SKOLIAD No. 90

Robert Bilinski

Wow! My first year as Skoliad Editor has just finished. What a roller-coaster ride that was! I needed some adjusting to the task, and I learned a lot more than I thought I would need for the job. Editors are under-appreciated. Based on what I learned in my small part of ***CRUX with MAYHEM***, I would like to thank Jim Totten for an astounding job as Editor-in-Chief of the whole magazine. With that said, I would like to thank the rest of the team for the warm welcome I received. They also chipped in when the bank of contests was running low. Now, with the contacts I have made, I think I can run smoothly with a set rotation of contests. You should keep an eye on the section, since it should hold a surprise or two in the rest of my term.

My new challenge for the year will be getting a regular supply of solutions from young and energetic students. I appeal to all teachers out there who run contest training sessions to remind your students that they can get published in these pages. Then remind all the other teachers you know that they can use contest material in their classrooms!

Which brings me to the most important part of this first year as editor: I discovered a bunch of ingenious and sharp-witted students without whom this job would have little meaning. Thanks guys!

It is now the time to list our top performers in this year's Skoliad (each of whom won a past volume of Mathematical Mayhem):

February	Angela Park and Alex Wice
March	Alex Remorov
April	Alex Wice
May	Alan Guo and Eric Zhang
September	Alex Remorov
October	Geoffrey Siu and Bobby Xiao
November	Khartik Natarajan
December	Carl O'Connor and Geoffrey Siu

There was an unsung solver: Angela Park. This 8^{th} grader's solutions were submitted during the transition between editors. Her 13 good solutions (out of 13 sent) never had a chance to be shown! We hope to hear more from Angela in the new year.

I would also like to thank Mariannella Ouellet for her submission and for being a fun student!

Please send your solutions to the problems in this edition by ***1 June, 2006***. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

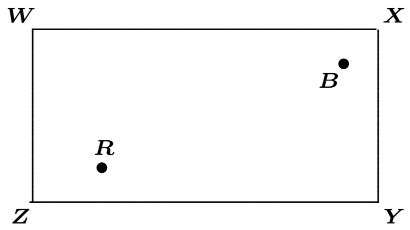
Les question de ce Skoliad provienent du Concours de Mathématiques des Maritimes 2005. Mes remerciements à David Horrocks de l'Université de l'Île du Prince Edouard.

Concours de Mathématiques des Maritimes 2005

3 mars 2005

1. Une jardinière possède une tondeuse à siège et une tondeuse poussée. La tonte de la pelouse lui prend trois heures avec la tondeuse poussée et soixante-quinze minutes avec la tondeuse à siège. Une journée, la jardinière tond une portion de la pelouse avec la tondeuse à siège et le reste avec la tondeuse poussée. Si cela lui prend quatre-vingt-seize minutes, quelle fraction de la pelouse a-t-elle tondue avec la tondeuse à siège ?

2. Les bandes (côtés intérieurs) d'une table de billard forment le rectangle $WXYZ$ comme dans le diagramme ci-dessous. La bande WZ a cinq pieds de long. La bande WX a dix pieds de long. Une boule rouge (R) est située à un pied de YZ et à deux pieds de WZ. Une boule bleue (B) est située à un pied de WX et à un pied de XY. Nous voulons frapper la boule bleue pour qu'elle frappe la bande YZ, avec l'angle d'incidence égal à l'angle de réflexion, pour ensuite frapper la boule rouge. Quel point sur la bande YZ devons-nous viser ?



3. Trois étudiants jouent un jeu où le perdant de chaque partie doit doubler l'argent de chacun des deux autres joueurs. Après trois parties, chaque joueur a perdu une fois et possède \$24. Combien d'argent possédait chaque étudiant au début du jeu ?

4. Quels sont les entiers a pour lesquels l'équation $x^3 - 13x + a = 0$ possède trois racines entières ?

5. Dans le triangle ABC, l'angle A est droit. Soit x la longueur du côté AB, y celle de AC. Soit D le point de BC tel que $\angle DAC = 30°$. Trouver, en termes de x et y, la longueur de AD.

6. Évaluer la somme suivante.

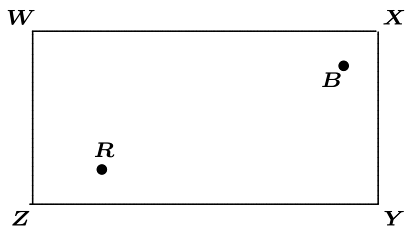
$$\frac{1}{1^4 + 1^2 + 1} + \frac{2}{2^4 + 2^2 + 1} + \cdots + \frac{2005}{2005^4 + 2005^2 + 1}\,.$$

2005 Maritime Mathematics Competition

March 3, 2005

1. A gardener owns a riding lawn mower and a push mower. It takes her 3 hours to cut the entire lawn with the push mower but only 75 minutes with the riding mower. One particular day, she cuts a portion of the lawn with the push mower and the rest with the riding lawn mower. If the total time to mow the lawn was 96 minutes, what fraction of the lawn was cut with the riding mower?

2. Suppose that W, X, Y, and Z are the corners of a rectangular pool table as shown in the diagram below. Side WZ is 5 feet in length and side WX is 10 feet long. A red ball is placed 1 foot from side YZ and 2 feet from side WZ, and a blue ball is placed 1 foot from both the sides WX and XY. Suppose that the blue ball is shot towards the side YZ where it will bounce off the edge, with angle of incidence equal to angle of reflection. At what point on the side YZ should the blue ball strike if it is to hit the red ball upon rebounding?



3. Three students play a game with the understanding that the loser is to double the money of each of the other two. After three games, each has lost once and each has \$24. How much did each student have to start?

4. Find all integers a for which the equation $x^3 - 13x + a = 0$ has three integer roots.

5. Triangle ABC is right-angled at A. Let x and y denote the lengths of the sides AB and AC, respectively. Suppose that the point D on BC is such that $\angle DAC = 30^\circ$. Determine the length of AD in terms of x and y.

6. Evaluate the following sum.

$$\frac{1}{1^4+1^2+1} + \frac{2}{2^4+2^2+1} + \cdots + \frac{2005}{2005^4+2005^2+1}\,.$$

Next we give the solutions to the second batch of problems from the 4th annual CNU Regional High School Mathematics Contest [2005 : 193-196].

4th Annual CNU Regional High School Mathematics Contest
Saturday December 6, 2003

17. The number of positive integer solutions for $2(x+y) = xy + 7$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution by Geoffrey Siu, student, London Central Secondary School, London, ON.

The given equation is equivalent to $xy - 2x - 2y + 7 = 0$, or

$$(x-2)(y-2) \;=\; -3.$$

For integer solutions, we need $x-2$ and $y-2$ to be integer factors of -3. Thus, one factor is ± 1 and the other is ∓ 3. Since we are seeking positive solutions, we must have one factor equal to -1 and the other equal to 3. Hence, there are two positive integer solutions, namely $(1,5)$ and $(5,1)$.

20. The minimum of $S = x^2 + 2xy + 3y^2 + 2x + 6y + 4$ is

(A) 4 (B) 1 (C) 0 (D) -1

Solution by the editor.

Completing the square, we get

$$S \;=\; x^2 + 2x(y+1) + 3(y+1)^2 + 1 \;=\; (x+y+1)^2 + 2(y+1)^2 + 1\,.$$

Hence, S is the sum of non-negative quantities (because of the squares). We will have a minimum when the squares are 0; that is, when $x+y+1 = 0$ and $y+1 = 0$. Solving these equations gives $y = -1$ (from the second equation) and $x = 0$ (substituting into the first equation). Thus, the minimum value of S is 1 (when $x = 0$ and $y = -1$).

Also solved by Geoffrey Siu, student, London Central Secondary School, London, ON.

22. If $3\sin\theta + 4\cos\theta = 5$, then $\tan\theta$ is

(A) 1 (B) -1 (C) $\frac{3}{4}$ (D) $\frac{4}{3}$

Solution by the editor.

The equation is equivalent to $\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta = 1$. Squaring this yields $\frac{9}{25}\sin^2\theta + \frac{16}{25}\cos^2\theta + \frac{24}{25}\sin\theta\cos\theta = 1$. Combining with $\sin^2\theta + \cos^2\theta = 1$, we get $\frac{16}{25}\sin^2\theta + \frac{9}{25}\cos^2\theta - \frac{24}{25}\sin\theta\cos\theta = 0$, or $(\frac{4}{5}\sin\theta - \frac{3}{5}\cos\theta)^2 = 0$. From this, it follows that $\tan\theta = \frac{3}{4}$.

Also solved by Geoffrey Siu, student, London Central Secondary School, London, ON.

24. Si $f(x+y) = f(xy)$ et $f(7) = 7$, alors $f(49) =$

(A) 49 (B) 14 (C) 7 (D) 1

I. *Solution par Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

Puisque $x + y \neq xy$, il faut que la fonction soit constante pour que $f(x+y) = f(xy)$ peu importe la valeur de x et de y. En sachant que $f(7) = 7$, nous savons que $f(49) = 7$.

II. *Solution by Geoffrey Siu, student, London Central Secondary School, London, ON.*

Let $y = 1$. Then $f(x+1) = f(x)$. Hence, by induction, $f(n) = f(7)$ for $n \geq 7$ and $f(49) = 7$.

26. Soit $a = 1! + 2! + 3! + \cdots + 2003!$. Le chiffre des unités de a vaut

(A) 9 (B) 5 (C) 3 (D) 0

Solution par Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.

À partir de 5, toutes les valeurs de $x!$ se termineront par 0 puisqu'elles vont toutes contenir 5×2 dans leur développement $(1 \times \underline{2} \times 3 \times 4 \times \underline{5} \times \ldots)$. Ainsi, la seule chose que l'on a à calculer est

$$1! + 2! + 3! + 4! \;=\; 1 + 2 + 6 + 24 \;=\; 33\,.$$

La réponse est 3.

Solutioné aussi par Geoffrey Siu, student, London Central Secondary School, London, ON.

27. Find the sum of all the positive integers less than 45 that are not divisible by 3.

(A) 600 (B) 625 (C) 650 (D) 675

Identical solutions by Carl O'Connor, student, Collège Montmorency, Laval, QC; and Geoffrey Siu, student, London Central Secondary School, London, ON.

We find the sum of positive integers less than or equal to 45 and subtract the sum of multiples of 3 between 0 and 45 (inclusive):

$$\frac{45(46)}{2} - 3\,\frac{15(16)}{2} \;=\; 1035 - 360 \;=\; 675\,.$$

28. Quel est le plus petit entier k tel que $2x(kx-4) - x^2 + 6 = 0$ n'ait pas de solution réelle ?

(A) -1 (B) 2 (C) 3 (D) 4

Une solution identique soumise par Carl O'Connor, étudiant, Collège Montmorency, Laval, QC; et Geoffrey Siu, étudiant, London Central Secondary School, London, ON.

Écrivons cette équation comme $(2k - 1)x^2 - 8x + 6 = 0$. Pour que cette équation n'ait pas de solution réelle, il faut que $\Delta = b^2 - 4ac < 0$. Il faut donc que $(-8)^2 - (4)(2k-1)(6) < 0$. Si nous isolons k, nous arrivons à $k > \frac{11}{6}$. Le plus petit entier supérieur à $\frac{11}{6}$ est 2.

29. The number of distinct positive integer factors of 30^4 is

(A) 100 (B) 125 (C) 123 (D) 30

Identical solutions by Geoffrey Siu, student, London Central Secondary School, London, ON; and Carl O'Connor, student, Collège Montmorency, Laval, QC.

Note that $30 = 2 \times 3 \times 5$. Any factor of 30^4 must contain only the three prime factors of 30, each repeated 0, 1, 2, 3, or 4 times (as in the Pólya's Paragon column [2005 : 146–147]), for a total of $5 \times 5 \times 5 = 125$ factors.

30. La somme des racines de $f(x) = x(2x+3)(4x+5) + (6x+7)(8x+9)$ est

(A) $\frac{-35}{4}$ (B) $\frac{35}{4}$ (C) -70 (D) 70

Solution par Geoffrey Siu, étudiant, London Central Secondary School, London, ON.

Si les racines d'une cubique $f(x)$ avec premier coefficient 1 sont a, b et c, alors on a :

$$f(x) = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc\,.$$

Ici, en développant, on a

$$f(x) = 8x^3 + 70x^2 + 125x + 63 = 8\left(x^3 + \tfrac{35}{4}x^2 + \tfrac{125}{8}x + \tfrac{63}{8}\right)\,.$$

La somme des racines est $\frac{-35}{4}$.

32. Une ligne L a une pente de -2 et passe par le point $(r, -3)$. Une seconde ligne K, perpendiculaire à L en (a, b), passe par le point $(6, r)$. Le valeur de a est

(A) r (B) $\frac{2r}{5}$ (C) $2r - 3$ (D) 1

Solution par Geoffrey Siu, étudiant, London Central Secondary School, London, ON.

La ligne L a l'équation $y = -2x + b$. En substituant le point $(r, -3)$, on obtient $b = 2r - 3$. Donc L a l'équation $y = -2x + 2r - 3$.

La ligne K est perpendiculaire à L, donc sa pente est $\frac{1}{2}$, et son équation est $y = \frac{1}{2}x + k$. En substituant le point $(6, r)$, on obtient $k = r - 3$. Ainsi, K a l'équation $y = \frac{1}{2}x + r - 3$.

Au point (a, b), on a $b = -2a + 2r - 3$ (puisque L passe par (a, b)) et $b = \frac{1}{2}a + r - 3$ (puisque K passe par (a, b)). Donc, $-2a + 2r - 3 = \frac{1}{2}a + r - 3$, ou bien $a = \frac{2r}{5}$.

Une solution incorrecte a été soumise.

38. A box contains 11 balls, numbered 1, 2, ..., 11. If 6 balls are drawn simultaneously at random, what is the probability that the sum of the numbers drawn is odd?

(A) $\frac{100}{231}$ (B) $\frac{115}{231}$ (C) $\frac{1}{2}$ (D) $\frac{118}{231}$

Solution by Geoffrey Siu, student, London Central Secondary School, London, ON.

Since we must draw an odd number of odd-numbered balls in order to get an odd sum, we must draw 1 odd-numbered ball (and 5 even), 3 odd (and 3 even), or 5 odd (and 1 even).

There are 6 odd-numbered balls and 5 even-numbered balls. Hence, there are

$$\binom{6}{1}\binom{5}{5} + \binom{6}{3}\binom{5}{3} + \binom{6}{5}\binom{5}{1} = 236$$

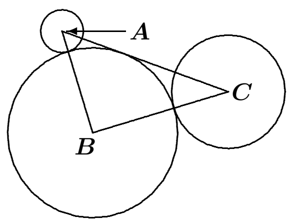
ways to get an odd sum. In total, there $\binom{11}{6} = 462$ ways of choosing 6 balls. Thus, the probability is $\frac{236}{462} = \frac{118}{231}$.

3. (Questions en équipe) Trouver le maximum de $f(x) = \left(\frac{1}{2}\right)^{x^2-2x}$.

Solution par Geoffrey Siu, étudiant, London Central Secondary School, London, ON.

On sait que $g(x) = \left(\frac{1}{2}\right)^x$ est une fonction décroissante, donc on veut le minimum de $x^2 - 2x = (x-1)^2 - 1$. Le minimum de $x^2 - 2x$ est donc -1 et le maximum de $f(x) = \left(\frac{1}{2}\right)^{x^2-2x}$ est $f(1) = \left(\frac{1}{2}\right)^{1^2-2} = 2$.

10. (Team Round) In the diagram, the circle with centre A has radius 1, and the big circle with centre B has radius 4. The third circle has centre C, and the big circle touches the two other circles. Also, $\angle ABC$ is a right angle and the line AC touches the big circle. Find the radius of the circle with centre C.



Solution by Geoffrey Siu, student, London Central Secondary School, London, ON.

Let D be the of tangency of AC with the big circle. Then $\triangle ABD$ is a right triangle, since AC is tangent to the big circle. By the Pythagorean Theorem, $AD = \sqrt{5^2 - 4^2} = 3$.

Clearly $\triangle ABD$ is similar to $\triangle ACB$. Hence,

$$BC = \frac{AB^2}{AD} = \frac{25}{3}.$$

If E is the point of tangency between the big circle and the circle centred at C, then E lies on BC and $EC = BC - 4 = \frac{13}{3}$ is the radius of the circle centred at C.

11. (Questions en équipe) Un homme voyage en automobile à une vitesse moyenne de 50 miles par heure. Il revient par le même chemin à une vitesse moyenne de 30 miles par heure. Quelle est la vitesse moyenne pour le voyage ?

I. *Solution par Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

Soit x la disance en miles et y le temps de route en heure (pour aller seulement). Ainsi, $\frac{x}{y}$ est la vitese moyenne. Donc, on a $\frac{x}{50} + \frac{x}{30} = 2y$, puisqu'il a fait 2 fois la distance x (une fois à 50 miles/h et l'autre à 30 miles/h). Puisque $30x + 50x = 300y$, donc $x = 37,5y$ ou bien $\frac{x}{y} = 37,5$. La vitesse moyenne est donc 37,5 miles par heure.

II. *Solution by Geoffrey Siu, student, London Central Secondary School, London, ON.*

Let the length of a one way trip be x miles. Then the time (in hours) for the trip out is $\frac{x}{50}$ and the time for the trip back is $\frac{x}{30}$. His average speed is

$$\frac{d_{\text{total}}}{t_{\text{total}}} = \frac{2x}{(x/50) + (x/30)} = 37.5 \text{ miles/h}.$$

12. (Questions en équipe) Résoudre algébriquement pour x :

$$\left(\log_{10} x^2\right)^2 = \log_{10}(x^4).$$

Solution par Geoffrey Siu, étudiant, London Central Secondary School, London, ON.

Écrivons l'équation comme $\left(\log_{10} x^2\right)^2 = 2\log_{10} x^2$. Soit $a = \log_{10} x^2$. On obtient que $a^2 = 2a$, ou bien $a = 0$ ou $a = 2$. Ce qui nous donne $x = \pm 1$ ou $x = \pm 10$.

Une solution incomplète a été soumise.

That brings us to the end of another issue, and another year.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier mai 2006**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

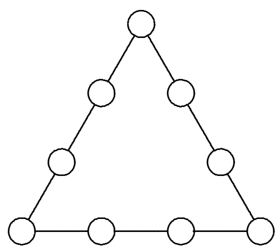
Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M219. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

Écrire chacun des nombres 1, 2, 3, 4, 5, 6, 7, 8, 9 dans exactement un des cercles de telle sorte que :

1. les sommes des quatre nombres sur chaque côté du triangle soient égales ; et
2. les sommes des carrés des quatre nombres sur chaque côté du triangle soient égales.



M220. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

Montrer comment numéroter les faces d'un dé octaédrique avec les nombres de 1 à 8 de telle sorte que la somme des nombres sur les quatre faces aboutissant à chacun des sommets soit toujours la même.

M221. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Montrer qu'un carré de 5×5 peut être recouvert par trois carrés de 4×4.

M222. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

On suppose que les côtés et l'hypoténuse d'un triangle rectangle mesurent respectivement $30a + 40b$, $40a + 30b$ et $50a + kb$, où a, b et k sont des entiers positifs. Trouver les plus petites valeurs possibles pour a, b et k.

M223. *Proposé par Larry Rice, Université de Waterloo, Waterloo, ON.*

Il y a exactement deux façons d'écrire la fraction $\frac{3}{10}$ comme la somme de deux nombres rationnels positifs de numérateur 1, à savoir comme $\frac{1}{10} + \frac{1}{5}$ ou $\frac{1}{20} + \frac{1}{4}$.

Déterminer de combien de façons on peut écrire $\frac{3}{2006}$ comme somme de deux nombres rationnels positifs et de numérateur 1.

M224. *Proposé par Robert Bilinski, Collège Montmorency, Laval, QC.*

Soit $A(-1, 1)$ et $B(3, 9)$ deux points sur la parabole d'équation $y = x^2$. Choisisons un autre point $M(m, m^2)$ sur la parabole, situé entre A et B. Soit H le point sur le segment joignant A à B ayant la même abscisse que M.

Montrer que si la longueur de MH est k unités, alors le triangle AMB a une aire de $2k$ unités-carré. Cette relation est-elle encore valable si M n'est pas entre A and B ?

M225. *Proposé par Zun Shan, Normal University, Chine, et Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

On considère la suite $\{x_n\}$ définie par $x_1 = 1/2005$ et $x_{n+1} = x_n + x_n^2$ pour $n \geq 1$. Poser

$$S = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{2005}}{x_{2006}}.$$

Déterminer $\lfloor S \rfloor$, la partie entière de S.

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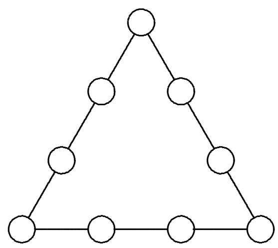
M219. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Place each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in exactly one of the circles in such a way that:

1. the sums of the four numbers on each side of the triangle are equal; and
2. the sums of the squares of the four numbers on each side of the triangle are equal.



M220. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Show how to number the faces of an octahedral die using the numbers 1 though 8 in such a way that the sum of the numbers on the four faces joining at each vertex is always the same.

M221. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Prove that a 5×5 square can be covered by three 4×4 squares.

M222. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Suppose that $30a + 40b$ and $40a + 30b$ are the sides of a right triangle and that $50a + kb$ is the hypotenuse, where a, b, and k are positive integers. Find the smallest possible values of a, b, and k.

M223. *Proposed by Larry Rice, University of Waterloo, Waterloo, ON.*

The fraction $\frac{3}{10}$ can be written as the sum of two positive rational numbers with numerator 1 in exactly two ways, namely as $\frac{1}{10} + \frac{1}{5}$ and $\frac{1}{20} + \frac{1}{4}$.

Determine the number of ways that $\frac{3}{2006}$ can be expressed as the sum of two positive rational numbers with numerator 1.

M224. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

Let $A(-1,1)$ and $B(3,9)$ be two points on the parabola $y = x^2$. Take another point $M(m, m^2)$ on the parabola lying between A and B. Let H be the point on the line segment joining A to B that has the same x–coordinate as M.

Show that if the length of MH is k units, then triangle AMB has area $2k$ square units. Does this relationship still hold if M is not between A and B?

M225. *Proposed by Zun Shan, Normal University, China; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Define a sequence $\{x_n\}$ by $x_1 = 1/2005$ and $x_{n+1} = x_n + x_n^2$ for $n \geq 1$. Set

$$S = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{2005}}{x_{2006}}.$$

Determine $\lfloor S \rfloor$, the greatest integer not exceeding S.

Mayhem Solutions

M151. (Revisited) *Proposed by Babis Stergiou, Chalkida, Greece.*

Let a, b, c be real numbers with $abc = 1$. Prove that

$$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \ \geq \ 2(a^2b + b^2c + c^2a)\,.$$

[*Ed.* Vedula N. Murty, Dover, PA, USA has observed that the inequality is incorrect. His counter-example is $a = 2$, $b = -1/2$, $c = -1$. To prove the inequality, it must be assumed that a, b, and c are positive.]

Solution by Vedula N. Murty, Dover, PA, USA.

Assume $a \geq b \geq c > 0$. Then

$$a^3+b^3+c^3-(a^2b+b^2c+c^2a) \ = \ (a-b)(a^2-c^2)+(b-c)(b^2-c^2) \ \geq \ 0\,,$$

and hence,

$$a^3 + b^3 + c^3 \ \geq \ a^2b + b^2c + c^2a\,. \qquad (1)$$

Since $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a} > 0$, we can substitute $\frac{1}{c}$ for a, $\frac{1}{b}$ for b, and $\frac{1}{a}$ for c in (1) to obtain

$$\frac{1}{c^3} + \frac{1}{b^3} + \frac{1}{a^3} \ \geq \ \frac{1}{c^2b} + \frac{1}{b^2a} + \frac{1}{a^2c}\,.$$

Since $abc = 1$, this can be written as

$$(bc)^3 + (ca)^3 + (ab)^3 \ \geq \ a^2b + b^2c + c^2a\,. \qquad (2)$$

Adding (1) and (2), we obtain the proposed inequality.

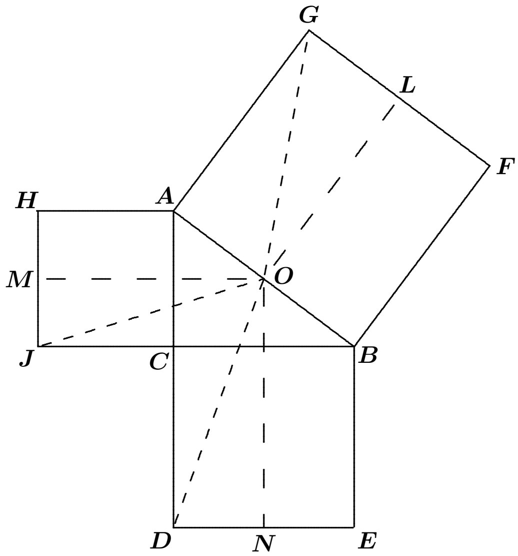
Also solved by Arkady Alt, San Jose, CA, USA; Mihály Bencze, Brasov, Romania; and Peter Smoczynski, Thompson Rivers University, Kamloops, BC.

M152. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A right-angled triangle has the property that, when a square is drawn externally on each side of the triangle, the six vertices of the squares that are not vertices of the triangle are concyclic. Characterize such triangles.

Solution by the proposer.

Since the six outer vertices of the squares are concyclic, the outer sides of the squares are chords of the same circle Γ. The perpendicular bisectors of these sides are therefore concurrent. (They are also the perpendicular bisectors of the sides of $\triangle ABC$; hence, concur at the circumcentre O of $\triangle ABC$.)



We must also have $OJ = OD = OG$, since each is a radius of Γ. Let the sides of $\triangle ABC$ be a, b, and c, as usual. Then,

$$\begin{aligned} OJ^2 &= OM^2 + MJ^2 = \left(b + \frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 \\ &= b^2 + ab + \tfrac{1}{4}(a^2 + b^2)\,, \\ OD^2 &= ON^2 + ND^2 = \left(a + \frac{b}{2}\right)^2 + \left(\frac{a}{2}\right)^2 \\ &= a^2 + ab + \tfrac{1}{4}(a^2 + b^2)\,, \\ \text{and} \quad OG^2 &= OL^2 + LG^2 = c^2 + \left(\frac{c}{2}\right)^2 \\ &= \tfrac{5}{4}c^2 = \tfrac{5}{4}(a^2 + b^2) = a^2 + b^2 + \tfrac{1}{4}(a^2 + b^2)\,. \end{aligned}$$

Setting these equal, we get

$$b^2 + ab = a^2 + ab = a^2 + b^2\,.$$

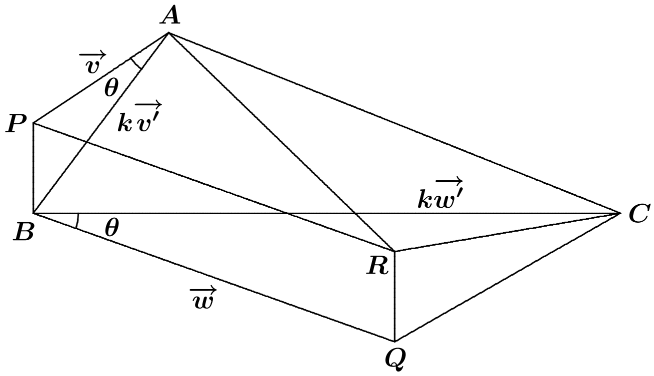
This is true if and only if $a = b$. Thus, the right triangles with the desired property are precisely the isosceles right triangles.

M153. Correction. *Proposed by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Two similar triangles APB and BQC are erected externally on a triangle ABC. If R is a point such that $PBQR$ is a parallelogram, show that triangles ARC and APB are similar.

[*Ed:* In the original problem, the word "similar" was mistyped as "congruent".]

Solution by Geoffrey A. Kandall, Hamden, CT, USA.



