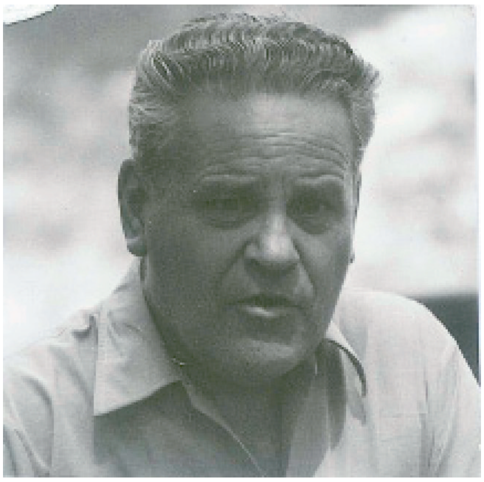
Contributor Profiles:

Jordi Dou Mas de Xexas



Jordi Dou was born in 1911 in Olot, Girona (Spain). He attended Primary and Secondary School at *Escoles Pies d'Olot* finishing his studies in 1928. Next, he studied at the *Universitat de Barcelona*, getting a Bachelors degree in Architecture and Mathematics.

In 1941, Jordi obtained a Doctoral degree in Architecture. He then entered the *Ajuntament de Barcelona* as chief of the section of *Valoracions*, and later the *Patrimoni* until his retirement.

During all this time he taught Mathematics, at the high school level in the *Institut Milà i Fontanals* in Barcelona and at the university level in the School of Architecture and in the Mathematics Faculty in Barcelona. He also was an active contributor in the field of mathematical problem-solving. His contributions, mainly in geometry, appeared in the *American Mathematical Monthly*, ***Crux Mathematicorum***, *Mathematics Magazine*, and *L'Escaire*. In ***Crux Mathematicorum*** he contributed from the inception of the journal until his retirement; he made contributions to the *Monthly* over a span of 44 years; he served as a member of the Board of Editors of *L'Escaire*. He also was one of the leaders accompanying the Spanish team at the International Mathematics Olympiad held in Paris in 1983.

There is no doubt that Jordi Dou was one of the best and most prolific of Spanish problemists of the second half of the 20^{th} century. Many other Spanish mathematicians have since followed in his footsteps, sharing his enthusiasm for problems. Regular contributions by Spaniards can be found currently in ***CRUX with MAYHEM*** and other journals with a problems column.

Dou has been retired since the early 1980s and resides at his home in Olot, Girona. He remained active as a problem-poser and problem-solver until 2001.

SKOLIAD No. 92

Robert Bilinski

Please send your solutions to the problems in this edition by ***September 1, 2006***. A copy of MATHEMATICAL MAYHEM Vol. 2 will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our items in this issue come from the 5th annual CNU Regional Mathematics Contest. Only a selection of the problems has been included. Thanks go to R. Porsky, C.N.U., Newport News, VA.

5^e Concours Annuel CNU Régional de Mathématique du Secondaire

Samedi, le 13 Novembre 2004

8. Combien de lait à 4% de matières grasses doit-on ajouter à du lait à 1% pour obtenir 12 gallons de lait à 2% ?

(A) 3 gallons (B) 4 gallons (C) 8 gallons (D) 9 gallons

12. Sachant que $f(x) = x + 2$ et $g(x) = \sqrt[3]{x}$, trouvez $f^{-1} \circ g^{-1}(2)$.

(A) 8 (B) -6 (C) 2 (D) 6

15. Un segment de 1 cm est une corde d'un de deux cercles concentriques et est tangent au plus petit des deux. Quelle est l'aire de la région annulaire entre les deux cercles ?

(A) $\frac{\pi}{6}$ cm^2 (B) $\frac{\pi}{4}$ cm^2 (C) $\frac{\pi}{3}$ cm^2 (D) $\frac{\pi}{2}$ cm^2

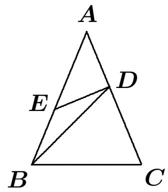
16. Résoudre l'équation $8^{\frac{1}{6}} + x^{\frac{1}{3}} = \dfrac{7}{3 - \sqrt{2}}$.

(A) 24 (B) 27 (C) 32 (D) 64

18. Soit $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ une fonction qui vérifie $f(x) + f\left(\frac{1}{x}\right) = 3x$. Quelle est la somme des valeurs de x pour lesquelles on a $f(x) = 1$?

(A) 1 (B) 2 (C) -1 (D) -2

21. Dans $\triangle ABC$ dans la figure à droite, $AB = AC$, $BC = BD$, $AD = DE = EB$. Quelle est $\angle$A ? (Note : La figure n'est pas à l'échelle.)

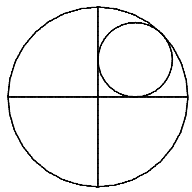


(A) $30°$ (B) $36°$ (C) $45°$ (D) $54°$

32. Soit n le nombre de manières que 10 dollars peuvent être changés en 10 sous et 25 sous avec chacun d'eux utilisé au moins une fois. Alors n vaut :

(A) 18 (B) 38 (C) 21 (D) 19

34. Deux perpendiculaires qui se coupent au centre d'un cercle de rayon 1, séparent le cercle en quatre parties. Un plus petit cercle est inscrit dans une de ces parties comme dans la figure. Quel est le rayon du plus petit cercle ?



(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\sqrt{2}-1$

5th Annual CNU Regional High School Mathematics Contest

Saturday, November 13, 2004

8. How much milk with 4% fat should be added to milk with 1% fat to obtain 12 gallons of milk with 2% fat?

(A) 3 gallons (B) 4 gallons (C) 8 gallons (D) 9 gallons

12. Given $f(x) = x + 2$ and $g(x) = \sqrt[3]{x}$, find $f^{-1} \circ g^{-1}(2)$.

(A) 8 (B) -6 (C) 2 (D) 6

15. A line segment 1 cm long is a chord of the larger of two concentric circles, and is tangent to the smaller of the two circles. What is the area of the annular region between the two circles?

(A) $\frac{\pi}{6}$ cm^2 (B) $\frac{\pi}{4}$ cm^2 (C) $\frac{\pi}{3}$ cm^2 (D) $\frac{\pi}{2}$ cm^2

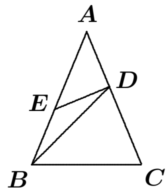
16. Solve the equation $8^{\frac{1}{6}} + x^{\frac{1}{3}} = \dfrac{7}{3-\sqrt{2}}$.

(A) 24 (B) 27 (C) 32 (D) 64

18. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be any function such that $f(x) + f\left(\frac{1}{x}\right) = 3x$. What is the sum of the values of x for which $f(x) = 1$?

(A) 1 (B) 2 (C) -1 (D) -2

21. In $\triangle ABC$ in the figure to the right, $AB = AC$, $BC = BD$, $AD = DE = EB$. What is $\angle A$? (Note: the figure is not drawn to scale.)

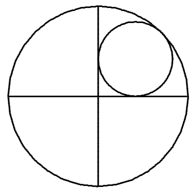


(A) 30° (B) 36° (C) 45° (D) 54°

32. Let n be the number of ways that 10 dollars can be changed into dimes and quarters with at least one of each being used. Then n equals:

(A) 18 (B) 38 (C) 21 (D) 19

34. Two perpendicular lines intersecting at the centre of a circle of radius 1, divide the circle into four parts. A smaller circle is inscribed in one of those parts, as shown. What is the radius of the smaller circle?



(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\sqrt{2}-1$

Next we give the solutions to the twenty-first annual W.J. Blundon Mathematics contest [2005 : 353–355].

The 21st W.J. Blundon Mathematics Contest
Sponsored by the CMS and the Department of Mathematics and Statistics, Memorial University
February 18, 2004

1. A farmer spent exactly \$100 to buy 100 animals. Cows cost \$10, sheep \$3 and pigs 50 cents each. How many of each did he buy?

Solution by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK, modified by the editor.

Suppose the farmer buys x cows, y sheep, and z pigs. This means that

$$\begin{aligned} x+y+z &= 100\,, & (1)\\ \text{and}\quad 10x+3y+.5z &= 100\,, \end{aligned}$$

which can be multiplied by 2 to get

$$20x+6y+z = 200\,. \qquad (2)$$

Subtracting (1) from (2) yields $19x+5y=100$.

As we know, x and y must be positive integers. Also, we know that the last digit of a multiple of 19 can be any digit from 0 to 9. However, the last digit of a multiple of 5 can only be 0 or 5. Thus, if we want to make a sum which is 100, then the last digit of $19x$ and the last digit of $5y$ must be the same, either 0 or 5.

First we suppose that both last digits are 0. Since $19x$ ends in the digit 0, so does x. If $x=0$, then we must have $y=20$, which means that

$z = 80$. If $x \geq 10$, then y is negative, which is impossible.

Next we suppose that both last digits are 5. Since $19x$ ends in the digit 5, so does x. If $x = 5$, then $y = 1$ and $z = 94$. If $x \geq 15$, then y is negative, which is impossible.

From the above, we can see there are only two possibilities, namely (i) 0 cows, 20 sheep, and 80 pigs, or (ii) 5 cows, 1 sheep, and 94 pigs.

2. Show that if a three-digit number is divisible by 3, then the sum of its digits is divisible by 3.

Solution by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK.

Let this three-digit number be $100X + 10Y + Z$, which can also be written as $99X + 9Y + (X + Y + Z)$. Clearly, $99X + 9Y$ is divisible by 3. Then, if our three-digit number is divisible by 3, we must have $X + Y + Z$ also divisible by 3.

3. Consider the points $A(1, 0)$, $B(3, 0)$, $C(3, 5)$, and $D(1, 4)$. Find an equation of the line through the origin that divides the quadrilateral $ABCD$ into two parts of equal area.

Solution by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK, modified by the editor.

Since we have two parallel sides only (the vertical ones), $ABCD$ is a trapezoid. Let the equation of the line we seek be $y = ax$, because it passes through the origin. Let the points of intersection with $ABCD$ be $(1, m)$ and $(3, n)$.

Since $AD = 4$, $BC = 5$, and $AB = 2$, the area of the trapezoid $ABCD$ is $2\left(\frac{5+4}{2}\right) = 9$. Then, since the line $y = ax$ divides the trapezoid into equal parts, we see that $m + n = 9/2$. Also, $m/n = 1/3$, using similar triangles.

Solving these equations, we get $m = 9/8$ and $n = 27/8$. Thus, line $y = ax$ passes through $(1, 9/8)$ and $(3, 27/8)$, which gives us $a = 9/8$. The equation of the line is $y = \frac{9}{8}x$.

4. Find all real solutions to the equation $1 + x + x^2 + x^3 = x^4 + x^5$.

Solution by the editor.

The given equation can be rewritten as

$$\begin{aligned} 1 + x + x^2 + x^3 - x^4 - x^5 &= 0\,, \\ (1 + x)(1 + x^2 - x^4) &= 0\,. \end{aligned}$$

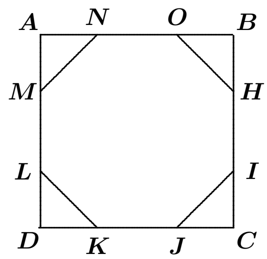
Hence, either $x = -1$ or $x^4 - x^2 - 1 = 0$. Applying the quadratic formula to the second equation, we see that $x^2 = \dfrac{1 \pm \sqrt{5}}{2}$, which means that

$x = \pm\sqrt{\frac{1+\sqrt{5}}{2}}$, since $\frac{1-\sqrt{5}}{2} < 0$.

One incorrect solution was received.

5. Find the exact area of the regular octagon formed by cutting equal isosceles right triangles from the corners of a square with sides of length one unit.

Solution by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK.



Label the vertices of the octagon and the square as shown. Set $x = AM = AN$. Then $MN = x\sqrt{2}$. Because the octagon is regular, we also have $ON = x\sqrt{2}$. Since $AB = 1$, we have $AN + NO + OB = 1$, which means that $x\sqrt{2} + x + x = 1$. From this, we get $x = \frac{2-\sqrt{2}}{2}$.

The total area of the four right triangles is $4 \times \frac{1}{2}\left(\frac{2-\sqrt{2}}{2}\right)^2 = 3 - 2\sqrt{2}$. Hence, the area of the octagon is $1^2 - (3 - 2\sqrt{2}) = 2\sqrt{2} - 2$.

6. If A, B, and C are angles of a triangle, prove that

$$\cos C = \sin A \sin B - \cos A \cos B\,.$$

Solution by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK.

$$\begin{aligned}
\cos C &= \cos(180^\circ - A - B) \\
&= -\cos(A + B) \\
&= -(\cos A \cos B - \sin A \sin B) \\
&= \sin A \sin B - \cos A \cos B\,.
\end{aligned}$$

7. If $a + b + c = 0$ and $abc = 4$, find $a^3 + b^3 + c^3$.

Solution by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK.

We can treat a, b, c as the roots of a cubic equation. Since $a+b+c = 0$ and $abc = 4$, this cubic equation is $x^3 + Zx - 4 = 0$, for some Z.

Substituting a, b, c into the equation, we get

$$\begin{aligned}
a^3 &= 4 - Za\,, \\
b^3 &= 4 - Zb\,, \\
c^3 &= 4 - Zc\,.
\end{aligned}$$

Adding these equations, we get $a^3 + b^3 + c^3 = 12 - Z(a + b + c)$. Since $a + b + c = 0$, we find that $a^3 + b^3 + c^3 = 12$.

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

We obtain the answer 12 immediately if we use the identity

$$a^3 + b^3 + c^3 - 3abc \;=\; (a+b+c)(a^2+b^2+c^2-ab-bc-ca)\,.$$

8. (a) If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, find $\log_5 12$.
(b) Solve $x^{\log_{10} x} = 100x$.

Solution to part (a) *by the editor.*

(a) $\log_5 12 = \dfrac{\log_{10} 12}{\log_{10} 5} = \dfrac{\log_{10} 3 + 2\log_{10} 2}{\log_{10} 10 - \log_{10} 2} = \dfrac{2a+b}{1-a}$.

Solution to part (b) *by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK.*

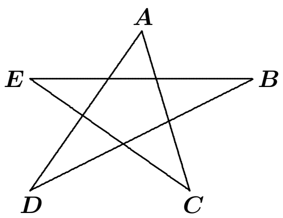
(b) Taking the base 10 logarithm on both sides, we get

$$(\log_{10} x)^2 \;=\; \log_{10}(100x) \;=\; \log_{10} x + 2\,.$$

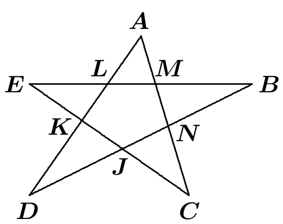
Making the substitution $y = \log_{10} x$, we get $y^2 - y - 2 = 0$; that is, $(y+1)(y-2) = 0$. Then $y = -1$ or $y = 2$. In other words, $\log_{10} x = -1$ or $\log_{10} x = 2$, which gives $x = \frac{1}{10}$ or $x = 100$. We check that these are indeed solutions of the original equation.

Part (a) *also solved by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK.*

9. In the figure below, find the sum of the angles A, B, C, D, and E.



Solution by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK.



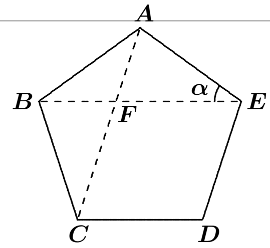
Label the vertices of the inner pentagon in a clockwise manner with the letters J, K, L, M, and N, so that J is the intersection of BD and CE. In any pentagon, the sum of all the interior angles is $(5-2)\cdot 180^\circ = 540^\circ$. We know that

$$\begin{aligned}
\angle A + \angle C + \angle AKC &= 180^\circ\,,\\
\angle B + \angle D + \angle BLD &= 180^\circ\,,\\
\angle C + \angle E + \angle CME &= 180^\circ\,,\\
\angle D + \angle A + \angle DNA &= 180^\circ\,,\\
\angle E + \angle B + \angle EJB &= 180^\circ\,.
\end{aligned}$$

If we now add all five of these equations, we get

$$\begin{aligned} 2(\angle A + \angle B + \angle C + \angle D + \angle E) + 540^\circ &= 5 \cdot 180^\circ\,, \\ \text{or} \quad \angle A + \angle B + \angle C + \angle D + \angle E &= 180^\circ\,. \end{aligned}$$

10. Let $ABCDE$ be a regular pentagon with each side of length 1. The length of BE is τ, and the angle FEA is α. Find τ and $\cos\alpha$.



Solution by the editor.

Because $ABCDE$ is a regular pentagon, the interior angles all have the same measure, namely $(5-2)\cdot 180^\circ/5 = 108^\circ$. Since triangle ABE is isosceles, we have

$$\alpha \;=\; \angle ABE \;=\; \angle AEB \;=\; (180^\circ - 108^\circ)/2 \;=\; 36^\circ\,.$$

Similarly, $\angle BAC = \angle BCA = 36^\circ$. Thus, triangle ABF is isosceles and $\angle AFB = 180^\circ - 2(36^\circ) = 108^\circ$. Then $\angle AFE = 72^\circ$, and we see that triangle EAF is isosceles with $EF = EA = 1$. Furthermore, triangles ABE and FAB are similar. Therefore,

$$\frac{AB}{BE} = \frac{FA}{AB}\,,$$

which simplifies to $FA \cdot BE = AB^2 = 1$.

Letting $x = FA = BF$, we have $BE = BF + FE = x + 1$. The relation $FA \cdot BE = 1$ becomes $x(x+1) = 1$, or $x^2 + x - 1 = 0$, which gives us $x = \dfrac{-1 \pm \sqrt{5}}{2}$. Since $x > 0$, we see that $x = \dfrac{-1+\sqrt{5}}{2}$. Therefore,

$$\tau \;=\; BE \;=\; x + 1 \;=\; \frac{1+\sqrt{5}}{2}\,.$$

Also solved by Jia-xi Sun, student, Walter Murray Collegiate Institute, Saskatoon, SK.

The solutions to the BC Colleges High School Mathematics Contest, Junior Final Round will appear next month.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Jia-Xi Sun! Congratulations! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). ~~The Assistant Mayhem Editor is Jeff Hooper (Acadia~~ University). The other staff members are John Grant McLoughlin (University of New Brunswick), Ian VanderBurgh (University of Waterloo), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier juillet 2006**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M232. *Proposé par Nicholas Buck, College of New Caledonia, Prince George, CB, et John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

On peut recouvrir un échiquier standard de 8 lignes par 8 colonnes avec 32 dominos, chaque domino couvrant deux cases adjacentes. Supposons qu'on enlève au hasard deux cases. Si le nouvel échiquier obtenu ne peut plus être recouvert par 31 dominos, quelle est la probabilité pour que :

1. les deux cases enlevées soient dans la même ligne ?
2. les deux cases enlevées se touchent en un sommet (diagonalement, horizontalement ou verticalement) ?

