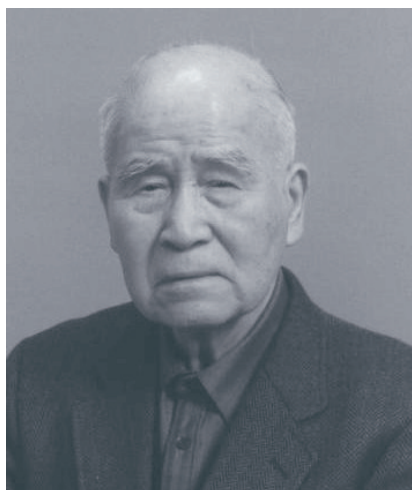


Contributor Profiles:

Toshio Seimiya



Toshio Seimiya was born in Tokyo on March 30, 1910, which means he will have celebrated his 96th birthday by the time this issue appears. When he was fourteen years old, he learned about the Theorem of Pythagoras, and promptly discovered new proofs. At the age of sixteen, he discovered the following theorem, a generalization of Simson's Theorem, which became known as Seimiya's Theorem.

Theorem. Let P and Q be points on the circumcircle of $\triangle ABC$, and let D , E , and F be the reflections of P with respect to the lines BC , CA , and AB , respectively. Let X , Y , and Z be the intersections of

QD , QE , and QF with BC , CA , and AB , respectively. Then X , Y , and Z are collinear.

In 1931 Seimiya entered the Tokyo Imperial University (now Tokyo University). When he graduated in 1934, he became a mathematics teacher at the military academy, from which he retired in 1945. In 1949 he was appointed as Professor at Tokyo Gakugei University, where he remained until retiring in 1973, after which he was named Professor Emeritus.

While he may have retired in 1973, Seimiya has never really left the world of mathematics, specifically geometry. As Editor-in-Chief of **CRUX with MAYHEM**, I always look forward to a letter from him. Each letter contains either a new set of proposals or one or more solutions to earlier problems posed by others. The proposals he submits to us invariably elicit glowing remarks from our Problems Editors such as "How does he do it?!", or "We have yet again a wonderful set of original and very interesting problems from Seimiya!" On the other hand, his solutions to the proposals of others are always correct and elegantly presented. Our Problems Editors need to continually exercise caution that we do not simply use his solutions all the time, in order to be fair to all solvers.

I am sure I speak on behalf of all lovers of geometry when I say "May you live long, Toshio, and continue to submit problems for the rest of the world to enjoy!"

SKOLIAD No. 93

Robert Bilinski

Please send your solutions to the problems in this edition by **October 1, 2006**. A copy of **MATHEMATICAL MAYHEM Vol. 3** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Nos questions proviennent du Concours de l'Association Mathématique du Québec 2004 (niveau secondaire). Nous remercions Véronique Hussin, Université de Montréal qui s'occupe des concours de l'AMQ du secondaire.

Concours de l'Association Mathématique du Québec (niveau secondaire) 5 février 2004

1. (Les vases d'eau salée.) Deux vases, A et B , d'une capacité de six litres chacun, contiennent chacun quatre litres d'eau salée, selon les concentrations suivantes : A contient 5% de sel et B contient 10% de sel. On vide un litre d'eau salée du vase A dans le vase B puis on mélange. On vide ensuite un litre du vase B dans le vase A , puis on mélange à nouveau. Quelle concentration de sel (en pourcentage) chacun des vases A et B contiennent-ils maintenant ?

2. (La multiplication de Koallo.) Koallo habite le joli village d'Oloko, au Nigéria. Comme il aime les mathématiques, il a remarqué récemment, qu'avec une correspondance appropriée entre les chiffres et les lettres et en multipliant par 11 le nom de son village, il obtenait son nom ! Êtes-vous capable de faire comme lui ? Plus précisément, pouvez-vous trouver les chiffres différents que doivent représenter les lettres O , L , K et A pour que l'équation $OLOKO \times 11 = KOALLO$ soit vraie. Attention, $OLOKO$ doit être vu comme un nombre de cinq chiffres et non comme le produit $O \times L \times O \times K \times O$. Il en va de même pour $KOALLO$.

3. (Les nombres de Fibonacci dans des triangles de Pythagore.) La suite : 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... est la célèbre suite de Fibonacci. On voit que si on commence avec 1 et 2, les termes qui suivent sont toujours obtenus comme la somme des deux nombres précédents de la suite. Ainsi, par exemple, on a $3 = 2 + 1$, $5 = 3 + 2$, $8 = 5 + 3$. Un triangle de Pythagore est, quant à lui, un triangle rectangle dont la longueur de chacun des côtés est un nombre entier.

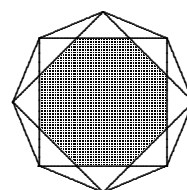
On remarque alors que si nous prenons quatre termes consécutifs de la suite de Fibonacci, quelques opérations simples nous permettent de former des triangles de Pythagore. Par exemple, soit les quatre nombres 3, 5, 8 et 13, alors un premier côté x du triangle est obtenu en prenant deux fois le produit des deux nombres du milieu ($x = 2 \times 5 \times 8 = 80$), le deuxième côté y est obtenu en multipliant le premier et le dernier des quatre nombres

($y = 3 \times 13 = 39$) et le dernier côté z est égal à la somme des carrés des deux nombres du milieu ($z = 5^2 + 8^2 = 89$). Ainsi, on a bien obtenu un triangle de Pythagore car on a $80^2 + 39^2 = 89^2$.

- (a) Vérifiez que cela marche aussi si on prend les nombres 2, 3, 5 et 8.
 (b) Pouvez-vous montrer que cela marche tout le temps ? Plus précisément, si a, b, c et d désignent quatre nombres consécutifs de la suite de Fibonacci et que l'on pose $x = 2bc$, $y = ad$ et $z = b^2 + c^2$, montrer que x, y et z forment les côtés entiers d'un triangle rectangle.

4. (Que de chiffres !) Trouvez le nombre de chiffres et la somme des chiffres de l'entier $16^8 \times 5^{30}$.

5. (L'octogone.) Si on relie entre eux les sommets d'un octogone régulier qui ont un sommet voisin en commun, on obtient au centre de la figure un nouvel octogone régulier, décalé et plus petit que le premier (en gris sur le dessin). Si l'aire de l'octogone initial est 1, quelle est l'aire du nouvel octogone ?



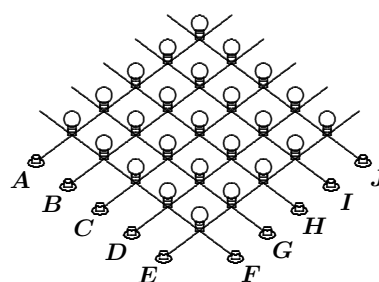
Indice : lorsque deux figures sont semblables, le rapport de leurs aires est égal au carré du rapport de leurs côtés homologues.

6. (Et hop ... sans calculatrice !) Expliquez pourquoi l'égalité suivante est vraie

$$\frac{2004^2}{2003 \times 2005} + \frac{2005^2}{2004 \times 2006} + \dots + \frac{3004^2}{3003 \times 3005} = 1001 + \frac{1}{2} \left(\frac{1}{2003} + \frac{1}{2004} - \frac{1}{3004} - \frac{1}{3005} \right).$$

Indice : obtenir une décomposition appropriée de chacun des termes $\frac{2004^2}{2003 \times 2005}$, $\frac{2005^2}{2004 \times 2006}$, etc. et additionner le tout.

7. (Les ampoules de Raoul.) Raoul se confectionne un circuit électrique formé de vingt-cinq ampoules disposées en carrés et de dix interrupteurs, notés de A à J , comme sur le dessin. S'il appuie sur un interrupteur, alors les cinq ampoules situées sur la ligne de cet interrupteur voient leur état inversé : celles qui sont allumées s'éteignent tandis que celles qui sont éteintes s'allument.



- (a) Montrer que, quelque soit l'état initial des ampoules (certaines ampoules peuvent être allumées tandis que d'autres, non), il est toujours possible de manipuler les interrupteurs de telle sorte que, dans chacune des dix rangées correspondantes, il y ait toujours plus d'ampoules allumées qu'éteintes.
 (b) Est-il toujours possible d'allumer toutes les ampoules en même temps ? Que votre réponse soit oui ou non, il faut donner la preuve de ce que vous avancez.

Contest of the Association of Quebec Mathematics (Secondary Level) February 5, 2004

1. (Salt-water vases.) Two vases, A and B , each having a capacity of 6 litres, are filled with salt-water solutions in the following concentrations: A contains 5% salt and B contains 10% salt. We empty a litre of the solution in A into B , and we mix the resulting contents of B . We then empty a litre of the solution in B into A , and we mix the resulting contents of A . What concentration of salt (in percent) does each vase contain?

2. (Koallo's multiplication.) Koallo lives in the quaint Nigerian village of Oloko. Since he likes mathematics, he recently noticed that, with an appropriate correspondence between numbers and letters, multiplying the name of his village by 11 yielded his own name! Are you able to do the same? More precisely, can you find the different digits that the letters O , L , K , and A must represent in order that the equation $OLOKO \times 11 = KOALLO$ is true? Watch out, $OLOKO$ is understood to be a five-digit number and not the product $O \times L \times O \times K \times O$. The same goes for $KOALLO$.

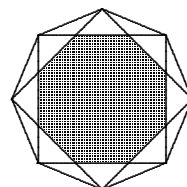
3. (Fibonacci numbers in Pythagorean triangles.) The sequence: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... is the famous Fibonacci sequence. We see that if we start with 1 and 2, the other numbers in the sequence are obtained by adding together the two previous numbers. Thus, for example, $3 = 1 + 2$, $5 = 2 + 3$, $8 = 5 + 3$. A Pythagorean triangle is a right triangle all of whose sides are integers.

If we take any four consecutive Fibonacci numbers, we can use a few simple operations to form a Pythagorean triangle. For example, if we take the four numbers 3, 5, 8, and 13, then the first side x of the Pythagorean triangle is obtained by taking twice the product of the two middle numbers ($x = 2 \times 5 \times 8 = 80$), the second side y is obtained by multiplying the first and last of the four numbers ($y = 3 \times 13 = 39$) and the last side z is equal to the sum of the squares of the two middle numbers ($z = 5^2 + 8^2 = 89$). We have thus obtained a Pythagorean triangle, since we have $80^2 + 39^2 = 89^2$.

- (a) Show that this process also works if we use the numbers 2, 3, 5, and 8.
 (b) Can you show that it works always? More precisely, if a , b , c , and d are four consecutive numbers of the Fibonacci sequence and if we let $x = 2bc$, $y = ad$ and $z = b^2 + c^2$, show that x , y , and z are the integer sides of a right triangle.

4. (That figures!) Find the number of digits and the sum of the digits for the integer $16^8 \times 5^{30}$.

5. (The octagon.) If we draw a line between two vertices of a regular octagon which have a common adjacent vertex, we obtain a new regular octagon in the centre of the figure, smaller than the first (in grey in the drawing). If the initial octagon has area 1, what is the area of the new octagon?



Hint: When two figures are similar, the ratio of their areas is equal to the square of the ratio of the corresponding sides.

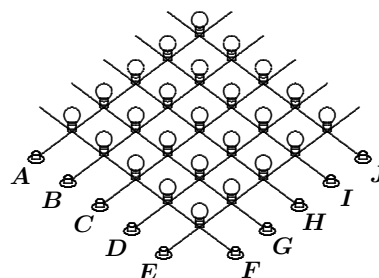
6. (Presto . . . without calculators!) Explain why the following equality holds.

$$\frac{2004^2}{2003 \times 2005} + \frac{2005^2}{2004 \times 2006} + \cdots + \frac{3004^2}{3003 \times 3005}$$

$$= 1001 + \frac{1}{2} \left(\frac{1}{2003} + \frac{1}{2004} - \frac{1}{3004} - \frac{1}{3005} \right).$$

Hint: Decompose each of the terms $\frac{2004^2}{2003 \times 2005}$, $\frac{2005^2}{2004 \times 2006}$, etc. appropriately, then add them all.

7. (Raoul's light bulbs.) Raoul has built an electrical network containing 25 bulbs arranged in a square with ten switches, as in the drawing. If he pushes a switch, then the five bulbs in that row change states: those that were lit are no longer lit, and vice versa.



- Show that, whatever the initial state of the bulbs (some lit, others not), it is always possible to work the switches so as to produce a state in which each of the ten rows has more bulbs lit than not lit.
- Is it always possible to arrive at a state where they are all lit? Whatever your answer (yes or no), you must prove it.

Next we give the rest of the solutions to the 2005 BC Colleges Junior High School Mathematics Contests, Final Round. [2005 : 261–270].

BC Colleges High School Mathematics Contest 2005 Junior Final Round, Part A Friday, May 6, 2005

1. Deux opérations \star and \diamond sont définies par les deux tables suivantes :

\star	1	2	3
1	1	3	2
2	1	3	1
3	3	3	1

\diamond	1	2	3
1	4	2	3
2	3	6	5
3	2	6	4

Par exemple, $1 \diamond 2 = 2$. La valeur de $2 \diamond (3 \star 3)$ est :

- (A) 6 (B) 5 (C) 4 (D) 3 (E) 2

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Il suffit de regarder les tableaux et de faire les opérations, $3 \star 3 = 1$ et $2 \diamond 1 = 3$. La réponse est D.

2. Trois personnes laissent leurs manteaux au vestiaire. Lorsqu'ils sortent, trois manteaux sont distribués au hasard parmi eux. La probabilité qu'*aucun* d'eux ne reçoive son manteau est :

- (A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{5}{6}$

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

La probabilité que le premier ne reçoive pas son manteau est de $\frac{2}{3}$, la probabilité qu'un des deux autres ne reçoive pas son manteau est de 1 puisque c'est la première personne qui l'a et la probabilité que le dernier ne reçoivent pas le sien est de $\frac{1}{2}$.

Donc, la probabilité recherchée est $\frac{2}{3} \times 1 \times \frac{1}{2} = \frac{1}{3}$. La réponse est B.

3. Un épicier utilise une balance à plateaux où des poids peuvent être placés sur n'importe lequel d'entre eux indépendamment de l'objet pesé. L'épicier a trois poids qui peuvent être utilisés pour peser précisément n'importe quel poids en kilogrammes allant de 1 kg à 13 kg. Les trois poids de l'épicier sont :

- (A) 2, 5, 6 (B) 3, 4, 6 (C) 1, 5, 7 (D) 2, 4, 7 (E) 1, 3, 9

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Il suffit d'essayer de faire chaque nombre de 1 à 13 en additionnant et soustrayant les chiffres proposés. Pour la bonne solution (1, 3, 9), on a :

$$\begin{array}{llll} 1 = 1 & 2 = 3 - 1 & 3 = 3 & 4 = 3 + 1 \\ 5 = 9 - 3 - 1 & 6 = 9 - 3 & 7 = 9 - 3 + 1 & 8 = 9 - 1 \\ 9 = 9 & 10 = 9 + 1 & 11 = 9 + 3 - 1 & 12 = 9 + 3 \\ 13 = 9 + 3 + 1 \end{array}$$

La réponse est E.

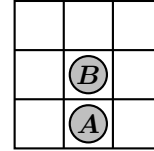
4. Le nombre de cartes qui doivent être pigés d'un paquet normal de 52 cartes pour être sûr d'avoir au moins deux aces ou trois d'une même sorte est :

- (A) 9 (B) 13 (C) 27 (D) 49 (E) 50

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

On peut piger 2 cartes de chaque sortes, donc 8. La 9^e carte est obligatoirement une des 4 sortes, donc on a assurément au moins 3 cartes d'une même sorte en 9 cartes. La réponse est A.

5. The game of Solitaire JumpIt is played on a 3×3 grid. A single player places two or more game discs on the grid. If two discs, A and B , are adjacent horizontally, vertically, or diagonally and there is an open space on the side of B away from A , then A can jump B and disc B is removed. (See the diagram.) The player makes jumps as long as possible. The player wins if he or she can continue until only one disc remains. The maximum number of discs that can be placed on the grid in a way that the player still wins is:



- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Official solution, modified by the editor.

Any disc placed in a corner cannot be jumped. Thus, there are at most 5 positions on the board where a disc can be placed and subsequently be jumped, namely the 5 positions not in the corners. This allows us to place a 6th disc in one of the corners and use it to jump the others. This can be done by cyclically jumping along the edges and one final jump across the central square. This shows that the player can win with 6 discs properly placed.

One can easily see that placing any further discs in the corners will not lead to a win. These discs can only jump other discs and can never be jumped because they will always be in a corner, with the result that the game will end with at least two discs remaining on the board.

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

6. The product $\left(1 + \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{n-1}\right) \left(1 + \frac{1}{n}\right)$ is equal to:

- (A) 1 (B) $\frac{1}{n}$ (C) $\frac{n+1}{n}$ (D) -1 (E) None of these

Official solution.

Since the factors in the product alternate between addition and subtraction, and the first and last terms both have an addition, there is an odd number of factors. Each pair of adjacent factors gives:

$$\left(1 + \frac{1}{i}\right) \left(1 - \frac{1}{i+1}\right) = 1 + \frac{1}{i} - \frac{1}{i+1} + \frac{1}{i(i+1)} = 1 + \frac{i+1-i-1}{i(i+1)} = 1.$$

Since the product of each such pair is 1, the only factor that remains is the last one. Therefore, the product equals $1 + \frac{1}{n} = \frac{n+1}{n}$. The answer is C.

An incomplete solution was received.

7. Le nombre d'entiers entre 500 et 600 dont la somme des chiffres est 12 est :

- (A) 6 (B) 7 (C) 8 (D) 10 (E) 12

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Nous partons avec un 5 dans notre addition, puisque c'est le chiffre des centaines. Donc, il faut trouver des couples de chiffres qui donnent 7

additionnés ensemble. Ces couples pourront être utilisés deux fois puisque l'on peut changer la position des chiffres dans le nombre. Les couples (x, y) tels que $x + y = 7$ sont : $(0, 7)$, $(1, 6)$, $(2, 5)$, $(3, 4)$ et leur contraire, soit 8 couples au total. Il y a donc 8 nombres entre 500 et 600 dont la somme des chiffres vaut 12.

8. Un point entier du plan (x, y) est tel que x et y sont entiers. Le nombre de points entiers qui gisent à l'intérieur ou sur la frontière de la région limitée par la parabole $y = x^2$ et la ligne $y = 50$ est :

- (A) 470 (B) 485 (C) 490 (D) 750 (E) 765

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC, modifiée par le rédacteur.

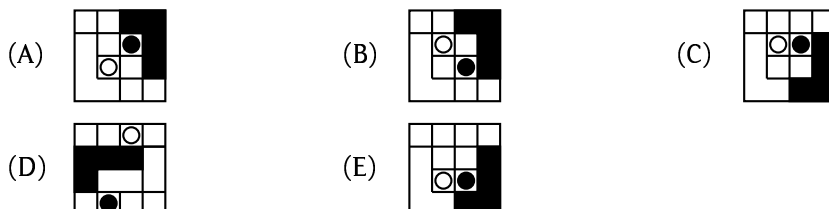
Commençons par trouver les points entiers sur la courbe $y = x^2$ entre $y = 0$ et $y = 50$: $(0, 0)$, $(1, 1)$, $(2, 4)$, $(3, 9)$, $(4, 16)$, $(5, 25)$, $(6, 36)$ et $(7, 49)$, et leurs équivalents pour les valeurs négatives de x . Nous savons déjà qu'il y a 51 points entiers sur l'axe des y dans la région. Pour une valeur de $x = k \neq 0$, le nombre de points entiers dans la région sur la droite verticale $x = k$ vaut $51 - k^2$. Pour chaque valeur de x différente de 0 sur le domaine étudié, on compte 2 fois les points entiers pour considérer les valeurs négatives correspondantes de x (sauf pour $x = 0$). Ainsi, le nombre de points est :

$$51 + 2(50 + 47 + 42 + 35 + 26 + 15 + 2) = 485.$$

La réponse est B.

