SKOLIAD No. 84

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Please send your solutions to the problems in this edition by ***1 June, 2005***. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (∗) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

Our items this issue come from a contest organized by the South african Mathematical Society and sponsored by the Actuarial Society of South Africa.

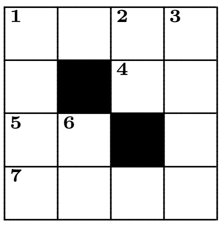
South African Interprovincial Mathematics Olympiad 2004

Team Paper, Juniors: 60 minutes allowed

1. (∗) Five bags of rice are weighed two at a time, in all possible combinations. The ten weights are 72, 73, 76, 77, 79, 80, 81, 83, 84, and 87. What are the weights of the five bags?

2. (∗) Delete 60 digits from the number $1\,2\,3\,4\,5\,6 \ldots 38\,39\,40$ in such a way as to make the resulting number as small as possible.

3. Solve the crossnumber puzzle:

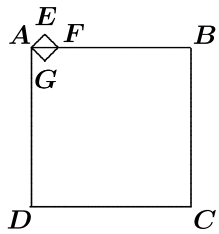


Across	**Down**
1. Cube of a prime	1. Square of a prime
4. Square	2. Three times cube root of 1 Across
5. Square	3. Square of a prime
7. Cube	6. Twice cube root of 7 Across

4. Find the sum of the digits of $10^{2004} - 2004$.

5. An urn contains 100 balls of different colours: namely, 10 white, 10 black, 12 yellow, 14 blue, 24 green, and 30 red. What is the minimum number of balls that must be drawn from the urn without looking if you want to be certain that at least 15 of the balls drawn are of the same colour?

6. In the diagram at right, square $ABCD$ has side 24 cm and square $AEFG$ has side 2 cm. What is the length of CE in cm?



7. All the positive integers, starting with 1, are written in order, namely,

$$1234567891011121314151 6\ldots$$

Find the digit appearing in the 206 788$^{\text{th}}$ position.

8. How many times does the number 2 appear when the product

$$1002 \cdot 1003 \cdot 1004 \cdots 2004$$

is expanded into its prime factors?

9. In the addition below, digits have been replaced by letters in a one-to-one fashion. Given that $D = 5$, work out the original numbers.

D	O	N	A	L	D
G	E	R	A	L	D
R	O	B	E	R	T

10. Consider a square having 16 cells each containing a plus sign or a minus sign. Suppose we change all the signs in a given row (or column), doing this several times until the number of minus signs is a minimum. A table that has the property that any such change does not decrease the number of minus signs is called a *minimal table*, and the number of minus signs in a minimal table is called the *characteristic* of the table. Find all possible values of the characteristic.

Olympiade Mathématique Interprovinciale d'Afrique du Sud 2004

Épreuve en Équipe, Junior : 60 minutes permises

1. (∗) Cinq sacs de riz sont pesés deux à la fois dans toutes les combinaisons. Les dix pesées obtenues sont 72, 73, 76, 77, 79, 80, 81, 83, 84 et 87. Combien pèsent chacun des sacs ?

2. (∗) Enlever 60 chiffres du nombre 1 2 3 4 5 6 ... 38 39 40 pour que le nombre résultant soit le plus petit possible.

3. Résoudre le nombre croisé :

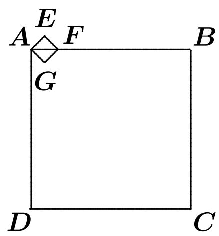
1		2	3
	■	4	
5	6	■	
7			

Horizontal	**Vertical**
1. Cube d'un premier	1. Carré d'un premier
4. Carré	2. Trois fois la racine cubique de 1 Horizontal
5. Carré	3. Carré d'un premier
7. Cube	6. Double de la racine cubique de 7 Horizontal

4. Trouver la somme des chiffres de $10^{2004} - 2004$.

5. Une urne contient 100 balles de couleurs différentes, soit 10 blanches, 10 noires, 12 jaunes, 14 bleues, 24 vertes et 30 rouges. Quelle est le nombre minimal de balles pigées à l'aveugle que l'on doit enlever pour s'assurer qu'il y a 15 balles de la même couleur parmi les balles tirées.

6. Dans le dessin à droite, le carré $ABCD$ a 24 cm de côté alors que $AEFG$ en a 2 cm. Que mesure CE en cm ?



7. Tous les entiers positifs à partir de 1 sont écrits en ordre, soit :

$$12345678910111213141516\ldots$$

Quel est le $206\,788^{\text{ieme}}$ chiffre ?

8. Combien de fois le nombre 2 apparaît quand

$$1002 \cdot 1003 \cdot 1004 \cdots 2004$$

est décomposé en facteurs premiers ?

9. Dans l'addition suivante, les chiffres ont été remplacés par des lettres une à une. Sachant que $D = 5$, trouver les nombres originaux.

D	O	N	A	L	D
G	E	R	A	L	D
R	O	B	E	R	T

10. Considérer un carré ayant 16 cases contenant chacune des plus et des moins. Supposons que l'on change tous les signes dans une rangée (ou une colonne) à répétition jusqu'à ce que le nombre de moins soit minimal. Une table qui a la propriété de ne pas pouvoir se faire réduire le nombre de moins est appelée *minimale* et le nombre de moins s'y trouvant est la *caractéristique* de la table. Trouver toutes les caractéristiques possibles.

Next we give the solutions to the Nova Scotia Math League championship [2004 : 321–322].

Nova Scotia Math League Game 4: Group Questions

1. A *lattice point* is a point (x, y), where the coordinates are both integers. For example, $(3, -4)$ and $(5, 0)$ are lattice points, but $(2, 4.58)$ is not.

Determine the number of lattice points on the circumference of the circle $x^2 + y^2 = 25$.

Solution by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON, modified by the editor.

If a lattice point has coordinates (m, n) and lies on the circumference of the circle $x^2 + y^2 = 25$, then $m^2 + n^2 = 25$. Since m and n are integers and $m^2 \geq 0$, it follows that n^2 is in $\{0, 1, 4, 9, 16, 25\}$. By symmetry, this also holds for m^2.

- If $n^2 = 1$ or $n^2 = 4$, then m^2 is not a perfect square ($m^2 = 24$ or $m^2 = 21$) and yields no lattice points on the circle.
- If $n^2 = 0$, then $n = 0$ and $m^2 = 25$, which yields $m = \pm 5$. Thus, we get the two lattice points $(-5, 0)$ and $(5, 0)$.
- If $n^2 = 9$, then $n = \pm 3$ and $m^2 = 25 - 9 = 16$, which yields $m = \pm 4$. Hence, we get the four lattice points $(-4, -3)$, $(-4, 3)$, $(4, -3)$, and $(4, 3)$.
- If $n^2 = 16$, then $n = \pm 4$ and $m^2 = 25 - 16 = 9$, which yields $m = \pm 3$. Thus, we get the four lattice points $(-3, -4)$, $(-3, 4)$, $(3, -4)$, and $(3, 4)$.
- Finally, if $n^2 = 25$, then $n = \pm 5$ and $m^2 = 0$, which means that $m = 0$. This gives the two lattice points $(0, -5)$ and $(0, 5)$.

Therefore, there are 12 lattice points in total on the circle.

One incorrect solution was received.

2. Let $S = \frac{2^2-1}{2^2} \times \frac{3^2-1}{3^2} \times \frac{4^2-1}{4^2} \times \cdots \times \frac{2004^2-1}{2004^2}$. Express S as a fraction, reduced to lowest terms.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

$$\begin{aligned} S &= \frac{\prod_{k=2}^{2004}(k+1)\prod_{k=2}^{2004}(k-1)}{\prod_{k=2}^{2004}k\prod_{k=2}^{2004}k} = \frac{\prod_{k=3}^{2005}k\prod_{k=1}^{2003}k}{\prod_{k=2}^{2004}k\prod_{k=2}^{2004}k} \\ &= \frac{2005}{2\cdot 2004} = \frac{2005}{4008}\,. \end{aligned}$$

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

3. A *palindrome* is a number that reads the same forwards and backwards, such as 8338 and 50705.

Let A and B be four-digit palindromes, and let C be a five-digit palindrome. If $A+B=C$, determine all possible values of C.

Solution by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON, modified by the editor.

Let the palindrome A be $abba$, where a and b are digits with $a \neq 0$. Similarly, let the palindrome B be $cddc$, where c and d are digits with $c \neq 0$. Since A and B are four-digit numbers, their sum, $A+B=C$, is at most 19998. Therefore, the left-most digit of C must be 1. Let the palindrome C be $1fgf1$, where f and g are digits.

Since the units digit of C is 1, we must have $a+c=1$ or $a+c=11$. But only $a+c=11$ is possible, because $a \neq 0$ and $c \neq 0$. This means that $f=1$ or $f=2$ (look at the thousands column of the addition).

The addition now looks like

			1	
	a	b	b	a
+	c	d	d	c
1	f	g	f	1

where $f=1$ or $f=2$ and the first row represents "carries".

Case 1: $f=1$.

Because $a+c=11$ and $f=1$, we see that $b+d$ ends with 0 (look at the tens column). If $b+d=10$ and $a+c=11$, then, looking at the left-most f in C, we see that f must be 2, a contradiction. Hence, $b+d=0$, from which we have $g=0$, which gives us $C=11011$.

Case 2: $f = 2$.

Since $a + c = 11$ and $f = 2$, then $b + d$ ends with 1. Thus, $b + d = 1$ or $b + d = 11$. If $b + d = 1$ and $a + c = 11$, then, looking at the left-most f in C, we see that f has to be 1, a contradiction. Hence, $b + d = 11$, from which we have $g = 2$, which gives us $C = 12221$.

Therefore, the possible palindromes are 11011 and 12221.

One incomplete solution was received.

4. The circle with equation $x^2 + y^2 = 1$ intersects the line $y = 7x + 5$ at two distinct points, A and B. Let O be the centre of the circle. Find the measure of $\angle AOB$.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

It is easy to find A and B as $\left(-\frac{3}{5}, \frac{4}{5}\right)$ and $\left(-\frac{4}{5}, -\frac{3}{5}\right)$, respectively. The distance between these points is $\sqrt{2}$, and each of them is a distance 1 from O. Hence, AOB is a triangle with sides $AO = BO = 1$ and $AB = \sqrt{2}$. By the converse of the Pythagorean Theorem, $\angle AOB = \pi/2$.

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

5. Determine all integers x such that $(x^2 - 3x + 1)^{x+1} = 1$.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON, modified by the editor.

The given equation has the form $b^y = 1$, where $b = x^2 - 3x + 1$ and $y = x + 1$. The equation $b^y = 1$ holds for integers b and y in the following cases:

(a) $b = 1$. Then $x^2 - 3x = 0$; whence, $x = 0$ or $x = 3$.

(b) $b = -1$ and y is even. Then $(x-2)(x-1) = 0$ and $x+1$ even; whence, $x = 1$.

(c) $y = 0$ and $b \neq 0$. Then $x = -1$.

Therefore, the solutions are $x = -1$, $x = 0$, $x = 1$, and $x = 3$.

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

6. Let $f(a, b)$ denote the sum of the integers between a and b, inclusive. For example, $f(1, 5) = 1+2+3+4+5 = 15$ and $f(3, 6) = 3+4+5+6 = 18$.

Determine the value of $f(133333, 533333)$.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Let $a = 133333$. Then we have the arithmetic series

$$a + (a + 1) + \cdots + (a + 400000) .$$

The sum of this series is

$$\begin{aligned} 400001a + 200000(400001) &= (400001)(333333) \\ &= 3(44444511111) \\ &= 133333533333\,. \end{aligned}$$

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

7. A hexagon and an equilateral triangle have equal perimeters. If the area of the hexagon is $6\sqrt{3}$ square units, what is the area of the triangle?

Solution by the editors.

The problem intended the hexagon to be regular. Let a be the length of a side of the hexagon. Then the equilateral triangle has side-length $2a$. When we subdivide the hexagon into 6 small equilateral triangles of side-length a we see that each of them must have area $\sqrt{3}$. Since the equilateral triangle of side-length $2a$ has twice the base and twice the altitude of the smaller triangle, it must have 4 times the area. Thus, the area of the original triangle is $4\sqrt{3}$.

Two incorrect solutions were received.

8. Determine all values of x for which

$$(1999x - 99)^3 = (1234x - 56)^3 + (765x - 43)^3\,.$$

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON, modified by the editor.

Factoring the sum of cubes on the right side of the equation, we find that one factor is $(1999x - 99)$; thus, one solution of the equation is $x = 99/1999$.

Now we rewrite the equation as

$$(1999x - 99)^3 - (1234x - 56)^3 = (765x - 43)^3\,.$$

Factoring the difference of cubes on the left side, we find that one factor is $(765x - 43)$; thus, one solution is $x = 43/765$.

Finally, we rewrite the equation as

$$(1999x - 99)^3 - (765x - 43)^3 = (1234x - 56)^3\,.$$

Factoring again, we see that $x = 56/1234$ is a solution.

Since the original equation is cubic in x, it has only 3 roots, and we have them all.

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

9. Find all solutions (x, y) in real numbers to

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{5}{6}\,, \\ x^2y + xy^2 &= 30\,. \end{aligned}$$

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Let $p = x + y$ and $q = xy$. The first equation gives us $6p = 5q$. The second equation gives us $pq = 30$. Solving, we first obtain $p = 5q/6$ and then $q^2 = 36$. Therefore, we have $q = \pm 6$. If $q = 6$, then $p = 5$; if $q = -6$, then $p = -5$. Solving for x and y, we get the solutions $(x, y) = (3, 2)$, $(2, 3)$, $(-6, 1)$, and $(1, -6)$.

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

10. Find the number of real solutions to the equation $\sin(x) = x/315$.

Solution by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

If $x = 0$, we have $\sin x = 0 = x/315$, which means that $(0, 0)$ is a solution.