M233. *Proposé par Richard K. Guy, Université de Calgary, Calgary, AB.*

Pouvez-vous placer huit entiers distincts, choisis parmi les nombres de 0 à 12, aux sommets d'un cube de telle sorte que les douzes arêtes aient les différences 1, 2, ..., 12 entre leurs extrémités ?

Trouver une manière de le faire ou montrer que c'est impossible.

M234. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Soit J un nombre de deux chiffres sans communs diviseurs autres que 1. En permutant ces deux chiffres, on obtient un nombre I qui est p% plus grand que J. Trouver toutes les valeurs possibles de p, p étant un nombre naturel plus petit que 100.

M235. *Proposé par Ron Lancaster, Université de Toronto, Toronto, ON.*

Résoudre l'équation

$$2^x + 2^{x+1} + \cdots + 2^{x+2006} = 4^x + 4^{x+1} + \cdots + 4^{x+2006}\,.$$

M236. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

Un voyageur débarque sur une île inconnue et constate qu'elle est habitée par des chevaliers qui ne disent que la vérité, et par des valets qui ne disent que des mensonges. Un jour, il rencontra trois habitants qu'on appellera A, B et C, et leur demanda "Combien y a-t-il de chevaliers parmi vous trois ?"

A donna une réponse que ne comprit pas le voyageur, mais que les deux autres entendirent. Questionné sur ce que A avait dit, B répondit "A a dit qu'il y avait un chevalier parmi nous."

"Ne croyez pas B," s'exclama C, "c'est un menteur !"

À quel groupe appartiennent B et C ?

M237. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Soit ABC un triangle isocèle avec $AB = AC$, et dans lequel les longueurs des côtés sont des entiers sans commun diviseur autre que 1. Le centre I du cercle inscrit divise la bissectrice intérieure AD de façon que $\frac{AI}{ID} = \frac{25}{24}$. Déterminer le rayon du cercle inscrit du triangle ABC.

. . . .

M232. *Proposed by Nicholas Buck, College of New Caledonia, Prince George, BC, and John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

A standard checkerboard with 8 rows and 8 columns can be covered using 32 dominoes, each covering two adjacent squares. Suppose that two squares are randomly removed. If the resulting board cannot be covered by 31 dominoes, what is the probability that:

1. the two squares removed were in the same row?

2. the two squares removed shared a common vertex (diagonally, horizontally, or vertically)?

M233. *Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

Can you place eight distinct integers selected from 0 to 12 at the vertices of a cube so that the twelve edges have the differences $1, 2, \ldots, 12$ between their end-points?

Either find a way to do this, or prove that it is impossible.

M234. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let J be a two-digit number in which the two digits share no common divisors other than 1. Reversing the digits of J produces I, a number that is $p\%$ greater than J. Find all possible values of p, if p is a natural number less than 100.

M235. *Proposed by Ron Lancaster, University of Toronto, Toronto, ON.*

Solve

$$2^x + 2^{x+1} + \cdots + 2^{x+2006} \;=\; 4^x + 4^{x+1} + \cdots + 4^{x+2006}\,.$$

M236. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

A traveller to a strange island discovers that it is inhabited by knights who can only make true statements and knaves who can only make false statements. One day a traveller encountered three inhabitants, whom we will call A, B, and C, and asked, "How many knights are there among you three?"

A made an answer, which the traveller missed, but which was understood by the other two. When B was asked what A said, B responded, "A said that there is one knight among us."

"Don't believe B," exclaimed C, "he is lying."

What are B and C?

M237. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let ABC be an isosceles triangle with $AB = AC$, and let the lengths of the sides be integers with no common divisor other than 1. The incentre I divides the internal angle bisector AD such that $\dfrac{AI}{ID} = \dfrac{25}{24}$. Determine the radius of the incircle of $\triangle ABC$.

Mayhem Solutions

M175. *Proposed by the Mayhem Staff.*

A set S consists of five positive integers. Show that it is always possible to find a subset of S containing three elements such that the sum of the elements in the subset is a multiple of 3.

Solution by Geneviève Lalonde, Massey, ON.

We can classify all integers as being in one of three groups:

1. One less than a multiple of three.
2. A multiple of three.
3. One more than a multiple of three.

Among the five numbers in the set S, we must either have at least one number in each group, or else no number in at least one of the groups.

If we have at least one in each group, then we have numbers of the form $3a - 1$, $3b$, and $3c + 1$ in our collection, and the sum of these is $3a - 1 + 3b + 3c + 1 = 3(a + b + c)$, a multiple of 3.

Suppose that none of the numbers is in (at least) one of the groups. Then, since we have five numbers and two groups, at least one group must have (at least) three numbers in it. If we have three numbers in the first group, then those numbers have the form $3a - 1$, $3b - 1$, and $3c - 1$; if we have three numbers in the second group, then those numbers have the form $3a$, $3b$, and $3c$; and if we have three numbers in the third group, then those numbers have the form $3a + 1$, $3b + 1$, and $3c + 1$. In each case, the sum of the three numbers is a multiple of 3.

M176. *Proposed by the Mayhem Staff.*

How many points (x, y) with positive integral coordinates lie on the curve defined by $x^2 + y^2 + 2xy - 2005x - 2005y - 2006 = 0$?

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

We rearrange the terms in the equation to produce

$$\begin{aligned} (x+y)^2 - 2005(x+y) - 2006 &= 0\,, \\ (x+y+1)(x+y-2006) &= 0\,. \end{aligned}$$

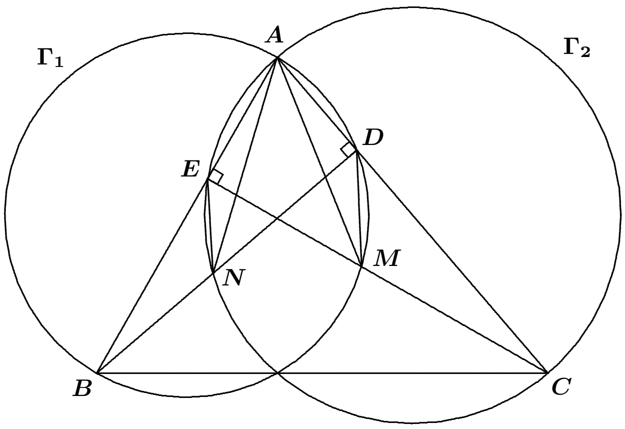
Setting the first factor, $x + y + 1$, equal to zero does not produce any points on the curve having positive integral coordinates. However, setting the second factor, $x + y - 2006$, equal to zero produces 2005 points $(1, 2005)$, $(2, 2004)$, …, $(2005, 1)$ with positive integral coordinates.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Alper Cay, Uzman Private School, Kayseri, Turkey.

M177. *Proposed by Babis Stergiou, Chalkida, Greece.*

Let ABC be a triangle with angle A acute. Let BD and CE be two of its altitudes, with D on AC and E on AB. On diameters AB and AC construct circles Γ_1 and Γ_2, respectively. Let M be the point of intersection of Γ_1 and CE, and let N be the point of intersection of Γ_2 and BD, where M and N are interior points of $\triangle ABC$. Prove that $AM = AN$.

Solution by Alper Cay, Uzman Private School, Kayseri, Turkey, modified by the editor.



Since $AENC$ is a cyclic quadrilateral, we see that $\angle ENA = \angle ECA$. Similarly, since quadrilateral $ADMB$ is cyclic, we have $\angle AMD = \angle ABD$. Since $\angle BDC = \angle BEC = 90°$, quadrilateral $EBCD$ is also cyclic; whence, $\angle EBD = \angle ECD$. Thus,

$$\angle ENA = \angle ECA = \angle ECD = \angle EBD = \angle ABD = \angle AMD\,.$$

This implies that $\triangle AEN$ is similar to $\triangle ANB$ and $\triangle ADM$ is similar to $\triangle AMC$. Hence,

$$\frac{AE}{AN} = \frac{AN}{AB} \quad \text{and} \quad \frac{AD}{AM} = \frac{AM}{AC}\,.$$

These equations can be rewritten as $AN^2 = AE\cdot AB$ and $AM^2 = AD\cdot AC$.

Since $\triangle ABD$ is similar to $\triangle AEC$, we also have

$$\frac{AE}{AD} = \frac{AC}{AB}\,.$$

Therefore,

$$\frac{AN^2}{AM^2} = \frac{AE \cdot AB}{AD \cdot AC} = \frac{AC}{AB} \cdot \frac{AB}{AC} = 1\,;$$

whence, $AM = AN$.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC.

M178. *Proposed by the Mayhem Staff.*

Show that, if $10a + b$ is a multiple of 7, then $a - 2b$ must also be a multiple of 7.

Solution by Eric Zhang, Lisgar Collegiate Institute, Ottawa, ON.

If $10a + b$ is divisible by 7, then we can write $10a + b = 7n$ where n is an integer. By rearranging the above as $b = 7n - 10a$ and substituting this into $a - 2b$, we obtain

$$a - 2b \;=\; a - 2(7n - 10a) \;=\; 21a - 14n\,,$$

which is divisible by 7.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA; Alper Cay, Uzman Private School, Kayseri, Turkey; and "Jack" Sang Hoon Jeong, Lauren Sapikowski, and Andrew West, Washington and Lee University, Lexington, VA, USA. Several solvers pointed out that the problem did not specify integer values for a and b and showed that the statement fails if a and b are not integers.

M179. *Proposed by the Mayhem Staff.*

(a) Find all two-digit numbers that increase by 75% when their digits are reversed.

(b) Show that no three-digit numbers increase by 75% when their digits are reversed.

Solution by Luis Vega Lastra, I.E.S. Torres Quevedo, Santander, Cantabria, Spain.

(a) Let a be the first digit of the number and let b be the second digit, where $0 < a < 10$ and $0 \le b < 10$. Then

$$\begin{aligned} \tfrac{7}{4}(10a + b) &= 10b + a\,, \\ 70a + 7b &= 40b + 4a\,, \\ 66a &= 33b\,, \\ b &= 2a\,. \end{aligned}$$

Thus, all numbers where the second digit is twice the first are solutions. These numbers are 12, 24, 36, and 48.

(b) Let a be the first digit of the number, let b be the second digit, and let c be the third digit, where $0 < a < 10$ and $0 \le b,\, c < 10$. Then

$$\begin{aligned} \tfrac{7}{4}(100a + 10b + c) &= 100c + 10b + a\,, \\ 700a + 70b + 7c &= 400c + 40b + 4a\,, \\ 696a + 30b - 393c &= 0\,, \\ 232a + 10b &= 131c\,. \end{aligned} \qquad (1)$$

Considering this equation $\mod 10$, we find that $c \equiv 2a \pmod{10}$; that is, $c = 2a + 10k$ for some integer k. Substituting for c in equation (1), we get

$$\begin{aligned} 232a + 10b &= 262a + 1310k\,, \\ 10b &= 30a + 1310k\,, \\ b &= 3a + 131k\,. \end{aligned}$$

Since a and b are integers such that $0 < a < 10$ and $0 \leq b < 10$, we must have $k = 0$. Then $b = 3a$ and $c = 2a$.

Thus, all three-digit numbers abc with $c = 2a$ and $b = 3a$ increase by 75% when their digits are reversed. The numbers are 132, 264, and 396.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; and James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

M180. *Proposed by the Mayhem Staff.*

(a) Show that $121_{(b)}$ is a perfect square in any base $b > 2$.

(b) Determine the smallest value of b for which $232_{(b)}$ is a perfect square.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

(a) We first observe that $121_{(b)} = (1 \times b^2) + (2 \times b) + 1 = (b+1)^2$. Thus, $121_{(b)}$ is a perfect square in any base $b > 2$. (Base 2 is not allowed because the digit 2 appears in the number.)

(b) Let $x^2 = (2 \times b^2) + (3 \times b) + 2$. Rewrite this as $2b^2 + 3b + 2 - x^2 = 0$, and solve for b. Then

$$b = \frac{-3 \pm \sqrt{9 - (4 \times 2 \times (2 - x^2))}}{4} = \frac{-3 \pm \sqrt{8x^2 - 7}}{4}\,,$$

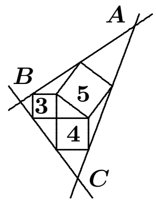
which implies that $8x^2 - 7$ must be a perfect square. We have the following cases:

- $x = 1$. Then $8x^2 - 7 = 1$, a perfect square, and $b = -1$ or $b = -\frac{1}{2}$.
- $x = 2$. Then $8x^2 - 7 = 25$, a perfect square, and $b = -2$ or $b = \frac{1}{2}$.
- $x = 4$. Then $8x^2 - 7 = 121$, a perfect square, and $b = -\frac{7}{2}$ or $b = 2$.
- $x = 11$. Then $8x^2 - 7 = 961$, a perfect square, and $b = -\frac{17}{2}$ or $b = 7$.

Since b must be an integer with $b > 3$ (since one of the digits is 3), the smallest possible value is $b = 7$. This could also be verified by substituting $b = 4, 5, \ldots$ into $2b^2 + 3b + 2$.

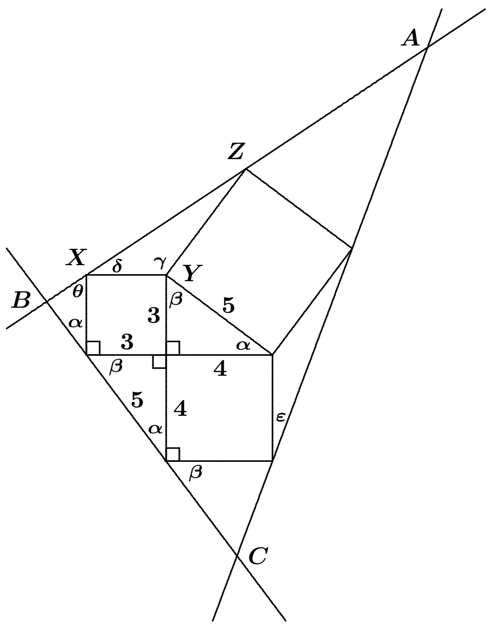
Also solved by Alper Cay, Uzman Private School, Kayseri, Turkey. One incorrect solution was received.

M181. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*



A 3–4–5 right triangle has squares drawn outward on each of its sides. Lines are drawn through the outer corners of these squares, as shown. These lines form a triangle ABC. Determine whether $\triangle ABC$ is also a right triangle.

Solution by Lisa Smith, Laura Anne Taylor, and Derrick Moldenhauer, Washington and Lee University, Lexington, VA, USA, modified by the editor.



Label the diagram as shown at right. Then $\angle B = \beta + \delta$ and $\angle C = \alpha + \varepsilon$. Therefore,

$$\begin{aligned}\angle B + \angle C &= (\alpha+\beta) + \delta + \varepsilon \\ &= 90^\circ + \delta + \varepsilon \\ &> 90^\circ\,.\end{aligned}$$

It follows that $\angle A < 90^\circ$.

By the Law of Cosines in $\triangle XYZ$, we have

$$\begin{aligned} XZ^2 & \\ &= 3^2 + 5^2 - 2(3)(5)\cos\gamma \\ &= 34 - 30\cos\gamma\,.\end{aligned}$$

Since $\gamma = 180^\circ - \beta$, we get

$$\begin{aligned} XZ^2 &= 34 + 30\cos\beta \\ &= 34 + 30\left(\tfrac{3}{5}\right) \;=\; 52\,.\end{aligned}$$

Thus, $XZ = 2\sqrt{13}$. Again by the Law of Cosines in $\triangle XYZ$, we have

$$YZ^2 \;=\; 5^2 \;=\; 52 + 3^2 - 2(3)\big(2\sqrt{13}\big)\cos\delta\,,$$

which gives us $\cos\delta = 3/\sqrt{13}$. Then $\sin\delta = 2/\sqrt{13}$, and hence,

$$\begin{aligned}\cos B &= \cos(\beta+\delta) \;=\; \cos\beta\cos\delta - \sin\beta\sin\delta \\ &= \left(\frac{3}{5}\right)\left(\frac{3}{\sqrt{13}}\right) - \left(\frac{4}{5}\right)\left(\frac{2}{\sqrt{13}}\right) \;=\; \frac{1}{5\sqrt{13}} \;>\; 0\,.\end{aligned}$$

Therefore, $\angle B < 90^\circ$.

By similar calculations, it can be shown that $\cos C = \dfrac{23}{5\sqrt{73}} > 0$, and thus, $\angle C < 90^\circ$.

Therefore, $\triangle ABC$ is not a right triangle.

Problem of the Month

Ian VanderBurgh

Some of the most interesting problems are those which require very little specialized knowledge but do require very careful thinking. Here is such a problem. It uses concepts which can be understood by students as early as Grade 8.

Problem (1995 American High School Mathematics Examination)

A list of five positive integers has mean 12 and range 18. The mode and median are both 8. How many different values are possible for the second largest element in the list?

(The American High School Mathematics Examination (AHSME) was the precursor to the current AMC→12. The 1995 AHSME was the 46^{th} sitting of this competition.)

Before we launch into an analysis of this problem, we should make sure that we understand the concepts in the problem. Given a list of numbers,

- the mean is the usual "average" of the numbers in the list (if we wanted to be fancy, we would refer to this as the "arithmetic mean"),
- the range is the difference between the smallest and largest numbers in the list,
- the median is the "middle" number in the list when the numbers are arranged in ascending order (thankfully, in this problem, our list has an odd number of terms, so we don't have to worry about what happens if there is an even number of terms), and
- the mode is the most frequently ocurring number in the list.

Solution

For convenience, we will arrange our list in ascending order. Since we have a list of five positive integers whose median is 8, the third number in the list is 8.

Since the mean of the five numbers in the list is 12, the sum of the five numbers must be $5 \times 12 = 60$. Since the mode of the five numbers is 8, the number 8 must occur more frequently than any other number in the list. This tells us that 8 occurs 2, 3, 4, or 5 times in the list. (We will have to remember to check later that, in the case where 8 occurs twice, no other number occurs twice, so that 8 is the unique mode.)

Can 8 occur five times? No. The mean of the list must be 12, and if the list consisted of five 8s, its mean would be 8.

Can 8 occur four times? No. If 8 occurs four times and the sum of the numbers is 60, then the fifth number must be 28. But then the range of the list is $28 - 8 = 20$, not 18 as prescribed.

Therefore, 8 occurs either two or three times in the list. Since the mean is 12, the largest number in the list is not 8. This tells us that the list has one of the following forms (with the values in ascending order):

(i)_, 8, 8, 8, _; (ii) 8, 8, 8, _, _; (iii) _, 8, 8, _, _; (iv) _, _, 8, 8, _.

We can now use the fact that the range must be 18 to assign variable values, x and y not equal to 8, to the unknown numbers in the lists:

(i) x, 8, 8, 8, $x + 18$; (ii) 8, 8, 8, x, 26;
(iii) x, 8, 8, y, $x + 18$; (iv) x, y, 8, 8, $x + 18$.

