left( 1 + \exp \left( \frac{1}{n} \sum_{k=1}^n \ln a_k \right) \right) \right) \\ &= \ln \left( \ln \left( 1 + \prod_{k=1}^n a_k^{1/n} \right) \right) \\ &= \ln \left( \ln \left( 1 + \sqrt[n]{\prod_{k=1}^n a_k} \right) \right). \end{aligned} \quad (3)$$

From (1), (2), and (3), we conclude that

$$\prod_{k=1}^n \ln(1 + a_k)^{1/n} \leq \ln \left( 1 + \sqrt[n]{\prod_{k=1}^n a_k} \right),$$

from which the result follows immediately.

Equality holds if and only if  $a_1 = a_2 = \cdots = a_n$ .

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer. All the solutions are essentially the same as the one featured above.

**3100.** [2005 : 545, 548] Proposed by Michel Bataille, Rouen, France.

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfy  $f(xf(y)) = yf(x)$  for all real numbers  $x$  and  $y$ .

- (a) Show that  $f$  is an odd function.  
 (b) Determine  $f$ , given that  $f$  has exactly one discontinuity.

(a) *Solution by Joel Schlosberg, Bayside, NY, USA.*

Setting  $x = y = 0$  in the given condition shows that  $f(0) = 0$ . Note that the zero function, namely  $f(x) = 0$  for all  $x$ , satisfies the functional equation; it is odd, as desired (and continuous on  $\mathbb{R}$ ). We now suppose that  $f$  is a non-zero function; therefore, we assume that there is some  $a \in \mathbb{R}$  with  $f(a) \neq 0$ . If  $f(y) = f(z)$ , then

$$yf(a) = f(af(y)) = f(af(z)) = z(f(a)).$$

Cancelling  $f(a)$  from both sides shows that  $y = z$ . Thus,  $f$  is one-to-one. Choosing any  $z \neq 0$ , we have

$$f(zf(xy)) = xyf(z) = xf(zf(y)) = f(zf(y)f(x)).$$

Since  $f$  is one-to-one, we see that  $zf(xy) = zf(x)f(y)$ , and

$$f(xy) = f(x)f(y). \quad (1)$$

Substituting  $x = y = 1$  in (1), we get  $f(1) = (f(1))^2$ . Since  $f$  is one-to-one, we must have  $f(1) \neq f(0) = 0$ ; thus, by cancellation,

$$f(1) = 1.$$

Substituting  $x = y = -1$  in (1), we find  $1 = f(1) = (f(-1))^2$ ; hence,  $f(-1) = \pm 1$ . Since  $f$  is one-to-one, we must have  $f(-1) \neq f(1) = 1$ . Therefore,

$$f(-1) = -1.$$



Finally, we substitute  $y = -1$  in (1) to get

$$f(-x) = -f(x);$$

thus,  $f$  is an odd function, proving (a).

(b) *Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

Using  $f(1) = 1$  from part (a) and setting  $x = 1$  in the given condition, we deduce that  $f(f(y)) = yf(1) = y$  for all  $y$ ; that is,  $f$  is its own inverse. We now assume that  $f$  has exactly one discontinuity. Since  $f$  is odd, this discontinuity must be at 0. Since  $f$  is one-to-one and continuous on  $(0, \infty)$ , it is either strictly increasing or strictly decreasing on that half line.

Suppose first that  $f$  is strictly increasing on  $(0, \infty)$ . Let  $x \in (0, \infty)$ . If  $f(x) > x$ , then  $f(f(x)) > f(x)$ ; since  $f$  is its own inverse, this implies that  $x > f(x)$ , a contradiction. On the other hand, if  $f(x) < x$ , then  $f(f(x)) < f(x)$ , implying that  $x < f(x)$ , again a contradiction. Thus, for all  $x \in (0, \infty)$ , we must have  $f(x) = x$ . But since  $f$  is odd, this implies that  $f(x) = x$  for all  $x \in \mathbb{R}$ , a contradiction since  $f$  has a discontinuity at  $x = 0$ . Hence,  $f$  is not strictly increasing on  $(0, \infty)$ .

Finally, suppose  $f$  is strictly decreasing on  $(0, \infty)$ . We will show that  $f(x) = 1/x$  for all  $x \in (0, \infty)$ . Let  $x \in (0, \infty)$  and set  $y = f(x)$ . Note that  $f(xy) = f(xf(x)) = xf(x) = xy$ . If  $y < 1/x$ , then  $xy < 1$ , which implies that  $xy = f(xy) > f(1) = 1$ , a contradiction. If  $y > 1/x$ , then  $xy > 1$ , which implies that  $xy = f(xy) < f(1) = 1$ , again a contradiction. Thus,  $y = 1/x$ ; whence,  $f(x) = 1/x$  for all  $x \in (0, \infty)$ . Since  $f$  is odd and  $f(0) = 0$ , we have

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHARLES R. DIMINNIE and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Curtis also solved part (a) and Schlosberg also solved part (b).*

*Part (b) follows quickly from known results. Diminnie and Zarnowski referred to  $f(xy) = f(x)f(y)$  (which appears in our solution to part (a)) as Cauchy's Power Equation; they referenced [1] for the fact that, when  $f(x)$  is continuous for  $x > 0$ , the only solutions are  $f(x) = 0$  and  $f(x) = x^c$ . Janous stated that it is well known that if  $f$  is any odd involution which is continuous on the positive reals, then necessarily  $f(x) = 1/x$  or  $f(x) = -(1/x)$ ; he provided reference [2].*

#### References

- [1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989, pp. 29–30.
- [2] J. Dhombres, *Some Aspects of Functional Equations*, Chulalongkorn University Press, Bangkok, 1979.