The number of solutions is the number of points of intersection of the curve $y = \sin x$ with the line $y = x/315$. We know that $\sin x$ is periodic with period 2π and that $-1 \le \sin x \le 1$. Hence, if the line $y = x/315$ is to intersect $y = \sin x$, we need to have $-1 \le x/315 \le 1$; that is, $-315 \le x \le 315$. Since $315 \approx 100.27\pi$, we can write these inequalities as $-100.27\pi \le x \le 100.27\pi$.

There is no solution to the equation for $-100.27\pi \le x \le -100\pi$, since we then have $-1 \le x/315 \le -0.997$ and $-0.75 \le \sin x \le 0$. The same goes for $100\pi \le x \le 100.27\pi$. We can also see that there will be 2 solutions in each of the 100 intervals $-100\pi \le x \le -98\pi$, $-98\pi \le x \le -96\pi$, $\dots$, $-2\pi \le x \le 0$, $0 \le x \le 2\pi$, $\dots$, $98\pi \le x \le 100\pi$. But this counts the solution $x = 0$ twice; thus, we have $2 \times 100 - 1 = 199$ solutions.

Also solved by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

That brings us to the end of another Skoliad. Continue to send in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Amy Cameron (Carleton University), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier août 2005**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M163. Correction. *Proposé par Équipe de Mayhem.*

Montrer qu'il est possible de mettre des entiers non négatifs sur les faces de deux pipés (non nécessairement les mêmes pour les deux) de sorte que tous les résultats $1, 2, 3, \ldots, 12$ soient également probables quand on lance les pipés.

M182. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Si a, b et c sont des nombres positifs tels que $a+b+c=1$, montrer que

$$(1+a)(1+b)(1+c) \;\geq\; 8(1-a)(1-b)(1-c)\,.$$

M183. *Proposé par l'Équipe de Mayhem.*

Dans le tableau à droite, on appelle deux lettres *voisines* si elle sont adjacentes horizontalement, verticalement ou diagonalement. Partant d'une lettre "M" quelconque sur le pourtour du tableau, trouver le nombre de possibilités d'épeler le mot "MATH" en bougeant exclusivement entre lettres voisines.

M	M	M	M	M	M	M
M	A	A	A	A	A	M
M	A	T	T	T	A	M
M	A	T	H	T	A	M
M	A	T	T	T	A	M
M	A	A	A	A	A	M
M	M	M	M	M	M	M

M184. *Proposé par l'Équipe de Mayhem.*

Trouver toutes les solutions (a, b) de l'équation $ab - 24 = 2a$, où a et b sont des entiers positifs.

M185. *Proposé par Neven Jurič, Zagreb, Croatie.*

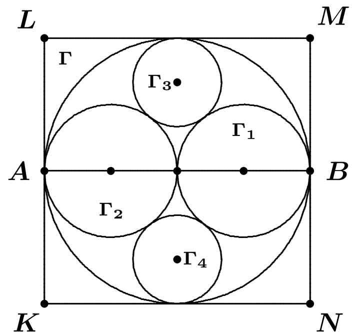
Un lac a la forme d'un triangle avec des côtés de longueur a, b et c. Vues d'un hélicoptère en position stationnaire au-dessus du lac, les lignes de visée des trois sommets sont perpendiculaires deux à deux. A quelle hauteur au-dessus du lac se trouve l'hélicoptère ?

M186. *Proposé par l'Équipe de Mayhem.*

On désigne par $\lfloor x \rfloor$ le plus grand entier plus petit ou égal à x. Par exemple, $\lfloor 2.5 \rfloor = 2$ et $\lfloor -7.4 \rfloor = -8$. Sachant que $\sum_{i=1}^{n} \left\lfloor \sqrt{i} \right\rfloor = 217$, trouver la valeur de n.

M187. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Un cercle Γ de rayon $2r$ est inscrit dans un carré $KLMN$. Le segment AB est un diamètre de ce cercle, A et B étant les points milieu de côtés opposés du carré. Deux cercles Γ_1 et Γ_2 de rayons r sont centrés sur AB et sont extérieurement tangents l'un à l'autre, chacun étant intérieurement tangent à Γ. Deux cercles Γ_3 et Γ_4 sont extérieurement tangents Γ_1 et Γ_2 et intérieurement tangents à Γ.



Avec une règle et un compas, construire la tangente commune à Γ_1 et Γ_3 en minimisant l'usage du compas.

Quel est le nombre minimal qui a été fait de l'usage du compas ?

. . . .

M163. Correction. *Proposed by the Mayhem Staff.*

Show that it is possible to put non-negative integer values on the faces of two dice (not necessarily the same on both dice) so that, when the dice are tossed, the outcomes 1, 2, 3, . . . , 12 are equally probable.

M182. *Proposed by Babis Stergiou, Chalkida, Greece.*

If a, b, c are positive numbers, such that $a + b + c = 1$, prove that

$$(1 + a)(1 + b)(1 + c) \;\geq\; 8(1 - a)(1 - b)(1 - c) .$$

M183. *Proposed by the Mayhem Staff.*

In the array at right, two letters are called *neighbouring* letters if they are adjacent to each other horizontally, vertically, or diagonally. Starting from any letter "M" on the outside of the array, find the number of ways of spelling "MATH" by moving only between neighbouring letters.

M	M	M	M	M	M	M
M	A	A	A	A	A	M
M	A	T	T	T	A	M
M	A	T	H	T	A	M
M	A	T	T	T	A	M
M	A	A	A	A	A	M
M	M	M	M	M	M	M

M184. *Proposed by the Mayhem Staff.*

Find all solutions (a, b) for the equation $ab - 24 = 2a$, where a and b are positive integers.

M185. *Proposed by Neven Jurič, Zagreb, Croatia.*

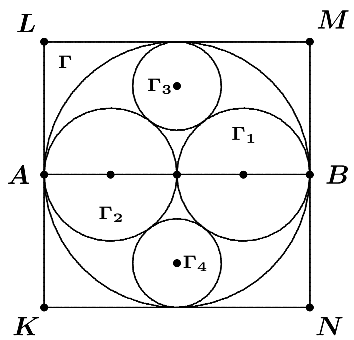
A lake has the shape of a triangle with sides of length a, b, and c. From a helicopter, which is hovering in a stationary position above the lake, the lines-of-sight to the three vertices of the triangle are pairwise perpendicular. How high is the helicopter above the lake?

M186. *Proposed by the Mayhem Staff.*

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.5 \rfloor = 2$ and $\lfloor -7.4 \rfloor = -8$. Given that $\sum_{i=1}^{n} \left\lfloor \sqrt{i} \right\rfloor = 217$, determine the value of n.

M187. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A circle Γ of radius $2r$ is inscribed in a square $KLMN$. Line segment AB is a diameter of this circle, where A and B are mid-points of opposite sides of the square. Two circles Γ_1 and Γ_2 of radii r have centres on AB and are externally tangent to one another, and each is internally tangent to Γ. Two circles Γ_3 and Γ_4 are externally tangent to Γ_1 and Γ_2 and internally tangent to Γ.



Construct the common tangent to Γ_1 and Γ_3 using straight edge and compass with a minimum use of the compass.

What is the minimum number of times that the compass has to be used?

Mayhem Solutions

M122. *Proposed by the Mayhem Staff.*

In a certain province, vehicle licence plates each have exactly three letters followed by three digits. We are told that to produce such a licence plate, it costs $\$n$ for each digit $n > 0$ and \$10 for each digit 0. For letters, the costs are proportional to the position of the letter in the alphabet, namely, \$1 for A, \$2 for B, and so on, up to \$26 for Z.

(a) Find the cost of producing an entire set of licence plates (that is, from AAA 000 to ZZZ 999).

(b) Determine how many plates would cost exactly \$100.

Solution by Geneviève Lalonde, Massey, ON.

(a) In a set of licence plates, each digit occurs the same number of times, as does each letter. Therefore, we can work with averages. The average cost of a digit is $\frac{1}{10}(\$1 + \$2 + \cdots + \$10) = \frac{1}{10}\left(\frac{10 \times \$11}{2}\right) = \$5.50$, and the average cost of a letter is $\frac{1}{26}(\$1+\$2+\cdots+\$26) = \frac{1}{26}\left(\frac{26 \times \$27}{2}\right) = \$13.50$. Thus, the average cost of a licence plate is $\$5.50 \times 3 + \$13.50 \times 3 = \$57$. Since the total number of licence plates is $26^3 \times 10^3 = 17\ 576\ 000$, the total cost of all of them is $17\ 576\ 000 \times \$57 = \$1\ 001\ 832\ 000$.

(b) Consider any licence plate that costs exactly \$100. Since a letter costs at most \$26, the maximum cost of 3 letters is \$78, implying that the numbers cost at least \$22. Similarly, the maximum cost of the numbers is \$30, which means that the letters cost at least \$70. For $k \in \{0, 1, \ldots, 8\}$, if the letters cost $\$(70 + k)$, then the numbers cost $\$(30 - k)$. These are the only possibilities. We now need to determine the number of ways to choose the 3 letters and 3 numbers for each $k \in \{0, 1, \ldots, 8\}$.

Since both sets of costs above are of the form $3n - 8$, $3n - 7$, $\ldots$, $3n$ ($n = 26$ for the letters and $n = 10$ for the numbers), let us next examine all the possible ways to sum three numbers to $3n - \ell$, when the numbers can all range from 1 to n. We claim that the number of ways is $\frac{1}{2}(\ell + 1)(\ell + 2)$. The following table demonstrates this for $\ell = 0, 1, 2, 3$. [*Ed:* We leave it to the reader to prove this in general.]

Sum	Possibilities	#	Sum	Possibilities	#
$3n$	n, n, n	1	$3n - 1$	$n, n, n - 1$	3
$3n - 2$	$n, n, n - 2$	3	$3n - 3$	$n, n, n - 3$	3
	$n, n - 1, n - 1$	3		$n, n - 1, n - 2$	6
				$n - 1, n - 1, n - 1$	1
	Total	6		Total	10

Now, we simply calculate the number of ways to achieve the allowable costs for letters and numbers.

Letters	# Ways	Numbers	# Ways	Total # Ways
$70	45	$30	1	45
$71	36	$29	3	108
$72	28	$28	6	168
$73	21	$27	10	210
$74	15	$26	15	225
$75	10	$25	21	210
$76	6	$24	28	168
$77	3	$23	36	108
$78	1	$22	45	45
				1287

Thus, there are 1287 licence plates that cost exactly $100.

M123. *Proposed by the Mayhem Staff.*

In what base is 221 a factor of 1215?

Solution by Mikel Díez, 3° ESO student, IES Sagasta, Logroño, Spain.

The number 1215 in base a has the value $a^3 + 2a^2 + a + 5$ and 221 in base a has the value $2a^2 + 2a + 1$. Now,

$$a^3 + 2a^2 + a + 5 \;=\; (2a^2 + 2a + 1)\left(\tfrac{1}{2}a + \tfrac{1}{2}\right) + \left(-\tfrac{1}{2}a + \tfrac{9}{2}\right).$$

Since $1215_{(a)}$ must be a multiple of $221_{(a)}$, the remainder, $-\frac{1}{2}a + \frac{9}{2}$, must be 0 and $\frac{1}{2}a + \frac{1}{2}$ must be an integer. Therefore, $a = 9$.

Also solved by Robert Bilinski, Outremont, QC.

M124. *Proposé par l'Équipe de Mayhem.*

Sans l'aide d'une table, trouver si 2003 est un nombre premier.

Solution par Robert Bilinski, Outremont, QC.

Il suffit d'essayer la division par tous les entiers premiers de 2 jusqu'à $\sqrt{2003} \approx 44.75$, soit 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43. Si on ne s'en souvient pas (la mémoire est-elle une liste ?), on peut se faire un crible d'Ératosthène rapidement. Puisqu'il n'y a aucun nombre dans cette liste qui donne un quotient entier, on conclut que 2003 est premier.

M125. *Proposed by the Mayhem Staff.*

List all of the positive integers less than 122003 that are both perfect squares and perfect cubes.

Solution by Mikel Díez, 3° ESO student, IES Sagasta, Logroño, Spain.

In order to be both perfect squares and perfect cubes they must be perfect sixth powers (and, conversely, every perfect sixth power is both a perfect square and a perfect cube). The only solutions are:

$$1^6 = 1\,,\quad 2^6 = 64\,,\quad 3^6 = 729\,,\quad 4^6 = 4096\,,$$
$$5^6 = 15625\,,\quad 6^6 = 46656\,,\quad 7^6 = 117649\,.$$

Also solved by Robert Bilinski, Outremont, QC.

M126. *Proposed by the Mayhem Staff.*

Given that the letters A, B, C, D, E, F represent distinct decimal digits, find the values of these letters so that

$$ABC \times DEF = 232323\,.$$

Here, ABC and DEF represent 3-digit numbers.

Combination of solutions by Robert Bilinski, Outremont, QC; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

The numbers are 851 and 273. They are obtained by factoring 232323:

$$232323 = 3 \cdot 7 \cdot 13 \cdot 23 \cdot 37\,,$$

and then examining all the possible groupings of the factors that yield two 3-digit numbers.

Also solved by Doug Newman, Lancaster, CA, USA.

M127. *Proposed by Ovidiu Gabriel Dinu, Balcesti, Valcea, Romania.*

Prove that if $a, b \in \mathbb{R}$ and $a - b = 1$, then $a^3 - b^3 \geq \frac{1}{4}$.

Solution by the Austrian IMO Team.

We know that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$. Using $a - b = 1$, the given inequality is equivalent to

$$a^2 + ab + b^2 \geq \tfrac{1}{4}\,.$$

Furthermore, we can write a as $a = 1 + b$, yielding

$$(1+b)^2 + (1+b)b + b^2 \geq \tfrac{1}{4}\,.$$

This inequality is equivalent to each of the following:

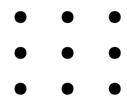
$$\begin{aligned} (1 + 2b + b^2) + (b + b^2) + b^2 &\geq \tfrac{1}{4}\,, \\ 3b^2 + 3b + \tfrac{3}{4} &\geq 0\,, \\ 3\left(b + \tfrac{1}{2}\right)^2 &\geq 0\,. \end{aligned}$$

Because a square number is always positive, the last line is true. Thus, the given inequality is true.