Let us now examine each of these possibilities.

In case (i), for the sum of the numbers in the list to be 60, we must have $x + 8 + 8 + 8 + (x + 18) = 60$; that is, $2x + 42 = 60$, or $x = 9$. But $x \leq 7$, since the list is in ascending order. Thus, this case is impossible.

In case (ii), for the sum of the numbers in the list to be 60, we must have $x = 10$. This gives us one possible list: 8, 8, 8, 10, 26.

In case (iii), for the sum of the numbers in the list to be 60, we must have $x + 8 + 8 + y + (x + 18) = 60$; that is, $2x + y = 26$. Also, for the list to be in ascending order, we must have $x \leq 7$, $y \geq 9$, and $y \leq x + 18$. Using these three restrictions together with $2x + y = 26$ and the fact that all of the elements are positive integers, we obtain the possibilities

$$(x, y) \in \{(3,20),\ (4,18),\ (5,16),\ (6,14),\ (7,12)\}.$$

This gives us five new possible lists.

In case (iv), again we must have $2x + y = 26$. But, for the list to be in ascending order, we must also have $x \leq y < 8$, which implies that $2x + y \leq 21$. Hence, this case is impossible.

Therefore, there are six possible lists:

$$\begin{array}{lll} 8, 8, 8, 10, 26; & 3, 8, 8, 20, 21; & 4, 8, 8, 18, 22; \\ 5, 8, 8, 16, 23; & 6, 8, 8, 14, 24; & 7, 8, 8, 12, 25; \end{array}$$

We can check quickly that each of these six lists has the desired properties. Thus, we obtain six possibilities for the second largest element, namely 10, 12, 14, 16, 18, and 20.

That was a good exercise in organized thinking while applying some of our basic statistical knowledge. It's always amazing how such simple concepts can create complicated problems!

Pólya's Paragon

Placing Greater Value on Place Value

John Grant McLoughlin

Place value and counting are the pillars of primary mathematical learning experiences. This Paragon focuses on the role of place value in a variety of contexts related to mathematical problem-solving. We will work exclusively with base 10. Anyone who enjoys the ideas will find fertile ground for exploration in extending the scope to other bases.

To begin, let us suppose that a list of positive integers is provided in a multiple-choice format. One of them is a perfect square, and the question asks which one. The numbers appear much too large for the square roots to be readily known. What should we do?

A reasonable course of action would be to consider the units digit in each number. Suppose that the units digits in the five choices given are 2, 3, 6, 7, and 8, respectively. The perfect square can now be identified as the number ending in 6. Why? Because all perfect squares end in 0, 1, 4, 5, 6, or 9. This is evident from the following table, which shows how the units digit of x^2 depends on the units digit of x, for any positive integer x. (For example, if $x = 12$, with units digit 2, then $x^2 = 144$, with units digit 4.)

units digit of x	0	1	2	3	4	5	6	7	8	9
units digit of x^2	0	1	4	9	6	5	6	9	4	1

To understand why the units digit of x^2 depends only on the units digit of x and not on the other digits, we write $x = 10a + b$, where b is the units digit of x. Then $x^2 = (10a + b)^2 = 100a^2 + 20ab + b^2$. The integers $100a^2$ and $20ab$ each have 0 as their units digit. It follows that the units digit of x^2 is the same as the units digit of b^2, which depends only on b. (For the benefit of anyone familiar with modular arithmetic, we note that what we are describing is the calculation of x^2 modulo 10.)

Readers are encouraged to discover for themselves which digits may appear as the units digit in perfect cubes and higher powers.

Let us shift our attention to divisibility tests. Two well-known tests state that a number is divisible by 3 or 9 if and only if the sum of the digits in the number is divisible by 3 or 9, respectively. To understand why these tests work, let us consider the case of divisibility by 9 for a four-digit number $abcd$, by which we really mean the number $1000a + 100b + 10c + d$. We may remove groups of 9 from this number without changing the remainder when the number is divided by 9. For example, the integer $100b$ may be reduced by $99b$ without changing the remainder upon division by 9. (Keep in mind that the concern is not with the quotient but only with the remainder.) Working in this manner, we derive the test in the following form:

$$\begin{array}{rrrrrrr} & 1000a & + & 100b & + & 10c & + \; d \\ - & (999a & + & 99b & + & 9c) & \\ \hline & a & + & b & + & c & + \; d \end{array}$$

We find that the sum of the digits will produce the same remainder, upon division by 9, as the original number. Divisibility by 3 works in a similar manner. Notice that both divisibility tests can be understood by writing a number in expanded form using place value.

A less familiar divisibility test is the test for divisibility by 11. Here the sums of alternate digits are calculated and the difference between these sums is found. If this difference is divisible by 11, then the original number is divisible by 11. For example, to see that 2451735 is divisible by 11, we note that $(2+5+7+5)-(4+1+3)=11$, which is obviously divisible by 11.

Let us see why this divisibility test works. Consider a four-digit number, $abcd$, expressed in terms of its place value expansion. Groups of 11 may be removed without changing the remainder upon division by 11. The curious observation here is the relationship between powers of 10 and multiples of 11. Observe that 1 and 100 are each one greater than a multiple of 11, whereas 10 and 1000 are each one less than a multiple of 11. (In particular, 1000 is one less than $1001 = 7\cdot 11\cdot 13$.) This pattern continues for higher powers of 10. Thus,

$$\begin{array}{rrrrrrrr} & 1000a & + & 100b & + & 10c & + & d \\ - & (1001a & + & 99b & + & 11c) & & \\ \hline & -a & + & b & - & c & + & d \end{array}$$

Now I wish to leave the reader with two challenges to consider.

1. The digits 0, 1, 5, and 6 all have perfect squares ending in their own digit. We say that such numbers are *automorphic*. Are there any two-digit automorphic numbers? Such numbers must end in 0, 1, 5, or 6; that is, they must be of the form $(10a+b)$, where $a \in \{1, 2, \ldots, 9\}$ and $b \in \{0, 1, 5, 6\}$. Determine all such numbers.

2. All rational numbers can be expressed as decimals that either repeat (as with $1/3$), or terminate (as with $1/10$). Consider any rational number of the form b/c, where b and c are positive integers with $b < c$ such that the greatest common divisor of b and c is 1 (so that the fraction is in lowest terms). How can one readily determine, without carrying out the decimal division, whether the decimal equivalent of the rational number will terminate?

It is hoped that readers will contribute ideas and feedback pertaining to the content of this feature. Feel free to contact me via email (`johngm@unb.ca`) or through correspondence with the editorial board of the journal.

THE OLYMPIAD CORNER

No. 252

R.E. Woodrow

As a first problem set we give the 9^{th} and 10^{th} grades of the Romanian Mathematical Olympiad. My thanks go to Andy Liu, Canadian Team Leader to the IMO in Japan, for collecting the set for our use.

ROMANIAN MATHEMATICAL OLYMPIAD

9^{th} Grade

1. Find positive integers a and b such that, for every x, $y \in [a, b]$, we have $\dfrac{1}{x} + \dfrac{1}{y} \in [a, b]$.

2. An integer $n \geq 2$ is called *friendly* if there exists a family A_1, A_2, $\ldots$, A_n of subsets of the set $\{1, 2, \ldots, n\}$ such that:

(i) $i \notin A_i$ for every $i \in \{1, 2, \ldots, n\}$;

(ii) $i \in A_j$ if and only if $j \notin A_i$, for every distinct i, $j \in \{1, 2, \ldots, n\}$;

(iii) $A_i \cap A_j$ is non-empty for every i, $j \in \{1, 2, \ldots, n\}$.

Prove: (a) 7 is a friendly number, and (b) n is friendly if and only if $n \geq 7$.

3. Prove that the mid-points of the altitudes of a triangle are collinear if and only if the triangle is right.

4. Let P be a plane. Prove that there exists no function $f : P \to P$ such that for every convex quadrilateral $ABCD$, the points $f(A)$, $f(B)$, $f(C)$, $f(D)$ are the vertices of a concave quadrilateral.

10^{th} Grade

1. Let $OABC$ be a tetrahedron such that $OA \perp OB \perp OC \perp OA$, let r be the radius of its inscribed sphere, and let H be the orthocentre of triangle ABC. Prove that $OH \leq r\big(\sqrt{3} + 1\big)$.

2. The complex numbers z_1, z_2, $\ldots$, z_5, have the same non-zero modulus, and $\sum\limits_{i=1}^{5} z_i = \sum\limits_{i=1}^{5} z_i^2 = 0$. Prove that z_1, z_2, $\ldots$, z_5 are the complex coordinates of the vertices of a regular pentagon.

3. Let a, b, c be the complex coordinates of the vertices A, B, C of a triangle. It is known that $|a| = |b| = |c| = 1$ and that there exists $\alpha \in \big(0, \frac{\pi}{2}\big)$ such that $a + b\cos\alpha + c\sin\alpha = 0$. Prove that $1 < [ABC] \leq \frac{1+\sqrt{2}}{2}$, where $[XYZ]$ is the area of XYZ.

4. A finite set A of complex numbers has the property that $z \in A$ implies $z^n \in A$ for every positive integer n.

(a) Prove that $\sum_{z \in A} z$ is an integer.

(b) Prove that, for every integer k, there is a set A which fulfills the above condition with $\sum_{z \in A} z = k$.

As a second set of problems we give the problems of the 16th Korean Mathematical Olympiad, written in April, 2003. Thanks go to Andy Liu, Canadian Team Leader, for collecting them for the *Corner*.

16th KOREAN MATHEMATICAL OLYMPIAD

April 12-13, 2003

Time: 9 hours

1. The computers in a computer lab are connected by cables as follows: Each computer is directly connected to exactly three other computers via cables. There is at most one cable joining two computers and any pair of computers in the lab can exchange data. (Two computers A and B can exchange data if there exists a sequence of computers starting from A and ending at B in which two computers next to each other in the sequence are directly joined by a cable.)

Let k be the smallest number of computers in the lab whose removal results in leaving just one computer in the lab or a pair of computers not able to exchange data any more. Let ℓ be the smallest number of cables whose deletion results in the existence of two computers that cannot exchange data any more. Show that $k = \ell$.

2. Let $ABCD$ be a rhombus with $\angle A < 90^\circ$. Let its two diagonals AC and BD meet at a point M. A point O on the line segment MC is selected such that $O \neq M$ and $OB < OC$. The circle centred at O passing through points B and D meets the line AB at point B and a point X (where $X = B$ when the line AB is tangent to the circle) and meets the line BC at point B and a point Y. Let the lines DX and DY meet the line segment AC at P and Q, respectively. Express the value of $\dfrac{OQ}{OP}$ in terms of t when $\dfrac{MA}{MO} = t$.

3. Show that there exist no integers x, y, z with $x \neq 0$ satisfying

$$2x^4 + 2x^2y^2 + y^4 \;=\; z^2\,.$$

4. Suppose that the incircle of $\triangle ABC$ is tangent to the sides AB, BC, CA at points P, Q, R, respectively. Prove the following inequality:

$$\frac{BC}{PQ} + \frac{CA}{QR} + \frac{AB}{RP} \;\geq\; 6\,.$$

5. Answer the following where m is a positive integer.

(a) Prove that if $2^{m+1}+1$ divides $3^{2^m}+1$, then $2^{m+1}+1$ is a prime.

(b) Is the converse of (a) true?

6. Let m and n be relatively prime positive integers satisfying $6 \leq 2m < n$. Consider n distinct points on a circle. Join one of these n points, say P, by a line segment to the m^{th} point Q counterclockwise from P, then join Q by a line segment to the m^{th} point R counterclockwise from Q, and so on. Repeat this process until no new line segment is added. Denote by I the number of intersections among these line segments inside the circle (excluding those on the circle).

(a) Find an expression for the maximum of I in terms of m and n when the locations of the n points change.

(b) Show that the inequality $I \geq n$ holds regardless of the locations of the n points. Also show that n points can be located so that $I = n$ if $m = 3$ and n is even.

As a third set we give the X National Mathematical Olympiad of Turkey written in December 2002. Thanks again go to Andy Liu for collecting the problems for our use.

X NATIONAL MATHEMATICAL OLYMPIAD OF TURKEY

Day 1 – 14 December 2002

(Time: 4.5 hours)

1. Let $n \geq 2$ be an integer, and let $(a_1, a_2, \ldots, a_n)$ be a permutation of $1, 2, \ldots, n$. For each $k \in \{1, 2, \ldots, n\}$, a_k apples are placed at the point k on the real axis. Children named A, B, C are assigned respective points x_A, x_B, $x_C \in \{1, 2, \ldots, n\}$. For each $k \in \{1, 2, \ldots, n\}$, the children whose points are closest to k divide a_k apples equally among themselves. We call (x_A, x_B, x_C) a *stable configuration* if no child's total share can be increased by assigning a new point to this child and not changing the points of the other two. Determine the values of n for which a stable configuration exists for some distribution $(a_1, a_2, \ldots, a_n)$ of the apples.

2. Two circles are externally tangent to each other at a point A and internally tangent to a third circle Γ at points B and C. Let D be the mid-point of the secant of Γ which is tangent to the smaller circles at A. Show that A is the incentre of the triangle BCD if the centres of the circles are not collinear.

3. Graph Airlines (GA) operates flights between some of the cities of the Republic of Graphia. There are GA flights between each city and at least three different cities, and it is possible to travel from any city to any other city in the Republic of Graphia using GA flights. GA decides to discontinue some of its flights. Show that this can be done in such a way that it is still possible to travel from any city to any other city using GA flights, yet at least $2/9$ of the cities have only one flight.

Day 2 – 15 December 2002

(Time: 4.5 hours)

4. Find all prime numbers p for which the number of ordered pairs of integers (x, y) with $0 \leq x,\ y < p$ satisfying the condition $y^2 \equiv x^3 - x \pmod{p}$ is exactly p.

5. In an acute triangle ABC with $|BC| < |AC| < |AB|$, the points D on side AB and E on side AC satisfy the condition $|BD| = |BC| = |CE|$. Show that the circumradius of the triangle ADE is equal to the distance between the incentre and the circumcentre of the triangle ABC.

6. Let n be a positive integer and $\mathbb{R}^n$ be the set of ordered n–tuples of real numbers. Let T denote the collection of $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for which there exists a permutation σ of $1, 2, \ldots, n$ such that $x_{\sigma(i)} - x_{\sigma(i+1)} \geq 1$ for each $i \in \{1, 2, \ldots, n-1\}$. Prove that there is a real number d satisfying the following condition:

For every $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, there exist $(b_1, b_2, \ldots, b_n) \in T$ and $(c_1, c_2, \ldots, c_n) \in T$ such that, for each $i \in \{1, 2, \ldots, n\}$,

$$a_i = \tfrac{1}{2}(b_i + c_i)\,, \quad |a_i - b_i| \leq d\,, \quad \text{and} \quad |a_i - c_i| \leq d\,.$$

Now we turn to the solutions on file for problems of the 2000 Kürschák Contest given in [2004 : 205].

1. For a positive integer n, consider the square in the Cartesian plane whose vertices are $A(0,0)$, $B(n,0)$, $C(n,n)$ and $D(0,n)$. The grid points of the integer lattice inside or on the boundary of this square are coloured either red or green in such a way that every unit square in the lattice has exactly two red vertices. How many such colourings are possible?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Colourings which satisfy the given condition will be called "good". We will prove that the number of good colourings is $2^{n+2} - 2$.

For $n = 1$, the number of good colourings is $\binom{4}{2} = 6 = 2^{n+2} - 2$.

Now consider $n \geq 2$. Let c_0, c_1, $\ldots$, c_n be the columns of the $(n+1) \times (n+1)$ grid described in the problem, and let r_0, r_1, $\ldots$, r_n be the rows. Note that in any good colouring, the lower-left unit square contains exactly two red vertices.

Let us form a good colouring. Suppose we colour red the points with coordinates $(0,0)$ and $(1,0)$. The colouring of the two columns c_0 and c_1 is then completely determined. We have two ways to colour the column c_2, depending on which of the points $(2,0)$ and $(2,1)$ we choose to colour red. (Either choice determines the colouring of the whole column c_2 since the colouring of c_1 is already determined). We use the same reasoning for the other columns, so that we obtain exactly 2^{n-1} good colourings in this case.

The same reasoning (with perhaps consideration of the rows instead of the columns) leads to the same number in each of the three other cases for which we colour red two adjacent lattice points in the lower-left unit square.

Now, let a_n be the number of good colourings of the $(n+1) \times (n+1)$ grid for which the two red vertices in the lower-left unit square are $(0,0)$ and $(1,1)$. Let us consider such a good colouring.

(a) If we colour red the point $(2,1)$, then we have two adjacent red vertices at $(1,1)$ and $(2,1)$. A good colouring of the whole grid leads to a good colouring of the $n \times n$ grid obtained by deleting c_0 and r_0, and in this $n \times n$ grid, the lower-left unit square has two adjacent red vertices (namely $(1,1)$ and $(2,1)$ in the 'old' numbering). As seen above, there are 2^{n-2} good colourings for this $n \times n$ grid. Each of them determines the colouring of r_0 (because the point $(2,1)$ is already red) and c_0 (because $(1,3)$ is already red) so as to obtain a good colouring of the original $(n+1) \times (n+1)$ grid.

(b) If we colour red the point $(1,2)$, the same reasoning as in (a) leads to 2^{n-2} good colourings again.

(c) Otherwise, we colour red the point $(2,2)$. This leads to a good colouring of the $n \times n$ grid obtained by deleting c_0 and r_0, where the red points of the lower-left unit square are $(0,0)$ and $(1,1)$ (in the 'new' numbering). There are a_{n-1} such good colourings. Each of these determines the colouring of r_0 and c_0. It follows that $a_n = 2 \times 2^{n-2} + a_{n-1}$. Since $a_1 = 1$, we easily deduce that $a_n = 2^n - 1$.

Similarly, there are $2^n - 1$ good colourings of the $(n+1) \times (n+1)$ grid for which the red vertices in the lower-left unit square are $(1,0)$ and $(0,1)$.

The total number of good colourings is $4 \cdot 2^{n-1} + 2(2^n - 1) = 2^{n+2} - 2$.

2. Let T be a point in the plane of the non-equilateral triangle ABC which is different from the vertices of the triangle. Let the lines AT, BT, and CT meet the circumcircle of the triangle at A_T, B_T, and C_T, respectively. Prove that there are exactly two points P and Q in the plane for which the triangles $A_PB_PC_P$ and $A_QB_QC_Q$ are equilateral. Prove, furthermore, that the line PQ passes through the circumcentre of the triangle ABC.

Solution by Michel Bataille, Rouen, France.

