SKOLIAD No. 97

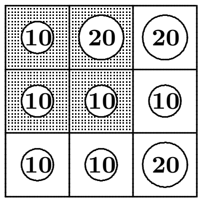
Robert Bilinski

Please send your solutions to the problems in this edition by ***May 1, 2007***. A copy of **MATHEMATICAL MAYHEM Vol. 7** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our contest this month comes from New Zealand. Thanks go to Warren Palmer, Otago University, Otago, New Zealand, for providing us with this contest and permitting us to publish it in Skoliad.

National Bank of New Zealand Competition 2000
(Grades 9 and above) 1 Hour allowed

1. **Grade 9 only**. In this problem we'll be placing various arrangements of 10¢ and 20¢ coins on the nine squares of a 3×3 grid. Exactly one coin will be placed in each of the nine squares. The grid has four 2×2 subsquares each containing a corner, the centre, and the two squares adjacent to these. One example is shown in the diagram.



(a) Find an arrangement where the totals of the four 2×2 subsquares are 40¢, 60¢, 60¢, and 70¢ in any order. (Draw a diagram showing your arrangement.)

(b) Find an arrangement where the totals of the four 2×2 subsquares are 50¢, 60¢, 70¢, and 80¢ in any order. (Draw a diagram showing your arrangement.)

For each part of the problem below, illustrate your answer with a suitable arrangement and an explanation of why no other suitable arrangement contains a larger (part (c)) or a smaller (part (d)) amount of money.

(c) What is the maximum amount of money which can be placed on the grid so that each of the 2×2 subsquares contains exactly 50¢?

(d) What is the minimum amount of money which can be placed on the grid so that the average of the amount of money in each of the 2×2 subsquares is exactly 60¢?

2. Humankind was recently contacted by three alien races: the Kweens, the Ozdaks, and the Merkuns. Little is known about these races except that Kweens always speak the truth while Ozdaks always lie. In any group of aliens a Merkun will never speak first. When it does speak, it tells the truth if the previous statement was a lie, and lies if the previous statement was truthful. Although the aliens can readily tell one another apart, of course to humans all aliens look the same.

A high-level delegation of three aliens has been sent to Earth to negotiate our fate. Among them is at least one Kween. On arrival they make the following statements (in order):

First Alien: The second alien is a Merkun.

Second Alien: The third alien is not a Merkun.

Third Alien: The first alien is a Merkun.

Which alien or aliens can you be certain are Kween?

3. (Note: In this question an "equal division" is one where the total weight of the two parts is the same.)

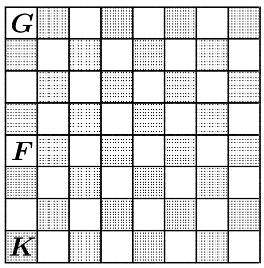
(a) Belinda and Charles are burglars. Among the loot from their latest caper is a set of 12 gold weights of 1g, 2g, 3g, and so on, through to 12g. Can they divide the weights equally between them? If so, explain how they can do it, and if not, why not?

(b) When Belinda and Charles take the remainder of the loot to Freddy the fence, he demands the 12g weight as his payment. Can Belinda and Charles divide the remaining 11 weights equally between them? If so, explain how they can do it, and if not, why not?

(c) Belinda and Charles also have a set of 150 silver weights of 1g, 2g, 3g, and so on, through to 150g. Can they divide these weights equally between them? If so, explain how they can do it, and if not, why not?

4. A chessboard is an 8×8 grid of squares. One of the chess pieces, the king, moves one square at a time in any direction, including diagonally.

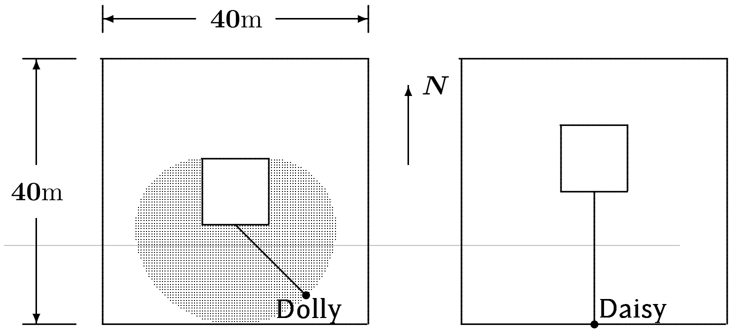
(a) A king (denoted by K in the diagram) stands on the lower left corner of a chessboard. It has to reach the square marked F in exactly three moves. Show that the king can do this in exactly four different ways.



(b) Assume that the king is placed back on the bottom left corner. In how many ways can it reach the upper left corner (marked G) in exactly seven moves?

5. (Note: For this question answers containing expressions such as $\frac{4\pi}{13}$ are acceptable. If you have a calculator you may use the button for π if you like.)

(a) The Jones family lives in a perfectly square house, 10m by 10m, which is placed exactly in the middle of a 40m by 40m lot, entirely covered (except for the house) in grass. They keep the family pet, Dolly the sheep, tethered to the middle of one side of the house on a 15m rope. What is the area of the part of the lawn (in m^2) in which Dolly is able to graze? (See shaded area.)

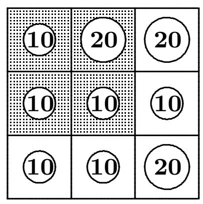


The Jones The Smiths

(b) The Jones' neighbours, the Smiths, have an identical lot to the Jones but their house is located five metres to the North of the centre. Their pet sheep, Daisy, is tethered to the middle of the southern side of the house on a 20m rope. What is the area of the part of the lawn (in m^2) in which Daisy is able to graze?

Compétition 2000 de la Banque Nationale de Nouvelle-Zélande

($9^{\text{ième}}$ année et plus) 1 Heure allouée



1. **$9^{\text{ième}}$ année seulement**. Dans ce problème nous allons placer dans une grille 3×3 divers arrangements de 10 sous et de 20 sous. Seulement une pièce peut-être placée dans chacune des 9 cases. La grille a 4 sous-grilles 2×2 chacune contenant un coin, le centre et deux carrés adjacents à ces derniers. Un exemple est montré dans le diagramme.

(a) Trouver un arrangement où les totaux des quatre sous-grilles 2×2 sont 40 sous, 60 sous, 60 sous et 70 sous dans n'importe quel ordre. (Dessiner votre arrangement).

(b) Trouver un arrangement où les totaux des quatre sous-grilles 2×2 sont 50 sous, 60 sous, 70 sous et 80 sous dans n'importe quel ordre. (Dessiner votre arrangement).

Pour chaque partie qui suit, illustrer votre réponse avec un arrangement et une explication de pourquoi aucun autre arrangement ne contient plus (sous-question (c)) ou moins (sous-question (d)) d'argent.

(c) Quel est le montant maximal d'argent qui peut être placé dans une grille pour que chacune des sous-grilles contienne exactement 50 sous ?

(d) Quel est le montant minimal d'argent qui peut être placé dans une grille pour que la moyenne des montants d'argent de chaque sous-grille soit exactement 60 sous ?

2. L'Humanité a récemment été contactée par trois races d'extra-terrestres : Les Kweens, les Ozdaks et les Merkuns. Peu est connu de ces races à part que les Kweens disent toujours la vérité alors que les Ozdaks mentent toujours. Dans n'importe quel groupement, un Merkun ne parlera jamais premier. Quand il prendra la parole, il dira la vérité si le précédant commentaire était un mensonge, et vice versa. Bien que les extra-terrestres puissent être capables de se distinguer l'un l'autre, ils se ressemblent tous aux yeux d'un humain.

Une délégation de haut niveau des 3 races a été envoyée sur Terre pour négocier notre sort. Parmi eux, il y a au moins un Kween. En arrivant, ils font les commentaires suivants (dans l'ordre) :

Le premier : Le second extra-terrestre est un Merkun.
Le second : Le troisième extra-terrestre n'est pas un Merkun.
Le troisième : Le premier extra-terrestre est un Merkun.

Lequel ou lesquels d'entre eux êtes-vous sûr sont des Kween ?

3. (Note : Dans cette question une "division égale" en est une où le poids total des deux parties est le même.)

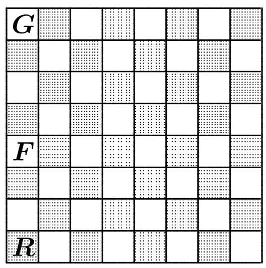
(a) Belinda et Charles sont des voleurs. Dans leur dernier butin, on retrouve un ensemble de 12 poids en or de 1g, 2g, 3g, etc jusqu'à 12g. Peuvent-ils diviser les poids également entre eux ? Si oui, expliquez comment, sinon, expliquez pourquoi pas.

(b) Quand Belinda et Charles prennent le restant de leur magot voir Freddy le receleur, celui-ci exige le poids de 12g en paiement. Belinda et Charles peuvent-ils alors séparer les 11 poids restants également entre eux ? Si oui, expliquez comment, sinon, expliquez pourquoi pas.

(c) Belinda et Charles ont aussi un ensemble de 150 poids en argent de 1g, 2g, 3g, etc jusqu'à 150g. Peuvent-ils diviser les poids également entre eux ? Si oui, expliquez comment, sinon, expliquez pourquoi pas.

4. Un échiquier est une grille de carrés 8×8. Une pièce d'échecs nommée le roi se déplace d'une case dans n'importe quelle direction, incluant les diagonales.

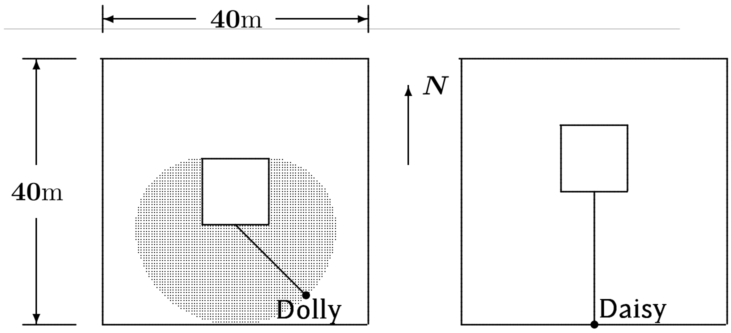
(a) Un roi (marqué par un R) se trouve dans le coin inférieur gauche de l'échiquier. Il doit rejoindre la case marquée par un F en trois coups exactement. Montrer que le roi peut faire cela de quatre manières différentes.



(b) Supposons que le roi revienne à la case en bas à gauche. De combien de manières différentes peut-il atteindre le coin en haut à gauche (marqué par un G) en sept coups exactement ?

5. (Note : Pour cette question, les réponses contenant des expressions de la forme $\frac{4\pi}{13}$ sont acceptées. Si vous avez une calculatrice, vous pouvez utiliser le bouton pour π si vous le voulez.)

(a) La famille des Jones vie dans une maison parfaitement carrée, de 10m par 10m, qui est placée exactement dans le centre d'un lot de 40m par 40m, couverte entièrement (à part pour la maison) par du gazon. Ils gardent un animal domestique, Dolly le mouton, attaché au milieu d'un côté de la maison avec une corde de 15m. Quelle est l'aire de la pelouse (en m^2) dans laquelle Dolly peut brouter ? (Voir la zone hachurée).



Les Jones   Les Smiths

(b) Les voisins des Jones, les Smiths, ont un lot identique aux Jones mais leur maison est située cinq mètres au Nord du centre. Leur mouton domestiqué, Daisy, est attachée au milieu du côté Sud de la maison avec une corde de 20m. Quelle est l'aire de la pelouse (en m^2) dans laquelle Daisy peut brouter ?

Next we give readers' solutions to the 5th annual CNU Regional Mathematics Contest [2006 : 66–68].

8. Combien de lait à 4% de matières grasses doit-on ajouter à du lait à 1% pour obtenir 12 gallons de lait à 2% ?

(A) 3 gallons (B) 4 gallons (C) 8 gallons (D) 9 gallons

Solution identique par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC; et Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC.

Soit x le nombre de gallons de lait 4% et y le nombre de gallons de lait 1%. Schématisons la situation par deux équations :

$$\begin{aligned} 4x + y &= 2 \cdot 12\,, \\ x + y &= 12\,. \end{aligned}$$

En résolvant ces équations, nous obtenons $x = 4$ et $y = 8$. Il faut donc ajouter 4 gallons de lait 4%, ainsi, la réponse est B.

12. Sachant que $f(x) = x + 2$ et $g(x) = \sqrt[3]{x}$, trouvez $f^{-1} \circ g^{-1}(2)$.

(A) 8 (B) -6 (C) 2 (D) 6

Solution par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC.

Si $g(x) = \sqrt[3]{x}$, alors la fonction réciproque correspondante est $g^{-1}(x) = x^3$. Similairement, si $f(x) = x + 2$, alors $f^{-1}(x) = x - 2$. Nous avons donc :

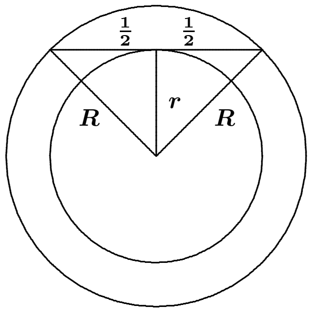
$$f^{-1} \circ g^{-1}(2) \;=\; f^{-1}(2^3) \;=\; 8 - 2 \;=\; 6\,.$$

La réponse est D.

15. Un segment de 1 cm est une corde d'un de deux cercles concentriques et est tangent au plus petit des deux. Quelle est l'aire de la région annulaire entre les deux cercles ?

(A) $\frac{\pi}{6}$ cm^2 (B) $\frac{\pi}{4}$ cm^2 (C) $\frac{\pi}{3}$ cm^2 (D) $\frac{\pi}{2}$ cm^2

Solution par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC.



D'après le diagramme et le Théorème de Pythagore, on a $R^2 - r^2 = \frac{1}{4}$. L'aire de la région annulaire entre les deux cercles est égale à la différence d'aire entre les deux cercles. En notant A cette aire, nous pouvons écrire :

$$A \;=\; \pi R^2 - \pi r^2 \;=\; \pi(R^2 - r^2) \;=\; \frac{\pi}{4}\,.$$

La réponse est B.

16. Résoudre l'équation $8^{\frac{1}{6}} + x^{\frac{1}{3}} = \dfrac{7}{3 - \sqrt{2}}$.

(A) 24 (B) 27 (C) 32 (D) 64

Solution par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC.

On a $8^{\frac{1}{6}} = \sqrt{2}$ et donc

$$\begin{aligned}
\left(\sqrt{2} + x^{\frac{1}{3}}\right)\left(3 - \sqrt{2}\right) &= 7\,,\\
3\sqrt{2} - 2 + x^{\frac{1}{3}}\left(3 - \sqrt{2}\right) &= 7\,,\\
x^{\frac{1}{3}} = \frac{9 - 3\sqrt{2}}{3 - \sqrt{2}} &= 3\,.
\end{aligned}$$

Ainsi, on a $x = 3^3 = 27$ et la réponse est B.

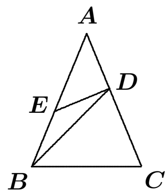
18. Soit $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ une fonction qui vérifie $f(x)+f\left(\frac{1}{x}\right) = 3x$. Quelle est la somme des valeurs de x pour lesquelles on a $f(x) = 1$?

(A) 1 (B) 2 (C) -1 (D) -2

Explication par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC, modifiée par le rédacteur.

Si on remplace x par $1/x$, on obtient $f(x) + f(1/x) = 3/x$. Il faut que $3/x = 3x$ pour tous les x dans le domaine $\mathbb{R} \setminus \{0\}$. Mais $3/x = 3x$ se résoud et donne $x = \pm 1$. Ceci signifie qu'il n'y a pas une fonction f définie sur $\mathbb{R} \setminus \{0\}$ qui vérifie $f(x) + f(1/x) = 3x$.

21. Dans $\triangle ABC$ dans la figure à droite, $AB = AC$, $BC = BD$, $AD = DE = EB$. Quelle est $\angle$A ? (Note : La figure n'est pas à l'échelle.)



(A) $30°$ (B) $36°$ (C) $45°$ (D) $54°$

Solution par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC, modifiée par le rédacteur.

Pour simplifier, on note $\angle A = \alpha$. Puisque $AD = DE$, alors on a $\angle AED = \angle A = \alpha$, et donc $\angle BED = 180° - \alpha$. Puisque $DE = EB$, on a $\angle EBD = \angle EDB$. Aussi, $\angle EBD + \angle EDB = 180° - \angle BED = \alpha$. Donc $\angle EBD = \angle EDB = \alpha/2$.

De plus, on a

$$\angle BDC \;=\; 180° - \angle ADB \;=\; \angle A + \angle ABD \;=\; \tfrac{3\alpha}{2}\,.$$

Mais $BD = BC$, alors $\angle C = \angle BDC = \frac{3}{2}\alpha$. En notant que $AB = AC$, on a $\angle B = \angle C = \frac{3}{2}\alpha$. Puisque $\angle A + \angle B + \angle C = 180°$, on obtient

$$\alpha + \tfrac{3}{2}\alpha + \tfrac{3}{2}\alpha \;=\; 180°\,,$$

d'où $\alpha = 45°$. La réponse est C.

32. Soit n le nombre de manières que 10 dollars peuvent être changés en 10 sous et 25 sous avec chacun d'eux utilisé au moins une fois. Alors n vaut :

(A) 18 (B) 38 (C) 21 (D) 19

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Par hypothèse, on a $1000 = 10x + 25y$.

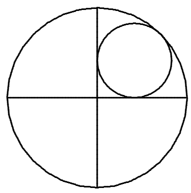
Puisque chaque pièce doit être utilisée une fois par hypothèse, la valeur minimale de x doit donc être supérieure ou égale à 1. Pour trouver la borne minimale, il suffit de trouver la plus petite valeur de x pour laquelle le rapport du coefficient de x et du coefficient de y $(10/25)$ multiplié par la valeur de x donne un nombre naturel. La première valeur de x pour laquelle $0.4x$ est un nombre naturel est 5. La borne inférieure est donc 5, et l'écart minimal entre les valeurs de x doit être de 5. Ainsi $x_{\text{min}} = 5$.

Puisque y doit être utilisé au moins une fois par hypothèse, nous trouvons la valeur maximale de x en soustrayant à la valeur de $x = 100$, qui satisfait l'équation si $y = 0$, la valeur de la plus petite division de x possible pour que l'équation soit résolvable avec un nombre naturel comme y (5, tel que trouvé précédemment). Ainsi, $x_{\text{max}} = 95$.

Le nombre de valeurs possibles de x naturel pour qu'une valeur de y naturel puisse satisfaire l'équation nous est donné en divisant la valeur maximale de x (95) par la plus petite division possible (5) pour obtenir $\frac{95}{5} = 19$. La réponse est D.

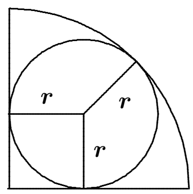
Aussi solutionné par JEAN-DAVID HOULE, étudiant, Cégep de Drummondville, Drummondville, QC.

34. Deux perpendiculaires qui se coupent au centre d'un cercle de rayon 1, séparent le cercle en quatre parties. Un plus petit cercle est inscrit dans une de ces parties comme dans la figure. Quel est le rayon du plus petit cercle ?



(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\sqrt{2} - 1$

Solution par Jean-David Houle, étudiant, Cégep de Drummondville, Drummondville, QC.



Notons r le rayon du plus petit cercle. On trace le rayon du plus petit cercle jusqu'au point de tangence avec la perpendiculaire horizontale. On fait de même avec la perpendiculaire verticale. Finalement, on trace une fois encore le rayon du plus petit cercle jusqu'au point de contact entre les deux cercles.