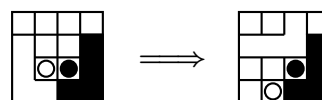
9. *Wot th'ell* is a game played on a 4×4 checker board. Both players have an L-shaped piece which covers four squares and a disc which covers one. The players alternate moves, one playing white pieces and the other playing black. A move consists of picking up the L-shaped piece, possibly turning it over, and placing it back on the board in a new position. Then the player removes his disc and puts it back on the board (the disc may be returned to where it came from). Neither the L-shaped piece nor the disc can be placed so that it covers any square that is already occupied by a disc or an L-shaped piece. A player who is unable to move loses.

The one of the following boards on which white can play and win is:



Official solution.

The answer is E. White can play from the position in E as shown to the right, after which black cannot place the L-shaped piece in any other position than its current position without covering a square that is already occupied by a disc or the other L-shaped piece.



10. Il y a une hauteur critique (qui est un nombre entier d'étages au-dessus du sol), telle qu'un oeuf largué de cette hauteur (ou au-dessus) va casser, mais que si on le largue d'une hauteur plus basse (quel qu'en soit le nombre de fois) ne cassera pas. On vous procure deux oeufs et on vous dit que cette hauteur est entre 1 et 37 étages (inclusivement). Vous voulez développer un plan qui permet de l'identifier de la manière la plus efficace. Évidemment, on pourrait commencer au premier étage et monter un étage à la fois. Cette stratégie pourrait prendre un coup si la hauteur critique est un étage, mais cette technique prendrait jusqu'à 37 coups si celui-ci est la hauteur critique. La stratégie optimale va nécessiter le nombre minimal de chutes. Dans le pire scénario, le nombre de chute nécessaire pour identifier la hauteur critique avec la stratégie optimale est :

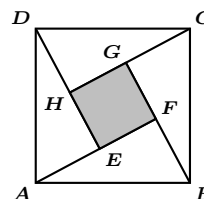
- (A) 8 (B) 9 (C) 12 (D) 19 (E) 35

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Ma stratégie consiste à lancer l'oeuf du 8^{ième}, 15^{ième}, 21^{ième}, 26^{ième}, 30^{ième}, 33^{ième}, 35^{ième} et 36^{ième} étages dans cet ordre. Dès que l'oeuf brise, on revient à l'étage le plus bas n'ayant pas été vérifiée et on remonte. De cette façon, il y a 8 pires scénarios possibles, au sens où la procédure sera la plus longue, soient quand l'étage où l'oeuf casse est l'un des suivants : 7, 14, 20, 25, 29, 32, 34, 37. La réponse est A.

BC Colleges High School Mathematics Contest 2005 Junior Final Round, Part B Friday, May 6, 2005

1. In the diagram, $ABCD$ is a square with side length 17 and the four triangles ABF , DAE , BCG , and CDH are congruent right triangles. Furthermore, $\overline{FB} = 8$. Find the area of the shaded quadrilateral $EFGH$.



Official solution.

Since $FB = 8$ and $AB = 17$, the Pythagorean Theorem gives $AF = 15$. The area of each of the four congruent triangles is $\frac{8 \times 15}{2} = 60$. Thus, the area of the shaded square $EFGH$ is $17^2 - 4(60) = 289 - 240 = 49$.

2. A party went to a restaurant for dinner. At the end of the meal they decided to split the bill evenly among them. If each contributed \$16, they found that they were \$4 short, while if each put in \$19, they had enough to pay the bill, 15% for the tip, and \$2 left over. How much was the bill, and how many were in the party?

Official solution.

Let n be the number in the party, and let b be the bill. We are given $16n = b - 4$ (or $b = 16n + 4$) and $19n = 1.15b + 2$, which together yield $19n = 1.15(16n + 4) + 2 = 18.4n + 6.6$. This implies that $0.6n = 6.6$, or $n = 11$. Then, $b = 16(11) + 4 = 176 + 4 = 180$. Hence, there were 11 in the party and the bill was \$180.

3. Find the number of solutions in integers (x, y) of the equation

$$x^2y^3 = 6^{12}.$$

Official solution.

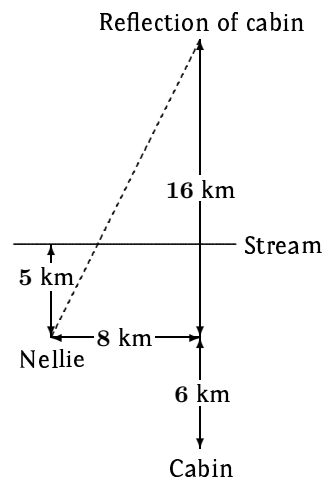
First, since $6^{12} > 0$ and $x^2 \geq 0$, we must have $x^2 > 0$ and $y > 0$. We are given that $x^2y^3 = 6^{12} = 2^{12}3^{12}$. Since x and y both divide $2^{12}3^{12}$, we must have $x = \pm 2^i3^j$ and $y = 2^k3^\ell$, where i, j, k , and ℓ are non-negative integers. Then $x^2 = 2^{2i}3^{2j}$ and $y^3 = 2^{3k}3^{3\ell}$. Therefore, $x^2y^3 = 2^{2i+3k}3^{2j+3\ell}$. We want $x^2y^3 = 2^{12}3^{12}$. Thus, $2i + 3k = 12$ and $2j + 3\ell = 12$. Solving these equations, we get $(i, k) \in \{(0, 4), (3, 2), (6, 0)\}$ and $(j, \ell) \in \{(0, 4), (3, 2), (6, 0)\}$.

Now, each of the three values of i can be paired with any of the three values of j . Once this is done, the values of k and ℓ are determined. Therefore, the number of solutions in positive integers is $3 \times 3 = 9$, and the total number of solutions is $9 \times 2 = 18$.

4. Nellie is 5 km south of a stream that flows due east. She is 8 km west and 6 km north of her cabin. She wishes to water her horse at the stream and then return to her cabin. What is the shortest distance that Nellie must travel?

Official solution.

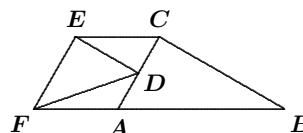
Nellie's shortest route will be a straight line to the stream followed by a straight line to her cabin. If we reflect in the stream the second part of her trip to the cabin, it is clear that the shortest distance will be the straight line from where Nellie starts to the reflection of the cabin. We use Pythagoras to calculate this distance as $\sqrt{8^2 + 16^2} = 8\sqrt{5}$.



5. In the diagram, triangle ABC is a 30° - 60° - 90° triangle with the right angle at vertex C , the 30° angle at vertex B , and side AB having length 20. Segment ED is perpendicular to side AC and D bisects AC . Segment EC is parallel to AB . Segment EF is perpendicular to ED and F is on the extension of AB .

(a) Find the length of segment ED .

(b) Find the length of segment DF .



Official Solution.

(a) In $\triangle ABC$, side AB is the hypotenuse; thus, $AC = 10$. Since D bisects AC , we have $CD = 5$. Since ED is perpendicular to AC and EC is parallel to AB , triangle CDE is a 30° - 60° - 90° triangle with the 30° angle at E and with hypotenuse EC . Then $EC = 10$ and $ED = 5\sqrt{3}$.

(B) Since segment EF is perpendicular to ED , it is parallel to AC . Furthermore, AF is parallel to EC , implying that $ACEF$ is a rhombus with all sides equal to 10. Applying Pythagoras's Theorem to right triangle DEF gives $DF = \sqrt{175} = 5\sqrt{7}$.

We received a late solution set with correct solutions to 7 of the 10 problems from the 21st W.J. Blundon Mathematics Contest. Unfortunately, it did not arrive in time for publication in the March Skoliad. Thank you to Natalia Desy, grade 8, SMP Xaverius, Palembang, Indonesia. We hope your next set of solutions arrives on time to be considered for publication.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Jean-François Désilets. Congratulations Jean-François! Continue sending in your contests and solutions.

ERRATUM

In the article entitled *On an Inequality from IMO 2005* by Vasile Cîrtoaje in the March 2006 issue, the editors neglected to make a correction to two conjectured inequalities. The right sides of inequalities (a) and (b) on page 106 should both be " ≤ 1 " instead of " ≥ 0 ". We apologize to the author for failing to make this correction.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is Jeff Hooper (Acadia University). The other staff members are John Grant McLoughlin (University of New Brunswick), Ian VanderBurgh (University of Waterloo), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Please send your solutions to the problems in this edition by 1 August 2006. Solutions received after this date will only be considered if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M238. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Let PQ be a chord of a parabola, and let R be the mid-point of PQ . Let S be a point on the parabola such that the tangent at S is parallel to PQ . The tangents at P and Q intersect at T . Show that R , S , and T are collinear.

M239. *Proposed by Aliyev Yakub, Baku State University, Baku, Azerbaijan.*

If $a, b, c > 0$, prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{(a+b+c)^2}{6abc}.$$

M240. *Proposed by the Mayhem Staff.*

Using each of the digits 0, 1, 2, 3, ..., 9 just once, find four perfect squares (greater than zero) such that one consists of four digits, one consists of three digits, one consists of two digits, and one consists of one digit. (**Note:** There is more than one solution. How many can you find?)

M241. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Three gunfighters, called Quick, Fast, and Slow, stand one at each vertex of an equilateral triangle. Quick is faster on the draw than Fast, and Fast is faster than Slow. If x intends to fire at y , we will say that x targets y . We will assume that if x fires at y , then y will be hit, and that if x and y both target each other, the one who is slower on the draw will be hit before he can fire. A combatant cannot fire once he has been hit.

In the first phase of the confrontation, each combatant targets one of the other two and fires a maximum of one round. No man knows how fast the other two are, and the targeting choices are made randomly and cannot be changed during the first phase.

If two combatants survive the first phase, they face each other in a second phase and the fastest draw wins. If only one combatant survives the first phase, he is the winner (and there is no second phase).

Find the probability that:

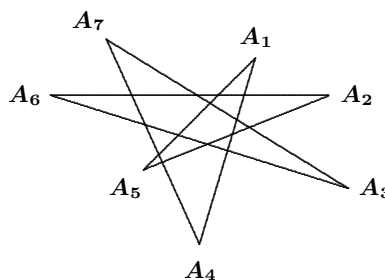
- (a) Quick survives; (b) Fast survives; (c) Slow survives.

M242. *Proposed by Houda Anoun, Bordeaux, France.*

For which natural numbers x is the number $x^4 + x^3 + x^2 + x + 1$ a perfect square?

M243. *Proposed by K.R.S. Sastry, Bangalore, India.*

In the 7-point star shown, no three lines are concurrent. Find the sum $A_1 + A_2 + \dots + A_7$.



.....

M238. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Soit PQ une corde d'une parabole et soit R le point milieu de PQ . Soit S un point sur la parabole tel que la tangente en S est parallèle à PQ . Si T désigne le point d'intersection des tangentes en P et Q , montrer que R, S et T sont colinéaires.

M239. *Proposé par Aliyev Yakub, Baku State University, Baku, Azerbaijan.*

Si $a, b, c > 0$, montrer que

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{(a+b+c)^2}{6abc}.$$

M240. *Proposé par l'Équipe de Mayhem.*

En utilisant une seule fois chacun des chiffres de 0 à 9, trouver quatre carrés parfaits (positifs) tels qu'il y en ait un de quatre chiffres, un de trois, un de deux et un dernier de un chiffre. (Note : Il y a plus d'une solution. Combien pouvez-vous en trouver?)

M241. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Trois mousquetaires, appelés Vif, Rapide et Lent, sont postés aux sommets d'un triangle équilatéral. Vif a la gâchette plus rapide que Rapide, qui lui est plus rapide que Lent. Si x s'apprête à tirer sur y , on dira que x vise y . On va supposer que si x tire sur y , alors y sera touché, et que si x et y se visent l'un l'autre, celui qui a la gâchette la moins rapide sera touché avant qu'il ne puisse tirer. Une fois touché, aucun des trois ne peut tirer.

Dans la première phase de la confrontation, chaque combattant vise l'un des deux autres et tire une cartouche au maximum. Aucun des participants ne sait lequel des deux autres est le plus rapide ; le choix des visées se fait au hasard sans possibilité de changement durant la première phase.

Si deux combattants survivent à la première phase, ils se font face et c'est le plus rapide qui gagne. S'il n'y a qu'un survivant au terme de la première phase, il est déclaré vainqueur (et il n'y a pas de seconde phase).

Trouver la probabilité que :

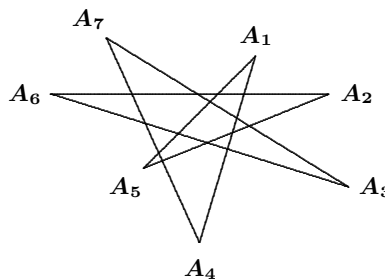
- (a) Vif survive ; (b) Rapide survive ; (c) Lent survive.

M242. *Proposé par Houda Anoun, Bordeaux, France.*

Pour quels nombres naturels x le nombre $x^4 + x^3 + x^2 + x + 1$ est-il un carré parfait ?

M243. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Dans l'étoile à 7 branches de la figure, il n'y a aucun sous-ensemble de trois droites concourantes. Trouver la somme $A_1 + A_2 + \dots + A_7$.



Mayhem Solutions

M182. *Proposed by Babis Stergiou, Chalkida, Greece.*

If a, b, c are positive numbers, such that $a + b + c = 1$, prove that

$$(1 + a)(1 + b)(1 + c) \geq 8(1 - a)(1 - b)(1 - c).$$

Solution by Titu Zvonaru, Comănești, Romania.

By the AM–GM Inequality we have

$$1 + a = a + b + c + a = a + b + a + c \geq 2\sqrt{(a + b)(a + c)}.$$

Hence,

$$1 + a \geq 2\sqrt{(1 - c)(1 - b)}. \quad (1)$$

Similarly,

$$1 + b \geq 2\sqrt{(1 - a)(1 - c)} \quad (2)$$

and

$$1 + c \geq 2\sqrt{(1 - b)(1 - a)}. \quad (3)$$

From (1), (2), and (3), the given inequality follows. Equality holds if and only if $a + b = a + c = b + c$; that is, if and only if $a = b = c = \frac{1}{3}$.

Also solved by the Austrian IMO-Team 2005: Yimin Ge, Peter Gila, Bernhard Kininger, Michael Moshammer, Jakob Preininger, Thomas Takacs.

M183. *Proposed by the Mayhem Staff.*

In the array at right, two letters are called *neighbouring* letters if they are adjacent to each other horizontally, vertically, or diagonally. Starting from any letter “M” on the outside of the array, find the number of ways of spelling “MATH” by moving only between neighbouring letters.

```

M M M M M M M
M A A A A A M
M A T T T A M
M A T H T A M
M A T T T A M
M A A A A A M
M M M M M M M

```

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

We will count the MATH words using the As, since the H is common to all words. The 4 corner As have 5 adjacent Ms and 1 adjacent T; thus, the 4 corners give us 20 occurrences of MATH. The 8 As adjacent to a corner A have 3 Ms and 2 Ts adjacent; whence, these 8 As give us 48 occurrences of MATH. The 4 middle As have 3 Ms and 3 Ts adjacent, which means that the 4 middle As give us 36 more occurrences of MATH. Therefore, we have $20 + 48 + 36 = 104$ occurrences of MATH in the array.

Also solved by Andrew Fischer and Frank Barlow (Humke's Raiders), Washington and Lee University, Lexington, VA, USA; and Titu Zvonaru, Comănești, Romania. One incorrect solution was received.

M184. *Proposed by the Mayhem Staff.*

Find all solutions (a, b) for the equation $ab - 24 = 2a$, where a and b are positive integers.

Solution by Geoffrey Siu, London Central Secondary School, London, ON.

Rearranging $ab - 24 = 2a$, we get $a(b - 2) = 24$. Thus, for positive integer solutions, a and $b - 2$ must be factors of 24. Hence, there are 8 solutions for (a, b) , namely

$$(1, 26), (2, 14), (3, 10), (4, 8), (6, 6), (8, 5), (12, 4), \text{ and } (24, 3).$$

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Titu Zvonaru, Comănești, Romania.

M185. *Proposed by Neven Jurič, Zagreb, Croatia.*

A lake has the shape of a triangle with sides of length a , b , and c . From a helicopter, which is hovering in a stationary position above the lake, the lines-of-sight to the three vertices of the triangle are pairwise perpendicular. How high is the helicopter above the lake?

Solution by Titu Zvonaru, Comănești, Romania.

Let ABC be the lake and let H be the helicopter. We denote $HA = x$, $HB = y$ and $HC = z$. Since $HA \perp HB$, $HB \perp HC$, and $HC \perp HA$, we have

$$\begin{aligned}x^2 + y^2 &= c^2, \\y^2 + z^2 &= a^2, \\z^2 + x^2 &= b^2.\end{aligned}$$

Solving the system, we obtain

$$x^2 = \frac{-a^2 + b^2 + c^2}{2}, \quad y^2 = \frac{a^2 - b^2 + c^2}{2}, \quad z^2 = \frac{a^2 + b^2 - c^2}{2}.$$

Hence, $\triangle ABC$ must be acute-angled. If h is the distance from H to the lake, we can write the volume of the tetrahedron $HABC$ in two ways:

$$\frac{1}{3}[ABC]h = \frac{1}{3}[HAB] \cdot HC,$$

which yields

$$h = \frac{xyz}{2[ABC]} = \frac{\sqrt{2(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}}{8\sqrt{s(s-a)(s-b)(s-c)}},$$

where $s = \frac{1}{2}(a + b + c)$ is the semi-perimeter of $\triangle ABC$.

M186. *Proposed by the Mayhem Staff.*

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . For example, $\lfloor 2.5 \rfloor = 2$ and $\lfloor -7.4 \rfloor = -8$. Given that $\sum_{i=1}^n \lfloor \sqrt{i} \rfloor = 217$, determine the value of n .

Solution by Geoffrey Siu, London Central Secondary School, London, ON.