We embed the figure in the complex plane and, for simplicity, denote a point or its complex representation by the same small letter. Without loss of generality, we suppose that a, b, and c lie on the unit circle Γ, so that $a\overline{a} = b\overline{b} = c\overline{c} = 1$. Note that the points p and q we seek cannot lie on Γ.

Let $a' \in \Gamma$ and $m \notin \Gamma$. The line aa' passes through m if and only if $m + aa'\overline{m} = a + a'$; that is, $a' = \overline{T_m(a)}$, where T_m denotes the Möbius transformation defined by $T_m(z) = \dfrac{1 - \overline{m}z}{m - z}$. As a result, for a', b', c' on Γ, the lines aa', bb' and cc' concur at m if and only if $a' = \overline{T_m(a)}$, $b' = \overline{T_m(b)}$ and $c' = \overline{T_m(c)}$.

The statement that $a'b'c'$ is equilateral is successively equivalent to

- $\dfrac{a' - c'}{a' - b} \in \{-\omega,\ -\omega^2\}$, where $\omega = \exp(2\pi i/3)$,
- $\overline{[T_m(m), T_m(a), T_m(b), T_m(c)]} \in \{-\omega,\ -\omega^2\}$, where $[\cdot,\cdot,\cdot,\cdot]$ denotes the cross ratio,
- $[T_m(m), T_m(a), T_m(b), T_m(c)] \in \{-\omega,\ -\omega^2\}$ (since $-\omega$ and $-\omega^2$ are conjugates),
- $[m, a, b, c] \in \{-\omega,\ -\omega^2\}$ (since T_m preserves the cross ratio).

The conclusion follows, since p and q are the two points given by $[p, a, b, c] = -\omega$, $[q, a, b, c] = -\omega^2$. Note that p, $q \neq \infty$ since ABC is not equilateral. Also, from $[q, a, b, c] = \overline{[p, a, b, c]}$, an easy calculation yields $q = 1/p$. Thus, 0, p, and q are collinear (p and q are even inverses in Γ).

Notes.

1. P and Q are called the isodynamic points of $\triangle ABC$. Various properties of these points can be found in R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960, pp. 295–7.

2. A nice reference for the use of complex numbers, cross ratios, and Möbius transformations is L. Hahn, *Complex Numbers and Geometry*, Mathematical Association of America, 1994.

3. For a slightly different solution of the problem, see *American Mathematical Monthly*, 109, December 2002, pp. 926–7.

3. Let k denote a non-negative integer. Assume that the integers $a_1, \ldots, a_n$ give at least $2k$ different remainders when divided by $n+k$. Prove that some of the integers add up to a number divisible by $n+k$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

It is well known that among any n integers we may find some which add up to a number divisible by n. Thus, we will assume that k is positive. In this case, we will prove a stronger version of the problem: If the integers $a_1, \ldots, a_n$ give at least $k+1$ different remainders when divided by $n+k$, then some of them add up to a number divisible by $n+k$.

First assume that $k > 1$. Without loss of generality, we suppose that $a_1, \ldots, a_{k+1}$ are distinct modulo $n+k$. Let $S = a_1 + \cdots + a_{k+1}$.

Now, we consider the three following groups:

- Type I: $a_1, \ldots, a_{k+1}$.
- Type II: $S - a_1, \ldots, S - a_{k+1}$.
- Type III: S, $S + a_{k+2}$, $S + a_{k+2} + a_{k+3}, \ldots, S + a_{k+2} + \cdots + a_n$.

There are $(k+1) + (k+1) + (n-k) = n+k+2$ numbers listed in the three groups above. Obviously, if one of these numbers is divisible by $n+k$ we are done. Otherwise, by the Pigeonhole Principle, at least two, say x and y, have the same remainder modulo $n+k$.

Since $a_1, \ldots, a_{k+1}$ are distinct modulo $n+k$, it follows that x and y cannot be both of type I nor both of type II. Clearly, if at least one of x and y has type III, we have the desired conclusion (by considering $y-x$ or $x-y$). Therefore, without loss of generality, we may assume that $x = a_p$ belongs to Type I and $y = S - a_q$ belongs to Type II.

If $p \neq q$, then $y - x$ leads to the desired conclusion (since $k+1 > 2$, we are sure that $y-x$ is a non-empty sum). If $p = q$, then we actually have three such pairs, say $(a_1, S-a_1)$, $(a_2, S-a_2)$, $(a_3, S-a_3)$. (Remember that we have $n+k+2$ numbers in the list for only $n+k-1$ possible remainders and that, according to the previous cases, we may assume that no three numbers have the same remainder modulo $n+k$. Therefore, at most two pairs would lead to at least $n+k$ distinct remainders.) It follows that $2a_1 \equiv 2a_2 \equiv 2a_3 \equiv S \pmod{n+k}$.

Since a_1, a_2, a_3 are distinct modulo $n+k$, we deduce that $n+k$ is even and $a_1 = a_2 = t(n+k)/2$ and $a_2 = a_3 + r(n+k)/2$ for some odd integers t and r. Thus, $a_1 \equiv a_3 \pmod{n+k}$, a contradiction. The conclusion follows.

Now, let us consider the case $k = 1$. If the integers give at least three distinct remainders modulo $n+1$, the proof above can be adapted word for word, since the proof used only the fact that S is a sum of at least three terms. Thus, we only have to consider the case where the given integers give two remainders modulo $n+1$.

Let us assume that, for some positive integer p, we have

$$\begin{array}{rrcl} & a_1 & \equiv & \cdots \equiv a_p \equiv r \pmod{n+1} \\ \text{and} & a_{p+1} & \equiv & \cdots \equiv a_n \equiv s \pmod{n+1}\ , \end{array}$$

with $r \not\equiv s \pmod{n+1}$. If $ir \equiv 0 \pmod{n+1}$ for some $i = 1, \ldots, p$, we are done. Otherwise, it follows that r, $2r$, $\ldots$, pr are distinct modulo $n+1$. We may assume that $js \not\equiv -is \pmod{n+1}$ for all $i = 1, \ldots, p$ and all $j = 1, \ldots, n-p$ (otherwise we are done). Then, there are only $n-p$ available remainders for the $n-p$ numbers s, $2s$, $\ldots$, $(n-p)s$, and one of them is 0. Thus, either one of these numbers is divisible by $n+1$ or at least two have the same remainder modulo $n+1$. In both cases, there exists $j \in \{1, 2, \ldots, np\}$ such that $js \equiv 0 \pmod{n+1}$, and we are done.

Now we move to the September 2004 number of the *Corner* and readers' solutions to some of the problems of the 11^{th} Japanese Mathematical Olympiad, Second Round, given in [2004 : 266–267].

1. An $m \times n$ chessboard is given. Each square is painted black or white in such a way that for every black square, the number of black squares adjacent to it is odd. Prove that the number of black squares is even. (Two squares are *adjacent* if they are different and have a common edge.)

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Consider the graph whose vertices are the black squares, any two joined by an edge if and only if they are adjacent. From the problem statement, each vertex has odd degree. But it is well known that in any finite simple graph the number of vertices with odd degree is even (this follows from the fact that the sum of the degrees is twice the number of edges). Thus, the number of vertices is even, and we are done.

2. A positive integer n is written in decimal notation as $a_m a_{m-1} \cdots a_1$; that is,

$$n \;=\; 10^{m-1}a_m + 10^{m-2}a_{m-1} + \cdots + a_1 \,,$$

where a_m, a_{m-1}, ..., $a_1 \in \{0, 1, \ldots, 9\}$ and $a_m \neq 0$. Find all n such that

$$n \;=\; (a_m + 1) \times (a_{m-1} + 1) \times \cdots \times (a_1 + 1) \,.$$

Solved by Houda Anoun, Bordeaux, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give the solution by Anoun (en français).

Soit n un entier dont la notation décimale est sous la forme de $a_m a_{m-1} \cdots a_1$ qui satisfait la condition

$$n \;=\; (a_m + 1) \times (a_{m-1} + 1) \times \cdots \times (a_1 + 1) \,. \tag{1}$$

Supposons qu'il existe $j \in \{1, \ldots, m\}$ tel que $a_j = 0$. On a $a_i + 1 \leq 10$ pour tout $i \in \{1, \ldots, m\}$. Si $a_j = 0$, il s'en suit que

$$(a_m + 1) \times (a_{m-1} + 1) \times \cdots \times (a_1 + 1) \;\leq\; 10^{m-1} \,.$$

C'est à dire que $n \leq 10^{m-1}$. Puisque $a_m \neq 0$, on déduit que $a_m = 1$ et $a_i = 0$ pour $i \in \{1, \ldots, m-1\}$. D'après (1) on a aussi $n = 2 \times 1 \cdots \times 1 = 2$. Ceci est absurde. On conclut que $a_i \neq 0$ pour tout $i \in \{1, \ldots, m\}$.

Maintenant supposons qu'il existe $j \in \{1, \ldots, m\}$ tel que $a_j = 9$. Donc $a_j + 1 = 10$ est un diviseur de n (par (1)). D'où $a_1 = 0$, chose qui est contradictoire avec le résultat précédent. On déduit que $a_i \in \{1, \ldots, 8\}$ pour tout $i \in \{1, \ldots, m\}$.

Puis on montre par récurrence que si $n \geq 3$, donc

$$\begin{aligned}(a_m + 1) \times (a_{m-1} + 1) \times \cdots \times (a_1 + 1) \\ <\;\; 10^{m-1}a_m + 10^{m-2}a_{m-1} + \cdots + a_1 \,.\end{aligned} \tag{2}$$

On considère d'abord le cas $m = 3$. On a $(a_2+1)\times(a_1+1) \leq 10a_2+a_1$, car cette inégalité est équivalent à $(9 - a_1) \times a_2 \geq 1$, qui est vrai. Or

$$\begin{aligned}(a_3 + 1) \times (a_2 + 1) \times (a_1 + 1) &\leq (a_3 + 1) \times (10a_2 + a_1) \\ &= (10a_2 + a_1) \times a_3 + 10a_2 + a_1 \\ &< 10^2 a_3 + 10a_2 + a_1 \,.\end{aligned}$$

Ceci vérifie (2) pour $m = 3$.

Puis supposons que (2) est appliqué pour $m = k$, où $k \in \{3, 4, \ldots, \}$. On a alors

$$\begin{aligned}&(a_{k+1} + 1) \times (a_k + 1) \times \cdots \times (a_1 + 1) \\ &\quad < (a_{k+1} + 1) \times \left(10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1\right) \\ &\quad = \left(10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1\right) \times a_{k+1} \\ &\qquad\qquad + 10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1 \\ &\quad < 10^k a_{k+1} + 10^{k-1}a_k + \cdots + a_1 \,.\end{aligned}$$

La preuve par récurrence est complète.

Si n vérifie la condition (1), alors il n'est pas composé au plus de deux chiffres. Il ne peut étre composé d'un seul chiffre a_1, car sinon on aura $a_1 = a_1 + 1$. Cherchons finalement les entiers n de la forme $10a_2 + a_1$ tels que $n = (a_2 + 1) \times (a_1 + 1)$. On a alors :

$$10a_2 + a_1 = (a_2 + 1) \times (a_1 + 1) \,.$$

D'où $(9 - a_1) \times a_2 = 1$, ceci implique que $a_1 = 8$ et $a_2 = 1$.

L'unique solution au problème est donc l'entier 18.

3. Three real numbers a, b, $c \geq 0$ satisfy

$$a^2 \leq b^2 + c^2, \qquad b^2 \leq c^2 + a^2, \qquad c^2 \leq a^2 + b^2 \,.$$

Prove the inequality

$$(a + b + c)(a^2 + b^2 + c^2)(a^3 + b^3 + c^3) \geq 4(a^6 + b^6 + c^6) \,.$$

When does equality hold?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Vedula N. Murty, Dover, PA, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We use Zhou's presentation.

By the Cauchy-Schwarz Inequality, we have

$$(a + b + c)(a^3 + b^3 + c^3) \geq (a^2 + b^2 + c^2)^2 \,.$$

Thus,

$$(a + b + c)(a^2 + b^2 + c^2)(a^3 + b^3 + c^3) \geq (a^2 + b^2 + c^2)^3 \,.$$

Therefore, it will suffice to prove that

$$(a^2+b^2+c^2)^3 \;\geq\; 4(a^6+b^6+c^6)\,.$$

We have

$$\begin{aligned}(a^2+b^2+c^2)^3 &= (a^6+b^6+c^6)+6(abc)^2\\ &\quad +3(a^2b^4+b^2c^4+c^2a^4+a^4b^2+b^4c^2+c^4a^2)\\ &= 4(a^6+b^6+c^6)+12(abc)^2\\ &\quad +3(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)\\ &\geq 4(a^6+b^6+c^6)\,,\end{aligned}$$

since $a^2 \leq b^2+c^2$, $b^2 \leq c^2+a^2$, and $c^2 \leq a^2+b^2$. Equality holds if and only if one of a, b, c is 0 and the other two are equal.

4. Let p be a prime number and m a positive integer. Show that there exists a positive integer n such that the decimal representation of p^n contains a string of m consecutive 0s.

Solutiond by Pierre Bornsztein, Maisons-Laffitte, France.

Case 1. $p \notin \{2,5\}$.

Then p is coprime to 10^{m+1}. Hence, by the Euler-Fermat Theorem, there exists $n>0$ such that $p^n \equiv 1 \pmod{10^{m+1}}$ (for example, $n=\phi(10^{m+1})$). It follows that the decimal expansion of p^n has the form $\overline{q}0\ldots01$, where $\overline{q}$ is the decimal expansion of some positive integer q and there are m consecutive 0s between $\overline{q}$ and 1.

Case 2. $p=2$.

As above, there exists $a>0$ such that $2^a \equiv 1 \pmod{5^{2m}}$. It follows that $2^{a+2m}-2^{2m} \equiv 0 \pmod{10^{2m}}$. Thus, $2^{a+2m}-2^{2m}$ contains a string of $2m$ consecutive 0s.

But, since $2^{2m}=4^m<10^m$, the decimal expansion of 2^{2m} does not use more than m digits. It follows that the decimal expansion of 2^{a+2m} contains a string of m consecutive 0s, as desired.

Case 3. $p=5$.

As above, there exists $a>0$ such that $5^a \equiv 1 \pmod{2^{4m}}$. It follows that $5^{a+4m}-5^{4m} \equiv 0 \pmod{10^{4m}}$. Thus, $5^{a+4m}-5^{4m}$ contains a string of $4m$ consecutive 0s.

But, since $5^{4m}<10^{3m}$, the decimal expansion of 5^{4m} does not use more than $3m$ digits. It follows that the decimal expansion of 5^{a+4m} contains a string of m consecutive 0s, as desired.

Next we turn to readers' solutions to problems of the 14th Mexican Mathematical Olympiad given in [2004 : 267–268].

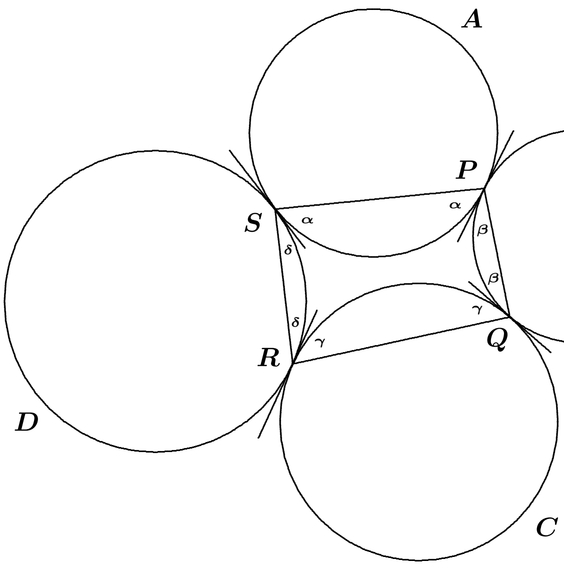
1. Let A, B, C, and D be circles such that (i) A and B are externally tangent at P, (ii) B and C are externally tangent at Q, (iii) C and D are externally tangent at R, and (iv) D and A are externally tangent at S. Assume that A and C do not intersect and that B and D do not intersect.

(a) Prove that P, Q, R, and S lie on a circle.

(b) Assume further that A and C have radius 2, B and D have radius 3, and the distance between the centres of A and C is 6. Determine the area of $PQRS$.

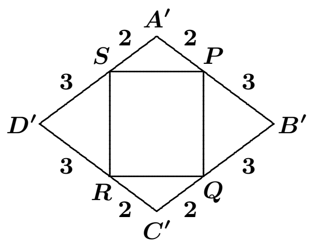
Solution by Geoffrey A. Kandall, Hamden, CT, USA.

(a)



Draw the common internal tangents at P, Q, R, and S, and label angles α, β, γ, and δ as shown in the diagram above. Since the sum of the internal angles in quadrilateral $PQRS$ is 360°, it follows that $\alpha+\beta+\gamma+\delta = 180^\circ$. Hence, opposite angles of $PQRS$ are supplementary. Thus, $PQRS$ is cyclic.

(b) Let A', B', C', D' be the centres of the circles A, B, C, D, and let $A'C'$ and $B'D'$ meet at E. Since $A'B'C'D'$ is a rhombus, the diagonals $A'C'$ and $B'D'$are the perpendicular bisectors of one another. Then, since the distance between A' and C' is 6, we have $A'E = 3$. Hence, $D'E = 4$ and $D'B' = 8$. Since $SP \parallel D'B' \parallel RQ$ and $SR \parallel A'C' \parallel PQ$, it follows that $PQRS$ is a rectangle.



We have

$$\frac{SP}{D'B'} = \frac{2}{5} \quad \text{and} \quad \frac{SR}{A'C'} = \frac{3}{5}.$$

Therefore, $SP = \frac{2}{5} \cdot 8 = \frac{16}{5}$ and $SR = \frac{3}{5} \cdot 6 = \frac{18}{5}$. Finally,

$$[PQRS] = SP \cdot SR = \frac{288}{25}.$$

2. A triangle like the one shown is constructed with the numbers from 1 to 2000 in the first row. Each number in the triangle, except those in the first row, is the sum of the two numbers above it. What number occupies the lowest vertex of the triangle? (Write your final answer as a product of primes.)