Also solved by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Robert Bilinski, Outremont, QC; and Doug Newman, Lancaster, CA, USA.

M128. *Proposed by Lobzang Dorji, Paro, Bhutan.*

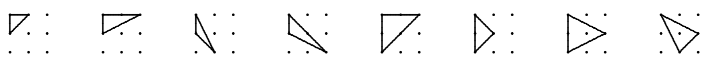
(a) Nine dots are uniformly spaced in a 3×3 square array as shown. Verify that 8 non-congruent triangles can be formed using three of the dots as vertices.



(b) Suppose that a 4×4 square array of dots is employed. How many non-congruent triangles could be formed using three of the dots as vertices?

Solution to part (a) *by Doug Newman, Lancaster, CA, USA.*

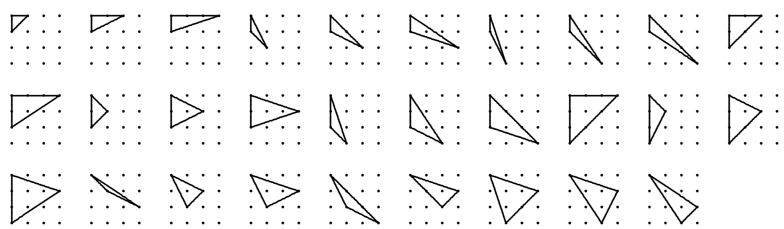
The 8 non-congruent triangles that can be formed are as follows:



Also solved by Robert Bilinski, Outremont, QC; and David Wagner, University of New Brunswick, Fredericton, NB.

Solution to part (b) *by Robert Bilinski, Outremont, QC; Doug Newman, Lancaster, CA, USA; and David Wagner, University of New Brunswick, Fredericton, NB.*

We list below the 29 non-congruent triangles.



[*Ed:* Note that by using reflections, rotations and translations, we may always position one vertex in the upper left corner of the grid. Also, we can always find a second vertex on or below the main diagonal (from upper left to lower right). The first nine triangles above have the second vertex 1 unit below the first; the next eight have the second vertex 2 units below the first; and the following four have the second vertex 3 units below the first. These are the only triangles with either a horizontal or vertical side. The remaining eight triangles systematically move the second vertex across and down the grid, keeping on or below the main diagonal.]

Wagner used another systematic approach to determine that the number of non-congruent triangles in a 5×5 grid is 79. Comments are welcomed concerning either a general term or a verification/proof that a solution is complete for an $n \times n$ grid.

M129. *Proposed by the Mayhem Staff.*

A die is tossed. If the die lands on '1' or '2', then one coin is tossed. If the die lands on '3', then two coins are tossed. Otherwise, three coins are tossed. Given that the resulting coin tosses produced no 'heads', what is the probability that the die landed on '1' or '2'.

Solution by Robert Bilinski, Outremont, QC.

Let A be the event that the number 1 or 2 appears on the die, B the event that the number 3 appears, and C the event that a number more than 3 appears. Thus, $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{6}$, and $P(C) = \frac{1}{2}$. Let NH be the event that no heads appear when the coins are tossed. Then we have

$$P(NH \mid A) = \frac{1}{2}, \quad P(NH \mid B) = \frac{1}{4}, \quad \text{and} \quad P(NH \mid C) = \frac{1}{8}.$$

Hence,

$$\begin{aligned} &P(NH) \\ &\quad = P(NH \mid A) \cdot P(A) + P(NH \mid B) \cdot P(B) + P(NH \mid C) \cdot P(C) \\ &\quad = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{8} \cdot \frac{1}{2} = \frac{13}{48}. \end{aligned}$$

But we want $P(A \mid NH)$.

$$P(A \mid NH) = \frac{P(NH \mid A) \cdot P(A)}{P(NH)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{13}{48}} = \frac{8}{13}.$$

One incorrect solution was also received.

Problem of the Month

Ian VanderBurgh, University of Waterloo

Problem (1983 Australian Mathematics Competition, Junior Division) Some unit cubes are assembled to form a larger cube and then some of the faces of this larger cube are painted. After the paint dries, the larger cube is dis-assembled into the unit cubes, and it is found that 45 of these have no paint on any of their faces. How many faces of the larger cube were painted?

The first thing that popped into my mind when I saw this problem was "Gee, it's too bad that they didn't tell us the size of the large cube!" Can we figure this out, though, from the information that we have been given?

Since we know that there are 45 unit cubes which are unpainted, the large cube must be made up of at least 45 unit cubes. What is the smallest size of large cube that is made up of at least 45 unit cubes? A $3 \times 3 \times 3$ cube would have $3^3 = 27$ unit cubes, and a $4 \times 4 \times 4$ cube would have $4^3 = 64$ cubes. Thus, the large cube must be at least $4 \times 4 \times 4$.

A good next question to ask is which unit cubes are sure to have no paint on them. We know that if we take away the outer layer of unit cubes, then none of the remaining unit cubes will have any paint on them. What do we know about these remaining unit cubes? They form a cube which has dimensions 2 less than the outer cube. We now have enough of a start to write a formal solution.

Solution. Since there are at least 45 unit cubes in the large cube, the large cube is at least $4 \times 4 \times 4$.

When the outer layer of unit cubes is removed, we are left with a new cube (composed of unit cubes) which has dimensions 2 less than that of the full large cube. Since this inner cube cannot contain more than 45 unit cubes (since there are only 45 unpainted cubes in total), it has to be smaller than $4 \times 4 \times 4$. Hence, the large cube must be smaller than $6 \times 6 \times 6$. Thus, the larger cube must be either $4 \times 4 \times 4$ or $5 \times 5 \times 5$.

Could it be $4 \times 4 \times 4$? It would then be made up of 64 unit cubes in total. Since there are 45 unpainted unit cubes, there must be 19 cubes with paint on them. But painting one face puts paint on 16 unit cubes, leaving us with only 3 more painted unit cubes. There is no way to paint only 3 more unit cubes by painting another full face. Hence, the cube cannot be $4 \times 4 \times 4$.

Thus, the larger cube must be $5 \times 5 \times 5$. It has 45 unpainted cubes and 80 painted cubes (since there are $5^3 = 125$ unit cubes). The cube we get after removing the outer layer is $3 \times 3 \times 3$, contributing 27 unpainted cubes. Thus, the outer layer of cubes has 80 painted cubes and 18 unpainted cubes.

If a face of the cube is left unpainted, how many unit cubes remain without paint as a result? Each face is a 5×5 square, and each face is next to 4 other faces. The cubes on the edge of the face may or may not have paint on them from other faces, depending on which of the adjacent faces are painted. An unpainted face could result in:

- 25 unpainted cubes, if no adjacent faces are painted;
- 20 unpainted cubes, if one adjacent face is painted;
- 15 or 16 unpainted cubes, if two adjacent faces are painted (why do we get two answers here?);
- 12 unpainted cubes, if three adjacent faces are painted;
- 9 unpainted cubes, if four adjacent faces painted.

Thus, if we leave two opposite faces unpainted and paint the other four faces, then each of the two unpainted faces has all four adjacent faces painted, giving a total of 18 unpainted cubes.

Therefore, 4 faces of the large $5 \times 5 \times 5$ are painted, leaving a total of 45 cubes with no paint on them.

We have managed to solve the problem, even without being told the size of the original cube. A little bit of logical thinking overcame that difficulty. This problem also has the nice feature that we could build a model and actually paint some faces. Of course, we could then watch the paint dry.

Pólya's Paragon

Fun With Numbers (Part 2)

Shawn Godin

When last we met, we looked at the multiplicative nature of numbers by exploring prime factorization and the function $d(n)$, the number of divisors of a number n. For homework, you were to create a table and look for some patterns. Here are some results you might have found (if you went on long enough):

$$\begin{aligned} 8 &= 2^3 \quad \text{and} \quad d(8) = 4\,, \\ 25 &= 5^2 \quad \text{and} \quad d(25) = 3\,, \\ 81 &= 3^4 \quad \text{and} \quad d(81) = 5\,. \end{aligned}$$

What is happening here? Last time we saw that $d(p) = 2$ when p is a prime number. Do you see a more general rule now? Try to come up with a conjecture before you read on.

Here are some more results:

$$\begin{aligned} 6 &= 2 \times 3 \quad \text{and} \quad d(6) = 4 = 2 \times 2\,, \\ 18 &= 2 \times 3^2 \quad \text{and} \quad d(18) = 6 = 2 \times 3\,, \\ 40 &= 2^3 \times 5 \quad \text{and} \quad d(40) = 8 = 4 \times 2\,, \\ 360 &= 2^3 \times 3^2 \times 5 \quad \text{and} \quad d(360) = 24 = 4 \times 3 \times 2\,. \end{aligned}$$

Can you see what is happening here? Can you combine it with your earlier conjecture to come up with one that takes care of all cases?

Here is one more set of results:

$$\begin{array}{lllll} 40 = 8 \times 5\,, & d(8) = 4, & d(5) = 2, & \text{and} & d(40) = 8\,; \\ 12 = 4 \times 3\,, & d(4) = 3, & d(3) = 2, & \text{and} & d(12) = 6\,; \\ 450 = 18 \times 25, & d(18) = 6, & d(25) = 3, & \text{and} & d(450) = 18\,; \\ 120 = 12 \times 10, & d(12) = 6, & d(10) = 4, & \text{and} & d(120) = 12\,; \\ 500 = 25 \times 20, & d(25) = 3, & d(20) = 6, & \text{and} & d(500) = 12\,. \end{array}$$

What pattern do you find here? What is different in the last two lines above? Try to come up with a conjecture before you read on.

Let's try to put all this together.

Theorem. For any number n, if its prime factorization is given by

$$n \;=\; p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_n^{e_n}\,,$$

then

$$d(n) \;=\; (e_1 + 1) \times (e_2 + 1) \times (e_3 + 1) \times \cdots \times (e_n + 1)\,.$$

We will not prove the theorem, but rather try to understand it using an example. Consider the case $n = 2^4 \times 3^2 \times 5^5 = 450000$. According to the theorem, $d(450000) = (4+1) \times (2+1) \times (5+1) = 5 \times 3 \times 6 = 90$. Thus, 450000 has 90 divisors (including the trivial divisors 1 and 450000).

On the other hand, any number that is a divisor of 450000 has to be of the form $2^a \times 3^b \times 5^c$, where $0 \le a \le 4$, $0 \le b \le 2$, and $0 \le c \le 5$. (Why?) How many numbers of this form can we generate? We can "create" a divisor of 450000 by picking exponents for the primes 2, 3, and 5. For the prime 2, we may choose the exponent to be 0, 1, 2, 3, or 4; thus, the number of possible choices is 5 $(= 4 + 1)$. Similarly, there are $3 = 2 + 1$ choices for the exponent of 3, and there are $6 = 5 + 1$ choices for the exponent of 5. Now, for each power of 2, we have 3 choices for the power of 3, and for each of these choices there are 6 choices for the power of 5. Putting all of these together, we get $5 \times 3 \times 6 = 90$ ways to create our divisor of 450000.

It turns out that the function $d(n)$ has a special property: if two numbers a and b are *relatively prime*, meaning that they have no common divisor except 1, then $d(a \times b) = d(a) \times d(b)$. Mathematicians refer to this property of $d(n)$ by saying that $d(n)$ is a *multiplicative function*. Notice that the equation $d(a \times b) = d(a) \times d(b)$ is not true when a and b have a common divisor greater than 1. For completeness, we add that a function f satisfying $f(a \times b) = f(a) \times f(b)$ for all a and b is called *totally multiplicative*. For example, the function $f(x) = x^2$ is totally multiplicative.

Now that we have seen the importance of expressing a number by its prime factorization, the next thing to do is to figure out how to break a number down into primes. The long way is to check by division if certain primes divide a number, but there are some easier ways. Next issue we will look at a powerful technique that will allow us to consider some divisibility rules, among other things.

For homework, consider the table created below

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49
50	51	52	53	54	55	56
57	58	59	60	61	62	63
64	65	66	67	68	69	70
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Examine the table, looking for interesting patterns. Look at what happens when you perform operations on numbers in the table. We will see how this simple table can be used next time.

Iterating Möbius Functions with Rational Coefficients, Part I

Kun-Chieh Wang

We will determine all the possible periods of a periodic sequence of functions obtained by iterating a Möbius function with rational coefficients.

1. Introduction

A *Möbius function* is a function of the form

$$f(z) \;=\; \frac{az+b}{cz+d}\,, \tag{1}$$

where z is a complex variable and the coefficients a, b, c, and d are complex numbers such that $ad \neq bc$. The coefficients form a 2×2 matrix

$$A \;=\; \left[\begin{array}{cc} a & b \\ c & d \end{array}\right].$$

This matrix is not uniquely determined by f; it can be multiplied by any non-zero complex number. The *determinant* of the matrix A is defined to be $\det(A) = ad - bc$. Our assumption that $ad \neq bc$ is equivalent to the condition $\det(A) \neq 0$. In other words, the matrix A is supposed to be invertible.

Note that the identity function $f(z) = z$ is a Möbius function whose coefficient matrix is the 2×2 *identity matrix* $I = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$.