On voit donc que la distance du centre du cercle jusqu'à celui du petit cercle est de $\sqrt{r^2 + r^2} = \sqrt{2}r$, et celle du centre du petit cercle jusqu'au point de contact avec le grand cercle est de r. En sachant que le rayon du grand cercle est de 1, on a $\sqrt{2}r + r = 1$ ou bien $r = 1/(\sqrt{2} + 1) = \sqrt{2} - 1$. La réponse est D.

That brings us to the end of another issue. This month's winner of a past volume of Mayhem is Jean-David Houle, student, Cégep de Drummondville, Drummondville, QC. Congratulations Jean-David !

Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Please send your solutions to the problems in this edition by **1 March 2007**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M263. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let a, b, and n be integers such that $(a^2+b^2)/5 = n$. Prove that $n = c^2 + d^2$ for some integers c and d.

M264. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

Given 1001 real numbers placed around a circle such that each number is the arithmetic mean of its neighbours or else its two neighbours are equal, prove that all the numbers are equal.

M265. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given triangle ABC and $DE \parallel BC$, with $D \in AB$ and $E \in AC$. Drop perpendiculars from D and E to BC, meeting BC at F and K, respectively. If $\dfrac{[ABC]}{[DEKF]} = \dfrac{32}{7}$, determine the ratio $\dfrac{|AD|}{|DB|}$.

M266. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A pair of two-digit numbers has the following properties :

1. The sum of the four digits is 17.
2. The sum of the two numbers is 89.
3. The product of the four digits is 210.
4. The product of the two numbers is 1924.

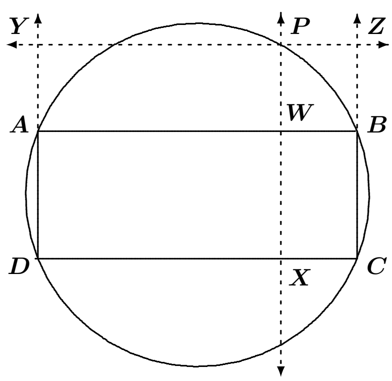
Determine the two numbers.

M267. *Proposed by the Mayhem Staff.*

Find a quintic polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 7 repeated k times, then $f(n)$ consists of the digit 7 repeated $5k+3$ times. (For example, $f(77) = 7777777777777$.) Compare with M256 [2006 : 265].

M268. *Proposed by the Mayhem Staff.*

Rectangle $ABCD$ is inscribed in a circle Γ and P is a point on Γ. Lines parallel to the sides of the rectangle are drawn through P and meet one pair of sides at points W and X and the extensions of the other pair of sides at Y and Z. Prove that the line through W and Y is perpendicular to the line through X and Z.



M263. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soit a, b et n des entiers tels que $(a^2+b^2)/5 = n$. Montrer que $n = c^2+d^2$ pour des entiers c et d.

M264. *Proposé par Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

Etant donné 1001 nombres réels placés autour d'un cercle de sorte que, ou bien chaque nombre est la moyenne arithmétique de ses voisins, ou alors ses deux voisins sont égaux, montrer que tous les nombres sont égaux.

M265. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Dans un triangle ABC, on dessine une parallèle DE au côté BC, avec $D \in AB$ et $E \in AC$. Soit respectivement F et K les pieds des perpendiculaires abaissées de D et E sur BC. Si $\frac{[ABC]}{[DEKF]} = \frac{32}{7}$, trouver le rapport $\frac{|AD|}{|DB|}$.

M266. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Une paire de nombres à deux chiffres possède les propriétés suivantes :

1. La somme des quatre chiffres est 17.
2. La somme des deux nombres est 89.
3. Le produit des quatre chiffres est 210.
4. Le produit des deux nombres est 1924.

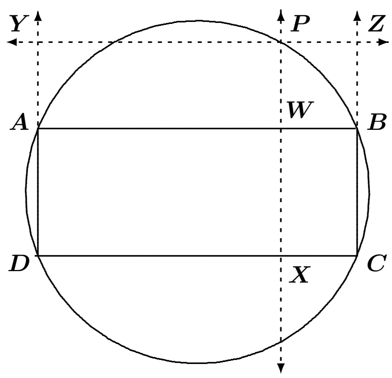
Déterminer ces deux nombres.

M267. *Proposé par l'Équipe de Mayhem.*

Trouver un polynôme quintique $f(x)$ tel que, si n est un entier positif formé du chiffre 7 répété k fois, alors $f(n)$ est formé du chiffre 7 répété $5k+3$ fois. (Par exemple, $f(77) = 7777777777777$.) Voir aussi M256 [2006 : 266].

M268. *Proposé par l'Équipe de Mayhem.*

Soit $ABCD$ un rectangle inscrit dans un cercle Γ et P un point sur Γ. Les droites par P et parallèles aux côtés du rectangle coupent une paire de côtés aux points W et X, et les extensions de l'autre paire de côtés en Y et Z. Montrer que la droite par W et Y est perpendiculaire à la droite par X et Z.



Mayhem Solutions

M213. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Set $S = (2+1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)+1$. Evaluate $S^{\frac{1}{1024}}$ without using a calculator.

Solution by John DeLeon, Angelo State University, San Angelo, TX.

Multiply S by $(2-1)$. Then

$$\begin{aligned} S &= (2-1)(2+1)(2^2+1)(2^4+1)\cdots(2^{1024}+1) + 1 \\ &= (2^2-1)(2^2+1)(2^4+1)\cdots(2^{1024}+1) + 1 \\ &= (2^4-1)(2^4+1)\cdots(2^{1024}+1) + 1 \\ &\vdots \\ &= (2^{1024}-1)(2^{1024}+1) + 1 \\ &= (2^{2048}-1) + 1 = 2^{2048}. \end{aligned}$$

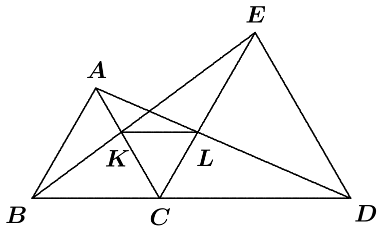
Therefore, $S^{\frac{1}{1024}} = 2^{\frac{2048}{1024}} = 2^2 = 4$.

Also solved by JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; ALPER CAY, Uzman Private School, Kayseri, Turkey; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; and VEDULA N. MURTY, Dover, PA, USA.

M214. *Proposed by Babis Stergiou, Chalkida, Greece.*

Two equilateral triangles ABC and CDE are on the same side of line BCD. If BE intersects AC at K and DA intersects CE at L, prove that KL is parallel to BD.

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.



Since $\angle BCA = \angle ECD = 60^\circ$, we see that $\angle ACE = 60^\circ$, from which it follows that $\angle BCE = \angle ACD$. Since $AC = BC$ and $CD = CE$, we see that $\triangle ACD$ and $\triangle BCE$ are congruent (SAS). Thus, $\angle CAD = \angle CBE$, and we see that $\triangle ACL$ is congruent to $\triangle BCK$ (AAS). From this, we have $CK = CL$. Therefore, $\triangle CKL$ is isosceles. Since $\angle ACE = 60^\circ$, we see that $\angle CKL = \angle CLK = 60^\circ = \angle BCA$. Hence, KL is parallel to BD.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; and GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina. There was one incorrect solution received.

M215. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Find a rational number s such that $s^2 + 5$ and $s^2 - 5$ are both squares of rational numbers.

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

Let p_1 and p_2 be positive rational numbers such that $p_1^2 = s^2 + 5$ and $p_2^2 = s^2 - 5$. Then $(p_1 + p_2)(p_1 - p_2) = p_1^2 - p_2^2 = (s^2 + 5) - (s^2 - 5) = 10$. If we set $a = p_1 + p_2$, then we have $p_1 + p_2 = a$ and $p_1 - p_2 = 10/a$. The solution to this linear system of equations is $p_1 = (a^2 + 10)/2a$ and $p_2 = (a^2 - 10)/2a$. We may then express s^2 in terms of a:

$$s^2 \;=\; p_1^2 - 5 \;=\; \left(\frac{a^2 + 10}{2a}\right)^2 - 5 \;=\; \frac{100 + a^4}{4a^2}\,.$$

This implies that $100 + a^4 = (2as)^2$ is the square of a rational number.

Clearly, $100 + a^4 > (a^2)^2$. Hence, we may write $100 + a^4 = (a^2 + b)^2$ with $b > 0$. Then $100 = 2a^2b + b^2$, which yields $a = \sqrt{(100 - b^2)/(2b)}$. By inspection, testing over the natural numbers $1, 2, \ldots, 9$, we find that for $b = 8$ we get $a = 3/2$; whence, $s = 41/12$.

Also solved by JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and VEDULA N. MURTY, Dover, PA, USA.

Bruening attached a note regarding this problem. It was solved by Fibonacci as part of a mathematical tournament at the court of Frederick II. Not only can it be found in books (for example, Eves' An Introduction to the History of Mathematics, *pp. 263, 284), but a version of it also appeared recently (March 2005) as problem #771 in the Problems Section of the* College Mathematics Journal (CMJ), *of which Bruening is a co-editor. According to the solution of that* CMJ *problem, there are an infinite number of solutions to this problem.*

M216. *Proposed by K.R.S. Sastry, Bangalore, India.*

A Heron triangle has integer sides and area. Two sides of a Heron triangle are 442 and 649. If its area is 132396, find its perimeter.

Solution by Vedula N. Murty, Dover, PA, USA.

Let A denote the angle between the two sides whose lengths are 442 and 669. Then $\frac{1}{2}(442)(649)\sin A = 132396$. This gives $\sin A = 12/13$. Hence, $\cos A = \pm 5/13$. Let a denote the length of the unknown side. Then we have $a^2 = (442)^2 + (649)^2 - 2(442)(649)\cos A$. Substituting $5/13$ for $\cos A$, we get a non-integer value for a, which is not allowed if the triangle is a Heron triangle. Using $\cos A = -5/13$, we obtain $a = 915$. Thus, the perimeter is equal to $442 + 669 + 915 = 2006$.

Also solved by JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; ALPER CAY, Uzman Private School, Kayseri, Turkey; ESTHER MARÍA GARCÍA-CABALLERO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA.

M217. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Let a, b, c be integers such that 2005 divides into both $ab+9b+81$ and $bc+9c+81$. Prove that 2005 also divides into $ca+9a+81$.

Solution by the Mayhem Staff.

Let $w = ca + 9a + 81$. We want to prove that 2005 divides w. Since 2005 divides $ab + 9b + 81$, we have

$$ab + 9b + 81 \; = \; 2005k \tag{1}$$

for some integer k. The prime divisors of 2005 are 5 and 401, neither of which divides 81. It then follows from (1) that neither 5 nor 401 divides b. On the other hand, both 5 and 401 divide bw, since

$$\begin{aligned} bw &= abc + 9ab + 81b \\ &= c(ab + 9b + 81) - 9(bc + 9c + 81) + 9(ab + 9b + 81)\,, \end{aligned}$$

which is divisible by 2005 (using the information given in the problem).

Therefore, both 5 and 401 must divide w. Thus, 2005 divides w.

There were 2 incomplete solutions received.

M218. *Proposed by Neven Jurič, Zagreb, Croatia.*

Compute the sum

$$\sum_{k=1}^{99} \frac{1}{k\sqrt{k+1} + (k+1)\sqrt{k}}\,.$$

Solution by Esther María García-Caballero, Universidad de Jaén, Jaén, Spain.

By rationalizing the denominators, we obtain a telescoping series:

$$\begin{aligned} &\sum_{k=1}^{99} \frac{1}{k\sqrt{k+1} + (k+1)\sqrt{k}} \\ &\quad = \sum_{k=1}^{99} \frac{(k+1)\sqrt{k} - k\sqrt{k+1}}{\left((k+1)\sqrt{k} + k\sqrt{k+1}\,\right)\left((k+1)\sqrt{k} - k\sqrt{k+1}\,\right)} \\ &\quad = \sum_{k=1}^{99} \frac{(k+1)\sqrt{k} - k\sqrt{k+1}}{(k+1)^2k - k^2(k+1)} \;=\; \sum_{k=1}^{99} \frac{(k+1)\sqrt{k} - k\sqrt{k+1}}{k(k+1)} \\ &\quad = \sum_{k=1}^{99} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \;=\; \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{100}} \;=\; \frac{9}{10}\,. \end{aligned}$$

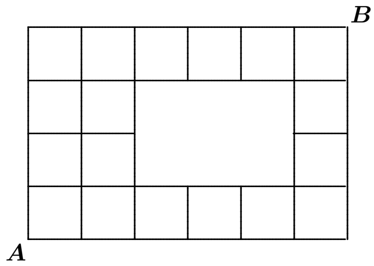
Also solved by ALPER CAY, Uzman Private School, Kayseri, Turkey; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; and VEDULA N. MURTY, Dover, PA, USA.

Problem of the Month

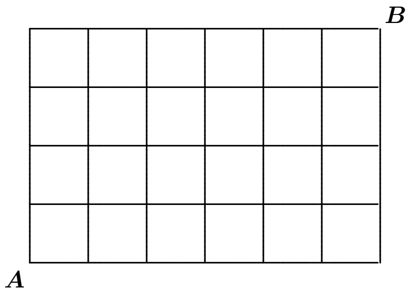
Ian VanderBurgh

This month, a variation on a classic problem:

Problem (2005 Small c Contest) In the road-map shown in the diagram, each line segment represents a street which can only be travelled along in either the rightwards or upwards direction. How many paths are there from point A to point B?



This is an neat twist on a classic problem whose text is identical, but whose diagram is a complete grid, with no "hole" in it:

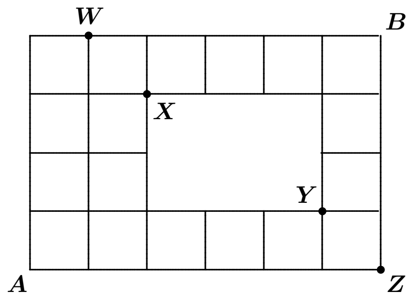


To solve this old chestnut, the insight is to notice that every possible path from A to B consists of exactly 6 moves to the right and 4 moves up. Also, each possible ordering of 6 moves to the right and 4 moves up gives a different path. If we use "R" to denote a move to the right and "U" to denote a move up, then possible paths include URUUURRRRR, RURURURURR, and so on. Thus, we have transformed the problem of counting paths into one of arranging letters (6 Rs and 4 Us).

How many arrangements of 6 Rs and 4 Us are there? Since we are arranging 10 objects, 6 of one type which are identical and 4 of another type which are identical, there are $\frac{10!}{6!4!} = \frac{10(9)(8)(7)}{4(3)(2)(1)} = 210$ arrangements, and hence, 210 possible paths in this classic problem. (Alternatively, we could think of choosing 6 of the 10 places in which to put the Rs, leaving the remaining 4 places for the Us; there are $\binom{10}{6} = 210$ such ways.)

How does this help with this month's Problem? At the very least, this gives us an upper bound for the number of paths in the problem that we want to solve—there certainly cannot be any more paths with some streets in the middle removed. It also gives us an idea of how to start. Here is a first approach that models the idea above.

Solution 1: It is difficult to get a handle on how to count these paths until we break up the paths into four categories. Consider the intersections in the diagram labelled W, X, Y, and Z.



Any path from A to B must pass through EXACTLY one of the four points W, X, Y, or Z. (You may need to stare at the diagram for a little while to convince yourself of this.) Why does this help? It helps because we can break up the paths from A to B into four disjoint categories—those from A to B passing through each of W, X, Y, and Z. We count the paths in this way by counting the number of paths from A to the particular intermediate point, and then from that point to B.

To get from A to W, we make 1 move to the right and 4 moves up. There are $\binom{5}{1} = 5$ such paths. To get from W to B, we make 5 moves to the right. There is only 1 such path. Thus, there are $5 \times 1 = 5$ paths from A to B through W.

To get from A to X, we make 2 moves to the right and 3 moves up. There are $\binom{5}{2} = 10$ such paths. To get from X to B, we make 4 moves to the right and 1 move up. There are $\binom{5}{4} = 5$ such paths. Thus, there are $10 \times 5 = 50$ paths from A to B through X.

To get from A to Y, we make 5 moves to the right and 1 move up. There are $\binom{6}{5} = 6$ such paths. To get from Y to B, we make 1 move to the right and 3 moves up. There are $\binom{4}{1} = 4$ such paths. Thus, there are $6 \times 4 = 24$ paths from A to B through Y.

To get from A to Z, we make 6 moves to the right. There is only 1 such path. To get from Z to B, we make 4 moves up. There is only 1 such path. Thus, there is $1 \times 1 = 1$ path from A to B through Z.

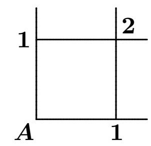
Therefore, in total there are $5 + 50 + 24 + 1 = 80$ paths from A to B.

That's a pretty insightful method. Taking a problem that seems difficult to get a handle on and dividing it up into separate problems that are relatively easy to deal with is always a good idea—figuring out how to divide it up was the real trick here.

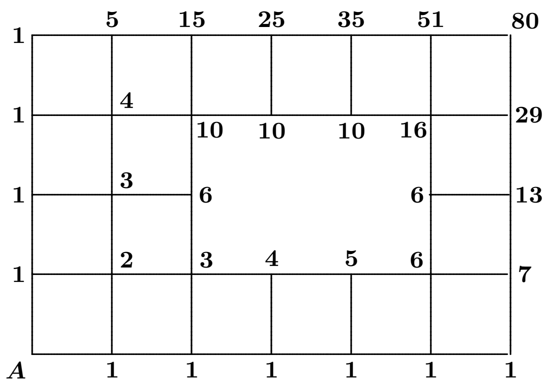
Have I led you far enough down the, er, path? Unfortunately, this relation to the "classic" problem and its method of solution may have blinded us to an easier solution.

Solution 2: If we choose any intersection, how many paths are there to get to it? Well, to get to any intersection, we must come from directly to the left or directly below. Thus, the total number of paths to any intersection is the sum of the number of paths to the intersection immediately to the left and the number of paths immediately below.

As an example, let's consider the bottom left corner. To get to the intersection directly above A, there is only one path. To get to the intersection directly to the right of A, there is only one path. To get to the intersection 1 move up and 1 move right of A, there are 2 paths—1 each through each of the two previously mentioned intersections.



We can follow this line of reasoning to fill in the number of paths to any intersection on the grid (using a slightly enlarged version of the grid so that we can fit the numbers on it!).



Hence, there are 80 paths from A to B.

Now that was much easier. And it's much easier to generalize—if we had a much bigger grid with a whole bunch of "holes" in the middle, we could generalize this very nicely. This is something that is always worth thinking about when solving a problem—will our method generalize to more complicated situations?

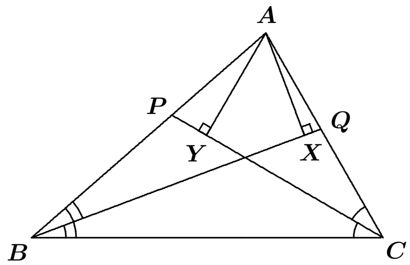
Pólya's Paragon

Remarkable Bisections

Bruce Shawyer

In a recent article [1] in *Mathematics Today* (the journal of the Institute of Mathematics and its Applications), Michael Carding (formerly a high school principal in Cheshire, England) described a visit to India. There, he met a thirteen year old female student who just loved mathematics. She posed him a problem:

In $\triangle ABC$, the perpendiculars from A to the internal bisectors of angles B and C meet those bisectors at X and Y. She claimed that XY is parallel to CB, but had not been able to prove it. She also stated that neither her friends nor her teachers had been able to prove it.