Let $\theta = \angle PAB = \angle QBC$, and let $X \mapsto X'$ be the linear transformation that rotates each vector counterclockwise by an angle θ while preserving its length.

Let $\overrightarrow{v} = \overrightarrow{AP}$ and $\overrightarrow{w} = \overrightarrow{BQ}$. Then there exists $k > 0$ such that $\overrightarrow{AB} = k\overrightarrow{v'}$ and $\overrightarrow{BC} = k\overrightarrow{w'}$. We then have $\overrightarrow{AR} = \overrightarrow{AP} + \overrightarrow{PR} = \overrightarrow{v} + \overrightarrow{w}$ and $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = k\overrightarrow{v'} + k\overrightarrow{w'} = k\overrightarrow{AR'}$. Hence, $\triangle ARC$ is similar to $\triangle APB$ with ratio of similitude $\lambda = AC : AB$. These triangles are congruent if and only if $\lambda = 1$; that is, if and only $AB = AC$.

Also solved by Houda Anoun, Bordeaux, France; and Robert Bilinski, Collège Montmorency, Laval, QC.

M154. *Proposed by the Mayhem Staff.*

Eight rooks are placed randomly on different squares of a chessboard. What is the probability that none of the rooks is under attack by another rook?

Solution by Geneviève Lalonde, Massey, ON.

There are $\binom{64}{8}$ ways to place 8 rooks. If we want to place them such that none is under attack by another, we must have one in each row and column. The number of ways to do that can be determined in various ways. Here are two different ways.

1. There are 64 squares on which the first rook can be placed. The next rook has to be located in the region formed by removing the row and column occupied by the first rook. A little checking shows that there are $49 = 7^2$ choices for the second rook, then $36 = 6^2$ for the third, and so on. Thus, the total number of ways to place them is $64 \cdot 49 \cdots 4 \cdot 1 = (8!)^2$. But we have counted the arrangements of rooks as though the rooks were distinguishable, whereas, in fact, they are indistinguishable. Therefore, we must divide our total by 8!, the number of permutations of the 8 rooks. The total number of arrangements is thus $(8!)^2/8! = 8!$.

2. We can fill the columns of the board sequentially. We have 8 choices of spots for a rook in the first column, then 7 spots in the second column (since we must avoid the row occupied by the rook in the first column), etc. Thus, there are 8! ways to place the rooks.

Now, the probability that no rook will be under attack is $8!/\binom{64}{8}$.

Also solved by Houda Anoun, Bordeaux, France; and Roger He, student, Prince of Wales Collegiate, St. John's, NL. One incorrect solution was received.

M155. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine the cubic $x^3 + px + q = 0$, given that

(i) it has one repeated root, and

(ii) p and q are integers such that q is the smallest permissible positive integral multiple of p.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

Let a be the repeated root of the cubic, and let c be the other root. Then

$$x^3 + px + q \;=\; (x-a)^2(x-c) \;=\; x^3 - (2a+c)x^2 + (2ac+a^2)x - a^2c\,.$$

Then $2a + c = 0$, $2ac + a^2 = p$, and $-a^2c = q$. These three equations imply that $c = -2a$, $p = -3a^2$, and $q = 2a^3$. We have $q = -\frac{2}{3}ap = kp$, where k is a positive integer such that $q = kp$ is the smallest permissible positive integral multiple of p. Since $-2a = 3k$, we see that a is an integral multiple of $\frac{1}{2}$, which together with $p = -3a^2$ implies that a is an integer. Then the smallest permissible positive integral value of k is 2, which implies that $a = -3$, $p = -27$, and $q = -54$. Thus, $x^3 - 27x - 54$ is the cubic polynomial.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; Peter Smoczynski, Thompson Rivers University, Kamloops, BC; and Marcie Fairchild, Daniel Mills, Laura Steil, and Willie Ward, students, Samford University, Birmingham, AL, USA.

M156. *Proposed by the Mayhem Staff.*

Solve for x where $0 \le x < 2\pi$:

$$2^{1+3\cos x} - 10(2)^{-1+2\cos x} + 2^{2+\cos x} - 1 \;=\; 0\,.$$

Solution by Marcie Fairchild, Daniel Mills, Laura Steil, and Willie Ward, students, Samford University, Birmingham, AL, USA.

Let $y = 2^{\cos x}$. Then the equation becomes $2y^3 - 5y^2 + 4y - 1 = 0$. Factoring gives $(y-1)^2(2y-1) = 0$. Thus, $2^{\cos x} = 1$ or $2^{\cos x} = \frac{1}{2}$. Then $\cos x = 0$ or $\cos x = -1$. Thus, the solutions to the equation are $x = \frac{\pi}{2}$, $x = \pi$, and $x = \frac{3\pi}{2}$.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; and James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

M157. *Proposed by Neven Jurič, Zagreb, Croatia.*

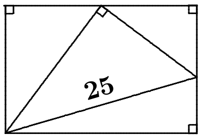
The formulas $a = x^2 - y^2$, $b = 2xy$, $c = x^2 + y^2$ are known to be useful for producing integer solutions of the equation $a^2+b^2 = c^2$. Are there similar formulas for integer solutions of the equation $a^2 + ab + b^2 = c^2$?

Solution by the proposer.

It can be verified that $a = x^2 - y^2$, $b = y^2 + 2xy$ and $c = x^2 + xy + y^2$ satisfy the equation. Furthermore, if x and y are integers, then a, b, and c are also integers.

M158. *Proposed by K.R.S. Sastry, Bangalore, India.*

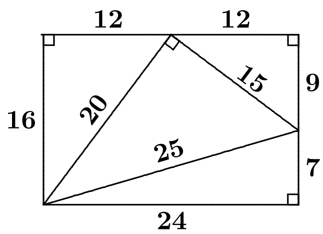
(a) Determine the lengths of the sides of the various right triangles in the figure, given that the lengths are integers.



(b) Find an integer-valued expression in place of the number 25 so that an infinite number of different rectangles may be generated, each with a right triangle inscribed as shown, such that the lengths of the sides of the various right triangles are integers.

Solution to part (a) by Robert Bilinski, Collège Montmorency, Laval, QC.

(a) Since $25^2 = 20^2 + 15^2 = 7^2 + 24^2$, we have the sides of the two right triangles with hypotenuse 25. Now, $15^2 = 9^2 + 12^2$ and $20^2 = 12^2 + 16^2$, while neither 24^2 nor 7^2 can be written as a sum of squares of any two integers. Hence, 7 and 24 must be the sides of the triangle which has its right angle in the bottom right corner in the figure. The rest of the configuration follows, as shown in the figure. The width of the rectangle is 24 and the height is 16.

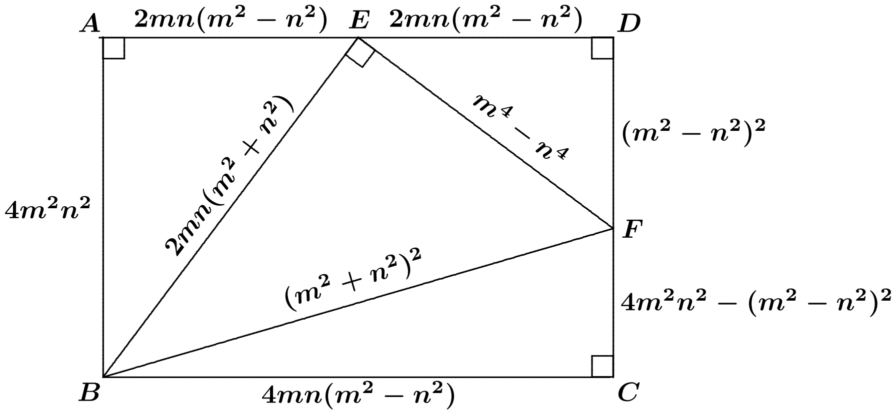


Solution to part (b) by the proposer.

(b) We label points A, B, C, D, E, F as shown in the figure below. For natural numbers m and n with $m > n > 0$ and $2mn > m^2 - n^2$, we set

$$AB = 4m^2n^2\,, \qquad BF = (m^2+n^2)^2\,, \qquad BC = 4mn(m^2-n^2)\,,$$
$$CF = 4m^2n^2 - (m^2-n^2)^2 = (2mn + m^2 - n^2)(2mn - m^2 + n^2)\,,$$
$$FD = (m^2-n^2)^2\,, \qquad DE = EA = 2mn(m^2-n^2)\,.$$

Then $EF = (m^2 - n^2)(m^2 + n^2) = m^4 - n^4$ and $BE = 2mn(m^2 - n^2)$.



M159. *Proposé par l'Équipe de Mayhem.*

Est-il possible d'arranger les nombres 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 le long d'un cercle de façon que la somme de deux voisins quelconques soit un nombre premier ?

Solution par Robert Bilinski, Collège Montmorency, Laval, QC.

Oui, on trouve les cycles

$$2,\ 5,\ 8,\ 3,\ 10,\ 1,\ 6,\ 7,\ 4,\ 9\,;$$
$$2,\ 5,\ 8,\ 9,\ 10,\ 1,\ 6,\ 7,\ 4,\ 3\,;$$
$$2,\ 5,\ 8,\ 3,\ 4,\ 1,\ 6,\ 7,\ 10,\ 9\,;$$
$$2,\ 5,\ 8,\ 3,\ 10,\ 7,\ 6,\ 1,\ 4,\ 9\,;$$
$$2,\ 5,\ 8,\ 9,\ 4,\ 1,\ 6,\ 7,\ 10,\ 3.$$

On y parvient par une étude des sommes possibles à partir des 10 nombres permis et de leur implication par rapport aux positions relatives de ces nombres. Par exemple, on sait que

$$1+2 = 3\,, \qquad 1+4 = 5\,, \qquad 1+6 = 7\,, \qquad 1+10 = 11\,,$$

indiquent que 1 peut être à côté de 2, 4, 6 ou 10.

M160. *Proposé par l'Équipe de Mayhem.*

Comme stratège, Napoléon était fier de son armée qu'il voulait très disciplinée, entraînée et mobile. Dans une de ses batailles contre les Alliés, il décida de varier l'arrangement de ses fantassins. Au début, ceux-ci furent groupés en 540 rangées égales ; puis il les groupa en 105 rangées égales, pour finalement les grouper en 216 rangées égales. Le nombre de soldats était le plus petit nombre rendant possibles de tels arrangements.

Combien y en avait-il ?

Solution par Robert Bilinski, Collège Montmorency, Laval, QC.

Soit n le nombre de soldats dans l'armée de Napoléon. Ainsi, il existe des nombres a, b et c tels que :

$$n = \begin{cases} 540a &= 2^2 \times 3^3 \times 5 \times a \\ 105b &= 3 \times 5 \times 7 \times b \\ 216c &= 2^3 \times 3^3 \times c \end{cases}$$

Puisque n est minimal, on prend $n = 2^3 \times 3^3 \times 5 \times 7 = 7560$ soldats.

M161. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Triangle ABC is isosceles, with $AB = AC$ and $BC = 1$. On the sides AB and AC are points P and Q, respectively, such that $PQ \parallel BC$ and the distance from PQ to BC is 1. On the segments AP and AQ are points R and S, respectively, such that $RS \parallel PQ$.

If $PQ/RS = (1+\sqrt{5})/2$ (the golden ratio), find the area of the trapezoid $PQSR$.

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

We need the height of the triangle to answer this question numerically. Let h_{ABC}, h_{APQ}, h_{ARS}, and h_{PQRS} be the heights of triangles ABC, APQ, ARS, and trapezoid $PQRS$, respectively. If $h_{ABC} = h$, then $h_{APQ} = h-1$. Using similar triangles, we get

$$PQ = \frac{h-1}{h} \qquad \text{and} \qquad RS = \frac{2(h-1)}{h(1+\sqrt{5})} .$$

Now $h_{ARS} = \dfrac{2(h-1)}{1+\sqrt{5}}$ and

$$h_{PQRS} = h_{APQ} - h_{ARS} = \frac{(h-1)(\sqrt{5}-1)}{1+\sqrt{5}} .$$

Thus,

$$\begin{aligned} [PQRS] &= \frac{1}{2} \cdot h_{PQRS} \cdot (PQ+RS) \\ &= \frac{1}{2} \cdot \frac{(h-1)(\sqrt{5}-1)}{1+\sqrt{5}} \cdot \left(\frac{h-1}{h} + \frac{2(h-1)}{h(1+\sqrt{5})} \right) \\ &= \frac{(h-1)^2(3+\sqrt{5})(\sqrt{5}-1)}{2h(1+\sqrt{5})^2} = \frac{(h-1)^2}{h(1+\sqrt{5})} . \end{aligned}$$

M162. *Proposed by Neven Jurič, Zagreb, Croatia.*

Rods 1 m in length are used to build a rigid cubic framework. Twelve rods are needed to build a cube of side 1 m. By fitting together 8 of these unit cubes, a cubic framework can be constructed that has side 2 m. By using 27 of the unit cubes, a cubic framework can be constructed that has side 3 m.

Each time the unit cubes are combined, there are duplicate rods along the edges where the cubes fit together. The duplicates may be removed (leaving one rod where there were previously two) and re-used elsewhere.

How large a cubic framework can be created in the manner described above if the total length of the available rods would connect the earth to the moon, assuming a distance of 384 000 km between them?

Solution by the proposer.

Suppose that there are n rods in each row and column of each layer of the cubic framework. Then each layer contains $2n(n+1)$ rods. There are $n+1$ such layers. Combined, the $n+1$ layers contain $2n(n+1)^2$ rods. There are $(n+1)^2$ rods between consecutive layers, and there are n such inter-layer spaces. Hence, together they contain a total of $n(n+1)^2$ rods. Therefore, the number of metre-long rods in the entire cubic framework is

$$2n(n+1)^2 + n(n+1)^2 = 3n(n+1)^2 \,.$$

The desired greatest cubic framework that can be formed from these rods has an edge n_{max} metres long, where n_{max} is the greatest n which satisfies the condition $3n(n+1)^2 \leq 384\,000\,000$.

Checking, we find that:

$$\begin{array}{rrcl} & 3 \times 503 \times 504^2 & < & 384\,000\,000 \\ \text{and} & 3 \times 504 \times 505^2 & > & 384\,000\,000\,. \end{array}$$

Thus, $n_{\text{max}} = 503$, and the edge of the required framework is 503 m.

Problem of the Month

Ian VanderBurgh, University of Waterloo

This month, we have our second annual two-for-one holiday special. The two problems that we will consider both relate to neighbours, a good holiday theme. I have left the first problem in its original multiple choice format to demonstrate an important point about solving multiple choice problems. For the second problem, I have withheld the multiple choice possibilities in a Scrooge-like manner.

Problem 1. (1995 Australian Mathematics Competition, Junior Division)

We wish to make a string of letters which contains every possible three-letter combination of Ps and Qs somewhere (that is, it must contain all these: PPP, PPQ, PQP, PQQ, QPP, QPQ, QQP and QQQ). An example of such a string, of length 18, is $PPPPQPQQPPQPQQPQQQ$. What is the length of the shortest such string?

(A) 8 (B) 10 (C) 12 (D) 16 (E) 18

Solution. The best way to start this type of problem is to fiddle around for a few minutes and see what you can come up with. Suppose we try starting with $PPPQQQ$. (Why this starting place? I tried this because I knew that PPP and QQQ had to appear somewhere, and I tried to get them out of the way early.) This takes care of 4 of the three-letter combinations, namely, PPP, PPQ, PQQ, QQQ. Take a minute and try to extend this string to obtain the remaining four combinations using as few letters as possible.

How did you do? I used 11 letters in total – $PPPQQQPPQPQ$. Does this help us answer the original question? Yes, it tells us that the answer cannot be 12, 16, or 18, since we have found a string of length 11 which works. Hence, the answer is either 8 or 10. (Eliminating answers is very helpful when trying to answer multiple choice questions.)

If you look at the three-letter combinations in the string I found, you will see that one combination is actually used twice, namely PPQ. This suggests that a shorter string might be possible.

Now we will try to figure out how short such a string could possibly be. The strings we are looking for must have all 8 different three-letter combinations. In the best-case scenario, these 8 different combinations would begin in positions 1 through 8. The three-letter combination starting in position 8 would occupy positions 8, 9, and 10; thus, the string must be at least 10 letters long. This tells us that 8 cannot be the minimum.

Thus, the answer to the original question must be 10, since we have eliminated all of the other possible answers. We have answered the question without actually finding a string of length 10 which works! This is a pretty powerful idea: in multiple choice contests, one can sometimes obtain the correct answer to the question without actually answering the question.

That being said, it is probably a good idea to try to find such a string of length 10. (We would have had to do this if the problem had not been multiple choice.) One such string is $PPPQPQQQPP$. Can you find others?

Problem 2. (1995 Australian Mathematics Competition, Junior Division)

A class at school contains 19 students who stand in a line next to one another each morning at assembly. The students decide that they would like to stand beside different classmates each morning, so that any pair who stand next to each other one morning will not stand next to each other on the next morning or any morning after that. What is the maximum number of days for which this can happen?

Solution. Here we are asked for a maximum. In a similar way to the previous problem, we need to find a large number of days that does work, and explain why we can't find a larger number of days that also works. We will initially try to get a sense of the range where the answer is likely to be found.

Focus on one of the students (call her Melissa). On any given day, if Melissa is not at the end of the line, she will have 2 student neighbours. (On any given day, 17 of the 19 students are not at either end of the line; thus, it is reasonable to consider this case.) Since there are 19 students, Melissa can have 18 different neighbours, each at most once. Thus, the maximum is probably around 9 days (2 different neighbours on each of 9 days gives 18 neighbours).

After day 9 there will have to be at least one student who has never been on the end of the line, because at most 18 people (2 per day) can have been on the end, and there are 19 people. This student will have had 18 different neighbours in those 9 days and will have no possible new neighbours for day 10. Therefore, the maximum is at most 9 days.

It looks like 9 days might be the answer. Can we actually get to 9 days? Here some of our trial-and-error skills come into play. Since we are trying to get to 9 days (a pretty large number of days), we should certainly try to be systematic in our trial and error; otherwise we could get hopelessly lost in checking who was whose neighbour on which day.

Let us number the students as they are lined up on the first day:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19.$$

A logical second-day line-up to try is

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 2, 4, 6, 8, 10, 12, 14, 16, 18.$$

Here we took every other student from the line-up on the first day. Alternatively, we could think of placing the 19 numbers around a circle and taking every other number starting from 1. If we continue by taking every third student, then every fourth, and so, until eventually taking every ninth number starting with 1, we will get 9 days with no repeated neighbours. Try this out! The configuration on the last day will be

$$1, 10, 19, 9, 18, 8, 17, 7, 16, 6, 15, 5, 14, 4, 13, 3, 12, 2, 11.$$

Be sure to convince yourself that there are indeed no repeated neighbours over the 9 days.

We have shown that the maximum is at most 9 days, and we are able to get 9 days to work. Therefore, the answer is 9 days.

See you in the New Year!

Pólya's Paragon

It Ain't So Complex (Part 4)

Shawn Godin

In this issue, we wrap up our treatment of complex numbers by looking at some applications to geometry. These problems all appeared in last month's homework; let's see how you did.

First we will show how to use complex numbers to prove the Triangle Inequality. In Part 2 (October issue, p. 371), we saw that arrows representing z_1, z_2, and $z_1 + z_2$ can be arranged into a triangle with sides of length $|z_1|$, $|z_2|$, and $|z_1 + z_2|$. We also saw that $z\overline{z} = |z|^2$. Thus,

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \;=\; (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= |z_1|^2 + z_1\overline{z_2} + \overline{z_1}z_2 + |z_2|^2 \,. \end{aligned}$$

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then

$$\begin{aligned} z_1\overline{z_2} &= a_1a_2 + b_1b_2 + i(a_2b_1 - a_1b_2) \\ \text{and} \quad \overline{z_1}z_2 &= a_1a_2 + b_1b_2 + i(a_1b_2 - a_2b_1) \,. \end{aligned}$$

Thus, $z_1\overline{z_2} + \overline{z_1}z_2 = 2(a_1a_2 + b_1b_2) = 2\mathfrak{Re}(z_1\overline{z_2}) = 2\mathfrak{Re}(\overline{z_1}z_2)$. Therefore,

$$\begin{aligned} |z_1 + z_2|^2 &= |z_1|^2 + 2\mathfrak{Re}(z_1\overline{z_2}) + |z_2|^2 \;\leq\; |z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \;=\; (|z_1| + |z_2|)^2 \,. \end{aligned}$$

Since $|z_1 + z_2| \geq 0$ and $|z_1| + |z_2| \geq 0$, when we take square roots of both sides of the above inequality, we must have $|z_1 + z_2| \leq |z_1| + |z_2|$, which is the Triangle Inequality.

We can go one step further by writing

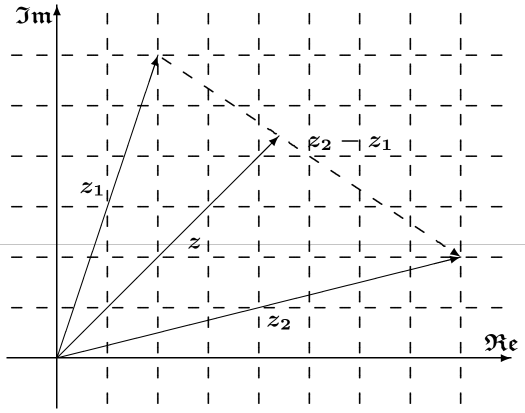
$$|z_1| \;=\; |(z_1 + z_2) + (-z_2)| \;\leq\; |z_1 + z_2| + |-z_2| \;=\; |z_1 + z_2| + |z_2| \,.$$

That is, $|z_1| - |z_2| \leq |z_1 + z_2|$. We can use a similar argument for z_2 to see that $|z_2| - |z_1| \leq |z_1 + z_2|$. These last two inequalities can be combined to yield

$$\big||z_1| - |z_2|\big| \;\leq\; |z_1 + z_2| \,.$$

This is another form of the Triangle Inequality.

Next we will prove that the medians of a triangle meet at a common point. But first, we need to convince ourselves of the following fact. For any two unequal complex numbers, z_1 and z_2, drawn on the complex plane, the complex number given by $z = (1 - t)z_1 + tz_2$, $0 < t < 1$ lies on the line segment joining z_1 and z_2 and divides it in the ratio $t : (1 - t)$.



In the diagram above we see that, if we take a complex number z which satisfies our conditions, then $z = z_1 + t(z_2 - z_1) = (1 - t)z_1 + tz_2$.

Thus, if we consider a triangle whose vertices are the three complex numbers z_1, z_2, z_3 in the complex plane, then $\frac{1}{2}(z_2 + z_3)$ corresponds to the mid-point of the side with vertices at z_2 and z_3. Thus, a point on the median has co-ordinates

$$z \;=\; (1 - t)z_1 + t \cdot \frac{z_2 + z_3}{2}\,.$$

If we choose a point which is $\frac{2}{3}$ of the way from the vertex z_3 to the mid-point of z_1z_2, it will have co-ordinates

$$z \;=\; \frac{z_1 + z_2 + z_3}{3}\,.$$

From the symmetry of this expression, we see that the same could be said for the point which is $\frac{2}{3}$ of the way from the vertex z_2 to the mid-point of z_1z_3 and also for the point which is $\frac{2}{3}$ of the way from the vertex z_1 to the mid-point of z_2z_3. Thus, these three points coincide; that is, the medians of a triangle intersect at a point which divides the medians in the ratio $2 : 1$. This point is called the *centroid* or *centre of gravity* of the triangle.

We want to draw your attention to one last point. In the solutions to last month's homework you were pointed in the direction of looking at n^{th} roots of complex numbers. If you continue your investigation, you will find that the n such roots, when plotted on the complex plane, are the vertices of a regular n–gon centred at the origin.

Well, that wraps up our discussion of complex numbers. Until February, happy problem solving.

Mayhem Year End Wrap Up

Shawn Godin

Another beautiful Sunday morning in October signals another year's end here at Mayhem (editors work in a completely different time zone!) Mayhem continues to grow and evolve, thanks, for the most part, to our readers. It is you who submit problems and solutions and help us to keep going and keep growing. To all our regulars, and to the new names we have seen over the year, thanks, and I look forward to hearing from you in 2006.

At this point I need to thank two members of the Mayhem Staff who help make it possible for me to make it through the year without ending up in a room with padded walls! First, I must thank Mayhem Assistant Editor, JOHN GRANT McLOUGHLIN. John's keen eye and thoughtful suggestions always make an issue much better than when it leaves my computer. John is leaving the Mayhem staff as Assistant Editor, although he will still be part of the Crux Board in his capacity of Book Reviews Editor. I will miss his detailed editorial comments that make each Mayhem section that much better. Secondly, I must thank IAN VANDERBURGH. Ian has just completed his first full volume in the Mayhem family. His regular column, *Problem of the Month*, is a great addition to the Mayhem features. On top of that, Ian is still young, which means that we can look forward to reading his columns for at least the next 20 years! (You should always read the fine print, Ian!)

I also want to thank a number of people who toil behind the scenes: ED BARBEAU, ROBERT BILINSKI, RICHARD HOSHINO, RON LANCASTER, PAUL OTTAWAY, LARRY RICE, BRUCE SHAWYER, and GRAHAM WRIGHT. They were there when I needed them and their contributions were always appreciated, although not always acknowledged. Thanks everyone!

All the best of the season to all our readers and contributors! I hope you have a great year in 2006. We are planning to put the Mayhem section totally online, with free access to all in 2006. We hope that our Mayhem family will continue to grow as a result. Happy problem solving! See you in 2006.

THE OLYMPIAD CORNER

No. 250

R.E. Woodrow

We begin this number of the *Corner* with the problems of the 38th Mongolian Mathematical Olympiad, Final Round. We thank Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for the *Corner*.

38th MONGOLIAN MATHEMATICAL OLYMPIAD
Final Round, May 2002
10th Grade

1. Let n and k be natural numbers. Find the least possible value for the cardinality of a set A that satisfies the following condition: There exist subsets A_1, ..., A_n of A such that any union of k of the A_i is equal to A, but any union of $k-1$ of them is not equal to A.

2. For a natural number p, one can move between two integer points in a plane when the distance between the points is p. Find all primes p for which the point $(2002, 38)$ can be reached from the point $(0, 0)$ using permitted moves.

3. The incircle of triangle ABC with $AB \neq BC$ touches sides BC and AC at points A_1 and B_1, respectively. The segments AA_1 and BB_1 meet the incircle at A_2 and B_2, respectively. Prove that the lines AB, A_1B_1, and A_2B_2 are concurrent.

4. Given are 131 distinct natural numbers, each with prime divisors not exceeding 42. Prove that four of them can be chosen whose product is a perfect square.

5. Let a_0, a_1, ... be an infinite sequence of positive numbers. Show that $1 + a_n > \sqrt[n]{2}\, a_{n-1}$ for infinitely many positive integers n.

6. Let A_1, B_1, and C_1 be the respective mid-points of the sides BC, AC, and AB of triangle ABC. Take a point K on the segment C_1A_1 and a point L on the segment A_1B_1 such that

$$\frac{C_1K}{KA_1} = \frac{BC + AC}{AC + AB} \quad \text{and} \quad \frac{A_1L}{LB_1} = \frac{AC + AB}{AB + BC}\,.$$

Let $S = BK \cap CL$. Show that $\angle C_1A_1S = \angle B_1A_1S$.

Next we present the problems of the 19th Balkan Mathematical Olympiad written in Antalya, Turkey, April 2002. My thanks again go to Bill Sands, chair of the International Olympiad Committee of the Canadian Mathematical Society for collecting these questions for our use.

19th BALKAN MATHEMATICAL OLYMPIAD
Antalya, Turkey
April 27, 2002

1. Let A_1, A_2, $\dots$, A_n $(n \geq 4)$ be points in the plane such that no three of them are collinear. Some pairs of distinct points among A_1, A_2, $\dots$, A_n are connected by line segments in such a way that each point is connected to at least three others. Prove that there exists $k > 1$ and distinct points X_1, X_2, $\dots$, $X_{2k} \in \{A_1, A_2, \dots, A_n\}$ such that for each $1 \leq i \leq 2k-1$, X_i is connected to X_{i+1} and X_{2k} is connected to X_1.