Let f and g be Möbius functions with respective coefficient matrices

$$A \;=\; \left[\begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array}\right] \quad\text{and}\quad B \;=\; \left[\begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array}\right].$$

The composite function $f \circ g$ is defined by $f \circ g(z) = f\big(g(z)\big)$. Thus,

$$f \circ g(z) \;=\; \frac{a_1\left(\dfrac{a_2z+b_2}{c_2z+d_2}\right)+b_1}{c_1\left(\dfrac{a_2z+b_2}{c_2z+d_2}\right)+d_1} \;=\; \frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)}\,.$$

We see that $f \circ g$ is another Möbius function, and that its coefficient matrix is the matrix product AB. (As a composite function, $f \circ g$ may be undefined at certain points, but we will always assume that continuous extensions have been made wherever possible to maximize the domain. This will allow us to say, for example, that if $f(z) = 1/z$, then $f \circ f$ is the identity function.)

Given a Möbius function f, we define a sequence $\{f_k\}_{k=0}^{\infty}$ of Möbius functions, where f_0 is the identity function, $f_1 = f$, and $f_k = f \circ f_{k-1}$ for $k \geq 2$. It is easy to prove inductively that the coefficient matrix of f_k is A^k, where A is the coefficient matrix of f.

The sequence $\{f_k\}$ is said to be *periodic* if there is a positive integer n such that $f_n = f_0$. The smallest such integer n is then called the *period* of the sequence. Thus, if n is the period, there is no positive integer $m < n$ for which $f_m = f_0$. Actually, we need only rule out the positive divisors m of n. If ℓ is another positive integer for which $f_\ell = f_0$, then the greatest common divisor of n and ℓ also has this property.

From now on, we assume that the coefficients of f are rational. Then each function f_k has rational coefficients. The sequence $\{f_k\}$ is periodic if and only if $A^n = \alpha I$ for some positive integer n and some non-zero rational number α. Moreover, if $A^m \neq \beta I$ for any positive divisor m of n and any non-zero rational number β, then the sequence is of period n. We will determine all the possible periods.

2. Case Studies

The identity function $f(z) = z$ is periodic of period 1, and it is the only function with this property.

To find examples with a period of 2, we calculate

$$A^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix}.$$

We see that A^2 is a rational multiple of I if and only if $a + d = 0$. Two obvious examples are $f(z) = -z$ and $f(z) = 1/z$.

To simplify subsequent computations, we note that, if $c = 0$, then $d \neq 0$, and f is a linear function. Clearly, a linear function cannot generate a periodic sequence of functions of period greater than 2. Henceforth, we assume that $c \neq 0$. Without loss of generality, we may assume that $c = 1$.

Let $g(z) = z - d$. The inverse function for g is $g^{-1}(z) = z + d$. Define $h(z) = g^{-1}(f(g(z)))$. Let B and C be the coefficient matricies for g and h respectively. Then $B^{-1}AB = C$. Explicitly,

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 1 & d \end{bmatrix}\begin{bmatrix} 1 & -d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a+d & b-ad \\ 1 & 0 \end{bmatrix}.$$

We draw the attention of the reader to the form of the bottom row of C.

Since $C^k = (B^{-1}AB)^k = B^{-1}A^kB$, we see that C^k is a rational multiple of I if and only if A^k is. Therefore, if $\{f_k\}$ is a periodic sequence, then $\{h_k\}$ is a periodic sequence with the same period. Thus, we can replace f by h. It follows that we may assume not only $c = 1$, but also $d = 0$. This may cause some examples of periodic sequences of a certain period to be lost, but we cannot lose them all. Since all we need is one example for each possible period, our assumption will cause no problem. Henceforth, we take $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$.

For $n = 3$, we have

$$A^3 = \begin{bmatrix} a^3 + 2ab & b(a^2+b) \\ a^2 + b & ab \end{bmatrix}.$$

If this is a rational multiple of I, we must have $a^2+b = 0$. Then $A^3 = -a^3 I$. Taking $a = 1$, we obtain the following example of a function for which $\{f_k\}$ has period 3:

$$f(z) = \frac{z-1}{z}.$$

For $n = 4$, we have

$$A^4 = \begin{bmatrix} a^4 + 3a^2b + b^2 & ab(a^2+2b) \\ a(a^2 + 2b) & b(a^2+b) \end{bmatrix}.$$

If this is a rational multiple of I, we must have $a(a^2 + 2b) = 0$. However, $a = 0$ leads to a sequence of period 2. Hence, we must have $a^2 + 2b = 0$. Then $A^4 = -\frac{a^4}{4}I$. Taking $a = 1$, we obtain an example of a function for which $\{f_k\}$ has period 4:

$$f(z) = \frac{2z-1}{2z}.$$

It looks as if all values might be possible, but now we come up against the first negative case.

For $n = 5$, we have

$$A^5 = \begin{bmatrix} a^5 + 4a^3b + 3ab^2 & b(a^4 + 3a^2b + b^2) \\ a^4 + 3a^2b + b^2 & ab(a^2+2b) \end{bmatrix}.$$

If this is a rational multiple of I, we must have $a^4 + 3a^2b + b^2 = 0$. This is a quadratic equation in a^2/b. It follows easily from the Quadratic Formula that there are no rational solutions.

For $n = 6$, we have

$$A^6 = \begin{bmatrix} a^6 + 4a^4b + 4a^2b^2 + b^3 & ab(a^2+b)(a^2+3b) \\ a(a^2+b)(a^2+3b) & a^4b + 3a^2b^2 + b^3 \end{bmatrix}.$$

If this is a rational multiple of I, we must have $a(a^2 + b)(a^2 + 3b) = 0$. However, $a = 0$ leads to a sequence of period 2 while $a^2 + b = 0$ leads to a sequence of period 3. Hence, $a^2 + 3b = 0$. Then $A^4 = -\frac{a^4}{4}I$. Taking $a = 1$, we obtain a function for which $\{f_k\}$ has period 6:

$$f(z) = \frac{3z-1}{3z}.$$

The next case is $n = 7$, but we suspect that, like the case $n = 5$, it would not work. Thus, we will skip over it.

For $n = 8$, we have

$$A^8 = \begin{bmatrix} a^8 + 7a^6b + 15a^4b^2 + 10a^2b^3 + b^4 & ab(a^2+2b)(a^4+4a^2b+2b^2) \\ a(a^2+2b)(a^4+4a^2b+2b^2) & a^6b + 5a^4b^2 + 6a^2b^3 + b^4 \end{bmatrix}.$$

If this is a rational multiple of I, we must have

$$a(a^2+2b)(a^4+4a^2b+2b^2) = 0\,.$$

However, $a = 0$ leads to a sequence of period 2 while $a^2 + 2b = 0$ leads to a sequence of period 4. Hence, $a^4 + 4a^2b + 2b^2 = 0$. This is a quadratic equation in a^2/b. It follows easily from the Quadratic Formula that there are no rational solutions. Thus, the case $n = 8$ does not work either.

We now skip ahead to a case that is more likely to work than those passed over. For $n = 12$, we have

$$A^{12} = \begin{bmatrix} \begin{matrix} a^{12} + 11a^{10}b + 45a^8b^2 + 84a^6b^3 \\ +70a^4b^4 + 19a^2b^5 + b^6 \end{matrix} & \begin{matrix} ab(a^2+b)(a^2+2b)(a^2+3b) \\ (a^4+4a^2b+b^2) \end{matrix} \\ \\ \begin{matrix} a(a^2+b)(a^2+2b)(a^2+3b) \\ (a^4+4a^2b+b^2) \end{matrix} & \begin{matrix} a^{10}b + 9a^8b^2 + 28a^6b^3 \\ +35a^4b^4 + 15a^2b^5 + b^6 \end{matrix} \end{bmatrix}.$$

If this is a rational multiple of I, we must have

$$a(a^2+b)(a^2+2b)(a^2+3b)(a^4+4a^2b+b^2) = 0\,.$$

However, $a = 0$ leads to a sequence of period 2, $a^2 + b = 0$ leads to a sequence of period 3, $a^2 + 2b = 0$ leads to a sequence of period 4, and $a^2 + 3b = 0$ leads to a sequence of period 6. Hence, $a^4 + 4a^2b + b^2 = 0$. This is a quadratic equation in a^2/b. It follows easily from the Quadratic Formula that there are no rational solutions.

At this point, we make a bold conjecture that the only possible values for the period of a periodic sequence $\{f_k\}$ are 1, 2, 3, 4, and 6. In Part II, we will prove this conjecture.

Acknowledgement:

The author thanks Prof. J.N. Hsu of Taiwan for suggesting this topic for investigation.

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THE OLYMPIAD CORNER

No. 244

R.E. Woodrow

We start by giving the Compositions de Mathématiques of the Concours Général des Lyceés, 2001, Classe terminale S. Again my thanks go to Chris Small, Canadian Team Leader to the 42^{nd} IMO, for obtaining them for us.

CONCOURS GÉNÉRAL DES LYCÉES Session de 2001
COMPOSITION DE MATHÉMATIQUES
Class terminale S – Durée : 5 heures

Les premières questions de chacune des quatre parties de ce problème sont indépendantes des autres parties. Il n'est donc pas obligatoire de commencer son étude dans l'ordre indiqué. Les candidats peuvent admettre les résultats d'une question, à condition de l'indiquer clairement sur la copie.

On appelle *trio* tout triplet de nombres réels (a,b,c) non tous nuls et vérifiant la relation :

$$ab+bc+ca \;=\; 0\,.$$

Lorsque $a+b+c=1$, on dit que le trio (a,b,c) est un trio *réduit*.

Les coordonnées sont rapportées à un repère orthonormal direct $(0,\overrightarrow{I},\overrightarrow{J},\overrightarrow{K})$ de l'espace.

Première partie

On note C l'ensemble des points de coordonnées (a,b,c) où (a,b,c) est un trio. On note Γ l'ensemble des points de coordonnées où (a,b,c) est un trio réduit. On note $\mathcal{P}$ le plan d'équation $x+y+z=1$.

1. Existe-t-il des trios (a,b,c) tels que $a+b+c=0$?

2. Montrer que C est une réunion de droites passant par O et privées de ce point.

3. Montrer que Γ est l'intersection d'un plan et d'une sphère de centre O. Quelle est la nature géométrique de Γ?

4. Donner la nature géométrique de C et l'illustrer par un croquis.

5. Soit L un point fixé de Γ. Montrer que le volume V du tétrèdre $OLL'L''$, où L' et L'' sont deux points distincts de Γ et différents de L, est maximal lorsque les arêtes issues de O sont deux à deux orthogonales et déterminer alors les coordonnées de L' et L'' en fonction de celles de L.

6. Montrer que le produit abc admet un maximum et un minimum lorsque le point de coordonnées (a, b, c) décrit Γ. Préciser les trios réduits réalisant ces extrémums.

Deuxième partie

Dans cette partie et les suivantes, un trio (a, b, c) est dit *rationnel* lorsque a, b et c sont des ombres rationnels (éléments de l'ensemble $\mathbb{Q}$) ; il est dit *entier* lorsque a, b et c sont des nombres entiers relatifs (éléments de l'ensemble $\mathbb{Z}$) ; enfin un trio entier est dit *primitif* si a, b et c n'admettent que 1 et -1 comme diviseurs communs.

1. Déterminer la nature de l'ensemble H_1 des points de coordonnées $(x, y, 1)$ tels que $(x, y, 1)$ soit un trio. Montrer que le point Ω_1 de coordonnées $(-1, -1, 1)$ est un centre de symétrie de H_1. Quels sont les points de H_1 à coordonnées entières ?

2. Pour tout entier naturel non nul h, on note Z_h l'ensemble des trois entiers (a, b, c) tels que $c = h$. Déterminer Z_h pour $h = 1$ et $h = 2$.

3. Montrer que Z_h est un ensemble fini et exprimer le nombre $H(h)$ de ses éléments en fonction de celui des diviseurs de h^2 dans $\mathbb{Z}$. Montrer que 4 divise $N(h) - 2$.

4. Pour tout entier naturel non nul h, on note $N'(h)$ le nombre de trios entiers (a, b, c) tels que l'un au moins des entiers a, b ou c soit égal à h. Exprimer $N'(h)$ en fonction de $N(h)$ selon la parité de h.

5. Montrer qu'à tout trio entier (a, b, c) on peut associer un triplet (r, s, t) d'entiers tels que r et s soient premiers entre eux, s positif ou nul, et tels que l'on ait :

$$a = r(r+s)t\,, \quad b = s(r+s)t\,, \quad c = -rst\,.$$

Énoncer et démontrer une réciproque. Pour quels trios (a, b, c) le triplet (r, s, t) n'est-il pas unique ?

6. Déterminer les triplets (r, s, t) ainsi associés aux trios primitifs. En déduire que si (a, b, c) est un trio primitif, alors $|abc|$, $|a + b|$, $|c|$ et $|c + a|$ sont des carrés d'entiers.

7. Pour tout entier naturel non nul h, on note $P(h)$ le nombre de trios primitifs (a, b, c) tels que $c = h$. Montrer que $P(h)$ est une puissance de 2. Pour quels entiers h a-t-on $P(h) = N(h)$? Expliciter une suite d'entiers (h_n) telle que la suite $\big(P(h_n)/N(h_n)\big)$ converge vers zéro.

8. Soit $(a, b, 1)$ un trio. Montrer qu'il existe deux suites (x_n) et (y_n) convergeant respectivement vers a et b et telles que, pour tout n, $(x_n, y_n, 1)$ soit un trio rationnel.

9. Soit (a, b, c) un trio réduit. Montrer qu'il existe trois suites (x_n), (y_n) et (z_n) convergeant respectivement vers a, b et c et telles que, pour tout n, (x_n, y_n, z_n) soit un trio rationnel réduit.

Troisième partie

On note j le nombre complexe $e^{2i\pi/3}$, c'est-à-dire $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Pour tout trio $T = (a, b, c)$ on note $\widehat{T} = (a, c, b)$, $S(T) = a + b + c$ et $z(T) = a + bj + cj^2$.

1. Calculer le module de $z(T)$ en fonction de $S(T)$. Peut-on avoir $z(T) = 0$? Calculer le cosinus et le sinus d'un argument θ de $z(T)$ en fonction de a, b et c.