For any positive integers i and j , if $j^2 \leq i < (j+1)^2$, then $\lfloor \sqrt{i} \rfloor = j$. Thus,

$$\sum_{i=j^2}^{(j+1)^2-1} \lfloor \sqrt{i} \rfloor = \sum_{i=j^2}^{(j+1)^2-1} j = j((j+1)^2 - j^2) = j(2j+1).$$

Hence,

$$\begin{aligned} \sum_{i=1}^{48} \lfloor \sqrt{i} \rfloor &= \sum_{i=1}^3 \lfloor \sqrt{i} \rfloor + \sum_{i=4}^8 \lfloor \sqrt{i} \rfloor + \sum_{i=9}^{15} \lfloor \sqrt{i} \rfloor \\ &\quad + \sum_{i=16}^{24} \lfloor \sqrt{i} \rfloor + \sum_{i=25}^{35} \lfloor \sqrt{i} \rfloor + \sum_{i=36}^{48} \lfloor \sqrt{i} \rfloor \\ &= 1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + 5 \cdot 11 + 6 \cdot 13 = 203, \end{aligned}$$

which is 14 short. Then

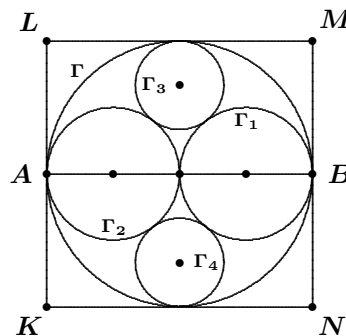
$$\sum_{i=1}^{50} \lfloor \sqrt{i} \rfloor = \sum_{i=1}^{48} \lfloor \sqrt{i} \rfloor + \sum_{i=49}^{50} \lfloor \sqrt{i} \rfloor = 203 + 2 \cdot 7 = 217.$$

Thus, the required value of n is 50.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; Alper Cay, Uzman Private School, Kayseri, Turkey; Chris Dadak, Washington and Lee University, Lexington, VA, USA; and Titu Zvonaru, Comănești, Romania.

M187. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A circle Γ of radius $2r$ is inscribed in a square $KLMN$. Line segment AB is a diameter of this circle, where A and B are mid-points of opposite sides of the square. Two circles Γ_1 and Γ_2 of radii r have centres on AB and are externally tangent to one another, and each is internally tangent to Γ . Two circles Γ_3 and Γ_4 are externally tangent to Γ_1 and Γ_2 and internally tangent to Γ .

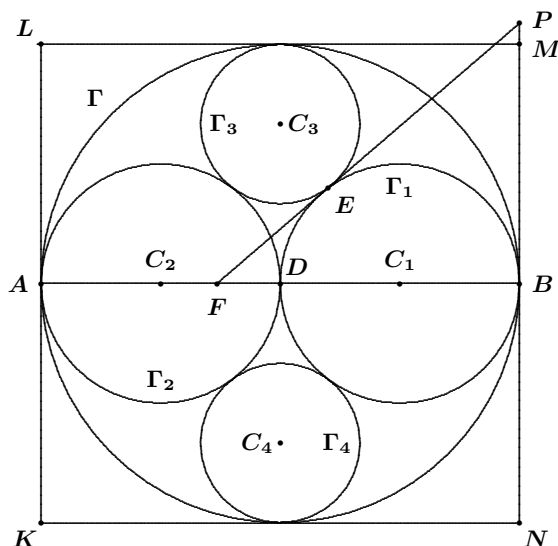


Construct the common tangent to Γ_1 and Γ_3 using straight edge and compass with a minimum use of the compass.

What is the minimum number of times that the compass has to be used?

Solution by Geoffrey Siu, London Central Secondary School, London, ON.

We will prove that the common tangent to Γ_1 and Γ_3 passes through M . This implies that the tangent can be drawn without a compass by joining M to the intersection of Γ_1 and Γ_3 .



Let C_i be the centre of Γ_i for $i = 1, 2, 3, 4$, and let D be the centre of Γ . Let the radius of Γ_3 be a . By symmetry, $C_3D \perp AB$. By applying the Pythagorean Theorem to $\triangle C_3DC_1$, we get

$$r^2 + (2r - a)^2 = (a + r)^2,$$

which gives $a = \frac{2}{3}r$.

Let E be the point of tangency of Γ_1 and Γ_3 . Let F be the point of intersection of the common tangent to Γ_1 and Γ_3 with AB , and let P be the point of intersection of this tangent with the line through M and N (see figure). By ASA congruence, we have $\triangle C_3DC_1 \cong \triangle FEC_1$. Thus,

$$\begin{aligned} FC_1 &= C_3C_1 = \frac{5}{3}r, \\ FB &= \frac{5}{3}r + r = \frac{8}{3}r, \\ \text{and } FE &= C_3D = 2r - \frac{2}{3}r = \frac{4}{3}r. \end{aligned}$$

By AAA similarity, $\triangle FEC_1$ is similar to $\triangle FBP$. Hence,

$$\begin{aligned} \frac{PB}{EC_1} &= \frac{FB}{FE}, \\ PB &= \frac{FB \cdot EC_1}{FE} = \frac{\left(\frac{8}{3}r\right)(r)}{\left(\frac{4}{3}r\right)} = 2r. \end{aligned}$$

Therefore, P is located at M .

We conclude that the common tangent to Γ_1 and Γ_3 passes through M .

Problem of the Month

Ian VanderBurgh

On competitions which are traditionally hard, straightforward problems are unexpected. This actually makes these easier problems harder because we do not tend to look for an easy solution. Here is one such problem.

Problem (2004 William Lowell Putnam Mathematical Competition)

Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N , but by the end of her season, $S(N)$ was more than 80% of N . Was there necessarily a moment in between when $S(N)$ was exactly 80% of N ?

The Putnam Competition is a North American mathematics contest for undergraduate university students, written at the beginning of December. The competition consists of two 3-hour sessions, each comprised of six problems, with each problem marked out of 10. Often, the median mark of the 3000 to 4000 competitors is 0 or 1 out of 120. (The marks tend to be low for two reasons: many of the problems are hard, and the problems are marked in a way that does not tend to give many part marks.) Many of the problems can be approached with advanced secondary school mathematics, and very few require much material beyond the level of first-year undergraduate mathematics.

Some Putnam problems are quite approachable, like this one.

A good first step for this problem is to try and quickly find an example that shows how she can get from below 80% to above 80% without passing through exactly 80%. Any luck? I didn't think so. This is what I attempted, and try as I might, I always hit 80%. So it appears that there is necessarily a moment when $S(N)$ is exactly 80% of N .

Solution

We prove this by contradiction, assuming that there is no such moment. If there is no such moment, we can make the following observation: there must be a point in time when the ratio is less than 80% after a particular free throw and above 80% after the next free throw. This is true because there must be a first time when the ratio is above 80%.

Let us then suppose that after n free throws, Shanille has made fewer than 80% of her attempts, say m attempts made, and after $n + 1$ free throws, she has made more than 80% of her attempts. Did she make the $(n + 1)^{\text{st}}$ free throw? Yes, she must have; otherwise, her percentage made would have decreased, not increased. Thus, after $n + 1$ free throws, she has made $m + 1$ of her attempts.

Since 80% can be written as $\frac{4}{5}$, we have

$$\frac{m}{n} < \frac{4}{5} \quad \text{and} \quad \frac{m+1}{n+1} > \frac{4}{5}.$$

Rearranging, we obtain $5m < 4n$ and $5m + 5 > 4n + 4$; that is,

$$0 < 4n - 5m < 1.$$

But n and m are integers; thus, we cannot have $0 < 4n - 5m < 1$. This is a contradiction.

Therefore, our assumption is wrong, so there must be a point where $S(n)$ is exactly 80% of N .

This is a pretty fast solution to a problem that, based on its source, looks like it could be quite complicated.

As I was thinking about this problem, I was a bit disconcerted, because it did not seem like this was the sort of thing that should be true in general. That got me wondering if there was something special about 80%. So I tried to see what happened with 50%. Again, I could not go from less than 50% to greater than 50% without going through exactly 50%.

Then I tried 60%. We can see that 3 out of 6 free throws is less than 60%, that 4 out of 7 is less than 60%, but 5 out of 8 is greater than 60%. (Here, we assume that Shanille makes her 7th and 8th free throws.) Hence, we indeed can go from one side of 60% to the other without going through exactly 60%. What is going on here?

The percentages 80%, 50% and 60% can be written as the following fractions in lowest terms: $\frac{4}{5}$, $\frac{1}{2}$ and $\frac{3}{5}$. Why do these fractions behave differently?

Let's go back to our solution and try to trace the contradiction. The contradiction emerged when we had an integer which had to be "sandwiched" between 0 and 1, which cannot happen. The "0" occurs pretty naturally in the proof, from looking at the last instant where the fraction m/n is less than the given percentage. The "1", on the other hand, comes from the difference between the numerator and denominator in the representation of the percentage as a fraction (in our solution, the "1" comes from $5 - 4$). That's the distinction between the fractions $\frac{4}{5}$, $\frac{1}{2}$, and $\frac{3}{5}$. The first two have the property that their numerator is one less than their denominator, while the third does not have this property.

It appears that, in general, there must be a moment when the percentage is exactly the given percentage if the given percentage can be written as a fraction in lowest terms whose numerator is one less than its denominator. (Can you prove this?)

Now that was really interesting! We managed to figure out what was going on in a more general situation than the one we were given, and this generalization led us to understand the original problem more fully. This is an interesting example of problem modification—taking a problem and trying to understand it and then change it slightly. This is a favourite technique of contest and problems creators worldwide.

THE OLYMPIAD CORNER

No. 253

R.E. Woodrow

We begin this issue with the problems of the Final Round of the Japanese Mathematical Olympiad 2003. My thanks go to Andy Liu, Canadian Team Leader to the IMO in Japan, for collecting them for our use.

JAPAN MATHEMATICAL OLYMPIAD 2003

February 11, 2003

Time: 4 hours

1. A point P lies in a triangle ABC . The edge AC meets the line BP at Q , and AB meets CP at R . Suppose that $AR = RB = CP$ and $CQ = PQ$. Find $\angle BRC$.

2. We have two distinct positive integers a and b , with b a multiple of a . When written in decimal form, each of a and b consists of $2n$ digits, with the most significant digit non-zero. Furthermore, the n most significant digits of a are identical to the n least significant digits of b , and vice versa, as in $n = 2$, $a = 1234$, $b = 3412$ (although this example does not satisfy the other condition, that b be a multiple of a). Determine a and b .

3. Find the greatest real number k such that, for any positive a, b, c with $a^2 > bc$,

$$(a^2 - bc)^2 > k(b^2 - ca)(c^2 - ab).$$

4. Let p and q be relatively prime integers with $q \geq 2$. A list of integers $(r, a_1, a_2, \dots, a_n)$, with $|a_i| \geq 2$ for each $i = 1, \dots, n$, is said to be an expansion of p/q if

$$\frac{p}{q} = r + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}.$$

For example, $(-1, -3, 2, -2)$ is an expansion of $-10/7$ since

$$\frac{-10}{7} = -1 + \frac{1}{-3 + \frac{1}{2 + \frac{1}{-2}}}.$$

Now define the *weight* of an expansion $(r, a_1, a_2, \dots, a_n)$ to be the product

$$(|a_1| - 1)(|a_2| - 1) \cdots (|a_n| - 1).$$

For example, the weight of the expansion $(-1, -3, 2, -2)$ of $-10/7$ is 2.

Show that the sum of the weights of all expansions of p/q is q .

5. In a plane determine the greatest possible integer n such that one can place n distinct points with no three collinear, and colour each of them with either red, green, or yellow so that:

- inside each triangle with all vertices red, there is at least one green point;
- inside each triangle with all vertices green, there is at least one yellow point; and
- inside each triangle with all vertices yellow, there is at least one red point.

Next we turn to the First Round of the Hungarian National Olympiad 2002–2003, for the special mathematical classes, First Round and Final Round. Thanks go to Andy Liu, Canadian Team Leader to the IMO in Japan, for submitting these for our use.

**HUNGARIAN MATHEMATICAL OLYMPIAD
2002–2003
National Olympiad for the Special Mathematical Classes
First Round**

1. A rectangular brick has volume $V = x \text{ cm}^3$, and surface area $S = y \text{ cm}^2$. Find the minimal volume for which $x = 10y$.

2. The triangle H is tiled by the subtriangles H_1, H_2, \dots, H_n . The inradius of H is r , and the inradii of H_1, H_2, \dots, H_n are r_1, r_2, \dots, r_n , respectively. Prove that $r \leq r_1 + r_2 + \dots + r_n$.

3. Let ABC be a triangle. We drop a perpendicular from A to the internal bisectors starting from B and C , their feet being A_1 and A_2 . In the same way we define B_1, B_2 and C_1, C_2 . Prove that

$$2(A_1A_2 + B_1B_2 + C_1C_2) = AB + BC + CA.$$

4. We have a small and a large box. In the small box there are 20 red and 20 white balls; in the large one there are 1005 red and 995 white balls. We choose a box, draw a ball from it and look at it. Without putting it back, we again choose a box and draw a ball from it. Find the best strategy in order to maximize the probability of having at least one red ball among the two chosen balls.

5. Is it possible to find 2002 distinct positive integers such that if x and y are any two of them, then $|x - y| = \gcd(x, y)$?

Final Round

1. A set H of points in the plane is called *nice* if any three points of H has an axis of symmetry. Prove the following statements:
 - (a) A nice set is not necessarily symmetrical.
 - (b) If a nice set H has exactly 2003 points, then all of them must lie on a line.

2. We colour the vertices of a 2003-gon with red, blue, and green such that neighbours cannot have the same colour. In how many ways can we accomplish this?

3. Let t be a fixed positive integer. Let $f_t(n)$ denote the number of integers k such that $1 \leq k \leq n$ and $\binom{k}{t}$ is odd. (If $1 \leq k < t$, then $\binom{k}{t} = 0$.) Prove that if n is a sufficiently great power of 2, then $\frac{f_t(n)}{n} = \frac{1}{2^r}$, where r is an integer determined by t and independent of n .

As a final contest set for this number we give the problems of the 2002 Kürschák Competition. Thanks again go to Andy Liu for saving these for our use.

— 2002 KÜRSCHÁK COMPETITION —

1. We have an acute-angled triangle which is not isosceles. We denote its orthocentre, circumcentre, and incentre by M , K , and O , respectively. Prove that if a vertex of the triangle is on the circumcircle of MKO , then there must be another vertex on the circumcircle.

2. The Fibonacci sequence is defined by the following recursion: $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$. Suppose that the positive integers a and b satisfy:

$$\min \left\{ \frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n} \right\} \leq \frac{a}{b} \leq \max \left\{ \frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n} \right\}.$$

Prove that $b \geq f_{n+1}$.

3. Prove that one can distribute all the sides and diagonals of a convex 3^n -gon into groups of three segments such that in each group the three segments form a triangle.

Now we return to solutions from our readers to problems from the September 2004 number of the *Corner* and the Midi Finale of the 26th Olympiade Mathématique Belge [2004 : 268–269].

1. Les parallélogrammes $ABCD$ et $AEFG$ sont tels que E appartient à la droite BC et D à la droite FG . Comparer les aires de ces deux parallélogrammes. Sont-elles égales ? L'une est-elle toujours plus grande que l'autre ? Si oui, laquelle ?

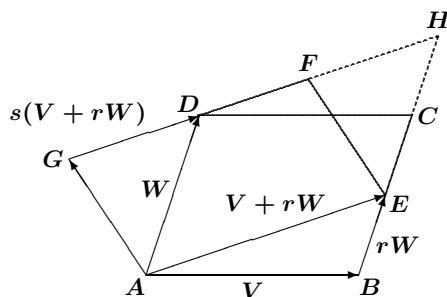
Solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Geoffrey A. Kandall, Hamden, CT, USA. First, we give Kandall's solution.

The areas are equal. Let $V = \overrightarrow{AB}$ and $W = \overrightarrow{AD}$. Then there exist scalars r and s such that $\overrightarrow{BE} = rW$ and $\overrightarrow{GD} = s(V + rW)$. Hence,

$$\overrightarrow{AG} = \overrightarrow{AD} + \overrightarrow{DG} = \overrightarrow{AD} - \overrightarrow{GD} = -sV + (1 - rs)W.$$

Then

$$\begin{aligned} [AEFG] &= |(V + rW) \times (-sV + (1 - rs)W)| \\ &= |(1 - rs)(V \times W) - rs(W \times V)| \\ &= |(1 - rs)(V \times W) + rs(V \times W)| = |V \times W| = [ABCD]. \end{aligned}$$



Next, we give the solution of Bilinski.

Let $H = BC \cap FG$. Then $AE \parallel HD$ (since $AE \parallel GF$ and the points G, D, F , and H are collinear) and $AD \parallel EH$ (since $AD \parallel BC$ and B, E, C , and H are collinear). Therefore, $AEHD$ is a parallelogram.

Parallelograms $ABCD$ and $AEHD$ have equal areas because the side AD is common to both parallelograms and the height from AD to BC in $ABCD$ is the same as the height from AD to EH in $AEHD$. Similarly, parallelograms $AEHD$ and $AEFG$ have equal areas because they have the common side AE and the same height perpendicular to AE . Thus, $ABCD$ and $AEFG$ have equal areas.

2. Trouver tous les entiers x pour lesquels \sqrt{x} et $\sqrt{x - \sqrt{x}}$ sont eux-mêmes des entiers.

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Pierre Bornshtein, Maisons-Laffitte, France. We give Bornshtein's write-up.

Soit $x \geq 0$ une éventuelle solution du problème. De la première condition, on déduit que $x = a^2$ pour un certain entier $a \geq 0$. La deuxième condition s'écrit alors : $a^2 - a = b^2$ pour un certain entier $b \geq 0$.

Mais alors, si $a \geq 2$, on a $(a-1)^2 < b^2 < a^2$, ce qui est impossible. Donc $a = 0$ ou $a = 1$; c'est à dire, $x = 0$ ou $x = 1$.

Réciproquement, $x = 0$ et $x = 1$ sont bien des solutions du problème.

3. Déterminer tous les triplets (x, y, z) constitués d'entiers naturels qui satisfont à l'équation

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1.$$

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Pierre Bornshtein, Maisons-Laffitte, France. We give Bornshtein's solution.

Soit (x, y, z) une éventuelle solution du problème. Alors

$$yz + 2xz = xyz + 3xy. \quad (1)$$

Si $x \geq 3$ et $y \geq 3$, alors $y(x-1) \geq 3(x-1) \geq 2x$. Par suite :

$$xyz + 3xy > xyz = yz + (x-1)yz \geq yz + 2xz,$$

qui contredit (1). Donc $x < 3$ ou $y < 3$.

Supposons d'abord que $y \geq 3$. Alors $x = 1$ ou $x = 2$.

Si $x = 2$, on veut résoudre $yz + 4z = 2yz + 6y$, ou $yz + 6y = 4z$. Donc $(y-4)(z+6) = -24$. Or $y-4 \geq -1$ et $z+6 \geq 7$, donc $y-4 = -1$ et $z+6 = 24$. Cela conduit à $(x, y, z) = (2, 3, 18)$, qui est effectivement une solution.

Si $x = 1$, on veut résoudre $2z = 3y$, ce qui conduit directement à $(x, y, z) = (1, 2a, 3a)$ où $a \geq 2$ (car $y \geq 3$) est un entier. Réciproquement, on vérifie facilement que ces triplets sont bien solutions.

Si $y = 2$, l'équation devient $3x = z$, d'où les triplets solutions $(b, 2, 3b)$ où $b > 0$ est un entier (notons que l'on récupère le cas $a = 1$ pour les triplets du paragraphe précédent).

Si $y = 1$, l'équation devient $3x = z + xz$, c'est à dire $(x+1)(3-z) = 3$, avec $x+1 \geq 2$. Cela conduit à $x+1 = 3$ et $3-z = 1$; c'est à dire, $x = z = 2$. Réciproquement, le triplet $(2, 1, 2)$ est bien solution.

Finalement, les solutions sont les triplets de la forme $(1, 2a, 3a)$, $(a, 2, 3a)$ où $a > 0$ est un entier, ainsi que $(2, 1, 2)$ et $(2, 3, 18)$.