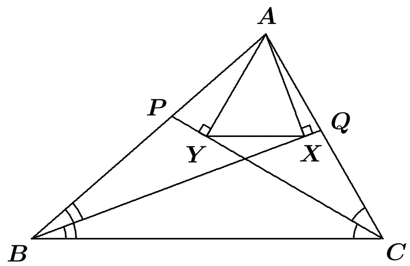
This was obviously a challenge to Mr. Carding. The article states that Mr. Carding claimed jet-lag and duties to cover his immediate inadequacy, and stated that they have resolved the problem via e-mail. No solution is given in the article.

Here, we examine this problem and give a solution which has a surprising ending.

When first thinking about a problem with bisected angles, it is natural to consider angles! For example, what are the sizes of angles PAY, QAX, and XAY? It is also tempting to draw the line segment XY. Also, if $XY \parallel BC$, then we see the z–Theorem coming into play.

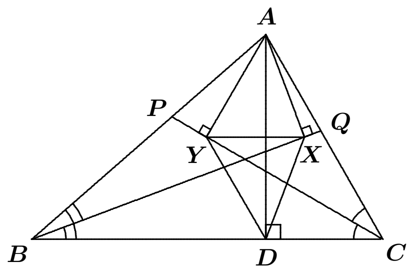
A little elementary calculation shows that $\angle PAY = A + \frac{1}{2}C - 90^\circ$ (use $\angle APY = B + \frac{1}{2}C$, so that $\angle PAY = 90^\circ - B - \frac{1}{2}C$ and $\angle A + \angle B + \angle C = 180^\circ$). Similarly, we see that $\angle QAX = A + \frac{1}{2}B - 90^\circ$. Then $\angle XAY = \frac{1}{2}(B + C)$.

If $XY \parallel BC$, then we would have $\angle AYX = 90^\circ - \frac{1}{2}C$ and $\angle AXY = 90^\circ - \frac{1}{2}B$. Conversely, if we can prove that either of these equations holds, then $XY \parallel BC$. Can we prove this to be true?



As in all interesting geometry problems, some construction is needed. I always told my geometry students (at university, in particular) to make as many constructions as seemed to be possibly useful. Later, when a solution has been found, not all of the constructions made will be necessary. Then, a new diagram should be drawn and the old one discarded.

The useful constructions here turn out to be drawing the altitude AD from A to BC and the line segments DX and DY. From this, we can see that quadrilaterals $AYDC$ and $AXDB$ are cyclic, which gives lots of equalities amongst angles. In particular,



$$\begin{aligned} \angle ADY &= \angle ACY = \tfrac{1}{2}C \\ &= \angle YCD = \angle YAD . \end{aligned}$$

Therefore, $\triangle AYD$ is isosceles, and thus, $AY = YD$. Similarly, $AX = XD$.

This means that $AXDY$ is a very special quadrilateral with the property that the diagonals are perpendicular and XY bisects AD. Thus, it follows that $XY \parallel BC$, and the theorem is proved.

The surprising extra result is that XY bisects the altitude AD. With minor adjustments, this proof is valid for non-acute triangles.

Now, how did I come to think of this proof? To be honest, in preparing a diagram to work with, I resorted to computer algebra. I chose coordinates for A, B and C, worked out the coordinates of the incentre to get the equations of BX and CY, next the equations of AX and AY, and then the coordinates of X and Y.

I had chosen BC along the x–axis. I was very surprised to discover that the y–coordinates of X and Y were not only equal, but were equal to one half of the y–coordinate of A. This then gave me the clue that I had to prove $AX = XD$ and $AY = YD$. The rest is the proof above.

Now, examining the figure leads to the following result.

Given triangle ABC, let AD be an altitude. Draw the perpendicular bisector of AD and the semicircle ADC. Let Y be the point where these intersect. Then CY is a bisector of $\angle ACD$. This is the internal bisector of $\angle ACB$ if D lies between B and C, and the external bisector of $\angle ACB$ otherwise. We have discovered another way to bisect an angle!

Reference

[1] Culture Shock for Mathematics and Science, *Mathematics Today*, Vol. 42, No. 4, 2006, pp. 129–131.

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THE OLYMPIAD CORNER

No. 257

R.E. Woodrow

We begin this issue with problems from the Belarus Mathematical Olympiad 2002. My thanks go to Andy Liu, Canadian Team Leader to the 2003 IMO in Japan, for collecting the problems for our use.

BELARUS MATHEMATICAL OLYMPIAD 2002
Final Round, Category A

1. (*D. Dudko*) Each cell of an $n \times n$ $(n \geq 3)$ square table contains one of the unit vectors $\uparrow$, $\downarrow$, $\leftarrow$, $\rightarrow$ (placed parallel to a side of the table so that the middle of the vector coincides with the centre of the cell in which it is located). On each move, a beetle creeps from one cell to another in accordance with the vector's direction. If the beetle starts from any cell, then it comes back to this cell after some number of moves. The vectors are directed so that they do not allow the beetle to leave the table.

Is it possible that the sum of all vectors in any row, except for the first and last rows, is equal to the vector that is parallel to this row, and the sum of all vectors in any column, except for the first and last columns, is equal to the vector that is parallel to this column?

2. (*D. Bazylev*) Let

$$P(x) \;=\; (x+1)^p(x-3)^q \;=\; x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n\,,$$

where p and q are positive integers.

(a) Given that $a_1 = a_2$, prove that $3n$ is a perfect square.

(b) Prove that there exist infinitely many pairs (p, q) of positive integers p and q such that the equality $a_1 = a_2$ is valid for the polynomial $P(x)$.

3. (*E. Barabanov*) Two triangles are said to be *twins* if one of them is an image of the other one under a parallel projection.

Prove that two triangles are twins if and only if either at least one side of one of them equals a side of the other or the triangles have equal segments that connect the corresponding vertices with some points on the opposite sides which divide these sides in the same ratio.

4. (*V. Kolbun*) Positive numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ satisfy the condition $a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots + b_n = 1$.

Find the smallest possible value of the sum

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \dots + \frac{a_n^2}{a_n+b_n}.$$

5. (*I. Voronovich*) Let m, n, k be positive integers such that $m > n > k$. A $1 \times m$ strip of paper is divided into 1×1 cells. Bill and Pit place numbers 0 and 1 in the cells so that the sum of the numbers in any n consecutive cells is equal to k. After the task is performed, it turns out that the sum $S(B)$ of all numbers put on the strip by Bill is different from the sum $S(P)$ of Pit.

Find the largest possible value of $|S(B) - S(P)|$.

6. (*A. Romanenko, D. Zmeikov*)

(a) A positive integer is called *nice* if it can be represented as an arithmetic mean of some (not necessarily distinct) positive integers each of which is a non-negative power of 2.

Prove that all positive integers are nice.

(b) A positive integer is called *ugly* if it cannot be represented as an arithmetic mean of pairwise *distinct* positive integers each of which is a non-negative power of 2.

Prove that there exist infinitely many ugly positive integers.

7. (*E. Barabanov*) Does there exist a surjective function $f : \mathbb{R} \to \mathbb{R}$ such that the expression $f(x+y) - f(x) - f(y)$ takes exactly two values 0 and 1 for various real x and y?

8. (*I. Voronovich*) Find the area of the convex pentagon $ABCDE$, given that $AB = BC$, $CD = DE$, $\angle ABC = 150°$, $\angle CDE = 30°$, and $BD = 2$.

Final Round, Category B

1. (*I. Zhuk*) The diagonals AC and BD of the convex quadrilateral $ABCD$ are perpendicular and intersect at point O. Circles S_1, S_2, S_3, S_4 with centres O_1, O_2, O_3, O_4 are inscribed in the triangles AOB, BOC, COD, DOA, respectively. Prove that:

(a) the sum of the diameters of S_1, S_2, S_3, S_4 is less than or equal to

$$(2 - \sqrt{2})(AC + BD),$$

(b) $O_1O_2 + O_2O_3 + O_3O_4 + O_4O_1 < 2(\sqrt{2} - 1)(AC + BD)$.

2. (*D. Bazylev*) Given

$$\frac{\sin a}{\sin b} = \frac{\sin c}{\sin d} = \frac{\sin(a-c)}{\sin(b-d)},$$

with a, b, c, $d \in (0, \pi)$, prove that $a = b$ and $c = d$.

3. (*I. Akulich*) A rabbit is at one of the vertices of a cube. The rabbit is hunted by M hunters who do not see it or its path. Per round, each hunter selects one of the vertices (at his own discretion) and fires his gun. If the rabbit is at one of the selected vertices, it is killed. If not, then it either jumps to a neighbouring vertex (two vertices are neighbouring if they are connected by an edge) or stays at the same vertex. The hunt continues in this manner until the rabbit is killed.

Find the number N of shots required to kill the rabbit if the number of hunters is:

(a) $M = 3$; (b) $M = 5$; (c) $M = 4$.

4. (*I. Voronovich*) Pairwise distinct positive integers a, b, c, d, e, f, g, h, and n satisfy the equalities $n = ab + cd = ef + gh$.

Find the smallest possible value of n.

5. (*I. Voronovich*) The quadrilateral $ABCD$ is cyclic and has the property that $AB = BC = AD + CD$. Given that $\angle BAD = \alpha$ and that the diagonal $AC = d$, find the area of the triangle ABC.

6. (*A. Mirotin, E. Mirotin*) Let $\mathbb{Q}_1 = \{x \in \mathbb{Q} \mid x \geq 1\}$. The function $f : \mathbb{Q}_1 \to \mathbb{R}$ satisfies the inequality $|f(x+y) - f(x) - f(y)| < \varepsilon$ for all x, $y \in \mathbb{Q}_1$, where $\varepsilon > 0$ is a real number.

Prove that there exists a real number q such that, for all $x \in \mathbb{Q}_1$,

$$\left|\frac{f(x)}{x} - q\right| < 2\varepsilon .$$

7. (*E. Barabanov*) We say that a triangle and a rectangle are *twin* if they have the same perimeter and the same area.

Prove that, for a given rectangle, there exists a twin triangle if the rectangle is not a square and the ratio of the bigger side of the rectangle to its smaller side is at least $\lambda - 1 + \sqrt{\lambda(\lambda-2)}$, where $\lambda = 3\sqrt{3}/2$.

8. (*E. Barabanov, V. Kaskevich*) Five tennis players take part in a doubles tournament. Let $R(A)$ be the rating of player A. It is known that the ratings of the players are distinct. Moreover, the pairwise sums of the ratings of different pairs are distinct. If $R(A) + R(B) > R(C) + R(D)$, then pair (A, B) wins against (C, D), pair (C, D) is disbanded, and a new pair (C, E) or (D, E) plays the next game against (A, B). The tournament lasts until some pairs that have already met before, have to meet again.

Find the largest possible number of the games in this tournament.

Final Round, Category C

1. (*A. Mirotin*) (a) There are $k \geq 3$ positive integers such that no two of them are coprime while any three of them are coprime. Determine all possible values of k.

(b) Does there exist an infinite set of positive integers satisfying the same condition?

2. (*I. Zhuk*) Prove that a right-angled triangle can be inscribed in the parabola $y = x^2$ so that its hypotenuse is parallel to the x–axis if and only if the altitude from the right angle is equal to 1. (A triangle is inscribed in a parabola if all three vertices of the triangle are on the parabola.)

3. (*I. Zhuk*) The diagonals A_1A_4, A_2A_5, and A_3A_6 of the convex hexagon $A_1A_2A_3A_4A_5A_6$ meet at a point K. Given that $A_2A_1 = A_2A_3 = A_2K$, $A_4A_3 = A_4A_5 = A_4K$, and $A_6A_5 = A_6A_1 = A_6K$, prove that the hexagon is cyclic.

4. (*V. Kaskevich*) Some cells of an $n \times m$ (n, $m \geq 2$) board are white and the others are black. A beetle, initially outside the board, creeps onto one of the outermost cells and recolours it in the opposite colour. The beetle may then move from cell to cell by creeping from its current cell to a neighbouring cell (a cell sharing a side with the given cell). Upon entering a cell, the beetle recolours it in the opposite colour. Finally, the beetle leaves the board.

Does there exist a route for the beetle that results in all cells of the board being coloured black?

5. (*I. Akulich*) Is it possible, for some positive integer N, to write the numbers N, N^2, and N^3 using each of the digits 0 through 9 exactly once?

6. (*A. Shamruk*) Distinct points A_0, A_1, ..., A_{1000} on one side of an angle and distinct points B_0, B_1, ..., B_{1000} on the other side are spaced so that $A_0A_1 = A_1A_2 = \cdots = A_{999}A_{1000}$ and $B_0B_1 = B_1B_2 = \cdots = B_{999}B_{1000}$. Find the area of the quadrilateral $A_{999}A_{1000}B_{1000}B_{999}$ if the areas of the quadrilaterals $A_0A_1B_1B_0$ and $A_1A_2B_2B_1$ are equal to 5 and 7, respectively.

7. (*I. Voronovich*) A quadrilateral $ABCD$ is cyclic with $AB = 2AD$ and $BC = 2CD$. Given that $\angle BAD = \alpha$, and diagonal $AC = d$, find the area of the triangle ABC.

8. (*S. Mazanik*) There are N teams in a volleyball tournament. Each team plays exactly one game against each other team. A team receives 1 point for a win and 0 points for a loss (there are no draws in a volleyball tournament). It is known that, among any four teams, at least two have the same number of points in the games involving these four teams.

Find the largest possible value of N.

As a second block of problems for your puzzling pleasure we give 11 problems selected from the Thai Mathematical Olympiad Examination 2002. Thanks again go to Andy Liu, Canadian Team Leader to the 2003 IMO in Japan, for obtaining them for our use.

THAI MATHEMATICAL OLYMPIAD EXAMINATION 2002
Selected Problems

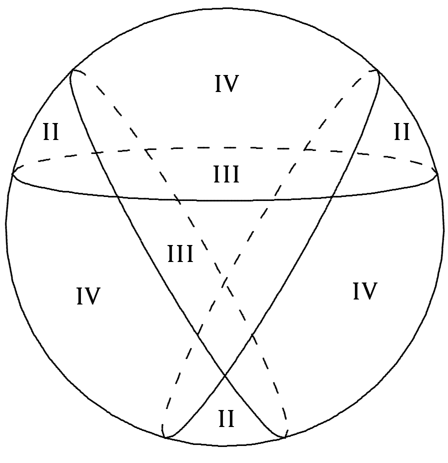
1. Three circles are pairwise tangent (internally or externally) at the points A, B, C. The distances between pairs of centres are 3, 3, and 4 units. If the triangle ABC is not isosceles, find its perimeter.

2. Draw 3 great circles on a sphere which divide the surface of the sphere into regions as follows:

- 3 identical regions bounded by 2 arcs of the circles (type II);
- 2 identical regions bounded by 3 arcs of the circles (type III);
- 3 identical regions bounded by 4 arcs of the circles (type IV).

How many different ways can one paint each region with one of three colours so that no two adjacent regions are painted with the same colour?

[*Ed.:* The circles cannot be "great" circles in order for the surface of the sphere to be divided in this manner.]



3. Find the maximum real number K such that

$$\frac{1}{ka+b}+\frac{1}{kb+a}\geq\frac{K}{a+b}$$

for all a, $b>0$ and all $k\in[0,\pi]$.

4. Let x_1 and x_2 be consecutive integers (that is, $x_2 = x_1 + 1$). For each integer $n \geq 3$, let x_n be the remainder when $x_{n-1}^2 + x_{n-2}^2$ is divided by 7. If $x_{2545} = 1$, determine the value of x_4.

5. Let $ABCDEF$ be a convex hexagon such that $\triangle BCD$, $\triangle DEF$ and $\triangle FAB$ have equal area. Suppose that $AB = BC$, $CD = DE$, $EF = FA$ and $\angle B + \angle D + \angle F = 360^\circ$. Show that there is an interior point O and 3 vertices of the hexagon $ABCDEF$ such that the 3 line segments from O to the chosen vertices cut the hexagon $ABCDEF$ into 3 parts of equal area.

6. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a non-constant polynomial with integer coefficients. Assume that $p(-1) = 0$ and $p(\sqrt{2})$ is an integer. Show that there is an integer k such that $p(k) + a_k$ is even.

7. Find all integers n with the property that both $n + 2002$ and $n - 2002$ are perfect squares.

8. In how many ways can 8 pawns be placed on a 4×4 chessboard subject to the following conditions?

(i) Each pawn is placed in a square, and each square holds no more than one pawn.

(ii) The number of pawns in each row and in each column is even.

(iii) No two rows have the same pattern, and no two columns have the same pattern.

9. Find the greatest integer which divides

$$(a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$$

for any integers a, b, c, d.

10. A toy set consists of four pieces: a cat robot, a cat control, a mouse robot, and a mouse control. Whenever the two robots are put at different places on the floor, the cat robot will chase, while the mouse robot will run away, both in the direction of the straight line joining them, with the speeds set by the respective controls.

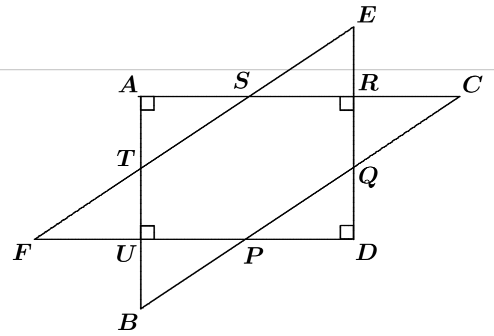
Three boys each have a toy set. They place their cat-mouse robots on the circumference of a circular ground. The three cats T_1, T_2, T_3 are placed at the same point and allowed simultaneously to chase the three mice J_1, J_2, J_3, respectively. It turns out that each cat catches its respective mouse at the same instant. The distances (in centimeters) between mouse robots are:

Distances between	J_1 and J_2	J_2 and J_3	J_1 and J_3
at the start	32	32	90
at the capture	245	125	370

Find, up to two decimal places, the ratio of the average speed of T_1 to that of T_3.

11. Two right triangles ABC and DEF have $\angle A = \angle D = 90^\circ$ and have the same area. The triangles intersect at points P, Q, R, S, T, U as pictured, with $\angle U = \angle R = 90^\circ$, $[ATS] = [DPQ]$, and $[PQRSTU] = \frac{2}{3}[ABC]$ (where $[XYZ]$ denotes the area of a polygon XYZ).

Let $\alpha = [QCR] + [RES] + [FTU] + [UBP]$. Find $\alpha/[AUDR]$.



We turn to our file of readers' solutions, beginning with problems of the 2001–2002 British Mathematical Olympiad Round 1, given in the September 2005 number of the *Corner* ([2005 : 287]).

1. Find all positive integers m, n, where n is odd, that satisfy

$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}.$$

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; and Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's write-up.

In the following, all letters stand for positive integers.

Suppose m and n satisfy the given conditions; that is, n is odd and

$$12n + 48m = mn. \tag{1}$$

Then $4 \mid mn$ and, since $(4, n) = 1$, we have $4 \mid m$; that is, $m = 4t$. Then (1) reduces to

$$3n + 48t = tn. \tag{2}$$

From (2), we have $n \mid 48t$. Since $48t = 16 \cdot 3t$ and $(n, 16) = 1$, we have $n \mid 3t$; that is, $3t = nu$. Then (2) reduces to

$$9 + 48u = nu. \tag{3}$$

Therefore, $u \mid 9$; that is, $u = 1$, $u = 3$, or $u = 9$. Then (3) yields $n = 57$, $n = 51$, or $n = 49$, respectively. From (1), we get $m = 76$, $m = 204$, or $m = 588$, respectively.