2. Soit z_0 un nombre complexe non nul. Déterminer les trios $T = (a, b, c)$ tels que $z(T) = z_0$.

3. Étant donnés deux trios T_1 et T_2, montrer qu'il existe un unique trio, noté $T_1 * T_2$, vérifiant $S(T_1 * T_2) = S(T_1)S(T_2)$ et $z(T_1 * T_2) = z(T_1)z(T_2)$. Calculer $T_1 * T_2$ en fonction de T_1 et T_2. Que peut-on dire d'un argument de $z(T_1 * T_2)$? Que peut-on dire d'un argument de $z(T_1 * \widehat{T_1})$?

4. Si T_1 et T_2 sont réduits, en est-il de même de $T_1 * T_2$? Si T_1 et T_2 sont entiers, en est-il de même de $T_1 * T_2$? Si T_1 et T_2 sont primitifs, en est-il de même de $T_1 * T_2$?

5. Comparer les trios $T_1 * T_2$ et $T_2 * T_1$, $(T_1 * T_2) * T_3$ et $T_1 * (T_2 * T_3)$, T_1 et $T_1 * (1, 0, 0)$.

6. Étant donnés les trios T_1 et T_2, résoudre l'équation $T_1 * T = T_2$ où le trio T est l'inconnue.

7. Étant-donné un trio T, on définit une suite de trios (T_n) par $T_0 = (1, 0, 0)$ et $T_{n+1} = T * T_n$. Calculer $S(T_n)$. Étant donné un entier p, résoudre l'équation $T_1 * T = T_2$ où le trio T est l'inconnue.

Quatrième partie

On note A l'ensemble des entiers m non nuls tels qu'il existe deux entiers u, v tels que $m = u^2 + 3v^2$. On note A' l'ensemble des nombres complexes z non nuls tels qu'il existe deux entiers u, v tels que $z = u + iv\sqrt{3}$ (on remarquera que $|z|^2 = u^2 + 3v^2$). On note B l'ensemble des entiers n non nuls tels qu'il existe deux entiers r, s tels que $n = r^2 + rs + s^2$.

1. Montrer que le produit de deux éléments de A' appartient à A', puis que le produit de deux éléments de A appartient à A.

2. Montrer que, si p est un nombre premier élément de A, alors $p = 3$ ou 3 divise $p - 1$.

3. Montrer que $A = B$ (on pourra notamment remarquer que

$$r^2 + rs + s^2 = (r+s)^2 - (r+s)s + s^2)\,.$$

4. Montrer que 4 divise les éléments pairs de A et que les quotients appartiennent à A, puis que tout élément de A est produit d'un élément impair de A par une puissance de 4.

5. (a) Soit s'il en existe, un entier impair $m = u^2 + 3v^2$ tel que les entiers u et v soient premiers entre eux et qui admet un diviseur premier p n'appartenant pas à A. Montrer qu'il existe alors un plus petit entier strictement positif n_0 tel que $n_0 p$ appartienne à A. Montrer que n_0 est impair.

(b) Établir l'existence de deux entiers u' et v' inférieurs en valeur absolue à $p/2$ tels que p divise $u' - u$ et $v' - v$. Montrer que p divise l'entier non nul $u'^2 + 3v'^2$ et que $n_0 < p$.

(c) Établir l'existence de deux entiers non nuls premiers entre eux u_0 et v_0 tels que $n_0 p = u_0^2 + 3v_0^2$.

(d) Établir l'existence de deux entiers u_1 et v_1 inférieurs en valeur absolue à $n_0/2$ tels que n_0 divise $u_1 - u_0$ et $v_1 - v_0$. Montrer que n_0 divise l'entier non nul $u_1^2 + 3v_1^2$ que l'on notera $n_0 n_1$.

(e) En déduire qu'un tel entier m ne peut pas exister (on pourra considérer l'entier $n_0^2 n_1 p$).

6. Montrer que tout élément de A s'écrit $m = C^2 \cdot p_1 \cdots p_k$ où C est un entier naturel non nul et les p_i des nombres premiers distincts éléments de A.

7. (a) Soient p un nombre premier tel que 3 divise $p - 1$, et K l'ensemble des triplets (x, y, z) où les entiers x, y et z sont strictement compris entre 0 et p, et tels que p divise $(xyz - 1)$. Montrer que K possède $(p-1)^2$ éléments, et que 3 divise le nombre d'éléments de K ne vérifiant pas $x = y = z$.

(b) En déduire qu'il existe un entier x strictement compris entre 1 et p tel que p divise $x^2 + x + 1$, puis que p appartient à A. Décrire les éléments de A.

8. Soit D l'ensemble des entiers d tels qu'il existe un trio entier (a, b, c) vérifiant $a + b + c = d$ et $abc \neq 0$. Montrer, grâce à la question 5 de la deuxième partie, que tout élément de D possède un diviseur premier élément de A. Réciproquement, que peut-on dire d'un entier non nul admettant un diviseur premier élément de A ?

9. En déduire les éléments de D compris au sens large entre 2001 et 2010.

Next we turn to solutions from our readers to problems of the 2000 Chinese Mathematical Olympiad given [2002 : 482–483].

1. In triangle ABC, $a \le b \le c$, where $a = BC$, $b = CA$ and $c = AB$, the circumradius is R and the inradius is r. What can be said about $\angle C$ if $a + b - 2R - 2r$ is

(a) positive? (b) zero? (c) negative?

Solved by Pierre Bornsztein, Maisons-Laffitte, France. We give the comment by Mohammed Aassila, Strasbourg, France.

This is problem 2690 [2001 : 534]. A solution appeared in [2002 : 549].

2. The sequence $\{a_n\}$ is defined by $a_1 = 0$, $a_2 = 1$ and

$$a_n = \frac{n}{2}a_{n-1} + \frac{n(n-1)}{2}a_{n-2} + (-1)^n\left(1 - \frac{n}{2}\right)$$

for $n \ge 3$. Simplify

$$a_n + 2\binom{n}{1}a_{n-1} + 3\binom{n}{2}a_{n-2} + \cdots + (n-1)\binom{n}{n-2}a_2 + n\binom{n}{n-1}a_1 .$$

Solution by Mohammed Aassila, Strasbourg, France.

First we show by induction that $a_n = (-1)^n + na_{n-1}$ for all $n \ge 2$. The relation is easily checked for $n = 2$. Consider any fixed integer $n \ge 3$. If $a_{n-1} = (-1)^{n-1} + (n-1)a_{n-2}$, then

$$\begin{aligned} a_n &= \frac{n}{2}a_{n-1} + \frac{n(n-1)}{2}a_{n-2} + (-1)^n\left(1 - \frac{n}{2}\right) \\ &= (-1)^n + \tfrac{1}{2}na_{n-1} + \tfrac{1}{2}n\big((-1)^{n-1} + (n-1)a_{n-2}\big) \\ &= (-1)^n + \tfrac{1}{2}na_{n-1} + \tfrac{1}{2}na_{n-1} = (-1)^n + na_{n-1} , \end{aligned}$$

as desired. Now, by induction again,

$$a_n = n! - \frac{n!}{1!} + \frac{n!}{2!} + \cdots + (-1)^n\frac{n!}{n!} ,$$

which is the number of derangements of $(1, 2, \ldots, n)$.

5. Find all positive integers n for which there are k integers n_1, n_2, ..., n_k, each greater than 3, such that

$$n = n_1 n_2 \cdots n_k = \sqrt[2^k]{2^{(n_1-1)(n_2-1)\cdots(n_k-1)}} - 1 .$$

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Aassila's write-up.

Let n be a positive integer, and let n_1, n_2, ..., n_k be integers, each greater than 3. Suppose that the equation in the problem is satisfied. Then

$$n = n_1 \cdots n_k = 2^m - 1 , \tag{1}$$

where

$$m = \frac{(n_1-1)(n_2-1)\cdots(n_k-1)}{2^k} = \left(\frac{n_1-1}{2}\right)\left(\frac{n_2-1}{2}\right)\cdots\left(\frac{n_k-1}{2}\right). \qquad (2)$$

From (1), we see that m is a positive integer, since $2^m = n+1$ is a positive integer. We also see that n is odd and each n_i is odd. Since $n_i > 3$, we must have $n_i \geq 5$, for each i.

Lemma. For any integer $j \geq 10$, we have $2^j - 1 > j^3$.

Proof. We proceed by induction on j. The inequality is true for $j = 10$, since $2^{10} - 1 = 1023 > 10^3 = 1000$. Next, we assume that $2^j - 1 > j^3$ for some fixed integer $j \geq 10$. Let us prove that $2^{j+1} - 1 > (j+1)^3$. Since $j \geq 10$, we have

$$\left(\frac{j+1}{j}\right)^3 < \left(\frac{5}{4}\right)^3 < 2.$$

Thus, $2^{j+1} - 1 > 2^{j+1} - 2 = 2(2^j - 1) > 2j^3 > (j+1)^3$. ■

Now, let us return to our problem. Suppose that $m \geq 10$. Using the lemma and (2), we have

$$2^m - 1 > m^3 = \left(\frac{n_1-1}{2}\right)^3 \cdots \left(\frac{n_k-1}{2}\right)^3. \qquad (3)$$

For each i, since $n_i \geq 5$, we must have

$$\left(\frac{n_i-1}{2}\right)^3 \geq 4 \cdot \frac{n_i-1}{2} > n_i.$$

Using (3), we deduce that $2^m - 1 > n_1 n_2 \cdots n_k$, which contradicts (1).

Thus, there are no solutions with $m \geq 10$. It is easy to check that the only positive integer $m \leq 9$ for which (1) and (2) can be satisfied is $m = 3$, which gives $n = 7$.

Now we look at readers' solutions to problems of the 2000 Russian Mathematical Olympiad given in [2002 : 483–484].

1. Prove that there exist ten different real numbers $a_1, a_2, \ldots, a_{10}$ such that the equation

$$(x-a_1)(x-a_2)\cdots(x-a_{10}) = (x+a_1)(x+a_2)\cdots(x+a_{10})$$

has exactly 5 different real roots.

Solution by Mohammed Aassila, Strasbourg, France.

Let a_1, a_2, $\dots$, a_{10} be distinct real numbers such that a_1, $\dots$, a_5 are positive, $a_6 = 0$, $a_7 + a_8 = 0$, and $a_9 + a_{10} = 0$. For $k \in \{6, 7, 8, 9, 10\}$, the factor $x - a_k$ occurs on both sides of the given equation and, hence, a_k is a real root of the equation.

For $x \notin \{a_6, a_7, a_8, a_9, a_{10}\}$, the equation reduces to

$$(x - a_1)(x - a_2)\dots(x - a_5) \;=\; (x + a_1)(x + a_2)\dots(x + a_5)\,. \qquad (1)$$

If $x > 0$, then, for each $k \in \{1, 2, 3, 4, 5\}$,

$$|x - a_k| \;=\; \max\{x - a_k, a_k - x\} \;<\; x + a_k \;=\; |x + a_k|\,,$$

and (1) cannot hold. By symmetry, (1) cannot hold if $x < 0$. Neither does (1) hold if $x = 0$. Therefore, (1) has no real roots. Hence, the equation in the problem statement has no real roots besides a_6, a_7, a_8, a_9, a_{10}.

3. Let a_1, a_2, $\dots$, a_{2000} be real numbers such that

$$a_1^3 + a_2^3 + \dots + a_n^3 \;=\; (a_1 + a_2 + \dots + a_n)^2$$

for all n, $1 \le n \le 2000$. Prove that every element of the sequence is an integer.

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bornsztein's solution.

First we note that if $a_i = i$ for $i = 1, 2, \dots, n$, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \;=\; \sum_{i=1}^{n} i^3 \;=\; \left(\frac{n(n+1)}{2}\right)^2 \;=\; \left(\sum_{i=1}^{n} i\right)^2 \;=\; (a_1 + a_2 + \dots + a_n)^2\,.$$

Lemma. If $a_i = i$ for $i = 1, 2, \dots, n$ and $\sum\limits_{i=1}^{n+1} a_i^3 = \left(\sum\limits_{i=1}^{n+1} a_i\right)^2$, then $a_{n+1} \in \{n+1, -n, 0\}$.

Proof: Starting with the given relation $\sum\limits_{i=1}^{n+1} a_i^3 = \left(\sum\limits_{i=1}^{n+1} a_i\right)^2$, we get

$$a_{n+1}^3 + \sum_{i=1}^{n} i^3 \;=\; \left(a_{n+1} + \sum_{i=1}^{n} i\right)^2 \;=\; a_{n+1}^2 + 2a_{n+1}\left(\frac{n(n+1)}{2}\right) + \left(\sum_{i=1}^{n} i\right)^2\,,$$

which simplifies to $a_{n+1}(a_{n+1} + n)\big(a_{n+1} - (n+1)\big) = 0$. The conclusion of the lemma follows. ∎

For each positive integer k, let $\mathcal{P}_k$ be the claim:

If a_1, a_2, ..., a_k are real numbers such that $\sum\limits_{i=1}^{n} a_i^3 = \left(\sum\limits_{i=1}^{n} a_i\right)^2$ for all integers n such that $1 \le n \le k$, then a_1, a_2, ..., a_k are integers.

We will prove, by induction on k, that $\mathcal{P}_k$ holds for each k. First note that $\mathcal{P}_1$ holds, since the equation $a_1^3 = a_1^2$ implies that $a_1 \in \{0, 1\}$. Let $k \ge 1$ be a fixed integer, and suppose that $\mathcal{P}_i$ holds for $i = 1, 2, \ldots, k$. Let $a_1, a_2, \ldots, a_{k+1}$ be real numbers such that $\sum\limits_{i=1}^{n} a_i^3 = \left(\sum\limits_{i=1}^{n} a_i\right)^2$ for all integers n such that $1 \le n \le k+1$.

Case 1. There exists $i \in \{1, 2, \ldots, k+1\}$ such that $a_i = 0$.