4. (a) Quelles sont toutes les valeurs prises par le reste (par défaut) de la division de a^3 par 7 lorsque a est un entier naturel quelconque ?

(b) Les entiers naturels a, b et c sont tels que $a^3 + b^3 + c^3$ est divisible par 7. Que vaut alors le reste de la division de $a \cdot b \cdot c$ par 7 ?

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; and Pierre Bornsztejn, Maisons-Laffitte, France. We give Bornsztejn's write-up.

(a) On vérifie sans astuce que, selon la congruence de a modulo 7, celle de a^3 est respectivement :

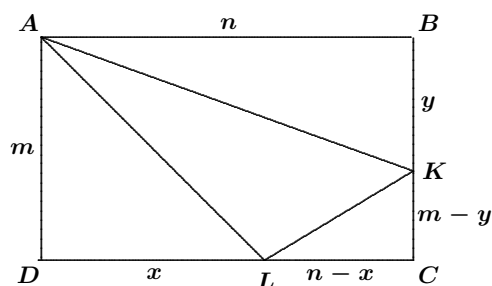
$$\begin{aligned} a &\equiv 0, 1, 2, 3, 4, 5, 6 \\ a^3 &\equiv 0, 1, 1, 6, 1, 6, 6. \end{aligned}$$

(b) D'après le tableau ci-dessus, on constate que si $a^3 + b^3 + c^3$ est divisible par 7, c'est que modulo 7, on a $(a, b, c) \equiv (0, 0, 0)$ ou $(0, 1, 6)$ (à l'ordre près). Dans les deux cas, on a $abc \equiv 0 \pmod{7}$.

Next we look at solutions to problems of the Maxi Finale of the 26th Olympiade Mathématique Belge, given [2004 : 269].

1. Étant donné un rectangle $ABCD$, déterminer deux points K et L respectivement sur $[BC]$ et sur $[CD]$ tels que les triangles ABK , AKL et ADL aient la même aire.

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; and Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's solution.



Assume $[ABK] = [AKL] = [ADL]$. Let $AD = m$, $AB = n$, $DL = x$, and $BK = y$. Then $[ADL] = \frac{1}{2}mx$, $[ABK] = \frac{1}{2}ny$, and

$$[AKL] = mn - \frac{1}{2}mx - \frac{1}{2}ny - \frac{1}{2}(m - y)(n - x) = \frac{1}{2}mn - \frac{1}{2}xy.$$

From $[ABK] = [ADL]$ we obtain $mx = ny$; that is, $\frac{x}{n} = \frac{y}{m} = t$, say. Since $[ABK] = [AKL]$, it follows that $ny = mn - xy$. Putting $x = nt$ and $y = mt$, we see easily that $t^2 + t - 1 = 0$. Hence, $t = \frac{1}{2}(\sqrt{5} - 1)$.

Conversely, if $t = \frac{1}{2}(\sqrt{5} - 1)$, it is a simple matter to verify that $[ABK] = [AKL] = [ADL]$. Thus, the points K and L are determined by the conditions

$$\frac{BK}{BC} = \frac{DL}{DC} = \frac{\sqrt{5} - 1}{2}.$$

2. Trouver toutes les solutions du système suivant d'inconnues réelles x, y, u et v .

$$\begin{aligned}x^2 + y^2 &= 1, & xu + yv &= 1, \\u^2 + v^2 &= 1, & xu - yv &= \frac{1}{2}.\end{aligned}$$

Résolu par Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; et Pierre Bornsztein, Maisons-Laffitte, France. Nous présentons la solution par Bornsztein.

Soit x, y, u, v des réels tels que

$$x^2 + y^2 = 1 \quad (1)$$

$$u^2 + v^2 = 1 \quad (2)$$

$$xu + yv = 1 \quad (3)$$

$$xu - yv = \frac{1}{2}. \quad (4)$$

Les équations (1) et (2) sont équivalentes à

$$x = \cos \theta, \quad y = \sin \theta, \quad u = \cos \alpha, \quad v = \sin \alpha,$$

où $\theta, \alpha \in [0, 2\pi)$.

L'équation (3) s'écrit alors $\cos(\theta - \alpha) = 1$; c'est à dire, $\theta = \alpha$ (puisque $-2\pi < \theta - \alpha < 2\pi$); c'est à dire, $x = u$ et $y = v$.

Mais alors, l'équation (4) s'écrit : $x^2 - y^2 = \frac{1}{2}$. À l'aide de (1), il vient directement $x^2 = \frac{3}{4}$ et $y^2 = \frac{1}{4}$.

Finalement, les quadruplets (x, y, u, v) solutions sont :

$$\begin{aligned}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}\right), & \quad \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \\ \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{2}\right), & \quad \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).\end{aligned}$$

4. Les entiers $a_0, a_1, a_2, \dots, a_{100}$ satisfont aux conditions suivantes :

$$\begin{aligned}a_1 &> a_0 \geq 0, \\ a_{k+2} &= 3a_{k+1} - 2a_k \quad (\text{pour } k = 0, 1, 2, \dots, 98).\end{aligned}$$

Comparer les nombres a_{100} et 2^{99} . Sont-ils égaux? L'un est-il toujours plus grand que l'autre? Si oui, lequel?

Résolu par Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; et Pierre Bornsztein, Maisons-Laffitte, France. Nous présentons la solution par Bornsztein.

On va prouver que $a_{100} > 2^{99}$.

Notons tout d'abord que puisque a_0 et a_1 sont entiers, la condition $a_1 - a_0 > 0$ donne $a_1 - a_0 \geq 1$. De $a_{k+2} = 3a_{k+1} - 2a_k$, on déduit qu'il

existe deux constantes λ et μ telles que, pour $k = 0, 1, \dots, 100$, on ait $a_k = \lambda \cdot 1^k + \mu \cdot 2^k$. En utilisant cette relation pour $k = 0$ et $k = 1$, il vient $a_k = 2a_0 - a_1 + (a_1 - a_0)2^k$. En particulier :

$$\begin{aligned} a_{100} &= 2a_0 - a_1 + (a_1 - a_0)2^{100} \\ &= (a_1 - a_0)2^{99} + (a_1 - a_0)(2^{99} - 1) + a_0 \\ &\geq 2^{99} + (2^{99} - 1) + a_0 \geq 2^{99} + a_0 \geq 2^{99}, \end{aligned}$$

d'où la conclusion annoncée.

Now we turn to the October 2004 *Corner* and our readers' solutions to a problem of the XVII Argentinian Mathematical Olympiad [2004 : 342–344].

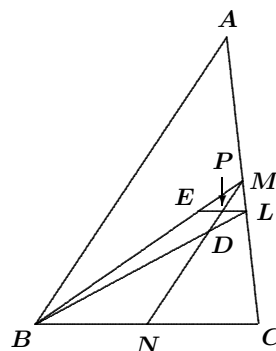
2. Given a triangle ABC with side AB greater than BC , let M be the mid-point of AC , and let L be the point at which the bisector of $\angle B$ cuts side AC . A straight line is drawn through M parallel to AB , cutting the bisector BL at D , and another straight line is drawn through L parallel to BC , cutting the median BM at E . Show that ED is perpendicular to BL .

Solved by Michel Bataille, Rouen, France; and Babis Stergiou, Chalkida, Greece. First we present Stergiou's solution.

Let N be the mid-point of BC . The line through M parallel to AB intersects BC at N and bisects EL at a point P . Triangle PDL is isosceles, since

$$\begin{aligned} \angle PLD &= \angle LBC = \frac{1}{2}\angle B \\ \text{and } \angle PDL &= \angle ABL = \frac{1}{2}\angle B. \end{aligned}$$

We then have $DP = PL = PE$, which implies that $\triangle DEL$ is right-angled at D ; that is, $DE \perp BL$.



Next we give Bataille's solution.

Let $a = BC$, $b = CA$, and $c = AB$, as usual.

First, we note that L is between A and C , and $\frac{LC}{LA} = \frac{BC}{BA} = \frac{a}{c}$. Thus,

$$c\overrightarrow{LC} + a\overrightarrow{LA} = \vec{0}. \quad (1)$$

Hence, $c(\overrightarrow{BC} - \overrightarrow{BL}) + a(\overrightarrow{BA} - \overrightarrow{BL}) = \vec{0}$, which gives

$$(a + c)\overrightarrow{BL} = c\overrightarrow{BC} + a\overrightarrow{BA}. \quad (2)$$

Also, using (1),

$$(a + c)(\overrightarrow{LM} + \overrightarrow{MC}) = (a + c)\overrightarrow{LC} = a(\overrightarrow{LC} - \overrightarrow{LA}) = a\overrightarrow{AC} = 2a\overrightarrow{MC};$$

whence $\overrightarrow{ML} = \frac{c-a}{c+a} \overrightarrow{MC}$. Since $LE \parallel BC$, we see that $\overrightarrow{ME} = \frac{c-a}{c+a} \overrightarrow{MB}$. Similarly, using (1) again, we get $\overrightarrow{LM} = \frac{c-a}{2c} \overrightarrow{LA}$, and since $MD \parallel AB$, we have $\overrightarrow{MD} = \frac{c-a}{2c} \overrightarrow{AB}$. Now,

$$\begin{aligned} \frac{a-c}{2(a+c)} (\overrightarrow{BC} + \overrightarrow{BA}) &= \frac{a-c}{a+c} \overrightarrow{BM} = \overrightarrow{ME} = \overrightarrow{MD} + \overrightarrow{DE} \\ &= \frac{c-a}{2c} \overrightarrow{AB} + \overrightarrow{DE}, \end{aligned}$$

which easily yields

$$\frac{2c(a+c)}{a-c} \overrightarrow{DE} = c \overrightarrow{BC} - a \overrightarrow{BA}. \quad (3)$$

Now we note that the right sides of (2) and (3) are orthogonal vectors:

$$(c \overrightarrow{BC} + a \overrightarrow{BA}) \cdot (c \overrightarrow{BC} - a \overrightarrow{BA}) = c^2 a^2 - a^2 c^2 = 0.$$

It follows that \overrightarrow{BL} and \overrightarrow{DE} are orthogonal. Thus, $ED \perp BL$.

Next we look at readers' solutions to problems of the XXI Albanian Mathematical Olympiad 2000, 12th Form, third round [2004 : 343–344].

1. (a) Prove the inequality

$$\frac{(1+x_1)(1+x_2)\cdots(1+x_n)}{1+x_1x_2\cdots x_n} \leq 2^{n-1}, \quad \forall x_1, x_2, \dots, x_n \in [1, +\infty).$$

(b) When does the equality hold?

Solved by Pierre Bornsztejn, Maisons-Laffitte, France; and Vedula N. Murty, Dover, PA, USA. We give Bornsztejn's solution.

For each $A \subseteq \{1, 2, \dots, n\}$, let $p(A) = \prod_{i \in A} x_i$. Then

$$(1+x_1)(1+x_2)\cdots(1+x_n) = \sum_A p(A).$$

Note that for all $a, b \geq 1$, we have $a+b \leq 1+ab$, with equality if and only if $a=1$ or $b=1$. It follows that for all $A \subseteq \{1, 2, \dots, n\}$, we have

$$p(A) + p(\overline{A}) = \prod_{i \in A} x_i + \prod_{i \notin A} x_i \leq 1 + \prod_{i=1}^n x_i.$$

Now we obtain the desired inequality as follows:

$$\begin{aligned} (1+x_1)(1+x_2)\cdots(1+x_n) &= \sum_A p(A) = \frac{1}{2} \sum_A [p(A) + p(\bar{A})] \\ &\leq \frac{1}{2} \left(1 + \prod_{i=1}^n x_i\right) \sum_A 1 = (1+x_1x_2\cdots x_n)2^{n-1}. \end{aligned}$$

Equality holds if and only if, for all $A \subseteq \{1, 2, \dots, n\}$, we have $p(A) = 1$ or $p(\bar{A}) = 1$. Therefore, equality holds if and only if at most one of the x_i is greater than 1.

2. Consider the sequence $x_1, x_2, \dots, x_n, \dots$ such that $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2+x_n}$ for all $n > 1$. Find

$$(a) \lim_{n \rightarrow \infty} x_n; \quad (b) \lim_{n \rightarrow \infty} 4^n(2-x_n).$$

Solution by Michel Bataille, Rouen, France.

Note that $x_1 = 2 \cos(\pi/4)$. Assume that $x_n = 2 \cos(\pi/2^{n+1})$ for some positive integer n . Then

$$x_{n+1} = \sqrt{2 + 2 \cos\left(\frac{\pi}{2^{n+1}}\right)} = \sqrt{4 \cos^2\left(\frac{\pi/2^{n+1}}{2}\right)} = 2 \cos\left(\frac{\pi}{2^{n+2}}\right).$$

It follows (by induction) that $x_n = 2 \cos(\pi/2^{n+1})$ for all positive integers n . Then $\lim_{n \rightarrow \infty} x_n = 2$.

Furthermore, for all $n \in \mathbb{N}$, we have

$$2 - x_n = 2 \left(1 - \cos\left(\frac{\pi}{2^{n+1}}\right)\right) = 4 \sin^2\left(\frac{\pi}{2^{n+2}}\right).$$

Since $\sin x \sim x$ as $x \rightarrow 0$, we readily obtain $2 - x_n \sim 4\pi^2/4^{n+2}$ as $n \rightarrow \infty$. The result, $\lim_{n \rightarrow \infty} 4^n(2-x_n) = \frac{1}{4}\pi^2$, follows at once.

3. Prove that, if $0 < a < b < \frac{\pi}{2}$, then

$$(a) \frac{a}{b} < \frac{\sin a}{\sin b}; \quad (b) \frac{\sin a}{\sin b} < \frac{\pi a}{2b}.$$

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

(a) Let $I = \left(0, \frac{\pi}{2}\right)$ and $f(x) = \frac{\sin x}{x}$. Then $f'(x) = \frac{\cos x}{x^2}(x - \tan x)$ for all $x \in I$. It is well known that $x < \tan x$ for all $x \in I$, so that $f'(x) < 0$, which means that f is decreasing on I . Therefore, if $0 < a < b < \frac{\pi}{2}$, then $f(a) > f(b)$, which is equivalent to $\frac{a}{b} < \frac{\sin a}{\sin b}$.

(b) It is well known that $\frac{2}{\pi}x < \sin x < x$ if $x \in \left(0, \frac{\pi}{2}\right)$. Therefore, if $0 < a < b < \frac{\pi}{2}$, then $\sin a < a$ and $\frac{2}{\pi}b < \sin b$. Thus, $\frac{\sin a}{\sin b} < \frac{\pi a}{2b}$.

5. Let a, b, c be the sides of a triangle, and let α, β, γ be the angles opposite the sides a, b, c , respectively.

- (a) Prove that $\gamma = 2\alpha$ if and only if $c^2 = a(a + b)$.
- (b) Find all triangles such that a, b, c are natural numbers, b is a prime, and $\gamma = 2\alpha$.

Solved by Michel Bataille, Rouen, France; and Vedula N. Murty, Dover, PA, USA (part (a) only). We present Bataille's solution.

- (a) First, suppose that $c^2 = a(a + b)$. Then, using the Law of Cosines,

$$c^2(a - b) = a(a^2 - b^2) = a(c^2 - 2bc \cos \alpha).$$

This simplifies to $c = 2a \cos \alpha$. Since $c \sin \alpha = a \sin \gamma$ (from the Law of Sines), we obtain $\sin \gamma = \sin 2\alpha$. It follows that $\gamma = 2\alpha$ or $\gamma = \pi - 2\alpha$. In the latter case, since we also have $\gamma = \pi - \alpha - \beta$, we find that $\alpha = \beta$, which implies that $a = b$. Then $c^2 = 2a^2$ and $\triangle ABC$ is isosceles and right-angled with hypotenuse c ; thus $\alpha = \pi/4$ and $\gamma = \pi/2$. In both cases, $\gamma = 2\alpha$.

Conversely, if $\gamma = 2\alpha$, then $\sin \gamma = 2 \sin \alpha \cos \alpha$, or (using the Law of Sines and the Law of Cosines)

$$\frac{c}{a} = \frac{\sin \gamma}{\sin \alpha} = 2 \cos \alpha = 2 \left(\frac{b^2 + c^2 - a^2}{2bc} \right),$$

which yields $c^2(b - a) = a^2(b^2 - a^2)$. If $a \neq b$, we obtain $c^2 = a(a + b)$. If $a = b$, then $\alpha = \beta$. Since $\gamma = 2\alpha$, it follows that $\triangle ABC$ is isosceles and right-angled with $c^2 = 2a^2$, and $c^2 = a(a + b)$ is still true.

(b) Suppose that $\gamma = 2\alpha$. From the hypotheses and part (a), we see that $c^2 - a^2 = ab$; thus $c > a$. Let $d = \gcd(a, c)$, and let a' and c' be the coprime integers defined by $a = da'$ and $c = dc'$. Then $d(c' - a')(c' + a') = ba'$. Since b is prime, $b \mid d$ or $b \mid (c' - a')$ or $b \mid (c' + a')$. If $b \mid d$ or $b \mid (c' - a')$, then $b \mid (c - a)$, which implies that $b \leq c - a$, contradicting the triangle inequality. Thus, $c' + a' = kb$ for some positive integer k .

Now $kd(c' - a') = a'$, implying that k divides a' . Since k also divides $c' + a'$, we see that k divides both c' and a' . It follows that $k = 1$, $b = a' + c'$, and $d(c' - a') = a'$. As a result, $d \mid a'$ and $dc' = a'(d + 1)$. Since a' and c' are coprime, the latter shows that $a' \mid d$. Hence, $a' = d$ and $c' = d + 1$. We finally have $b = 2d + 1$, $a = d^2$, and $c = d(d + 1)$. Note that $d > 1$, since $b < a + c$, and hence, $b \geq 5$.

Conversely, let $b = 2d + 1$ be a prime with $b \geq 5$, and let $a = d^2$ and $c = d(d + 1)$. Then $c - a < b < c + a$ is easily checked (since $d > 1$), and thus, there exists a triangle with sides a, b , and c . Moreover, the relation $c^2 = a(a + b)$ is obtained at once, and (a) shows that $\gamma = 2\alpha$ in this triangle.

In conclusion, the suitable triangles are those with sides $b = 2d + 1$, $a = d^2$, and $c = d(d + 1)$ for some prime $b \geq 5$.

Next we look at solutions from our readers to problems of the 1st Round of the Mathematics Competitions in Finland 2000–2001 given [2004 : 344].

3. Determine all positive integers m and n such that

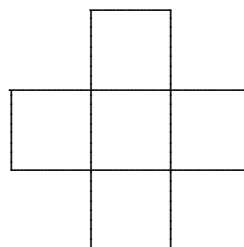
$$m^2 - n^2 = 270.$$

Solved by Pierre Bornsztejn, Maisons-Laffitte, France; Vedula N. Murty, Dover, PA, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

We show that, more generally, the equation $m^2 - n^2 = 2k$ has no integer solutions if k is odd.

Suppose m and n are positive integers such that $m^2 - n^2 = 2k$, where k is an odd integer. Clearly, m and n have the same parity. Hence, $m + n$ and $m - n$ have the same parity. Since $(m + n)(m - n) = m^2 - n^2 = 2k$ is even, both $m + n$ and $m - n$ must be even, which is impossible, since $4 \nmid 2k$.