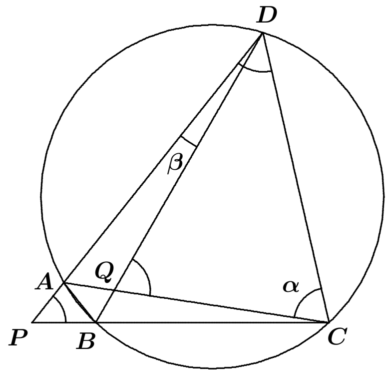
Thus, we find that the only possible pairs (m, n) are $(76, 57)$, $(204, 51)$, and $(588, 49)$. All three satisfy the required conditions.

2. The quadrilateral $ABCD$ is inscribed in a circle. The diagonals AC and BD meet at Q. The sides DA, extended beyond A, and CB, extended beyond B, meet at P.

Given that $CD = CP = DQ$, prove that $\angle CAD = 60^\circ$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Geoffrey A. Kandall, Hamden, CT, USA. We give the solution by Amengual Covas.

We set $\alpha = \angle QCD$ and $\beta = \angle BDA$. Then



$$\begin{aligned} \angle DQC &= \angle QCD = \alpha\,, \\ \angle PCA &= \angle BCA \\ &= \angle BDA = \beta\,, \\ \angle CAD &= \angle DAQ \\ &= \angle DQC - \angle QDA \\ &= \alpha - \beta\,, \\ \angle CDQ &= 180^\circ - 2\alpha\,, \end{aligned}$$

and

$$\angle APC = \angle DPC = \angle CDP = \angle CDQ + \angle BDA = (180^\circ - 2\alpha) + \beta\,.$$

By the External Angle Theorem applied to $\triangle APC$ at A,

$$\begin{aligned} \angle CAD &= \angle APC + \angle PCA = (180^\circ - 2\alpha + \beta) + \beta \\ &= 180^\circ - 2(\alpha - \beta) = 180^\circ - 2 \cdot \angle CAD\,. \end{aligned}$$

Hence, $\angle CAD = 60^\circ$.

3. Find all positive real solutions to the equation

$$x + \left\lfloor \frac{x}{6} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor,$$

where $\lfloor t \rfloor$ denotes the largest integer less than or equal to the real number t.

Solved by Michel Bataille, Rouen, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

Clearly x must be an integer. Let $x = 6k+r$, where k and r are integers and $0 \le r \le 5$. Then the given equation becomes

$$7k + r = 7k + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{2r}{3} \right\rfloor, \quad \text{or} \quad r = \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{2r}{3} \right\rfloor.$$

We can easily check that this equation is satisfied for $r \in \{0, 2, 3, 4, 5\}$, but not for $r = 1$. Thus, the set of all real solutions is $\{x \in \mathbb{Z} \mid x \not\equiv 1 \pmod 6\}$, and the set of all positive real solutions is $\{x \in \mathbb{N} \mid x \not\equiv 1 \pmod 6\}$.

4. Twelve people are seated around a circular table. In how many ways can six pairs of people engage in handshakes so that no arms cross? (Nobody is allowed to shake hands with more than one person at once.)

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We use Bornsztein's write-up.

More generally, let C_n be the number of ways $2n$ people around a circular table can all engage in handshakes so that no arms cross.

Let us define $C_0 = 1$. It is easy to see that $C_1 = 1$.

For $n \geq 2$, let A_1, A_2, ..., A_{2n} be the people, in clockwise order. Let A_j be the person who shakes hands with A_1. Since everyone is engaged in handshakes and no arms cross, there must be an even number of people on each side of the line A_1A_j. Thus, j is even, say $j = 2k$, with $1 \leq k \leq n$. On one side of the line A_1A_j there are $2(k-1)$ people, and on the other side are $2(n-k)$ people. In each of the groups, each person has to engage in a handshake with another person from the same group. Therefore, the number of ways the handshakes can occur is $C_{k-1}C_{n-k}$.

The total number of ways for the people to engage in handshakes is

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}\,. \qquad (1)$$

From this relation, since $C_0 = C_1 = 1$, it is easy to obtain $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, and $C_6 = 132$, which is the answer to the given problem.

Note: From (1), since $C_0 = 1$, we see that the numbers C_n are the well-known Catalan Numbers, and $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

5. Let f be a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of non-negative integers, which has the following properties:

(a) $f(n+1) > f(n)$ for each $n \in \mathbb{Z}^+$,

(b) $f(n + f(m)) = f(n) + m + 1$ for all $m, n \in \mathbb{Z}^+$.

Find all possible values of $f(2001)$.

Solution by Michel Bataille, Rouen, France.

We show that the only function f satisfying (a) and (b) is given by $f(n) = n + 1$ for all $n \in \mathbb{Z}^+$. It follows that the only possible value of $f(2001)$ is 2002.

Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfies conditions (a) and (b) and let $a = f(0)$. Note that (b) with $m = n = 0$ yields $f(a) = a + 1$.

First, we show by induction that, for all $n \in \mathbb{Z}^+$,

$$f(n) = (n+1)a\,. \qquad (1)$$

Clearly, (1) holds for $n = 0$. Now, assume that $f(k) = (k+1)a$ for some $k \in \mathbb{Z}^+$. Then, on the one hand,

$$f(a+k+1) \;=\; f\big(k+f(a)\big) \;=\; f(k)+a+1\,,$$

and on the other hand,

$$f(a+k+1) \;=\; f\big(k+1+f(0)\big) \;=\; f(k+1)+0+1 \;=\; f(k+1)+1\,.$$

Therefore, $f(k+1) = f(k)+a = (k+2)a$, which completes the induction step and the proof of (1). Note that a must be positive (for condition (a) to hold).

Conversely, a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by $f(n) = (n+1)a$, where a is some positive integer, clearly satisfies condition (a), while condition (b) becomes $\big(n+(m+1)a+1\big)a = (n+1)a+m+1$, or $(m+1)a^2 = m+1$. Thus, condition (b) holds if and only if $a = 1$.

In conclusion, only $f(n) = n+1$ satisfies both (a) and (b).

Next we look at solutions from our readers to problems of Round 2 of the 2001-2002 British Mathematical Olympiad given in [2005 : 288].

1. The altitude from one of the vertices of an acute-angled triangle ABC meets the opposite side at D. From D, perpendiculars DE and DF are drawn to the other two sides. Prove that the length of EF is the same whichever vertex is chosen.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Amengual Covas.

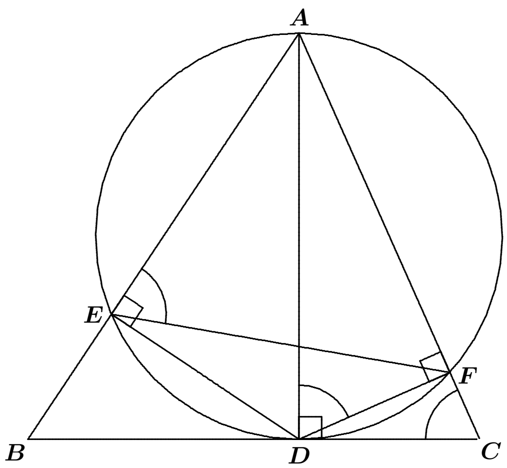
Suppose that D is the foot of the altitude from A. Since $\angle AED$ and $\angle DFA$ are right angles, the quadrilateral $AEDF$ may be inscribed in a circle. Therefore,

$$\angle AEF \;=\; \angle ADF \;=\; 90^\circ - \angle FAD \;=\; 90^\circ - \angle CAD \;=\; \angle BCA\,,$$

making $\triangle AFE$ similar to $\triangle ABC$. Then $\dfrac{EF}{BC} = \dfrac{AF}{AB}$; that is,

$$EF \;=\; \frac{AF\cdot BC}{AB}\,. \tag{1}$$

We have $AF = AD\cdot\sin\angle ADF = AD\cdot\sin\angle BCA = AD\cdot AB/(2R)$ and $AD\cdot BC = 2S$, where R and S denote the circumradius and the area of $\triangle ABC$, respectively. On substituting into (1), we find that $EF = S/R$. Since this quantity depends only on R and S, the same expression would be obtained if we had chosen the altitude from B or C.



2. A conference hall has a round table with n chairs. There are n delegates to the conference. The first delegate chooses his or her seat arbitrarily. Thereafter, the $(k+1)^{\text{st}}$ delegate sits k places to the right of the k^{th} delegate, for $1 \leq k \leq n-1$. (In particular, the second delegate sits next to the first.) No chair can be occupied by more than one delegate.

Find the set of values n for which this is possible.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's version.

This is possible if and only if $n = 2^p$ for some non-negative integer p.

This problem is equivalent to problem #2 of the Tournament of the Towns, Spring 1985, Senior Level. A solution may be found in *International Mathematics Tournament of the Towns 1984–1989*, edited by P.J. Taylor, AMT Publishing. Another solution may be found in *Cours d'arithmétique*, exercice 158 at: `http://www.animath.fr/cours/arithm.pdf`.

3. Prove that the sequence defined by $y_0 = 1$ and

$$y_{n+1} \;=\; \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right) \qquad (\text{for } n \geq 0)$$

consists only of integers.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; and Pavlos Maragoudakis, Pireas, Greece. We give Krimker's write-up, modified by the editor.

Let $\{F_n\}_{n=1}^{\infty}$ be the Fibonacci sequence. We will show that $y_n = F_{2n+1}$ for all $n \geq 0$. This implies that y_n is an integer for all $n \geq 0$.

We recall Binet's Formula:

$$F_n \;=\; \tfrac{1}{\sqrt{5}}\left(\phi_1^n - \phi_2^n\right),$$

where $\phi_1 = \frac{1}{2}(1+\sqrt{5})$ and $\phi_2 = \frac{1}{2}(1-\sqrt{5})$. Note that $\phi_1^2 = \frac{1}{2}(3+\sqrt{5})$ and $\phi_2^2 = \frac{1}{2}(3-\sqrt{5})$. Also $\phi_1\phi_2 = -1$.

We claim that the sequence $\{F_{2n+1}\}_{n=0}^{\infty}$ satisfies the same recurrence relation as y_n. Indeed, we have

$$5F_{2n+1}^2 - 4 \;=\; \left(\phi_1^{2n+1} - \phi_2^{2n+1}\right)^2 + 4\phi_1^{2n+1}\phi_2^{2n+1} \;=\; \left(\phi_1^{2n+1} + \phi_2^{2n+1}\right)^2,$$

and hence,

$$\begin{aligned} &\tfrac{1}{2}\left(3F_{2n+1} + \sqrt{5F_{2n+1}^2 - 4}\right) \\ &\qquad = \tfrac{3}{2\sqrt{5}}\left(\phi_1^{2n+1} - \phi_2^{2n+1}\right) + \tfrac{1}{2}\left(\phi_1^{2n+1} + \phi_2^{2n+1}\right) \\ &\qquad = \tfrac{1}{\sqrt{5}}\left(\tfrac{3+\sqrt{5}}{2}\right)\phi_1^{2n+1} - \tfrac{1}{\sqrt{5}}\left(\tfrac{3-\sqrt{5}}{2}\right)\phi_2^{2n+1} \\ &\qquad = \tfrac{1}{\sqrt{5}}\phi_1^{2n+3} - \tfrac{1}{\sqrt{5}}\phi_2^{2n+3} \;=\; F_{2n+3} \;=\; F_{2(n+1)+1}, \end{aligned}$$

which proves the claim.

When $n = 0$, we have $F_{2(0)+1} = F_1 = 1 = y_0$. Since the recurrence relation has a unique solution satisfying the given initial condition, we must have $y_n = F_{2n+1}$ for all $n \geq 0$.

4. Suppose that $B_1, \ldots, B_N$ are N spheres of unit radius arranged in space so that each sphere touches exactly two others externally. Let P be a point outside all these spheres, and let the N points of contact be $C_1, \ldots, C_N$. The length of the tangent from P to the sphere B_i $(1 \leq i \leq N)$ is denoted by t_i. Prove that the product of the quantities t_i is not more than the product of the distances PC_i.

Solution by Michel Bataille, Rouen, France.

Without loss of generality, we suppose that the spheres are numbered so that $C_1, C_2, \ldots, C_{N-1}, C_N$ are the respective points of contact of B_1 with B_2, of B_2 with B_3, ..., of B_{N-1} with B_N, and of B_N with B_1.

First consider the spheres B_1 and B_2, and let O_1 and O_2 be their respective centres. Since C_1 is the mid-point of O_1O_2, the Parallelogram Identity (applied to the parallelogram with sides PO_1 and PO_2) yields

$$4PC_1^2 + O_1O_2^2 \;=\; 2(PO_1^2 + PO_2^2),$$

and hence, $4PC_1^2 + 4 = 2(1 + t_1^2 + 1 + t_2^2)$. Thus,

$$PC_1^2 \;=\; \frac{t_1^2 + t_2^2}{2} \;\geq\; t_1t_2$$

(where the inequality results from the AM–GM Inequality).

In a similar way, we get $PC_2^2 \geq t_2t_3, \ldots, PC_{N-1}^2 \geq t_{N-1}t_N$, and $PC_N^2 \geq t_Nt_1$. By multiplication,

$$PC_1^2 \cdot PC_2^2 \cdots PC_{N-1}^2 \cdot PC_N^2 \;\geq\; t_1^2 \cdot t_2^2 \cdots t_{N-1}^2 \cdot t_n^2 .$$

As a result, $t_1t_2 \cdots t_N \leq PC_1PC_2 \cdots PC_N$.

We now give readers' solutions to some problems from the 15th Korean Mathematical Olympiad given at [2005 : 288–289].

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x-y) = f(x) + xy + f(y)$ for every $x \in \mathbb{R}$ and every $y \in \{f(x) \mid x \in \mathbb{R}\}$, where $\mathbb{R}$ is the set of all real numbers.

Solution by Michel Bataille, Rouen, France.

The functions $\theta : x \mapsto 0$ and $\phi : x \mapsto -x^2/2$ are the solutions for f.

It is readily checked that θ and ϕ are solutions. We show that there is no other solution.

Let f be a function such that

$$f(x-y) \;=\; f(x) + xy + f(y)\,, \tag{1}$$

for every $x \in \mathbb{R}$ and $y \in \{f(x) \mid x \in \mathbb{R}\}$. Let $a = f(0)$. Taking $x = 0$ in (1), we get

$$f(-y) \;=\; a + f(y)\,. \tag{2}$$

Next we take $x = y$ in (1) to get

$$f(y) \;=\; \tfrac{1}{2}(a - y^2)\,. \tag{3}$$

Equations (2) and (3) hold whenever $y \in f(\mathbb{R}) = \{f(x) \mid x \in \mathbb{R}\}$.

For $x \in \mathbb{R}$ and $y \in f(\mathbb{R})$, we have $xy - f(x-y) = -f(y) - f(x)$ (from (1)), and therefore,

$$f\big(xy - f(x-y)\big) \;=\; f\big(-f(y) - f(x)\big)\,. \tag{4}$$

Using (1) and (3), the left side L of (4) may be rewritten as

$$\begin{aligned} L &= f(xy) + xyf(x-y) + \tfrac{1}{2}a - \tfrac{1}{2}(f(x-y))^2 \\ &= f(xy) + \tfrac{1}{2}a + \tfrac{1}{2}f(x-y)\big(2xy - f(x-y)\big) \\ &= f(xy) + \tfrac{1}{2}a + \tfrac{1}{2}\big(xy + f(x) + f(y)\big)\big(xy - f(x) - f(y)\big) \\ &= f(xy) + \tfrac{1}{2}a + \tfrac{1}{2}x^2y^2 - \tfrac{1}{2}\big(f(x)\big)^2 - \tfrac{1}{2}\big(f(y)\big)^2 - f(x)f(y)\,. \end{aligned}$$

Using (1), (2), and (3), the right side R of (4) may be rewritten as

$$\begin{aligned} R &= f\big(-f(y)\big) - f(x)f(y) + f\big(f(x)\big) \\ &= a + f\big(f(y)\big) - f(x)f(y) + f\big(f(x)\big) \\ &= a + \tfrac{1}{2}a - \tfrac{1}{2}\big(f(y)\big)^2 - f(x)f(y) + \tfrac{1}{2}a - \tfrac{1}{2}\big(f(x)\big)^2 \\ &= 2a - \tfrac{1}{2}(f(y))^2 - f(x)f(y) - \tfrac{1}{2}(f(x))^2\,. \end{aligned}$$

From $L = R$, we deduce that $f(xy) + \frac{1}{2}x^2y^2 = \frac{3}{2}a$. With $x = 0$, we obtain $a = \frac{3}{2}a$; hence, $a = 0$. Thus, $f(xy) = -\frac{1}{2}x^2y^2$. If $f \neq \theta$, there exists $b \neq 0$ with $b \in f(\mathbb{R})$, and then, for all $r \in \mathbb{R}$,

$$f(r) \;=\; f\left(b \cdot \frac{r}{b}\right) \;=\; -\frac{1}{2} \cdot b^2 \cdot \frac{r^2}{b^2} \;=\; -\frac{1}{2} \cdot r^2\,.$$

Therefore, $f = \phi$, and the proof is complete.

3. The following facts are known in a mathematics contest:

(a) The number of problems tested was $n \geq 4$.

(b) Each problem was solved by exactly four contestants.

(c) For each pair of problems, there is exactly one contestant who solved both problems.

Assuming the number of contestants is greater than or equal to $4n$, find the minimum value of n for which there always exists a contestant who solved all the problems.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The minimum value of n is $n = 14$.

We number the problems $1, 2, \ldots, n$. Now, consider the graph G whose vertices are the students, two being joined by an edge if and only if they solved a common problem. From (c), our graph is simple (no multiple edges), and from (b), to each problem corresponds a complete subgraph K_4 with four vertices. We colour each of these associated subgraphs with colours $c_1, \ldots, c_n$ (the subscript is the same for the problem and the colour). From (c), these K_4 subgraphs are pairwise disjoint in terms of edges, and any two have exactly one vertex in common.

We claim that if there exists a vertex, say A, which belongs to at least five of these K_4, then this vertex is common to all the K_4 (which means that this student solved all the problems). Indeed, if one of the K_4 does not have vertex A, then it has to share a different vertex with each of the five K_4 which do have A as vertex. But five vertices is a bit too much for a K_4; hence, we have a contradiction.

Now assume that $n \geq 14$. Let us consider an arbitrary K_4, say K. It has to share a vertex with at least 13 other K_4. Thus, from the pigeon-hole principle, K shares the same vertex at least four times. This leads to a set of five K_4 which have a common vertex. From above, this ensures that a student solved all the problems.

To complete the solution, it remains to find a distribution of 52 students and 13 problems satisfying the conditions in the problem, but such that no student solved all the problems. (Note that such a configuration can also be used for $n \leq 13$.) We will give a configuration with 13 students who solved problems. To get 52 students, just add 39 students who solved nothing. Rows are problems and columns are students. A "$\circ$" means that the problem has been solved by the student.

○	○	○	○									
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			○			○		○		○		

4. For $n \geq 3$, let $S = a_1 + a_2 + \cdots + a_n$ and $T = b_1 b_2 \cdots b_n$ for positive real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, where the numbers b_i are pairwise distinct.

(a) Find the number of distinct real zeroes of the polynomial

$$f(x) \;=\; (x-b_1)(x-b_2)\cdots(x-b_n)\sum_{j=1}^{n}\frac{a_j}{x-b_j}\,.$$

(b) Prove the inequality

$$\frac{1}{n-1}\sum_{j=1}^{n}\left(1-\frac{a_j}{S}\right)b_j \;>\; \left(\frac{T}{S}\sum_{j=1}^{n}\frac{a_j}{b_j}\right)^{\frac{1}{n-1}}\,.$$

Solution by Michel Bataille, Rouen, France, modified by the editor.