Delete a_i from the sequence $a_1, a_2, \ldots, a_{k+1}$. The remaining k numbers in the sequence satisfy the hypothesis of $\mathcal{P}_k$. Since $\mathcal{P}_k$ holds, each of these numbers must be an integer. Thus, all of $a_1, a_2, \ldots, a_{k+1}$ are integers.

Case 2. There exists $i \in \{1, 2, \ldots, k+1\}$ such that $a_{i+1} = -a_i$.

Then $k \ge 2$. The fact that $\mathcal{P}_i$ holds implies that $a_1, a_2, \ldots, a_i$ are integers. Then $a_{i+1} = -a_i$ is an integer. Now delete a_i and a_{i+1} from the sequence $a_1, a_2, \ldots, a_{k+1}$. The remaining $k-1$ numbers in the sequence satisfy the hypothesis of $\mathcal{P}_{k-1}$. Since $\mathcal{P}_{k-1}$ holds, each of these numbers must be an integer. Thus, all of $a_1, a_2, \ldots, a_{k+1}$ are integers.

Case 3. For each $i \le k+1$, we have $a_i \ne 0$ and $a_i \ne -a_{i-1}$.

Then an easy induction, using the lemma above, shows that $a_i = i$ for all $i \in \{1, 2, \ldots, k+1\}$. Thus, all of $a_1, a_2, \ldots, a_{k+1}$ are integers.

This ends the induction step, and we are done.

5. Prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \ < \ x\,,y \ \le \ 1\,.$$

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Aassila's write-up.

The problem should read: prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \le \frac{2}{\sqrt{1+xy}} \quad \text{for } 0 \le x,\, y \le 1.$$

If $x = 0$, then the inequality reduces to $1 + \dfrac{1}{\sqrt{1+y^2}} \le 2$, which is true, since $y > 0$. By symmetry, the inequality is also true if $y = 0$.

Now, suppose that $0 < x \le 1$ and $0 < y \le 1$. Let $u \ge 0$ and $v \ge 0$ be such that $x = e^{-u}$ and $y = e^{-v}$. Then the inequality becomes

$$\frac{1}{\sqrt{1+e^{-2u}}} + \frac{1}{\sqrt{1+e^{-2v}}} \le \frac{2}{\sqrt{1+e^{-(u+v)}}}\,;$$

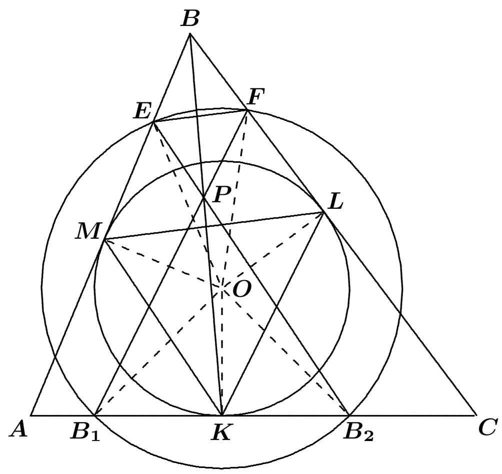
that is,

$$\frac{f(u)+f(v)}{2} \;\le\; f\left(\frac{u+v}{2}\right),$$

where $f(x) = \dfrac{1}{\sqrt{1+e^{-2x}}}$. Since $f''(x) = \dfrac{1-2e^{2x}}{(1+e^{-2x})^{5/2}e^{4x}}$, the function f is concave on an interval containing $[0,\infty)$. Therefore, the above inequality is true.

6. The incircle of triangle ABC with centre O touches the side AC at K. Another circle with the same centre intersects each side at two points. The points of intersection on AC are B_1 and B_2, with B_1 closer to A. E is the point of intersection on AB closer to B, and F is the point of intersection on BC closer to B. Let P be the point of intersection of B_2E and B_1F. Prove that B, K, and P are collinear.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let the incircle meet BC and BA at L and M, respectively. Let r be the radius of the incircle, and let r' be the radius of the other circle centred at O. Since $OK = OL = OM = r$ and $OB_1 = OB_2 = OE = OF = r'$, and since $\angle OKB_1 = \angle OKB_2 = \angle OLF = \angle OME = 90^\circ$, we have

$$\triangle OKB_1 \;\cong\; \triangle OKB_2 \;\cong\; \triangle OLF \;\cong\; \triangle OME\,.$$

Hence, $KB_1 = KB_2 = LF = ME$. Since $AK = AM$ and $KB_2 = ME$, we have $AK : KB_2 = AM : ME$. Thus, $MK \parallel EB_2$; that is, $MK \parallel EP$. Then, by symmetry, $LK \parallel FP$.

Similarly, since $BM = BL$ and $EM = FL$, we obtain $ML \parallel EF$. In triangles KLM and PFE, we have $KM \parallel PE$, $KL \parallel PF$, and $ML \parallel EF$. It follows that KP, LF, and ME are concurrent. Therefore, B, K, and P are collinear.

Now we turn to problems from the February 2003 number of the *Corner*. We give readers' solutions to the 2000 Korean Mathematical Olympiad, given in [2003 : 22–23].

1. Prove that for any prime p, there exist integers x, y, z, and w such that $x^2 + y^2 + z^2 - wp = 0$ and $0 < w < p$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the version of Sinefakopoulos.

For $p = 2$, we set $x = y = w = 1$ and $z = 0$.

For p odd, we note that the numbers

$$1 + 0^2\,, \quad 1 + 1^2\,, \quad \ldots\,, \quad 1 + \left(\frac{p-1}{2}\right)^2$$

leave $(p + 1)/2$ distinct remainders upon division by p, as do the numbers

$$-\,0^2\,, \quad -\,1^2\,, \quad \ldots\,, \quad -\left(\frac{p-1}{2}\right)^2\,.$$

Indeed, if $0 \le a,\, b \le (p - 1)/2$, then p divides $a^2 - b^2 = (a - b)(a + b)$ if and only if p divides $a + b$ or $a - b$, which is impossible if $a \ne b$.

Thus, there are at least two integers $0 \le x,\, y \le (p - 1)/2$ such that

$$1 + x^2 \;\equiv\; -y^2 \pmod{p}\,.$$

Hence, $x^2 + y^2 + 1 = wp$ for some integer w. Since

$$0 \;<\; x^2 + y^2 + 1 \;\le\; 2\left(\frac{p-1}{2}\right)^2 + 1 \;<\; p^2\,,$$

we have $0 < w < p$, as desired.

2. Determine all functions f from the set of real numbers to itself such that for every x and y,

$$f(x^2 - y^2) \;=\; (x - y)\bigl(f(x) + f(y)\bigr)\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We feature Bornsztein's version.

The solution is the set of linear functions passing through the origin.

Let f be a function satisfying the given functional equation. Setting $x = y = 0$ in this equation leads to $f(0) = 0$. Then, setting $y = -x$ gives $0 = f(0) = 2x(f(x) + f(-x))$, from which we deduce that f is an odd function.

For all x and y, we have

$$f(x^2 - y^2) \;=\; (x-y)\big(f(x)+f(y)\big)$$

and also (replacing y by $-y$ and noting that $f(-y) = -f(y)$),

$$f(x^2 - y^2) \;=\; (x+y)\big(f(x)-f(y)\big)\,.$$

Hence,

$$(x-y)\big(f(x)+f(y)\big) \;=\; (x+y)\big(f(x)-f(y)\big)\,;$$

that is, $xf(y) = yf(x)$. Setting $y = 1$, we get $f(x) = xf(1)$. Thus, f is linear.

Conversely, it is easy to verify that any linear function passing through the origin is a solution of the problem.

4. Let p be a prime such that $p \equiv 1 \pmod 4$. Evaluate

$$\sum_{k=1}^{p-1}\left(\left\lfloor\frac{2k^2}{p}\right\rfloor\right) - 2\left(\left\lfloor\frac{k^2}{p}\right\rfloor\right)\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the write-up of Aassila.

Let $\{x\}$ denote the fractional part of x; that is, $\{x\} = x - \lfloor x\rfloor$. Then

$$\begin{aligned}\left\lfloor\frac{2k^2}{p}\right\rfloor - 2\left\lfloor\frac{k^2}{p}\right\rfloor &= \left(\frac{2k^2}{p} - \left\{\frac{2k^2}{p}\right\}\right) - 2\left(\frac{k^2}{p} - \left\{\frac{k^2}{p}\right\}\right)\\ &= 2\left\{\frac{k^2}{p}\right\} - \left\{\frac{2k^2}{p}\right\}\,.\end{aligned}$$

If $\left\{\frac{k^2}{p}\right\} < \frac{1}{2}$, then

$$2\left\{\frac{k^2}{p}\right\} - \left\{\frac{2k^2}{p}\right\} \;=\; 2\left\{\frac{k^2}{p}\right\} - 2\left\{\frac{k^2}{p}\right\} \;=\; 0\,.$$

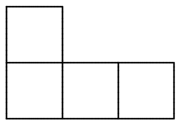
If $\left\{\frac{k^2}{p}\right\} \ge \frac{1}{2}$ then

$$2\left\{\frac{k^2}{p}\right\} - \left\{\frac{2k^2}{p}\right\} \;=\; 2\left\{\frac{k^2}{p}\right\} - \left(2\left\{\frac{k^2}{p}\right\} - 1\right) \;=\; 1\,.$$

Hence, $\sum\limits_{k=1}^{p-1}\left(\left\lfloor\frac{2k^2}{p}\right\rfloor - 2\left\lfloor\frac{k^2}{p}\right\rfloor\right)$ is equal to the number of $k \in [1, p-1]$ such that $\left\{\frac{k^2}{p}\right\} \geq \frac{1}{2}$; that is, the number of non-zero residues k modulo p such that k^2 is congruent to some number in $\left[\frac{p+1}{2}, p-1\right]$. Thus,

$$\sum_{k=1}^{p-1}\left(\left\lfloor\frac{2k^2}{p}\right\rfloor - 2\left\lfloor\frac{k^2}{p}\right\rfloor\right) = \frac{p-1}{2}\,.$$

5. Prove that an $m \times n$ rectangle can be constructed using copies of the following shape if and only if mn is a multiple of 8.



Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Tracy Walker, student, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Walker and Wang.

To make the conclusion correct, we must add the condition that $m > 1$ and $n > 1$.

Call a figure of the given shape a *tetramino*. We will prove that an $m \times n$ board admits a *perfect cover* by tetraminos if and only if $m > 1$, $n > 1$, and $8 \mid mn$.

We first show necessity. Suppose an $m \times n$ board has been covered by exactly t tetraminos. Then $mn = 4t$, which implies that $4 \mid mn$. Suppose, to the contrary, that $8 \nmid mn$. Then t is odd.

We now colour the rows of the board in an alternating manner so that all the odd numbered rows are coloured white and all the even numbered rows are coloured black. Since at least one of m and n is even, the total number of (unit) white squares must be even (and the total number of black squares is also even). It is also clear that a tetramino, regardless of its position on the board, must contain either three white squares and one black square (type I) or three black squares and one white square (type II). Let t_1 and t_2 denote the number of tetraminos of types I and II, respectively. Then $t_1 + t_2 = t$. Since t is odd, we deduce that t_1 and t_2 have opposite parity. The total number of white squares is $3t_1 + t_2$, which is odd. We have a contradiction.

We now establish the sufficiency. First note that a 2×4 board and a 3×8 board both admit perfect covers by tetraminos, as shown by Figure 1 and Figure 2, respectively:

1	2	2	2
1	1	1	2

Figure 1

1	1	3	3	3	4	6	6
1	2	3	4	4	4	5	6
1	2	2	2	5	5	5	6

Figure 2

Now consider an $m \times n$ board where $m > 1$, $n > 1$, and $8 \mid mn$. By considering the *transpose* of the board, if necessary, we may assume that either (i) $2 \mid m$ and $4 \mid n$ or (ii) $8 \mid n$. In case (i), a perfect cover may be obtained using $\frac{mn}{8}$ copies of Figure 1.

For case (ii), we invoke a well-known result of Frobenius which states that, if a, $b \in \mathbb{N}$ such that $(a,b) = 1$, then any natural number $m \geq (a-1)(b-1)$ can be written as $m = ra + sb$ where r, $s \in \mathbb{N} \cup \{0\}$. In particular, if $a = 2$ and $b = 3$, then any natural number m, except 1, can be written as $m = 2r + 3s$ for some non-negative integers r and s. Then it is clear that $\frac{nr}{4}$ copies of Figure 1 together with $\frac{ns}{8}$ copies of Figure 2 would provide a perfect cover.

6. The real numbers a, b, c, x, y, and z are such that $a > b > c > 0$ and $x > y > z > 0$. Prove that

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)} \geq \frac{3}{4}.$$

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give Bornsztein's solution.

We will prove the given inequality under the slightly weaker hypothesis that $a \geq b \geq c > 0$ and $x \geq y \geq z > 0$. For convenience, we let S denote the expression on the left side of the inequality.