4. A number of cross-shaped pieces, as shown, are placed on an 8×8 chessboard in such a way that the squares of the pieces and the squares of the chessboard are aligned and the pieces do not overlap each other. We say that the board has been filled if no more pieces can be placed on the board satisfying the conditions above. Determine the smallest possible number of pieces with which the board can be filled.



Solved by Pierre Bornsztejn, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use Bornsztejn's write-up.

First, note that a piece can be placed in the upper-left corner (Figure 1) if no other previously placed piece has its centre in the upper-left 4×4 square. Thus, if the board is filled, there is at least one piece whose centre square lies in the upper-left 4×4 square. Since similar observations hold for the other corners, we must have at least 4 pieces to fill the board.

On the other hand, Figure 2 shows that the board can be filled with 4 pieces. Thus, the minimum number is 4.

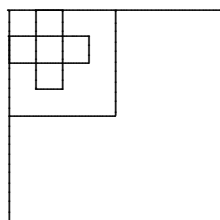


Figure 1

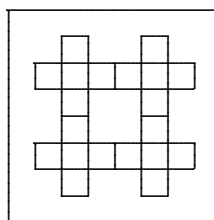


Figure 2

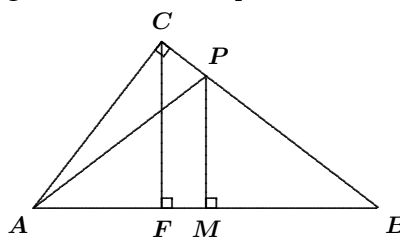
Next we move to solutions to problems of the Final Round of the Mathematics Competitions in Finland, 2000–2001 given [2004 : 245].

1. Let ABC be a right triangle with hypotenuse AB and altitude CF , where F lies on AB . The circle through F centred at B and another circle of the same radius centred at A intersect on the side CB . Determine $FB : BC$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let the circle through F centred at B intersect BC at P . Let M be the foot of the perpendicular from P to AB .

Since the circle with radius BF centred at A passes through P , we have $AP = BF = BP$. Thus, triangle APB is isosceles, which implies that M is the



mid-point of AB . From similar triangles PBM and CBF , we get $\frac{BP}{MB} = \frac{BC}{FB}$; from similar triangles ABC and CBF , we get $\frac{AB}{BC} = \frac{BC}{FB}$. Hence,

$$\frac{FB}{BC} = \frac{FB}{MB} \cdot \frac{MB}{BC} = \frac{BP}{MB} \cdot \frac{1}{2} \frac{AB}{BC} = \frac{BC}{FB} \cdot \frac{1}{2} \frac{BC}{FB}.$$

It follows that $\left(\frac{FB}{BC}\right)^3 = \frac{1}{2}$; that is, $\frac{FB}{BC} = \frac{1}{\sqrt[3]{2}}$.

2. Two non-intersecting curves have equations $y = ax^2 + bx + c$ and $y = dx^2 + ex + f$, where $ad < 0$. Prove that there exists a straight line having no points in common with the two curves.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bornsztein's argument, modified by the editor.

Let $f(x) = ax^2 + bx + c$ and $g(x) = dx^2 + ex + f$. With no loss of generality, we may assume that $a < 0$ and $d > 0$. Then the graph of f is concave and the graph of g is convex. Thus, $f(x) < g(x)$ for all $x \in \mathbb{R}$.

For any $x_0 \in \mathbb{R}$, the line L_1 tangent to the graph of f where $x = x_0$ has slope $2ax_0 + b$, and the line L_2 tangent to the graph of g where $x = x_0$ has slope $2dx_0 + e$. Choose $x_0 = \frac{e-b}{2(a-b)}$ so that $L_1 \parallel L_2$. Then L_1 lies strictly below L_2 , because $f(x_0) < g(x_0)$. Furthermore, the graph of f lies below L_1 , since this graph is concave, and the graph of g lies above L_2 , since this graph is convex.

It follows that any line strictly between L_1 and L_2 (and parallel to these lines) is a solution of the problem.

3. The positive integers a , b , and c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{41}{42}.$$

Solved by Pierre Bornshtein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

Let $S = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Due to complete symmetry, we may assume that $a \leq b \leq c$. Clearly, $a \geq 2$.

If $a \geq 4$, then $S \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} < \frac{41}{42}$. If $a = 3$, then $\frac{1}{b} + \frac{1}{c} < \frac{2}{3}$. Since $b = c = 3$ clearly yields a contradiction, we must have $b \geq 3$ and $c \geq 4$. Then $S \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{4} = \frac{11}{12} < \frac{41}{42}$.

It remains to consider the case when $a = 2$. Since $\frac{1}{b} + \frac{1}{c} < \frac{1}{2}$, we have $b \geq 3$. If $b = 3$, then $\frac{1}{c} < \frac{1}{6}$ yields $c \geq 7$, and hence, $S \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$. Now suppose that $b \geq 4$. Since $b = c = 4$ clearly yields a contradiction, we must have $b \geq 4$ and $c \geq 5$. Then $S \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20} < \frac{41}{42}$.

We see that in all cases, $S \leq \frac{41}{42}$, and that equality holds if and only if $(a, b, c) = (2, 3, 7)$.

4. In the weekly State Lottery, a sequence of seven numbers is picked at random. Each number may be any of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Determine the probability that the sequence is composed of only five different numbers.

Solution by Pierre Bornshtein, Maisons-Laffitte, France.

I assume that the question asks for exactly five different numbers and not for at most five.

First note that there are 10^7 possible sequences of seven numbers.

Let us count the sequences composed of only five different numbers. Among these five digits, either one is used three times and the four others are used once each (case 1), or two of them are used twice and the three others are used once (case 2).

Case 1. There are $\binom{10}{1}$ choices for the digit used three times, say α , and there are $\binom{9}{4}$ choices for the other four digits. There are $\binom{7}{3}$ choices for placing α three times and $4!$ ways to use the other four digits in the remaining spots. Thus, there are $\binom{10}{1} \binom{9}{4} \binom{7}{3} 4! = 1058400$ sequences in this case.

Case 2. There are $\binom{10}{2}$ choices for the digits used twice, say α and β , and there are $\binom{8}{3}$ choices for the other three digits. There are $\binom{7}{2}$ choices for placing α twice, $\binom{5}{2}$ choices for the placing β twice once α has been placed, and $3!$ ways to use the other three digits in the remaining spots. Thus, there are $\binom{10}{2} \binom{8}{3} \binom{7}{2} \binom{5}{2} 3! = 3175200$ sequences in this case.

Altogether, there are $1058400 + 3175200 = 4233600$ sequences with exactly five different digits. Thus, the desired probability is $\frac{4233600}{10^7} = \frac{1323}{3125}$.

Remark. If the problem intended at most five digits, we would proceed as follows.

If a sequence uses exactly 6 different digits, then one is used twice and the others are used once each. There are $\binom{10}{1}$ choices for the digit used twice, say α , and $\binom{9}{5}$ choices for the other five digits. There are $\binom{7}{2}$ choices for the placing α twice and $5!$ ways to use the other five digits in the remaining spots. This gives $\binom{10}{1} \binom{9}{5} \binom{7}{2} 5! = 343980$ sequences in this case.

If a sequence uses exactly 7 different digits, then each of them is used only once. There are $\binom{10}{7}$ choices for these digits and $7!$ ways to use them. This gives $\binom{10}{7} = 604800$ sequences.

Altogether, there are $343980 + 604800 = 948780$ sequences using more than five different numbers. It follows that the probability that a sequence uses at most five different numbers is $1 - \frac{948780}{10^7} = \frac{452561}{500000}$.

5. Determine $n \in \mathbb{N}$ such that $n^2 + 2$ divides $2 + 2001n$.

Solution by Pierre Bornsstein, Maisons-Laffitte, France.

Let n be a solution. Then $2 + 2001n \geq n^2 + 2$, and hence, $n \leq 2001$. Since $2 + 2001n \equiv 2 \pmod{3}$ and $n^2 + 2$ divides $2 + 2001n$, we cannot have $n^2 + 2 \equiv 0 \pmod{3}$. It follows that $n \equiv 0 \pmod{3}$. Then

$$n^2 + 2 \equiv 2 \pmod{3}. \quad (1)$$

Now we note that the following expressions are integers:

$$\frac{2 + 2001n}{n^2 + 2} \times n - 2001 = \frac{2n - 4002}{n^2 + 2},$$

$$\frac{2 + 2001n}{n^2 + 2} \times 2 - \frac{2n - 4002}{n^2 + 2} \times 2001 = \frac{8008006}{n^2 + 2}.$$

Thus, $n^2 + 2$ must divide $8008006 = 2 \times 19 \times 83 \times 2539$.

It follows that $n^2 + 2 = 2^a \times 19^b \times 83^c \times 2539^d$ for some integers $a, b, c, d \in \{0, 1\}$. In order to satisfy (1), we must have $(a, c) = (1, 0)$ or $(a, c) = (0, 1)$. Direct checking leads to $n = 0, 6, 9$, or 2001 .

Conversely, it is straightforward to verify that $n = 0, 6, 9$, or 2001 are solutions of the problem.

Finally, we look at material in our files from readers about problems given in the November 2004 number of the *Corner*. First we consider the Final Round of the 37th Mongolian Mathematical Olympiad [2004 : 413].

2. Prove that, if ABC is an acute-angled triangle, then

$$\frac{a^2 + b^2}{a + b} \cdot \frac{b^2 + c^2}{b + c} \cdot \frac{c^2 + a^2}{c + a} \geq 16 \cdot R^2 \cdot r \cdot \frac{m_a}{a} \cdot \frac{m_b}{b} \cdot \frac{m_c}{c}.$$

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

Using the formulas $4m_a^2 = 2(b^2 + c^2) - a^2$ and

$$w_a^2 = \frac{bc(b+c+a)(b+c-a)}{(b+c)^2} = \frac{4bcs(s-a)}{(b+c)^2},$$

it can be shown that

$$m_a \leq w_a \cdot \frac{b^2 + c^2}{2bc}. \quad (1)$$

[*Ed.*: We omit Alt's proof of (1) because the same argument was given recently by Alt in his (featured) solution to problem 2963 [2005 : 350–351], to which we refer the reader.]

Substituting $w_a = \frac{2\sqrt{bcs(s-a)}}{b+c}$ in (1), we get

$$m_a \leq \frac{b^2 + c^2}{b+c} \sqrt{\frac{s(s-a)}{bc}}.$$

Letting K denote the area of $\triangle ABC$, we have $K = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron's formula). We will also use the formulas $abc = 4RK = 4Rrs$. We have

$$\begin{aligned} 16R^2r \prod_{\text{cyclic}} \frac{m_a}{a} &\leq 16R^2r \prod_{\text{cyclic}} \frac{b^2 + c^2}{a(b+c)} \sqrt{\frac{s(s-a)}{bc}} \\ &= \frac{16R^2rs}{(abc)^2} \sqrt{s(s-a)(s-b)(s-c)} \prod_{\text{cyclic}} \frac{b^2 + c^2}{b+c} \\ &= \frac{16R^2rs}{(4RK)(4Rrs)} (K) \prod_{\text{cyclic}} \frac{b^2 + c^2}{b+c} = \prod_{\text{cyclic}} \frac{b^2 + c^2}{b+c}, \end{aligned}$$

as required.

That completes the *Corner* for this issue. Send me your Olympiad contests as well as your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

The Mathematics of Oz: Mental Gymnastics From Beyond the Edge

By Clifford A. Pickover, Cambridge University Press, 2002.

ISBN 0-5210167-8-9, hardcover, 351+xvii pages, US\$34.99.

Reviewed by **Jill Milne**, Teacher Candidate, Ontario Institute for Studies in Education, University of Toronto, Toronto, ON.

The premise of this book is that the wicked but not quite evil space alien Dr. Oz has kidnapped Dorothy and Toto, the plan being to test the intelligence and problem-solving abilities of Dorothy (and sometimes Toto) to determine whether earth should be invaded or not. Dorothy is told that Earth will not be invaded if she and Toto can show that Earthlings are just too smart to take over easily. Hence the mathematical problems.

The book has a simple plot that develops with a few twists depending on the path taken (you are offered a choice of the order in which you do the problems). There are 108 problems in total, each rated with one to four stars based on difficulty. I have found the rating to be an accurate assessment in general, although some of the one-star problems, such as number 20, "Salty Number Cycle", deserve extra points for being clever beyond what one would expect from the rating, while problem 100, "Venusian Number Bush", with a two-star rating, is not much harder than a Sudoku problem from the "Globe and Mail" newspaper.

While all of the problems can be done with paper and pencil, some could be tackled (and that is the operative word here—tackled) with a computer program. However, unless the programmer in question is a seasoned expert, it is far easier to do them by hand. Problem 54, "The Lunatic Ferris Wheel", can best be drawn using an old fashioned Spirograph. Rated with a deserved four stars, the solutions are really neat and surprising.

Problem 55, "the Ultimate Spindle", has a four-star rating and, should the reader decide to program it, might have been justifiably given another star. This is the sort of application where I would want to use APL2 to generate input for Geometer's Sketchpad[®], get a good supply of desk candy, and put the "Danger!! Do not disturb the feral Coder" sign on the door.

My personal pick for classroom use would be problem 43, "Ramanujan Congruences and the Quest for Transcendence". With modifications and a less intimidating title, I think it would be a valuable exercise for almost any high school class. One of the charming features of the book is the quote that introduces each problem. In all cases the quote offers a very apropos comment on the meaning of the problem, and sometimes after you have spent what seems like forever solving it, you realize that the quote even offers a hint. But be warned. Finding out what the hint might mean is harder than the problem itself!

Soviet Union Mathematical Olympiad 1961–1968

By Anton Cherney and Andy Liu, Intuitive Math. Press, Edmonton, 2003.

ISBN 0-9731123-0-1, paperback, 135+vi pages, US\$15.00.

Reviewed by **John Grant McLoughlin**, University of New Brunswick, Fredericton, NB.

This book features contest papers from the Soviet Union Mathematical Olympiad from 1961 to 1968. (The contest was actually called the *Russian Mathematical Olympiad* from 1961 to 1966, but the authors avoided using this name in the title due to potential confusion with another contest.) There are three contest papers from each year: Grade Eight, Grade Nine, and Grade Ten. The core of the book contains the complete contest papers consisting of five problems each. An additional set of five problems at each level is included in 1968 due to the expansion of the contest to include an oral round. Detailed solutions to the contest papers appear in subsequent sections of the book. Appendix C contains all 120 problems arranged by topic (rather than by level or year) into 24 sets of five problems each. The range of topics includes four related to number theory, four related to combinatorics or graph theory, eight related to geometry, and an assortment of others including chessboard problems, equalities, and sequences.

The quality of problems and solutions makes this a worthwhile addition to the libraries of keen problem-solvers at the secondary level, or others interested in contests. The material is much harder than the grade level may suggest to those unfamiliar with contest expectations in Eastern Europe. The following two problems offer insight into the level of difficulty while bringing attention to the major concern with the book—typographical errors.

(1963 #4 Grade Eight)

Each diagonals [sic] of a convex quadrilateral bisects its area. Prove that the quadrilateral is a parallelogram.

(1965 #4 Grade Nine)

Is it possible [sic] to choose 1965 points inside a unit square such that every rectangle with area $1/200$ and sides parallel to the sides of the square contains at least one of these points?

The review copy of the book included an “Errata” insert with neither of the above problems noted. Several additional errors were identified while reading the book. Few errors impacted upon the understanding of the problems. But the quality of the resource would be improved by correcting such errors in future printings.

The book, representing the compilations of N.B. Vasilyev (to whose memory the book is dedicated) and A.A. Yegorov, as translated and edited by Anton Cherney and Andy Liu, is intended as the first of a series of four volumes containing translations of the problems up to 1992. The layout and the content are likely to continue to be strengths, as this book offers rich problems and solutions in an accessible format. I recommend the book to those who enjoy collections of contest problems.

Variations on an IMO Inequality

Gerhard J. Woeginger

In July 2004, the 45th International Mathematical Olympiad for high-school students (IMO 2004) was held in Athens, Greece. It brought together 486 participants from 85 countries. As in previous years, the problem set consisted of six problems. The fourth IMO problem read as follows:

[IMO problem 2004/4]:

Let $n \geq 3$ be an integer, and let x_1, \dots, x_n be positive real numbers that satisfy

$$(x_1 + x_2 + \dots + x_n) \left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right) < n^2 + 1. \quad (1)$$

Prove that for all i, j, k with $1 \leq i < j < k \leq n$, the numbers x_i, x_j, x_k are the side lengths of a non-degenerate triangle.

An equivalent formulation of the conclusion in this problem is that $x_i < x_j + x_k$ holds for any three pairwise distinct indices i, j, k .

The generalized result

In this note, we will prove and discuss the following generalization of the above IMO problem:

Generalization. Let $m \geq 1$ and $n \geq m + 1$ be integers, and let $\alpha > 1/m$ be a real number. Let x_1, \dots, x_n be positive real numbers that satisfy

$$\sum_{i=1}^n x_i \sum_{i=1}^n \frac{1}{x_i} < \left(n - m - 1 + \sqrt{m^2 + 1 + m^2 \alpha + 1/\alpha} \right)^2. \quad (2)$$

Then, for any set $I \subset \{1, 2, \dots, n\}$ with $|I| = m$ and for any index $\ell \in \{1, \dots, n\} \setminus I$, the inequality $x_\ell < \alpha \sum_{i \in I} x_i$ holds.

Here are three remarks on this generalization.

1. We may deduce from the Cauchy Inequality that the left side in (1) and (2) is at least n^2 . Therefore, the two bounds on the corresponding right sides must be greater than or equal to n^2 . Note that both right sides actually are very close to n^2 .

2. The condition $\alpha > 1/m$ imposed on α is a mild and reasonable condition. Otherwise, for $\alpha \leq 1/m$, the desired conclusion $x_\ell < \alpha \sum_{i \in I} x_i$ could

never hold true, because the maximum number x_ℓ will always be greater than or equal to the arithmetic mean of the smallest m numbers.

3. The bound on the right side of (2) is the best possible. We justify this in the following way: Let $x_i = 1$ for $1 \leq i \leq m$, let $x_{m+1} = \alpha m$, and let $x_i = m \sqrt{\alpha(\alpha+1)/(\alpha m^2 + 1)}$ for $m+2 \leq i \leq n$. Then the left side of (2) becomes equal to the right side, and we get the smallest possible violation of (2). Furthermore, in this case the conclusion does not hold any more, since $x_{m+1} = \alpha \sum_{i=1}^m x_i$.

Setting $m = 2$ and $\alpha = 1$ in the generalization yields the following corollary:

Corollary. Let x_1, \dots, x_n be positive real numbers that satisfy

$$\sum_{i=1}^n x_i \sum_{i=1}^n \frac{1}{x_i} < (n - 3 + \sqrt{10})^2. \quad (3)$$

Then $x_i < x_j + x_k$ holds for any three pairwise distinct indices i, j, k .

The corollary demonstrates that without losing the conclusion in the IMO problem, the bound $B_1(n) := n^2 + 1$ in inequality (1) may be replaced by the weaker bound $B_2(n) := (n - 3 + \sqrt{10})^2$. For $n = 3$ both bounds coincide (since $B_1(3) = B_2(3) = 10$), whereas for all $n \geq 4$ we have $B_2(n) > B_1(n)$. The extremal example introduced above specializes to $x_1 = x_2 = 1$, $x_3 = 2$, and $x_i = \sqrt{8/5}$ for $4 \leq i \leq n$. It shows that any further relaxation of the bound $B_2(n)$ would make the conclusion invalid.