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
& 3 & & 5 & & 7 & & 9 & \\
& & 8 & & 12 & & 16 & & \\
& & & 20 & & 28 & & & \\
& & & & 48 & & & &
\end{array}$$

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let p be the number of integers written on the first row. Let a_n be the first number (on the left) appearing on the n^{th} row. An easy induction shows that the n^{th} row is

$$a_n,\ a_n + 2^{n-1},\ a_n + 2 \cdot 2^{n-1},\ \ldots,\ a_n + (p-n)2^{n-1}.$$

It follows immediately that $a_{n+1} = 2a_n + 2^{n-1}$. Since $a_1 = 1$, we deduce that $a_n = (n+1)2^{n-2}$. The problem asks for a_{2000}. We have

$$a_{2000} = 2001 \times 2^{1998} = 2^{1998} \times 3 \times 23 \times 29.$$

3. Given a set A of positive integers, a set A' is constructed consisting of all elements of A as well as all positive integers that can be obtained as follows: some elements of A are chosen, without repetition, and for each of them a sign ($+$ or $-$) is chosen; the signed numbers are then added and the result is placed in A'. For example, if $A = \{2, 8, 13, 20\}$, then two elements of A' are 8 and 14 (since 8 belongs to A and $14 = 20 + 2 - 8$). From A', a set A'' is constructed in the same fashion as A' is constructed from A. What is the minimum number of elements that A must have if A'' is to contain all integers from 1 to 40 (including 1 and 40)?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The minimum is 3 and it is achieved for $A = \{1, 5, 25\}$ (for example). Indeed, in that case, $A' = \{1, 4, 5, 6, 19, 20, 21, 24, 25, 26, 29, 30, 31\}$. From that, it is routine to verify that $\{1, 2, \ldots, 40\} \subset A''$.

Now, let $0 < x < y$ and $A = \{x, y\}$. Thus, $A' = \{x, y - x, y, y + x\}$ (there may be some repetitions). Note that if we select $k \geq 1$ elements from A', we may construct 2^k sums from them, not necessary positive. More precisely, it is clear that if we may obtain the sum s from these k elements, then we may also obtain $-s$ by reversing all the signs.

Since $|A'| \leq 4$, we deduce that we can construct at most

$$2 \times \binom{4}{1} + 4 \times \binom{4}{2} + 8 \times \binom{4}{3} + 16 \times \binom{4}{4} \;=\; 80$$

sums, not necessarily positive or distinct. From above, we see that there are at most 40 positive sums. But these sums are not pairwise distinct, since x can be obtained from $\{x\}$ and from $\{y + x,\, y\}$. Thus, there are fewer than 40 positive sums, so that we cannot obtain all the integers from 1 to 40.

It follows that 3 cannot be improved.

5. An $n \times n$ square is divided into unit squares and painted black and white in a checkerboard pattern. The following operation may be performed on the board: choose a sub-rectangle whose side lengths are both odd or both even, but not both 1, and reverse the colours of the unit squares in this rectangle (that is, black squares become white and white squares become black).

Find all values of n for which it is possible to make all unit squares the same colour by a finite sequence of operations.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that this is possible for all positive n except $n = 2$.

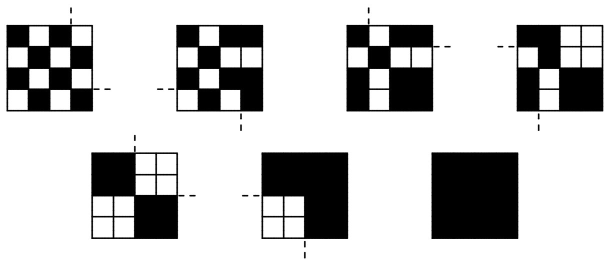
There is nothing to do for $n = 1$. It is impossible for $n = 2$, since the only subrectangle available is the whole square itself, which does not lead to the desired colouring.

Assume first that $n \geq 3$ is odd. By choosing $1 \times n$ rectangles corresponding to the columns c_1, c_3, $\dots$, c_n (from left to right), each of the rows of the whole square becomes monochromatic. Next, choosing appropriate rows as subrectangles leads to a monochromatic whole square. It follows that each odd $n \geq 3$ is a solution of the problem.

Now suppose that n is even. Let $n = 2^a b$, where $a \geq 1$ and $b \geq 3$, with b odd. By dividing the $n \times n$ square into 2^{2a} subsquares of size $b \times b$, we may use the result above to give the same colour, say black, to all these subsquares. This leads to a monochromatic $n \times n$ square. Thus, n is a solution.

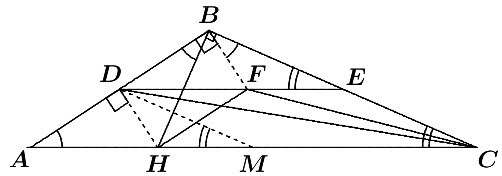
Following the same reasoning, we may prove that if $n = 4$ is a solution, then $n = 2^a$ is also a solution for all $a \geq 2$. Therefore, it only remains to prove that $n = 4$ is a solution.

But the following sequence of 6 operations does the job, where we have used dotted lines to mark the rectangles involved at each stage:



6. Let ABC be a triangle with $\angle B > 90^\circ$ such that, for some point H on AC, we have $AH = BH$, and BH is perpendicular to BC. Let D and E be the mid-points of AB and BC, respectively. Through H a parallel to AB is drawn, intersecting DE at F. Prove that $\angle BCF = \angle ACD$.

Solved by Michel Bataille, Rouen, France; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's treatment.



Since D and E are the mid-points of AB and BC, respectively, we have $DE \parallel AC$. Let M be the mid-point of AC; then $DM \parallel BC$. These imply that

$$\angle BED = \angle BCA = \angle DMA. \qquad (1)$$

Since $DF \parallel AH$ and $HF \parallel AD$, quadrilateral $DAHF$ is a parallelogram. Thus, $HF = AD = DB$; hence, quadrilateral $BDHF$ is a parallelogram, and we have $BF \parallel DH$. Since $AH = BH$ and $AD = BD$, we get $HD \perp AB$. Thus, $\angle DBF = \angle ADH = 90^\circ$.

Since $\angle HBE = 90^\circ$, we have $\angle DBF = \angle HBE$; that is,

$$\angle DBH + \angle HBF = \angle HBF + \angle FBE.$$

Thus, $\angle DBH = \angle FBE$. Since $AH = BH$, we have $\angle DAH = \angle DBH$, and hence,

$$\angle FBE = \angle DAH. \qquad (2)$$

It follows from (1) and (2) that $\triangle FBE \sim \triangle DAM$. Thus,

$$EF : DM = BE : AM = EC : MC. \qquad (3)$$

Since $\angle FEC = 180^\circ - \angle BEF = 180^\circ - \angle DMA = \angle DMC$, it follows (in view of (3)) that $\triangle FEC \sim \triangle DMC$. Hence, $\angle ECF = \angle MCD$; that is, $\angle BCF = \angle ACD$.

That completes the *Corner* for this issue. This is now Olympiad Season. Send me your Olympiad materials, as well as your nice solutions and generalizations to problems featured in the *Corner*.

BOOK REVIEWS

John Grant McLoughlin

Chinese Mathematics Competitions and Olympiads 1981-1993
By A. Liu, Australian Mathematics Trust Enrichment Series, V. 13, 1998
ISBN 1-87642-000-6, paperback, 194 pages, AUs$40.00.
Chinese Mathematics Competitions and Olympiads 1993-2001
By A. Liu, Australian Mathematics Trust Enrichment Series, V. 22, 2005
ISBN 1-87642-016-2, paperback, 173 pages, AUs$40.00.
Reviewed by **Georg Gunther**, *Sir Wilfred Grenfell College (MUN), Corner Brook, NL*

In 1985, China sent its first team to the International Mathematical Olympiad (IMO). Since then, she has dominated this event, ranking first among the ninety participating nations twelve out of the past fifteen years. Selection to the Chinese team is based on two annual events. The first is the Chinese National High School Mathematics Competition, held each October. The second is the Chinese Mathematical Olympiad, written at the conclusion of the annual Winter Math Camp held in January.

There has, for years, been great interest within the English-speaking world in finding out what China does in order to achieve such consistently outstanding results at the IMO. It was this interest that prompted the Australian Mathematics Trust to collaborate with Dr. Andy Liu to produce these two volumes. Dr. Liu is eminently qualified for the task. He has been intimately involved with international mathematics competitions for a quarter of a century, serving as one of the leaders of the American IMO team from 1981–84, and as the Canadian IMO team leader in 2000 and 2003. For his many services over the years in this capacity, he was awarded, in 1996, the David Hilbert International Award from the World Federation of National Mathematics Competitions.

A glance through the two volumes immediately reveals a far-ranging and astonishing selection of mathematics problems covering all the traditional areas of high school competition, including algebra, number theory, combinatorics, plane and solid geometry, complex numbers, trigonometry, and functional equations. There are also numerous problems that defy classification into any one category. Almost all of the problems are unusual and non-standard in a variety of ways, making these two volumes a particularly rich and rewarding source for anyone interested in Olympiad-type problems.

The problems are of three types: multiple choice, answer only, and full solution. By far the majority require full solutions. All of the problems, even the multiple-choice, are challenging and require careful thought and analysis.

The books are well organized. Each volume begins with a number of contests and Olympiads. The bulk of each of the books consists of detailed answers and solutions to all of the problems. One thing that makes these books such a valuable resource for students is the fact that, in many cases, a number of alternate solutions are presented to the same problem. As well, many of the solutions include extensive commentary, suggesting other possible approaches, or providing some intriguing connections to other problems. When diagrams are needed, they are well drawn and well labelled, making the solutions easy to comprehend.

These volumes are a rich source of serious and profound mathematics. Like all of their companion volumes in the Australian Mathematics Trust Enrichment Series, they are a valuable addition to the growing literature devoted to the mathematics of Olympiad-level competitions. Teachers and professors will have no trouble finding problems to challenge their brightest students. Students engaged in self-study will be able to learn a great deal of good mathematics by studying the problems in these two books.

Cunning Combination Problems & Other Puzzles
By Ivan Moscovich, Sterling Publishing, New York, NY, 2005
ISBN 1-4027-2346-6, paperbound, 128 pages, CDN$13.95.
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB.*

Moscovich writes in the introduction: "The popular fun and pedagogic aspects of recreational mathematics overlap considerably, and there is no clear boundary between recreational and 'serious' mathematics." The book reiterates this claim on many levels. Playful activities abound with a blend of about one hundred challenges including assortments of covering, colouring, cutting, filling-in, and spatial orientation puzzles. The mathematics underlying many problems is quite advanced, whereas, the instructions are not. Readers are invited to play wherever they happen to open the book.

The colourful puzzles and solutions attract the attention of the puzzler. Other features also struck this reviewer. Usually several problems on particular themes followed one another, thus developing an appreciation for a branch of recreational math. For example, I am familiar with some MacMahon colouring puzzles, though I never read of Percy MacMahon (1854-1929) prior to getting a glimpse of his contribution through an excerpt in this book. A series of problems growing out of this contribution are subsequently presented. While many of the families of puzzles are familiar, such as magical figures, Moscovich manages to add clever wrinkles that delight the recreational mathematician. One such example is the magic wheel, a figure consisting of three concentric circles each containing the same number of digits. The challenge is to turn the three circles separately such that the sum along each spoke of the wheel is identical.

Overall, this book is an affordable and worthwhile addition to the collections of those who enjoy recreational mathematics. The book would serve well as a welcome introductory book to a curious problem solver or high school student. I expect the same can be said of other books in the Mastermind Collection written by Moscovich, some of which may appear in future reviews.

On an Inequality from IMO 2005

Vasile Cîrtoaje

The third problem from the IMO 2005 states the following:
Let x, y, z be positive numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5} \;\geq\; 0\,.$$

In his recent paper [2], Nikolai Nikolov stated and proved the following interesting generalization: *If x_1, x_2, …, x_n are positive real numbers with $\prod\limits_{i=1}^{n} x_i \geq 1$, then*

$$\sum_{\text{cyclic}} \frac{x_1^\alpha - x_1}{x_1^\alpha + x_2 + \cdots + x_n} \begin{cases} \geq\; 0\,, & \text{for } \alpha \geq 1, \\ \leq\; 0\,, & \text{for } -\dfrac{1}{n-1} \leq \alpha \leq 1. \end{cases}$$

In this article we will show that the inequalities above are straightforward consequences of the following more general inequalities.

Proposition. Let x_1, x_2, …, x_n be positive real numbers with $\prod\limits_{i=1}^{n} x_i \geq 1$.

(a) If $\alpha > 1$, then

$$\sum_{\text{cyclic}} \frac{x_1^\alpha}{x_1^\alpha + x_2 + \cdots + x_n} \;\geq\; 1\,. \tag{1}$$

(b) If $\alpha > 1$, then

$$\sum_{\text{cyclic}} \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} \;\leq\; 1\,. \tag{2}$$

(c) If $n \geq 3$ and $-\dfrac{2}{n-2} \leq \alpha < 1$, then

$$\sum_{\text{cyclic}} \frac{x_1^\alpha}{x_1^\alpha + x_2 + \cdots + x_n} \;\leq\; 1\,. \tag{3}$$

(d) If $-1-\dfrac{2}{n-1} \leq \alpha < 1$, then

$$\sum_{\text{cyclic}} \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} \;\geq\; 1\,. \tag{4}$$

The inequalities (1) and (3) were published in [1] for $\prod_{i=1}^{n} x_i = 1$, as a generalization of an inequality from IMO 1996. Because of this, we will only sketch the proofs of these here. As for the inequality (2), we posted it (without solution) on the Mathlinks Site – IMO 2005 Mexico Forum [3] in July 2005.

Proof of (a): First we observe that it suffices to consider only the case $\prod_{i=1}^{n} x_i = 1$. In order to show this, let $r = \sqrt[n]{x_1 x_2 \cdots x_n}$ and $y_i = x_i/r$ for $i = 1, 2, \ldots, n$. Note that $r \geq 1$ and $\prod_{i=1}^{n} y_i = 1$. Thus, inequality (1) becomes

$$\sum_{\text{cyclic}} \frac{y_1^\alpha}{y_1^\alpha + \frac{y_2 + \cdots + y_n}{r^{\alpha-1}}} \geq 1\,,$$

and we see that it suffices to prove the inequality for $r = 1$; that is, for $\prod_{i=1}^{n} x_i = 1$.

Under the assumption $\prod_{i=1}^{n} x_i = 1$, we will show that there exists a suitable real number p such that

$$\frac{x_1^\alpha}{x_1^\alpha + x_2 + \cdots + x_n} \geq \frac{x_1^p}{x_1^p + x_2^p + \cdots + x_n^p}\,. \tag{5}$$

If this claim is valid, then adding (5) to the analogous inequalities for $x_2, \ldots, x_n$ will yield (1). Inequality (5) is equivalent to

$$x_2^p + \cdots + x_n^p \geq (x_2 \cdots x_n)^{\alpha-p}(x_2 + \cdots + x_n)\,.$$

Choosing

$$p = \frac{(n-1)\alpha + 1}{n}\,, \quad p > 1$$

reduces the inequality to the homogeneous inequality

$$x_2^p + \cdots + x_n^p \geq (x_2 \cdots x_n)^{\frac{p-1}{n-1}}(x_2 + \cdots + x_n)\,.$$

Since

$$(x_2 \cdots x_n)^{\frac{p-1}{n-1}} \leq \left(\frac{x_2 + \cdots + x_n}{n-1}\right)^{p-1}$$

(by the AM–GM Inequality), it is enough to show that

$$\frac{x_2^p + \cdots + x_n^p}{n-1} \geq \left(\frac{x_2 + \cdots + x_n}{n-1}\right)^p\,.$$

This inequality follows easily by applying Jensen's Inequality to the convex function $f(x) = x^p$. Equality in (1) occurs if and only if $x_i = 1$ for $1 \leq i \leq n$.

Proof of (b): We will consider two cases.

Case 1. $1 < \alpha \leq n+1$.

Since $\prod_{i=1}^{n} x_i \geq 1$ implies that $\sum_{i=1}^{n} x_i \geq n$ by the AM–GM Inequality, it suffices to prove (2) for $\sum_{i=1}^{n} x_i \geq n$. As above we may consider $\sum_{i=1}^{n} x_i = n$. Indeed, if we set $r = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $y_i = x_i/r$ for $1 \leq i \leq n$, then $r \geq 1$ and $\sum_{i=1}^{n} y_i = 1$, and the inequality (2) becomes

$$\sum_{\text{cyclic}} \frac{y_1}{r^{\alpha-1}y_1^{\alpha} + y_2 + \cdots + y_n} \leq 1\,,$$

and we see that it suffices to prove the last inequality for $r = 1$; that is, for $\sum_{i=1}^{n} x_i = n$.

Write the inequality (2) in the form

$$\frac{x_1}{x_1^{\alpha} - x_1 + n} + \frac{x_2}{x_2^{\alpha} - x_2 + n} + \cdots + \frac{x_n}{x_n^{\alpha} - x_n + n} \leq 1\,.$$

For any $x > 0$, by Bernoulli's Inequality, we have

$$x^{\alpha} = \big(1 + (x-1)\big)^{\alpha} \geq 1 + \alpha(x-1)\,,$$

and hence, $x^{\alpha} - x + n \geq n - \alpha + 1 + (\alpha - 1)x > 0$. Consequently, it suffices to show that

$$\sum_{i=1}^{n} \frac{x_i}{n - \alpha + 1 + (\alpha-1)x_i} \leq 1\,.$$

This inequality clearly holds for $\alpha = n+1$. For $\alpha < n+1$, using

$$\frac{(\alpha-1)x_i}{n-\alpha+1+(\alpha-1)x_i} = 1 - \frac{n-\alpha+1}{n-\alpha+1+(\alpha-1)x_i}\,,$$

it may be rewritten as

$$\sum_{i=1}^{n} \frac{1}{n-\alpha+1+(\alpha-1)x_i} \geq 1\,.$$

Setting $y_i = n - \alpha + 1 + (\alpha-1)x_i$ for $i = 1, 2, \ldots, n$, we have $y_i > 0$ and $\sum_{i=1}^{n} y_i = n^2$. The inequality reduces to

$$\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n} \geq 1\,,$$

which is an immediate consequence of the well-known inequality

$$(y_1 + y_2 + \cdots + y_n)\left(\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}\right) \;\geq\; n^2\,.$$

Case 2. $\alpha \geq n - \dfrac{1}{n-1}$.