From $a \geq b \geq c > 0$ and $x \geq y \geq z > 0$, we deduce that

$$a^2x^2 \;\geq\; b^2y^2 \;\geq\; c^2z^2\,. \tag{1}$$

Moreover,

$$0 \;<\; by+cz \;\leq\; cz+ax \;\leq\; ax+by$$

and

$$0 \;<\; bz+cy \;\leq\; cx+az \;\leq\; ay+bx\,.$$

Then

$$\begin{aligned}\frac{1}{(by+cz)(bz+cy)} &\geq \frac{1}{(cz+ax)(cx+az)}\\ &\geq \frac{1}{(ax+by)(ay+bx)} \;>\; 0\,.\end{aligned} \tag{2}$$

From (1), (2), and Chebyshev's Inequality, it follows that

$$\begin{aligned}S \;\geq\; &\tfrac{1}{3}(a^2x^2+b^2y^2+c^2z^2)\,\cdot\\ &\cdot\left(\frac{1}{(by+cz)(bz+cy)} + \frac{1}{(cz+ax)(cx+az)} + \frac{1}{(ax+by)(ay+bx)}\right).\end{aligned}$$

Using the AM–HM Inequality, we get

$$\frac{1}{3}\left(\frac{1}{(by+cz)(bz+cy)}+\frac{1}{(cz+ax)(cx+az)}+\frac{1}{(ax+by)(ay+bx)}\right)$$
$$\geq \frac{3}{(by+cz)(bz+cy)+(cz+ax)(cx+az)+(ax+by)(ay+bx)}\,.$$

Hence,

$$S \;\geq\; \frac{3(a^2x^2+b^2y^2+c^2z^2)}{S'}\,, \tag{3}$$

where

$$\begin{aligned} S' &= (by+cz)(bz+cy)+(cz+ax)(cx+az)+(ax+by)(ay+bx) \\ &= a^2(xy+xz)+b^2(yz+yx)+c^2(zx+zy) \\ &\qquad +(ab+ac)x^2+(bc+ba)y^2+(ca+cb)z^2\,. \end{aligned}$$

Note that $xy+xz \leq \frac{1}{2}(x^2+y^2)+\frac{1}{2}(x^2+z^2) = x^2+\frac{1}{2}(y^2+z^2)$, with equality if and only if $x = y = z$. Using this and similar inequalities, we deduce that

$$\begin{aligned} S' &\leq a^2\left(x^2+\tfrac{1}{2}y^2+\tfrac{1}{2}z^2\right)+b^2\left(\tfrac{1}{2}x^2+y^2+\tfrac{1}{2}z^2\right) \\ &\qquad +c^2\left(\tfrac{1}{2}x^2+\tfrac{1}{2}y^2+z^2\right)+\left(a^2+\tfrac{1}{2}b^2+\tfrac{1}{2}c^2\right)x^2 \\ &\qquad\quad +\left(\tfrac{1}{2}a^2+b^2+\tfrac{1}{2}c^2\right)y^2+\left(\tfrac{1}{2}a^2+\tfrac{1}{2}b^2+c^2\right)z^2 \\ &= (a^2+b^2+c^2)(x^2+y^2+z^2)+a^2x^2+b^2y^2+c^2z^2\,, \end{aligned}$$

with equality if and only if $x = y = z$ and $a = b = c$. Using Chebyshev's Inequality again, we get

$$\begin{aligned} S' &\leq 3(a^2x^2+b^2y^2+c^2z^2)+a^2x^2+b^2y^2+c^2z^2 \\ &= 4(a^2x^2+b^2y^2+c^2z^2)\,. \end{aligned} \tag{4}$$

From (3) and (4), we deduce that $S \geq \frac{3}{4}$. Equality occurs if and only if $x = y = z$ and $a = b = c$.

We conclude this number of the *Corner* with the only solution we have on file to problems of the 2000 Vietnamese Mathematical Olympiad given [2003 : 24–25]. Here is an opportunity for readers to rise to the challenge of filling in the gaps!

4. For every integer $n \geq 3$ and any given angle α in $(0,\pi)$, let $P_n(x) = x^n\sin\alpha - x\sin n\alpha + \sin(n-1)\alpha$.

(a) Prove that there is only one polynomial of the form $f(x) = x^2+ax+b$ such that for every $n \geq 3$, $P_n(x)$ is divisible by $f(x)$.

(b) Prove that there does not exist a polynomial $g(x)$ of the form $g(x) = x+c$ such that for every $n \geq 3$, $P_n(x)$ is divisible by $g(x)$.

Solved by Michel Bataille, Rouen, France.

First we observe that

$$\begin{aligned}
&P_{n+1}(x) - xP_n(x) \\
&\quad = x^2 \sin n\alpha - x\big(\sin(n+1)\alpha + \sin(n-1)\alpha\big) + \sin n\alpha \\
&\quad = x^2 \sin n\alpha - 2x \sin n\alpha \cos\alpha + \sin n\alpha \\
&\quad = (\sin n\alpha)(x^2 - 2x\cos\alpha + 1)\,.
\end{aligned}$$

(a) Let $f(x) = x^2 + ax + b$, and suppose that $f(x)$ divides $P_n(x)$ for all $n \geq 3$. Then $f(x)$ divides $P_{n+1}(x) - P_n(x) = (\sin n\alpha)(x^2 - 2x\cos\alpha + 1)$ for all $n \geq 3$. Choosing n such that $\sin n\alpha \neq 0$ ($n = 3$ or $n = 4$ will do), we see that necessarily $f(x) = x^2 - 2x\cos\alpha + 1$.

Conversely, since

$$\begin{array}{rcccl}
P_n(x) & = & xP_{n-1}(x) & + & \big(\sin(n-1)\alpha\big)f(x) \\
P_{n-1}(x) & = & xP_{n-2}(x) & + & \big(\sin(n-2)\alpha\big)f(x) \\
\vdots & & \vdots & & \vdots \\
P_2(x) & = & 0 & + & (\sin\alpha)f(x)\,,
\end{array}$$

we have $P_n(x) = f(x)\big(\sin(n-1)\alpha + x\sin(n-2)\alpha + \cdots + x^{n-2}\sin\alpha\big)$. Thus, $f(x)$ divides every $P_n(x)$ with $n \geq 3$.

(b) Suppose $g(x) = x + c$ divides $P_n(x)$ for all $n \geq 3$. In particular, $g(x)$ divides $P_3(x)$. From (a), we have $P_3(x) = (\sin\alpha)f(x)(x + 2\cos\alpha)$. If we assume that c is a real number, then $g(x)$ does not divide $f(x)$, since $f(x)$ has two non-real roots, $e^{i\alpha}$ and $e^{-i\alpha}$. Therefore, $g(x)$ must divide $x + 2\cos\alpha$, and we must have $c = -2\cos\alpha$.

Since $g(x)$ divides $P_4(x)$, it follows that $P_4(-2\cos\alpha) = 0$, which yields $3 - 4\sin^2\alpha = 0$. Then $\alpha = \frac{\pi}{3}$ or $\alpha = \frac{2\pi}{3}$. But in each of these cases, $-2\cos\alpha$ is not a root of $P_5(x)$. Thus, $g(x) = x + c$ cannot divide all $P_n(x)$ (if c is real).

That completes the *Corner* for this issue. Send me your Olympiad contest materials and your nice solutions and generalizations to problems in the *Corner*.

BOOK REVIEWS

John Grant McLoughlin

How to Solve It: A New Aspect of Mathematical Method (2^{nd} *Ed.*)
by George Pólya (with a new foreword by John Conway), published by Princeton University Press, 2004
ISBN 0-691-11966-X, paper, 288 pages, US$16.95.
Reviewed by **Curt Crane**, *Cobequid Educational Centre, Truro, NS.*

Originally published in 1945, the Princeton Science Library has recently decided to re-release George Pólya's signature piece, *How to Solve It*. This decision comes at a good time, since there isn't another book on the market quite like it. While problems are intermixed into the commentary, the book really is not about the problems themselves. Rather, the approach that one takes in solving these problems is the focus of the book.

What is the unknown? What are the data? What is the condition? Pólya repeats this mantra throughout as a guideline for attacking any problem the intrepid mathematician may encounter. He outlines a set of guidelines and practices that readers may use to tackle any problem they encounter, ranging from isolating the unknown value to searching for a related problem.

The book's primary purpose is as a tool for the mathematics teacher. The first part of the book deals entirely with some approaches and techniques which teachers may use in the classroom to help develop the problem-solving skills of their students. Pólya uses the following problem as an example: "Find the diagonal of a rectangular parallelepiped of which the length, the width, and the height are all known." While intrepid mathematics teachers may be able to solve this basic problem on their own, this is not Pólya's concern. Instead, he discusses methods the mathematics teacher may use to assist a student in determining how to tackle the problem. Through the use of sample dialogues, teachers are shown how to help students transform the problem into something they can solve, without explicitly telling the students the steps involved. It is this problem-solving process which Pólya cherishes.

Much of the book is a dictionary of heuristic terminology, which should be of interest to the reader. Unfortunately, this is presented in alphabetical order; as such, it tends to lack flow and cohesion (Pólya himself recommends that it not be read too quickly). Nevertheless, the section is rife with sample problems. One example from the *Reductio ad absurdum* definition is to write numbers using each of the ten digits exactly once, in such a way that the sum of the numbers is exactly 100. For example, one could arrange the ten digits as follows: $19 + 28 + 30 + 7 + 6 + 5 + 4 = 99$. However, this total falls one short of 100. After several failed attempts, the students may begin to suspect that there is more to this problem than they originally suspected; is it indeed even possible to arrange the digits such that they sum to 100? Is there a way to prove that this is not possible?

The book concludes with 20 mathematics problems at the level of a high school student, or possibly a good middle school student, complete with optional hints and solutions to further emphasize Pólya's method of problem solving. For example, question #4 states "To number the pages of a bulky volume, the printer used 2989 digits. How many pages has the volume?" The hints suggest that the reader find a problem related to this one; for example, how many digits would be required to create a volume of 9 pages? 99 pages? By solving these simpler problems first, the original problem becomes clearer.

While the manner in which the book is organized can make it difficult to read at times, *How to Solve It* should be both enjoyable and informative for mathematics teachers everywhere. Students and other mathematicians may also be able to learn from Pólya's approach to problem solving.

USA and International Mathematical Olympiads 2003
Edited by Titu Andreescu and Zuming Feng, published by the Mathematical Association of America, 2004
ISBN 0-88385-817-7, paperbound, 104 pages, US$26.95.
Reviewed by ***Ian VanderBurgh****, University of Waterloo, Waterloo, ON.*

This book has solid mathematical value but perhaps little practical value. It contains problems, hints, and solutions for the United States of America Mathematical Olympiad (USAMO), the International Mathematical Olympiad (IMO), and the Team Selection Test for the US IMO team, all from the year 2003. There is also a glossary, a list of problem credits, and a summary of results from the USAMO and IMO from the last several years.

There is no doubt that there is good mathematics in this book. Each problem has a hint, and many problems have multiple solutions, some of which are, not surprisingly, quite ingenious. The solutions do contain a few typos to stumble over, and, for my money, are written and laid out in such a way as to make them quite difficult to understand.

While students who are preparing for olympiads could definitely get some value from these problems, I do have two additional reservations. First, there are only 18 problems in this book and the problems are hard enough so as to be accessible to relatively few. Second, all of these problems and at least one solution to most of them are easily obtainable with a few minutes searching on the Internet.

Thus, unless you have a keen interest in seeing hints to these problems, in seeing multiple solutions, or reading about the recent history of the USAMO and US performance at the IMO, then I would suggest looking for a compilation of more problems which are harder to come by free of charge.

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er octobre 2005**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

3013. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Un trapèze isocèle $ABCD$ avec $AB = BC = CD$ est inscrit dans un cercle Γ. Soit E un point variable sur l'arc AD de Γ ne contenant pas B et C. Soit F et G les intersections respectives de EB avec AC et de EC avec BD.

(a) Montrer que l'aire du quadrilatère $FBCG$ est constante.

(b) Montrer que $[EFG] : [EAD] = BC : AD$, où $[PQR]$ désigne l'aire du triangle PQR.

3014. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Dans un quadrilatère convex $ABCD$, soit O l'intersection des diagonales AC et BD, et soit M et N les points milieu respectifs de AC et BD. On suppose que $[OAB] + [OCD] = [OBC]$, où $[PQR]$ désigne l'aire du triangle PQR. Montrer que AN, DM et BC sont concourants.

3015. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Dans un triangle ABC, on suppose que $BC < AB$ et $BC < AC$. Les bissectrices extérieures des angles ABC et ACB coupent respectivement AC et AB en D et E. Si I désigne le centre du cercle inscrit, montrer que

$$\frac{BD}{CE} = \frac{(AI^2 - BI^2)CI}{(AI^2 - CI^2)BI}\,.$$

3016. *Proposé par Neven Jurič̌, Zagreb, Croatie.*

Deux sphères de rayon r sont extérieurement tangentes. Trois sphères de rayon R sont extérieurement tangentes l'une à l'autre. Si chacune des trois sphères de rayon R sont en même temps extérieurement tangentes aux deux sphères de rayon r, exprimer R en fonction de r.

3017. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Dans un quadrilatère $ABCD$, soit P, Q, R et S les points sur les côtés AB, BC, CD et DA, respectivement, de telle sorte que

$$\frac{AP}{PB}\cdot\frac{BQ}{QC}\cdot\frac{CR}{RD}\cdot\frac{DS}{SA} = 1\,.$$

Soit O l'intersection de PR et QS. Montrer que

$$DS\cdot\frac{AP}{PB}+AS\cdot\frac{DR}{RC} = AD\cdot\frac{SO}{OQ}\,.$$

3018. *Proposé par Christopher J. Bradley, Bristol, GB.*

Dans un triangle ABC, les céviennes AL, BM et CN sont concourantes en K, et les céviennes AU, BV et CW sont concourantes en T. On suppose que MN coupe VW en P, que NL coupe WU en Q et que LM coupe UV en R.

(a) Montrer que QAR, RBP et PCQ sont des droites.

(b) Montrer que AP, BQ et CR sont soit concourantes, soit parallèles.

(c) Etant donné T, quelles sont les conditions sur K pour que les droites dans (b) soient parallèles ?

3019. *Proposé par Emilio Fernández Moral, IES Sagasta, Logroño, Espagne.*

Soit q et s des entiers tels que $q \geq 1$ et $0 \leq s \leq q-1$. Montrer que

$$\sum_{k=0}^{s}(-1)^{s+k}2^{2k}\binom{q-1+k}{2k}\binom{q-1-k}{s-k} = \binom{2q-1}{2s}$$

et

$$\sum_{k=0}^{s}(-1)^{s+k}2^{2k+1}\binom{q+k}{2k+1}\binom{q-1-k}{s-k} = \binom{2q}{2s+1}\,.$$

3020. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit $A_1A_2\cdots A_n$ un polygone régulier inscrit dans le cercle Γ, et soit P un point intérieur de Γ. Les droites PA_1, PA_2, $\dots$, PA_n coupent Γ pour la seconde fois en B_1, B_2, $\dots$, B_n, respectivement.