2. The sequence a_1, a_2, $\dots$, a_n, $\dots$ is defined by

$$a_1 = 20, \qquad a_2 = 30, \qquad a_{n+2} = 3a_{n+1} - a_n, \quad \text{for } n > 1.$$

Find all positive integers n for which $1 + 5a_na_{n+1}$ is a perfect square.

3. Two circles with different radii intersect at two points A and B. The common tangents of these circles are MN and ST, where the points M and S are on one of the circles, and N and T are on the other.

Prove that the orthocentres of the triangles AMN, AST, BMN, and BST are the vertices of a rectangle.

4. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$

$$2n + 2001 \leq f\big(f(n)\big) + f(n) \leq 2n + 2003.$$

($\mathbb{N}$ is the set of all positive integers).

As a third set, we give the problems of the Bulgarian Mathematical Olympiad, Final Round, written May 17-18, 2003. My thanks go to Andy Liu of the University of Alberta, Canadian Team Leader to the IMO 2003 in Japan, for collecting them for our use.

BULGARIAN MATHEMATICAL OLYMPIAD
Final Round, May 17-18, 2003

1. Let x_1, x_2, $\dots$, x_5 be real numbers. Find the least positive integer n with the following property: if some n distinct sums of the form $x_p + x_q + x_r$ (with $1 \leq p < q < r \leq 5$) are equal to 0, then $x_1 = x_2 = \cdots = x_5 = 0$.

2. Let H be an arbitrary point on the altitude CP of the acute triangle ABC. The lines AH and BH intersect BC and AC in M and N, respectively.

(a) Prove that $\angle NPC = \angle MPC$.

(b) Let O be the common point of MN and CP. An arbitrary line through O meets the sides of the quadrilateral $CNHM$ in D and E. Prove that $\angle EPC = \angle DPC$.

3. Given the sequence $\{y_n\}_{n=1}^{\infty}$ defined by $y_1 = y_2 = 1$ and

$$y_{n+2} = (4k-5)y_{n+1} - y_n + 4 - 2k\,, \qquad n \geq 1\,,$$

find all integers k such that every term of the sequence is a perfect square.

4. A set A of positive integers is called *uniform* if, after any of its elements is removed, the remaining ones can be partitioned into two subsets with equal sums of their elements. Find the least positive integer $n > 1$ such that there exists a uniform set A with n elements.

5. Let a, b, c be rational numbers such that $a+b+c$ and $a^2+b^2+c^2$ are equal integers. Prove that the number $a \cdot b \cdot c$ can be written as the ratio of a perfect cube and a perfect square which are relatively prime.

6. Determine all polynomials $P(x)$ with integer coefficients such that, for any positive integer n, the equation $P(x) = 2^n$ has an integer root.

To add to your puzzling pleasure over the seasonal break we give the problems of the Chinese Mathematical Olympiad 2003. Thanks again go to Andy Liu for collecting them for our use.

CHINESE MATHEMATICAL OLYMPIAD 2003

1. Let I be the incentre of triangle ABC, and let B_1 and C_1 be the mid-points of AC and AB, respectively. The extension of B_1I cuts C_1B at B_2, and the extension of C_1I cuts the extension of B_1C at C_2. Further, B_2C_2 cuts BC at K. Let H be the orthocentre of triangle ABC, and let A_1 be the circumcentre of triangle HBC. Prove that A, I, and A_1 are collinear if and only if triangles BKB_2 and CKC_2 have equal area.

2. Determine the maximum size of a subset S of $\{1, 2, \ldots, 100\}$ such that, for any a and b in S, there exist c and d in S, distinct from a and b, with c relatively prime to both a and b, and d relatively prime to neither a nor b.

3. Let $n \geq 2$ be a fixed integer. For $i = 1, 2, \ldots, n$, let θ_i be such that $0 < \theta_i < \frac{\pi}{2}$ and

$$\tan\theta_1 \tan\theta_2 \cdots \tan\theta_n = 2^{n/2}\,.$$

Determine, in terms of n, the smallest positive number λ such that

$$\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n \;\leq\; \lambda\,.$$

4. Determine all triples (a,m,n) of positive integers such that $a \geq 2$, $m \geq 2$, and $a^n + 203$ is a multiple of $a^m + 1$.

5. Ten candidates with different qualifications apply for a single job vacancy. The employer has decided on the following protocol. The candidates will be arranged in random order and interviewed one by one, at which time their qualifications will be assessed. The first three candidates will be rejected automatically. Thereafter, the first candidate better qualified than any of the first three will be hired. If there is no such candidate, then the tenth and last candidate will be hired.

Let the candidates be ranked from 1 to 10 in descending order of qualifications. For $1 \leq k \leq 10$, let A_k denote the number of the $10!$ possible orders of interviews which would lead to the hiring of the k^{th}-ranked candidate. Prove that

(a) $A_1 > A_2 > \cdots > A_8 = A_9 = A_{10}$;

(b) the probability that the hired candidate is ranked 1, 2, or 3 is greater than $\frac{7}{10}$;

(c) the probability that the hired candidate is ranked 8, 9, or 10 is not greater than $\frac{1}{10}$.

6. Let c_1, c_2, c_3, and c_4 be positive numbers such that $c_1c_2 + c_3c_4 = 1$. Let (x_1,y_1), (x_2,y_2), (x_3,y_3), and (x_4,y_4) be points on the circle $x^2+y^2=1$. Prove that

$$(c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4)^2 + (c_1x_4 + c_2x_3 + c_3x_2 + c_4x_1)^2 \\ \leq\; 2\left(\frac{c_1^2+c_2^2}{c_1c_2} + \frac{c_3^2+c_4^2}{c_3c_4}\right)\,.$$

Next we turn to the March 2004 number of the *Corner* and readers' solutions to problems of the Hungary-Israel Binational Mathematical Competition 2001, Individual Competition [2004 : 82].

1. Find positive integers x, y, z such that $x > z > 1999 \cdot 2000 \cdot 2001 > y$ and $2000x^2 + y^2 = 2001z^2$.

Solution by Michel Bataille, Rouen, France.

If (x,y,z) is such a triple of positive integers, then

$$\frac{2000(x-z)}{z-y} \;=\; \frac{z+y}{x+z}\,.$$

Denoting this rational by $\frac{m}{n}$ (with m, $n \in \mathbb{N}$), we have

$$n(y+z) \;=\; m(x+z) \qquad \text{and} \qquad 2000n(x-z) \;=\; m(z-y)\,.$$

These equations are satisfied by the integers

$$x \;=\; d(2000n^2 + 2mn - m^2)\,, \quad y \;=\; d(m^2 + 4000mn - 2000n^2)\,,$$
$$z \;=\; d(m^2 + 2000n^2)$$

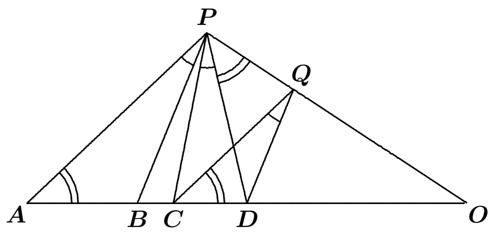
for any integer d. Taking $m = 1$, $n = 2$, and $d = 1999 \times 2000$, we find that

$$x \;=\; 1999 \times 2000 \times 8003\,, \quad y \;=\; 1999 \times 2000\,, \quad z \;=\; 1999 \times 2000 \times 8001\,.$$

These integers x, y, and z satisfy all the given conditions.

2. Points A, B, C, D lie on the line ℓ, in that order. Find the locus of points P in the plane for which $\angle APB = \angle CPD$.

Solved by Michel Bataille, Rouen, France; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's version.



Let P be a point such that $\angle APB = \angle CPD$. Let Q be the point such that $QC \parallel PA$ and $QD \parallel PB$. Then $\triangle PAB \sim \triangle QCD$, implying that $\angle CQD = \angle APB = \angle CPD$. Thus, P, C, D, and Q are concyclic.

Case 1. $AB = CD$.

Then $\triangle PAB \equiv \triangle QCD$. Thus, $PB = QD$. Therefore, $PBDQ$ is a parallelogram. Hence, $PQ \parallel BD$; that is $PQ \parallel CD$. Since P, C, D, Q are conclycic, we have $PC = QD$. Hence, $PB = PC$. Thus, the locus of P is the perpendicular bisector of BC (excluding the mid-point of BC).

Case 2. $AB \neq CD$.

Let O be the point of intersection of PQ with AD. Since $PA \parallel QC$, we have $OA : OC = PA : QC$. Also, since $\triangle PAB \sim \triangle QCD$, we have $PA : QC = AB : CD$. Therefore, $OA : OC = AB : CD$, which is a constant ratio. Thus, O is a fixed point.

Since P, C, Q, D are concyclic, we have $\angle QPD = \angle QCD = \angle PAB$; that is, $\angle OPD = \angle PAO$, from which we get $OP^2 = OA \cdot OD$. Thus, $OP = \sqrt{OA \cdot OD}$, a constant. Therefore, the locus of P is the circle with centre O and radius $\sqrt{OA \cdot OD}$ (excluding the points of intersection with the line ℓ).

3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real x,

$$f\big(f(x)\big) \;=\; f(x) + x\,.$$

Solution by Pierre Bornsztein, Maisons-Laffitte, France, modified by the editor.

Note that $f(x) = \varphi x$ and $f(x) = -x/\varphi$ are solutions of the problem, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. We will prove that there is no other solution.

Let f be a solution. The function f must be injective, because if $f(a) = f(b)$, then $a = f\big(f(a)\big) - f(a) = f\big(f(b)\big) - f(b) = b$. Thus, f is a bijection from $\mathbb{R}$ onto $f(\mathbb{R})$. Since f is continuous, it follows that f is strictly monotonic on $\mathbb{R}$. Moreover, $f\big(f(0)\big) = f(0)$; whence, $f(0) = 0$.

Since f is monotonic, it follows that $\lim\limits_{x\to+\infty} f(x)$ is either a real number or $\pm\infty$. If $\lim\limits_{x\to+\infty} f(x) = L \in \mathbb{R}$, then, using the continuity of f, we obtain

$$f(L) - L \;=\; \lim_{x\to+\infty}\Big(f\big(f(x)\big) - f(x)\Big) \;=\; \lim_{x\to+\infty} x \;=\; +\infty\,,$$

which is absurd. We have a similar result for $\lim\limits_{x\to-\infty} f(x)$. Therefore, f is unbounded above and unbounded below, and $f(\mathbb{R}) = \mathbb{R}$.

Let f^0 denote the identity function on $\mathbb{R}$. For each positive integer n, let $f^{n+1} = f \circ f^n$ and $f^{-(n+1)} = f^{-1} \circ f^{-n}$. Since we are given that $f^2 = f^1 + f^0$, it follows that $f^{n+2} = f^{n+1} + f^n$ for all $n \in \mathbb{Z}$. Solving this difference equation, we get

$$f^n \;=\; \frac{g}{\sqrt{5}}\,\varphi^n + \frac{h}{\sqrt{5}}\left(-\frac{1}{\varphi}\right)^n\,,$$

where $g = \dfrac{1}{\varphi}f^0 + f^1$ and $h = \varphi f^0 - f^1$; that is, $g(x) = \dfrac{1}{\varphi}x + f(x)$ and $h(x) = \varphi x - f(x)$. Note that for all $x \in \mathbb{R}$, since $\varphi > 1$, we have

$$\lim_{n\to-\infty} \frac{g(x)}{\sqrt{5}}\,\varphi^n \;=\; 0 \qquad\text{and}\qquad \lim_{n\to+\infty} \frac{h(x)}{\sqrt{5}}\left(-\frac{1}{\varphi}\right)^n \;=\; 0\,.$$

Case 1. f is increasing.

We will show that $h(x) = 0$ for all $x \in \mathbb{R}$. Then we will have $f(x) = \varphi x$ for all $x \in \mathbb{R}$.

First consider any $x > 0$. Since $f(0) = 0$ and f is increasing, we deduce that $f^n(x) > 0$ for all $n \in \mathbb{Z}$. If $h(x) < 0$, then

$$\lim_{n\to-\infty} f^{2n}(x) \;=\; \lim_{n\to-\infty} \frac{h(x)}{\sqrt{5}}\left(-\frac{1}{\varphi}\right)^{2n} \;=\; -\infty\,,$$

which is impossible, since $f^n(x) > 0$ for all $n \in \mathbb{Z}$. Similarly, if $h(x) > 0$, then $\lim\limits_{n\to-\infty} f^{2n+1}(x) = -\infty$, and again we have a contradiction. Therefore, $h(x) = 0$.

Now consider any $x < 0$. In this case, we have $f^n(x) < 0$ for all $n \in \mathbb{Z}$. In much the same manner as above, we see that if $h(x) \neq 0$, then either $\lim\limits_{n\to-\infty} f^{2n}(x) = \infty$ or $\lim\limits_{n\to-\infty} f^{2n+1}(x) = \infty$, giving a contradiction. Thus, $h(x) = 0$.

Finally, we note that $h(0) = 0 - f(0) = 0$.

Case 2. f is decreasing.

By an argument similar to the argument in Case 1, but with $x \to +\infty$ instead of $x \to -\infty$, we see that $g(x) = 0$ for all $x \in \mathbb{R}$. Then $f(x) = -x/\varphi$ for all $x \in \mathbb{R}$.

4. Let $P(x) = x^3 - 3x + 1$. Find the polynomial Q whose roots are the fifth power of the roots of P.

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's write-up.

Let a, b, c be the roots of P. Then $a + b + c = 0$, $ab + bc + ca = -3$, and $abc = -1$. The required polynomial Q is given by

$$\begin{aligned} Q(x) &= (x - a^5)(x - b^5)(x - c^5) \\ &= x^3 - (a^5 + b^5 + c^5)x^2 + (a^5b^5 + b^5c^5 + c^5a^5)x - a^5b^5c^5 \\ &= x^3 - S_5x^2 + T_5x + 1\,, \end{aligned}$$

where $S_5 = a^5 + b^5 + c^5$ and $T_5 = a^5b^5 + b^5c^5 + c^5a^5$.

Generalizing our definition of S_5, we define $S_n = a^n + b^n + c^n$, for each positive integer n. Observe that $T_5 = \frac{1}{2}(S_5^2 - S_{10})$. Therefore, the determination of $Q(x)$ will be complete if we can calculate S_5 and S_{10}.

We have $S_1 = a+b+c = 0$ and $S_2 = (a+b+c)^2 - 2(ab+bc+ca) = 6$. Since a, b, and c are roots of P, they each satisfy the equation $x^3 = 3x - 1$. It follows that $S_{n+3} = 3S_{n+1} - S_n$ for all $n \geq 0$. Therefore,

$$\begin{aligned} S_3 &= 3 \times 0 - 3 \;=\; -3 \\ S_4 &= 3 \times 6 - 0 \;=\; 18 \\ S_5 &= 3 \times (-3) - 6 \;=\; -15 \\ S_6 &= 3 \times 18 - (-3) \;=\; 57 \\ S_7 &= 3 \times (-15) - 18 \;=\; -63 \\ S_8 &= 3 \times 57 - (-15) \;=\; 186 \\ S_{10} &= 3 \times 186 - (-63) \;=\; 621\,. \end{aligned}$$

Now we have $T_5 = \frac{1}{2}(S_5^2 - S_{10}) = \frac{1}{2}(15^2 - 621) = -198$. Thus, $Q(x) = x^3 + 15x^2 - 198x + 1$.

We also give the method used by Díaz-Barrero.

We consider the companion matrix of the polynomial $P(x)$, namely,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 0 \end{pmatrix},$$

for which the characteristic polynomial is $\det(xI - A) = x^3 - 3x + 1$. For each integer $k \geq 1$, the zeroes of the characteristic polynomial of A^k are the k^{th} powers of the zeroes of $P(x)$. Taking this into account, we find that $Q(x) = \det(xI - A^5)$. It is an easy exercise to see that

$$A^5 = \begin{pmatrix} -3 & 9 & -1 \\ 1 & -6 & 9 \\ -9 & 28 & -6 \end{pmatrix}.$$

Then

$$\begin{aligned} Q(x) = \det(xI - A^5) &= \begin{vmatrix} x+3 & -9 & 1 \\ -1 & x+6 & -9 \\ 9 & -28 & x+6 \end{vmatrix} \\ &= x^3 + 15x^2 - 198x + 1\,. \end{aligned}$$

5. A triangle ABC is given. The mid-points of sides AC and AB are B_1 and C_1, respectively. The centre of the incircle of $\triangle ABC$ is I. The lines B_1I, C_1I meet the sides AB, AC at C_2, B_2, respectively. Given that the areas of $\triangle ABC$ and $\triangle AB_2C_2$ are equal, what is $\angle BAC$?

[*Ed*. The line C_1I was originally stated in error as B_2I. It has been corrected above.]

Solved by Toshio Seimiya, Kawasaki, Japan; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya's solution.

We set $BC = a$, $CA = b$, $AB = c$, $AC_2 = x$ and $AB_2 = y$. Since $[AB_2C_2] = [ABC]$ (where $[PQR]$ denotes the area of triangle PQR), we have $xy = bc$.

Let D and E be the intersections of BI and CI with AC and AB, respectively. Since AI and CI are the bisectors of $\angle BAD$ and $\angle BCD$, we have

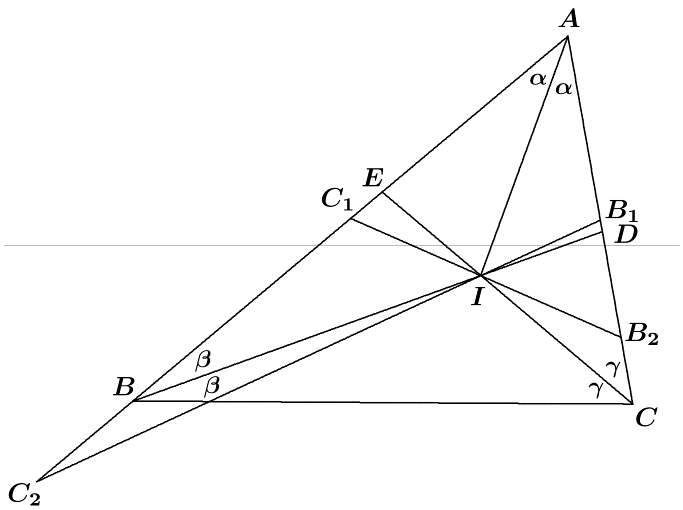
$$\frac{BI}{ID} = \frac{AB}{AD} = \frac{BC}{CD} = \frac{AB+BC}{AD+CD} = \frac{AB+BC}{AC} = \frac{a+c}{b}.$$

Similarly, $\dfrac{CI}{IE} = \dfrac{a+b}{c}$.

Since BD is the bisector of $\angle ABC$, we have

$$AD : DC = AB : BC = c : a\,.$$

From this, we get $AD = \dfrac{bc}{a+c}$. Similarly, $AE = \dfrac{bc}{a+b}$.



Case 1. $c = a$.

Then B_1 coincides with D, and therefore C_2 coincides with B. Thus, $x = c$ and $y = b$. Hence, B_2 coincides with C, and C_1 coincides with E. Thus, $AC = BC$; that is $b = a$. Thus, $a = b = c$. Hence, $\triangle ABC$ is equilateral, and we get $\angle BAC = 60°$.

Case 2. $a < c$.

Then $x > c$, and $y < b$, from which we get $a > b$. By Menelaus' Theorem for $\triangle ABD$, we have

$$\frac{AC_2}{C_2B} \cdot \frac{BI}{ID} \cdot \frac{DB_1}{B_1A} = 1\,. \tag{1}$$

Since $DB_1 = AD - AB_1 = \dfrac{bc}{a+c} - \dfrac{b}{2} = \dfrac{b(c-a)}{2(a+c)}$, we have $\dfrac{DB_1}{B_1A} = \dfrac{c-a}{a+c}$.

Now (1) becomes

$$\frac{x}{x-c} \cdot \frac{a+c}{b} \cdot \frac{c-a}{a+c} = 1\,;$$

whence, $\dfrac{x}{x-c} = \dfrac{b}{c-a}$. Thus, $x = \dfrac{bc}{b-c+a}$. Similarly, $y = \dfrac{bc}{c-b+a}$. Since $xy = bc$, we get

$$bc = \frac{b^2c^2}{(b-c+a)(c-b+a)}\,.$$

Then $a^2 - (b-c)^2 = bc$, or equivalently, $a^2 = b^2 - bc + c^2$. Since $a^2 = b^2 + c^2 - 2bc\cos\angle BAC$, we get $\cos\angle BAC = \frac{1}{2}$. Thus, $\angle BAC = 60°$.

Case 3. $a > c$.

As in case 2, we can prove that $\angle BAC = 60°$.

Now we present reader's solutions to the Team Competition of the Hungary-Israel Binational Mathematical Competition given [2004 : 82–83].

In the following questions, G_n is a simple undirected graph with n vertices, K_n is the complete graph with n vertices, $K_{n,m}$ is the complete bipartite graph with m vertices in one of the two partite sets and n vertices in the other, and C_n is a circuit with n vertices. The number of edges in the graph G_n is denoted $e(G_n)$.

1. The edges of K_n, $n \geq 3$, are coloured with n colours, and every colour appears at least once. Prove that there is a triangle whose sides are coloured with 3 different colours.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and the Samford University Problem Solving Group, Birmingham, AL, USA. We give the solution by the Samford Group.

Since K_3 is a triangle, if it is coloured with 3 different colours, then it is a tri-coloured triangle.

The rest of the proof is by induction. Suppose that, for some n, if K_n is coloured with at least n colours, then K_n contains a tri-coloured triangle.

Now, suppose K_{n+1} is coloured with $n+1$ colours. Let v be any vertex in K_{n+1}. Either v is, or is not, incident on at least two edges e_1 and e_2 such that e_1 is the only edge coloured c_1 and e_2 is the only edge coloured c_2.

If so, then let v_1 and v_2 be the vertices adjacent to v via e_1 and e_2, respectively. Then the triangle with vertices v, v_1 and v_2 is a tri-coloured triangle, as the colour of the edge connecting v_1 and v_2 cannot be either c_1 or c_2.

If not, then remove v and all its incident edges from K_{n+1}, creating a copy of K_n. Since v is incident on at most one edge e in K_{n+1} such that e is the only edge of a particular colour, the resulting copy of K_n is coloured with at least n different colours. By the induction hypothesis, then, there is a tri-coloured triangle in this copy of K_n and therefore in its supergraph K_{n+1}.

2. An integer $n \geq 5$ is given. If $e(G_n) \geq \dfrac{n^2}{4} + 2$, prove that there exist two triangles which have exactly one common vertex.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will use induction on n.

For $n = 5$, let A, B, C, D, E be the vertices. Since $\frac{5^2}{4} + 2 = 8.25$, we have at least 9 edges. Thus, at most one pair of vertices, say D and E, is not connected by an edge. It follows that the triangles ABD and ACE exist and we are done.

Now assume that the result holds for a given $n \geq 5$. Now consider a graph G_{n+1} with $e(G_{n+1}) \geq \dfrac{(n+1)^2}{4} + 2$.

If some vertex M has degree $d(M) \leq (2n+1)/4$, then, by deleting M

and all the edges with endpoint M, we obtain a subgraph G_n with

$$e(G_n) \ \geq \ e(G_{n+1}) - \frac{2n+1}{4} \ \geq \ \frac{(n+1)^2}{4} + 2 - \frac{2n+1}{4} \ \geq \ \frac{n^2}{4} + 2\,.$$

The induction hypothesis ensures us about the existence of two triangles in G_n, and hence in G_{n+1}, which have exactly one common vertex, and we are done in that case.

Therefore, we may assume that every vertex M has degree $d(M) > (2n+1)/4$. Since $2n+1$ is odd and $d(M)$ is an integer, it follows that $d(M) \geq (2n+2)/4 = (n+1)/2$. Let $n = 2p+e$, where $e = 0$ or $e = 1$, according to the parity of n. Then, for each vertex M, we have

$$d(M) \ \geq \ p+1\,. \tag{1}$$

Let A be a vertex of G_{n+1} with maximal degree $d(A) = k \geq p+1$. Let A be adjacent to $A_1, \ldots, A_k$, and not adjacent to $B_1, \ldots, B_{n-k}$ (if any).

Case 1. n is even ($n = 2p$).

Note that $k \geq p+1 \geq 4$.

Lemma 1. If MN is an edge, then there exists a vertex P such that MNP is a triangle.

Proof: From (1), M is adjacent to at least p vertices different from N, and N is adjacent to at least p vertices different from M. Since $2p = n > n-1$, it follows that there exists a vertex P which is adjacent to M and N, which proves the lemma. ∎

From the lemma, since AA_1 is an edge, we may assume that AA_1A_2 is a triangle.

Now AA_3 is an edge. Then if AA_3A_i is a triangle for some $i \geq 4$, the triangles AA_3A_i and AA_1A_2 lead to the desired conclusion. From now, we assume that A_3A_i is not an edge for $i \geq 4$.

From the lemma, it follows that A_3A_1 or A_3A_2 is an edge. With no loss of generality, we may assume that AA_1A_3 is a triangle.

Let $i \geq 4$. Since AA_i is an edge, then if AA_iA_j is a triangle for some $j \geq 2$ and $j \neq 3$, we deduce that the triangles AA_iA_j and AA_1A_3 give the desired conclusion. Otherwise, the only triangle with vertices A and A_i is AA_1A_i. In that case, A_1 is adjacent to A and to all A_i for $i \geq 2$. Thus, $d(A_1) \geq k = d(A)$.

From the maximality of k, we deduce that $d(A_1) = k$, which means that A_1 is adjacent to none of the B_i.

On the other hand, $d(A_2) \geq p+1 \geq 4$ and A_2 is not adjacent to A_i for $i \geq 3$. Then, A_2 must be adjacent to at least one of the B_i, say B_1. From the lemma, it follows that there exists a vertex P such that A_2B_1P is a triangle. This vertex P is neither A nor A_1, since they are not adjacent to B_1. Thus, AA_1A_2 and A_2B_1P is a pair of triangles with exactly one common vertex, and we are done.

Case 2. n is odd ($n = 2p + 1$).

Suppose first that $k < p + 2$. From the maximality of k and (1), it follows that $d(M) = k + 1$ for each vertex M of G_{n+1}. Thus,

$$e(G_{n+1}) = \frac{1}{2}\sum_{M} d(M) = \frac{1}{2}(n+1)(p+1) = \frac{(n+1)^2}{4} < \frac{(n+1)^2}{4} + 2,$$

a contradiction. Then,

$$k \geq p + 2. \tag{2}$$

Lemma 2. If AM is an edge, then there exists a vertex P such that AMP is a triangle.

Lemma 2 can be proved in exactly the same way as lemma 1.

Now assume, for a contradiction, that there are no pair of triangles which have exactly one common vertex.

Exactly as above, we deduce that $d(A_1) = d(A)$ and A_iA_j is not an edge for all $i, j \geq 2$. And that A_1 is adjacent to none of the B_i's.

For $i = 2, \ldots, k$, since $d(A_i) \geq p+1 \geq 3$, it follows that A_i is adjacent to at least $p-1$ of the B_j's. In particular, we must have $p - 1 \leq n - k$ which leads to $k \leq p + 2$. From (2), it follows that $k = p + 2$ and $n - k = p - 1$. Thus, A_iB_j is an edge for $i \geq 2$ and $j \geq 1$. That leads to

$$d(A_i) = p + 1 \quad \text{for} \quad i \geq 2. \tag{3}$$

From our hypothesis, no pair (B_i, B_j) is an edge; otherwise, the triangles $B_iB_jA_2$ and AA_1A_2 would have exactly one common vertex. Thus,

$$d(B_i) = k - 1 = p + 1 \quad \text{for} \quad i \geq 1. \tag{4}$$

Now, from (2), (3) and (4), we deduce that

$$\begin{aligned} e(G_{n+1}) &= \frac{1}{2}\left(d(A) + d(A_1) + \sum_{i=2}^{k} d(A_i) + \sum_{i=1}^{n-k} d(B_i)\right) \\ &= \frac{1}{2}\left((p+2) + (p+2)\sum_{i=2}^{p+2}(p+1) + \sum_{i=1}^{p-1}(p+1)\right) \\ &= \frac{(n+1)^2}{4} + 1, \end{aligned}$$

a contradiction. Thus, there are two triangles which have exactly one common vertex, and we are again done.