As above, we may assume that $\prod\limits_{i=1}^{n} x_i = 1$. Under this assumption it suffices to show that

$$\frac{(n-1)x_1}{x_1^\alpha + x_2 + \cdots + x_n} + \frac{x_1^p}{x_1^p + x_2^p + \cdots + x_n^p} \;\leq\; 1 \qquad (6)$$

for a suitable real number p, and to add this inequality to the analogous inequalities for $x_2, \ldots, x_n$. Set $t = \sqrt[n-1]{x_2 \cdots x_n}$. By the AM–GM Inequality, we have $x_2 + \cdots + x_n \geq (n-1)t$ and $x_2^p + \cdots + x_n^p \geq (n-1)t^p$. Thus, it suffices to show that

$$\frac{(n-1)x_1}{x_1^\alpha + (n-1)t} + \frac{x_1^p}{x_1^p + (n-1)t^p} \;\leq\; 1\,.$$

Since $x_1 = 1/t^{n-1}$, this inequality is equivalent to

$$(n-1)t^{n+q} \;-\; (n-1)t^q \;-\; t^{q-np} + 1 \;\geq\; 0\,,$$

where $q = (n-1)(\alpha-1)$. We will now show that the inequality holds for

$$p \;=\; \frac{(n-1)(\alpha-n-1)}{n}\,.$$

Indeed, for this value of p, the inequality successively becomes the following:

$$\begin{aligned}
(n-1)t^{n+q} - (n-1)t^q - t^{n(n-1)} + 1 \;&\geq\; 0\,,\\
(n-1)t^q(t^n-1) - (t^n-1)(t^{n^2-2n} + t^{n^2-3n} + \cdots + 1) \;&\geq\; 0\,,\\
(t^n-1)\big[(t^q - t^{n^2-2n}) + (t^q - t^{n^2-3n}) + \cdots + (t^q-1)\big] \;&\geq\; 0\,,
\end{aligned}$$

and we see that the last inequality is true for $q \geq n^2 - 2n$; that is, for $\alpha \geq n - 1/(n-1)$. Equality occurs in (2) if and only if $x_i = 1$ for $1 \leq i \leq n$.

Proof of (c): The first part of the proof is similar to the proof of part (a). Finally, we have to prove the inequality

$$x_2 + \cdots + x_n \;\geq\; (x_2 \cdots x_n)^{\frac{1-p}{n-1}}(x_2^p + \cdots + x_n^p)$$

for $p = \dfrac{(n-1)\alpha+1}{n}$, where $-\dfrac{1}{n-2} \leq p < 1$. For $p = -\dfrac{1}{n-2}$, the inequality reduces to

$$x_2 + \cdots + x_n \;\geq\; \sqrt[n-2]{x_3 \cdots x_n} + \cdots + \sqrt[n-2]{x_2 \cdots x_{n-1}}\,,$$

which can be proved by adding the inequalities

$$\begin{aligned}\frac{x_3+\cdots+x_n}{n-2} &\geq \sqrt[n-2]{x_3\cdots x_n}\,,\\ &\vdots\\ \frac{x_2+\cdots+x_{n-1}}{n-2} &\geq \sqrt[n-2]{x_2\cdots x_{n-1}}\,.\end{aligned}$$

For $-\dfrac{1}{n-2} < p < 1$, by the Weighted AM–GM Inequality, we have

$$\frac{1+(n-2)p}{1-p}x_2 + x_3 + \cdots + x_n \;\geq\; \frac{n-1}{1-p}\, x_2^p (x_2x_3\cdots x_n)^{\frac{1-p}{n-1}}\,.$$

Adding this inequality to the analogous ones for $x_3, \ldots, x_n$, we get the required inequality. Equality occurs in (3) if and only if $x_i = 1$ for $1 \leq i \leq n$.

Proof of (d): As above, it suffice to consider the case where $\prod\limits_{i=1}^{n} x_i = 1$. By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\sum_{\text{cyclic}} \frac{x_1}{x_1^\alpha + x_2 + \cdots + x_n} &\geq \frac{(x_1+x_2+\cdots+x_n)^2}{\sum\limits_{\text{cyclic}} x_1(x_1^\alpha + x_2 + \cdots + x_n)}\\ &= \frac{(x_1+x_2+\cdots+x_n)^2}{(x_1+x_2+\cdots+x_n)^2 + \sum\limits_{i=1}^{n} x_i^{1+\alpha} - \sum\limits_{i=1}^{n} x_i^2}\,.\end{aligned}$$

Thus, we still have to show

$$\sum_{i=1}^{n} x_i^2 \;\geq\; \sum_{i=1}^{n} x_i^{1+\alpha}\,.$$

Case 1. $-1 \leq \alpha < 1$.

We can prove the inequality using Chebyshev's Inequality and the AM–GM Inequality, as follows:

$$\begin{aligned}\sum_{i=1}^{n} x_i^2 &\geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i^{1-\alpha}\right)\left(\sum_{i=1}^{n} x_i^{1+\alpha}\right)\\ &\geq (x_1x_2\cdots x_n)^{(1-\alpha)/n}\sum_{i=1}^{n} x_i^{1+\alpha} \;=\; \sum_{i=1}^{n} x_i^{1+\alpha}\,.\end{aligned}$$

Case 2. $-1 - \dfrac{2}{n-1} \leq \alpha < -1$.

It is convenient to replace the numbers $x_1, x_2, \ldots, x_n$ by $x_1^{(n-1)/2}, x_2^{(n-1)/2}, \ldots, x_n^{(n-1)/2}$, respectively. (Note that their product is equal to 1.) We also use the substitution $q = (n-1)(1+\alpha)/2$, and note that $-1 \le q < 0$. Thus, we have to prove that

$$\sum_{i=1}^{n} x_i^{n-1} \ \ge \ \sum_{i=1}^{n} x_i^{q} \, ,$$

when $\prod_{i=1}^{n} x_i = 1$. Using the well-known Maclaurin Inequality:

$$\sum_{i=1}^{n} x_i^{n-1} \ \ge \ \sum_{\text{cyclic}} x_2 \cdots x_n \, ,$$

and Chebyshev's Inequality together with the AM–GM Inequality, we get the desired inequality

$$\begin{aligned} \sum_{i=1}^{n} x_i^{n-1} \ \ge \ \sum_{i=1}^{n} \frac{1}{x_i} \ &\ge \ \frac{1}{n}\Big(\sum_{i=1}^{n} x_i^{-1-q}\Big)\Big(\sum_{i=1}^{n} x_i^{q}\Big) \\ &\ge \ \sqrt[n]{(x_1x_2\cdots x_n)^{-1-q}} \sum_{i=1}^{n} x_i^{q} \ = \ \sum_{i=1}^{n} x_i^{q} \, . \end{aligned}$$

Now the proof is complete. Equality occurs in (4) if and only if $x_i = 1$ for $1 \le i \le n$.

Finally, we make the following three conjectures: *Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 + x_2 + \cdots + x_n \ge n$. Then, for any $\alpha > 1$, we have*

(a) $\displaystyle\sum_{\text{cyclic}} \frac{1}{x_1^{\alpha} + x_2 + \cdots + x_n} \ge 0$; (b) $\displaystyle\sum_{\text{cyclic}} \frac{x_1}{x_1^{\alpha} + x_2 + \cdots + x_n} \ge 0$;

(c) $\displaystyle\sum_{\text{cyclic}} \frac{x_1^{\alpha} - x_1}{x_1^{\alpha} + x_2 + \cdots + x_n} \ge 0$.

Acknowledgement. The author would like to express his gratitude to the referee for helpful suggestions.

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[1] Vasile Cîrtoaje, Some Generalizations of Two Inequalities from IMO, *Gazeta Matematica* A, 2(2001).

[2] Nikolai Nikolov, A Generalization of an Inequality from IMO 2005, `http://arxiv.org/PS_cache/math/pdf/0512/0512172.pdf`

[3] `http://www.mathlinks.ro/Forum/viewtopic.php?p=281573#p281573`.

Vasile Cîrtoaje
University of Ploiesti
Ploiesti, Romania
vcirtoaje@upg-ploiesti.ro

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er octobre 2006**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

3114. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit a, b et c trois nombres réels positifs tels que

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \;=\; 2\,.$$

Montrer que

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{1}{4c+1} \;\geq\; 1\,.$$

3115. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit a, b et c les longueurs respectives des côtés opposés aux sommets A, B et C du triangle ABC. Montrer que

$$\frac{\cos^3 A}{a}+\frac{\cos^3 B}{b}+\frac{\cos^3 C}{c} \;<\; \frac{a^2+b^2+c^2}{2abc}\,.$$

3116. *Proposé par Arkady Alt, San Jose, CA, USA.*

Si a, b et c sont trois nombres réels arbitraires, montrer que

$$\sum_{\text{cyclique}} a(b+c-a)^3 \;\leq\; 4abc(a+b+c)\,.$$

3117. *Proposé par Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Soit a, b et c les longueurs des côtés et s le demi-périmètre du triangle ABC. Montrer que

$$\sum_{\text{cyclique}} (a+b)\sqrt{ab(s-a)(s-b)} \;\leq\; 3abc\,.$$

3118. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit BE et CF les hauteurs du triangle acutangle ABC, avec E sur AC et F sur AB. Soit respectivement BK et CL les bissectrices intérieures des angles ABC et ACB, avec K sur AC et L sur AB. Désignons par I le centre du cercle inscrit du triangle ABC et par O celui de son cercle circonscrit. Montrer que E, F et I sont colinéaires si et seulement si K, L et O le sont.

3119. *Proposé par Michel Bataille, Rouen, France.*

Désignons respectivement par r et s le rayon du cercle inscrit du triangle ABC et son demi-périmètre. Montrer que

$$3\sqrt{3}\sqrt{\frac{r}{s}} \;\le\; \sqrt{\tan\left(\tfrac{1}{2}A\right)}+\sqrt{\tan\left(\tfrac{1}{2}B\right)}+\sqrt{\tan\left(\tfrac{1}{2}C\right)} \;\le\; \sqrt{\frac{s}{r}}\,.$$

3120. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle isocèle avec $AB = BC$, soit F le point milieu de AC. Soit $\alpha = \angle BAX$, où X est un point variable sur le rayon BF. Aussi longtemps que $\alpha \neq \pi/2$, les réflexions de la droite BF dans BA et XA se coupent en un point, disons M.

Trouver $\displaystyle\lim_{\alpha\to\pi/2} |\cos\alpha|\cdot CM$.

3121. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit n et r des entiers positifs. Montrer que

$$\left(\frac{1}{2^n}\sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\left[1-\frac{1}{2^{nr}}\binom{n}{k}^{r}\right]\right)^{r} \;\le\; \frac{r^r}{(r+1)^{r+1}}\,.$$

3122. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

On suppose que les triangles ABC et $A'B'C'$ ont un angle droit en A, respectivement en A'. Soit h_a, respectivement $h_{a'}$, les hauteurs sur les côtés a et a'. Si $b \ge c$ et $b' \ge c'$, montrer que

$$\sqrt{aa'}+2\sqrt{h_a h_{a'}} \;\le\; \sqrt{2}\left(\sqrt{bb'}+\sqrt{cc'}\right)\,.$$

3123. *Proposé par Joe Howard, Portales, NM, USA.*

Soit a, b et c les côtés d'un triangle. Montrer que

$$\frac{abc(a+b+c)^2}{a^2+b^2+c^2} \;\ge\; 2abc+\prod_{\text{cyclique}}(b+c-a)\,.$$

3124. *Proposé par Joe Howard, Portales, NM, USA.*

Soit a, b et c les côtés du triangle ABC dans lequel au plus un angle excède $\pi/3$, et soit r le rayon du cercle inscrit. Montrer que

$$\frac{\sqrt{3}(abc)}{a^2+b^2+c^2} \;\geq\; 2r\,.$$

3125. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

Soit m_a, h_a et w_a les longueurs respectives de la médiane, de la hauteur et de la bissectrice intérieure aboutissant sur le côté a du triangle ABC. On définit de manière analogue m_b, m_c, h_b, h_c, w_b et w_c. Soit R le rayon du cercle circonscrit au triangle ABC.

(a) Montrer que

$$\sum_{\text{cyclique}} \frac{b^2+c^2}{m_a} \;\leq\; 12R\,.$$

(b) Montrer que

$$\sum_{\text{cyclique}} \frac{b^2+c^2}{h_a} \;\geq\; 12R\,.$$

(c)$\star$ Déterminer le domaine des valeurs de

$$\frac{1}{R}\sum_{\text{cyclique}} \frac{b^2+c^2}{w_a}\,.$$

. . . .

3114. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, c be positive real numbers such that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \;=\; 2\,.$$

Prove that

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{1}{4c+1} \;\geq\; 1\,.$$

3115. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, c, be the lengths of the sides opposite the vertices A, B, C, respectively, in triangle ABC. Prove that

$$\frac{\cos^3 A}{a}+\frac{\cos^3 B}{b}+\frac{\cos^3 C}{c} \;<\; \frac{a^2+b^2+c^2}{2abc}\,.$$

3116. *Proposed by Arkady Alt, San Jose, CA, USA.*

For arbitrary real numbers a, b, c, prove that

$$\sum_{\text{cyclic}} a(b+c-a)^3 \;\le\; 4abc(a+b+c)\,.$$

3117. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let a, b, c be the lengths of the sides and s the semi-perimeter of $\triangle ABC$. Prove that

$$\sum_{\text{cyclic}} (a+b)\sqrt{ab(s-a)(s-b)} \;\le\; 3abc\,.$$

3118. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let BE and CF be altitudes of the acute-angled triangle ABC with E on AC and F on AB. Let BK and CL be the interior angle bisectors of $\angle ABC$ and $\angle ACB$, respectively, with K on AC and L on AB. Let I denote the incentre of $\triangle ABC$, and let O denote its circumcentre. Prove that E, F, and I are collinear if and only if K, L, and O are collinear.

3119. *Proposed by Michel Bataille, Rouen, France.*

Let r and s denote the inradius and semi-perimeter, respectively, of triangle ABC. Show that

$$3\sqrt{3}\sqrt{\frac{r}{s}} \;\le\; \sqrt{\tan\left(\tfrac{1}{2}A\right)}+\sqrt{\tan\left(\tfrac{1}{2}B\right)}+\sqrt{\tan\left(\tfrac{1}{2}C\right)} \;\le\; \sqrt{\frac{s}{r}}\,.$$

3120. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be an isosceles triangle with $AB = BC$, and let F be the mid-point of AC. Let $\alpha = \angle BAX$, where X is a variable point on the ray BF. As long as $\alpha \neq \pi/2$, the reflections of the line BF in BA and XA intersect. Let that point of intersection be denoted by M.

Find $\displaystyle\lim_{\alpha\to\pi/2} |\cos\alpha|\cdot CM$.

3121. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n and r be positive integers. Show that

$$\left(\frac{1}{2^n}\sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\left[1-\frac{1}{2^{nr}}\binom{n}{k}^{r}\right]\right)^{r} \;\le\; \frac{r^r}{(r+1)^{r+1}}\,.$$

3122. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $\triangle ABC$ and $\triangle A'B'C'$ have right angles at A and A', respectively, and let h_a and $h_{a'}$ denote the altitudes to the sides a and a', respectively. If $b \geq c$ and $b' \geq c'$, prove that

$$\sqrt{aa'} + 2\sqrt{h_a h_{a'}} \;\leq\; \sqrt{2}\left(\sqrt{bb'} + \sqrt{cc'}\right) .$$

3123. *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, c be the sides of a triangle. Show that

$$\frac{abc(a+b+c)^2}{a^2+b^2+c^2} \;\geq\; 2abc + \prod_{\text{cyclic}} (b+c-a) .$$

3124. *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, c be the sides of $\triangle ABC$ in which at most one angle exceeds $\pi/3$, and let r be its inradius. Show that

$$\frac{\sqrt{3}(abc)}{a^2+b^2+c^2} \;\geq\; 2r .$$

3125. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let m_a, h_a, and w_a denote the lengths of the median, the altitude, and the internal angle bisector, repectively, to side a in $\triangle ABC$. Define m_b, m_c, h_b, h_c, w_b, and w_c similarly. Let R be circumradius of $\triangle ABC$.

(a) Show that

$$\sum_{\text{cyclic}} \frac{b^2+c^2}{m_a} \;\leq\; 12R .$$

(b) Show that

$$\sum_{\text{cyclic}} \frac{b^2+c^2}{h_a} \;\geq\; 12R .$$

(c)$\star$ Determine the range of

$$\frac{1}{R}\sum_{\text{cyclic}} \frac{b^2+c^2}{w_a} .$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria from the list of solvers of 2972 and from the list of those resolving a conjecture of Bencze in 2984 [2006 : 51–52].

3008. [2005 : 45, 47] Corrected. *Proposed by Mihály Bencze and Marian Dinca, Romania.*

The convex polygon $A_1A_2\cdots A_n$ is inscribed in a circle Γ. Let G be the centroid of this polygon. For $k = 1, 2, \ldots, n$, denote by B_k the second point of intersection of the line A_kG with the circle Γ. Prove that

$$\sum_{k=1}^{n} GB_k \;\geq\; \sum_{k=1}^{n} GA_k\,.$$

Ed.: The published statement of the problem mistakenly had an equal sign instead of the proposer's intended inequality. All four correspondents produced a counterexample to the problem as it had appeared.

I. *Counterexample to the equality statement submitted independently by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Richard I. Hess, Rancho Palos Verdes, CA, USA; and Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $A_1A_2A_3$ be a small isosceles triangle having a circumcircle Γ with large radius R. Then $\sum\limits_{k=1}^{n} GA_k$ is small while $\sum\limits_{k=1}^{n} GB_k$ is approximately $2R$. These two sums are clearly not equal.

II. *Solution to the intended problem by Michel Bataille, Rouen, France.*

Note that G is interior to the polygon (by convexity), and hence, G is interior to Γ as well. Let O be the centre of Γ, and let R be the radius of Γ. The power p of G with respect to Γ satisfies $p = R^2 - GO^2 = GA_k \cdot GB_k$ for each $k = 1, 2, \ldots, n$. Invoking the AM–HM Inequality, we have

$$\sum_{k=1}^{n} GB_k \;=\; p\sum_{k=1}^{n}\frac{1}{GA_k} \;\geq\; \frac{n^2p}{\sum\limits_{k=1}^{n} GA_k}\,.$$

Thus, it is sufficient to show that

$$\left(\sum_{k=1}^{n} GA_k\right)^2 \;\leq\; n^2p\,. \tag{1}$$

Since G is the centroid of $A_1A_2\cdots A_n$, we have

$$\sum_{k=1}^{n} OA_k^2 \;=\; nOG^2 + \sum_{k=1}^{n} GA_k^2\,;$$

that is (noting that $OA_1 = \cdots = OA_n = R$),

$$\sum_{k=1}^{n} GA_k^2 \;=\; n(R^2 - GO^2) \;=\; np\,. \tag{2}$$

With the help of the Cauchy-Schwarz Inequality, we obtain

$$\left(\sum_{k=1}^{n} GA_k\right)^2 \;\le\; n\sum_{k=1}^{n} GA_k^2 \;=\; n^2p\,,$$

which is (1).

Remark. We also have $\sum\limits_{k=1}^{n} GB_k^2 \ge \sum\limits_{k=1}^{n} GA_k^2$. Indeed,

$$x^2\sum_{k=1}^{n} GA_k^2 - 2npx + \sum_{k=1}^{n} GB_k^2 \;=\; \sum_{k=1}^{n}(xGA_k - GB_k)^2 \;\ge\; 0$$

for all real numbers x. Hence, $(np)^2 \le \left(\sum\limits_{k=1}^{n} GA_k^2\right)\left(\sum\limits_{k=1}^{n} GB_k^2\right)$, or, from (2), $n^2p^2 \le np\sum\limits_{k=1}^{n} GB_k^2$. Therefore,

$$np \;=\; \sum_{k=1}^{n} GA_k^2 \;\le\; \sum_{k=1}^{n} GB_k^2\,.$$

Bataille also provided a counterexample to the claim of equality. The proposers correctly solved the intended problem.