(a) Without loss of generality, suppose that $b_1 < b_2 < \cdots < b_n$. Let

$$R(x) \;=\; \sum_{j=1}^{n}\frac{a_j}{x-b_j} \;=\; \frac{f(x)}{(x-b_1)(x-b_2)\cdots(x-b_n)}\,. \tag{1}$$

For each $j \in \{1, 2, \ldots, n\}$, the function $x \mapsto a_j/(x-b_j)$ is strictly decreasing on the intervals $(-\infty, b_j)$ and (b_j, ∞). Therefore, R is strictly decreasing on each of the intervals $(-\infty, b_1)$, (b_1, b_2), $\ldots$, (b_{n-1}, b_n), and (b_n, ∞). Furthermore, for each $j \in \{1, 2, \ldots, n\}$, since

$$\lim_{x\to b_j^-}\frac{a_j}{x-b_j} = -\infty \quad\text{and}\quad \lim_{x\to b_j^+}\frac{a_j}{x-b_j} = \infty\,,$$

we have $\lim\limits_{x\to b_j^-} R(x) = -\infty$ and $\lim\limits_{x\to b_j^+} R(x) = \infty$. Therefore, R vanishes exactly once on each interval (b_j, b_{j+1}). Since $R(x) < 0$ for $x \in (-\infty, b_1)$ and $R(x) > 0$ for $x \in (b_n, \infty)$, it follows that R has exactly $n-1$ real zeroes (and these are distinct).

From (1), we see that

$$f(x) \;=\; \sum_{j=1}^{n} a_j p_j(x)\,, \tag{2}$$

where $p_j(x) = \prod\limits_{i\neq j}(x-b_i)$. Thus, $f(x)$ is a polynomial of degree $n-1$ and has at most $n-1$ real zeroes. Each of the $n-1$ distinct real zeroes of R is also a zero of $f(x)$ (as we see from (1)). We conclude that $f(x)$ has exactly $n-1$ distinct real zeroes.

(b) Let $r_1, r_2, \ldots, r_{n-1}$ be the zeroes of $f(x)$. From (2), we see that the coefficient of x^{n-1} in $f(x)$ is S. Therefore,

$$f(x) \;=\; S(x-r_1)(x-r_2)\cdots(x-r_{n-1})\,. \tag{3}$$

Comparing the coefficients of x^{n-2} in (2) and (3), we obtain

$$-S(r_1 + r_2 + \cdots + r_{n-1}) \;=\; \sum_{j=1}^{n} a_j \Bigl(-\sum_{i\neq j} b_i\Bigr)\,,$$

and hence,

$$\begin{aligned} r_1 + r_2 + \cdots + r_{n-1} &= \frac{1}{S}\sum_{j=1}^{n} a_j \sum_{i\neq j} b_i \;=\; \frac{1}{S}\sum_{i=1}^{n} b_i \sum_{j\neq i} a_j \\ &= \sum_{i=1}^{n} b_i \left(1 - \frac{a_i}{S}\right)\,. \end{aligned}$$

Applying the AM–GM Inequality, we get

$$\begin{aligned} \frac{1}{n-1}\sum_{i=1}^{n} b_i \left(1 - \frac{a_i}{S}\right) &= \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1} \\ &> (r_1 r_2 \cdots r_{n-1})^{\frac{1}{n-1}}\,. \end{aligned} \tag{4}$$

(The inequality is strict because the r_j are not all equal.)

Setting $x = 0$ in (2) and (3) gives

$$\sum_{j=1}^{n} a_j \prod_{i\neq j}(-b_i) \;=\; S(-1)^{n-1} r_1 r_2 \cdots r_{n-1}\,;$$

that is, $r_1 r_2 \cdots r_{n-1} = \frac{T}{S}\sum\limits_{j=1}^{n} \frac{a_j}{b_j}$. This, together with (4), gives the desired inequality.

6. Let p_n be the n^{th} prime counting from the smallest prime 2 in increasing order. For example, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$.

(a) For a given $n \geq 10$, let r be the smallest integer satisfying

$$2 \;\leq\; r \;\leq\; n-2\,, \quad n - r + 1 \;<\; p_r\,,$$

and define $N_s = (s p_1 p_2 \cdots p_{r-1}) - 1$ for $s = 1, 2, \ldots, p_r$. Prove that there exists j, $1 \leq j \leq p_r$, such that none of $p_1, p_2, \ldots, p_n$ divides N_j.

(b) Using the result of (a), find all positive integers m for which

$$p_{m+1}^2 \;<\; p_1 p_2 \cdots p_m\,.$$

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

The inequality in (b) is known as Bonse's Inequality and is valid for $m \geq 4$. A complete proof, with the same reasoning as requested in the problem, can be found in J. Roberts, *Elementary Number Theory, a Problem Oriented Approach*, MIT Press, pp. 147 and 195–197.

Next we turn to the solution file for the 2002 Yugoslav Mathematical Olympiad given at [2005 : 373].

1. Let a, b, and c be positive numbers, and let n and k be positive integers. Prove the inequality:

$$\frac{a^{n+k}}{b^n} + \frac{b^{n+k}}{c^n} + \frac{c^{n+k}}{a^n} \;\geq\; a^k + b^k + c^k\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We present Bataille's write-up.

The proposed inequality is equivalent to

$$c^n a^{2n+k} + a^n b^{2n+k} + b^n c^{2n+k} \;\geq\; b^n c^n a^{n+k} + c^n a^n b^{n+k} + a^n b^n c^{n+k}\,,$$

or

$$c^n a^{n+k}(a^n - b^n) + a^n b^{n+k}(b^n - c^n) + b^n c^{n+k}(c^n - a^n) \;\geq\; 0\,.$$

Since the left side L remains the same after a circular permutation of a, b, c, we may consider only the cases $a \geq b \geq c$ and $a \geq c \geq b$.

In the former case, we use $c^n - a^n = (c^n - b^n) + (b^n - a^n)$ to get

$$L \;=\; c^n(a^n - b^n)(a^{n+k} - b^n c^k) + b^n(b^n - c^n)(a^n b^k - c^{n+k})\,.$$

The first term on the right side is non-negative because $a^n \geq b^n$ and $a^k \geq c^k$; the second term is non-negative because $b^n \geq c^n$, $a^n \geq c^n$, and $b^k \geq c^k$. Thus, $L \geq 0$.

In the latter case, similarly, we use $a^n - b^n = (a^n - c^n) + (c^n - b^n)$ to get

$$L \;=\; c^n(a^n - c^n)(a^{n+k} - b^n c^k) + a^n(c^n - b^n)(c^n a^k - b^{n+k})\,,$$

which is again the sum of two non-negative terms. Thus, $L \geq 0$ in all cases, and the result follows.

2. Let (f_n) be a sequence defined by: $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$, for $n \geq 1$. Prove that the area of the triangle with side lengths $\sqrt{f_{2n+1}}$, $\sqrt{f_{2n+2}}$, and $\sqrt{f_{2n+3}}$ equals $\frac{1}{2}$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

The statement as given is false. We will prove the correct statement: that the area A of the described triangle is $\frac{1}{2}\sqrt{f_{2n+1}f_{2n+2}}$.

For notational convenience, set $a = \sqrt{f_{2n+1}}$, $b = \sqrt{f_{2n+2}}$, and $c = \sqrt{f_{2n+3}}$. Then we have $a^2 + b^2 = c^2$. Denote the semiperimeter of the triangle by $s = \frac{1}{2}(a+b+c)$. Then

$$\begin{aligned} s(s-c) &= \tfrac{1}{4}(a+b+c)(a+b-c) \\ &= \tfrac{1}{4}\big((a+b)^2 - c^2\big) \;=\; \tfrac{1}{2}ab \\ \text{and}\quad (s-a)(s-b) &= \tfrac{1}{4}(b+c-a)(c+a-b) \\ &= \tfrac{1}{4}(c^2-(b-a)^2) \;=\; \tfrac{1}{2}ab\,. \end{aligned}$$

Hence, by Heron's Formula, we have

$$A \;=\; \sqrt{s(s-a)(s-b)(s-c)} \;=\; \tfrac{1}{2}ab \;=\; \tfrac{1}{2}\sqrt{f_{2n+1}f_{2n+2}}\,.$$

4. Does there exist a positive integer k such that the digits 3, 4, 5, and 6 do not appear in the decimal representation of the number $2002! \cdot k$?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Krimker.

This k exists for any $n \in \mathbb{N}$ in place of 2002.

Let $n! = 2^{\alpha_2}3^{\alpha_3}5^{\alpha_5}\cdots p^{\alpha_p}$ (with $\alpha_i \geq 0$) be the factorization of $n!$ into primes, and let $m = \frac{n!}{2^{\alpha_2}5^{\alpha_5}}$. Since $(m,10)=1$, applying the Euler-Fermat Theorem, we have $m \mid 10^{\phi(m)}-1$. Let $k = 5^{\alpha_2-\alpha_5}\frac{10^{\phi(m)}-1}{m}$. Then k is a positive integer (note that $\alpha_2 > \alpha_5$), and

$$n!\cdot k \;=\; 2^{\alpha_2}5^{\alpha_5}m5^{\alpha_2-\alpha_5}\frac{10^{\phi(m)}-1}{m} \;=\; 10^{\alpha_2}(10^{\phi(m)}-1)\,,$$

which has only the digits 9 and 0 in its decimal representation.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations.

In Memoriam:
Robert Barrington Leigh

We regret to inform our readership of the passing of a young Canadian mathematician, Robert Barrington Leigh. Robert was an undergraduate student at the University of Toronto, where he had three top-15 finishes in the Putnam competition. This followed a distinguished high school career: he received a bronze medal at the IMO in 2002 and 2003.

A full obituary appears in the November 2006 issue of the *Notes of the Canadian Mathematical Society*.

BOOK REVIEWS

John Grant McLoughlin

Winning Ways for Your Mathematical Plays, Second Edition
By E.R. Berlekamp, J.H. Conway, and R.K. Guy, published by A.K. Peters Ltd.
Volume 1, 2001, ISBN 1-56881-130-6, softcover, 296 pages, US$49.95;
Volume 2, 2003, ISBN 1-56881-142-X, softcover, 212 pages, US$39.00;
Volume 3, 2003, ISBN 1-56881-143-8, softcover, 275 pages, US$49.00;
Volume 4, 2004, ISBN 1-56881-144-6, softcover, 224 pages, US$39.00.
Reviewed by **Amar Sodhi**, *Sir Wilfred Grenfell College, Memorial University of Newfoundland, Corner Brook, NL.*

"Would you care for a stroll to the Nim club?" the Red Queen asked Alice. Tweedledee and Tweedledum appear early, but instead of being through the looking glass, one finds oneself in a wonderland of combinatorial game theory. Berlekamp, Conway, and Guy write in a fast-paced, irreverent style, but one quickly realises that the four-volume set of *Winning Ways For Your Mathematical Plays* is a book written by serious mathematicians for those who are seriously interested in mathematics.

Each volume is named after a suit in a standard deck of cards, but cards, dice, or any other randomiser play no part in this work. The two-person games considered in the first three volumes are geared towards helping the Go player rather than the Bridge or Backgammon player in achieving winning ways. The fourth volume deals mainly with puzzles.

In the first volume, aptly named *Spade-Work*, the reader is introduced to the game of Hackenbush. In Hackenbush, each player takes turns removing edges from a graph following a well-defined set of rules. A player who cannot move loses the game. Hackenbush and the other games considered here have similarities to the archetypical combinatorial game, Nim. So it is not surprising that nim-heaps, nim-sequences and an arithmetic (fittingly called "Nimber theory") play a role in the objective of this volume, which is determining the advantage, in terms of number-of-moves, the starting player has in any particular game. The number-of-moves is not necessarily a number, and this is where the mathematical fun begins. Pencil and paper are needed, more to gain an understanding of the terminology and methodology used to analyse a game than in the actual playing of a game.

The rules of the game are relaxed somewhat in Volume 2, entitled *Change of Heart*. Misère versions of the games played in Volume 1 and "loopy" games are now considered. As in Volume 1, many interesting games are introduced, but it is still the mathematical development rather than the actual games that holds the fascination. It is not until Volume 3, entitled *Games in Clubs*, that the playing of a game takes centre stage. Here the reader will find detailed analyses of popular childhood games such as Dots and Boxes and Fox and Geese, as well as Conway's game of Sprouts. The Go player, however, is left to her own devices.

For those readers who prefer puzzles to games, Volume 4 will hold the most interest. This last volume, *Solitaire Diamonds*, consists of a chapter on peg solitaire, a chapter devoted to puzzles (including a section on Rubik's Cube), and a chapter on Conway's game of Life.

Winning Ways for Your Mathematical Plays is stimulating and entertaining reading for those who want to enjoy the playground of mathematics at a higher level than mere diversion. Avid readers of Martin Gardner (to whom *Winning Ways* is dedicated) will gain further insights about familiar topics, but this work is not the best choice for the reader who uses mathematical musings as a form of relaxation. On the other hand, a graduate or undergraduate student who wants to learn quickly the rudiments of combinatorial game theory may find the rather more formal *On Number and Games* by John Conway more to his liking.

The appeal of *Winning Ways for Your Mathematical Plays* is that high school students who are suitably inclined, mathematicians (amateur or otherwise), or computer scientists will certainly be ready and motivated to immerse themselves in the field of combinatorial game theory upon actively participating in this work.

USA and International Mathematical Olympiads 2004
Edited by Titu Andreescu, Zuming Feng, and Po-Shen Loh, published by the Mathematical Association of America, 2005.
ISBN 0-88385-819-3, paperbound, 100+viii pages, US$31.95.
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB.*

This annual publication has a core featuring problems, hints, and formal solutions to problems from three events: the USA Mathematical Olympiad (USAMO), the International Mathematical Olympiad (IMO), and the IMO Team Selection Test. The IMO from which the problems in this volume are taken is the 45^{th} IMO, held in Athens, Greece. The core is supplemented by other sections, including *Problem Credits* and an appendix containing IMO and USAMO results from each year from 2000 to 2004 inclusive, in addition to cumulative IMO results for this period.

Once I got over my surprise at the increase of over 25% in the price of the book compared to last year, three features of the book struck me as particularly praiseworthy. First, there is a rich variety of solutions, often three or more for one problem. Second, the 81 titles listed as *Further Reading* offer a strong base for anyone interested in advanced contest problems and related mathematical topics. Finally, the *Glossary* covers a range of theorems, inequalities, definitions, and trigonometric identities, explaining all terms that appear in boldface in either the *Hints* or *Formal Solutions* sections of the book. Many problems books could be improved by the inclusion of such a glossary.

Hexagons and Inequalities

Yakub N. Aliyev

The popular author Lewis Carroll is also famous for his problems. Problem 71 in [2] pages 18 and 108–109 states,

> In a given Triangle place a Hexagon having its opposite sides equal and parallel, and three of them lying along the sides of the Triangle, and such that its diagonals intersect in a given Point.

The problem has been generalized for the case when the given point is not inside the triangle [5]. We will look at another way to modify the problem, where the main diagonals of our hexagons will each be parallel to one of the sides.

Problem 1. Let M be a given point in the plane of triangle ABC. Construct the lines A_1B_2, B_1C_2, and C_1A_2 meeting at M such that for $i = 1$ and $i = 2$, A_i lies on BC, B_i lies on CA, C_i lies on AB, and moreover, $A_1B_2 \parallel A_2B_1$, $B_1C_2 \parallel B_2C_1$, $C_1A_2 \parallel C_2A_1$ (as in Figure 1).

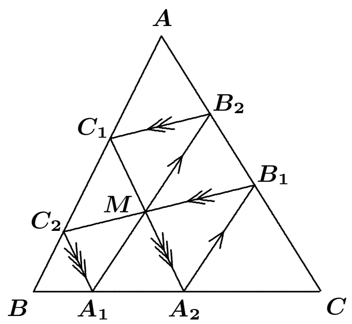

Figure 1: Problem 1.

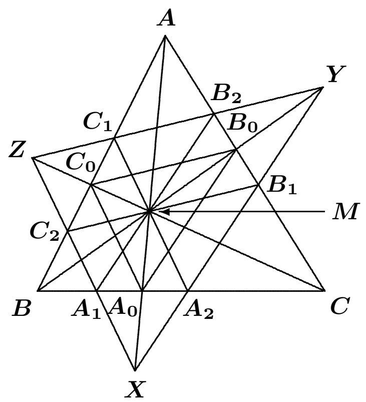

Figure 2: Solution to Problem 1.

Solution. *Analysis.* Let lines C_2A_1, A_2B_1, and B_2C_1 intersect at X, Y, and Z as in Figure 2, and let A_0, B_0, and C_0 be the mid-points of A_1A_2, B_1B_2, and C_1C_2, respectively. Because the lines joining these mid-points are the mid-lines of trapezoids, we therefore have

$$A_0B_0 \parallel A_1B_2 \parallel A_2B_1\,, \quad B_0C_0 \parallel B_1C_2 \parallel B_2C_1\,, \quad C_0A_0 \parallel C_1A_2 \parallel C_2A_1\,.$$

Thus, triangles A_1ZB_2 and $A_0C_0B_0$ have corresponding sides parallel; by Desargues' Theorem, A_1A_0, B_2B_0, and ZC_0 are concurrent. In other words, C lies on ZC_0. But, M also lies on ZC_0 (since ZC_2MC_1 is a parallelogram).

This means that CM passes through C_0. Similarly, AM passes through A_0 and BM passes through B_0.

Construction. We are to construct the points A_i, B_i, and C_i. Extend the lines AM, BM, and CM to intersect the sides of $\triangle ABC$ at A_0, B_0, and C_0, respectively. Next construct the parallel to A_0C_0 through M, which intersects BA and BC at C_1 and A_2, respectively. Analogously, draw the parallel through M to B_0A_0 (and to B_0C_0) to find A_1 and B_2 (and B_1 and C_2). Desargues' Theorem then implies that the joins of appropriate pairs of these points form parallel lines as desired.

Although the figures and discussion apply to the case where M is inside the triangle, the construction for other locations of M will be essentially the same.

Problem 2. Construct the point M for which the hexagon $A_1A_2B_1B_2C_1C_2$ of Problem 1 is inscribed in a circle.

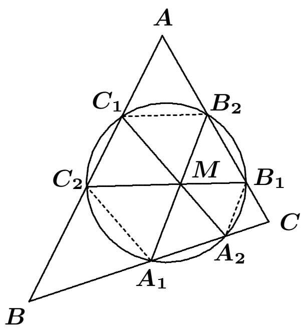

Figure 3: Problem 2.

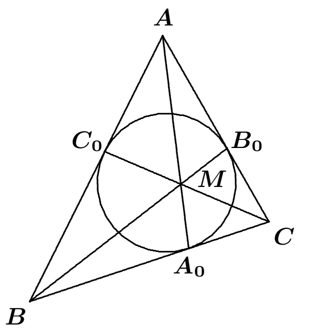

Figure 4: Solution to Problem 2.

Solution. *Analysis*. We assume that $A_1A_2B_1B_2C_1C_2$ is cyclic (Figure 3.) From $C_1A_2 \parallel C_2A_1$ we deduce that line segments $C_1C_2 = A_1A_2$. Since the secants from B satisfy $BC_2 \cdot BC_1 = BA_1 \cdot BA_2$, these two equations give us $BC_2 = BA_1$. Analogously, $AC_1 = AB_2$ and $CB_1 = CA_2$. Recalling that A_0, B_0, and C_0 are the mid-points of A_1A_2, B_1B_2, and C_1C_2, respectively, we finally obtain

$$BC_0 = BA_0\,, \quad AC_0 = AB_0\,, \quad CB_0 = CA_0\,.$$

It is then easy to prove that A_0, B_0, and C_0 are the points where the incircle of $\triangle ABC$ touches the sides.