## YEAR END FINALE

This issue brings to a close my fourth year as Editor-in-Chief of *CRUX with MAYHEM*. How time flies!

The Canadian Mathematical Society (CMS) is planning to identify a set of “favourite” CRUX problems from over the years. The Editorial Board of *CRUX with MAYHEM* has suggested that the readers be consulted. Consequently, I am appealing to you, our readers, to identify some of your favourite problems that have graced our pages over all the years since the inception of *Crux Mathematicorum* (or even its precursor, *Eureka*). I would appreciate receiving lists of your favourites before **February 28, 2007**.

This year we introduced a new feature, Contributor Profiles, which seems to be popular with our readers, if I am to believe the feedback I have received. We will print more profiles as time and space permit.

Some of you may have noticed a glitch in the Mayhem section of the October issue. The diagrams in the solution of M210, while understandable, did not appear as they should. This was discovered only after printing and did not appear in the proofs. We still do not know what caused the problem, but we have decided to be more diligent in future before the actual printing occurs to see that it does not happen again.

Now I want to thank the many individuals who contribute so much to *CRUX with MAYHEM*, without whose contributions the journal would certainly suffer. The first person to thank, as always, is BRUCE CROFOOT, my Associate Editor. Despite the fact that Bruce has recently married and started a family, he continues to devote much valuable time scrutinizing several drafts of each issue, section by section.

Secondly, I wish to thank SHAWN GODIN, who after six years as Mayhem Editor, is stepping down. I have enjoyed working with Shawn since I became Editor-in-Chief, and will miss his regular emails. I wish Shawn all the best in his future endeavours and want him to know that his return to the CRUX family in the future would be welcomed with open arms! JEFF HOOPER, who has been working with Shawn over the past year as Assistant Mayhem Editor, will be the new Mayhem Editor effective in the new year. Part way through this past year, we also welcomed three new people to the Mayhem family, namely MARK BREDIN, MONIKA KHBEIS, and ERIC ROBERT; their role is to determine correctness of the solutions submitted to the Mayhem problems, and to select which solution(s) is (are) to be featured for each problem.

There are many others whom I wish to thank for their particular contributions. These include ILIYA BLUSKOV, CHRIS FISHER, MARIA TORRES, EDWARD WANG, and BRUCE SHAWYER for their regular and timely service in assessing the solutions; BRUCE GILLIGAN, for ensuring that *CRUX with MAYHEM* has quality articles; JOHN GRANT McLOUGHLIN, for ensuring that we have book reviews that are appropriate to our readership; ROBERT WOODROW for overseeing the *Olympiad Corner*; and ROBERT BILINSKI, who does likewise with the *Skoliad*.

The task of providing us with timely translations of our Problems still rests on the shoulders of JEAN-MARC TERRIER and MARTIN GOLDSTEIN. I want to thank them for their efforts, and for always coming through even when I have given them very little time for turn-around. We often reword the English problem after they have translated it into French, as their translation often improves the wording of the original problem. That kind of attention to detail is more than I have a right to ask for, and I am most grateful.

I want to thank all the proofreaders. MOHAMMED AASSILA, BILL SANDS, and STAN WAGON (whom I inadvertently forgot to acknowledge last year) assist the editors with this task. The quality of the work of all these people is a vital part of what makes *CRUX with MAYHEM* what it is. Thank you, one and all.

Thanks also go to Thompson Rivers University and my colleagues in the Department of Mathematics and Statistics for their continued understanding and support. Special thanks go to SUSAN HOWIE, secretary to our department, for all that she does to give me more time to edit.

Also, the  $\text{\LaTeX}$  expertise of JOANNE CANAPE at the University of Calgary and TAO GONG at Wilfrid Laurier University is much appreciated.

Thanks to GRAHAM WRIGHT, the Managing Editor, who keeps me on the right track (and adhering to deadlines!), and to the University of Toronto Press, and TAMI EHRLICH in particular, who continue to print a high-quality product.

The online version of *CRUX with MAYHEM* continues to grow, thanks in large part to JUDI BORWEIN at Dalhousie University.

Last but not least, I send my thanks to you, the readers. Without you, *CRUX with MAYHEM* would not be possible. I would just like to point out something to both new solvers and old: Please ensure that your name and address is on EVERY problem or proposal, and that each starts on a fresh sheet of paper. Otherwise, there may be filing errors, resulting in a submitted solution or proposal being lost.—

I wish everyone the compliments of the season, and a very happy, peaceful, and prosperous year 2007.