Proof of the generalized result

Our main tool will be the Cauchy Inequality in the following form: For all positive real numbers a_1, \dots, a_k and b_1, \dots, b_k , we have

$$\sum_{i=1}^k a_i \sum_{i=1}^k b_i \geq \left(\sum_{i=1}^k \sqrt{a_i b_i} \right)^2. \quad (4)$$

Now suppose, for the sake of contradiction, that the $m+1$ numbers x_1, \dots, x_{m+1} satisfy $x_{m+1} \geq \alpha \sum_{i=1}^m x_i$. We will show that under these circumstances inequality (2) does not hold.

To this end, we define $S = \sum_{i=1}^m x_i$, $T = \sum_{i=1}^m \frac{1}{x_i}$, and $z = x_{m+1}/S$. Note that our assumption yields $z \geq \alpha > 1/m$, and that the Cauchy Inequality (4)

yields $S \cdot T \geq m^2$. This leads to the following useful inequality:

$$\begin{aligned}
& \sum_{i=1}^{m+1} x_i \sum_{i=1}^{m+1} \frac{1}{x_i} \\
&= \left(\sum_{i=1}^m x_i \sum_{i=1}^m \frac{1}{x_i} \right) + x_{m+1} \frac{1}{x_{m+1}} + \sum_{i=1}^m \left(\frac{x_{m+1}}{x_i} + \frac{x_i}{x_{m+1}} \right) \\
&= S \cdot T + 1 + x_{m+1} T + \frac{S}{x_{m+1}} \geq m^2 + 1 + x_{m+1} \frac{m^2}{S} + \frac{S}{x_{m+1}} \\
&= m^2 + 1 + m^2 z + \frac{1}{z} \geq m^2 + 1 + m^2 \alpha + \frac{1}{\alpha}. \tag{5}
\end{aligned}$$

Here the final inequality follows from $z \geq \alpha$, and from the fact that the function $f(z) = m^2 z + 1/z$ for $z > 1/m$ is an increasing function.

Next, in the Cauchy Inequality (4), we set $a_i = x_{m+1+i}$ for $i = 1, \dots, n - m - 1$, and $a_{n-m} = \sum_{i=1}^{m+1} x_i$. Furthermore, we set

$b_i = 1/x_{m+1+i}$ for $i = 1, \dots, n - m - 1$, and $b_{n-m} = \sum_{i=1}^{m+1} \frac{1}{x_i}$. Note that (5) provides a lower bound on $a_{n-m} b_{n-m}$. Then the Cauchy Inequality yields

$$\begin{aligned}
\sum_{i=1}^n x_i \sum_{i=1}^n \frac{1}{x_i} &= \sum_{i=1}^{n-m} a_i \sum_{i=1}^{n-m} b_i \geq \left(n - m - 1 + \sqrt{a_{n-m} b_{n-m}} \right)^2 \\
&\geq \left(n - m - 1 + \sqrt{m^2 + 1 + m^2 \alpha + 1/\alpha} \right)^2.
\end{aligned}$$

To summarize, whenever there are $m+1$ numbers that violate the conclusion of the generalized problem, then also the inequality in (2) is violated. This completes the argument.

Related results

We close this note by discussing several results that all are related to the special case $m = 1$. By setting $m = 1$ in the generalization we get the following corollary:

Corollary. Let $\alpha > 1$ and x_1, \dots, x_n be positive real numbers such that

$$\sum_{i=1}^n x_i \sum_{i=1}^n \frac{1}{x_i} < \left(n - 2 + \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} \right)^2. \tag{6}$$

Then $x_i < \alpha x_j$ holds for any two indices i and j .

In this corollary, the conclusion holds for *any* pair of indices. In the following proposition, a similar conclusion holds for *some* pair of indices.

Proposition. Let $\alpha > 1$ and x_1, \dots, x_n be positive real numbers such that

$$\sum_{i=1}^n x_i \sum_{i=1}^n \frac{1}{x_i} < \frac{1}{\alpha^{n-1}} (1 + \alpha + \alpha^2 + \dots + \alpha^{n-1})^2. \quad (7)$$

Then there exist two indices i and j such that $x_i \leq x_j < \alpha x_i$.

Proof: Without loss of generality, let $x_1 \leq x_2 \leq \dots \leq x_n$. Now suppose for the sake of contradiction that the conclusion is violated. Then $x_{i+1} \geq \alpha x_i$ holds for $1 \leq i \leq n-1$. An easy induction yields $x_j/x_i \geq \alpha^{j-i}$ for all $j \geq i$. Since for $z \geq 1$ the function $f(z) = z + \frac{1}{z}$ is increasing, we conclude from this that $x_j/x_i + x_i/x_j \geq \alpha^{j-i} + \alpha^{i-j}$. This implies

$$\begin{aligned} \sum_{i=1}^n x_i \sum_{i=1}^n \frac{1}{x_i} &= n + \sum_{1 \leq i < j \leq n} \left(\frac{x_j}{x_i} + \frac{x_i}{x_j} \right) \geq n + \sum_{1 \leq i < j \leq n} (\alpha^{j-i} + \alpha^{i-j}) \\ &= \sum_{i=1}^n \alpha^i \sum_{i=1}^n \alpha^{-i} = \frac{1}{\alpha^{n-1}} (1 + \alpha + \alpha^2 + \dots + \alpha^{n-1})^2. \end{aligned}$$

Since the resulting inequality violates (7), we have reached the desired contradiction. ■

The example with $x_i = \alpha^i$ for $1 \leq i \leq n$ demonstrates that the bound in the right hand side of (7) is the best possible.

The following proposition gives a kind of reverse statement for the preceding two results. It is a straightforward reformulation of the well-known Schweitzer Inequality. And it is also a special case of the famous Kantorovich Inequality, which was stated and proved in 1948 by Leonid Vitaliyevich Kantorovich (the same Kantorovich who, in 1975, received the Nobel Prize for Economics).

Proposition. Let $\alpha > 1$ and x_1, \dots, x_n be positive real numbers such that $x_j < \alpha x_i$ holds for any two indices i and j . Then

$$\sum_{i=1}^n x_i \sum_{i=1}^n \frac{1}{x_i} < \frac{(\alpha + 1)^2}{4\alpha} n^2. \quad (8)$$

For even n , the bound in the right side of (8) is the best possible, as shown by the example $x_i = 1$ for $1 \leq i \leq n/2$ and $x_i = \alpha$ for $n/2 < i \leq n$. For odd n , the bound can be further decreased to $\frac{(\alpha + 1)^2}{4\alpha} n^2 - \frac{(\alpha - 1)^2}{4\alpha}$.

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PROBLEMS

Solutions to problems in this issue should arrive no later than 1 November 2006. An asterisk () after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

3113. Correction. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle and let a be the length of the side opposite the vertex A . If m_a is the length of the median from A to BC , and if R is the circumradius of $\triangle ABC$, prove that $m_a - R$ is positive, negative, or zero, according as $\angle A$ is acute, obtuse, or right-angled.

3126. *Proposed by Hidetoshi Fukugawa, Kani, Gifu, Japan.*

Let D be any point on the side BC of triangle ABC . Let Γ_1 and Γ_2 be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Let ℓ be the common external tangent to Γ_1 and Γ_2 which is different from BC . If P is the point of intersection of AD and ℓ , show that $AB = 2AP$.

3127. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let H be the foot of the altitude from A to BC , where BC is the longest side of $\triangle ABC$. Let R , R_1 , and R_2 be the circumradii of $\triangle ABC$, $\triangle ABH$, and $\triangle ACH$, respectively. Similarly, let r , r_1 , r_2 be the inradii of these triangles. Prove that

- (a) $R_1^2 + R_2^2 - R^2$ is positive, negative, or zero according as angle A is acute, obtuse, or right-angled.
- (b) $r_1^2 + r_2^2 - r^2$ is positive, negative, or zero according as angle A is obtuse, acute, or right-angled.

3128. *Proposed by K.R.S. Sastry, Bangalore, India.*

In triangle ABC , we have $AB = AC = 5$, $BC = 6$. Let E be a point on AC and F a point on AB such that $BE = CF$, $\angle EBC \neq \angle FCB$, and $\sin \theta = 5/13$, where $\theta = \angle EBC$. Let H be the point of intersection of BE and CF , and let K be the point on BC such that $HK \perp BC$.

Find the length of HK .

3129. Proposed by K.R.S. Sastry, Bangalore, India.

In $\triangle ABC$, the adjacent internal trisectors of the angles B and C meet at the point P , and the adjacent internal trisectors of the angles A and C meet at the point Q .

Characterize those triangles in which $AQ + BP = AB$.

3130. Proposed by Michel Bataille, Rouen, France.

Let A, B, C be the angles of a triangle. Show that

$$\begin{aligned} & (\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C) (\csc \frac{1}{2}A + \csc \frac{1}{2}B + \csc \frac{1}{2}C) \\ & - (\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C) \geq 6\sqrt{3}. \end{aligned}$$

3131. Proposed by Michel Bataille, Rouen, France.

The normal at M to a conic with focus F meets the focal axis at N . Let H and K be points on MF such that $HN \perp MF$ and $KN \perp MN$. If $\frac{1}{HN} = \frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right)$ and $NK = \sqrt{ab}$ (where $a > b > 0$), show that $KI = (a + b)/2$ for some significant point I on MN .

3132. Proposed by Mihály Bencze, Brasov, Romania.

Let $F(n)$ be the number of ones in the binary expression of the positive integer n . For example,

$$\begin{aligned} F(5) &= F(101_{(2)}) = 2, \\ F(15) &= F(1111_{(2)}) = 4. \end{aligned}$$

Let $S_k = \sum_{n=1}^{\infty} \frac{F^k(n)}{n(n+1)}$, where $F^k(n)$ is defined recursively by $F^1 = F$ and $F^k = F \circ F^{k-1}$ for $k \geq 2$.

- Prove that $S_1 = 2 \ln 2$.
- Prove that $\frac{18}{5} \ln 2 - \frac{1}{15} \leq S_2 \leq 4 \ln 2$.
- Prove that $\frac{218}{25} \ln 2 - \frac{7}{25} \leq S_3 \leq 11 \ln 2$.
- ★ Compute S_k .

3133. Proposed by Mihály Bencze, Brasov, Romania.

Let ABC be any triangle. Show that

$$\sum_{\text{cyclic}} \frac{1 + 2 \sin A - \cos 2A}{8 + 3 \cos \left(\frac{A}{2} \right) \cos \left(\frac{B-C}{2} \right) + \cos \left(\frac{3A}{2} \right) \cos \left(\frac{3(B-C)}{2} \right)} \leq 1.$$

3134. *Proposed by Mihály Bencze, Brasov, Romania.*

Let O be the circumcentre of $\triangle ABC$. Let D , E , and F be the mid-points of BC , CA , and AB , respectively; let K , M , and N be the mid-points of OA , OB , and OC , respectively. Denote the circumradius, inradius, and semiperimeter of $\triangle ABC$ by R , r , and s , respectively. Prove that

$$2(KD + ME + NF) \geq R + 3r + \frac{s^2 + r^2}{2R}.$$

3135. *Proposed by Marian Marinescu, Monbonnot, France.*

Let \mathbb{R}^+ be the set of non-negative real numbers. For all $a, b, c \in \mathbb{R}^+$, let $H(a, b, c)$ be the set of all functions $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$h(x) = h(h(ax)) + b(bx) + cx$$

for all $x \in \mathbb{R}^+$. Prove that $H(a, b, c)$ is non-empty if and only if $b \leq 1$ and $4ac \leq (1 - b)^2$.

3136. *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with circumcircle Γ ; let ℓ be a transversal which meets the line BC at L , the line CA at M , and the line AB at N . Let Γ_1 be the circle through A which is tangent to BC at L , and let Γ_2 and Γ_3 be similarly defined with respect to B and C . Let QR , RP , and PQ be the common chords of Γ and Γ_1 , Γ and Γ_2 , and Γ and Γ_3 , respectively. Prove that AP , BQ , and CR are concurrent.

3137. *Proposed by Tina Balfour and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Find all solutions in non-negative integers to the following Diophantine equations:

$$(a) \quad 5^m + 3^m = 2^k; \qquad (b) \quad \star \quad 5^m + 3^n = 2^k.$$

3138. *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA.*

Let a_1 be a non-zero real number, and define the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_{n+1} = n^2/a_n$ for $n \geq 1$. Prove that

$$\sum_{n=1}^N \frac{1}{a_n} = \left(\frac{1}{\pi a_1} + \frac{\pi a_1}{4} \right) \ln(N) + O(1).$$

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3113. Correction. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit a la longueur du côté opposé au sommet A d'un triangle ABC . Si m_a est la longueur de la médiane de A à BC , et si R est le rayon du cercle circonscrit du triangle ABC , montrer que $m_a - R$ est positif, négatif ou nul suivant que $\angle A$ est aigu, obtus ou droit.

3126. *Proposé par Hidetoshi Fukugawa, Kani, Gifu, Japon.*

Soit D un point sur le côté BC du triangle ABC . Soit respectivement Γ_1 et Γ_2 les cercles inscrits des triangles ABD et ACD . Soit ℓ la tangente extérieure commune à Γ_1 et Γ_2 et distincte de BC . Si P est le point d'intersection de AD et ℓ , montrer que $AB = 2AP$.

3127. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit H le pied de la hauteur de A à BC , où BC est le côté le plus long du triangle ABC . Soit respectivement R, R_1 et R_2 les rayons des cercles circonscrits des triangles ABC, ABH et ACH . De même, soit r, r_1 et r_2 les rayons des cercles inscrits de ces triangles. Montrer que

- (a) $R_1^2 + R_2^2 - R^2$ est positif, négatif ou nul, suivant que l'angle A est aigu, obtus ou droit.
- (b) $r_1^2 + r_2^2 - r^2$ est positif, négatif ou nul, suivant que l'angle A est aigu, obtus ou droit.

3128. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Soit ABC un triangle avec $AB = AC = 5$ et $BC = 6$. Soit E un point sur AC et F un point sur AB tels que $BE = CF$, $\angle EBC \neq \angle FCB$ et $\sin \theta = 5/13$, où $\theta = \angle EBC$. Soit H le point d'intersection de BE et CF , et soit K le point sur BC tel que $HK \perp BC$.

Trouver la longueur de HK .

3129. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Dans le triangle ABC , les droites trisectrices intérieures des angles B et C , adjacentes au côté BC , se coupent au point P , et celles des angles A et C , adjacentes au côté AC , se coupent au point Q .

Caractériser les triangles pour lesquels $AQ + BP = AB$.

3130. *Proposé par Michel Bataille, Rouen, France.*

Soit A, B et C les angles d'un triangle. Montrer que

$$\left(\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C\right) \left(\csc \frac{1}{2}A + \csc \frac{1}{2}B + \csc \frac{1}{2}C\right) - \left(\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C\right) \geq 6\sqrt{3}.$$

3131. *Proposé par Michel Bataille, Rouen, France.*

La normale en M à une conique de foyer F coupe l'axe focal en N . Soit H et K deux points sur MF tels que $HN \perp MF$ et $KN \perp MN$. Si $\frac{1}{HN} = \frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right)$ et $NK = \sqrt{ab}$ (où $a > b > 0$), montrer que $KI = (a + b)/2$ pour un certain point remarquable I sur MN .

3132. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $F(n)$ le nombre de 1 dans l'expression binaire de l'entier positif n . Par exemple,

$$\begin{aligned} F(5) &= F(101_{(2)}) = 2, \\ F(15) &= F(1111_{(2)}) = 4. \end{aligned}$$

Soit $S_k = \sum_{n=1}^{\infty} \frac{F^k(n)}{n(n+1)}$, où $F^k(n)$ est défini récursivement par $F^1 = F$ et $F^k = F \circ F^{k-1}$ pour $k \geq 2$.

- (a) Montrer que $S_1 = 2 \ln 2$.
- (b) Montrer que $\frac{18}{5} \ln 2 - \frac{1}{15} \leq S_2 \leq 4 \ln 2$.
- (c) Montrer que $\frac{218}{25} \ln 2 - \frac{7}{25} \leq S_3 \leq 11 \ln 2$.
- (d)★ Calculer S_k .

3133. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit ABC un triangle quelconque. Montrer que

$$\sum_{\text{cyclique}} \frac{1 + 2 \sin A - \cos 2A}{8 + 3 \cos \left(\frac{A}{2} \right) \cos \left(\frac{B-C}{2} \right) + \cos \left(\frac{3A}{2} \right) \cos \left(\frac{3(B-C)}{2} \right)} \leq 1.$$

3134. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit O le centre du cercle circonscrit au triangle ABC . Soit respectivement D , E et F les points milieu des côtés BC , CA et AB ; soit K , M et N les points milieu des côtés OA , OB et OC . On désigne respectivement par R , r et s le rayon de cercle circonscrit, celui du cercle inscrit et le semi-périmètre du triangle ABC . Montrer que

$$2(KD + ME + NF) \geq R + 3r + \frac{s^2 + r^2}{2R}.$$

3135. *Proposé par Marian Marinescu, Monbonnot, France.*

Soit \mathbb{R}^+ l'ensemble des nombres réels non négatifs. Pour tout a, b et $c \in \mathbb{R}^+$, soit $H(a, b, c)$ l'ensemble de toutes les fonctions $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ telles que

$$h(x) = h(h(ax)) + b(bx) + cx$$

pour tout $x \in \mathbb{R}^+$. Montrer que $H(a, b, c)$ est non vide si et seulement si $b \leq 1$ et $4ac \leq (1 - b)^2$.

3136. *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit ABC un triangle de cercle circonscrit Γ ; soit ℓ une transversale coupant la droite BC en L , la droite CA en M et la droite AB en N . Soit Γ_1 le cercle passant par A et tangent à BC en L , et soit Γ_2 et Γ_3 définis de manière analogue, relativement à B et C . Soit respectivement QR , RP et PQ les cordes communes de Γ et Γ_1 , de Γ et Γ_2 , et de Γ et Γ_3 . Montrer que AP , BQ et CR sont concourantes.

3137. *Proposé par Tina Balfour et Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Trouver toutes les solutions en entiers non négatifs des équations diophantiennes suivantes :

(a) $5^m + 3^m = 2^k$;

(b)★ $5^m + 3^n = 2^k$.

3138. *Proposé par Paul Bracken, Université du Texas, Edinburg, TX, USA.*

Soit a_1 un nombre réel non nul. On définit la suite $\{a_n\}_{n=1}^{\infty}$ par $a_{n+1} = n^2/a_n$ pour $n \geq 1$. Montrer que

$$\sum_{n=1}^N \frac{1}{a_n} = \left(\frac{1}{\pi a_1} + \frac{\pi a_1}{4} \right) \ln(N) + O(1).$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria from the list of solvers of 2972 and from the list of those resolving a conjecture of Bencze in 2984 [2006 : 51–52].

3017. [2005 : 106, 108] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a quadrilateral $ABCD$, let P, Q, R, S be points on the sides AB, BC, CD, DA , respectively, such that

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1.$$

Let O be the intersection of PR and QS . Prove that

$$DS \cdot \frac{AP}{PB} + AS \cdot \frac{DR}{RC} = AD \cdot \frac{SO}{OQ}.$$

Solution by Kin Fung Chung, student, University of Toronto, Toronto, ON.