Construction. Construct the incircle of triangle ABC and label the points A_0, B_0, and C_0 where the incircle touches the sides BC, CA, and AB, respectively. (See Figure 4.) Next, draw the lines AA_0, BB_0, CC_0, which intersect at the desired point M. The point M is, of course, the *Gergonne point* of $\triangle ABC$ (see [3], page 13).

Problem 3. With points X, Y, and Z defined as in the solution to Problem 1, construct the point M for which the hexagon $AYCXBZ$ is inscribed in a circle.

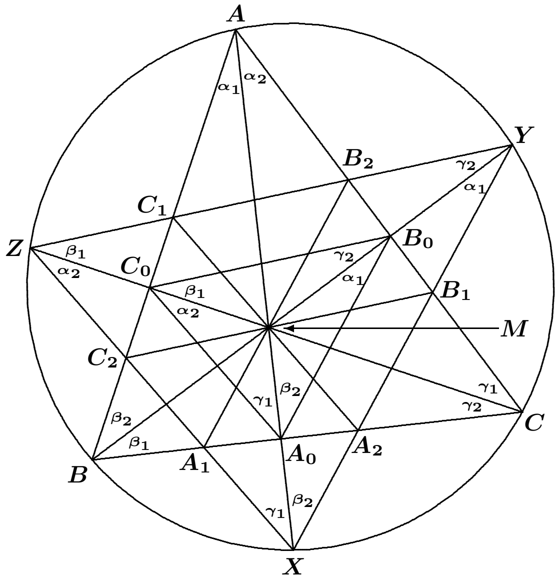

Figure 5: Solution to Problem 3.

Solution. *Analysis*. We suppose that $AYCXBZ$ is cyclic. Then, for the angles, we have (as in Figure 5)

$$\begin{array}{lclclclcl} \angle XAB &=& \angle XYB &=& \alpha_1\,, & \angle CAX &=& \angle CZX &= \alpha_2\,,\\ \angle YBC &=& \angle YZC &=& \beta_1\,, & \angle ABY &=& \angle AXY &= \beta_2\,,\\ \angle ZCA &=& \angle ZXA &=& \gamma_1\,, & \angle BCZ &=& \angle BYZ &= \gamma_2\,. \end{array}$$

Using the given parallel lines, we also have

$$\begin{array}{lclclclclcl} \angle A_0B_0B &=& \alpha_1\,, & \angle B_0C_0C &=& \beta_1\,, & \angle C_0A_0A &=& \gamma_1\,,\\ \angle CC_0A_0 &=& \alpha_2\,, & \angle AA_0B_0 &=& \beta_2\,, & \angle BB_0C_0 &=& \gamma_2\,. \end{array}$$

From the equality $\gamma_1 = \angle C_0A_0A = \angle ZCA = \angle C_0CA$, we deduce that the points A, C, A_0, and C_0 are concyclic. Thus, $\gamma_2 = \alpha_1$. Similarly, $\beta_1 = \alpha_2$ and $\beta_2 = \gamma_1$. But, we also know that

$$\angle ABC + \angle BCA + \angle CAB \;=\; \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 \;=\; 180^\circ\,.$$

Then $\alpha_1 + \beta_1 + \beta_2 = \gamma_2 + \alpha_2 + \gamma_1 = 90^\circ$. That is, considering $\triangle AA_0C$, we see that $\angle AA_0C = \alpha_1 + \beta_1 + \beta_2 = 90^\circ$. Consequently, the cevian AA_0 is perpendicular to the base BC and similarly, $BB_0 \perp AC$ and $CC_0 \perp AB$. It follows that M is the orthocentre of $\triangle ABC$.

Construction. Draw the altitudes AA_0, BB_0, and CC_0. Their intersection point is the desired point M. For the proof that M satisfies all the requirements, simply read the preceding analysis in reverse.

Remarks. We could also ask for our hexagons to be circumscribed about a circle. Other properties of hexagons with sides parallel to diagonals were investigated by S.I. Zetel in Problems 185–187 in [6], Ch. IV, pages 118–120.

We turn now to investigating the areas of triangles associated with the hexagons of Problem 1—hexagons that are inscribed in triangles and have diagonals that intersect in a point and are parallel to the nontriangular sides. Areas will be denoted by square brackets. We are interested in relationships among the following nine areas (as displayed in Figure 6):

$$\begin{array}{rclrclrcl} T_1 &=& [MC_1B_2]\,, & T_2 &=& [MA_1C_2]\,, & T_3 &=& [MB_1A_2]\,, \\ S_1 &=& [MA_1A_2]\,, & S_2 &=& [MB_1B_2]\,, & S_3 &=& [MC_1C_2]\,, \\ P_1 &=& [AB_2C_1]\,, & P_2 &=& [BC_2A_1]\,, & P_3 &=& [CA_2B_1]\,. \end{array}$$

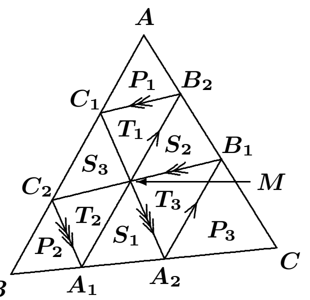

Figure 6: Definition of T_i, S_i, and P_i.

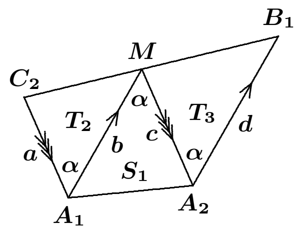

Figure 7: Lemma 1.

Lemma 1. $T_iT_j = S_k^2$, for $\{i, j, k\} = \{1, 2, 3\}$.

Proof: From the assumptions $A_1C_2 \parallel A_2M$ and $A_1M \parallel A_2B_1$, we get $\angle C_2A_1M = \angle A_2MA_1 = \angle MA_2B_1$, whose measure has been denoted by α in Figure 7. Let $a = A_1C_2$, $b = A_1M$, $c = A_2M$, $d = A_2B_1$, as in the figure. Then, by finding the areas T_2, T_3, and S_1 and substituting them into the proposed equation $T_2T_3 = S_1^2$, we obtain

$$\tfrac{1}{2}\,ab\sin\alpha \cdot \tfrac{1}{2}\,cd\sin\alpha \;=\; \tfrac{1}{4}\,b^2c^2\sin^2\alpha \qquad \text{if and only if} \qquad ad \;=\; bc\,,$$

which is true because of the similarity of triangles A_1C_2M and A_2MB_1. Similarly, we get $T_1T_3 = S_2^2$ and $T_1T_2 = S_3^2$. ■

Lemma 2. For $\{i, j, k\} = \{1, 2, 3\}$, we have $P_i = \dfrac{T_i^2}{S_j + S_k - T_i}$.

Proof: Because $A_1C_2 \parallel A_2C_1$, we can let h be the common height of triangles MA_1C_2, MC_1C_2, and MA_1A_2. Then, $S_1 = \frac{1}{2} \cdot h \cdot MA_2$, $S_3 = \frac{1}{2} \cdot h \cdot C_1M$, and $T_2 = \frac{1}{2} \cdot h \cdot C_2A_1$. Since $\triangle BC_2A_1$ is similar to $\triangle BC_1A_2$, we see that

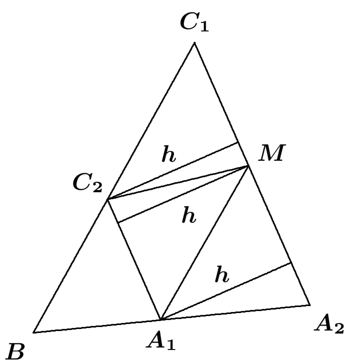

Figure 8: Lemma 2.

$$\frac{[BC_2A_1]}{[BC_1A_2]} = \frac{(C_2A_1)^2}{(C_1A_2)^2} = \left(\frac{C_2A_1}{C_1M + MA_2}\right)^2 = \left(\frac{T_2}{S_1 + S_3}\right)^2 .$$

By the definition of P_2, we therefore have

$$\frac{P_2}{P_2 + T_2 + S_1 + S_3} = \left(\frac{T_2}{S_1 + S_3}\right)^2 ;$$

whence, $P_2 = \dfrac{T_2^2}{S_1 + S_3 - T_2}$, as claimed. ■

From these two lemmas we will derive a pair of inequalities.

Inequality 1. $T_1 + T_2 + T_3 \geq S_1 + S_2 + S_3$.

Proof: This is just the AM–GM Inequality combined with Lemma 1:

$$\begin{aligned} T_1 + T_2 + T_3 &= \tfrac{1}{2}(T_2 + T_3) + \tfrac{1}{2}(T_1 + T_3) + \tfrac{1}{2}(T_1 + T_2) \\ &\geq \sqrt{T_2T_3} + \sqrt{T_1T_3} + \sqrt{T_1T_2} = \sqrt{S_1^2} + \sqrt{S_2^2} + \sqrt{S_3^2} . \end{aligned}$$

Inequality 2. $T_1 + T_2 + T_3 \leq \frac{1}{3}[ABC]$.

Proof: We want to show that

$$3(T_1 + T_2 + T_3) \leq T_1 + T_2 + T_3 + S_1 + S_2 + S_3 + P_1 + P_2 + P_3 .$$

By Lemma 2 this is equivalent to showing that

$$2(T_1+T_2+T_3) \leq \frac{T_1^2}{S_2 + S_3 - T_1} + \frac{T_2^2}{S_1 + S_3 - T_2} + \frac{T_3^2}{S_1 + S_2 - T_3} + S_1 + S_2 + S_3 .$$

Because all three denominators for the P_i are strictly positive, we can apply a variant of the Cauchy-Schwarz Inequality, namely

$$\begin{aligned} &\frac{(T_1 + T_2 + T_3)^2}{2(S_1 + S_2 + S_3) - (T_1 + T_2 + T_3)} \\ &\qquad \leq \frac{T_1^2}{S_2 + S_3 - T_1} + \frac{T_2^2}{S_1 + S_3 - T_2} + \frac{T_3^2}{S_1 + S_2 - T_3} , \end{aligned}$$

and reduce the proof to showing that

$$2(T_1 + T_2 + T_3) - (S_1 + S_2 + S_3) \leq \frac{(T_1 + T_2 + T_3)^2}{2(S_1 + S_2 + S_3) - (T_1 + T_2 + T_3)} .$$

To this end, we set $a = S_1 + S_2 + S_3$ and $b = T_1 + T_2 + T_3$, and we prove that

$$2b - a \leq \frac{b^2}{2a - b} .$$

This last task is equivalent to showing that

$$0 \leq 2a^2 - 5ab + 3b^2 = (3b - 2a)(b - a) .$$

Since $b \geq a$ (which is Inequality 1), the result is now clear.

Further inequalities of this type are to be found in [4] and [1].

Acknowledgements. The author would like to express very sincere thanks to the referee, who provided an extremely detailed critique of a first version of the present paper. The many helpful suggestions have led to a much clearer presentation.

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 May 2007**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

3176. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $A_1A_2\ldots A_{4n}$ be a planar polygon with perimeter $4n$, where n is a positive integer. Prove that this polygon can be covered by a circle with radius n.

3177. *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

Let P be any interior point of triangle $A_1A_2A_3$. Let T_1, T_2, T_3 denote the projections of P onto the sides A_2A_3, A_3A_1, A_1A_2, respectively, and let H_1, H_2, H_3 denote the orthocentres of triangles $A_1T_2T_3$, $A_2T_3T_1$, $A_3T_1T_2$, respectively. Prove that the lines H_1T_1, H_2T_2, H_3T_3 are concurrent.

3178. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all integers x, y, z such that $4^x+4^y+4^z$ is a perfect square.

3179. *Proposed by Michel Bataille, Rouen, France.*

A transversal of $\triangle ABC$ makes angles α, β, and γ with the lines BC, CA, and AB, respectively. Express the minimum and maximum values of

$$(\cos\alpha\cos\beta\cos\gamma)^2+(\sin\alpha\sin\beta\sin\gamma)^2$$

as functions of $p=\cos A\cos B\cos C$.

3180. *Proposed by Michel Bataille, Rouen, France.*

Find all positive real numbers a such that

$$\left(\sqrt{a}+3\right)^{\frac{1}{5}}+\left(\sqrt{5}-2\right)^{\frac{1}{3}} \;=\; \left(\sqrt{a}-3\right)^{\frac{1}{5}}+\left(\sqrt{5}+2\right)^{\frac{1}{3}}\,.$$

3181. *Proposed by Roger Zarnowski, Angelo State University, San Angelo, TX, USA.*

Show that for every integer $n\geq 2$, the equation $x^n+x^{-n}=1+x$ has a root in the interval $\left(1,1+\dfrac{1}{n}\right)$.

3182. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $a = BC$, $b = CA$, and $c = AB$ in $\triangle ABC$. Prove that

$$\frac{bc}{b+c}\sin^2\left(\frac{A}{2}\right) + \frac{ca}{c+a}\sin^2\left(\frac{B}{2}\right) + \frac{ab}{a+b}\sin^2\left(\frac{C}{2}\right) \;\le\; \frac{a+b+c}{8}\,.$$

3183. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let ABC be a triangle with inradius r and circumradius R. If s is the semiperimeter of the triangle, prove that

$$\sqrt{3}\,s \;\le\; r + 4R\,.$$

3184. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

For any real number x, let (x) denote the *fractional part* of x; that is, $(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x. Given $n \in \mathbb{Z}$, find all solutions of the equation

$$(x^2) - n(x) \;=\; 0\,.$$

3185. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let n be an integer, $n \ge 2$, and let a, b, and c be positive real numbers satisfying $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{a}{1-a^n} + \frac{b}{1-b^n} + \frac{c}{1-c^n} \;\ge\; \frac{(n+1)^{1+\frac{1}{n}}}{n}\,.$$

3186. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $f(x)$ be a function on an interval I which is convex for $x \ge a$ for some $a \in I$. Suppose that for all $x_1, x_2, \ldots, x_n \in I$ which satisfy $x_1 + x_2 + \cdots + x_n = na$, the following inequality holds:

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \;\ge\; f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\,.$$

Prove that this same inequality holds for all $x_1, x_2, \ldots, x_n \in I$ such that $x_1 + x_2 + \cdots + x_n \ge na$.

3187. *Proposed by Michel Bataille, Rouen, France.*

Let $ABCD$ be a planar quadrilateral which is not a parallelogram. Let C' and D' be the orthogonal projections onto the line AB of the points C and D, respectively. The perpendiculars from C to AD and from D to BC meet at P; the perpendiculars from C' to AD and from D' to BC meet at Q. Show that PQ is perpendicular to the line through the mid-points of AC and BD.

3176. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $A_1A_2\ldots A_{4n}$ un polygone plan de périmètre $4n$, n un entier positif. Montrer que ce polygone peut être recouvert par un disque de rayon n.

3177. *Proposé par Mihály Bencze et Marian Dinca, Brasov, Roumanie.*

Soit P un point arbitraire du triangle $A_1A_2A_3$. Soit respectivement T_1, T_2 et T_3 les projections de P sur les côtés A_2A_3, A_3A_1 et A_1A_2, et soit respectivement H_1, H_2 et H_3 les orthocentres des triangles $A_1T_2T_3$, $A_2T_3T_1$ et $A_3T_1T_2$. Montrer que les droites H_1T_1, H_2T_2 et H_3T_3 sont concourantes.

3178. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Déterminer tous les entiers x, y et z tels que $4^x + 4^y + 4^z$ soit un carré parfait.

3179. *Proposé par Michel Bataille, Rouen, France.*

Une transversale du triangle ABC découpe respectivement les angles α, β et γ avec les droites BC, CA et AB. Exprimer les valeurs maximale et minimale de

$$(\cos\alpha\cos\beta\cos\gamma)^2 + (\sin\alpha\sin\beta\sin\gamma)^2$$

comme fonctions de $p = \cos A\cos B\cos C$.

3180. *Proposé par Michel Bataille, Rouen, France.*

Trouver tous les nombres réels positifs a tels que

$$\left(\sqrt{a}+3\right)^{\frac{1}{5}} + \left(\sqrt{5}-2\right)^{\frac{1}{3}} \;=\; \left(\sqrt{a}-3\right)^{\frac{1}{5}} + \left(\sqrt{5}+2\right)^{\frac{1}{3}}\,.$$

3181. *Proposé par Roger Zarnowski, Angelo State University, San Angelo, TX, USA.*

Montrer que pour tout entier $n \geq 2$, l'équation $x^n + x^{-n} = 1 + x$ possède une racine dans l'intervalle $\left(1, 1+\dfrac{1}{n}\right)$.

3182. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit $a = BC$, $b = CA$ et $c = AB$ dans le triangle ABC. Montrer que

$$\frac{bc}{b+c}\sin^2\left(\frac{A}{2}\right) + \frac{ca}{c+a}\sin^2\left(\frac{B}{2}\right) + \frac{ab}{a+b}\sin^2\left(\frac{C}{2}\right) \;\leq\; \frac{a+b+c}{8}\,.$$

3183. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit respectivement r et R le rayon du cercle inscrit et celui du cercle circonscrit du triangle ABC. Si s désigne son demi-périmètre, montrer que

$$\sqrt{3}\,s \;\leq\; r + 4R\,.$$

3184. *Proposé par Fabio Zucca, Politecnico di Milano, Milano, Italie.*

Pour tout nombre réel x, on note (x) la *partie fractionnaire* de x ; c'est-à-dire, $(x) = x - \lfloor x \rfloor$, où $\lfloor x \rfloor$ est le plus grand entier ne dépassant pas x. Etant donné $n \in \mathbb{Z}$, trouver toutes les solutions de l'équation

$$(x^2) - n(x) \;=\; 0\,.$$

3185. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit n un entier, $n \geq 2$, et soit a, b et c trois nombres réels positifs tels que $a^2 + b^2 + c^2 = 1$. Montrer que

$$\frac{a}{1-a^n} + \frac{b}{1-b^n} + \frac{c}{1-c^n} \;\geq\; \frac{(n+1)^{1+\frac{1}{n}}}{n}\,.$$

3186. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit $f(x)$ une fonction définie dans un intervalle I et convexe pour $x \geq a$ pour un $a \in I$. Supposons que pour tous les x_1, x_2, $\ldots$, $x_n \in I$ satisfaisant $x_1 + x_2 + \cdots + x_n = na$, on ait

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \;\geq\; f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\,.$$

Montrer que la même inégalité est valide pour tous les x_1, x_2, $\ldots$, $x_n \in I$ tels que $x_1 + x_2 + \cdots + x_n \geq na$.

3187. *Proposé par Michel Bataille, Rouen, France.*

Soit $ABCD$ un quadrilatère plan qui ne soit pas un parallélogramme. Soit C' et D' les projections orthogonales respectives des points C et D sur la droite AB. Les perpendiculaires de C sur AD et de D sur BC se coupent en P ; les perpendiculaires de C' sur AD et de D' sur BC se coupent en Q. Montrer que PQ est perpendiculaire à la droite passant par les milieux de AC et BD.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3072. [2005 : 398, 401] *Proposed by Mohammed Aassila, Strasbourg, France.*

Find the smallest constant k such that, for any positive real numbers a, b, c, we have

$$abc\left(a^{125}+b^{125}+c^{125}\right)^{16} \;\le\; k\left(a^{2003}+b^{2003}+c^{2003}\right)\,.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

By setting $a=b=c=1$, we see that $k \ge 3^{15}$. We will show that the inequality holds when $k=3^{15}$; that is,

$$abc(a^{125}+b^{125}+c^{125})^{16} \;\le\; 3^{15}(a^{2003}+b^{2003}+c^{2003})\,. \qquad (1)$$

It will then follow that 3^{15} is the minimum possible value of k.