This ends the induction step and the proof.

3. If $e(G_n) \geq \frac{n\sqrt{n}}{2} + \frac{n}{4}$, prove that G_n contains C_4.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let $A_1, \ldots, A_n$ be the vertices, and for each i let d_i be the degree of A_i. Let $e = e(G_n)$. It is well known that $e = \frac{1}{2}\sum_{i=1}^{n} d_i$.

For each i, the number of undirected paths of the form MA_iP, where M and P are two distinct vertices, is $\binom{d_i}{2}$. On the other hand, there are $\binom{n}{2}$ pairs of two distinct vertices M and P. Thus, from the pigeon-hole principle, if $\sum_{i=1}^{n}\binom{d_i}{2} > \binom{n}{2}$, there is a pair (M, P) of vertices which belongs to at least two of the above paths of length 2, and in that case we have a C_4.

Therefore, if G_n does not contain a C_4, we must have $\sum_{i=1}^{n}\binom{d_i}{2} \le \binom{n}{2}$; that is,

$$\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i \;\le\; n(n-1)\,.$$

From the inequality between arithmetic and quadratic means, we have

$$\sum_{i=1}^{n} d_i^2 \;\ge\; \frac{1}{n}\left(\sum_{i=1}^{n} d_i\right)^2 \;=\; \frac{4e^2}{n}\,.$$

Thus, if G_n does not contain a C_4, we must have $\frac{4e^2}{n} - 2e \le n(n-1)$; that is, $4e^2 - 2ne - n^2(n-1) \le 0$. Treating the left side as a quadratic in e, we deduce that, if G_n does not contain a C_4, then $e \le \frac{n + n\sqrt{4n-3}}{4}$. This means that, if $e > \frac{n + n\sqrt{4n-3}}{4}$, then G_n contains a C_4. Since $\frac{n + n\sqrt{4n-3}}{4} < \frac{n\sqrt{n}}{2} + \frac{n}{4}$, we have a better result than desired.

4. (a) If G_n does not contain $K_{2,3}$, prove that $e(G_n) \le \frac{n\sqrt{n}}{\sqrt{2}} + n$.

(b) Given $n \ge 16$ distinct points $P_1, P_2, \ldots, P_n$ in the plane, prove that at most $n\sqrt{n}$ of the segments P_iP_j have unit length.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

(a) Let $A_1, \ldots, A_n$ be the vertices, and for each i let d_i be the degree of A_i. Let $e = e(G_n)$. It is well known that $e = \frac{1}{2}\sum_{i=1}^{n} d_i$.

For each i, the number of undirected paths of the form MA_iP, where M and P are two distinct vertices, is $\binom{d_i}{2}$. On the other hand, there are $\binom{n}{2}$ pairs of two distinct vertices (M, P). Thus, from the pigeon-hole principle, if $\sum_{i=1}^{n}\binom{d_i}{2} > 2\binom{n}{2}$ there is a pair (M, P) of vertices which belongs to at least three of the above paths of length 2, and in that case we have a $K_{2,3}$.

Thus, if G_n does not contain a $K_{2,3}$, we must have $\sum_{i=1}^{n}\binom{d_i}{2} \le 2\binom{n}{2}$; that is,

$$\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i \ \le\ 2n(n-1)\,.$$

From the inequality between arithmetic and quadratic means, we have

$$\sum_{i=1}^{n} d_i^2 \ \ge\ \frac{1}{n}\Bigl(\sum_{i=1}^{n} d_i\Bigr)^2 \ =\ \frac{4e^2}{n}\,.$$

Thus, if G_n does not contain a $K_{2,3}$, we must have $\frac{4e^2}{n} - 2e \le 2n(n-1)$; that is, $2e^2 - ne - n^2(n-1) \le 0$. Solving it as a quadratic expression in e, we deduce that, if G_n does not contain a $K_{2,3}$, then $e \le \dfrac{n+n\sqrt{8n-7}}{4}$. Since $\dfrac{n+n\sqrt{8n-7}}{4} < \dfrac{4n}{4} + \dfrac{n\sqrt{8n}}{4} = \dfrac{n\sqrt{n}}{\sqrt{2}} + n$, we have a better result than desired.

(b) Let us consider the graph G_n whose vertices are the n given points, any two connected by an edge if and only if the segment they form has unit length.

Suppose that G_n contains a $K_{2,3}$. Then there are two points, say A and B, which are connected to three other common points, say C, D and E. It follows that the circles with radii 1 and respective centres A and B intersect in at least three points, which is impossible. Thus, G_n does not contain a $K_{2,3}$.

From (a), we have $e \le \dfrac{n+n\sqrt{8n-7}}{4}$. Since $\dfrac{n+n\sqrt{8n-7}}{4} \le n\sqrt{n}$, we have a better result than desired, which holds even if $n \le 16$.

5. (a) Let p be a prime. Consider the graph whose vertices are the ordered pairs (x,y) with x, $y \in \{0, 1, 2, \ldots, p-1\}$, and whose edges join vertices (x,y) and (x',y') if and only if $xx' + yy' \equiv 1 \pmod{p}$. Prove that this graph does not contain C_4.

(b) Prove that for infinitely many values of n, there is a graph G_n that does not contain C_4 and satisfies $e(G_n) \ge \dfrac{n\sqrt{n}}{2} - n$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

(a) First note that if x, $y \in \{0, 1, \ldots, p-1\}$ and $x \equiv y \pmod{p}$, then $x = y$.

Let $A_1(a_1,b_1)$, $A_2(a_2,b_2)$, $A_3(a_3,b_3)$, and $A_4(a_4,b_4)$ be four pairwise distinct vertices from the graph. Suppose, for a contradiction, that $A_1A_2A_3A_4A_1$ is a C_4. Thus,

$$a_1a_2 + b_1b_2 \ \equiv\ 1 \pmod{p} \qquad (1)$$
$$a_2a_3 + b_2b_3 \ \equiv\ 1 \pmod{p} \qquad (2)$$
$$a_3a_4 + b_3b_4 \ \equiv\ 1 \pmod{p} \qquad (3)$$
$$a_4a_1 + b_4b_1 \ \equiv\ 1 \pmod{p} \qquad (4)$$

From (1) and (2), we deduce that

$$a_2(a_3 - a_1) + b_2(b_3 - b_1) \equiv 0 \pmod{p} . \tag{5}$$

From (3) and (4), we deduce that

$$a_4(a_3 - a_1) + b_4(b_3 - b_1) \equiv 0 \pmod{p} . \tag{6}$$

Case 1. p divides $a_3 - a_1$.

Then $a_3 = a_1$, and $b_3 \neq b_1$ (since $A_1 \neq A - 3$). From (5) and (6), it follows that p divides b_2 and b_4; that is, $b_2 = b_4 = 0$. Thus, from (1), we have $a_1 \neq 0 \pmod{p}$. From (1) and (4), we then have $a_1a_2 = a_1a_4 \pmod{p}$, which leads to $a_2 = a_4 \pmod{p}$ and to $a_2 = a_4$. Hence, $A_2 = A_4$, a contradiction.

Similarly, we prove that p divides none of the numbers $b_3 - b_1$, $a_2 - a_4$, and $b_2 - b_4$ (for the two last cases, we may start from similar relations, rather than (5) and (6)).

Case 2. p divides a_2.

Then from above and from (5), p divides b_2 and $a_2 = b_2 = 0$. In that case, $a_1a_2 + b_1b_2 \equiv 0 \pmod{p}$, which contradicts (1). Similarly, we prove that p divides none of the numbers a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4.

Now, it follows from (5) and (6) that

$$a_2b_2^{-1} \equiv (b_3 - b_1)(a_1 - a_3)^{-1} \equiv a_4b_4^{-1} \pmod{p} ;$$

therefore,

$$a_2b_4 \equiv a_4b_2 \pmod{p} . \tag{7}$$

Similarly,

$$a_1b_3 \equiv a_3b_1 \pmod{p} . \tag{8}$$

On the other hand, from (1) and (3), we deduce that

$$a_1a_2 - a_3a_4 + b_1b_2 - b_3b_4 \equiv 0 \pmod{p} .$$

Then, using (7) and (8), we have

$$\begin{aligned} 0 &\equiv b_3b_4(a_1a_2 - a_3a_4 + b_1b_2 - b_3b_4) \\ &\equiv a_3a_4b_1b_2 - b_3b_4a_3a_4 + b_3b_4(b_1b_2 - b_3b_4) \\ &\equiv (b_1b_2 - b_3b_4)(a_3a_4 + b_3b_4) \\ &\equiv b_1b_2 - b_3b_4 \pmod{p} \qquad \text{from (3)} . \end{aligned}$$

Thus, $b_1b_2 \equiv b_3b_4 \pmod{p}$. Similarly, we have $b_3b_2 \equiv b_1b_4 \pmod{p}$, and $a_1a_2 \equiv a_3a_4 \pmod{p}$ and $a_3a_2 \equiv a_1a_4 \pmod{p}$. It follows that $b_3b_2b_4^{-1} \equiv b_1 \equiv b_3b_4b_2^{-1} \pmod{p}$, which implies that $b_2 \equiv b_4 \pmod{p}$; that is, $b_2 = b_4$. Similarly, $a_2 = a_4$, and $A_2 = A_4$, a contradiction. Then, the graph contains no C_4.

(b) Let a, $b \in \{0, 1, 2, \ldots, p-1\}$ with $(a,b) \neq (0,0)$.

If $a \neq 0$, then a is invertible modulo p and, for any given y in $\{0, 1, 2, \ldots, p-1\}$, the equation $ax + by \equiv 1 \pmod{p}$ has a unique solution, namely $x = (1-by)a^{-1}$.

Thus, $A(a,b)$ is connected to each of the points $M\big((1-by)a^{-1}, y\big)$ for $y = 0, 1, \ldots, p-1$. It follows that $d(A) \geq p$.

If $a = 0$, then $b \neq 0$, and $A(0,b)$ is connected to each of the points $M\big(x, b^{-1}\big)$ for $x = 0, 1, \ldots, p-1$. Thus, $d(A) \geq p$. Then $d(A) \geq p$ for each point $A \neq 0$ $(0,0)$. It follows that the number e of edges of the graph defined in (a) satisfies

$$e = \frac{1}{2}\sum_{A\neq 0} d(A) \geq \frac{1}{2}(p^2-1)p\,.$$

Since the number n of vertices is $n = p^2$, we deduce that

$$e \geq \frac{n\sqrt{n}-\sqrt{n}}{2} \geq \frac{n\sqrt{n}}{2} - n\,.$$

Since we have proved that the graph contains no C_4, we are done.

Next we turn to solutions for problems of the Second Hong Kong (China) Mathematical Olympiad 1999 appearing [2004 : 83–84].

1. [5 marks] Determine all positive rational numbers $r \neq 1$ such that $r^{1/(r-1)}$ is rational.

Comment by Michel Bataille, Rouen, France. Solved by Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's comment.

This problem is known: a solution and references can be found in *Mathematics Magazine*, Vol. 69, No. 1, February 1996, p. 68.

2. [10 marks] Let I and O be the incentre and circumcentre, respectively, of $\triangle ABC$. Assume $\triangle ABC$ is not equilateral (so that $I \neq O$). Prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + CA$.

Solved by Michel Bataille, Rouen, France; and Toshio Seimiya, Kawasaki, Japan. We give Bataille's solution, adapted by the editor.

We will prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + CA$ (as required), and we will also prove that equality holds on one side of this equivalence if and only if it holds on the other side. We will use standard notation for the elements of the triangle ABC.

By the Cosine Law in $\triangle AIO$, we have

$$AO^2 = AI^2 + IO^2 - 2(AI)(IO)\cos\angle AIO\,.$$

Since $\angle AIO \leq 90^\circ$ if and only if $\cos\angle AIO \geq 0$, we deduce that

$$\angle AIO \;\leq\; 90^\circ \qquad \Longleftrightarrow \qquad AO^2 \leq AI^2 + IO^2\,. \tag{1}$$

Furthermore, equality occurs on one side of this equivalence if and only if it occurs on the other side.

We have

$$AI^2 \;=\; \frac{(s-a)^2}{\cos^2(A/2)} \;=\; \frac{(b+c-a)^2}{2(1+\cos A)} \;=\; \frac{bc(b+c-a)}{2s}\,,$$

where the last step makes use of the Cosine Law in $\triangle ABC$. Now $rs = \dfrac{abc}{4R}$ (since both expressions are equal to the area of $\triangle ABC$), and therefore,

$$AI^2 \;=\; \frac{2Rr}{a}(b+c-a)\,.$$

We also have $AO^2 = R^2$ and $IO^2 = R^2 - 2Rr$. Hence,

$$\begin{aligned} AI^2 + IO^2 \;&=\; \frac{2Rr}{a}(b+c-a) + (R^2 - 2Rr) \\ &=\; AO^2 + \frac{2Rr}{a}(b+c-2a)\,. \end{aligned}$$

Thus, $AI^2 + IO^2 \geq AO^2$ if and only if $b + c \geq 2a$, and equality in either of these inequalities implies equality in the other. Recalling (1), we obtain the desired conclusion.

3. [10 marks] Students have taken a test in each of n subjects ($n \geq 3$). It is known that, for any subject, exactly three students got the best score in the subject, and for any two subjects, exactly one student got the best score in both of the subjects. Determine the smallest n so that the above conditions imply that exactly one student got the best score in all n subjects.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that $n = 8$ is the desired minimum.

Consider the graph $\mathcal{G}$ whose vertices are the students, two of them being connected by an edge if and only if they share the best score in a subject. Since, for any subject, exactly three students got the best score in the subject, it follows that to each subject corresponds a unique triangle in the graph, and that each edge belongs to a triangle. (Whenever we refer to a "triangle", we will mean one of these triangles which correspond to the subjects.) Since, for any two subjects, exactly one student got the best score in both subjects, it follows that the graph is simple (no multiple edges). Thus, each edge belongs to exactly one triangle. Moreover, two triangles share exactly one vertex.

Now, we remark that if 4 triangles have a common vertex, then all the triangles have this vertex. Otherwise, a fifth triangle would have a common vertex with each of the 4 triangles, and these vertices would have to be distinct, which forces this fifth triangle to have more than 3 vertices.

Assume first that $n \geq 8$. Let T be an arbitrary triangle. It shares a vertex with at least 7 other triangles. The Pigeonhole Principle ensures that it shares the same vertex with at least three other triangles, so that we are in the situation described above.

To complete the proof, we now give an example with 7 subjects for which there is no student who got the best score in all the subjects. (In the table, a number denotes a subject, a letter denotes a student, and a cross denotes the best score.)

	A	B	C	D	E	F	G
1	×	×	×				
2	×			×	×		
3		×		×		×	
4			×	×			×
5	×					×	×
6		×			×		×
7			×		×	×	

4. [10 marks] Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x, y \in \mathbb{R}$,

$$f\big(x + yf(x)\big) \;=\; f(x) + xf(y)\,.$$

Solution by Michel Bataille, Rouen, France.

The solutions are the zero function and the identity function. Clearly, these functions satisfy the given condition (which we will denote by $\mathcal{C}$). Conversely, we show that any non-zero function f satisfying $\mathcal{C}$ is given by $f(x) = x$ for all real x. The proof goes through eleven steps.

(1) $f(0) = 0$.

This follows from $\mathcal{C}$ with $x = 1$ and $y = 0$.

(2) $f(x) = 0 \implies x = 0$.

If $f(x) = 0$, then $0 = f(x) = f\big(x + yf(x)\big) = xf(y)$, which implies that $x = 0$, since we can choose y such that $f(y) \neq 0$.

(3) $f(-1) = -1$.

This follows from step (2) and $\mathcal{C}$ with $x = y = -1$.

(4) $f\big(x - f(x)\big) = f(x) - x$ for all x.

Apply $\mathcal{C}$ with $y = -1$ and use (3).

(5) $f(1) = 1$.

Applying $\mathcal{C}$ with $x = 1 - f(1)$ and $y = 1$, and using (4), we have $f(0) = -(1 - f(1))^2$; then use (1).

(6) $f(t - 1) = f(t) - 1$ for all t.

Apply $\mathcal{C}$ with $x = 1$ and $y = t - 1$, and use (5).

(7) $f(u) = u \Longrightarrow f(tu) = uf(t)$ for all t.

Suppose that $f(u) = u$. Then

$$f(tu) \;=\; f\big(u + (t-1)f(u)\big) \;=\; f(u) + uf(t-1)\,,$$

using $\mathcal{C}$ with $x = u$ and $y = t - 1$. Using (6), we get

$$f(tu) \;=\; f(u) + u\big(f(t) - 1\big) \;=\; uf(t)\,.$$

(8) f is odd.

Apply (7) with $u = -1$, noting (3).

(9) $f(v) = -v \Longrightarrow f(tv) = -vf(t)$ for all t.

This is analogous to (7). Apply $\mathcal{C}$ with $x = v$ and $y = 1 - t$, and use (8) and (6) to get $f(1-t) = -f(t-1) = 1 - f(t)$.

(10) $f(x^2) = \big(f(x)\big)^2$ for all x.

If $f(x) = x$, the result follows from (7) with $u = t = x$. Otherwise, let $v = x - f(x)$. Then $v \neq 0$ and by (4) we have $f(v) = -v$. Now, (9) and $\mathcal{C}$ with $y = \frac{x}{v}$ together yield

$$\begin{aligned} f(x^2) &= f\big(vx + xf(x)\big) \;=\; -vf\Big(x + \frac{x}{v}f(x)\Big) \\ &= -vf(x) + x(-v)f\Big(\frac{x}{v}\Big) \;=\; -vf(x) + xf(x) \;=\; \big(f(x)\big)^2\,. \end{aligned}$$

(11) $f(r) \geq 0$ if $r \geq 0$.

Use $x = \sqrt{r}$ in (10).

Now, let x be any real number. If $x \geq f(x)$, then $f\big(x - f(x)\big) \geq 0$ by (11); hence, by (4), we have $f(x) \geq x$, and eventually $f(x) = x$. If $x \leq f(x)$, then $f\big(f(x) - x\big) \geq 0$ and, since f is odd, we have $-f\big(x - f(x)\big) \geq 0$. Therefore, $x - f(x) \geq 0$ and eventually $f(x) = x$. In either case, $f(x) = x$.

For an alternative solution, see *Mathematics Magazine*, Vol. 72, No. 3, June 1999, pp. 241–242.

Next we turn to problems of the 17th Balkan Mathematical Olympiad, appearing [2004 : 84].

1. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f\big(xf(x) + f(y)\big) \;=\; \big(f(x)\big)^2 + y\,,$$

for any real numbers x and y.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. We give the solution by Díaz-Barrero.

The functions $f(x) = x$ and $f(x) = -x$ are solutions. We claim that these are the only solutions.

Let $f:\mathbb{R}\to\mathbb{R}$ such that, for all real numbers x and y,

$$f\big(xf(x)+f(y)\big) \;=\; \big(f(x)\big)^2+y\,. \qquad (1)$$

Let $f(0)=a$. Setting $x=0$ in (1) yields

$$f\big(f(y)\big) \;=\; a^2+y \qquad (2)$$

for all $y\in\mathbb{R}$. This equation shows that f is a bijection. As a consequence, there exists b such that $f(b)=0$. Setting $x=b$ in (1), we get, for all $y\in\mathbb{R}$,

$$f\big(f(y)\big) \;=\; y\,. \qquad (3)$$

Comparing (2) and (3), we see that $a=0$ (and hence $b=0$). Then, substituting $y=0$ into (1), we get

$$f\big(xf(x)\big) \;=\; \big(f(x)\big)^2\,, \qquad (4)$$

for all $x\in\mathbb{R}$. Now, setting $x=f(t)$ in (4) gives

$$f\big(tf(t)\big) \;=\; t^2\,, \qquad (5)$$

for all $t\in\mathbb{R}$. Comparing (4) and (5), we get

$$\big(f(x)\big)^2 \;=\; x^2 \qquad (6)$$

for all $x\in\mathbb{R}$. Thus, for each $x\in\mathbb{R}$, we have either $f(x)=x$ or $f(x)=-x$.

Suppose there exist non-zero numbers α and β such that $f(\alpha)=-\alpha$ and $f(\beta)=\beta$. Then, taking $x=\alpha$ and $y=\beta$ in (1), we get

$$f(-\alpha^2+\beta) \;=\; \alpha^2+\beta\,,$$

which contradicts (6). We conclude that $f(x)=x$ for all $x\in\mathbb{R}$ or $f(x)=-x$ for all $x\in\mathbb{R}$.

2. Let ABC be a non-isosceles acute triangle, and let E be an interior point of the median AD, with D on BC. Let F be the orthogonal projection of E onto the line BC. Let M be an interior point of the segment EF, and let N and P be the orthogonal projections of M onto the lines AC and AB, respectively. Prove that the bisectors of angles PMN and PEN are parallel.

Solved by Toshio Seimiya, Kawasaki, Japan; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's write-up.

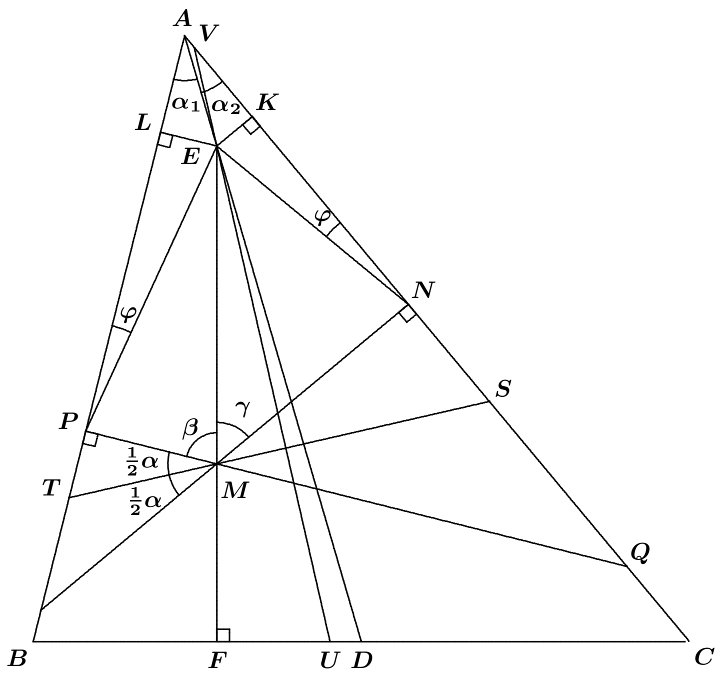
Since $\triangle ABC$ is not isosceles, we may assume that $AB<AC$.

As usual, we set $\alpha=\angle CAB$, $\beta=\angle ABC$, and $\gamma=\angle BCA$. We further set $\alpha_1=\angle BAD$ and $\alpha_2=\angle CAD$. Then

$$\sin\alpha_1:\sin\alpha_2 \;=\; \sin\beta:\sin\gamma \;=\; b:c\,. \qquad (1)$$

Let K and L be the points on AC and AB, respectively, such that $EK\perp AC$ and $EL\perp AB$. Then $EK=AE\sin\alpha_2$ and $EL=AE\sin\alpha_1$. Hence, using (1), we get

$$EK:EL \;=\; \sin\gamma:\sin\beta\,. \qquad (2)$$



Since quadrilateral $BFMP$ is cyclic, we have$\angle PME = \beta$. Similarly, $\angle NME = \gamma$ and $\angle NMQ = \alpha$, where Q is the point where the production of PM meets the line AC. Then $KN = EM \sin\gamma$ and $LP = EM \sin\beta$. This, together with (2), implies that $EK : EL = KN : LP$. Since we also have $\angle NKE = \angle PLE = 90^\circ$, it follows that

$$\triangle EKN \sim \triangle ELP . \tag{3}$$

Setting $\varphi = \angle ENK = \angle EPL$, we see that

$$\angle PEN = 360^\circ - (180^\circ - \alpha) - 2(90^\circ - \varphi) = 2\varphi + \alpha .$$

Let the bisector of $\angle PEN$ meet BC at U. Then $\angle UEN = \varphi + \frac{1}{2}\alpha$. Let the production of UE intersect AC at V. In $\triangle NEV$, we see that $\angle EVN + \angle VNE = \varphi + \frac{1}{2}\alpha$. Since $\angle VNE = \varphi$, we have $\angle EVN = \frac{1}{2}\alpha$. This implies that EU is parallel to the bisector of $\angle BAC$.

Let the bisector of $\angle NMQ$ intersect AC at S, and the production of SM intersect AB at T. It is easy to see that $\triangle ATS$ is isosceles, since both $\angle ATS$ and $\angle AST$ have the value $90^\circ - \frac{1}{2}\alpha$. Thus, the bisector of $\angle BAC$ is perpendicular to ST. The bisector of $\angle PMN$ is also perpendicular to ST, since it is perpendicular to the bisector of $\angle NMQ$.

Thus, the bisectors of angles PMN and PEN are both parallel to that of $\angle BAC$.

4. We say that a positive integer r is a *power* if it has the form $r = t^s$, for some integers $t \geq 2$ and $s \geq 2$. Show that, for any positive integer n, there exists a set A of n positive integers which satisfies the following conditions:

(i) Every element of A is a power.

(ii) For any k elements $r_1, r_2, \ldots, r_k$ from A (where $2 \leq k \leq n$), the number $\frac{r_1 + r_2 + \cdots + r_k}{k}$ is a power.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

First we prove the following lemma:

Lemma. For each integer $m \geq 1$ there exists an integer $a > 0$ such that each of the numbers $a, 2a, \ldots, ma$ is a power.

Proof: We use induction on m. For $m = 1$, simply choose $a = 4$. Let $m \geq 1$ be given, and assume that the result holds for this value of m. Then, there exists an integer $a > 0$ such that, for each $i = 1, 2, \ldots, m$, there exist two integers $x_i \geq 2$ and $\alpha_i \geq 2$ such that $ia = x_i^{\alpha_i}$.

Let $\alpha = \operatorname{lcm}\{\alpha_i \mid i = 1, 2, \ldots, m\}$ and let $b = a\big((m+1)a\big)^\alpha$. Let $i \in \{1, 2, \ldots, m\}$. Then there exists $\beta_i \in \mathbb{N}^*$ such that $\alpha = \alpha_i\beta_i$. Thus,

$$ib = x_i^{\alpha_i}\big((m+1)a\big)^{\alpha_i\beta_i} = \Big[x_i\big((m+1)a\big)^{\beta_i}\Big]^{\alpha_i}.$$

Moreover, $(m+1)b = ((m+1)a)^{\alpha+1}$. This proves the induction step. ∎

Now consider a positive integer n. Using the lemma with $m = n \times n!$, it follows that there exists a positive integer a such that each of the numbers $a, 2a, \ldots, n \times n!a$ is a power. Now let $A = \{n!a, 2n!a, \ldots, n \times n!a\}$. Clearly $|A| = n$ and each of the elements of A is a power. Now, let us give $r_1, r_2, \ldots, r_k$ from A, where $r_i = ax_in!$ with $x_i \in \{1, 2, \ldots, n\}$. Then

$$r_1 + r_2 + \cdots + r_k = an!(x_1 + x_2 + \cdots + x_k).$$

But $1 \leq x_1 + x_2 + \cdots + x_k \leq nk$. Thus, $\frac{r_1 + r_2 + \cdots + r_k}{k} = ra$, where $r = \frac{n!(x_1 + x_2 + \cdots + x_k)}{k}$ is a positive integer (since k divides $n!$) and $r \leq n \times n!$. From the construction, it follows that $\frac{r_1 + r_2 + \cdots + r_k}{k} = ra$ is a power, and we are done.