We rearrange the given condition, setting

$$\frac{x}{y} = \frac{DR \cdot QC}{AP \cdot BQ} = \frac{DS \cdot CR}{BP \cdot AS}.$$

Attach masses xBP, xAP, yDR, yCR at points A, B, C, D , respectively. We locate the centre of mass G of the system in two ways:

1. The centre of mass of A and B is at P , and the centre of mass of C and D is at R ; hence, G lies on PR .

2. Note that $xAP \cdot BQ = yDR \cdot CQ$ by the definition of x and y . This implies that the centre of mass of B and C is at Q . Similarly, the centre of mass of A and D is at S . Hence, G lies on SQ .

By the second step, we have

$$\begin{aligned} \frac{SO}{OQ} &= \frac{\text{mass at } Q}{\text{mass at } S} = \frac{xAP + yDR}{xBP + yCR} = \frac{\frac{x}{y}AP + DR}{\frac{x}{y}BP + CR} \\ &= \frac{\frac{DR \cdot QC}{BQ} + DR}{CR \left(\frac{DS}{AS} + 1 \right)} = \frac{\frac{DR \cdot QC}{BQ}}{CR \left(\frac{AD}{AS} \right)} + \frac{DR}{CR \left(\frac{AD}{AS} \right)} \\ &= \frac{DR}{CR} \cdot \frac{QC}{BQ} \cdot \frac{AS}{AD} + \frac{DR}{CR} \cdot \frac{AS}{AD} = \frac{DS}{AD} \cdot \frac{AP}{PB} + \frac{AS}{AD} \cdot \frac{DR}{RC}, \end{aligned}$$

which immediately yields the desired result.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEIZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Seimiya comments that we get a known theorem in the special case of his problem where $AP : PB = DR : RC$ and $BQ : QC = AS : SD$; in this case, the conclusion becomes $SO : OQ = AP : PB$. Compare this with the version found in Coxeter's Introduction to Geometry, exercise 2 on page 76: When all the points P on AB are related by a similarity to all the points P' on $A'B'$, the points dividing the segments PP' in the ratio $AB : A'B'$ (internally or externally) are distinct and collinear or else they all coincide. As a consequence of Seimiya's problem, we can divide the segments PP' in any fixed ratio, not just $AB : A'B'$, and the division points will still be collinear.

3021. [2005 : 107, 109] Proposed by Pierre Bornsztein, Maisons-Laffitte, France.

Let E be a finite set of points in the plane, no three of which are collinear and no four of which are concyclic. If A and B are two distinct points of E , we say that the pair $\{A, B\}$ is *good* if there exists a closed disc in the plane which contains both A and B and which contains no other point of E . We denote by $f(E)$ the number of good pairs formed by the points of E .

Prove that if the cardinality of E is 1003, then $2003 \leq f(E) \leq 3003$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

More generally, we show that $f(E) = 3n - k - 3$ when $n = |E| \geq 3$ and k is the number of sides of the convex hull H of E .

Let G denote the graph with E as the set of vertices and all line segments connecting each good pair of points of E as edges. We first claim that no two edges of G can cross each other. Suppose edges AB and CD cross each other. Consider the quadrilateral $ACBD$. Since $\{A, B\}$ is a good pair, there is a circle passing through A and B and containing neither C nor D in its interior. Hence, $\angle C + \angle D < \pi$. Similarly, $\angle A + \angle B < \pi$. Hence, $\angle A + \angle B + \angle C + \angle D < 2\pi$, a contradiction. Hence, G is a planar graph.

Next, we claim that every side of H is an edge of G . Let AB be a side of the polygon H . Then one side of AB contains no points of E . Thus, we can draw a circle centred on that side of AB with radius sufficiently large so that it passes through A and B but contains no other points of E . Hence, $\{A, B\}$ is a good pair. That is, AB is an edge of G .

Note that, for each point A in E , if B is a point closest to A , then AB is an edge of G .

Next, suppose AB is any edge of G . Then there is a point C such that the closed disc Γ formed by the circumcircle of $\triangle ABC$ intersects no other points of E (since no four points are concyclic). By perturbing Γ very slightly, we see that all three sides of $\triangle ABC$ are edges of G . If AB is not an edge of H , then we similarly have another point D on a different side of AB from

C such that all three sides of $\triangle ABD$ are edges of G . This implies that the interior of H is triangulated by the edges of G .

Let m denote the number of faces of G , including the exterior one. Since each interior face is enclosed by 3 edges and the exterior face is enclosed by k edges, we have $3(m-1) + k = 2f(E)$. Thus, $m = \frac{2}{3}f(E) - \frac{1}{3}k + 1$, which together with Euler's Formula, $m - f(E) + n = 2$, yields $f(E) = 3n - k - 3$.

Since $3 \leq k \leq n$, we have $2n - 3 \leq f(E) \leq 3n - 6$. The proposed problem is the special case when $n = 1003$.

Also solved by JOEL SCHLOSBERG, Bayside, NY, USA (who also derived the same formula obtained by Zhou featured above); PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3022. [2005 : 107, 110] Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

Given $\triangle ABC$, let C' be any point on the side AB , and let M and N be points on the sides BC and AC , respectively, such that $C'M \parallel AC$ and $C'N \parallel BC$.

Prove that the area of $\triangle C'CN$ is the geometric mean of the areas of $\triangle AC'N$ and $\triangle C'BM$.

Essentially the same solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Since $CNC'M$ is a parallelogram, we have $[C'CN] = [C'CM]$ (where $[XYZ]$ denotes the area of XYZ). Hence,

$$\frac{[C'CN]^2}{[AC'N][C'BM]} = \frac{[C'CN]}{[AC'N]} \cdot \frac{[C'CM]}{[C'BM]} = \frac{CN}{NA} \cdot \frac{CM}{MB} = \frac{MC'}{NA} \cdot \frac{NC'}{MB}.$$

Since triangles ANC' and $C'MB$ have parallel sides, they are similar, which implies that $\frac{MC'}{NA} = \frac{MB}{NC'}$. Consequently,

$$\frac{[C'CN]^2}{[AC'N][C'BM]} = 1,$$

which is exactly what we wanted to prove.

Also solved by HOUDA ANOUN, Bordeaux, France; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; R. LAUMEN, Deurne, Belgium; RAFAEL MARTINEZ CALAFAT, I.E.S. La Plana, Castellon, Spain; XIAO LIANG QI, student, Memorial University of Newfoundland; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; M^a JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3023. [2005 : 107, 110] *Proposed by Bogdan Nica, McGill University, Montreal, QC.*

Find all integer solutions of the system:

$$\begin{aligned} a^c + b^c - 2 &= c^3 - c, \\ b^a + c^a - 2 &= a^3 - a, \\ c^b + a^b - 2 &= b^3 - b. \end{aligned}$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA; and Kin Fung Chung, student, University of Toronto, Toronto, ON, modified by the editor.

First, we claim that $3^n \geq n^3 - n + 3$ for all $n \geq 1$. The claim is true for $n = 1$. Assume that it is true for some $n > 1$. Then

$$\begin{aligned} 3^{n+1} - (n+1)^3 + (n+1) - 3 \\ &\geq 3(n^3 - n + 3) - (n+1)^3 + (n+1) - 3 \\ &= (n-1)(n-2)(2n+3) \geq 0. \end{aligned}$$

By induction, the claim is proved. We conclude from this claim that

$$m^n - 2 > n^3 - n \quad (*)$$

for all $m \geq 3$ and $n \geq 1$.

Now let (a, b, c) be a solution of the given system, where a, b, c are integers. It is clear that no two of a, b, c can be zero. By symmetry, we may suppose $a \leq b \leq c$.

If $a \geq 3$, then $3 \leq a \leq b \leq c$. From (*), we see that such a, b, c cannot be a solution of the system.

If $a \leq -2$, then $b^a + c^a - 2 \geq -4$ and $a^3 - a \leq -6$. For such values of a , the second equation of the original system has no solution.

If $a = -1$, then the second equation becomes $\frac{1}{b} + \frac{1}{c} - 2 = 0$. Thus, $b = c = 1$. But $a = -1, b = c = 1$ is not a solution of the first equation.

If $a = 0$, then $0 < b \leq c$. By (*) and the first and third equations of the system, we conclude that $(b, c) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, but none of these possible values of a, b, c are a solution of the system.

If $a = 1$, then $1 \leq b \leq c$, and the second equation becomes $b + c = 2$. Hence, $b = c = 1$, and $a = b = c = 1$ is a solution of the system.

If $a = 2$, then $2 \leq b \leq c$. The second equation becomes $b^2 + c^2 = 8$. Hence, $b = c = 2$, and $a = b = c = 2$ is a solution of the system.

Therefore, $a = b = c = 1$ and $a = b = c = 2$ are the only integer solutions of the system.

Also solved by JOEL SCHLOSBERG, Bayside, NY, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and the proposer. Two solutions were incomplete and one incorrect.

3024. [2004 : 107, 110] *Proposed by the late Murray S. Klamkin, University of Alberta, Edmonton, AB; and K.R.S. Sastry, Bangalore, India.*

Generalize the following identity so that it involves an n^{th} order determinant in place of a 3^{rd} order determinant, and prove your generalization:

$$\begin{vmatrix} -bc & b^2 + bc & c^2 + bc \\ a^2 + ca & -ca & c^2 + ca \\ a^2 + ab & b^2 + ab & -ab \end{vmatrix} = (bc + ca + ab)^3.$$

[*Ed.* Two different generalizations were proved in the solutions that were submitted for this problem. One of these is given in the first solution below, and the other appears in the second and third solutions.]

I. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, modified slightly by the editor.

The 3×3 matrix in the given determinant may be written in the form $M - (ab + bc + ca)I$, where I is the 3×3 identity matrix and

$$M = \begin{bmatrix} b + c \\ a + c \\ a + b \end{bmatrix} \cdot [a \ b \ c].$$

This suggests the following generalization involving x_1, x_2, \dots, x_n in place of a, b, c , for $n \geq 2$.

Let $S_1 = \sum_i x_i$ and $S_2 = \sum_{i < j} x_i x_j$ (the first and second symmetric functions of x_1, x_2, \dots, x_n). Let

$$U = \begin{bmatrix} S_1 - x_1 \\ S_1 - x_2 \\ \vdots \\ S_1 - x_n \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

and let $M = U \cdot V^{\text{T}}$ (where V^{T} denotes the transpose of V). Let I denote the $n \times n$ identity matrix.

Since the matrix M has rank one, it has an eigenspace of dimension $n - 1$ corresponding to the eigenvalue 0. Any vector in this eigenspace is also an eigenvector of $M - S_2 I$ corresponding to the eigenvalue $0 - S_2 = -S_2$. We also note that U is an eigenvector of M corresponding to the eigenvalue $V^{\text{T}} \cdot U = S_1^2 - \sum_i x_i^2 = 2S_2$. This implies that U is an eigenvector of $M - S_2 I$ corresponding to the eigenvalue $2S_2 - S_2 = S_2$. It follows that

$$\det(M - S_2 I) = (-S_2)^{n-1} S_2 = (-1)^{n-1} S_2^n.$$

II. Solution by Michel Bataille, Rouen, France.

Let x_1, x_2, \dots, x_n be n indeterminates. For each i , let $p_i = \prod_{\ell \neq i} x_\ell$, and for all distinct i, j, k , let $p_{i,j,k} = \prod_{\ell \neq i,j,k} x_\ell$. Then let $s = \sum_{\ell} p_\ell$ and $s_{i,j} = \sum_{\ell \neq i,j} p_{i,j,\ell}$ for $i \neq j$. (The index ℓ runs from 1 to n subject to the indicated restrictions. A product over an empty set of indices is equal to 1.)

Using these notations, we define a determinant $\Delta = \Delta(x_1, x_2, \dots, x_n)$ as follows:

$$\Delta = \begin{vmatrix} -p_1 & x_2^2 s_{1,2} + p_1 & \dots & x_n^2 s_{1,n} + p_1 \\ x_1^2 s_{2,1} + p_2 & -p_2 & \dots & x_n^2 s_{2,n} + p_2 \\ \vdots & \vdots & & \vdots \\ x_1^2 s_{n,1} + p_n & x_2^2 s_{n,2} + p_n & \dots & -p_n \end{vmatrix};$$

that is, $\Delta_{i,i} = -p_i$ and $\Delta_{i,j} = x_j^2 s_{i,j} + p_i$ for $i \neq j$. We will prove that $\Delta = (-1)^{n-1} (n-2) s^n$.

Multiply rows 1, 2, \dots , n in Δ by x_1, x_2, \dots, x_n , respectively, and then extract the factors x_1, x_2, \dots, x_n from columns 1, 2, \dots , n , respectively. Entries $\Delta_{i,i}$ remain unchanged, while each entry $\Delta_{i,j}$ for $i \neq j$ becomes $x_i x_j s_{i,j} + p_j = s - p_i$. Now, adding rows 2, 3, \dots , n to the first row gives

$$\Delta = \begin{vmatrix} (n-2)s & (n-2)s & \dots & (n-2)s \\ s - p_2 & -p_2 & \dots & s - p_2 \\ \vdots & \vdots & & \vdots \\ s - p_n & s - p_n & \dots & -p_n \end{vmatrix}.$$

Now we factor $(n-2)s$ from the first row (so that the first row becomes a row of 1s) and then subtract the first column from each of the other columns to get

$$\Delta = (n-2)s \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ s - p_2 & -s & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ s - p_n & 0 & 0 & \dots & -s \end{vmatrix}.$$

From here we conclude that $\Delta = (n-2)s(-s)^{n-1} = (-1)^{n-1} (n-2)s^n$.

III. Solution by the proposers.

First we give a short proof of the given identity. This will lead to a generalization.

In the given identity, replace a, b, c by $1/a, 1/b, 1/c$, respectively, and multiply both sides by $(abc)^4$ to obtain

$$\begin{vmatrix} -a^2 & a(b+c) & a(b+c) \\ b(c+a) & -b^2 & b(c+a) \\ c(a+b) & c(a+b) & -c^2 \end{vmatrix} = abc(a+b+c)^3.$$

Dividing the rows by a , b , and c , respectively, we get

$$\begin{vmatrix} -a & b+c & b+c \\ c+a & -b & c+a \\ a+b & a+b & -c \end{vmatrix} = (a+b+c)^3. \quad (1)$$

It will suffice to prove this identity.

In the determinant on the left side of (1), if we let $a+b+c=0$, then we get three identical columns. Hence, the determinant has the factor $(a+b+c)^2$. Since the determinant is symmetric and homogeneous of the third degree, the third factor has the form $k(a+b+c)$ for some constant k . Since the coefficient of a^3 in the determinant is 1, we must have $k=1$. This proves (1).

We will now generalize (1) (which is equivalent to generalizing the given identity). In place of a, b, c , we consider a_1, a_2, \dots, a_n . Let $S = \sum_i a_i$, and let D be the matrix whose entries are $D_{ii} = -a_i$ and $D_{ij} = S - a_i$ for $i \neq j$. We will show that $|D| = (-1)^{n-1}(n-2)S^n$.

If we set $S=0$ in D , we get n identical columns. Therefore, S^{n-1} is a factor of $|D|$. Since the determinant is a symmetric and homogeneous polynomial of degree n , the remaining factor is kS for some constant k . Setting $a_1=1$ and $a_i=0$ for $i \neq 1$, we find that $-k$ is the determinant $|B|$, where B is the $(n-1) \times (n-1)$ matrix whose entries are $B_{ii}=0$ and $B_{ij}=1$ for $i \neq j$. This determinant is a special case of the known determinant $|C| = \left(1 + \sum_i b_i\right) / \prod_i b_i$, where $C_{ii} = 1 + 1/b_i$ and $C_{ij} = 1$ for $i \neq j$. Setting $b_i = -1$ for all i , we get $k = (-1)^{n-1}(n-2)$.

Also solved by JOEL SCHLOSBERG, Bayside, NY, USA; MARIAN TETIVA, Birlad, Romania; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incomplete solution.

Joe Howard, Portales, NM, USA observed that the adjoint of the 3×3 matrix A in the given determinant—that is, the matrix $\text{adj } A$ which is the transpose of the matrix of cofactors of the entries of A —has the property that $\det(\text{adj } A) = (bc + ca + ab)^3 \det(A)$. He then noted that, since $\det(\text{adj } A) = (\det A)^{n-1}$ for any $n \times n$ matrix A , it follows immediately that $\det A = (bc + ca + ab)^3$ for the given matrix A .

3025. [2005 : 107, 110] Proposed by Neven Jurič, Zagreb, Croatia.

For each chess piece, we assign to each square of a chessboard a number which is the number of moves available to that piece from that square. The power of the piece is then defined to be the sum of all these numbers over all the squares of the chessboard.

Do there exist integers $m \geq 2$ and $b \geq 2$ such that, on an $m \times b$ chessboard, the power of a rook is equal to the sum of the powers of a bishop and a knight?

[Ed: In the solutions below, $P(R)$, $P(B)$, and $P(K)$ denote the power of a rook, a bishop, and a knight, respectively.]

I. *Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Yes, a 3×3 board is such an example.

Clearly, $P(R) = 9 \times 4 = 36$, and the arrays displayed below show that $P(B) = 20$ and $P(K) = 16$, where the number in a square is the number of moves available to a bishop (knight, respectively) from that square.

2	2	2
2	4	2
2	2	2

$$P(B) = 20$$

2	2	2
2	0	2
2	2	2

$$P(K) = 16$$

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Yes, an ordinary 8×8 chessboard provides such an example, as shown by the arrays depicted below. [Ed: Clearly, $P(R) = 64 \times 14 = 896$. Thus, $P(R) = P(B) + P(K)$.]

7	7	7	7	7	7	7	7
7	9	9	9	9	9	9	7
7	9	11	11	11	11	9	7
7	9	11	13	13	11	9	7
7	9	11	13	13	11	9	7
7	9	11	11	11	11	9	7
7	9	9	9	9	9	9	7
7	7	7	7	7	7	7	7

$$P(B) = 560$$

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

$$P(K) = 336$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

In general, on an $m \times b$ board with $m \geq b \geq 3$, it is clear that $P(R) = mb(m+b-2)$. The formulas $P(B) = 2b(b-1)(3m-b-1)/3$ and $P(K) = 8bm - 12(m+b) + 16$ were obtained by both Hess and Janous. Janous also arrived at the 8×8 solution by considering square boards and equating $P(R)$ with $P(B) + P(K)$. In addition to the 3×3 and 8×8 solutions, Hess obtained two more solutions, namely 4×3 and 6×4 boards. He claimed without proof that these are the only solutions (up to transposing the board).