Due to symmetry, we may assume without loss of generality that $a \ge b \ge c$. Then, applying Chebyshev's Inequality, we get

$$\begin{aligned}\frac{a^{2003}+b^{2003}+c^{2003}}{3} &= \frac{a^3\cdot a^{2000}+b^3\cdot b^{2000}+c^3\cdot c^{2000}}{3}\\ &\ge \left(\frac{a^3+b^3+c^3}{3}\right)\left(\frac{a^{2000}+b^{2000}+c^{2000}}{3}\right)\,.\end{aligned}$$

By the AM–GM Inequality, we have $a^3+b^3+c^3 \ge 3abc$, and by the Power Mean Inequality,

$$\frac{a^{2000}+b^{2000}+c^{2000}}{3} \;\ge\; \left(\frac{a^{125}+b^{125}+c^{125}}{3}\right)^{16}\,.$$

Combining our results, we get

$$\frac{a^{2003}+b^{2003}+c^{2003}}{3} \;\ge\; abc\left(\frac{a^{125}+b^{125}+c^{125}}{3}\right)^{16}\,,$$

from which (1) follows.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Several solvers gave generalizations. Janous used the same approach as in our featured solution to prove that if x_1, x_2, $\ldots$, x_n are positive real numbers, where $n \ge 2$, and if p and q are positive real numbers such that $q-1 \ge -n/p$, then the smallest value of the constant C for which the inequality

$$\left(\prod_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i^p\right)^q \;\leq\; C\left(\sum_{i=1}^{n} x_i^{pq+n}\right)$$

is valid in general, is $C = n^{q-1}$.

3073. [2005 : 335, 337] *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let x, y, z be positive real numbers. Prove that

$$\frac{1}{x+y+z+1} - \frac{1}{(x+1)(y+1)(z+1)} \;\leq\; \frac{1}{8},$$

and determine when there is equality.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $t = x+y+z+3$. Then $t > 3$ and, by the AM–GM Inequality, we have

$$(x+1)(y+1)(z+1) \;\leq\; \left(\frac{t}{3}\right)^3 \;=\; \frac{t^3}{27}.$$

Hence,

$$\begin{aligned} &\frac{1}{x+y+z+1} - \frac{1}{(x+1)(y+1)(z+1)} - \frac{1}{8} \\ &\quad\leq\; \frac{1}{t-2} - \frac{27}{t^3} - \frac{1}{8} \;=\; \frac{8(t^3-27t+54) - t^3(t-2)}{8t^3(t-2)} \\ &\quad=\; \frac{-t^4+10t^3-216t+432}{8t^3(t-2)} \;=\; \frac{-(t-6)^2(t^2+2t-12)}{8t^3(t-2)} \;\leq\; 0, \end{aligned}$$

since $t > 3$. Equality holds if and only if $t = 6$ and $x = y = z$; that is, if and only if $x = y = z = 1$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina, MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; TOM LEONG, Brooklyn, NY, USA; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; STAN WAGON, Macalester College, St. Paul, MN, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Comănești, Romania, and BOGDAN IONIȚĂ, Bucharest, Romania; and the proposer.

Leong remarked that the two-variable analogue of finding the maximum value of $\frac{1}{x+y+1} - \frac{1}{(x+1)(y+1)}$ for x, $y > 0$ has an interesting answer—the maximum occurs at $x = y = \frac{1}{2}(1+\sqrt{5})$, the golden ratio.

Janous showed that the given inequality also holds when x, y, $z > -1$ as long as $x+y+z \geq \sqrt{13}-4$. He also considered the more general problem of establishing inequalities of the type $F(x,y,z;a,b) \leq 1/c$, where

$$F(x,y,z;a,b) \;=\; \frac{1}{x+y+z+a} - \frac{1}{(x+b)(y+b)(z+b)},$$

for positive real numbers x, y, z, and constants a, b, c. In particular, he proved that

(i) $F(x,y,z;2,1) \le \frac{1}{2}(5\sqrt{5}-11)$ *with equality if and only if* $x=y=z=\frac{1}{2}(1+\sqrt{5})$.

(ii) $F(x,y,z;16,2) \le \frac{4}{125}$ *with equality if and only if* $x=y=z=3$.

(iii) $F(x,y,z;30,3) \le \frac{143}{2058} - \frac{31\sqrt{93}}{6174}$ *with equality if and only if* $x=y=z=\frac{\sqrt{93}-3}{2}$.

Wagon expressed the opinion that this kind of problem should not be published since "there are well-known algorithms, now included in modern software, that solve problems such as these". We invite our readers to express their opinions about whether ***CRUX with MAYHEM*** *should continue to publish such problems.*

3074. [2005 : 399, 401] *Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania.*

Let $f : \left[0, \frac{1}{2005}\right] \to \mathbb{R}$ be a function such that

$$f\left(x+y^2\right) \;\ge\; y + f(x)\,,$$

for all real x and y with $x \in \left[0, \frac{1}{2005}\right]$ and $x+y^2 \in \left[0, \frac{1}{2005}\right]$. Give an example of such a function, or show that no such function exists.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA, modified by the editor.

The fact that no such function exists follows from this more general result. For any $\varepsilon > 0$ and $r > 1$, no function $f : [0,\varepsilon] \to \mathbb{R}$ can satisfy $f(x+y^r) \ge y + f(x)$ for all real x and y with $x \in [0,\varepsilon]$ and $x+y^r \in [0,\varepsilon]$.

For the purpose of contradiction, let us assume that such a function f exists. Hence, $f(\varepsilon) - f(0) > 0$, because $f(\varepsilon) = f(0+\varepsilon) = \varepsilon^{\frac{1}{r}} + f(0)$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum\limits_{n=1}^{\infty} \frac{1}{n^r} = L$ converges, there exists N sufficiently large so that $\left(\frac{\varepsilon}{L}\right)^{\frac{1}{r}} \sum\limits_{n=1}^{N} \frac{1}{n} > f(\varepsilon) - f(0)$. On the other hand, if $R = \varepsilon - \frac{\varepsilon}{L} \sum\limits_{n=1}^{N} \frac{1}{n^r} > 0$, then $R \in [0,\varepsilon]$, and $\sum\limits_{n=1}^{N} \frac{\varepsilon}{Ln^r} \in [0,\varepsilon]$.

We now observe that if $x_1 + x_2 + \cdots + x_k \in [0,\varepsilon]$, then

$$\begin{aligned} f(x_1+x_2+\cdots+x_k) &\ge x_1^{\frac{1}{r}} + f(x_2+x_3+\cdots+x_k) \\ &\ge x_1^{\frac{1}{r}} + x_2^{\frac{1}{r}} + f(x_3+x_4+\cdots+x_k) \\ &\;\vdots \\ &\ge \sum_{i=1}^{k-1} x_i^{\frac{1}{r}} + f(x_k)\,. \end{aligned}$$

Applying this property yields

$$\begin{aligned} f(\varepsilon) &= f\Big(0 + \sum_{n=1}^{N} \frac{\varepsilon}{Ln^r} + R\Big) \;\ge\; R^{\frac{1}{r}} + f\Big(0 + \sum_{n=1}^{N} \frac{\varepsilon}{Ln^r}\Big) \\ &\ge R^{\frac{1}{r}} + \left(\frac{\varepsilon}{L}\right)^{\frac{1}{r}} \sum_{n=1}^{N} \frac{1}{n} + f(0) \;>\; f(\varepsilon)\,, \end{aligned}$$

a contradiction.

Also solved by MICHEL BATAILLE, Rouen, France; TOM LEONG, Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

3075. [2005 : 399, 401] *Proposed by Mihály Bencze, Brasov, Romania.*

Solve the following equation where x is a positive real number:

$$(8^x - 5^x)(7^x - 2^x)(6^x - 4^x) + (9^x - 4^x)(8^x - 3^x)(5^x - 2^x) \;=\; 105^x\,.$$

Solution by David Kaspar, student, Arizona State University, Tempe, AZ, USA.

It can be easily checked that $x = 1$ is a solution. We claim that this is the only solution.

Since all quantities in the given equation are strictly positive for $x > 0$, we divide by $105^x = (3 \cdot 5 \cdot 7)^x$ and obtain the equivalent equation

$$\left(\left(\tfrac{8}{5}\right)^x - 1^x\right)\left(1^x - \left(\tfrac{2}{7}\right)^x\right)\left(2^x - \left(\tfrac{4}{3}\right)^x\right)$$
$$+ \left(\left(\tfrac{9}{7}\right)^x - \left(\tfrac{4}{7}\right)^x\right)\left(\left(\tfrac{8}{3}\right)^x - 1^x\right)\left(1^x - \left(\tfrac{2}{5}\right)^x\right) \;=\; 1\,.$$

Now, if $a > b > 0$, then $a^x - b^x$ is strictly increasing for $x \geq 0$, and is equal to zero at $x = 0$. Products and sums of non-negative strictly increasing functions are also strictly increasing functions. Thus, the left side above is a strictly increasing function on $[0, \infty)$. If it is equal to 1 at $x = 1$, then it is not equal to 1 for any other non-negative value of x.

Also solved by ANGELO STATE PROBLEM GROUP, Angelo State University, San Angelo, TX, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3076. [2005 : 457, 460] *Proposed by Vedula N. Murty, Dover, PA, USA.*

If x, y, z are non-negative real numbers and a, b, c are arbitrary real numbers, prove that

$$\big(a(y+z) + b(z+x) + c(x+y)\big)^2 \;\geq\; 4(xy+yz+zx)(ab+bc+ca)\,.$$

(Note: If we impose the conditions that $x + y + z = 1$ and that a, b, c are positive, then the above is equivalent to

$$ax + by + cz + 2\sqrt{(xy+yz+zx)(ab+bc+ca)} \;\leq\; a + b + c\,,$$

which is problem #8 of the 2001 Ukrainian Mathematical Olympiad, given in the December 2003 issue of ***CRUX with MAYHEM*** [2003 : 498]. The solution of the Ukrainian problem appears in [2005 : 443].)

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; and G. Tsintsifas, Thessaloniki, Greece, modified by the editor.

If $x+y+z=0$, then, since x, y, and z are non-negative, we must have $x=y=z=0$, and the inequality is obvious.

If $a+b+c=0$, then

$$0 \;=\; (a+b+c)^2 \;=\; a^2+b^2+c^2+2(ab+bc+ca)\,;$$

and hence $ab+bc+ca \le 0$, and again the inequality is obvious.

Without loss of generality, we may assume that $x+y+z=1$ and $a+b+c=1$. We can also assume that $ab+bc+ca \ge 0$, since the inequality is obvious otherwise. Then, by the AM–GM Inequality, we have

$$\begin{aligned}
&2\sqrt{(xy+yz+zx)(ab+bc+ca)}\\
&\le\; xy+yz+zx+ab+bc+ca\\
&=\; \frac{(x+y+z)^2-(x^2+y^2+z^2)}{2}+\frac{(a+b+c)^2-(a^2+b^2+c^2)}{2}\\
&=\; \frac{1-(x^2+y^2+z^2)}{2}+\frac{1-(a^2+b^2+c^2)}{2}\\
&=\; 1-\frac{(a^2+x^2)}{2}-\frac{(b^2+y^2)}{2}-\frac{(c^2+z^2)}{2}\\
&\le\; 1-ax-by-cz \;=\; (x+y+z)(a+b+c)-ax-by-cz\\
&=\; a(y+z)+b(z+x)+c(x+y)\,,
\end{aligned}$$

and the inequality follows.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; EDWARD DOOLITTLE, University of Regina, Regina, SK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; JOEL SCHLOSBERG, Bayside, NY, USA; STAN WAGON, Macalester College, St. Paul, MN, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer. There was one incorrect submission.

Doolittle proved that the equality holds if and only if the vectors (a,b,c) and (x,y,z) are linearly dependent or $y=z=0$ and $b+c=0$ (or cyclic permutations of the latter). Janous showed that the condition for x, y, and z can be weakened to: $xy+yz+zx \ge 0$ and $y \ne -z$ (or $z \ne -x$ or $x \ne -y$).

3077. [2005 : 457, 460] *Proposed by Arkady Alt, San Jose, CA, USA.*

In $\triangle ABC$, we denote the sides BC, CA, AB as usual by a, b, c, respectively. Let h_a, h_b, h_c be the lengths of the altitudes to the sides a, b, c, respectively. Let d_a, d_b, d_c be the signed distances from the circumcentre of $\triangle ABC$ to the sides a, b, c, respectively. (The distance d_a, for example, is positive if and only if the circumcentre and vertex A lie on the same side of the line BC.)

Prove that

$$\frac{h_a+h_b+h_c}{3} \;\le\; d_a+d_b+d_c\,.$$

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Let w_a, w_b, and w_c be the internal bisectors of angles A, B, and C, respectively. The desired inequality follows from the known stronger inequality $w_a + w_b + w_c \le 3(d_a + d_b + d_c)$ (item 12.3 in [1]) and the obvious inequalities $h_a \le w_a$, $h_b \le w_b$, and $h_c \le w_c$.

References

[1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (second solution); MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was also one incorrect solution submitted.

3078. [2005 : 457, 460] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

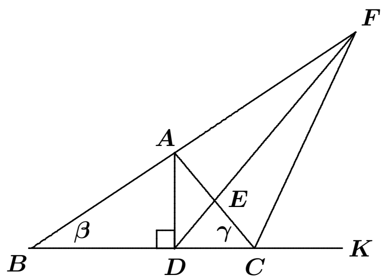
Let ABC be a triangle with $a > b$. Let D be the foot of the altitude from A to the line BC, let E be the mid-point of AC, and let CF be an external bisector of $\angle BCA$ with F on the line AB. Suppose that D, E, F are collinear.

(a) Determine the range of $\angle BCA$.

(b) Show that $c > b$.

(c) If $c^2 = ab$, determine the measures of the angles of $\triangle ABC$, and show that $\sin B = \cos^2 B$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.



Produce DC to K. Let $\alpha = \angle BAC$, $\beta = \angle ABC$ and $\gamma = \angle ACB$ in $\triangle ABC$.

(a) Note that $EA = ED = EC$, and thus $\angle EDC = \angle ECD = \gamma$. Also, $\angle EDC < \angle FCK = 90^\circ - \left(\frac{1}{2}\right)\gamma$. Hence, $90^\circ - \left(\frac{1}{2}\right)\gamma > \gamma$, so that $\gamma < 60^\circ$.

Whenever $\gamma < 60^\circ$, points D, E, and F are defined, no matter where A is. Also, F is farther from the line DC than A is, so that FA produced will meet the line DC at B. Thus, $\gamma \in (0^\circ, 60^\circ)$.

(b) Note that $\beta < \angle FDC = \gamma$. Hence, $c > b$.

(c) Assume that $c^2 = ab$. By Menelaus' Theorem, we have

$$\frac{CD}{DB} = \left(\frac{CE}{EA}\right)\cdot\left(\frac{AF}{FB}\right) = \frac{AF}{FB} = \frac{CA}{CB}$$

(since CF bisects $\angle ACK$). Hence, $\dfrac{CD}{DB} = \dfrac{b}{a}$. Thus, $\dfrac{\tan\beta}{\tan\gamma} = \dfrac{b}{a}$.

But, $\dfrac{\sin\beta}{\sin\gamma} = \dfrac{b}{c} = \dfrac{b}{\sqrt{ab}} = \sqrt{\dfrac{b}{a}}$. Hence, $\dfrac{\cos\beta}{\cos\gamma} = \sqrt{\dfrac{a}{b}}$. Therefore, $\cos\beta\sin\beta = \cos\gamma\sin\gamma$, or $\sin 2\beta = \sin 2\gamma$. Thus, $2\beta + 2\gamma = 180^\circ$, or $\beta+\gamma = 90^\circ$, giving $\alpha = 90^\circ$.

Then, $\sin\beta = \dfrac{AC}{BC} = \dfrac{b}{a}$ and $\cos\beta = \dfrac{BA}{BC} = \dfrac{\sqrt{ab}}{a} = \sqrt{\dfrac{b}{a}}$. Hence, $\sin\beta = \cos^2\beta$, as required.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

Woo observed that, since $\sin\beta = \cos^2\beta = 1 - \sin^2\beta$, *we have* $\sin\beta = g - 1 = g^{-1}$, *where* $g = \frac{1}{2}(1+\sqrt{5})$ *is the Golden Ratio. Thus,* $\beta = \arcsin(g^{-1})$ *and* $\gamma = \arccos(g^{-1})$.

3079. [2005 : 458, 460] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_1 \le x_2 \le \cdots \le x_n$. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^4 \le \frac{8(n-1)^3(n+1)(2n^2-3)}{15} \sum_{i,j=1}^{n} (x_i - x_j)^4 .$$

[*Ed:* The factor $(n-1)^3$ on the right side of the inequality was originally printed as $(n-1)^2$. This was noted by our featured solver and has been corrected above.]

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

By straightforward counting, we get the identity

$$n\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j| = 2\sum_{i=1}^{n}\sum_{j=1}^{n} |i-j||x_i - x_j| .$$

The sum on the right has $n(n-1)$ non-zero terms. Using the Power-Mean Inequality and then the Cauchy-Schwarz Inequality, we get

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j| &= 2(n-1)\frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} |i-j||x_i-x_j|}{n(n-1)} \\
&\le 2(n-1)\left(\frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} (i-j)^2(x_i-x_j)^2}{n(n-1)}\right)^{\frac{1}{2}} \\
&\le \frac{2\sqrt{n-1}}{\sqrt{n}}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(i-j)^4\right)^{\frac{1}{4}}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^4\right)^{\frac{1}{4}} .
\end{aligned}$$

Taking fourth powers and using the identity

$$\sum_{j=1}^{n}\sum_{i=1}^{n}(i-j)^4 \;=\; \frac{n^2(n^2-1)(2n^2-3)}{30}\,,$$

we obtain the desired result.

Also solved by the proposer.

Janous indicated that this proof applies to all positive integers $N > 1$; that is,

$$\left(\sum_{i,j=1}^{n}|x_i - x_j|\right)^N \;\le\; \left(\frac{2^N(n-1)^{N-2}}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}|i-j|^N\right)\left(\sum_{i,j=1}^{n}(x_i - x_j)^N\right).$$

For $N = 2$, the inequality is problem 5 of the 44th IMO in Tokyo.

3080. [2005 : 458, 461] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle, and let U and V be any two points in the plane of the triangle, but not on the sides of the triangle. Let $L = BU \cap CV$, $L' = BV \cap CU$, $M = CU \cap AV$, $M' = CV \cap AU$, $N = AU \cap BV$, $N' = AV \cap BU$. Prove that the triangles ABC, LMN are in perspective, as are triangles ABC, $L'M'N'$ and triangles LMN, $L'M'N'$. If the centres of these three perspectivities are P, P', P'', prove that P, P', P'' are collinear. Prove further that if U is the centroid G of $\triangle ABC$, then P'' is the mid-point of PP'.

[The proposer based this problem on an item he found in a century-old issue of the *Educational Times*, in which U and V are the centroid and the symmedian point of $\triangle ABC$. He verified that the result generalized to U and V being isogonal conjugates, and, using Cabri, he also found the result to be true for any two points in the plane of the triangle not on the sides. He adds "It is unlikely to be original, but is the sort of result that should not be lost to the world, and which solvers should enjoy."]

Solution by Edward Doolittle, University of Regina.

The triangle ABC can be mapped by an affine transformation to the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ in $\mathbb{R}^3$. Recall that every affine transformation preserves mid-points and centroids as well as incidence properties. We introduce homogeneous coordinates on the plane $x+y+z=1$ such that coordinates of the five given points are given by

$$a \;=\; (1,0,0)\,,\quad b \;=\; (0,1,0)\,,\quad c \;=\; (0,0,1)\,,$$
$$u \;=\; (u_1,u_2,u_3)\,,\quad v \;=\; (v_1,v_2,v_3)\,.$$

The condition that neither U nor V lies on a side of the triangle translates to the condition that none of u_i, v_i is 0. Note that the centroid G of ABC is mapped by the affine transformation to the point with homogeneous coordinates $g = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) = (1,1,1)$.