Next we turn to solutions from our readers to problems given in the April 2004 number of the *Corner* starting with the Israel Mathematical Olympiad 2001 given [2004 : 140–141].

1. Find all solutions of

$$\begin{aligned} x_1 + x_2 + \cdots + x_{2000} &= 2000, \\ x_1^4 + x_2^4 + \cdots + x_{2000}^4 &= x_1^3 + x_2^3 + \cdots + x_{2000}^3. \end{aligned}$$

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This is the same problem (with 2000 instead of 1997) as one of the Ukrainian Mathematical Olympiad problems from 1997. A solution appears in [2003 : 89–90].

2. Given 2001 real numbers x_1, x_2, $\ldots$, x_{2001} such that $0 \leq x_n \leq 1$ for each $n = 1, 2, \ldots, 2001$, find the maximum value of

$$\left(\frac{1}{2001}\sum_{n=1}^{2001} x_n^2\right) - \left(\frac{1}{2001}\sum_{n=1}^{2001} x_n\right)^2 .$$

Where is this maximum attained?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The given expression is convex in each of the variables. Thus, it reaches its maximum when all the x_ns are end-points of the interval of study; that is, $x_n \in \{0, 1\}$ for each n.

Let k be the number values n such that $x_n = 1$. Then the expression is

$$S = \frac{k}{2001} - \frac{k^2}{2001^2} = \frac{1}{2001^2}\left(\frac{2001^2}{4} - \left(k - \frac{2001}{2}\right)^2\right) .$$

Since k is an integer, we deduce that $S \leq \frac{1}{2001^2}\left(\frac{2001^2}{4} - \frac{1}{4}\right) = \frac{1001000}{4004001}$, and equality occurs if and only if $k = 1000$ or $k = 1001$.

3. We are given 2001 lines in the plane, no two of which are parallel and no three of which pass through a common point. These lines partition the plane into some regions (not necessarily finite) bounded by segments of these lines. These segments are called *sides*, and the collection of the regions is called a *map*. Two regions on the map are called *neighbours* if they share a side.

The set of intersection points of the lines is called the set of vertices. Two vertices are called *neighbours* if they are found on the same side.

A *legal colouring of the map* is a colouring of the regions (one colour per region) such that neighbouring regions have different colours.

A *legal colouring of the vertices* is a colouring of the vertices (one colour per vertex) such that neighbouring vertices have different colours.

(i) What is the minimum number of colours required for a legal colouring of the map?

(ii) What is the minimum number of colours required for a legal colouring of the vertices?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Though detailed, the statement is not really clear. I assume that in the graph theoretic wording, the regions are the faces of the representation of the planar graph defined by the set of lines (the 'elementary' regions), and that two intersection points are connected by an edge if and only if they are end-points of some segment on a line, but with no other point between them.

(i) We will answer the problem in the case of $n \geq 1$ lines, and prove that the minimum is 2.

Clearly, we need at least two colours, since there are at least two neighbouring regions.

We prove by induction on n that a legal 2-colouring does exist. It is obvious for $n = 1$.

Next, suppose that, for some given n, any map defined by n lines (as in the statement of the problem) has a legal 2-colouring.

Now consider a map defined by $n + 1$ lines, say $\ell_1, \ldots, \ell_{n+1}$. First, delete ℓ_{n+1}, and use a legal 2-colouring for the map defined by $\ell_1, \ldots, \ell_n$ as given by the induction hypothesis. Now restore ℓ_{n+1}. This line separates the plane into two half-planes, say Π_1 and Π_2. Some of the "old" regions may be separated by ℓ_{n+1} into two regions, one in each of the two half-planes. Then, keep the colours of all the regions belonging to Π_1. And give to each of the regions belonging to Π_2 the opposite colour to what it had before.

Since the initial colouring was legal, we do not have two neighbouring regions which belong to Π_1 (respectively, Π_2) with the same colour. And, with the interchanging of colours made in Π_2, we do not have two neighbouring regions (which formed an old region divided by ℓ_{n+1}), one in each of the half-planes, with the same colour.

Thus, we have found a legal 2-colouring for the map defined by the $n + 1$ lines. This ends the induction step and the proof.

(ii) We will answer the problem in the case of $n \geq 3$ lines.

First note that it is easy to see that there always will be at least one region which is a triangle. In fact, we may prove (see [1]) that there are at least $n - 2$ regions which are triangles. It follows that we need at least three colours for a legal colouring of the vertices.

Now we prove that a legal 3-colouring of the vertices does exist: Since there are a finite number of lines, there are a finite number, say k, of intersection points (namely, $k = n(n-1)/2$ since no two lines are parallel, and no three pass through the same point).

Thus, we may choose an orthogonal system of coordinates such that the intersection points are $M_1, \ldots, M_k$, with $M_i = (x_i, y_i)$ and $x_1 < \cdots < x_k$. Now we colour the M_is in the increasing order of their respective subscripts, as follows: Colour M_1 with colour c_1 and M_2 with colour c_2.

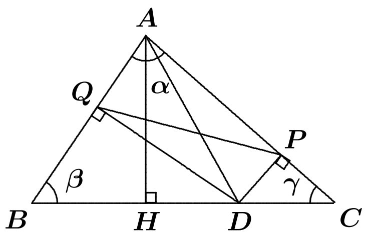
If, for $1 \leq i \leq k - 1$, the points $M_1, \ldots, M_i$ have been coloured with only three colours, and that 3-colouring is legal for the set of already coloured vertices, then we note that M_{i+1} has at most 4 neighbours, and among them at most two can have an x–coordinate smaller than x_{i+1}. Then, at least one colour is not used for the coloured neighbours of M_{i+1}. Thus, give any of these unused colours to M_{i+1}. This produces a legal 3-colouring for the set of vertices $M_1, \ldots, M_{i+1}$. Then, we construct by induction a legal 3-colouring of the set of vertices, and we are done.

Reference.

[1] St. Petersburg Contests, 1965–1984, p. 23.

4. The lengths of the sides of triangle ABC are 4, 5, 6. For any point D on one of the sides, drop the perpendiculars DP, DQ onto the other two sides (P, Q are on the sides). What is the minimal value of PQ?

Solved by Toshio Seimiya, Kawasaki, Japan; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya's solution.



First we consider an arbitrary acute triangle ABC. We set $a = BC$, $b = CA$, $c = AB$, $\alpha = \angle A$, $\beta = \angle B$, and $\gamma = \angle C$, and we denote by R the circumradius of $\triangle ABC$.

Suppose that D is on the side BC. Let H be the foot of the perpendicular from A to BC, and let P and Q be the feet of the perpendiculars from D to AC and AB, respectively. Since $AQDP$ is a cyclic quadrilateral, Ptolemy's Theorem implies that

$$PQ \;=\; AD\sin\angle PAQ \;=\; AD\sin\alpha \;\geq\; AH\sin\alpha\,.$$

Since

$$AH \;=\; AB\sin\beta \;=\; c\sin\beta \;=\; 2R\sin\gamma\cdot\sin\beta\,,$$

we see that $AH\sin\alpha = 2R\sin\alpha\sin\beta\sin\gamma$. Thus, the minimal value of PQ is $2R\sin\alpha\sin\beta\sin\gamma$.

If D is a point on either AB or AC, the minimal value of PQ is also $2R\sin\alpha\sin\beta\sin\gamma$. Thus, the minimal value of PQ for any point D on the perimeter of $\triangle ABC$ is $2R\sin\alpha\sin\beta\sin\gamma$; that is, $AH\sin\alpha$.

Now let $a = 6$, $b = 5$, and $c = 4$. Since $a > b > c$, we have $\alpha > \beta > \gamma$. By the Law of Cosines, $a^2 = b^2 + c^2 - 2bc\cos\alpha$, which means that $a^2 = 5^2 + 4^2 - 2\cdot 5\cdot 4\cos\alpha$. Hence, $40\cos\alpha = 5^2 + 4^2 - 6^2 = 5$, or $\cos\alpha = \frac{1}{8}$. Therefore, α is acute.

Since $\alpha > \beta > \gamma$, we see that $\triangle ABC$ is an acute triangle. Thus, the minimal value of PQ is $AH\sin\alpha$.

Since $\cos\alpha = \frac{1}{8}$, we have $\sin^2\alpha = 1 - \cos^2\alpha = 1 - \frac{1}{64} = \frac{63}{64}$. Since $AH\cdot BC = AB\cdot AC\sin\alpha$, we also have $AH\cdot a = bc\sin\alpha$; that is, $AH = \dfrac{bc\sin\alpha}{a}$. Hence,

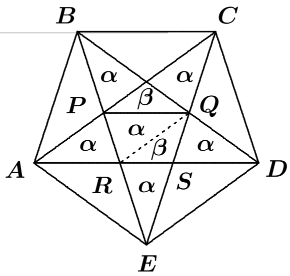
$$AH\sin\alpha \;=\; \frac{bc\sin^2\alpha}{a} \;=\; \frac{5\cdot 4}{6}\cdot\frac{63}{64} \;=\; \frac{105}{32}\,.$$

Therefore, the minimal value of PQ is $\frac{105}{32}$.

Next we turn to readers' solutions to problems of the 21st Brazilian Mathematical Olympiad 2001 given at [2004 : 141–142].

1. Let $ABCDE$ be a regular pentagon such that the star $ACEBD$ has area 1. Let P be the point of intersection of AC and BE, and let Q be the point of intersection of BD and CE. Find the area of $APQD$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.



Let R be the point of intersection of AD and BE, and S that of AD and CE. Each corner angle of a regular pentagon is trisected by the two diagonals passing through its vertex into three 36° angles. It follows that $PQ \parallel AR$ and $AP \parallel RQ$.

Let $\alpha = [PQR]$ and $\beta = [QRS]$. Then

$$[APQD] = 3\alpha + \beta = \tfrac{1}{2}(6\alpha + 2\beta) = \tfrac{1}{2}[ACEBD] = \tfrac{1}{2}.$$

5. There are n football teams in Tumbolia. A championship is to be organized in which each team plays against every other exactly once. Every match must take place on a Sunday, and no team can play more than once on the same day.

Find the least positive integer m for which it is possible to set up a championship lasting m Sundays.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is similar to one given at the Romanian Olympiad 1978, 10th class, final round. The answer is $m = n$ if n is odd, and $m = n - 1$ if n is even. A solution appears in: R. Honsberger, *More Mathematical Morsels*, pp. 80-82, Mathematical Association of America.

To complete this month's *Corner*, we give solutions to problems of the 49th Mathematical Olympiad of Lithuania 2000 given in [2004 : 142-143].

1. In a family there are four children of different ages, each age being a positive integer not less than 2 and not greater than 16. A year ago the square of the age of the eldest child was equal to the sum of the squares of the ages of the remaining children. One year from now the sum of the squares of the youngest and the oldest will be equal to the sum of the squares of the other two. How old is each child?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Titu Zvonaru, Comănești, Romania. We give the solution of Zvonaru.

Let x, y, z, t be the ages of the four children, where

$$2 \le x < y < z < t \le 16.$$

We have

$$(x-1)^2+(y-1)^2+(z-1)^2 \;=\; (t-1)^2 \qquad (1)$$

and $$(x+1)^2+(t+1)^2 \;=\; (y+1)^2+(z+1)^2\,. \qquad (2)$$

By taking the sum and difference of these equations and simplyifying, we get

$$2(y+z-t) \;=\; x^2+1 \qquad (3)$$

and $$y^2+z^2 \;=\; t^2+2x-1\,. \qquad (4)$$

Since $z < t$ and $y \leq 14$, it follows that $2(y+z-t) \leq 28$. Hence, $x^2+1 \leq 28$ from (3). It also follows from (3) that x is odd. Thus, we have $x \in \{3,\,5\}$.

If $x=3$, then (3) and (4) become

$$y+z-t \;=\; 5$$

and $$y^2+z^2 \;=\; t^2+5\,.$$

Then $y^2+z^2 = (y+z-5)^2+5$, which gives $yz-5y-5z+15=0$, or $(y-5)(z-5)=10$. Since $y<z$, we must have either $y-5=2$ and $z-5=5$, or $y-5=1$ and $z-5=10$. The first case gives $x=3$, $y=7$, $z=10$, and $t=12$, and the second gives $x=3$, $y=6$, $z=15$, and $t=16$.

If $x=5$, we get

$$y+z-t \;=\; 13$$

and $$y^2+z^2 \;=\; t^2+9\,.$$

Then $y^2+z^2=(y+z-13)^2+9$, which gives $yz-13y-13z+89=0$, or $(y-13)(z-13)=120$. But this equation has no solutions, since $y-13\leq 1$ and $z-13\leq 2$.

Therefore, the given problem has exactly two solutions, namely

$$(x,y,z,t) \;=\; (3,6,15,16) \qquad \text{and} \qquad (x,y,z,t) \;=\; (3,7,10,12)\,.$$

2. A sequence a_1, a_2, a_3, $\ldots$ is defined such that $a_n = n^2+n+1$ for all $n\geq 1$. Prove that the product of any two consecutive members of the sequence is itself a member of the given sequence.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Bataille's write-up.

It suffices to remark that

$$\begin{aligned} a_na_{n+1} &= (n^2+n+1)(n^2+3n+3) \;=\; n^4+4n^3+7n^2+6n+3 \\ &= (n+1)^4+(n+1)^2+1 \;=\; a_{(n+1)^2}\,. \end{aligned}$$

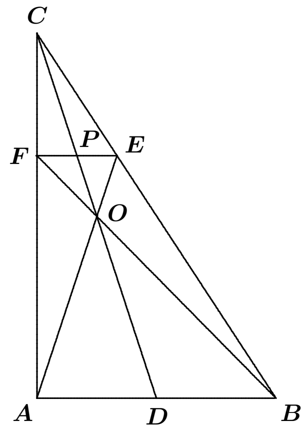
3. In the triangle ABC, the point D is the mid-point of the side AB. Point E divides BC in the ratio $BE:EC=2:1$. Given that $\angle ADC=\angle BAE$, determine $\angle BAC$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Toshio Seimiya, Kawasaki, Japan; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give the solution by Amengual Covas.

Let the lines CD and AE intersect at O. Let F be the intersection of BO and CA and let P be the intersection of EF and CD.

Then, by Ceva's Theorem, $\frac{CF}{FA}\cdot\frac{AD}{DB}\cdot\frac{BE}{EC}=1$. But we are assuming that $AD=DB$. Hence, $\frac{CF}{FA}=\frac{CE}{EB}$, which implies that EF is parallel to AB and

$$\angle EFC = \angle BAC\,. \tag{1}$$



Triangles OAD and OEP have equal corresponding angles and therefore are similar; and since $\triangle OAD$ is isosceles, so is $\triangle OEP$ with $OE=OP$.

Similarly, $\triangle CFE$ is similar to $\triangle CAB$. It follows that

$$\frac{EF}{AD} = \frac{EF}{\frac{1}{2}AB} = 2\cdot\frac{EF}{AB} = 2\cdot\frac{CE}{BC}$$

and

$$\frac{AE}{CD} = \frac{PD}{CD} = \frac{BE}{BC}$$

because $AE=AO+OE=DO+OP=PD$.

Since $BE : EC = 2 : 1$, we have $2\cdot\frac{CE}{BC}=\frac{BE}{BC}\left(=\frac{2}{3}\right)$. Thus, $\frac{EF}{AD}=\frac{AE}{CD}$ (from above). Therefore, triangles EFA and DAC are similar (SAS) with

$$\angle EFA = \angle DAC = \angle BAC\,. \tag{2}$$

Now $\angle EFA$ and $\angle EFC$ are supplementary angles and thus sum to 180°. It follows from (1) and (2) that $\angle BAC=90^\circ$.

4. Find all the triples of positive integers x, y, z with $x\le y\le z$ such that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

is a positive integer.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Bornsztein's solution.

Let x, y, z satisfy the conditions in the problem.

If $x\ge 4$, then y, $z\ge 4$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le\frac{3}{4}<1$. Thus, $x\in\{1,2,3\}$.

Case 1. $x = 1$.

Then $\frac{1}{y} + \frac{1}{z}$ is a positive integer. If $y \geq 3$, then $z \geq 3$ and $\frac{1}{y} + \frac{1}{z} < 1$. It follows that $y \in \{1, 2\}$. If $y = 1$, then $\frac{1}{z}$ is an integer, which forces $z = 1$. If $y = 2$, then $\frac{1}{z} = k - \frac{1}{2} = \frac{2k-1}{2}$, where k is a positive integer. Thus, $z = \frac{2}{2k-1}$, which forces $k = 1$ and $z = 2$.

Conversely, the triples $(1, 1, 1)$ and $(1, 2, 2)$ are solutions of the problem.

Case 2. $x = 2$.

Then $\frac{1}{y} + \frac{1}{z} = \frac{2k-1}{2}$, where k is a positive integer. Thus, $\frac{1}{y} + \frac{1}{z} \geq \frac{1}{2}$. If $y \geq 5$, then $z \geq 5$ and $\frac{1}{y} + \frac{1}{z} \leq \frac{2}{5} < \frac{1}{2}$, a contradiction. It follows that $y \in \{2, 3, 4\}$.

If $y = 4$, then, as above, $z = \frac{4}{4k-3}$, and hence, $k = 1$ and $z = 4$. If $y = 3$, then $z = \frac{6}{6k-5}$, implying that $k = 1$ and $z = 6$. If $y = 2$, then $\frac{1}{z} = k - 1$ is an integer, even though $z \geq 2$, which is absurd.

Conversely, the triples $(2, 4, 4)$ and $(2, 3, 6)$ are solutions of the problem.

Case 3. $x = 3$.

Then $\frac{1}{y} + \frac{1}{z} = k - \frac{1}{3} \geq \frac{2}{3}$, where k is a positive integer. If $y \geq 4$, then $z \geq 4$ and $\frac{1}{y} + \frac{1}{z} \leq \frac{1}{2} < \frac{2}{3}$, a contradiction. It follows that $y = 3$. Then $\frac{1}{2} = k - \frac{2}{3} = \frac{3k-2}{3}$. Thus, $z = \frac{3}{3k-2}$, which forces $k = 1$ and $z = 3$.

Conversely, the triple $(3, 3, 3)$ is a solution of the problem.

In conclusion, the solutions are $(1, 1, 1)$, $(1, 2, 2)$, $(2, 4, 4)$, $(2, 3, 6)$, and $(3, 3, 3)$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following equation for all real x and y:

$$(x + y)(f(x) - f(y)) = f(x^2) - f(y^2)\,.$$

Find: (a) one such function; (b) all such functions.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.

Any affine function $x \mapsto ax + b$ (for some real numbers a, b) clearly satisfies the functional equation. Conversely, we show that any solution is an affine function.

Let f be any solution. Set $b = f(0)$ and $g(x) = f(x) - b$ ($x \in \mathbb{R}$). It is readily seen that g is a solution as well and satisfies $g(0) = 0$. Taking $y = 0$

in the given equation (written for g) yields

$$(x+0)\big(g(x)-g(0)\big) \;=\; g(x^2)-g(0)\,;$$

that is, $xg(x) = g(x^2)$ for all real x. Substituting $-x$ for x gives us $-xg(-x) = g(x^2)$, and it follows that g is an odd function. Thus,

$$(x+y)\big(g(x)-g(y)\big) \;=\; g(x^2)-g(y^2) \;=\; (x-y)\big(g(x)+g(y)\big)\,.$$

Thus, $xg(y) = yg(x)$ for all x and y. As a result, $g(x)/x$ is constant on $\mathbb{R}\backslash\{0\}$ and $g(x) = xg(1)$ for all real numbers x. It follows that $f(x) = xg(1) + b$ and f is an affine function.

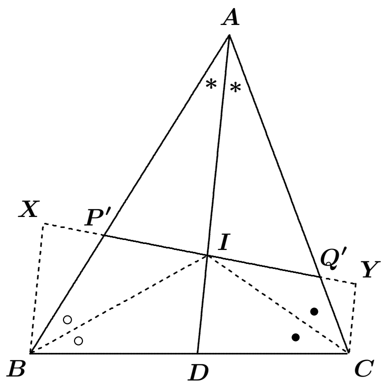
7. A line divides both the area and the perimeter of a triangle into two equal parts. Prove that this line passes through the incentre of the triangle. Does the converse statement always hold?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Toshio Seimiya, Kawasaki, Japan; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. First, we give Seimiya's solution.

Let ABC be a given triangle with incentre I. We may assume without loss of generality that the line in the problem intersects the sides AB and AC at P and Q, respectively.

We set $a = BC$, $b = CA$, and $c = AB$. Let D be the intersection of AI with BC. Since BI and CI are the bisectors of $\angle ABD$ and $\angle ACD$, respectively, we have

$$\frac{AI}{ID} \;=\; \frac{AB}{BD} \;=\; \frac{AC}{CD} \;=\; \frac{AB+AC}{BD+CD} \;=\; \frac{AB+AC}{BC} \;=\; \frac{b+c}{a}\,.$$



Let ℓ be a line through I which intersects sides AB and AC at P' and Q', respectively. Let X and Y be points on the line $P'Q'$ such that $BX \parallel AD$ and $CY \parallel AD$. Then

$$\frac{P'B}{AP'} \;=\; \frac{BX}{AI}\,, \qquad \text{and} \qquad \frac{Q'C}{AQ'} \;=\; \frac{CY}{AI}\,.$$

Since $BD : DC = AB : AC = c : b$, and $BX \parallel DI \parallel CY$, we get

$$b\cdot BX + c\cdot CY \;=\; (b+c)\cdot DI\,.$$

Hence,

$$b\,\frac{BX}{AI} + c\,\frac{CY}{AI} \;=\; (b+c)\frac{DI}{AI}\,;$$

that is,

$$b\,\frac{P'B}{AP'} + c\,\frac{Q'C}{AQ'} \;=\; (b+c)\frac{DI}{AI}\,.$$

Conversely, if P' and Q' are points on the sides AB and AC, respectively, and if

$$b\,\frac{P'B}{AP'} + c\,\frac{Q'C}{AQ'} \;=\; (b+c)\frac{ID}{AI}\,, \tag{1}$$

then P', Q', and I are collinear. (*Proof:* Let I' be the intersection of $P'Q'$ with AD. Then

$$b\,\frac{P'B}{AP'} + c\,\frac{Q'C}{AQ'} \;=\; (b+c)\frac{I'D}{AI'}\,,$$

from which we have $\frac{I'D}{AI'} = \frac{ID}{AI}$. Then I' coincides with I.)

Therefore, P, Q, and I are collinear if and only if (1) holds.

We set $x = AP$ and $y = AQ$. Then (1) becomes

$$b\,\frac{c-x}{x} + c\,\frac{b-y}{y} \;=\; (b+c)\frac{a}{b+c} \;=\; a\,;$$

that is,

$$bc\left(\frac{1}{x} + \frac{1}{y}\right) \;=\; a+b+c\,. \tag{2}$$

If PQ divides both the area and the perimeter of $\triangle ABC$ into two equal parts, then $xy = \frac{1}{2}bc$ and $x + y = \frac{1}{2}(a+b+c)$. Thus,

$$bc\left(\frac{1}{x} + \frac{1}{y}\right) \;=\; bc\,\frac{x+y}{x} \;=\; a+b+c\,,$$

and (2) holds. Therefore, PQ passes through I.

Next we consider converses.

I. If PQ passes through I and divides the area of $\triangle ABC$ into two equal parts, then PQ divides the perimeter of $\triangle ABC$ into two equal parts.

Since $bc\Big(\frac{1}{x} + \frac{1}{y}\Big) = a + b + c$ and $xy = \frac{1}{2}bc$, we obtain $x + y = \frac{a+b+c}{2}$. Thus, PQ divides the perimeter of $\triangle ABC$ into two equal parts.

II. If PQ passes through I and divides the perimeter of $\triangle ABC$ into two equal parts, then PQ divides the area of $\triangle ABC$ into two equal parts.

Since $bc\Big(\frac{1}{x} + \frac{1}{y}\Big) = a + b + c$ and $x + y = \frac{a+b+c}{2}$, we have $xy = \frac{1}{2}bc$. Thus, PQ divides the area of $\triangle ABC$ into two equal parts.

III. If PQ passes through I, then PQ divides both the area and the perimeter of $\triangle ABC$ into two equal parts.

This converse is not correct.

Next we give Bornsztein's generalization.

More generally, we will prove that the result holds for any convex polygon $\mathcal{P}$ into which we can inscribe a circle:

Let ℓ be a line which divides $\mathcal{P}$ into two sub-polygons, say $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $p(\mathcal{P}_1) = p(\mathcal{P}_2)$ and $[\mathcal{P}_1] = [\mathcal{P}_2]$ (where $p(\cdot)$ denotes the perimeter, and $[\cdot]$ denotes area). Let A and B be the points where ℓ meets $\mathcal{P}$.

Let I and r be the centre and the radius, respectively, of the incircle of $\mathcal{P}$ and let H be the orthogonal projection of I onto ℓ. With no loss of generality, we may assume that I belongs to the boundary of $\mathcal{P}_1$ (and $\mathcal{P}_2$) or is an interior point of $\mathcal{P}_1$ (and an exterior point of $\mathcal{P}_2$).

Then

$$[\mathcal{P}_1] \;=\; \tfrac{1}{2}r(p(\mathcal{P}_1) - AB) + \tfrac{1}{2}IH \cdot AB$$

and

$$[\mathcal{P}_2] \;=\; \tfrac{1}{2}r(p(\mathcal{P}_2) - AB) - \tfrac{1}{2}IH \cdot AB\,.$$

Since $p(\mathcal{P}_1) = p(\mathcal{P}_2)$ and $[\mathcal{P}_1] = [\mathcal{P}_2]$, we get $IH \cdot AB = 0$; that is, $I = H$. Then $I \in \ell$, as desired.

The converse does not hold: Consider an equilateral triangle ABC and the line through I parallel to the line BC, which meets the sides AB and AC in M and P, respectively.

Then $[AMP] = \frac{4}{9}[ABC]$, and $[BMPC] = \frac{5}{9}[ABC] \neq [AMP]$.

8. The equation $x^2 + y^2 + z^2 + u^2 = xyzu + 6$ is given. Find:

(a) at least one solution in positive integers;

(b) at least 33 such solutions;

(c) at least 100 such solutions.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Wang's solution.

For all $n \in \mathbb{N}$, $(x,y,z,u) = (1,2,n,n+1)$ satisfies the given equation since $x^2 + y^2 + z^2 + u^2 = xyzu + 6 = 2n^2 + 2n + 6$. Hence, there are infinitely many solutions in positive integers.

That completes the *Corner* for this issue and this volume of ***CRUX with MAYHEM***. Remember to send in your solutions promptly so that we can aim to keep the time between giving problems and looking at the solutions reasonably short. Also send me your Olympiad contest materials for the *Corner*.

BOOK REVIEWS

John Grant McLoughlin

Ants, Bikes, and Clocks: Problem Solving for Undergraduates
By William Briggs, published by Siam, 2004
ISBN 0-89871-574-1, softcover, vi+170 pages, US$42.00.
Reviewed by ***Robert Bilinski****, Collège Montmorency, Laval, QC.*

Ants, Bikes, and Clocks does not aim to teach mathematics, but rather to develop the art of problem solving. The technique used in each of the first ten chapters is to dissect a few choice problems while giving complete solutions. The student is then presented with a set of 26 problems (on average), some of which resemble the solved ones, but many of which do not. The eleventh chapter contains ten review problems in no particular order, and the twelfth contains complete solutions to odd-numbered problems.