3026. [2005 : 173, 176] *Proposed by Michel Bataille, Rouen, France.*

Let $a > 0$. Prove that

$$\frac{a^2 + 1}{e^a} + \frac{3a^2 - 1}{3e^{3a}} + \frac{5a^2 + 1}{5e^{5a}} + \frac{7a^2 - 1}{7e^{7a}} + \dots < \frac{\pi}{4}.$$

Composite of essentially the same solution by Edward Doolittle, University of Regina, Regina, SK; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $f(a)$ denote the expression on the left side of the inequality. Then

$$f(a) = a^2(e^{-a} + e^{-3a} + e^{-5a} + \dots) + (e^{-a} - \frac{1}{3}e^{-3a} + \frac{1}{5}e^{-5a} - \dots).$$

Since $e^{-a} < 1$, we can sum the series to get

$$f(a) = a^2 \left(\frac{e^{-a}}{1 - e^{-2a}} \right) + \tan^{-1}(e^{-a}) = \frac{1}{2}a^2 \operatorname{csch} a + \tan^{-1}(e^{-a}).$$

Then

$$\begin{aligned} f'(a) &= a \operatorname{csch} a - \frac{1}{2}a^2 \operatorname{csch} a \operatorname{coth} a - \frac{e^{-a}}{1 + e^{-2a}} \\ &= a \operatorname{coth} a \operatorname{sech} a - \frac{1}{2}a^2 \operatorname{sech} a \operatorname{coth}^2 a - \frac{1}{2} \operatorname{sech} a \\ &= -\frac{1}{2} \operatorname{sech} a (a^2 \operatorname{coth}^2 a - 2a \operatorname{coth} a + 1) \\ &= -\frac{1}{2} \operatorname{sech} a (a \operatorname{coth} a - 1)^2 < 0. \end{aligned}$$

Hence, f is strictly decreasing on $(0, \infty)$. Using L'Hôpital's Rule, we get

$$\lim_{a \rightarrow 0^+} f(a) = \lim_{a \rightarrow 0^+} \left(\frac{2a}{e^a + e^{-a}} \right) + \tan^{-1}(1) = \frac{\pi}{4}.$$

It follows that $f(a) < \frac{\pi}{4}$ for all $a > 0$.

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3027. [2005 : 173, 176] *Proposed by Geoffrey A. Kandall, Hamden, CT, USA.*

Let $ABCD$ be any quadrilateral, and let M be the mid-point of AB . On the sides CB , DC , and AD , equilateral triangles CBE , DCF , and ADG are constructed externally. Let N be the mid-point of EF and P be the mid-point of FG .

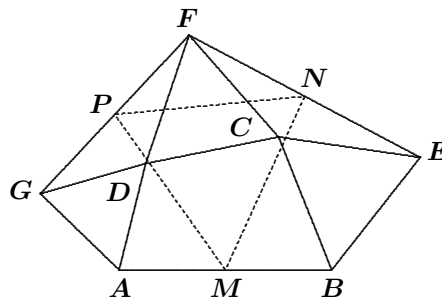
Prove that $\triangle MNP$ is equilateral.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Identify a point X with a complex number x . Let $\omega = e^{2\pi i/3}$. Then $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. Since triangles CBE , DCF , and ADG are equilateral and oriented counterclockwise, it follows that $c + \omega b + \omega^2 e = 0$, $d + \omega c + \omega^2 f = 0$, and $a + \omega d + \omega^2 g = 0$. Thus, $e = -\omega c - \omega^2 b$, $f = -\omega d - \omega^2 c$, and $g = -\omega a - \omega^2 d$. Then

$$\begin{aligned} 2n &= e + f \\ &= -\omega(c + d) - \omega^2(b + c) \\ \text{and } 2p &= f + g \\ &= -\omega(d + a) - \omega^2(c + d). \end{aligned}$$

We also have $2m = a + b$. Therefore,



$$\begin{aligned} 2(m + \omega n + \omega^2 p) &= a + b - \omega^2(c + d) - (b + c) - (d + a) - \omega(c + d) \\ &= -(1 + \omega + \omega^2)(c + d) = 0, \end{aligned}$$

which shows that $\triangle MNP$ is equilateral.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; EDWARD DOOLITTLE, University of Regina, Regina, SK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

All solutions made use of the same idea. It would be nice to see a purely geometric proof.

The editor was reminded of Napoleon's Theorem here, and wondered if there was a generalization where the apexes of the equilateral triangles are replaced by points on the perpendicular bisectors of the sides BC , CD and DA , at the same proportional distance with respect to the lengths of the respective sides. But this is not true, as shown by the simple example where the apexes are replaced by the centres of the squares on the sides and the mid-points of the sides. What is it about the equilateral triangles that makes this result true?

3028. [2005 : 174, 176] Proposed by Dorin Mărghidanu, Colegiul Național "A.I. Cuza", Corabia, Romania.

Let a_1, a_2, \dots, a_n be positive real numbers, and let $S_k = 1 + 2 + \dots + k$. Prove the following

$$1 + \frac{(a_1 a_2^2)^{\frac{1}{S_2}}}{a_1 + 2a_2} + \frac{(a_1 a_2^2 a_3^3)^{\frac{1}{S_3}}}{a_1 + 2a_2 + 3a_3} + \dots + \frac{(a_1 a_2^2 \dots a_n^n)^{\frac{1}{S_n}}}{a_1 + 2a_2 + \dots + na_n} \leq \frac{2n}{n+1}.$$

Solution by Edward Doolittle, University of Regina, Regina, SK.

By the Weighted AM-GM Inequality, we have

$$(a_1 a_2^2 \dots a_k^k)^{\frac{1}{S_k}} \leq \frac{a_1 + 2a_2 + \dots + ka_k}{S_k},$$

with equality if and only if $a_1 = a_2 = \dots = a_k$. Now, since

$$\frac{1}{S_k} = \frac{2}{k(k+1)} = \frac{2}{k} - \frac{2}{k+1},$$

we have

$$\begin{aligned} 1 + \frac{(a_1 a_2^2)^{\frac{1}{S_2}}}{a_1 + 2a_2} + \frac{(a_1 a_2^2 a_3^3)^{\frac{1}{S_3}}}{a_1 + 2a_2 + 3a_3} + \dots + \frac{(a_1 a_2^2 \dots a_n^n)^{\frac{1}{S_n}}}{a_1 + 2a_2 + \dots + na_n} \\ \leq \frac{1}{S_1} + \frac{1}{S_2} + \dots + \frac{1}{S_n} = \sum_{i=1}^n \left(\frac{2}{i} - \frac{2}{i+1} \right) = \frac{2n}{n+1}, \end{aligned}$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHÁLY BENCZE, Brasov, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer. All submitted solutions were essentially the same.

Mihály Bencze, Brasov, Romania, actually submitted the following generalization of this result. If a_k and α_k are positive for $k = 1, 2, \dots, n$, then

$$\sum_{k=1}^n \frac{(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k})^{\frac{1}{\alpha_1 + \alpha_2 + \cdots + \alpha_k}}}{\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_k a_k} \leq \sum_{k=1}^n \frac{1}{\alpha_1 + \alpha_2 + \cdots + \alpha_k}.$$

3029. [2005 : 174, 177] Proposed by Dorin Mărghidanu, Colegiul Național "A.I. Cuza", Corabia, Romania.

Let a_1, a_2, \dots, a_n be real numbers greater than -1 , and let α be any positive real number. Prove that if $a_1 + a_2 + \cdots + a_n \leq \alpha n$, then

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_n + 1} \geq \frac{n}{\alpha + 1}.$$

I. Composite of nearly identical solutions by the Austrian IMO-Team, 2005; Mihály Bencze, Brasov, Romania; Kin Fung Chung, student, University of Toronto, Toronto, ON; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; Li Zhou, Polk Community College, Winter Haven, FL, USA; and Titu Zvonaru, Comănești, Romania.

Since $a_i + 1 > 0$ for all i , we have, by the AM–HM Inequality,

$$\sum_{i=1}^n \frac{1}{a_i + 1} \geq \frac{n^2}{\sum_{i=1}^n (a_i + 1)} = \frac{n^2}{n + \sum_{i=1}^n a_i} \geq \frac{n^2}{n + \alpha n} = \frac{n}{\alpha + 1}.$$

Equality holds if and only if all the a_i s are equal.

II. Composite of very similar solutions by Michel Bataille, Rouen, France; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Edward Doolittle, University of Regina, Regina, SK; Joe Howard, Portales, NM, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA.

Since the function $f(x) = \frac{1}{x+1}$ is decreasing and strictly convex on the interval $(-1, \infty)$, we have, by Jensen's Inequality,

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{a_i + 1} = \frac{1}{n} \sum_{i=1}^n f(a_i) \geq f\left(\frac{1}{n} \sum_{i=1}^n a_i\right) \geq f(\alpha) = \frac{1}{\alpha + 1},$$

with equality if and only if $a_1 = a_2 = \dots = a_n = \alpha$. [Ed: Note that $-1 < \frac{1}{n} \sum_{i=1}^n a_i < \alpha$.]

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLU, Athens, Greece; and the proposer.

3030. [2005 : 174, 177] Proposed by Dorin Mărghidanu, Colegiul Național "A. I. Cuza", Corabia, Romania.

Show that, if a_1, a_2, \dots, a_n are positive real numbers, then

$$\frac{1}{a_1} + \frac{2}{(a_2)^{\frac{1}{2}}} + \frac{3}{(a_3)^{\frac{1}{3}}} + \dots + \frac{n}{(a_n)^{\frac{1}{n}}} \geq \frac{S_n}{(a_1 a_2 \dots a_n)^{\frac{1}{S_n}}}$$

where $S_n = 1 + 2 + \dots + n$.

Composite solution extracted from essentially the same solutions by those solvers marked with an asterisk (*) below.

By the AM–GM Inequality, we have

$$\begin{aligned} & \frac{1}{a_1} + \frac{2}{(a_2)^{\frac{1}{2}}} + \frac{3}{(a_3)^{\frac{1}{3}}} + \dots + \frac{n}{(a_n)^{\frac{1}{n}}} \\ & \geq (1 + 2 + \dots + n) \left(\frac{1}{a_1 (a_2)^{\frac{1}{2}} \dots (a_n)^{\frac{1}{n}}} \right)^{\frac{1}{1+2+\dots+n}} = \frac{S_n}{(a_1 a_2 \dots a_n)^{\frac{1}{S_n}}}, \end{aligned}$$

with equality if and only if all the a_i s are equal.

Solved by *ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; *the AUSTRIAN IMO TEAM, 2005; *MICHEL BATAILLE, Rouen, France; *MIHÁLY BENCZE, Brasov, Romania; *KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; *CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; EDWARD DOOLITTLE, University of Regina, Regina, SK; *OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; *JOHN G. HEUVER, Grande Prairie, AB; *WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; *JOHN LEONARD, University of Arizona, Tucson, AZ, USA; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; *JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Several solvers used the Weighted AM–GM Inequality with weights k/S_n on the numbers $1/(a_k)^{1/k}$, for $k = 1, 2, \dots, n$. This is actually equivalent to the argument given above, in this case.

3031. [2005 : 174, 177] *Proposed by Neven Jurič, Zagreb, Croatia.*

A quadruple (a, b, c, d) of positive integers is said to have the *Diophantine property* if each of the six integers $ab + 1$, $ac + 1$, $ad + 1$, $bc + 1$, $bd + 1$, $cd + 1$ is a perfect square. For example, each of the following nine quadruples has the Diophantine property:

$$\begin{array}{lll} (3, 5, 16, 1008), & (3, 8, 21, 2080), & (3, 16, 33, 6440), \\ (3, 21, 40, 10208), & (3, 33, 56, 22360), & (3, 40, 65, 31416), \\ (3, 56, 85, 57408), & (3, 65, 96, 75208), & (3, 85, 120, 122816). \end{array}$$

Find a general expression for the sequence of quadruples (a_n, b_n, c_n, d_n) which have the Diophantine property and for which the above examples represent the first terms.

Solution by Mercedes Sánchez Benito, Universidad Complutense, Madrid, Manuel Benito Muñoz and Emilio Fernández Moral, IES P. M. Sagasta, Logroño, Spain.

In problem 20 of his book Δ (the fourth book of *Arithmetica*), Diophantus of Alexandria proposes: “To find four numbers such that the product of any two increased by unity is a square.” He gives a solution in rational numbers: $\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}$.

Euler ([1], [2]) gives the following solution to Diophantus' problem in integers (a, b, c, d) :

Let a and b be such that $ab + 1 = p^2$ (with p an integer), $c = a + b + 2p$, and $d = 4p(p + a)(p + b)$. Then we also have

$$\begin{aligned} ac + 1 &= (p + a)^2, & bc + 1 &= (p + b)^2, & ad + 1 &= (2p(p + a) - 1)^2, \\ bd + 1 &= (2p(p + b) - 1)^2, & cd + 1 &= (2pc - 1)^2. \end{aligned}$$

The proposed sequence of quadruples (a_n, b_n, c_n, d_n) can be obtained from Euler's solution for $a = 3$ and p having integer values greater than 3, in increasing order, and such that $p^2 - 1 \equiv 0 \pmod{3}$. Therefore, the solution to the problem at hand is:

- $a_n = 3$ for $n = 1, 2, \dots$;
- $p_n = \begin{cases} 3m + 1 & \text{for } n = 2m - 1, \\ 3m + 2 & \text{for } n = 2m; \end{cases}$
- $b_n = \frac{p_n^2 - 1}{3} = \begin{cases} m(3m + 2) & \text{for } n = 2m - 1, \\ (m + 1)(3m + 1) & \text{for } n = 2m; \end{cases}$
- $c_n = a_n + b_n + 2p_n = \begin{cases} (m + 1)(3m + 5) & \text{for } n = 2m - 1, \\ (m + 2)(3m + 4) & \text{for } n = 2m; \end{cases}$
- $d_n = 4p_n(p_n + a_n)(p_n + b_n) = \begin{cases} 4(3m + 1)(3m + 4)(3m^2 + 5m + 1) & \text{for } n = 2m - 1, \\ 4(3m + 2)(3m + 5)(3m^2 + 7m + 3) & \text{for } n = 2m. \end{cases}$

If $p = 2$, we obtain the quadruple $(3, 1, 8, 120)$, which is not among the proposed ones. Note also that there are quadruples of positive integers with $a_n = 3$ and the Diophantine property which are not among the quadruples of the above solution: For example, the quadruple $(3, 5, 1008, 62496)$ has the Diophantine property, but cannot be obtained from Euler's solution.

Finally, we note that there are quadruples containing the number of this problem, 3031, and having the Diophantine property; we list three of them:

$$\begin{aligned} & (248, 1545, 3031, 4645441488) , \\ & (1545, 3031, 8904, 166786015280) , \\ & (3031, 5013, 15840, 962717421848) . \end{aligned}$$

References

- [1] T.L. Heath, *Diophantus of Alexandria*, 2nd ed., Dover, 1964, p. 181–182.
 [2] L.E. Dickson, *History of the Theory of Numbers*, Chelsea, 1971, vol. II, p. 517.

— Also solved by MICHEL BATAILLE, Rouen, France; EDWARD DOOLITTLE, University of Regina, Regina, SK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3032. [2005 : 174, 177] Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.

Let a, b, c be non-negative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca} \leq \frac{9}{2} .$$

Solution by the proposer, modified slightly by the editor.

Note first that the given inequality is successively equivalent to each of the following:

$$\begin{aligned} & 2(1-ab)(1-bc) + 2(1-bc)(1-ca) + 2(1-ca)(1-ab) \\ & \leq 9(1-ab)(1-bc)(1-ca) , \\ & 6 - 4(ab+bc+ca) + 2abc(a+b+c) \\ & \leq 9 - 9(ab+bc+ca) + 9abc(a+b+c) - 9a^2b^2c^2 , \\ & 0 \leq 3 - 5(ab+bc+ca) + 7abc(a+b+c) - 9a^2b^2c^2 , \\ & 0 \leq 3 - 5(ab+bc+ca) + 6abc(a+b+c) \\ & \quad + abc(a+b+c - 9abc) . \end{aligned} \tag{1}$$

By the AM–GM Inequality, we have

$$\begin{aligned} a+b+c - 9abc &= (a+b+c)(a^2+b^2+c^2) - 9abc \\ &\geq 3\sqrt[3]{abc} \cdot 3\sqrt[3]{a^2b^2c^2} - 9abc = 0 . \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned}
& 3 - 5(ab + bc + ca) + 6abc(a + b + c) \\
&= 3(a^2 + b^2 + c^2)^2 - 5(ab + bc + ca)(a^2 + b^2 + c^2) \\
&\quad + 6abc(a + b + c) \\
&= 3(a^4 + b^4 + c^4) + 6(a^2b^2 + b^2c^2 + c^2a^2) + abc(a + b + c) \\
&\quad - 5(ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2)) \\
&= [2(a^4 + b^4 + c^4) + 6(a^2b^2 + b^2c^2 + c^2a^2) - 4ab(a^2 + b^2) \\
&\quad - 4bc(b^2 + c^2) - 4ca(c^2 + a^2)] \\
&\quad + (a^4 + b^4 + c^4 + abc(a + b + c) \\
&\quad - ab(a^2 + b^2) - bc(b^2 + c^2) - ca(c^2 + a^2)) \\
&= [(a - b)^4 + (b - c)^4 + (c - a)^4] + a^2(a - b)(a - c) \\
&\quad + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0, \quad (3)
\end{aligned}$$

since $(a - b)^4 + (b - c)^4 + (c - a)^4 \geq 0$ and

$$a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0$$

is the well-known Schur's Inequality. Now (1) follows from (2) and (3).

We see that equality holds if and only if $a = b = c = \sqrt{3}/3$.

Also Solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incorrect solution.

Using the same proof presented above, the proposer actually proved the more general result that if a , b , and c are non-negative real numbers such that $a^2 + b^2 + c^2 = 1$, then, for all constants $k \geq 1$, we have

$$\frac{1}{k - ab} + \frac{1}{k - bc} + \frac{1}{k - ca} \leq \frac{9}{3k - 1}.$$

3033. [2005 : 175, 177] Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

Let I be the incentre of $\triangle ABC$, and let R and r be its circumradius and inradius, respectively. Prove that

$$6r \leq AI + BI + CI \leq \sqrt{12(R^2 - Rr + r^2)}.$$

I. Solution by Arkady Alt, San Jose, CA, USA.

[Ed: We give Alt's argument for the left inequality only.]

Let K and s be the area and the semiperimeter of the triangle. Using the well-known (or easy to prove) formulas

$$AI = \sqrt{\frac{bc(s - a)}{s}}, \quad BI = \sqrt{\frac{ca(s - b)}{s}}, \quad CI = \sqrt{\frac{ab(s - c)}{s}},$$

$abc = 4KR$, $K = sr$, $K = \sqrt{s(s-a)(s-b)(s-c)}$, and the AM-GM Inequality, we obtain

$$\begin{aligned} \frac{AI + BI + CI}{3} &\geq \sqrt[3]{AI \cdot BI \cdot CI} = \sqrt[3]{\frac{abc}{s^2} \sqrt{s(s-a)(s-b)(s-c)}} \\ &= \sqrt[3]{\frac{abcK}{s^2}} = \sqrt[3]{\frac{4RK^2}{s^2}} = \sqrt[3]{4Rr^2}. \end{aligned}$$

Thus, $AI + BI + CI \geq 3\sqrt[3]{4Rr^2}$. This inequality is stronger than the one proposed, because Euler's Inequality implies that $3\sqrt[3]{4Rr^2} \geq 6r$.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We give a solution "from the books". The inequality $AI + BI + CI \geq 6r$ is item 12.1 in [1]. On the other hand, item 12.2 in [1] is the inequality $AI + BI + CI \leq 2(R+r)$, which is stronger than the proposed one, because the well-known Euler's Inequality $R \geq 2r$ implies that $(R-2r)(2R-r) \geq 0$, and this is equivalent to $2(R+r) \leq \sqrt{12(R^2 - Rr + r^2)}$.

References

[1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; PANOS E. TSAOUSSOGLU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

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