The lines joining A, B, C to U and V have homogeneous coordinates given by the appropriate cross products:

$$\begin{aligned} au &= a \times u = [0, -u_3, u_2]\,, & av &= a \times v = [0, -v_3, v_2]\,, \\ bu &= b \times u = [u_3, 0, -u_1]\,, & bv &= b \times v = [v_3, 0, -v_1]\,, \\ cu &= c \times u = [-u_2, u_1, 0]\,, & cv &= c \times v = [-v_2, v_1, 0]\,. \end{aligned}$$

The coordinates of L, M, and N are also found by taking cross products:

$$\begin{aligned} l &= bu \times cv = (u_1v_1, u_1v_2, u_3v_1)\,, \\ m &= cu \times av = (u_1v_2, u_2v_2, u_2v_3)\,, \\ n &= au \times bv = (u_3v_1, u_2v_3, u_3v_3)\,, \end{aligned}$$

and the coordinates of L', M', and N' can be found by swapping u and v:

$$\begin{aligned} l' &= bv \times cu = (v_1u_1, v_1u_2, v_3u_1)\,, \\ m' &= cv \times au = (v_1u_2, v_2u_2, v_2u_3)\,, \\ n' &= av \times bu = (v_3u_1, v_2u_3, v_3u_3)\,. \end{aligned}$$

The coordinates of lines AL, BM, and CN can now be found:

$$\begin{aligned} al &= a \times l = [0, -u_3v_1, u_1v_2]\,, \\ bm &= b \times m = [u_2v_3, 0, -u_1v_2]\,, \\ cn &= c \times n = [-u_2v_3, u_3v_1, 0]\,. \end{aligned}$$

Since $al + bm + cn = [0, 0, 0]$, the lines are concurrent.

Homogeneous coordinates of the centre of perspective P can be found by taking the cross product of any two of the above lines; for example,

$$p = al \times bm = (u_1u_3v_1v_2, u_1u_2v_2v_3, u_2u_3v_1v_3)\,.$$

Interchanging u and v, we obtain similar results for AL', BM', and CN'; hence, these lines are concurrent through the point with coordinates

$$p' = al' \times bm' = (v_1v_3u_1u_2, v_1v_2u_2u_3, v_2v_3u_1u_3)\,.$$

Similar calculations give

$$\begin{aligned} ll' &= [u_1^2v_2v_3 - u_2u_3v_1^2, u_1u_3v_1^2 - u_1^2v_1v_3, u_1u_2v_1^2 - u_1^2v_1v_2]\,, \\ mm' &= [u_2u_3v_2^2 - u_2^2v_2v_3, u_2^2v_1v_3 - u_1u_3v_2^2, u_1u_2v_2^2 - u_2^2v_1v_2]\,, \\ nn' &= [u_2u_3v_3^2 - u_3^2v_2v_3, u_1u_3v_3^2 - u_3^2v_1v_3, u_3^2v_1v_2 - u_1u_2v_3^2]\,. \end{aligned}$$

Evaluating the determinant of the above quantities is quite an involved calculation. Happily, all we need here is the cross product of ll' and mm':

$$p'' = (u_1v_1(u_2v_3 + u_3v_2), u_2v_2(u_1v_3 + u_3v_1), u_3v_3(u_1v_2 + u_2v_1))\,.$$

Note that a common factor of $-(u_1v_2 - u_2v_1)^2$ has been removed from the components of p''; the factor is non-zero if and only if U, V, and C are not collinear. If the factor is zero, LL' and MM' are coincident, in which case we can take another pair of lines, say MM' and NN', to find P''. (Note that U and V cannot be collinear with more than one of A, B, and C.) Since the dot product of p'' and nn' gives 0, the lines LL', MM', and NN' are concurrent through P''. Alternatively, the fact that NN' passes through $P'' = LL' \cap MM'$ is a consequence of Pappus's Theorem: By definition, the triples N, U, M' and M, V, N' are collinear, and therefore their cross-joins L, L', and P'' are also collinear.

Next, we note that the three points P, P', and P'' are collinear because their coordinates satisfy $p + p' = p''$.

Finally, we must see how these points are related when U is the centroid of triangle ABC. In this case, the homogeneous coordinates of u are $(1, 1, 1)$ and we have

$$p = (v_1v_2, v_2v_3, v_3v_1)\,, \quad p' = (v_3v_1, v_1v_2, v_2v_3)\,, \quad \text{and} \quad p'' = p + p'\,.$$

The homogeneous coordinates of the mid-point P_2 of P and P' are found by multiplying the homogeneous coordinates of the latter two by a value which brings them to the plane $x + y + z = 1$ and then taking the arithmetic mean. For both p and p' the appropriate scaling factor is $\lambda = v_1v_2 + v_2v_3 + v_3v_1$, and we have

$$\begin{aligned} p &= \frac{1}{\lambda}(v_1v_2, v_2v_3, v_3v_1)\,, \\ p' &= \frac{1}{\lambda}(v_3v_1, v_1v_2, v_2v_3)\,, \\ p_2 &= \frac{1}{2\lambda}(v_1v_2 + v_3v_1, v_2v_3 + v_1v_2, v_3v_1 + v_2v_3)\,. \end{aligned}$$

Noting that P_2 has homogeneous coordinates equivalent to those of P'', we see that, in this case, P'' is the mid-point of P and P'.

Also solved by MICHEL BATAILLE, Rouen, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

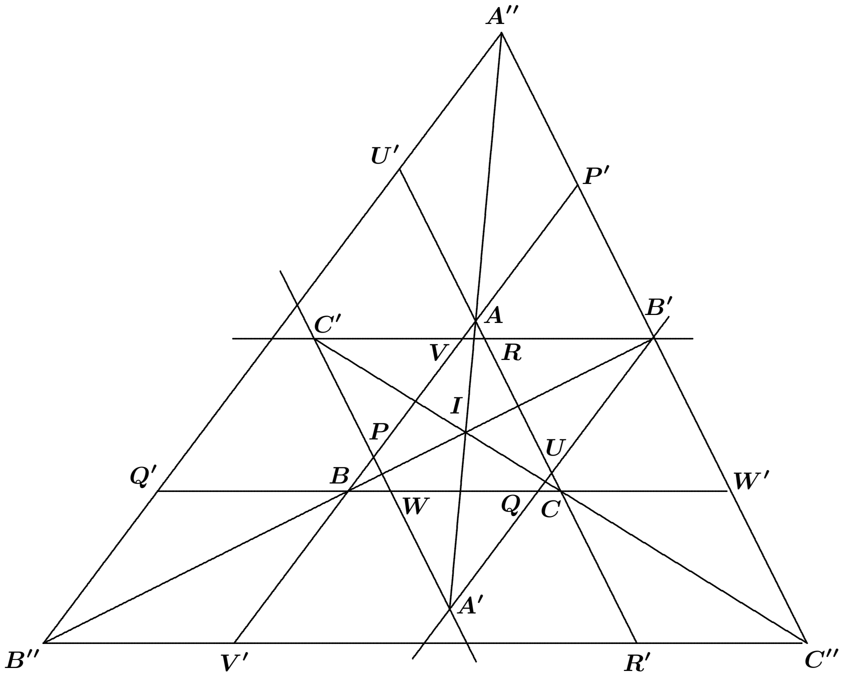
All solvers used coordinates in much the same way as in the featured solution.

3081. [2005 : 458, 461] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be an acute-angled triangle and let the altitudes from A, B, C to the opposite sides have lengths d, e, f, respectively. The circle centred at A with radius d meets the line segments AB and AC at P and U, respectively, and it meets the rays BA and CA at P' and U', respectively. Similarly, we define the points Q, V, Q', V' using the circle centred at B with radius e, and we define the points R, W, R', W' using the circle centred at C with radius f (where Q, Q', W, W' lie on the line BC). Let A' be the intersection of PW and QU, let B' be the intersection of QU and RV, and let C' be the intersection of RV and PW. Further, let A'' be the intersection of $P'W'$ and $Q'U'$, B'' the intersection of $Q'U'$ and $R'V'$, and C'' the intersection of $R'V'$ and $P'W'$.

Prove that the triangles ABC, $A'B'C'$, and $A''B''C''$ have a common incentre.

Solution by Michel Bataille, Rouen, France.



Let $BC = a$, $CA = b$, $AB = c$, as usual, and let $[ABC]$ denote the area of $\triangle ABC$. Then $AV = c - e$, $AR = b - f$, and $be = cf = 2[ABC]$; hence,

$$\frac{AV}{AB} = 1 - \frac{e}{c} = 1 - \frac{f}{b} = \frac{AR}{AC}.$$

Since V and R lie on the line segments AB and AC, respectively, it follows that VR is parallel to BC; that is, $B'C' \parallel BC$. Similarly, $A'C' \parallel AC$ and $A'B' \parallel AB$. Moreover, $PA' \parallel AU$ and $UA' \parallel AP$ imply that $APA'U$ is a parallelogram, while $AP = AU$ implies that this parallelogram is a rhombus. Thus, the diagonal AA' is the internal bisector of both $\angle PAU = \angle BAC$ and $\angle PA'U = \angle B'A'C'$. A similar result holds for BB' and CC'. The fact that $\triangle ABC$ and $\triangle A'B'C'$ have the same incentre I follows at once.

Analogously,

$$\frac{AV'}{AB} = 1 + \frac{e}{c} = 1 + \frac{f}{b} = \frac{AR'}{AC},$$

so that $R'V' \parallel BC$. In the same way, $Q'U' \parallel AB$ and $P'W' \parallel AC$. It follows that $AUA'P$ is the rhombus symmetric with $APA'U$ about the point A. Thus, A', A, A'' are collinear on the common internal bisector of $\angle BAC$, $\angle B'A'C'$, $\angle B''A''C''$. A similar result holds for B, B', B'' and for C, C', C''. Therefore, I is also the incentre of $\triangle A''B''C''$.

Also solved by JOEL SCHLOSBERG, *Bayside, NY, USA;* PETER Y. WOO, *Biola University, La Mirada, CA, USA; and the proposer.*

3082. [2005 : 458, 461] *Proposed by J. Walter Lynch, Athens, GA, USA.*

Suppose that four consecutive terms of a geometric sequence with common ratio r are the sides of a quadrilateral. What is the range of all possible values for r?

Solution by Robert P. Sealy, Mount Allison University, Sackville, NB, modified by the editor.

A necessary and sufficient condition for four given positive real numbers to be the lengths of the sides of a quadrilateral is that the largest of the four be less than the sum of the other three. We want the lengths of the sides to be a, ar, ar^2, and ar^3, for some $a > 0$ and $r > 0$. We consider two cases.

First suppose that $r \geq 1$. Then we require $a + ar + ar^2 > ar^3$; that is, $f(r) < 0$, where $f(r) = r^3 - r^2 - r - 1$. Note that $f'(r) = (3r+1)(r-1)$. Therefore $f(r)$ is increasing for $r \geq 1$. Since $f(1) < 0$, it follows that f has only one real zero for $r \geq 1$. By Cardano's Formula, we find that $r = c$, where

$$c \;=\; \tfrac{1}{3}\left(\left(19+3\sqrt{33}\right)^{1/3} + \left(19-3\sqrt{33}\right)^{1/3} + 1\right) \;\approx\; 1.84\,. \qquad (1)$$

We have $f(r) < 0$ for $1 \leq r < c$.

Now suppose that $0 < r < 1$. We require $ar^3 + ar^2 + ar > a$; that is, $g(r) > 0$, where $g(r) = r^3 + r^2 + r - 1$. Note that $g(r) = -r^3 f(1/r)$. Therefore $g(r) > 0$ if and only if $f(1/r) < 0$. Using the case we have already proved, we find that $g(r) > 0$ for $1/c < r < 1$. By Cardano's Formula,

$$1/c \;=\; \tfrac{1}{3}\left(\left(3\sqrt{33}+17\right)^{1/3} - \left(3\sqrt{33}-17\right)^{1/3} - 1\right) \;\approx\; 0.54\,. \qquad (2)$$

Combining our two cases, we obtain the interval $(1/c, c)$ as the range of all possible values of r.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; DOUGLASS L. GRANT, Cape Breton University, Sydney, NS; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; and the proposer.

Grant claims that the end-points of the range of r in this problem are "closely connected with the Misiurewicz Points of the Mandelbrot Set". Janous asks for a more direct method of proving that the two expressions involving radicals in the numbered equations above are reciprocals of one another. Is there a "good" way to see this?

3083. [2005 : 459, 461] *Proposed by Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON.*

Let n be a natural number, and suppose that the generalized Newton's binomial coefficient $\binom{\frac{1}{2}}{n}$ is written as a reduced fraction p/q where p and q are integers with $q > 0$. Show that $q = 2^k$ for some k with $0 \leq k \leq 2n-1$.

Composite of similar solutions by Michel Bataille, Rouen, France; James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA; and Joel Schlosberg, Bayside, NY, USA.

Note first that $\binom{\frac{1}{2}}{1} = \frac{1}{2}$ and $\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = -\frac{1}{2^3}$.

For $n \geq 3$, we have

$$\begin{aligned}\binom{\frac{1}{2}}{n} &= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!} = (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!} \\ &= (-1)^{n-1}\frac{(2n-2)!}{2^n\cdot 2\cdot 4\cdot 6\cdots(2n-2)\cdot n!} \\ &= (-1)^{n-1}\frac{(2n-2)!}{2^{2n-1}(n-1)!n!} = (-1)^{n-1}\frac{1}{2^{2n-1}n}\binom{2n-2}{n-1}.\end{aligned}$$

Hence, it suffices to prove that n divides $\binom{2n-2}{n-1}$. We observe that $n\binom{2n-1}{n} = (2n-1)\binom{2n-2}{n-1}$. Since n and $2n-1$ are relatively prime, it follows that n divides $\binom{2n-2}{n-1}$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; EDWARD DOOLITTLE, University of Regina, Regina, SK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, Brooklyn, NY, USA (2 solutions); PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposers.

Curtis, Doolittle, Janous, Leong, and Schlosberg all noted that

$$\binom{\frac{1}{2}}{n} = (-1)^{n-1}\frac{C_{n-1}}{2^{2n-1}},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ *is the* n^{th} *Catalan number. Leong also showed that*

$$\binom{\frac{1}{2}}{n} = \frac{(-1)^{n-1}}{2^{2n-1}}\left[\binom{2n-2}{n-1} - \binom{2n-2}{n}\right].$$

3084. [2005 : 459, 462] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$\sum_{k=1}^{n} x_k = 0 \quad \text{and} \quad \sum_{k=1}^{n} x_k^4 = 1.$$

Prove that

$$\left(\sum_{k=1}^{n} kx_k\right)^4 \leq \frac{n^3(n^2-1)(3n^2-7)}{240}.$$

Solution by Bin Zhao, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China, modified by the editor.

The Cauchy-Schwarz Inequality applied twice yields

$$\begin{aligned}\left(\sum_{k=1}^{n} kx_k\right)^4 &= \left(\sum_{k=1}^{n} kx_k - \tfrac{1}{2}(n+1)\sum_{k=1}^{n} x_k\right)^4 \\ &= \left(\sum_{k=1}^{n}\left(k-\tfrac{1}{2}(n+1)\right)x_k\right)^4 \\ &\le \left(\sum_{k=1}^{n}\left(k-\tfrac{1}{2}(n+1)\right)^2\right)^2\left(\sum_{k=1}^{n} x_k^2\right)^2 \\ &\le \left(\sum_{k=1}^{n}\left(k^2-(n+1)k+\tfrac{1}{4}(n+1)^2\right)\right)^2 n\left(\sum_{k=1}^{n} x_k^4\right) \\ &= \frac{n^3(n+1)^2(n-1)^2}{144} \le \frac{n^3(n^2-1)(3n^2-7)}{240}\,.\end{aligned}$$

We have actually proved a stronger inequality than the one proposed, since the inequality in the last line is strict for $n \ge 3$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Zhao also noted that Holder's Inequality may be applied to get

$$\begin{aligned}\left(\sum_{k=1}^{n} kx_k\right)^4 &= \left(\sum_{k=1}^{n}\left(k-\tfrac{1}{2}(n+1)\right)x_k\right)^4 \\ &\le \left(\sum_{k=1}^{n}\sqrt[3]{\left(k-\tfrac{1}{2}(n+1)\right)^4}\right)^3\left(\sum_{k=1}^{n} x_k^4\right) = \left(\sum_{k=1}^{n}\sqrt[3]{\left(k-\tfrac{1}{2}(n+1)\right)^4}\right)^3,\end{aligned}$$

where equality is possible.

3086. [2005 : 459, 462] *Proposed by Mihály Bencze, Brasov, Romania.*

If $a_k > 0$ for $k = 1, 2, \ldots, n$, prove that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) \ge \frac{1}{n}\left(\sqrt[3]{\frac{a_1}{a_2}}+\sqrt[3]{\frac{a_2}{a_3}}+\cdots+\sqrt[3]{\frac{a_n}{a_1}}\right)^3 \ge n^2\,.$$

Solution by Joel Schlosberg, Bayside, NY, USA.

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) &= (a_1+a_2+\cdots+a_n)\left(\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_1}\right) \\ &\ge \left(\sqrt{\frac{a_1}{a_2}}+\sqrt{\frac{a_2}{a_3}}+\cdots+\sqrt{\frac{a_n}{a_1}}\right)^2. \qquad (1)\end{aligned}$$

By the Power-Mean Inequality, we have

$$\left(\frac{\sqrt{\frac{a_1}{a_2}}+\sqrt{\frac{a_2}{a_3}}+\cdots+\sqrt{\frac{a_n}{a_1}}}{n}\right)^2 \geq \left(\frac{\sqrt[3]{\frac{a_1}{a_2}}+\sqrt[3]{\frac{a_2}{a_3}}+\cdots+\sqrt[3]{\frac{a_n}{a_1}}}{n}\right)^3,$$

yielding

$$\left(\sqrt{\frac{a_1}{a_2}}+\sqrt{\frac{a_2}{a_3}}+\cdots+\sqrt{\frac{a_n}{a_1}}\right)^2 \geq \frac{1}{n}\left(\sqrt[3]{\frac{a_1}{a_2}}+\sqrt[3]{\frac{a_2}{a_3}}+\cdots+\sqrt[3]{\frac{a_n}{a_1}}\right)^3. \quad (2)$$

From (1) and (2), the left inequality follows immediately. The right inequality is an immediate consequence of the AM–GM Inequality, since

$$\frac{1}{n}\left(\sqrt[3]{\frac{a_1}{a_2}}+\sqrt[3]{\frac{a_2}{a_3}}+\cdots+\sqrt[3]{\frac{a_n}{a_1}}\right)^3 \geq \frac{1}{n}\left(n\sqrt[3n]{\frac{a_1}{a_2}\cdot\frac{a_2}{a_3}\cdots\frac{a_n}{a_1}}\right)^3 = n^2.$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; EDWARD DOOLITTLE, University of Regina, Regina, SK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, Brooklyn, NY, USA; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer.

Using arguments similar to those given in the solution above, Janous showed that the left inequality can be replaced by the more general result that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) \geq n^{2-\alpha}\left(\sum_{k=1}^{n}\left(\frac{a_k}{a_{\pi(k)}}\right)^{\frac{1}{\alpha}}\right)^{\alpha},$$

where $\alpha \geq 2$ is real constant and π is any permutation of $\{1, 2, \ldots, n\}$. The given left inequality is the special case when $\alpha = 3$ and $\pi(k) = k+1 \pmod{n}$ for all k, $1 \leq k \leq n$.