The chapters evolve nicely in complexity. The problems in the first chapter need little in the way of mathematics, but instead require attentive reading. (A sample is: "If Mr. Kerry's rooster laid an egg in Mr. Bush's yard, who owns the egg?".) The second chapter gives practice in translating various situations into mathematics (we find on page 8 a summary of Pólya's method), the third chapter contains the basic strategies to solve these situations, while the fourth gives examples of strategies to discover (crossing the river with the wolf, the lamb, and the cabbage, etc). In the next three chapters, we are exposed to the difficulties of ratios, percentages, the different means (arithmetic, geometric, harmonic), and problems involving motion and speed. The reader begins to need calculus in the seventh chapter. The problems have "nice" closed solutions. That is not the case in the eighth through tenth chapters, where the solutions mix theoretical methods (probability, sequences, trigonometry, series) and numerical methods (simulations, Newton's method, Taylor expansions).

In summary, the first ten chapters have varied problems (no exercises here) that range in complexity from pre-calculus to modern methods. The subjects of the problems are also varied, ranging from motion problems (trains, boats, ants) to the study of ice cream cones, poker, and population migrations between ecosystems. All problems are titled and, when possible, there are references to the sources. Each chapter concludes with a section entitled "Hints and Answers", while complete solutions to odd-numbered problems appear at the end of the book (around 40 pages of solutions compared to 120 pages of everything else). Students can try the problems, then check the answers, try to make corrections if necessary, and finally check the solutions provided at the end.

Students need a mathematical background comparable to high school completion to appreciate this book fully. They would find it even more beneficial with a bachelor's degree. Ideally, a student would have the aid of a teacher, since the examples are not numerous, but a strong student with an interest in problem solving and contests could handle this book alone.

On the other hand, a teacher reading this book would find a refreshing, concise source of problems. It is different from a typical textbook. Its "short and sweet" approach leaves a lot of room for a teacher wanting to create course notes to complement the book. It is a good read.

Luck, Logic and White Lies: The Mathematics of Games
By Jörg Bewersdorff, translated by David Kramer, published by AK Peters, 2004
ISBN 1-56881-210-8, paperback, 504 pages, US$49.00.
Reviewed by **Sarah K.M. Aldous**, *Lambeth Academy, London, UK.*

This fascinating and readable book has been translated from the German original, which is now in its third edition. The book contains three sections, each covering a different aspect of game theory. The first section addresses games of chance, including roulette, die tossing, and blackjack. The focus is on finding the probability of winning. The second section centres on combinatorial games. The well-known heap game, Nim, is studied, and some aspects of Go, Backgammon, and other games of intellect are also examined. The third section deals with classical strategic games and uses tools such as minimax, maximin, saddle points, and optimal playing strategies derived from linear optimization. The three sections contain about fifteen chapters each, and these are engaging and atomic. Chapters from different sections can be read independently.

This book would be beneficial in several circumstances. First, the book is a broad introduction to three distinct aspects of game theory and, as such, would make a good textbook for a one-term overview of game theory. However, it lacks problems except for a motivating question at the start of each chapter; further problems would have to be supplied by the instructor. Secondly, the book is a great teaching resource. It is clearly written and progresses smoothly from basic to more advanced concepts. Many of the forty-five short chapters can be read independently, and the mathematical levels vary widely. This makes it a helpful supply for enrichment lessons, for example, for students from middle school to university. A motivated student could select chapters to use as a self-study project. More information on topics of interest can be found in the further reading lists, provided at the end of most chapters instead of in a common bibliography.

Overall, the writing of Bewersdorff and Kramer is easy to understand, engaging, and stimulating. There are very few errors and the translation is rarely awkward. The text is sprinkled with anecdotes and notes, often humorous or insightful. The diversity of topics make this an interesting read and a rich resource.

On Lucas Numbers and Egyptian Fractions

John H. Jaroma

The *Lucas numbers*, $L_n : 2, 1, 3, 4, 7, 11, 18, 29, \ldots$ are generated by the companion Lucas sequence, $V_0 = 2$, $V_1 = 1$, and $V_{n+2} = V_{n+1} + V_n$ for $n \geq 0$. They represent the counterpart to the *Fibonacci numbers*, $F_n : 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$, which are produced by the Lucas sequence, $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = U_{n+1} + U_n$ for $n \geq 0$. More generally, the sequences of Lucas and Fibonacci numbers are subsets of the recurrence relation, $X_{n+2} = PX_{n+1} - QX_n$, where $n \geq 0$ and P and Q are non-zero integers. These had once attracted the attention of Fermat, Pell, and Euler. Nevertheless, it was Lucas who in 1878 undertook the first systematic study of them [6]. He also appended the name of "Fibonacci" to the notable sequence that now bears the Italian mathematician's name.

Letting P and Q be relatively prime integers, the *Lucas sequences* and *companion Lucas sequences* are respectively defined as

$$\begin{aligned} U_{n+2} &= PU_{n+1} - QU_n\,, \quad U_0 \;=\; 0\,, \quad U_1 \;=\; 1\,, \quad n \in \{0, 1, \ldots\} \\ V_{n+2} &= PV_{n+1} - QV_n\,, \quad V_0 \;=\; 2\,, \quad V_1 \;=\; P\,, \quad n \in \{0, 1, \ldots\}. \end{aligned}$$

As Lucas developed his theory for these sequences, the methods he used were sometimes cumbersome and his proofs occasionally contained mistakes. In 1913, Carmichael endeavored to correct the errors he found in the Lucas papers and succeeded in generalizing some of the results [1]. In 1930, the refined theory of the Lucas sequences was further extended by D.H. Lehmer in his Ph.D. thesis [5]. Lehmer introduced what he called the extended Lucas sequences by replacing the parameter P with $\sqrt{R}$, where R and Q are relatively prime integers. This change was motivated by the fact that the discriminant $P^2 - 4Q$ of the characteristic equation $X^2 - PX + Q = 0$ of the Lucas sequences cannot be congruent to either 2 or 3 modulo 4.

In spite of this apparent deficiency, the Lucas sequences exhibit many interesting properties, as well as have quite a few unsettled questions associated with them. For example, we do not know if there exists any Lucas sequence $\{U_n(P,Q)\}$ with an infinite number of prime terms. However, we do know that it is necessary for the index n to be prime in order for $U_n(P,Q)$ to be prime. Thus, when looking for, say, a new prime Fibonacci number $F_n = U_n(1,-1)$, we necessarily restrict ourselves to only those F_n where n is a prime. Along similar lines, we are uncertain as to the infinitude of prime terms in the Lucas numbers $L_n = V_n(1,-1)$ albeit we do know that prime terms occur only if the underlying index is either a prime or a power of two. In 1999, Dubner and Keller announced that L_{14449} is prime and provided a list of known, as well as probable Lucas primes with indices $n \leq 50\,000$ [2]. Alas, no prime appeared in the list with index 2^k greater than 16. Therefore, it is unknown if L_{16} is the largest prime Lucas number with index 2^k.

The terms L_{2^k} are of special interest, for every one of their factors is *primitive*; that is, they occur as divisors of the sequence for the very first time at L_{2^k}‡. In turn, this leads us to the rank of apparition of a prime. An enigma of sorts, the *rank of apparition* of a prime p is defined to be the non-negative index of the first term in the sequence that contains p as a divisor. Since every factor of L_{2^k} is primitive, it follows that every prime factor of L_{2^k} has rank of apparition equal to 2^k. Unfortunately, we do not have an efficient way of predicting what the prime factors of L_{2^k} are. If we did, then it would be easy to tell which L_{2^k} are prime and which are composite.

Furthermore, given an arbitrary p, we have no explicit formula for determining the rank of p in an underlying Lucas sequence. In fact, Lehmer tells us that such a formula is not to be expected [5]. We add that it has been suggested that the term "rank of apparition" is a misnomer (See, for example, [8].) Indeed, the first article of [6] begins with the sentence, "De l'apparition des nombres premiers dans les séries récurrentes de première espèce". Perhaps it is due to a poor translation, but today's generally accepted nomenclature for the index of the first term in which p occurs as a divisor is the "rank of apparition" of p. This is in contradiction, say, to [7], which gives "appearance" as the English equivalent to *l'apparition*.

Now, we establish a connection between the Lucas numbers L_{2^k} and an open problem in *Egyptian fractions*. Egyptian fractions are *unit fractions*; that is, fractions of the form $1/n$, where $n > 1$. They are called such because, with the exception of $2/3$, the ancient Egyptians insisted on conducting all of their calculations with them. Recently, in this journal, the author illustrated a universal formula for decomposing $4/n$ into $1/x + 1/y - 1/z$ [4]. The problem is loosely connected to the much more difficult conjecture of Erdős and Straus which states that the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{1}$$

is solvable in positive integers for $n \geq 4$. This open question is described by Guy in [3] and verified by Swett for $n < 1003162753$. When the work of Bernstein, Obláth, Yamamoto, and Rosati are considered, the equation holds except possibly when $n \equiv 1,\ 121,\ 169,\ 289,\ 361,\ 529 \pmod{840}$.

It is the purpose of this note to show that if the denominator n of (1) is restricted to the Lucas numbers with index equal to 2^k, $k \geq 2$, then $4/n$ can be expressed as the sum of exactly two Egyptian fractions. To accomplish this, we first make the observation that $L_4 = 7$, $L_8 = 47$, $L_{16} = 2207$, $L_{32} = 4870847$, $L_{64} = 23725150497407$, etc.

‡It also follows from divisibility properties of the Lucas sequences that every factor of U_p, V_p, and V_{2^k} for $k = \{0, 1, \ldots\}$ is also primitive. See [6], [1], or [8].

Remark. For $k \geq 2$, each L_{2^k} ends in 7; more specifically, it ends alternately in 07 and 47. A proof by induction follows by noting that $L_4 = 07$ and $L_8 = 47$, and then appealing to the identity $L_n^2 = L_{2n} + 2(-1)^n$. This is a special case of the more general formula, $V_{2n} = V_n^2 - 2Q^n$, where $n \geq 0$, which had been introduced by Lucas in [6]. The general formula is also found as IV.5 on page 57 of the (perhaps) more accessible [8].

We now state and prove the following theorem. It shows that every fraction of the form $4/L_{2^n}$, $n \geq 2$ can be expressed as the sum of exactly two unit fractions.

Theorem 1. Consider any integer of the form L_{2^n}, where $n \geq 2$. Then, there exist positive integers x and y such that $\frac{4}{L_{2^n}} = \frac{1}{x} + \frac{1}{y}$; in particular, letting $k = (L_{2^n} - 3)/4$, it follows that

$$\frac{4}{L_{2^n}} = \frac{1}{k+1} + \frac{1}{(L_{2^n})(k+1)}.$$

Proof: By the above remark, if $n \geq 2$, then either $L_{2^n} \equiv 07 \pmod{100}$ or $L_{2^n} \equiv 47 \pmod{100}$. Hence, $L_{2^n} \equiv 3 \pmod 4$. Thus, for some $k \in \mathbb{Z}$, we have $4/L_{2^n} = 4/(4k+3)$. Applying the Fibonacci-Sylvester Splitting Algorithm to $4/L_{2^n}$, we have

$$\frac{4}{4k+3} - \frac{1}{k+1} = \frac{4k+4-4k-3}{(4k+3)(k+1)} = \frac{1}{(4k+3)(k+1)}.$$

Since $L_{2^n} = 4k+3$, we have $k = \frac{L_{2^n} - 3}{4}$. Therefore, it follows that

$$\frac{4}{L_{2^n}} = \frac{1}{k+1} + \frac{1}{(L_{2^n})(k+1)}. \qquad \blacksquare$$

Examples.

1. $\frac{4}{L_4} = \frac{4}{7} = \frac{1}{2} + \frac{1}{(2)(7)} = \frac{1}{2} + \frac{1}{14}.$

2. $\frac{4}{L_8} = \frac{4}{47} = \frac{1}{12} + \frac{1}{(12)(47)} = \frac{1}{12} + \frac{1}{564}.$

3. $\frac{4}{L_{16}} = \frac{4}{2207} = \frac{1}{552} + \frac{1}{(552)(2207)} = \frac{1}{552} + \frac{1}{1218264}.$

Acknowledgment. The author thanks the referee whose suggestions helped to make this a more comprehensive paper.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er juin 2006**. *Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

3088. *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit P un point dans le plan d'un triangle ABC mais à l'extérieur des droites déterminées par ses côtés. Soit AD, BE et CF les Céviennes passant par P. Les droites par A parallèles à BE et CF coupent la droite BC en L et L' respectivement. On définit les points M, M', N et N' de manière analogue. Montrer que L, L', M, M', N, N' sont sur une même conique.

3089. *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit P un point dans le plan d'un triangle ABC mais à l'extérieur des droites déterminées par ses côtés. Soit AD, BE et CF les Céviennes passant par P. Les droites par E et F parallèles à AD coupent la droite BC en L et L' respectivement. On définit les points M, M', N et N' de manière analogue. Montrer que L, L', M, M', N, N' sont sur une même conique.

3090. *Proposé par Arkady Alt, San Jose, CA, USA.*

Trouver toutes les solutions réelles non négatives (x, y, z) du système suivant d'inégalités :

$$\begin{aligned} 2x(3-4y) &\geq z^2+1\,, \\ 2y(3-4z) &\geq x^2+1\,, \\ 2z(3-4x) &\geq y^2+1\,. \end{aligned}$$

3091. *Proposé par Mihály Bencze et Marian Dinca, Roumanie.*

Soit $A_1A_2\cdots A_n$ un polygone convexe possédant un cercle inscrit et un cercle circonscrit. On désigne par B_1, B_2, $\ldots$, B_n les points de tangence du cercle inscrit respectifs par A_1A_2, A_2A_3, $\ldots$, A_nA_1. Montrer que

$$\frac{2sr}{R} \;\leq\; \sum_{k=1}^{n} B_kB_{k+1} \;\leq\; 2s\cos\left(\frac{\pi}{n}\right)\,,$$

où R est le rayon du cercle circonscrit, r celui du cercle inscrit, s est le demi-périmètre du polygone $A_1A_2\cdots A_n$, et $B_{n+1} = B_1$.

3092. *Proposé par Vedula N. Murty, Dover, PA, USA.*

(a) Soit a, b et c trois nombres réels positifs tels que $a + b + c = abc$. Trouver la valeur minimale de $\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2}$.

[Comparer avec le problème ***CRUX with MAYHEM*** 2814 [2003 : 110; 2004 : 112].]

(b) Soit a, b et c trois nombres réels positifs tels que $a + b + c = 1$. Trouver la valeur minimale de

$$\frac{1}{\sqrt{abc}} + \sum_{\text{cyclique}} \sqrt{\frac{bc}{a}}\,.$$

3093. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit p_k le k-ième nombre premier. Montrer que la série suivante diverge :

$$\sum_{n=1}^{\infty} \frac{1}{n(p_{n+1} - p_n)}\,.$$

3094. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit $x_1, x_2, \ldots, x_n$ des nombres réels non négatifs, où $n \geq 3$. Soit $S = \sum\limits_{k=1}^{n} x_k$ et $P = \prod\limits_{k=1}^{n} (1 + x_k^2)$. Montrer que

(a) $P \leq \max\limits_{1 \leq k \leq n} \left\{ \left(1 + \frac{S^2}{k^2}\right)^k \right\}$;

(b) $P \leq \left(1 + \frac{S^2}{n^2}\right)^n$ si $S > 2\sqrt{2}(n-1)$;

(c) $P \leq 1 + S^2$ si $S \leq 2\sqrt{2}$.

3095. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit a, b, c, p et q des nombres naturels. Sachant que $\lfloor x \rfloor$ désigne la partie entière de x, montrer que

$$\min\left\{a, \left\lfloor \frac{c + pb}{q} \right\rfloor\right\} \;\leq\; \left\lfloor \frac{c + p(a+b)}{p+q} \right\rfloor\,.$$

3096. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit ABC un triangle de côtés a, b et c opposés respectivement aux sommets A, B et C. Montrer que

$$\sum_{\text{cyclique}} \frac{bc}{b+c} \sin^2\left(\frac{A}{2}\right) \;\leq\; \frac{a+b+c}{8}\,.$$

3097. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a et b deux nombres réels positifs telles que $a < b$. On définit $A(a,b) = \frac{a+b}{2}$ et $L(a,b) = \frac{b-a}{\ln b - \ln a}$. Montrer que

$$L(a,b) \;<\; L\left(\frac{a+b}{2}, \sqrt{ab}\right) \;<\; \left(A(\sqrt{a}, \sqrt{b})\right)^2 \;<\; A(a,b)\,.$$

3098★. *Proposé par D.Z. Djokovic, University of Waterloo, Waterloo, ON ; et Edward T.H. Wang et Kaiming Zhao, Université Wilfrid Laurier, Waterloo, ON.*

Soit n et k des entiers positifs quelconques. Désignons par S la suite de longueur $2n$ obtenue en entrelaçant les deux suites n, $n-1$, …, 2, 1 et -1, -2, …, $-(n-1)$, $-n$, et soit $\mathcal{F}$ l'ensemble de toutes les $\binom{2n-k+1}{k}$ sous-suites K de S ayant k termes non consécutifs de S. Montrer que

$$\sum_{K\in\mathcal{F}} P(K) \;=\; 0\,,$$

où $P(K)$ désigne le produit de tous les termes de la suite K.

Par exemple, si $n = 3$ et $k = 2$, donc $S = 3, -1, 2, -2, 1, -3$, et

$$\mathcal{F} \;=\; \big\{\{3,\,2\},\,\{3,\,-2\},\,\{3,\,1\},\,\{3,\,-3\},\,\{-1,\,-2\},\,\{-1,\,1\},\\ \{-1,\,-3\},\,\{2,\,1\},\,\{2,\,-3\},\,\{-2,\,-3\}\big\}\,;$$

par conséquent,

$$\sum_{K\in\mathcal{F}} P(K) \;=\; 6-6+3-9+2-1+3+2-6+6 \;=\; 0\,.$$

[Ce résultat a été obtenu lors de recherches en Algèbre de Lie. Les proposeurs ont une démonstration pour k impair. Ils espèrent qu'on puisse en trouver une démonstration élémentaire.]

3099. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a_1, a_2, …, a_n des nombres réels positifs. Montrer que

$$\prod_{k=1}^{n} \ln(1+a_k) \;\le\; \left(\ln\left(1+\sqrt[n]{\prod_{k=1}^{n} a_k}\right)\right)^n\,.$$

3100. *Proposé par Michel Bataille, Rouen, France.*

Soit $f:\mathbb{R}\to\mathbb{R}$ satisfaisant $f\big(xf(y)\big) = yf(x)$ pour tous les nombres réels x et y.

(a) Montrer que f est une fonction impaire.

(b) Déterminer f, sachant que f n'a qu'une seule discontinuité.

3088. *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle and P a point in the plane of this triangle, not lying on any of the three lines determined by its sides. Let AD, BE, and CF be the Cevians through the point P. The lines through A parallel to BE and CF meet the line BC at L and L', respectively. Points M, M', N, and N' are similarly defined. Prove that L, L', M, M', N, N' all lie on a conic.

3089. *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle and P a point in the plane of this triangle, not lying on any of the three lines determined by its sides. Let AD, BE, and CF be the Cevians through the point P. The lines through E and F parallel to AD meet the line BC at L and L', respectively. Points M, M', N, and N' are similarly defined. Prove that L, L', M, M', N, N' all lie on a conic.

3090. *Proposed by Arkady Alt, San Jose, CA, USA.*

Find all non-negative real solutions (x, y, z) to the following system of inequalities:

$$\begin{aligned} 2x(3-4y) &\geq z^2+1\,, \\ 2y(3-4z) &\geq x^2+1\,, \\ 2z(3-4x) &\geq y^2+1\,. \end{aligned}$$

3091. *Proposed by Mihály Bencze and Marian Dinca, Romania.*

Let $A_1A_2\cdots A_n$ be a convex polygon which has both an inscribed circle and a circumscribed circle. Let B_1, B_2, $\dots$, B_n denote the points of tangency of the incircle with sides A_1A_2, A_2A_3, $\dots$, A_nA_1, respectively. Prove that

$$\frac{2sr}{R} \;\leq\; \sum_{k=1}^{n} B_kB_{k+1} \;\leq\; 2s\cos\left(\frac{\pi}{n}\right)\,,$$

where R is the radius of the circumscribed circle, r is the radius of the inscribed circle, s is the semiperimeter of the polygon $A_1A_2\cdots A_n$, and $B_{n+1}=B_1$.

3092. *Proposed by Vedula N. Murty, Dover, PA, USA.*

(a) Let a, b, and c be positive real numbers such that $a+b+c=abc$. Find the minimum value of $\sqrt{1+a^2}+\sqrt{1+b^2}+\sqrt{1+c^2}$.

[Compare with ***CRUX with MAYHEM*** problem 2814 [2003 : 110; 2004 : 112].]

(b) Let a, b, and c be positive real numbers such that $a+b+c=1$. Find the minimum value of

$$\frac{1}{\sqrt{abc}}+\sum_{\text{cyclic}}\sqrt{\frac{bc}{a}}\,.$$

3093. *Proposed by Mihály Bencze, Brasov, Romania.*

Let p_k be the k^{th} prime. Show that the following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n(p_{n+1}-p_n)}\,.$$

3094. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers, where $n \geq 3$. Let $S = \sum\limits_{k=1}^{n} x_k$ and $P = \prod\limits_{k=1}^{n} (1+x_k^2)$. Prove that

(a) $P \leq \max\limits_{1\leq k\leq n} \left\{ \left(1+\frac{S^2}{k^2}\right)^k \right\}$;

(b) $P \leq \left(1+\frac{S^2}{n^2}\right)^n$ if $S > 2\sqrt{2}(n-1)$;

(c) $P \leq 1+S^2$ if $S \leq 2\sqrt{2}$.

3095. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, c, p, and q be natural numbers. Using $\lfloor x \rfloor$ to denote the integer part of x, prove that

$$\min\left\{a, \left\lfloor \frac{c+pb}{q} \right\rfloor\right\} \;\leq\; \left\lfloor \frac{c+p(a+b)}{p+q} \right\rfloor\,.$$

3096. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let ABC be a triangle with sides a, b, c opposite the angles A, B, C, respectively. Prove that

$$\sum_{\text{cyclic}} \frac{bc}{b+c} \sin^2\left(\frac{A}{2}\right) \;\leq\; \frac{a+b+c}{8}\,.$$

3097. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a and b be two positive real numbers such that $a < b$. Define $A(a,b) = \frac{a+b}{2}$ and $L(a,b) = \frac{b-a}{\ln b - \ln a}$. Prove that

$$L(a,b) \;<\; L\left(\frac{a+b}{2}, \sqrt{ab}\right) \;<\; \left(A\left(\sqrt{a}, \sqrt{b}\right)\right)^2 \;<\; A(a,b)\,.$$

3098★. *Proposed by D.Z. Djokovic, University of Waterloo, Waterloo, ON; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON.*

Let n and k be any positive integers such that $k \leq n$. Let S denote the sequence of length $2n$ obtained by interlacing the two sequences $n, n-1, \ldots, 2, 1$ and $-1, -2, \ldots, -(n-1), -n$, and let $\mathcal{F}$ be the set of all $\binom{2n-k+1}{k}$ subsequences K of S which have length k and do not contain any pair of consecutive terms of S. Prove that

$$\sum_{K\in\mathcal{F}} P(K) \;=\; 0\,,$$

where $P(K)$ is the product of all k terms of the sequence K.

For example, if $n=3$ and $k=2$, then $S = 3, -1, 2, -2, 1, -3$, and

$$\begin{aligned}\mathcal{F} \;=\; \big\{&\{3,\,2\},\,\{3,\,-2\},\,\{3,\,1\},\,\{3,\,-3\},\,\{-1,\,-2\},\,\{-1,\,1\},\\ &\{-1,\,-3\},\,\{2,\,1\},\,\{2,\,-3\},\,\{-2,\,-3\}\big\}\,;\end{aligned}$$

hence,

$$\sum_{K\in\mathcal{F}} P(K) \;=\; 6-6+3-9+2-1+3+2-6+6 \;=\; 0\,.$$

[This result was obtained as a by-product of some research in Lie Algebra. The proposers have a proof for odd k. They hope that an elementary proof can be found.]

3099. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n} \ln(1+a_k) \;\leq\; \left(\ln\left(1+\sqrt[n]{\prod_{k=1}^{n} a_k}\right)\right)^{n}\,.$$

3100. *Proposed by Michel Bataille, Rouen, France.*

Let $f:\mathbb{R}\to\mathbb{R}$ satisfy $f\big(xf(y)\big) = yf(x)$ for all real numbers x and y.

(a) Show that f is an odd function.

(b) Determine f, given that f has exactly one discontinuity.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologise for omitting the name of JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain from the list of solvers of 2938.

2927★. [2004 : 172, 174] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \;\geq\; \frac{3(ab+bc+ca)}{a+b+c}\,.$$

Comments by Arkady Alt, San Jose, CA, USA.

The solution of this problem in [2005 : 179–180] is correct; however, an editorial comment at the end of the solution is not correct. It states "Janous supplied a chain of generalizations. First he proved that

$$\sum_{\text{cyclic}}\frac{a^3(b+c)}{b^3+c^3} \;\geq\; 2\sum_{\text{cyclic}}\frac{a^3}{b^2+c^2} \;\geq\; a+b+c\,.$$

He then extended the sharper inequality $\sum\limits_{\text{cyclic}}\frac{a^3}{b^2+c^2} \geq \frac{a+b+c}{2}$ by replacing the left side by $\sum\limits_{\text{cyclic}}\frac{a^{\lambda+1}}{b^\lambda+c^\lambda}$, where $\lambda \geq 0$."

Unfortunately, the inequality

$$\sum_{\text{cyclic}}\frac{a^3\,(b+c)}{b^3+c^3} \;\geq\; 2\sum_{\text{cyclic}}\frac{a^3}{b^2+c^2}$$

is not correct; in fact, the reverse inequality holds. Therefore, the inequality $2\sum\limits_{\text{cyclic}}\frac{a^3}{b^2+c^2} \;\geq\; a+b+c$ is not sharper than $\sum\limits_{\text{cyclic}}\frac{a^3(b+c)}{b^3+c^3} \;\geq\; a+b+c\,.$ Indeed,

$$\frac{a^3\,(b+c)}{b^3+c^3} \;=\; \frac{a^3}{b^2-bc+c^2} \;\leq\; \frac{a^3}{b^2-\frac{1}{2}(b^2+c^2)+c^2} \;=\; \frac{2a^3}{b^2+c^2}\,,$$

so that

$$\sum_{\text{cyclic}}\frac{a^3\,(b+c)}{b^3+c^3} \;\leq\; 2\sum_{\text{cyclic}}\frac{a^3}{b^2+c^2}\,,$$

as claimed.

[*Ed.:* Janous' solution contained a small error which was overlooked by the editor.]

2985. [2004 : 431, 434] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Let a and b be real numbers, and let

$$L \;=\; \lim_{n\to\infty} n\left(1-\frac{a}{n}-\frac{b\ln(n+1)}{n}\right)^{n}.$$

Prove that

$$L \;=\; \begin{cases} \infty & \text{if } b<1,\\ e^{-a} & \text{if } b=1,\\ 0 & \text{if } b>1. \end{cases}$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

As n increases without bound, we have

$$\begin{aligned} ne^{n\ln(1-a/n-(b\ln(n+1))/n)} &= ne^{-n\sum\limits_{k=1}^{\infty}\left(\frac{a+b\ln(n+1)}{n}\right)^k} \\ &= ne^{-a-b\ln(n+1)+O\left(\frac{\ln^2(n+1)}{n}\right)} \\ &= \frac{n}{(n+1)^b}e^{-a}\left(1+O\left(\frac{\ln^2(n+1)}{n}\right)\right), \end{aligned}$$

from which the conclusion follows.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

2986. [2004 : 431, 434] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Given three points A, B, C in the plane such that none of the angles of triangle ABC exceeds 165°, what is the minimum number of circular arcs needed to construct the circumcircle, using only a compass and no straightedge? (The answer should be fewer than 22.)

Combination of the proposer's solution and a solution by August Adler from 1890.