3009. [2005 : 45,48] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

With I the incentre of $\triangle ABC$, let the angle bisectors BI and CI meet the opposite sides at B' and C', respectively. Prove that $AB'\cdot AC'$ is greater than, equal to, or less than AI^2 according as $\angle A$ is greater than, equal to, or less than 90°.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Since $\cos\dfrac{A}{2} = \dfrac{s-a}{AI}$ and $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$, we have

$$AI^2 \;=\; \frac{bc(s-a)}{s}\,.$$

Since B' divides side $b = AC$ in the ratio $c : a$, we have $AB' = \dfrac{bc}{a+c}$.

Similarly, $AC' = \dfrac{bc}{a+b}$. Hence,

$$AB' \cdot AC' - AI^2 \;=\; \frac{bc}{s(a+b)(a+c)}\big(bcs - (s-a)(a+b)(a+c)\big)\,.$$

Since $2s = a+b+c$, the factor $\big(bcs - (s-a)(a+b)(a+c)\big)$ on the right side can be rewritten as

$$\frac{1}{2}\big(bc\,(a+b+c) - (-a+b+c)(a+b)(a+c)\big) \;=\; \frac{a}{2}\left(a^2 - b^2 - c^2\right)\,.$$

Thus,

$$AB' \cdot AC' - AI^2 \;=\; \frac{abc}{2s(a+b)(a+c)}\left(a^2 - b^2 - c^2\right)\,.$$

We conclude that $AB' \cdot AC'$ is greater than, equal to, or less than AI^2 according as a^2 is greater than, equal to, or less than $b^2 + c^2$, which holds according as $\angle A$ is greater than, equal to, or less than 90°.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VEDULA N. MURTY, Dover, PA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3010. [2005 : 45, 48] *Proposed by Mihály Bencze and Marian Dinca, Romania.*

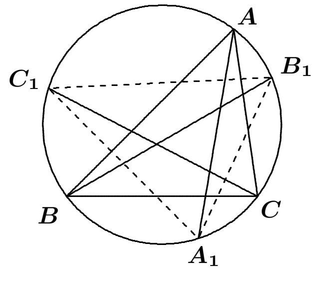
Let ABC be a triangle inscribed in a circle Γ. Let A_1, B_1, $C_1 \in \Gamma$ such that

$$\frac{\measuredangle A_1AB}{\measuredangle CAB} \;=\; \frac{\measuredangle B_1BC}{\measuredangle ABC} \;=\; \frac{\measuredangle C_1CA}{\measuredangle BCA} \;=\; \lambda\,,$$

where $0 < \lambda < 1$. Let the inradius and semiperimeter of $\triangle ABC$ be denoted by r and s, respectively; let the inradius and semiperimeter of $\triangle A_1B_1C_1$ be denoted by r_1 and s_1, respectively. Prove that

1. $s_1 \geq s$;
2. $r_1 \geq r$;
3. $[A_1B_1C_1] \geq [ABC]$, where $[PQR]$ denotes the area of triangle PQR.

Solution by Michel Bataille, Rouen, France.



We assume that A_1 is on the arc $\widehat{BC}$ which does not contain A, and we make similar assumptions for B_1 and C_1. Let the angles of $\triangle ABC$ be denoted by A, B, and C. Similarly, the angles of $\triangle A_1B_1C_1$ will be denoted by A_1, B_1, and C_1. We also let R denote the radius of Γ.

1. Note first that $\angle C_1A_1A = \angle C_1CA = \lambda C$. Similarly,

$$\begin{aligned} \angle AA_1B_1 &= \angle ABB_1 = \angle ABC - \angle B_1BC \\ &= \angle ABC - \lambda\angle ABC = (1-\lambda)B\,. \end{aligned}$$

Hence, $A_1 = \lambda C + (1-\lambda)B$. Since the sine function is concave on $(0,\pi)$, we have $\sin A_1 \geq \lambda \sin C + (1-\lambda)\sin B$, with analogous inequalities for $\sin B_1$ and $\sin C_1$. It follows that

$$\begin{aligned} s_1 &= R(\sin A_1 + \sin B_1 + \sin C_1) \\ &\geq R\big[\lambda\sin C + (1-\lambda)\sin B + \lambda\sin A \\ &\qquad\qquad + (1-\lambda)\sin C + \lambda\sin B + (1-\lambda)\sin A\big] \\ &= R(\sin A + \sin B + \sin C) = s\,. \end{aligned}$$

2. First recall that

$$\begin{aligned} r_1 &= 4R\sin\Big(\frac{A_1}{2}\Big)\sin\Big(\frac{B_1}{2}\Big)\sin\Big(\frac{C_1}{2}\Big) \\ \text{and}\qquad r &= 4R\sin\Big(\frac{A}{2}\Big)\sin\Big(\frac{B}{2}\Big)\sin\Big(\frac{C}{2}\Big)\,. \end{aligned}$$

Since the function $f(x) = \ln(\sin x)$ is concave on $\left(0,\frac{\pi}{2}\right)$, we obtain

$$\begin{aligned} \ln\left(\frac{r_1}{4R}\right) &= f\left(\frac{A_1}{2}\right) + f\left(\frac{B_1}{2}\right) + f\left(\frac{C_1}{2}\right) \\ &= f\left(\lambda\,\frac{C}{2} + (1-\lambda)\,\frac{B}{2}\right) + f\left(\lambda\,\frac{A}{2} + (1-\lambda)\,\frac{C}{2}\right) \\ &\qquad + f\left(\lambda\,\frac{B}{2} + (1-\lambda)\,\frac{A}{2}\right) \\ &\geq \lambda\,f\left(\frac{C}{2}\right) + (1-\lambda)\,f\left(\frac{B}{2}\right) + \lambda\,f\left(\frac{A}{2}\right) + (1-\lambda)\,f\left(\frac{C}{2}\right) \\ &\qquad + \lambda\,f\left(\frac{B}{2}\right) + (1-\lambda)\,f\left(\frac{A}{2}\right) \\ &= f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) = \ln\left(\frac{r}{4R}\right)\,, \end{aligned}$$

and $r_1 \geq r$ follows.

3. From parts 1 and 2 above, we have

$$[A_1B_1C_1] = r_1s_1 \geq rs = [ABC]\,.$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

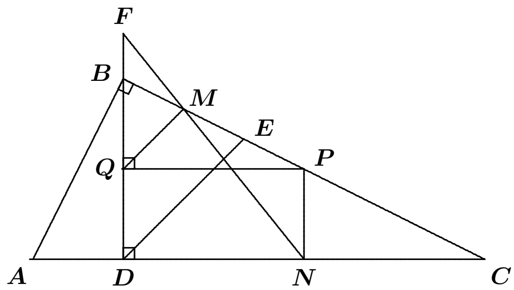
3011. [2005 : 45, 48] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a right-angled triangle with right angle at B. Let D be the foot of the perpendicular from B to AC, and let E be the intersection of the bisector of $\angle BDC$ with BC. Let M and N be the mid-points of BE and DC, respectively, and let F be the intersection of MN with BD. Prove that $AD = 2BF$.

Combination of solutions by Mathieu Guay-Paquet, student, McGill University, Montreal, QC; and Li Zhou, Polk Community College, Winter Haven, FL, USA, modified by the editor.

We will prove a slightly more general statement: if $DC = r\,NC$ and $BE = r\,ME$, for some $r > 1$, then $AD = r\,BF$. The statement of the original problem is the case $r = 2$.

Let P and Q be points on the segments BC and BD such that $\dfrac{BC}{PC} = \dfrac{BD}{QD} = r$. Then $QP \parallel DC$ and $PN \parallel BD$, so that $\dfrac{BD}{PN} = r$.



Since $\dfrac{BE}{ME} = r$, then QM is parallel to DE. Since triangles BQP and BDC are similar, QM is the bisector of $\angle BQP$, and therefore,

$$\frac{BM}{PM} = \frac{BQ}{PQ}.$$

Also, triangles BQP and ADB are similar, so that

$$\frac{BQ}{PQ} = \frac{AD}{BD}.$$

Finally, using the similar triangles BFM and PNM, we obtain

$$\frac{BF}{PN} = \frac{BM}{PM} = \frac{BQ}{PQ} = \frac{AD}{BD} = \frac{AD}{r \cdot PN}.$$

Therefore, $AD = r\,BF$, as claimed.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; the AUSTRIAN IMO TEAM 2005; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ROBERT BILINSKI, Collège Montmorency, Laval, QC; ALPER CAY, Uzman Private School, Kayseri, Turkey; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande

Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; R. LAUMEN, Deurne, Belgium; RAFAEL MARTINEZ CALAFAT, I.E.S. La Plana, Castellon, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Comănești, Romania; and the proposer.

3012. [2005 : 45, 48] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Triangles DBC, ECA, and FAB are constructed outwardly on $\triangle ABC$ such that $\angle DBC = \angle ECA = \angle FAB$ and $\angle DCB = \angle EAC = \angle FBA$. Prove that

$$AF + FB + BD + DC + CE + EA \;\geq\; AD + BE + CF\,.$$

When does equality hold?

Nearly identical solutions by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.

Using Ptolemy's Theorem on the quadrangle $ABDC$, we get

$$AB \cdot CD + AC \cdot BD \;\geq\; AD \cdot BC\,,$$

which implies that

$$\frac{AB \cdot CD}{BC} + \frac{AC \cdot BD}{BC} \;\geq\; AD\,.$$

Since the given conditions imply the similarity of triangles BCD, CAE, and ABF, we have $\dfrac{AB \cdot CD}{BC} = BF$ and $\dfrac{AC \cdot BD}{BC} = CE$. Therefore,

$$BF + CE \;\geq\; AD\,.$$

Similarly, $AF + CD \geq BE$ and $AE + BD \geq CF$. By adding these last three inequalities we get the desired inequality.

Equality holds holds in Ptolomy's Theorem if and only if the vertices of the given quadrangle lie on a circle in the prescribed order. If equality holds for the relation given in the problem, then $ABDC$, $BCEA$, and $CAFB$ must all be cyclic. Then

$$\angle BAC \;=\; 180^\circ - \angle BDC \;=\; 108^\circ - \angle CEA \;=\; \angle ABC\,.$$

Similarly, $\angle BCA = \angle ABC = \angle CAB$, from which we see that $\triangle ABC$ is equilateral. Also,

$$\angle BDC \;=\; \angle CEA \;=\; \angle AFB \;=\; 180^\circ - \angle ACB \;=\; 120^\circ\,.$$

Thus, equality holds if and only if $\triangle ABC$ is equilateral and $\angle BDC = 120^\circ$.

A similar analysis provides an equivalent condition; namely, equality holds if and only if $\triangle ABC$ and $\triangle DEF$ are concyclic and equilateral.

Also solved by MICHEL BATAILLE, Rouen, France; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

With only a little more effort Woo proved a somewhat stronger result: Of all hexagons $AFBDCE$ with $AD + BE + CF$ equal to some constant, the minimum perimeter occurs if and only if (i) *AD, BE, and CF each bisect the angles of the hexagon at each end-point,* (ii) *opposite sides of the hexagon are parallel,* (iii) *each angle of the hexagon is $120°$, and* (iv) *the perimeter of $AFBDCE$ equals $AD + BE + CF$.*

3013. [2005 : 105, 108] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Isosceles trapezoid $ABCD$ with $AB = BC = CD$ is inscribed in the circle Γ. On the arc AD of Γ which does not contain B and C, let E be a variable point. Let F and G be the respective intersections of EB with AC and of EC with BD.

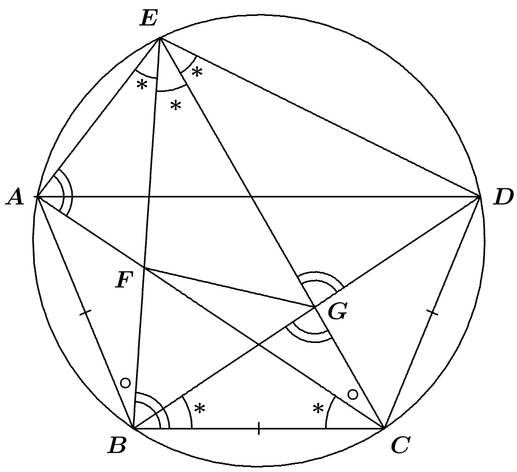
(a) Prove that the area of quadrilateral $FBCG$ is constant.

(b) Prove that $[EFG] : [EAD] = BC : AD$, where $[PQR]$ denotes the area of triangle PQR.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.

Since $AB = BC$, the arcs AB and BC are equal, from which we see that $\angle AEF = \angle BEC$. Also, $\angle FAE = \angle CAE = \angle CBE$. Thus, $\triangle AEF$ is similar to $\triangle BEC$. Similarly, $\triangle GED$ is similar to $\triangle BEC$.

Since $\angle AEB = \angle ACB$, we see that $\triangle BCF$ is similar to $\triangle AEF$; since $\angle CBD = \angle CED$ and $\angle BGC = \angle EGD$, we also see that $\triangle GBC$ is similar to $\triangle GED$. Hence, triangles AEF, BCF, GBC, and GED are similar.



(a) Since triangles BCF and GBC are similar, we have

$$\frac{BF}{GC} = \frac{CF}{BC},$$

or $BF \cdot BC = GC \cdot CF$. Since $BC = BA$, we have $BF \cdot BA = GC \cdot CF$. Also, $\angle ABE = \angle ACE$, so that

$$[ABF] = \tfrac{1}{2} \cdot BF \cdot BA \cdot \sin \angle ABE = \tfrac{1}{2} \cdot GC \cdot CF \cdot \sin \angle ACE = [FCG].$$

Therefore,

$$[FBCG] = [FBC] + [FCG] = [FBC] + [ABF] = [ABC],$$

which is a constant.

(b) Since triangles AEF and GED are similar, we have

$$\frac{AE}{GE} = \frac{EF}{ED},$$

or $AE \cdot ED = GE \cdot EF$. Therefore,

$$\frac{[EFG]}{[EAD]} = \frac{\frac{1}{2} \cdot GE \cdot EF \cdot \sin \angle BEC}{\frac{1}{2} \cdot AE \cdot ED \cdot \sin \angle AED} = \frac{\sin \angle BEC}{\sin \angle AED}.$$

Let R be the radius of the circle Γ. Since $\triangle BEC$ and $\triangle AED$ are both inscribed in the circle Γ, we have $BC = 2R \sin \angle BEC$ and $AD = 2R \sin \angle AED$, by the Sine Law. Therefore,

$$\frac{[EFG]}{[EAD]} = \frac{BC}{AD}.$$

Also solved by MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; Mª JESúS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3014. [2005 : 105, 108] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a convex quadrilateral $ABCD$, let O be the intersection of the diagonals AC and BD, and let M and N be the mid-points of AC and BD, respectively. Suppose that $[OAB] + [OCD] = [OBC]$, where $[PQR]$ denotes the area of triangle PQR. Prove that AN, DM, and BC are concurrent.

I. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Since the problem is affine-invariant, we may assume that $\triangle OBC$ is an isosceles right triangle with a right angle at O and that $OB = OC = 1$. Introduce coordinates such that $O = (0,0)$, $B = (0,1)$, $C = (1,0)$,

$A = (-a, 0)$, $D = (0, -d)$. Then $M = \left(\frac{1-a}{2}, 0\right)$ and $N = \left(0, \frac{1-d}{2}\right)$. The condition $[OAB] + [OCD] = [OBC]$ is equivalent to $a + d = 1$; thus, $M = \left(\frac{d}{2}, 0\right)$ and $N = \left(0, \frac{a}{2}\right)$. The lines AN and DM have equations $2y - x = a$ and $2x - y = d$, respectively. Solving these equations for x and y, we see that the intersection of the two lines is $\left(\frac{a+2d}{3}, \frac{2a+d}{3}\right)$, which clearly lies on the line BC, namely $x + y = 1$. The result follows.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let A' and D' be the points where AN and DM intersect BC. We shall prove that A' coincides with D'. Let a, b, c, d denote the lengths OA, OB, OC, OD, respectively. Since $\frac{a}{c} = \frac{[OAB]}{[OBC]}$ and $\frac{d}{b} = \frac{[OCD]}{[OBC]}$, the given condition implies that

$$\frac{a}{c} + \frac{d}{b} = 1\,. \tag{1}$$

From Menelaus' Theorem (applied to $\triangle OBC$ and transversal ANA'),

$$\begin{aligned}\frac{BA'}{A'C} &= \frac{BN}{NO} \cdot \frac{OA}{AC} = \frac{\frac{1}{2}(b+d)}{\frac{1}{2}(b+d)-d} \times \frac{a}{a+c} \\ &= \frac{1+\frac{d}{b}}{1-\frac{d}{b}} \cdot \frac{\frac{a}{c}}{1+\frac{a}{c}} = \frac{1+\frac{d}{b}}{1+\frac{a}{c}}\,,\end{aligned}$$

using (1). Similarly,

$$\frac{CD'}{D'B} = \frac{1+\frac{a}{c}}{1-\frac{a}{c}} \cdot \frac{\frac{d}{b}}{1+\frac{d}{b}} = \frac{1+\frac{a}{c}}{1+\frac{d}{b}}\,.$$

Hence, $\frac{BA'}{A'C} \cdot \frac{CD'}{D'B} = 1$. Thus, A' and D' coincide.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; JOEL SCHLOSBERG, Bayside, NY, USA; MARIAN TETIVA, Bîrlad, Romania; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Note that the two featured solutions are essentially the same; the second was included for those readers who prefer to avoid coordinates. All solvers provided reversible arguments, but only Tetiva and Zvonaru made the converse explicit: $[OAB] + [OCD] = [OBC]$ *if and only if* AN, DM, BC *are concurrent.*