Jim Totten

## Crux Mathematicorum with Mathematical Mayhem

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Marian Dinca 3008, 3010, 3091	K.R.S. Sastry 3024
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Scott Brown K11	María Mercedes Sánchez Benito 3031
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Mathieu Guay-Paquet 3011	
Richard I. Hess 3008	
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**Arkady Ah** 3004, 3028, 3029, 3034, 3039, 3040, 3041, 3042, 3044, 3047, K01, K02, K03, K10, 3062, 3065, 3070, 3071, 3072, 3073, 3076, 3077, 3086, 3087, 3090, 3092, 3095, 3096, 3097, 3099  
**Miguel Amengual Covas** 3011, 3014, 3087  
**Houda Anoun** 3022  
**Šefket Arslanagić** 3003, 3004, 3009, 3011, 3027, 3028, 3029, 3033, 3072, 3073, 3075, 3076, 3077, 3078, 3086, 3087, 3090, 3091, 3092, 3095, 3096, 3097, 3099, 3100  
**Dionne T. Bailey** 3026, 3039, 3040, 3062  
**Michel Bataille** 3002, 3003, 3004, 3007, 3009, 3011, 3012, 3013, 3014, 3015, 3017, 3022, 3023, 3026, 3027, 3031, 3032, 3033, 3034, 3036, 3038, 3039, 3044, 3046, 3047, 3050, 3051, 3052, 3054, 3056, K01, K03, K10, K12, K14, 3062, 3070, 3072, 3073, 3074, 3076, 3077, 3078, 3080, 3082, 3086, 3087, 3088, 3089, 3090, 3091, 3092, 3095, 3096, 3097, 3099, 3100  
**Brian D. Beasley** 3044  
**Francisco Bellot Rosado** 3003, 3007, 3011, 3015  
**Mihály Bencze** 3004, 3007, 3008, 3010, 3028, 3039, 3040, 3051, 3052, 3058, K04, K10, K11, 3062, 3075, 3079, 3084, 3086, 3087, 3091, 3093, 3097, 3099  
**Manuel Benito** K02, K03, K14, K15  
**Robert Bilinski** 3003, 3011  
**Pierre Bornstein** 3021  
**Paul Bracken** 3056  
**Christopher J. Bradley** 3018, 3050, 3080, 3081, 3088, 3089  
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**James T. Bruening** 3046  
**Elsie M. Campbell** 3026, 3039, 3040, 3062  
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**Kin Fung Chung** 3013, 3014, 3015, 3027, 3028, 3034, 3036  
**Oscar Ciarri** 3049, K02, K03, K14, K15  
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**Paul Deiermann** 3011, 3022  
**José Luis Díaz-Barrero** 3044, 3049, K10  
**Charles R. Diminnie** 3026, 3028, 3030, 3039, 3040, 3047, 3062, 3086, 3092, 3099, 3100  
**Marian Dinca** 3008, 3010, 3091  
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**Joe Howard** 3028, 3033, 3038, 3047, 3051, K03, K09, 3071  
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**Robert B. Israel** K02, 3063  
**Walther Janous** 2972, 2984, 3001, 3002, 3003, 3004, 3005, 3006, 3007, 3009, 3010, 3011, 3013, 3014, 3015, 3016, 3017, 3018, 3020, 3022, 3025, 3026, 3027, 3028, 3031, 3032, 3034, 3036, 3038, 3040, 3041, 3045, 3046, 3047, 3050, 3051, 3053, 3055, 3056, 3057, K01, K02, K03, K06, K09, 3059, 3062, 3064, 3065, 3068, 3069, 3070, 3072, 3073, 3075, 3076, 3077, 3078, 3082, 3083, 3084, 3086, 3090, 3091, 3093, 3094, 3095, 3096, 3097, 3099, 3100  
**Rongzheng Jiao** 3051, 3062, 3069, 3071, 3076, 3077, 3087  
**D. Kipp Johnson** 3082  
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**Bob Serkey** 3011, 3022, 3040  
**D.J. Smeenk** 3003, 3011, 3013, 3015, 3022, 3040, 3078  
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**G. Tsintsifas** 3077, 3086, 3087  
**Pham Van Thuan** 3001, 3002, 3005, 3051, 3057, K03  
**M<sup>a</sup> Jesús Villar Rubio** 3013, 3015, 3022, 3040, 3046  
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**Stan Wagon** 3040, 3073, 3076  
**Peter Y. Woo** 3002, 3003, 3005, 3007, 3009, 3010, 3011, 3012, 3013, 3015, 3016, 3017, 3018, 3020, 3021, 3022, 3034, 3040, 3043, 3044, 3045, 3050, 3053, 3055, 3057, 3058, 3062, 3063, 3077, 3080, 3081, 3083, 3086, 3087, 3088, 3089, 3096, 3097, 3099, 3100  
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