Preliminary comments. The geometry of compasses was developed independently by Georg Mohr in Denmark (1672) and Lorenzo Mascheroni in Italy (1797). They showed that if a required point can be constructed from a given set of points by straightedge and compass, then it could be constructed by compass alone. According to Howard Eves ([1], Section 4.5 page 169), in 1890 the Viennese geometer August Adler (1863 – 1923) published a different approach based on inversions in circles. Among other things, Adler showed how to construct a circle through three given non-collinear points using just 10 circles and no lines ([1], page 173, Problem 2; or [2], page 25).

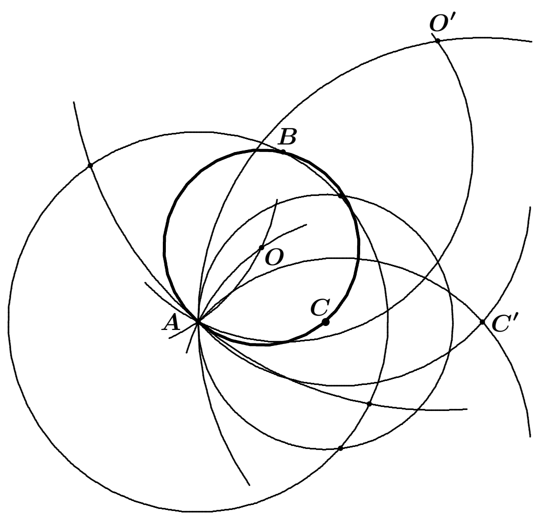
This beats the solution by our problem's proposer, whose construction uses 14 circles. Furthermore, it seems (unless this editor is missing something) as if the efficient construction will succeed for any given triangle ABC. There is probably no simple way to determine whether 10 circles is actually the minimum required, but it compares favorably with the familiar compass-and-straightedge construction that uses four circles and two lines. For Adler's solution, the following basic constructions are needed. The symbol $O(r)$ will represent the circle with centre O and radius r.

Construction (a). Construct the inverse of P in the circle $O(r)$ when P satisfies $OP > r/2$.
Solution. Three circles are needed: Draw $P(OP)$ cutting the given circle at Q and R, then draw $Q(r)$ and $R(r)$. The last two circles pass through O and meet again at P', the inverse of P. (*Proof:* $\triangle OPQ \sim \triangle OQP'$ (isosceles triangles with a shared angle at O). Hence, $OP/OQ = OQ/OP'$.)

Construction (b). Construct the reflection of P in line QR.
Solution. Two circles are needed: Draw $Q(QP)$ and $R(RP)$. These circles intersect in P and its reflection P'. (*Proof:* The triangles QPR and $QP'R$ are congruent.)

Adler's construction is based on the following fact: *The centre of circle ABC is inverted in circle $A(AB)$ to the reflection of A in line BC'* (where C' is the image of C under the inversion). This easily proved fact can be found in most geometry texts concerned with inversion such as [1] (Theorem 3.7.4, page 127), and [2] (page 9).

Solution to the problem. Label the triangle so that $\angle CAB$ is acute and $BA > CA > \frac{1}{2}BA$. (This can be arranged because a triangle must have at least two acute angles, and at most one edge could be less than half the largest edge.)



1. Draw circle $A(AB)$.
2. Construct (by Construction (a)) the inverse C' of C in circle $A(AB)$. (Construction (a) can be used here because we have taken $CA > \frac{1}{2}BA$. Note that because $BA > CA$, C' is outside of the circle of inversion.)
3. Construct (by (b)) the reflection O' of A in line BC'. (Note that because C' is outside circle $A(AB)$ and $\angle C'AB = \angle CAB$ is acute, the reflection of the centre A in secant BC' will also be outside the circle.)
4. Construct (by (a)) the inverse O of O' in circle $A(AB)$. (Construction (a) can be used because O' is outside the circle, which implies that $AO' > \frac{1}{2}AB$.)
5. Draw the circle $O(OA)$.

The circle $O(OA)$ is the circumcircle of ABC as a consequence of the preliminary fact stated above. The number of circles required using this construction is $1 + 3 + 2 + 3 + 1 = 10$.

References

[1] Howard Eves, *A Survey of Geometry*, revised edition, Allyn and Bacon, 1972.

[2] Dan Pedoe, *Circles, A Mathematical View*, Dover, 1979.

No other solutions were submitted.

2987. [2004 : 431, 434] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

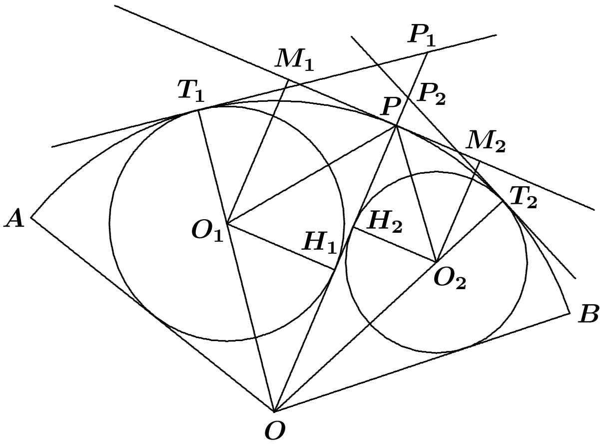
Let Γ be a circle with centre O, and let A, B be two points on the circle. The points A, O, and B determine a minor arc $\widehat{AB}$ with $0 < \theta \leq \pi$, where $\theta = \angle AOB$. Let P be any fixed point in the interior of the minor arc $\widehat{AB}$, let O_1 be the centre of the circle which is tangent to OA, OP, and the minor arc $\widehat{AB}$ at the point T_1, and let O_2 be the centre of the circle which is tangent to OB, OP, and the minor arc $\widehat{AB}$ at the point T_2.

(a) Prove that $\frac{\pi}{2} - \frac{\theta}{4} < \angle O_1PO_2 < \frac{\pi}{2}$.

(b) Prove that the lines O_2T_1, O_1T_2, and OP are concurrent.

(c)★ Let S be the intersection of the lines AT_1 and BT_2; let K be the intersection of the lines AT_2 and BT_1; let H be the intersection of the tangent lines to the arc $\widehat{AB}$ at A and B; and let T be the intersection of the tangent lines to the arc $\widehat{AB}$ at T_1 and T_2. Prove that the points S, K, H, and T are collinear.

Solution by Michel Bataille, Rouen, France.



(a) Let M_1 and M_2 be the projections of O_1 and O_2, respectively, onto the tangent to Γ at P, and let H_1 and H_2 be the projections of O_1 and O_2, respectively, onto the line OP. Let r be the radius of Γ, and let r_1 and r_2 be the radii of the circles γ_1 and γ_2 with centres O_1 and O_2, respectively, described in the statement of the problem. Then $O_1H_1 = r_1$, $O_2H_2 = r_2$, and

$$\begin{aligned} \overrightarrow{PO_1} \cdot \overrightarrow{PO_2} &= \left(\overrightarrow{PH_1} + \overrightarrow{H_1O_1}\right) \cdot \left(\overrightarrow{PH_2} + \overrightarrow{H_2O_2}\right) \\ &= \overrightarrow{PH_1} \cdot \overrightarrow{PH_2} + \overrightarrow{H_1O_1} \cdot \overrightarrow{H_2O_2} \\ &= PH_1 \cdot PH_2 - r_1 r_2 \,. \end{aligned}$$

But $PH_1 = O_1M_1 > r_1$ and $PH_2 - O_2M_2 > r_2$; hence, $\overrightarrow{PO_1} \cdot \overrightarrow{PO_2} > 0$ and we have $\angle O_1PO_2 < \pi/2$.

Let P_1 and P_2 be the points where the line OP meets the tangents to Γ at T_1 and T_2, respectively. Since P lies between H_1 and P_1, we have

$$\angle O_1PH_1 \;>\; \angle O_1P_1H_1 \;=\; \tfrac{1}{2}\angle T_1P_1H_1 \;=\; \frac{1}{2}\left(\frac{\pi}{2} - \angle T_1OP\right) .$$

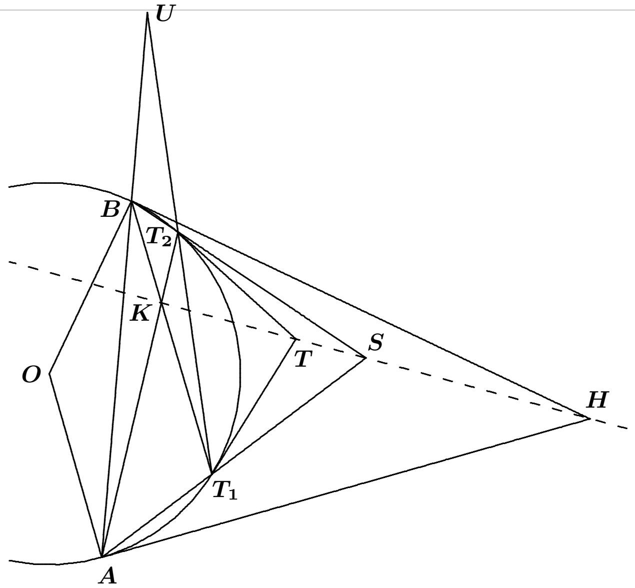
Similarly, $\angle O_2PH_2 > \frac{1}{2}\left(\frac{\pi}{2} - \angle T_2OP\right)$, and it follows that

$$\angle O_1PO_2 \;=\; \angle O_1PH_1 + \angle O_2PH_2 \;>\; \frac{\pi}{2} - \frac{1}{2}(\angle T_1OP + \angle T_2OP) \;=\; \frac{\pi}{2} - \frac{\theta}{4} \,.$$

(b) Let I be the point of intersection of OP and O_1O_2. Then $\triangle IO_1H_1$ and $\triangle IO_2H_2$ are similar, which means that $\dfrac{IO_2}{O_2I} = \dfrac{r_2}{r_1}$. Since $OT_1 = r$ and $O_1T_1 = r_1$, we have $\dfrac{T_1O_1}{OT_1} = -\dfrac{r_1}{r}$. Similarly, $\dfrac{T_2O}{O_2T_2} = -\dfrac{r}{r_2}$. It then follows that

$$\frac{IO_2}{O_1I} \cdot \frac{T_1O_1}{OT_1} \cdot \frac{T_2O}{O_2T_2} \;=\; 1 \,.$$

The desired concurrency results from the converse of Ceva's Theorem (O_2T_1, O_1T_2, and OP cannot be parallel since O_1 and T_2 are on opposite sides of the line OP.



(c) The quadrilateral AT_12T_2B is inscribed in Γ and its diagonal triangle is SKU, where U denotes the point of intersection of T_1T_2 and AB. (See the diagram above.) It is well known (for example, see [1]) that the line KS is the polar of U with respect to Γ. In addition, the polar of H is AB, which passes through U; hence, the polar of U passes through H. Similarly, since T_1T_2 is the polar of T, we see that T is on the polar of U. Finally, S, K, H, and T are collinear: they are all on the polar of U with respect to Γ.

Reference.

[1] C.V. Durell, *Modern Geometry*, MacMillan, 1952, p. 102.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

Zhou uses Pascal's Hexagon Theorem to prove part (c). *He uses $AT_1T_1BT_2T_2$ to see that S, T, and K are collinear, and AAT_1BBT_2 to see that H, S, and K are collinear.*

2988★. [2004 : 502, 506] *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let x, y, z be non-negative real numbers satisfying $x + y + z = 1$. Prove or disprove:

(a) $xy^2 + yz^2 + zx^2 \ \geq\ \frac{1}{3}(xy + yz + zx)$;

(b) $xy^2 + yz^2 + zx^2 \ \geq\ xy + yz + zx - \frac{2}{9}$.

How do the right sides of (a) and (b) compare?

Solution by Pavlos Maragoudakis, Lefkogia, Crete, Greece.

First we compare the right sides of (a) and (b). We claim that

$$\tfrac{1}{3}(xy+yz+zx) \;\geq\; xy+yz+zx-\tfrac{2}{9}\,.$$

This inequality is equivalent to

$$3(xy+yz+zx) \;\leq\; 1\,,$$

which is true, because it can be written successively as

$$\begin{aligned} 3(xy+yz+zx) &\leq (x+y+z)^2\,, \\ xy+yz+zx &\leq x^2+y^2+z^2\,, \\ \tfrac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right) &\geq 0\,. \end{aligned}$$

(a) The inequality is not true. For example, if $(x,y,z)=\left(\frac{3}{4},\frac{1}{4},0\right)$, then

$$xy^2+yz^2+zx^2 \;=\; \tfrac{3}{64} \;<\; \tfrac{4}{64} \;=\; \tfrac{1}{3}(xy+yz+zx)\,.$$

(b) This inequality is true. Since $(3y-1)^2\geq 0$, we have $y^2\geq\frac{2}{3}y-\frac{1}{9}$; thus, $xy^2\geq\frac{2}{3}xy-\frac{1}{9}x$. Similarly, $yz^2\geq\frac{2}{3}yz-\frac{1}{9}y$ and $zx^2\geq\frac{2}{3}zx-\frac{1}{9}z$. Adding the last three inequalities, we obtain

$$\begin{aligned} xy^2+yz^2+zx^2 &\geq \tfrac{2}{3}(xy+yz+zx)-\tfrac{1}{9}(x+y+z) \\ &= \tfrac{2}{3}(xy+yz+zx)-\tfrac{1}{9}\,. \end{aligned}$$

Thus,

$$xy^2+yz^2+zx^2 \;\geq\; \tfrac{2}{3}(xy+yz+zx)-\tfrac{1}{9}\,.$$

Note that the inequality $3(xy+yz+zx)\leq 1$ (proven above) can be written as

$$\tfrac{2}{3}(xy+yz+zx)-\tfrac{1}{9} \;\geq\; xy+yz+zx-\tfrac{2}{9}\,.$$

Consequently,

$$xy^2+yz^2+zx^2 \;\geq\; xy+yz+zx-\tfrac{2}{9}\,.$$

Also solved by ARKADY ALT, San Jose, CA, USA (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAJE, University of Ploiesti, Romania; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; MARIAN TETIVA, Bîrlad, Romania; B.J. VENKATACHALA, Indian Institute of Science, Bangalore, India; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Comănești, Romania. There was also one incorrect solution submitted.

Alt and Cîrtoaje have proposed related problems (generalizations) of the inequality of part (b). Wagon has pointed out that the problem is easily solvable by Mathematica.

2989. [2004 : 502, 506] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that if $0 < a < b < d < \pi$ and $a < c < d$ satisfy $a + d = b + c$, then

$$\frac{\cos(a-d) - \cos(b+c)}{\cos(b-c) - \cos(a+d)} < \frac{ad}{bc}\,.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

We have

$$\frac{\cos(a-d) - \cos(b+c)}{\cos(b-c) - \cos(a+d)} = \frac{\cos(a-d) - \cos(a+d)}{\cos(b-c) - \cos(b+c)} = \frac{\sin a \sin d}{\sin b \sin c}\,.$$

Since $\sin x > 0$ for $x \in (0, \pi)$, the desired inequality is equivalent to

$$\frac{\sin a \sin d}{ad} < \frac{\sin b \sin c}{bc}\,,$$

or

$$\ln\left(\frac{\sin a}{a}\right) + \ln\left(\frac{\sin d}{d}\right) < \ln\left(\frac{\sin b}{b}\right) + \ln\left(\frac{\sin c}{c}\right)\,.$$

Consider the function $f(x) = \ln\left(\frac{\sin x}{x}\right)$. Differentiating yields $f''(x) = \frac{\sin^2 x - x^2}{x^2 \sin^2 x}$. Since it is well known that $\sin x < x$ when $x > 0$, we have $f''(x) < 0$ for $x \in (0, \pi)$. Therefore, f is strictly concave in the interval $(0, \pi)$.

The given conditions imply that the vector (d, a) majorizes the vector (c, b) (or (b, c), whichever is in non-increasing order). Hence, by Karamata's Majorization Inequality, we have $f(a) + f(d) < f(b) + f(c)$, from which the result follows immediately.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

The proofs given by almost all the solvers are similar to the one featured above, although, besides Zhao, only Zhou explicitly quoted the Majorization Inequality.

2990. [2004 : 502, 506] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Given are an ellipse with centre O and a focus F, a line ℓ, and a point P. Construct with straightedge alone the line passing through the point P perpendicular to the line ℓ. (If a circle with its centre is given instead of an ellipse, then the construction is given by the well-known Poncelet–Steiner Construction Theorem.)

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

1. Draw the major axis OF to construct vertices V_1 and V_2.

2. The solution to 2740 [2003 : 246] shows how to use Pascal's Theorem to construct the tangent to a conic at a given point when given just four other points of the conic. The tangents ℓ_1, ℓ_2 at V_1, V_2 are perpendicular to the major axis.

3. Since $\ell_1 \parallel \ell_2$, we can construct the mid-point M of two points K and L chosen on ℓ_1 (see [2003 : 327, Figure 3]).

4. Using K, L, M, we can construct the minor axis through O (and thus, the minor vertices) and also the line ℓ_3 through F and parallel to ℓ_1. (See [2003 : 326, construction II(b)].)

5. Suppose ℓ_3 intersects the ellipse at R and S. Draw RO and SO to intersect the ellipse again at R' and S'. Then $R'S'$ intersects the major axis at the other focus F'. The problem is now reduced to 2944 [2005 : 248–249].

Also solved by MICHEL BATAILLE, Rouen, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2991. [2004 : 502, 506] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be an integer, $n \geq 3$. For all $z_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, prove

$$(n-1)\left|\sum_{i=1}^{n} z_i^3 - 3\sum_{1\le i<j<k\le n} z_iz_jz_k\right| \le \left|\sum_{i=1}^{n} z_i\right| \sum_{1\le i<j\le n}\Big(|z_i - z_j|^2 + (n-3)|z_iz_j|\Big).$$

[*Ed:* This problem had a typo, which has been corrected above. The term $|z_iz_j|$ originally appeared as $|z_i + z_j|$.]

Essentially the same solution to the corrected problem by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.

$$\begin{aligned}
(n-1)\Bigg(\sum_{i=1}^{n} z_i^3 &- 3\sum_{1\le i<j<k\le n} z_iz_jz_k\Bigg) \\
&= (n-1)\left(\sum_{i=1}^{n} z_i\right)\left(\sum_{i=1}^{n} z_i^2 - \sum_{1\le j<k\le n} z_jz_k\right) \\
&= \left(\sum_{i=1}^{n} z_i\right)\left(\sum_{1\le i<j\le n}(z_i-z_j)^2 - (n-3)\sum_{1\le j<k\le n} z_jz_k\right) \\
&= \left(\sum_{i=1}^{n} z_i\right)\sum_{1\le i<j\le n}\Big((z_i-z_j)^2 - (n-3)z_iz_j\Big).
\end{aligned}$$

By taking the modulus of both sides of this equation and using the Triangle Inequality on the right side, we get the desired result.

YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON showed that the original inequality was false.

2992. [2004 : 503, 506] *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Let Q be a point interior to $\triangle ABC$. Let M, N, P be points on the sides BC, CA, AB, respectively, such that $MN \parallel AQ$, $NP \parallel BQ$, and $PM \parallel CQ$. Prove that

$$[MNP] \;\le\; \tfrac{1}{3}[ABC]\,,$$

where $[XYZ]$ denotes the area of triangle XYZ.

Solution by the proposer.

Since $NP \parallel BQ$, then $[NPQ] = [NPB]$. Thus,

$$\frac{AN}{AC} \;=\; \frac{[ABN]}{[ABC]} \;=\; \frac{[APN]+[NPB]}{[ABC]} \;=\; \frac{[APN]+[NPQ]}{[ABC]} \;=\; \frac{[APQN]}{[ABC]}\,.$$

Similarly, $\dfrac{BP}{BA} = \dfrac{[BMQP]}{[ABC]}$ and $\dfrac{CM}{CB} = \dfrac{[CNQM]}{[ABC]}$. Hence,

$$\frac{AN}{AC}+\frac{BP}{BA}+\frac{CM}{CB} \;=\; \frac{[APQN]+[BMQP]+[CNQM]}{[ABC]} \;=\; \frac{[ABC]}{[ABC]} \;=\; 1\,.$$

In addition, we have

$$\frac{[NPQ]}{[ABC]} \;=\; \frac{[NPB]}{[ABC]} \;=\; \frac{[NPB][ABN]}{[ABN][ABC]} \;=\; \frac{BP}{BA}\cdot\frac{AN}{AC}\,.$$

Similarly, $\dfrac{[PMQ]}{[ABC]} = \dfrac{CM}{CB}\cdot\dfrac{BP}{BA}$ and $\dfrac{[MNQ]}{[ABC]} = \dfrac{AN}{AC}\cdot\dfrac{CM}{CB}$. Applying the well-known (and easy-to-prove) inequality

$$ab+bc+ca \;\le\; \tfrac{1}{3}(a+b+c)^2\,,$$

we obtain

$$\begin{aligned}\frac{[MNP]}{[ABC]} &= \frac{[MNQ]}{[ABC]}+\frac{[NPQ]}{[ABC]}+\frac{[PMQ]}{[ABC]}\\ &= \frac{AN}{AC}\cdot\frac{CM}{CB}+\frac{BP}{BA}\cdot\frac{AN}{AC}+\frac{CM}{CB}\cdot\frac{BP}{BA}\\ &\le \frac{1}{3}\left(\frac{AN}{AC}+\frac{BP}{BA}+\frac{CM}{CB}\right)^2 \;=\; \frac{1}{3}\,,\end{aligned}$$

as desired. Equality holds if and only if

$$\frac{AN}{AC} \;=\; \frac{BP}{BA} \;=\; \frac{CM}{CB} \;=\; \frac{1}{3}\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

2993★. [2004 : 503, 507] *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let x, y, z be non-negative real numbers satisfying $x + y + z = 1$. Prove or disprove:

(a) $\dfrac{x}{xy+1} + \dfrac{y}{yz+1} + \dfrac{z}{zx+1} \ \geq\ \dfrac{9}{10}$;

(b) $\dfrac{x}{y^2+1} + \dfrac{y}{z^2+1} + \dfrac{z}{x^2+1} \ \geq\ \dfrac{9}{10}$.

How do the left sides of (a) and (b) compare?

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $A(x,y,z)$ and $B(x,y,z)$ denote the expressions on the left sides of the inequalities (a) and (b), respectively. We have

$$A\left(0,\tfrac{2}{3},\tfrac{1}{3}\right) \ =\ \tfrac{29}{33} \ <\ \tfrac{9}{10} \ <\ \tfrac{14}{15} \ =\ B\left(0,\tfrac{2}{3},\tfrac{1}{3}\right)$$

and

$$A\left(0,\tfrac{1}{3},\tfrac{2}{3}\right) \ =\ \tfrac{31}{33} \ >\ \tfrac{9}{10} \ >\ \tfrac{35}{39} \ =\ B\left(0,\tfrac{1}{3},\tfrac{2}{3}\right) .$$

Thus, both inequalities (a) and (b) are invalid and no unconditional inequality between $A(x,y,z)$ and $B(x,y,z)$ exists.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAJE, University of Ploiesti, Romania; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAVLOS MARAGOUDAKIS, Lefkogia, Crete, Greece; B.J. VENKATACHALA, Indian Institute of Science, Bangalore, India; STAN WAGON, Macalester College, St. Paul, MN, USA; and ALEX WICE, student, Leaside High School, Toronto, ON.

Wagon has pointed out that the problem is easily solvable by Mathematica.

At least one solver has unsuccessfully attempted to solve the natural question: What are the largest constants a and b such that $A(x,y,z) \geq a$ and $B(x,y,z) \geq b$ for all non-negative real numbers x, y, and z with $x+y+z=1$? The moderator of this problem believes that the answers are as follows:

(a) $a \approx 0.8784148933$*; or more precisely,* $a = f(x_0)$*, where*

$$f(x) \ =\ 1 - x + \frac{x}{1+x-x^2}$$

and x_0 is the only root of the equation $x^3 - 2x^2 - 2x + 2 = 0$ on the interval $(0,1)$ ($x_0 \approx 0.6888921825$).

(b) $b \approx 0.8941074569$*; or more precisely,* $b = g(x_0)$*, where*

$$g(x) \ =\ 1 - x + \frac{x}{2-2x+x^2}$$

and x_0 is the only root of the equation $x^3 - 3x^2 + 6x - 2 = 0$ on the interval $(0,1)$ ($x_0 \approx 0.4039283620$).

Proving or disproving these claims will be left to the readers.

2994. [2004 : 507] *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, c be non-negative real numbers satisfying $a+b+c=3$. Show that

(a) $\displaystyle \frac{a^2}{b+1}+\frac{b^2}{c+1}+\frac{c^2}{a+1} \;\geq\; \frac{3}{2}$;

(b) $\displaystyle \frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1} \;\geq\; \frac{3}{2}$;

(c) $\displaystyle \frac{a^2}{b^2+1}+\frac{b^2}{c^2+1}+\frac{c^2}{a^2+1} \;\geq\; \frac{3}{2}$;

(d) $\displaystyle \frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1} \;\geq\; \frac{3}{2}$.

Composite of solutions by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Babis Stergiou, Chalkida, Greece.

(a) By the Cauchy-Schwarz Inequality, we have

$$\big((b+1)+(c+1)+(a+1)\big)\left(\frac{a^2}{b+1}+\frac{b^2}{c+1}+\frac{c^2}{a+1}\right) \;\geq\; (a+b+c)^2\,,$$

and thus, $6\left(\dfrac{a^2}{b+1}+\dfrac{b^2}{c+1}+\dfrac{c^2}{a+1}\right) \geq 9$, from which the result follows.

(b) For real numbers x, y, z and positive real numbers u, v, w, we have, by the Cauchy-Schwarz Inequality,

$$(u+v+w)\left(\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\right) \;\geq\; (x+y+z)^2\,.$$

Hence,

$$\begin{aligned}\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1} &= \frac{a^2}{ab+a}+\frac{b^2}{bc+b}+\frac{c^2}{ca+c}\\ &\geq \frac{(a+b+c)^2}{ab+bc+ca+3} \;=\; \frac{9}{ab+bc+ca+3}\,.\end{aligned}$$

Therefore, it suffices to show that

$$ab+bc+ca \;\leq\; 3\,, \qquad (1)$$

which holds since

$$\begin{aligned}9-3(ab+bc+ca) &= (a+b+c)^2-3(ab+bc+ca)\\ &= a^2+b^2+c^2-ab-bc-ca \;\geq\; 0\,.\end{aligned}$$

(c) Applying the result in (b) with a, b, c replaced by $\dfrac{3a^2}{a^2+b^2+c^2}$, $\dfrac{3b^2}{a^2+b^2+c^2}$, $\dfrac{3c^2}{a^2+b^2+c^2}$, respectively (whose sum is 3), we have

$$\frac{3a^2}{a^2+4b^2+c^2}+\frac{3b^2}{a^2+b^2+4c^2}+\frac{3c^2}{4a^2+b^2+c^2} \;\geq\; \frac{3}{2}\,. \qquad (2)$$

But

$$9 \;=\; (a+b+c)^2 \;=\; a^2+b^2+c^2+2ab+2bc+2ca \;\leq\; 3(a^2+b^2+c^2)\,, \quad (3)$$

and thus, from (2) and (3), we see that

$$\frac{3a^2}{3b^2+3}+\frac{3b^2}{3c^2+3}+\frac{3c^2}{3a^2+3} \;\geq\; \frac{3}{2}\,,$$

from which the result follows immediately.

(d) We have

$$\frac{a}{b^2+1} \;=\; a-\frac{ab^2}{b^2+1} \;\geq\; a-\frac{ab^2}{2b} \;=\; a-\frac{ab}{2}\,.$$

Similarly,

$$\frac{b}{c^2+1} \;\geq\; b-\frac{bc}{2}\,, \qquad \text{and} \quad \frac{c}{a^2+1} \;\geq\; c-\frac{ca}{2}\,.$$

Hence,

$$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1} \;\geq\; a+b+c-\frac{ab+bc+ca}{2} \;\geq\; 3-\frac{3}{2} \;=\; \frac{3}{2}$$

by (1) above.