3015. [2005 : 105, 108] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a triangle ABC with incentre I, suppose that $BC < AB$ and $BC < AC$. The exterior bisectors of $\angle ABC$ and $\angle ACB$ intersect AC and AB at D and E, respectively. Prove that

$$\frac{BD}{CE} = \frac{(AI^2 - BI^2)CI}{(AI^2 - CI^2)BI}.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Using the Sine Law on $\triangle AIB$, we get $\dfrac{AI}{BI} = \dfrac{\sin\left(\frac{1}{2}B\right)}{\sin\left(\frac{1}{2}A\right)}$. Similarly, $\dfrac{BI}{CI} = \dfrac{\sin\left(\frac{1}{2}C\right)}{\sin\left(\frac{1}{2}B\right)}$ and $\dfrac{AI}{CI} = \dfrac{\sin\left(\frac{1}{2}C\right)}{\sin\left(\frac{1}{2}A\right)}$. The right side of the desired equality transforms to

$$\begin{aligned}
\frac{(AI^2 - BI^2)\cdot CI}{(AI^2 - CI^2)\cdot BI} &= \frac{\dfrac{AI^2}{BI^2} - 1}{\dfrac{AI^2}{CI^2} - 1}\cdot\frac{BI}{CI} = \frac{\dfrac{\sin^2\left(\frac{1}{2}B\right)}{\sin^2\left(\frac{1}{2}A\right)} - 1}{\dfrac{\sin^2\left(\frac{1}{2}C\right)}{\sin^2\left(\frac{1}{2}A\right)} - 1}\cdot\frac{\sin\left(\frac{1}{2}C\right)}{\sin\left(\frac{1}{2}B\right)} \\
&= \frac{\sin^2\left(\frac{1}{2}B\right) - \sin^2\left(\frac{1}{2}A\right)}{\sin^2\left(\frac{1}{2}C\right) - \sin^2\left(\frac{1}{2}A\right)}\cdot\frac{\sin\left(\frac{1}{2}C\right)}{\sin\left(\frac{1}{2}B\right)} \\
&= \frac{\cos A - \cos B}{\cos A - \cos C}\cdot\frac{\sin\left(\frac{1}{2}C\right)}{\sin\left(\frac{1}{2}B\right)} \\
&= \frac{\sin\left(\frac{1}{2}(A+B)\right)\sin\left(\frac{1}{2}(B-A)\right)}{\sin\left(\frac{1}{2}(A+C)\right)\sin\left(\frac{1}{2}(C-A)\right)}\cdot\frac{\sin\left(\frac{1}{2}C\right)}{\sin\left(\frac{1}{2}B\right)} \\
&= \frac{\sin\left(\frac{1}{2}(B-A)\right)}{\sin\left(\frac{1}{2}(C-A)\right)}\cdot\frac{\sin\left(\frac{1}{2}C\right)\cos\left(\frac{1}{2}C\right)}{\sin\left(\frac{1}{2}B\right)\cos\left(\frac{1}{2}B\right)} \\
&= \frac{\sin\left(\frac{1}{2}(B-A)\right)}{\sin\left(\frac{1}{2}(C-A)\right)}\cdot\frac{\sin C}{\sin B}.
\end{aligned}$$

On the other hand, since BC is the shortest side of $\triangle ABC$, the points A, C, D and A, B, E lie in that order on their respective lines. Considering $\triangle BDC$, we have

$$\angle BDC = \angle ACB - \angle CBD = C - \frac{\pi - B}{2} = C - \frac{A+C}{2} = \frac{C-A}{2}.$$

Applying the Sine Law to $\triangle BCD$ we then get

$$\frac{BD}{BC} = \frac{\sin C}{\sin\left(\frac{1}{2}(C-A)\right)}.$$

Similarly,

$$\frac{CE}{BC} = \frac{\sin B}{\sin\left(\frac{1}{2}(B-A)\right)}.$$

Dividing the two expressions gives

$$\frac{BD}{CE} = \frac{\sin\left(\frac{1}{2}(B-A)\right)}{\sin\left(\frac{1}{2}(C-A)\right)} \cdot \frac{\sin C}{\sin B}.$$

Thus, we obtain an expression for the left side of the desired equality, which is the same as the one obtained for the right side. This completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; Mª JESúS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3016. [2005 : 105, 108]
Proposed by Neven Jurič, Zagreb, Croatia.

Two spheres of radius r are externally tangent to each other. Three spheres of radius R are all externally tangent to each other. If each of the three spheres of radius R are at the same time externally tangent to the two spheres of radius r, express R in terms of r.

Solution by Michel Bataille, Rouen, France.

We show that $R = 6r$.

Let O_1, O_2, and O_3 be the centres of the three spheres of radius R, and let and I_1 and I_2 be the centres of the two spheres of radius r. Let $S(O,\rho)$ denote the sphere with centre O and radius ρ. Since the spheres $S(O_1,R)$ and $S(I_1,r)$ are externally tangent, we have $O_1I_1 = R+r$. Similarly, $O_1I_2 = R+r$. It follows that O_1 lies in the perpendicular bisector plane P of the line segment I_1I_2. By a similar argument, the centres O_2 and O_3 lie in the same plane P. Since the spheres $S(O_1,R)$, $S(O_2,R)$, and $S(O_3,R)$ are externally tangent to each other, we have $O_1O_2 = O_2O_3 = O_3O_1 = 2R$. Hence, the triangle $O_1O_2O_3$ is an equilateral triangle in the plane P.

Now, let T be the orthogonal projection of I_1 and I_2 onto the plane P. Then T is the mid-point of I_1I_2, and we obtain

$$TO_1^2 \;=\; I_1O_1^2 - I_1T^2 \;=\; (R+r)^2 - r^2 \;=\; R^2 + 2Rr\,.$$

Similarly, we obtain the same expression for TO_2 and TO_3, showing that the point T is the circumcentre of $\triangle O_1O_2O_3$. Then $TO_1 = \frac{2}{3}\cdot\frac{2R\sqrt{3}}{2} = \frac{2R}{\sqrt{3}}$, and it follows that $4R^2/3 = R^2 + 2Rr$, or $R = 6r$, as claimed.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3018. [2005 : 106, 109]

Proposed by Christopher J. Bradley, Bristol, UK.

In $\triangle ABC$, the cevians AL, BM, and CN are concurrent at K, and the cevians AU, BV, and CW are concurrent at T. Suppose that MN meets VW at P, NL meets WU at Q, and LM meets UV at R.

(a) Prove that QAR, RBP, and PCQ are all straight lines.

(b) Prove that AP, BQ, and CR are either concurrent or parallel.

(c) Given T, under what conditions on K are the lines in (b) parallel?

I. *Solution to parts* (a) *and* (b) *by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Since parts (a) and (b) of the problem deal with projective properties, we may assume that line BC is the line at infinity. Furthermore, we may assume that the lines AB and AC are perpendicular. Since AN, MK, and VT all go through B, they are now parallel. Similarly, AV, WT, and NK are now parallel. Thus, $AMKN$ and $AVTW$ are both rectangles.

Since QN and AK both go through L (which lies on the line at infinity), they are parallel. Similarly, QW and AT are parallel. Under a reflection in line AN, line WV goes to line WQ (since $\angle VWA = \angle VTA = \angle AWQ$), and NM goes to NQ; hence, the intersection P of VW and NM goes to Q under the reflection. Similarly, R is the image of P under reflection in AM. It is immediate now that A is the mid-point of QR, showing that QAR is a line. By similar arguments, RBP and PCQ are also lines. This completes part (a).

Line BQ is the line through Q parallel to AN, and CR is the line through R parallel to AM. These two lines meet at the point S such that $PQSR$ is a rectangle centred at A. Since PA passes through S, we see that AP, BQ, and CR all pass through S, and the three lines were originally (with the given line at infinity) either concurrent or parallel depending on whether S was originally at infinity. This completes part (b).

II. *Solution by Michel Bataille, Rouen, France.*

Using areal coordinates relative to (A, B, C), let $K = \ell A + mB + nC$ and $T = uA + vB + wC$, where ℓ, m, n, u, v, and w are real numbers such that $\ell + m + n = u + v + w = 1$. Then

$$\begin{array}{rclcrcl} (m+n)L &=& mB+nC\,, & & (v+w)U &=& vB+wC\,, \\ (n+\ell)M &=& nC+\ell A\,, & \text{and} & (w+u)V &=& wC+uA\,, \\ (\ell+m)N &=& \ell A+mB\,, & & (u+v)W &=& uA+vB\,. \end{array}$$

(a) The equations of the lines MN and VW are $xmn - yn\ell - z\ell m = 0$ and $xvw - ywu - zuv = 0$, respectively. Thus, their intersection is $P = (-\alpha, \beta, \gamma)$, where we set $\alpha = \ell u(mw - nv)$, $\beta = mv(nu - \ell w)$, and $\gamma = nw(\ell v - mu)$. In the same way we obtain $Q = (\alpha, -\beta, \gamma)$ and $R = (\alpha, \beta, -\gamma)$. Since

$$\begin{vmatrix} -\alpha & \alpha & 0 \\ \beta & -\beta & 0 \\ \gamma & \gamma & 1 \end{vmatrix} = 0\,,$$

P, Q, and C are collinear. Similarly, R, P, and B are collinear, as are Q, R, and A.

(b) Lines AP, BQ, and CR have equations $\gamma y = \beta z$, $\gamma x = \alpha z$, and $\beta x = \alpha y$, respectively. Since

$$\begin{vmatrix} 0 & \gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & -\alpha & 0 \end{vmatrix} = 0\,,$$

these lines are either concurrent or parallel.

(c) Lines AP, BQ, and CR are parallel when the solution of the system

$$\begin{aligned} \gamma y &= \beta z\,, \\ \gamma x &= \alpha z\,, \\ \beta x &= \alpha y\,, \end{aligned}$$

lies on the line at infinity, $x + y + z = 0$; that is, if and only if $\alpha + \beta + \gamma = 0$. In the original notation this equation is

$$\ell m w(u - v) + m n u(v - w) + n \ell v(w - u) \;=\; 0\,. \qquad (*)$$

There are two possibilities:

(i) If T is the centroid G of $\triangle ABC$, then $u = v = w$ and, consequently, $(*)$ is always satisfied. Therefore, if T is the centroid, the lines AP, BQ, and CR are parallel, whatever the position of K.

(ii) If $T \neq G$, equation $(*)$ says that K is on the conic with equation

$$\bigl(w(u - v)\bigr)xy + \bigl(u(v - w)\bigr)yz + \bigl(v(w - u)\bigr)zx \;=\; 0\,.$$

This conic is easily seen to pass through A, B, C, T, and G; it is therefore determined by these five points.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON (a second solution in which part (c) *was also solved); LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

In Zhao's solution to part (c)*, he writes the conic from Bataille's solution as*

$$\begin{vmatrix} 1 & 1 & 1 \\ yz & xz & xy \\ mn & \ell n & \ell m \end{vmatrix} = 0\,.$$

Since (yz, xz, xy) and $(mn, \ell n, \ell m)$ are the isotomic conjugates of (x, y, z) and (ℓ, m, n), respectively, the conic equation is equivalent to the condition that the centroid of $\triangle ABC$ is collinear with the isotomic conjugates of K and T.

3019. [2005 : 106, 109] *Proposed by Emilio Fernández Moral, IES Sagasta, Logroño, Spain.*

Let q and s be integers such that $q \geq 1$ and $0 \leq s \leq q-1$. Show that

$$\sum_{k=0}^{s}(-1)^{s+k}2^{2k}\binom{q-1+k}{2k}\binom{q-1-k}{s-k} \;=\; \binom{2q-1}{2s}$$

and

$$\sum_{k=0}^{s}(-1)^{s+k}2^{2k+1}\binom{q+k}{2k+1}\binom{q-1-k}{s-k} \;=\; \binom{2q}{2s+1}.$$

Solution by Michel Bataille, Rouen, France.

First, we prove the following identities:

- for integers r and m such that $0 \leq r \leq \lfloor m/2 \rfloor$,

$$\frac{1}{2^m}\sum_{j=r}^{\lfloor m/2\rfloor}\binom{m+1}{2j+1}\binom{j}{r} \;=\; \binom{m-r}{r}\frac{1}{2^{2r}}\,; \tag{1}$$

- for integers k and n such that $0 \leq r \leq n$,

$$\sum_{j=0}^{k}\binom{2n+1}{2j}\binom{n-j}{n-k} \;=\; 2^{2k}\binom{n+k}{2k}\,; \tag{2}$$

and

$$\sum_{j=0}^{k}\binom{2n+2}{2j+1}\binom{n-j}{n-k} \;=\; 2^{2k+1}\binom{n+k+1}{2k+1}. \tag{3}$$

We start with the following known identity:

$$\sum_{r=0}^{\lfloor m/2\rfloor}\binom{m-r}{r}\left(\frac{x}{4}\right)^r = \frac{1}{\sqrt{1+x}}\left(\left(\frac{1+\sqrt{1+x}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{1+x}}{2}\right)^{m+1}\right).$$

[See R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994, formula (5.74)]. Using the binomial theorem to expand the right hand side, we obtain

$$\begin{aligned}\sum_{r=0}^{\lfloor m/2\rfloor}\binom{m-r}{r}\left(\frac{x}{4}\right)^r &= \frac{1}{2^m}\sum_{j=0}^{\lfloor m/2\rfloor}\binom{m+1}{2j+1}(\sqrt{1+x})^{2j}\\ &= \frac{1}{2^m}\sum_{j=0}^{\lfloor m/2\rfloor}\binom{m+1}{2j+1}(1+x)^j\,,\end{aligned}$$

and the identity (1) follows by equating the coefficients of x^r on both sides. With $m = 2n$ and $r = n - k$, equation (1) becomes

$$\begin{aligned}\sum_{j=n-k}^{n} \binom{2n+1}{2j+1}\binom{j}{n-k} &= \binom{2n-(n-k)}{n-k} 2^{2n-2(n-k)} \\ \text{or} \quad \sum_{l=0}^{k} \binom{2n+1}{2l}\binom{n-1}{n-k} &= 2^{2k}\binom{n+k}{n-k} = 2^{2k}\binom{n+k}{2k},\end{aligned}$$

(where $l = n - j$) which is (2). With $m = 2n + 1$, identity (3) can be obtained similarly. Now, setting $q - 1 = n$, the first of the proposed identities becomes

$$\sum_{k=0}^{s} (-1)^k 2^{2k} \binom{n+k}{2k}\binom{n-k}{s-k} = (-1)^s \binom{2n+1}{2s}. \tag{4}$$

Using (2) together with familiar properties of the binomial coefficients, and interchanging the order of the double summation we then have:

$$\begin{aligned}&\sum_{k=0}^{s} (-1)^k 2^{2k} \binom{n+k}{2k}\binom{n-k}{s-k} \\ &\qquad = \sum_{k=0}^{s}\sum_{j=0}^{k} (-1)^k \binom{2n+1}{2j}\binom{n-j}{n-k}\binom{n-k}{n-s} \\ &\qquad = \sum_{j=0}^{s} \binom{2n+1}{2j} \sum_{k=j}^{s} (-1)^k \binom{n-j}{n-s}\binom{s-j}{k-j} \\ &\qquad = \sum_{j=0}^{s} \binom{2n+1}{2j}\binom{n-j}{n-s} \sum_{k=j}^{s} (-1)^k \binom{s-j}{k-j}.\end{aligned}$$

But, with $l = k - j$, we get

$$\begin{aligned}\sum_{k=j}^{s} (-1)^k \binom{s-j}{k-j} &= \sum_{l=0}^{s-j} (-1)^{j+l} \binom{s-j}{l} \\ &= \begin{cases} (-1)^j (1-1)^{s-j} = 0 & \text{if } j < s, \\ (-1)^s & \text{if } j = s, \end{cases}\end{aligned}$$

and identity (4) follows immediately.

The proof of the second identity can be obtained similarly, using (3) instead of (2).

Also solved by the proposer.

3020. [2005 : 106, 109] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $A_1A_2\cdots A_n$ be a regular polygon inscribed in the circle Γ, and let P be an interior point of Γ. The lines PA_1, PA_2, $\ldots$, PA_n intersect Γ for the second time at B_1, B_2, $\ldots$, B_n, respectively.

(a) Prove that $\sum\limits_{k=1}^{n}(PA_k)^2 \geq \sum\limits_{k=1}^{n}(PB_k)^2$.

(b) Prove that $\sum\limits_{k=1}^{n}PA_k \geq \sum\limits_{k=1}^{n}PB_k$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA, expanded by the editor.

Without loss of generality, we may assume that Γ is the unit circle centred at the origin O. We identify the vertices of the regular $n-$gon with the n^{th} roots of unity. Thus, we identify A_k with ω^k, where $\omega = e^{2\pi i/n}$, for $k = 1, 2, \ldots, n$. We let P be represented by z.

(a) Since $\omega\overline{\omega} = |\omega|^2 = 1$, we have $\omega^{-1} = \overline{\omega}$ and $\omega^{-k} = \overline{\omega}^k = \overline{\omega^k}$. Hence,

$$\begin{aligned}\sum_{k=1}^{n}(PA_k)^2 &= \sum_{k=1}^{n}\left|\omega^k - z\right|^2 = \sum_{k=1}^{n}\left(\omega^k - z\right)\left(\overline{\omega^k} - \overline{z}\right) \\ &= \sum_{k=1}^{n}\left(1 + |z|^2 - z\omega^{-k} - \overline{z}\omega^k\right) \\ &= n\left(1 + |z|^2\right) - z\sum_{k=1}^{n}\omega^{-k} - \overline{z}\sum_{k=1}^{n}\omega^k = n\left(1 + |z|^2\right)\end{aligned}$$

since $\sum\limits_{k=1}^{n}\omega^k = \sum\limits_{k=0}^{n-1}\omega^k = 0$ and $\sum\limits_{k=1}^{n}\omega^{-k} = \overline{\sum\limits_{k=1}^{n}\omega^k} = 0$.

Next, let EF denote the chord through P which is perpendicular to $\overline{OP}$. Then

$$PA_k \cdot PB_k = PE \cdot PF = (PE)^2 = (OE)^2 - (OP)^2 = 1 - |z|^2\,.$$

Hence,

$$\begin{aligned}(PA_k)^2 + (PB_k)^2 &= (PA_k + PB_k)^2 - 2PA_k \cdot PB_k \\ &= (A_kB_k)^2 - 2\left(1 - |z|^2\right) \\ &\leq 4 - 2\left(1 - |z|^2\right) = 2\left(1 + |z|^2\right).\end{aligned}$$

Therefore,

$$\sum_{k=1}^{n}(PA_k)^2 + \sum_{k=1}^{n}(PB_k)^2 \leq 2n\left(1 + |z|^2\right) = 2\sum_{k=1}^{n}(PA_k)^2\,,$$

from which $\sum_{k=1}^{n}(PA_k)^2 \geq \sum_{k=1}^{n}(PB_k)^2$ follows.

(b) By the Triangle Inequality, we have

$$\begin{aligned}\sum_{k=1}^{n} PA_k &= \sum_{k=1}^{n}\left|\omega^k - z\right| = \sum_{k=1}^{n}\left|1-\omega^{-k}z\right| \geq \left|n - z\sum_{k=1}^{n}\omega^{-k}\right| = n \\ &\geq \frac{1}{2}\sum_{k=1}^{n} A_kB_k = \frac{1}{2}\left(\sum_{k=1}^{n} PA_k + \sum_{k=1}^{n} PB_k\right),\end{aligned}$$

from which $\sum_{k=1}^{n} PA_k \geq \sum_{k=1}^{n} PB_k$ follows.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.