Also solved by ARKADY ALT, San Jose, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

Solutions to parts (a), (b), *and* (c) *were given by PAVLOS MARAGOUDAKIS, Lefkogia, Crete, Greece and YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON. VEDULA N. MURTY, Dover, PA, USA gave solutions to parts* (a) *and* (b). *STAN WAGON, Macalester College, St. Paul, MN, USA submitted comments claiming that he can prove all these inequalities by using FindInstance in Mathematica.*

2995. [2004 : 503, 507] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let $ABCD$ be a cyclic quadrilateral in which the diagonals AC and BD intersect at right angles at E. Let O be the centre of its circumscribing circle. Let P be the point of intersection of the tangent lines at A and B. Let Q, R, S be similarly defined for the pairs B and C, C and D, D and A, respectively. It is known that $PQRS$ is a cyclic quadrilateral.

Let T, U, V, W be the orthocentres of $\triangle AOB$, $\triangle BOC$, $\triangle COD$, $\triangle DOA$, respectively.

Let F, G, H, K be the orthocentres of $\triangle POQ$, $\triangle QOR$, $\triangle ROS$, $\triangle SOP$, respectively.

Prove that $TUVW$ and $FGHK$ are straight lines intersecting at right angles at E.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let M and N be the mid-points of AB and BC, respectively. Then $\triangle ATN \sim \triangle OBM$ and $\triangle ABE \sim \triangle OBN$. Hence, $\frac{AT}{AM} = \frac{OB}{OM}$ and $\frac{AE}{AB} = \frac{ON}{OB}$. Similarly, $\frac{CU}{CN} = \frac{OB}{ON}$ and $\frac{CE}{CB} = \frac{OM}{OB}$. Therefore,

$$\frac{AT}{CU} = \left(\frac{AM}{CN}\right)\left(\frac{ON}{OM}\right) = \left(\frac{AB}{CB}\right)\left(\frac{ON}{OM}\right) = \frac{AE}{CE},$$

which implies that T, E, U are collinear. Likewise, U, E, V are collinear, and V, E, W are collinear. Thus, $TUVWE$ is a straight line.

Now, it is well known [1] that PR and QS also intersect at E. Since $\triangle FBP \sim \triangle GCR$, we have

$$\frac{FP}{GR} = \frac{BP}{CR} = \frac{AP}{CR}.$$

Applying the Law of Sines to $\triangle APE$ and $\triangle CRE$, we get

$$\frac{AP}{\sin(\angle AEP)} = \frac{EP}{\sin(\angle EAP)} = \frac{EP}{\sin(\angle ADC)}$$

and

$$\frac{CR}{\sin(\angle CER)} = \frac{ER}{\sin(\angle ECR)} = \frac{ER}{\sin(\angle ABC)}.$$

Since $\angle ADC + \angle ABC = 180^\circ$, we have $\frac{AP}{CR} = \frac{EP}{ER}$. Hence, $\frac{FP}{GR} = \frac{EP}{ER}$, which implies that F, E, G are collinear. Likewise, G, E, H are collinear, and H, E, K are collinear. Thus, $FGHKE$ is a straight line.

Finally, we show that $FE \perp TE$. Note that $\triangle ABE \sim \triangle OQB$, so that $\frac{AE}{BE} = \frac{OB}{QB}$. Also, $\triangle AMT \sim \triangle OMB \sim \triangle QBF$, so that $\frac{AT}{AM} = \frac{OB}{OM}$ and $\frac{MB}{OM} = \frac{BF}{QB}$. Therefore, $\frac{AT}{BF} = \frac{AE}{BE}$, which implies that $\triangle AET \sim \triangle BEF$. Hence, $\angle AET = \angle BEF$. Thus, $ET \perp EF$.

Reference

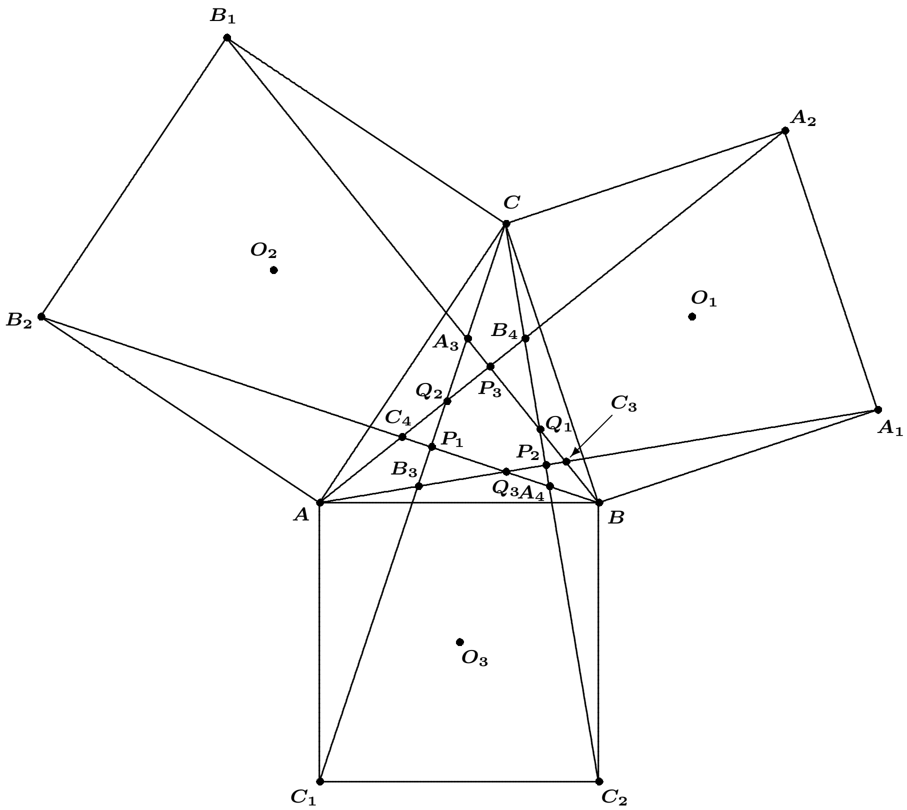
[1] I.M. Yaglom, *Geometric Transformations III*, MAA, 1973, p. 60.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2996. [2004 : 504, 508] *Proposed by Roland Eddy and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given $\triangle ABC$, draw squares AC_1C_2B, BA_1A_2C, CB_1B_2A outwards as indicated, with centres O_3, O_1, O_2, respectively. Let

$$\begin{array}{llll} A_3 = BB_1 \cap CC_1\,, & B_3 = CC_1 \cap AA_1\,, & C_3 = AA_1 \cap BB_1\,, \\ A_4 = BB_2 \cap CC_2\,, & B_4 = CC_2 \cap AA_2\,, & C_4 = AA_2 \cap BB_2\,, \\ P_1 = BB_2 \cap CC_1\,, & P_2 = CC_2 \cap AA_1\,, & P_3 = AA_2 \cap BB_1\,, \\ Q_1 = BB_1 \cap CC_2\,, & Q_2 = CC_1 \cap AA_2\,, & Q_3 = AA_1 \cap BB_2\,. \end{array}$$



Prove that:

(a) $AC \parallel A_3C_4 \parallel A_4C_3$;

(b) AQ_1, BQ_2, and CQ_3 are concurrent;

(c) $AA_1 \perp CC_2$;

(d) AP_1 bisects $\angle C_1P_1B_2$;

(e) A, P_1, and O_1 are collinear;

(f) AP_1, BP_2, and CP_3 are concurrent.

The proposers have proofs of all these results, but, except for (c), all are done using coordinate geometry. They would like to see nice synthetic proofs.

Combined solutions of Michel Bataille, Rouen, France; Joel Schlosberg, Bayside, NY, USA; Peter Y. Woo, Biola University, La Mirada, CA, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; Li Zhou, Polk Community College, Winter Haven, FL, USA; and Titu Zvonaru, Comăneşti, Romania.

Individual contributions will be indicated below only when they were unique. We prove the results in the order c, d, e, f, a, b.

(c) The 90° rotation about B takes triangle A_1AB to CC_2B. Thus, $AA_1 \perp CC_2$ as desired; as a bonus, these two segments have the same length. Similarly, using 90° rotations about A and C, we conclude that segment AA_1 is equal and perpendicular to CC_2, segment BB_1 is equal and perpendicular to AA_2, and segment CC_1 is equal and perpendicular to BB_2.

Lemma. $BA_1A_2CP_3P_2$, $CB_1B_2AP_1P_3$, and $AC_1C_2BP_2P_1$ are each cyclic hexagons; their centres are O_1, O_2, and O_3, respectively.

Proof: For the first hexagon, because of the right angles at A_1, C (given), and P_3 (by part (c)), the first circle has diameter BA_2 and passes through A_1, C, and P_3; similarly, because of the right angles at B, A_2 (given), and P_2 (by part (c)), A_1C is another diameter and that same circle passes through P_1. Analogous arguments work for the other two hexagons.

(d) By the lemma, $\angle AP_1C_1 = \angle ABC_1$ (since these are oriented angles in arc AC_1 of the third circle). Since $\angle ABC_1 = 45^\circ$ while (by part (c)) $\angle C_1P_1B_2 = 90^\circ$, AP_1 bisects $\angle C_1P_1B_2$, which is (d).

Alternatively [Woo], as in part (c), a rotation through 90° about A takes CC_1 to B_2B. Let P_1' be the image of P_1 under this rotation. It lies on B_2B (since P_1 lies on CC_1). Then $\angle P_1AP_1' = 90^\circ$ and $AP_1 = AP_1'$. Thus, $\angle AP_1B_2 = 45^\circ$, and again we see that AP_1 bisects $\angle C_1P_1B_2$.

(e) Since $\angle BP_1C = \angle BO_1C = 90^\circ$, we see that BO_1CP_1 is cyclic, which means that $\angle CP_1O_1 = BP_1O_1 = 45^\circ$. This, together with part (d), implies that O_1 lies on the bisector AP_1 of $\angle CP_1B = \angle C_1P_1B_2$. Thus, O_1 lies on AP_1, which is (e). Similarly, O_2 is on BP_2 and O_3 is on CP_3.

(f) [Zvonaru] Since, by part (e), the lines AP_1, BP_2, CP_3 coincide with AO_1, BO_2, CO_3, respectively, these lines concur at Vecten's point.

Alternatively, it is a familiar exercise to show that $AO_1 \perp O_2O_3$— just rotate segment O_3O_2 through 45° about A and dilate by the factor $\sqrt{2}$ to get BB_2, then rotate the image segment 45° about C and shrink by the factor $1/\sqrt{2}$ to get AO_1; it follows that the first and last segments are perpendicular. Similarly, $BO_2 \perp O_1O_3$ and $CO_3 \perp O_1O_2$. But we saw in part (e) that the lines AO_1 and AP_1 coincide, so that AP_1 lies along an altitude of $\triangle O_1O_2O_3$. Similarly, BP_2 and CP_3 lie along altitudes; whence, the three lines intersect in the orthocentre of $\triangle O_1O_2O_3$.

(a) We first show that AC is parallel to A_3C_4. Because of the right angles at P_1 and P_3, we see that $A_3C_4P_1P_3$ is cyclic. and so is $CB_1B_2AP_1P_3$

(by the lemma). Therefore,

$$\angle A_3C_4P_3 \;=\; \angle A_3P_1P_3 \;=\; \angle CP_1P_3 \;=\; \angle CAP_3\,.$$

Thus, $AC \parallel A_3C_4$.

It remains to show (for part (a)) that AC is parallel to A_4C_3. We isolate the relevant part of the figure and prove the following: *If O_2 is the intersection point of the diagonals B_2C and B_1A of the parallelogram ACB_1B_2 erected on the side AC of $\triangle ABC$, and P is any point on the line BO_2, and if we define A_4 to be the point where CP intersects BB_2, and C_3 to be where AP meets BB_1, then $AC \parallel A_4C_3$.* This is an immediate consequence of Desargues's Theorem since triangles B_1AC_3 and B_2CA_4 are perspective from the line BO_2; that is, corresponding sides B_1A and B_2C (which meet at O_2), AC_3 and CA_4 (which meet at P), and B_1C_3 and B_2A_4 (which meet at B) intersect in collinear points. We conclude that these triangles are perspective from a point. Since the joins B_1B_2 and AC of two pairs of the corresponding vertices are parallel, so is A_4C_3. Of course, this proves the second part of part (a), since $P = P_2$ is on BO_2 by part (e).

(b) [Bataille] We know that Q_3 is on the altitude from C of $\triangle ABC$—this is problem 2658 [2001 : 337; 2002 : 347], for which Toshio Seimiya provided a synthetic proof. Similarly, $BQ_2 \perp AC$, $AQ_1 \perp BC$, and, therefore, AQ_1, BQ_2, CQ_3 concur at the orthocentre of $\triangle ABC$.

Alternatively [Zhou], we know from part (c) that $A_3P_1 \perp BC_4$ and $C_4P_3 \perp BA_3$; it follows that the point Q_2, where A_3P_1 intersects C_4P_3, must be the orthocentre of $\triangle BA_3C_4$ and, therefore, $BQ_2 \perp A_3C_4$. Since $AC \parallel A_3C_4$ (by part (a)), we deduce that $BQ_2 \perp AC$. Likewise, $AQ_1 \perp BC$ and $CQ_3 \perp AB$. Therefore, we again see that AQ_1, BQ_2, CQ_3 concur at the orthocentre of $\triangle ABC$ and our task is complete.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, who, like the proposers, used coordinates.

2997. [2004 : 505, 509] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with incircle Γ. Suppose that Γ touches the sides BC, CA, AB at X, Y, Z, respectively. Let YZ meet BC at X'; let ZX meet CA at Y'; and let XY meet AB at Z'. Let P be any point on the line $X'Y'$. Suppose that AP meets BC at L, that BP meets CA at M, and that CP meets AB at N. Now let MN meet BC at U; let NL meet CA at V; and let LM meet AB at W.

Prove that UVW is a straight line, and that it is tangent to Γ.

[*Ed:* Bradley adds: "This problem is not original. It comes from a book of problems by Wolstenholme (St. John's College, Cambridge) dated in the 19^{th} Century, where the problem actually involves any conic touching the sides."]

Solution by Rudolf Fritsch, University of Munich, Munich, Germany.

The collinearity of U, V, and W is an immediate consequence of Desargues' Theorem: Since the triangles ABC and LMN are perspective from P, the intersection points of corresponding sides are collinear. The second property we are to prove is a special case of a theorem of classical projective geometry (See H.S.M. Coxeter, *The Real Projective Plane*; it is the dual of Theorem 7.71, page 185):

> *Let the conic Γ be inscribed in the triangle ABC, and let Q be the point of concurrence of the cevians through the points of tangency. The trilinear polar of Q is the locus of the points whose trilinear polars are tangents to Γ.*

If Γ is the incircle, then Q is the Gergonne point of the triangle. In the statement of the problem the line $X'Y'$ is the trilinear polar of the Gergonne point, and the line UVW is the trilinear polar of the point P as it moves along $X'Y'$. This completes the solution, but an explanation is perhaps required for those to whom the theorem is unfamiliar.

An explanation. In a projective plane, the *trilinear polarity* defined by a triangle ABC is an involutory correspondence between the points not on a side of the triangle and the lines not through a vertex. It is not a proper polarity since it is not defined on all points and lines of the plane. (See, for example, H.S.M. Coxeter, *Projective Geometry*, Section 3.4. The terminology is due to Poncelet; the correspondence is also called a *polarity with respect to a triangle*.) To find the polar of a point Q not on the sides of $\triangle ABC$, let X, Y, Z be the points where the cevians AQ, BQ, CQ intersect the opposite sides. If X', Y', Z' are the harmonic conjugates of X, Y, Z with respect to the vertices of the triangle (that is, X' is the harmonic conjugate of X with respect to B and C, and so forth), then X', Y', Z' lie on a line (by Desargues's Theorem) that we call the *trilinear polar of Q with respect to triangle ABC*. The dual process produces the *trilinear pole* of a given line not through a vertex.

The proof of the theorem is a straightforward exercise in projective geometry. Those without the necessary background can use projective coordinates. Choose a coordinate system such that the vertices A, B, C of the triangle have the coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, respectively, and the point Q has the coordinates $(1,1,1)$. The following claims are all easily verified. The trilinear polar of the point P with coordinates (x,y,z) is the line $[1/x, 1/y, 1/z]$: the cevians through (x,y,z) meet the opposite sides at

$$X = (0,y,z), \quad Y = (x,0,z), \quad Z = (x,y,0);$$

their harmonic conjugates are

$$X' = (0,-y,z), \quad Y' = (x,0,-z), \quad Z' = (-x,y,0),$$

which all lie on the line $[1/x, 1/y, 1/z]$.

Starting with the point $Q(1,1,1)$, the conic tangent to the sides of $\triangle ABC$ at $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ is

$$\Gamma : x^2 + y^2 + z^2 - 2(xy + yz + zx) \;=\; 0\,.$$

(To confirm this claim, it is, by symmetry, sufficient to verify that $x = 0$ meets the conic twice at the point satisfying $(y - z)^2 = 0$, which is $(0,1,1)$.) The polar of $(1,1,1)$ is the line $[1,1,1]$. A typical point P on this line is $(-p, p-1, 1)$, whose polar is the line $[-1/p, 1/(p-1), 1]$. In affine coordinates, this line,

$$y \;=\; \frac{p-1}{p}x + (1-p)\,,$$

meets the conic

$$2(x+y) \;=\; (x-y)^2 + 1$$

(which is the affine form of Γ, obtained by setting $z = 1$) with double contact at the point $(x,y) = \big(p^2, (1-p)^2\big)$. Therefore, the polar of every point on $L = [1,1,1]$ is tangent to Γ, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA (partial solution); and the proposer.

If, in the statement of our problem, we replace the circle Γ by an arbitrary conic that touches the sides of triangle ABC, then the cevians AX, BY, CZ will still concur at a point Q. Bataille, in his solution, explains that this concurrence is an immediate consequence of Brianchon's Theorem.

2998. [2004 : 505, 509] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer which is not a multiple of 3, and let A, B, C be $n \times n$ matrices with real entries that satisfy

$$A^2 + B^2 + C^2 \;=\; AB + BC + CA\,.$$

Prove that

$$\det\big((AB - BA) + (BC - CB) + (CA - AC)\big) \;=\; 0\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $\omega = e^{2\pi i/3}$ and let $M = A + \omega B + \omega^2 C$. Using $1 + \omega + \omega^2 = 0$ and $\overline{\omega} = \omega^2$, we get

$$\begin{aligned} M\overline{M} &= (A + \omega B + \omega^2 C)(A + \omega^2 B + \omega C) \\ &= A^2 + B^2 + C^2 + \omega^2(AB + BC + CA) + \omega(BA + CB + AC) \\ &= (1 + \omega^2)(AB + BC + CA) + \omega(BA + CB + AC) \\ &= -\omega(AB + BC + CA - BA - CB - AC)\,. \end{aligned}$$

Hence,

$$\begin{aligned}(-\omega)^n \det\big((AB-BA)+(BC-CB)+(CA-AC)\big) \\ = \det(M\overline{M}) = |\det M|^2 \in \mathbb{R}\,,\end{aligned}$$

from which the claim follows, since $(-\omega)^n \notin \mathbb{R}$.

Also solved by MICHEL BATAILLE, Rouen, France; and the proposer.

2999. [2004 : 505, 509] *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let m, n be positive integers. Prove that

$$\left(\frac{m+1}{m}\sum_{k=1}^{n}\frac{k}{n^{m+2}}(n^m-k^m)\right)^m < \frac{1}{m+1}\,.$$

Solution by Michel Bataille, Rouen, France.

Let $S_n = \sum\limits_{k=1}^{n}\dfrac{k}{n^m+2}(n^m-k^m)$. Then

$$S_n = \frac{1}{n^2}\sum_{k=1}^{n} k\left(1-\left(\frac{k}{n}^m\right)\right) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\,,$$

where $f(x) = x - x^{m+1}$. It is easy to verify that f reahces its maximum value of $f(x_0) = \dfrac{m}{(m+1)^{1+1/m}}$ if and only if $x_0 = \dfrac{1}{(m+1)^{1+1/m}}$.

If $n = 1$, we see that $S_n = S_1 = f(1) < f(x_0)$. If $n > 1$, then $f(k/n) \le f(x_0)$ for $k = 1, 2, \ldots, n$, and at least one of the numbers $f(k/n)$ is less than $f(x_0)$. Hence,

$$S_n < \frac{1}{n}(f(x_0)) = f(x_0)\,.$$

It follows that

$$\begin{aligned}\left(\frac{m+1}{m}\sum_{k=1}^{n}\frac{k}{n^{m+2}}(n^m-k^m)\right)^m &< \left(\frac{m+1}{m}f(x_0)\right)^m \\ &= \left(\frac{1}{(m+1)^{1/m}}\right)^m = \frac{1}{m+1}\,.\end{aligned}$$

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

3000. [2004 : 505, 509] *Proposed by Paul Dayao, Ateneo de Manila University, The Philippines.*

Let f be a continuous, non-negative, and twice-differentiable function on $[0,\infty)$. Suppose that $xf''(x)+f'(x)$ is non-zero and does not change sign on $[0,\infty)$. If x_1, x_2, …, x_n are non-negative real numbers and c is their geometric mean, show that

$$f(x_1)+f(x_2)+\cdots+f(x_n) \;\geq\; nf(c)\,,$$

with equality if and only if $x_1=x_2=\cdots=x_n$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Note that $xf''(x)+f'(x) = \frac{d}{dx}\big(xf'(x)\big)$. By hypothesis, we have either $\frac{d}{dx}\big(xf'(x)\big) < 0$ on $[0,\infty)$ or $\frac{d}{dx}\big(xf'(x)\big) > 0$ on $[0,\infty)$.

First suppose that $\frac{d}{dx}\big(xf'(x)\big) < 0$ on $[0,\infty)$. Then $xf'(x)$ is strictly decreasing on $[0,\infty)$. Thus, for $x>0$ we have $xf'(x) < 0f'(0) = 0$. Then $f'(x)<0$ for all $x>0$. In particular, $f'(1)<0$. Furthermore, for $x\geq 1$, we have $xf'(x) \leq 1f'(1) = f'(1)$; that is, $f'(x)\leq f'(1)/x$. Hence, for $x\geq 1$,

$$f(x)-f(1) \;=\; \int_1^x f'(t)dt \;\leq\; f'(1)\int_1^x \frac{1}{t}\,dt \;=\; f'(1)\ln x\,.$$

Since $f'(1)<0$, this inequality implies that $f(x)\to -\infty$ as $x\to\infty$, contradicting the requirement that f be non-negative.

Thus, the only possibility is that $\frac{d}{dx}\big(xf'(x)\big) > 0$ on $[0,\infty)$. Then $xf'(x)$ is strictly increasing on $[0,\infty)$, from which it follows that $f'(x)>0$ for all $x>0$. Thus, f is strictly increasing on $[0,\infty)$.

If $x_i=0$ for some i, then $c=0$ and

$$f(x_1)+f(x_2)+\cdots+f(x_n) \;\geq\; f(0)+f(0)+\cdots+f(0) \;=\; nf(0) \;=\; nf(c)\,.$$

Let us assume, therefore, that $x_i>0$ for all i. For each i, let $u_i=\ln x_i$. Let $g(u)=f(e^u)$. Then $g''(u) = e^u\big(e^uf''(e^u)+f'(e^u)\big)$. We see that $g''(u)>0$ for all real numbers u, which shows that g is strictly convex. Then Jensen's Inequality yields

$$g(u_1)+g(u_2)+\cdots+g(u_n) \;\geq\; ng\left(\frac{u_1+u_2+\cdots+u_n}{n}\right)\,,$$

with equality if and only if $u_1=u_2=\cdots=u_n$. Equivalently,

$$f(x_1)+f(x_2)+\cdots+f(x_n) \;\geq\; nf(c)\,,$$

with equality if and only if $x_1=x_2=\cdots=x_n$.

Also solved by MICHEL BATAILLE, Rouen, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete solution.

YEAR END FINALE

It is hard to believe that I have now completed three years as Editor-in-Chief of ***CRUX with MAYHEM***. Time sure flies when you're having fun!

We have received much favourable feedback on our Klamkin commemorative issue in September. Andy Liu wrote a fine tribute to Murray. It is unfortunate that we did not put such a tribute to Murray in ***CRUX with MAYHEM*** while he was alive.

We are now planning to identify individuals from time to time who have given a lot to ***CRUX with MAYHEM***, and feature them with a short biography and (hopefully) a picture. I, for one, look forward to putting faces to the many regular contributors to this magazine and finding out more about them personally. We are currently assembling a short list of CRUXers, and we hope to start this feature early in the New Year.

At this time, I need to thank the many individuals who contribute so much to ***CRUX with MAYHEM*** and without whose contributions the magazine would certainly suffer. The first person I need to thank, as always, is BRUCE CROFOOT, my Associate Editor. Bruce continues to put in vast amounts of time scrutinizing several drafts of each section. Secondly, I wish to thank SHAWN GODIN, who not only assembles all the material (including the determination of which solutions are correct and which to feature) for the *Mathematical Mayhem* section of ***CRUX with MAYHEM***, but also writes a monthly column.

There are many other people whom I wish to thank for their particular contributions. These include ILIYA BLUSKOV, RICK BREWSTER, CHRIS FISHER, EDWARD WANG, and BRUCE SHAWYER for their regular and timely service in assessing the solutions; BRUCE GILLIGAN, for ensuring that ***CRUX with MAYHEM*** has quality articles; JOHN GRANT McLOUGHLIN, for ensuring that we have book reviews that are appropriate to our readership; ROBERT WOODROW for overseeing the *Olympiad Corner*; and ROBERT BILINSKI, who is doing likewise with the *Skoliad*.

The task of providing us with timely translations of our Problems still rests on the shoulders of JEAN-MARC TERRIER and MARTIN GOLDSTEIN. I want to thank them for their efforts, and for always coming through even when I have given them very little time for turn-around. We often reword the English problem after they have translated it to French, as their translation often improves the wording of the original problem. That kind of attention is more than I have a right to ask for, and I am most grateful.

I want to thank all the proofreaders. MOHAMMED AASSILA and BILL SANDS assist the editors with this task. The quality of the work of all these people is a vital part of what makes ***CRUX with MAYHEM*** what it is. Thank you one and all.

Thanks also go to Thompson Rivers University (formerly the University College of the Cariboo) and my colleagues in the Department of Mathematics and Statistics for their continued understanding and support, and for believing that my work on this journal is important enough to reduce my teaching load sufficiently to allow me to do it. Special thanks go to SUSAN HOWIE, secretary to our department, for all that she does to give me more time to edit.

Also, the LaTeX expertise of JOANNE CANAPE at the University of Calgary and TAO GONG at Wilfrid Laurier University is much appreciated.

Thanks to GRAHAM WRIGHT, the Managing Editor, who keeps me on the right track (and adhering to deadlines!), and to the University of Toronto Press, and TAMI EHRLICH in particular, who continue to print a high-quality product.

The online version of ***CRUX with MAYHEM*** continues to grow. Thanks go to JUDI BORWEIN at Dalhousie University for putting all the material on the Canadian Mathematical Society website.

Last but not least, I send my thanks to you, the readers. Without you, ***CRUX with MAYHEM*** would not be what it is. We are receiving over 200 problem proposals each year now, and we can publish only 100 of these in each volume. Of course, we receive hundreds of solutions, as you will see in the index that follows. Every year, we receive solutions from new readers. This is very gratifying. We hope that these new solvers will become regular solvers and proposers of new problems. Please ensure that your name and address is on EVERY problem or proposal, and that each starts on a fresh sheet of paper. Otherwise, there may be filing errors, resulting in a submitted solution or proposal being lost. We need your ARTICLES, PROPOSALS, and SOLUTIONS to keep ***CRUX with MAYHEM*** alive and well. Keep them coming!

I wish everyone the compliments of the season, and a very happy, peaceful, and prosperous year 2006.

Jim Totten

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Austrian IMO Team 2004 2910

Southwest Missouri State University Problem-Solving Group 2965