(a) Montrer que $\sum\limits_{k=1}^{n}(PA_k)^2 \geq \sum\limits_{k=1}^{n}(PB_k)^2$.

(b) Montrer que $\sum\limits_{k=1}^{n}PA_k \geq \sum\limits_{k=a}^{n}PB_k$.

3021. *Proposé par Pierre Bornsztein, Maisons-Laffitte, France.*

Soit E un ensemble fini de points du plan tel qu'aucun triplet ne soit sur une même droite et qu'aucun quadruplet ne soit sur un même cercle. Si A et B sont deux points distincts de E, on dit que le couple $\{A, B\}$ est *bon* s'il existe un disque fermé dans le plan, contenant les points A et B à l'exclusion des autres. On désigne par $f(E)$ le nombre de bons couples formés par les points de E.

Montrer que si la cardinalité de E est 1003, alors $2003 \leq f(E) \leq 3003$.

3022. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Dans un triangle ABC, soit C' un point quelconque sur le côté AB, et soit M et N des points sur les côtés BC et AC, respectivement, de sorte que $C'M \parallel AC$ and $C'N \parallel BC$.

Montrer que l'aire du triangle $C'CN$ est la moyenne géométrique des aires des triangles $AC'N$ et $C'BM$.

3023. *Proposé par Bogdan Nica, Université McGill, Montréal, QC.*

Trouver toutes les solutions entières du système :

$$\begin{aligned} a^c + b^c - 2 &= c^3 - c\,, \\ b^a + c^a - 2 &= a^3 - a\,, \\ c^b + a^b - 2 &= b^3 - b\,. \end{aligned}$$

3024. *Proposé par feu Murray S. Klamkin, Université de l'Alberta, Edmonton, AB ; et K.R.S. Sastry, Bangalore, Inde.*

Généraliser, avec preuve à l'appui, l'identité ci-dessous à un déterminant d'ordre n au lieu d'un déterminant d'ordre 3 :

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ca & -ca & c^2+ca \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (bc+ca+ab)^3\,.$$

3025. *Proposé par Neven Jurič, Zagreb, Croatie.*

Pour chaque pièce d'un jeu d'échecs, on attribue à chaque case d'un échiquier un nombre qui est égal au nombre de mouvements permis à cette pièce à partir de cette case. La *puissance* de la pièce est alors définie comme la somme de tous ces nombres sur toutes les cases de l'échiquier.

Existe-t-il des entiers $m \geq 2$ et $b \geq 2$ de sorte que, sur un échiquier $m \times b$, la puissance d'une tour est égale à la somme des puissances d'un fou et d'un cavalier ?

3013. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Isosceles trapezoid $ABCD$ with $AB = BC = CD$ is inscribed in the circle Γ. On the arc AD of Γ which does not contain B and C, let E be a variable point. Let F and G be the respective intersections of EB with AC and of EC with BD.

(a) Prove that the area of quadrilateral $FBCG$ is constant.

(b) Prove that $[EFG] : [EAD] = BC : AD$, where $[PQR]$ denotes the area of triangle PQR.

3014. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a convex quadrilateral $ABCD$, let O be the intersection of the diagonals AC and BD, and let M and N be the mid-points of AC and BD, respectively. Suppose that $[OAB] + [OCD] = [OBC]$, where $[PQR]$ denotes the area of triangle PQR. Prove that AN, DM, and BC are concurrent.

3015. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a triangle ABC with incentre I, suppose that $BC < AB$ and $BC < AC$. The exterior bisectors of $\angle ABC$ and $\angle ACB$ intersect AC and AB at D and E, respectively. Prove that

$$\frac{BD}{CE} = \frac{(AI^2 - BI^2)CI}{(AI^2 - CI^2)BI}.$$

3016. *Proposed by Neven Jurič, Zagreb, Croatia.*

Two spheres of radius r are externally tangent to each other. Three spheres of radius R are all externally tangent to each other. If each of the three spheres of radius R are at the same time externally tangent to the two spheres of radius r, express R in terms of r.

3017. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a quadrilateral $ABCD$, let P, Q, R, S be points on the sides AB, BC, CD, DA, respectively, such that

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1.$$

Let O be the intersection of PR and QS. Prove that

$$DS \cdot \frac{AP}{PB} + AS \cdot \frac{DR}{RC} = AD \cdot \frac{SO}{OQ}.$$

3018. *Proposed by Christopher J. Bradley, Bristol, UK.*

In $\triangle ABC$, the cevians AL, BM, and CN are concurrent at K, and the cevians AU, BV, and CW are concurrent at T. Suppose that MN meets VW at P, NL meets WU at Q, and LM meets UV at R.

(a) Prove that QAR, RBP, and PCQ are all straight lines.

(b) Prove that AP, BQ, and CR are either concurrent or parallel.

(c) Given T, under what conditions on K are the lines in (b) parallel?

3019. *Proposed by Emilio Fernández Moral, IES Sagasta, Logroño, Spain.*

Let q and s be integers such that $q \geq 1$ and $0 \leq s \leq q-1$. Show that

$$\sum_{k=0}^{s}(-1)^{s+k}2^{2k}\binom{q-1+k}{2k}\binom{q-1-k}{s-k} = \binom{2q-1}{2s}$$

and

$$\sum_{k=0}^{s}(-1)^{s+k}2^{2k+1}\binom{q+k}{2k+1}\binom{q-1-k}{s-k} = \binom{2q}{2s+1}.$$

3020. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $A_1A_2\cdots A_n$ be a regular polygon inscribed in the circle Γ, and let P be an interior point of Γ. The lines PA_1, PA_2, $\ldots$, PA_n intersect Γ for the second time at B_1, B_2, $\ldots$, B_n, respectively.

(a) Prove that $\sum\limits_{k=1}^{n}(PA_k)^2 \geq \sum\limits_{k=1}^{n}(PB_k)^2$.

(b) Prove that $\sum\limits_{k=1}^{n} PA_k \geq \sum\limits_{k=a}^{n} PB_k$.

3021. *Proposed by Pierre Bornsztein, Maisons-Laffitte, France.*

Let E be a finite set of points in the plane, no three of which are collinear and no four of which are concyclic. If A and B are two distinct points of E, we say that the pair $\{A, B\}$ is *good* if there exists a closed disc in the plane which contains both A and B and which contains no other point of E. We denote by $f(E)$ the number of good pairs formed by the points of E.

Prove that if the cardinality of E is 1003, then $2003 \leq f(E) \leq 3003$.

3022. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given $\triangle ABC$, let C' be any point on the side AB, and let M and N be points on the sides BC and AC, respectively, such that $C'M \parallel AC$ and $C'N \parallel BC$.

Prove that the area of $\triangle C'CN$ is the geometric mean of the areas of $\triangle AC'N$ and $\triangle C'BM$.

3023. *Proposed by Bogdan Nica, McGill University, Montreal, QC.*

Find all integer solutions of the system:

$$\begin{aligned} a^c + b^c - 2 &= c^3 - c\,, \\ b^a + c^a - 2 &= a^3 - a\,, \\ c^b + a^b - 2 &= b^3 - b\,. \end{aligned}$$

3024. *Proposed by the late Murray S. Klamkin, University of Alberta, Edmonton, AB; and K.R.S. Sastry, Bangalore, India.*

Generalize the following identity so that it involves an n^{th} order determinant in place of a 3^{rd} order determinant, and prove your generalization:

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ca & -ca & c^2+ca \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (bc+ca+ab)^3\,.$$

3025. *Proposed by Neven Jurič, Zagreb, Croatia.*

For each chess piece, we assign to each square of a chessboard a number which is the number of moves available to that piece from that square. The *power* of the piece is then defined to be the sum of all these numbers over all the squares of the chessboard.

Do there exist integers $m \geq 2$ and $b \geq 2$ such that, on an $m \times b$ chessboard, the power of a rook is equal to the sum of the powers of a bishop and a knight?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of CARL LIBIS, University of Rhode Island, Kingston, RI, USA from the list of solvers of 2862(a).

2771★. [2002 : 399; 2003 : 405] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all pairs of positive integers a and b such that

$$(a+b)^b = a^b + b^a .$$

A new solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain. Their earlier solution [2003 : 406–407] made use of the Fermat-Wiles Theorem. The solution below uses only elementary number theory. The initial part is essentially the same as before.

Clearly, $(a,1)$ is a solution for all positive integers a. We show that these are the only solution pairs.

Assuming that the equation holds for $b > 1$, we have

$$a^b + b^a = (a+b)^b = \sum_{k=0}^{b} \binom{b}{k} a^{b-k} b^k > a^b + b^b ,$$

and thus, $a > b > 1$. Let $d = \gcd(a,b)$. Set $a_1 = a/d$ and $b_1 = b/d$. Then $a_1 > b_1$ and $\gcd(a_1, b_1) = 1$, and the given equation becomes

$$(a_1 + b_1)^b = a_1^b + d^{a-b} b_1^a , \qquad (1)$$

or equivalently (applying the Binomial Theorem and dividing by b_1^2),

$$d a_1^{b-1} + \binom{b}{2} a_1^{b-2} + \cdots + \binom{b}{b-1} a_1 b_1^{b-3} + b_1^{b-2} = d^{a-b} b_1^{a-2} . \qquad (2)$$

First, suppose that $b_1 = 1$. Then $d = b > 1$ and $a_1 > b_1 = 1$, and equation (1) becomes

$$(a_1 + 1)^d = a_1^d + d^{d(a_1 - 1)} . \qquad (3)$$

(i) If $a_1 = 2$, then equation (3) becomes $3^d = 2^d + d^d$. But this is not possible, since $3^d > 2^d + d^d$ when $d = 1$ or $d = 2$, and $3^d < 2^d + d^d$ when $d \geq 3$.

(ii) If $a_1 = 3$, then equation (3) becomes $4^d = 3^d + (d^2)^d$. But this is impossible, since $4^d < 3^d + (d^2)^d$ for all $d > 1$.

(iii) If $a_1 > 3$, then equation (3) cannot hold because $a_1 + 1 < d^{a_1 - 1}$ for all $d > 1$.

We conclude that $b_1 \neq 1$.

Next suppose that $b_1 = 2$. Then $b = 2d$. Furthermore, a_1 is odd (since $\gcd(a_1, b_1) = 1$), and equation (1) can be rewritten as

$$(a_1 + 2)^{2d} \;=\; a_1^{2d} + 2^{a_1 d} d^{d(a_1 - 2)} \,. \tag{4}$$

(i) If $a_1 = 3$, then equation (4) becomes $25^d = 9^d + 8^d d^d$. We can easily check that this equation does not hold for $d = 1$, $d = 2$, or $d = 3$. It does not hold for $d > 3$, since $25 < 8d$ when $d > 3$.

(ii) If $a_1 = 5$, then equation (4) becomes $49^d = 25^d + 32^d d^{3d}$. But, in fact, $49^d < 25^d + 32^d d^{3d}$ for all $d \geq 1$.

(iii) If $a_1 > 6$, then equation (4) cannot hold because $(a_1 + 2)^2 < 2^{a_1}$.

We conclude that $b_1 \neq 2$.

Now suppose that $b_1 = 2^\ell$ with $\ell > 1$. Then $a_1 > 2^\ell$, and equation (2) becomes

$$da_1^{b-1} + 2^{\ell-1} d(2^\ell d - 1) a_1^{b-2} + \binom{b}{3} 2^\ell a_1^{b-3} + \cdots + 2^{\ell(b-2)} \;=\; 2^{\ell(a-2)} d^{a-b} \,.$$

The first term on the left side of this equation is $da_1^{b-1} = da_1^{2^\ell d - 1}$. This term must be even, since the other terms in the equation all contain factors of 2. But a_1 is odd, since b_1 is even and $\gcd(a_1, b_1) = 1$. Therefore, $a_1^{2^\ell d - 1}$ is odd. This means that d is even, say $d = 2d_1$. Now equation (1) becomes

$$\left((a_1 + 2^\ell)^{2^{\ell-1} d_1}\right)^4 \;=\; \left(a_1^{2^{\ell-1} d_1}\right)^4 + \left(2^{\ell a_1 d_1} d^{d_1 (a_1 - 2^\ell)}\right)^2 \,.$$

But this equation cannot hold because the equation $x^4 - y^4 = z^2$ has no non-trivial integer solutions, as is well known.

It only remains for us to rule out the case where b_1 is a multiple of some odd prime p. In this case, we must have $a_1 > b_1 \geq 3$ and $\gcd(a_1, p) = 1$ (since $\gcd(a_1, b_1) = 1$). Since p divides b_1, it follows from equation (2) that p divides da_1^{b-1}. Since a_1 and p are relatively prime, we conclude that d is a multiple of p. Let $d = p^k d_1$ and $b_1 = p^\ell b_2$, where d_1 and b_2 are each relatively prime to p, and k and ℓ are positive integers. Then $b = p^{k+\ell} d_1 b_2$, and equation (2) becomes

$$p^k d_1 a_1^{b-1} + \binom{b}{2} a_1^{b-2} + \binom{b}{3} a_1^{b-3} p^\ell b_2 + \cdots + \binom{b}{b-1} a_1 p^{\ell(b-3)} b_2^{b-3}$$
$$+ p^{\ell(b-2)} b_2^{b-2} \;=\; p^{\ell(a-2) + k(a-b)} b_2^{a-2} d_1^{a-b} \,. \tag{5}$$

In the above equation, each term $t_m = \binom{b}{m} a_1^{b-m} p^{\ell(m-2)} b_2^{m-2}$ for $m \geq 2$ is divisible by the prime power p^{k+1}. To see this, there are two cases to consider.

Case (i): $p^j < m < p^{j+1}$ for some non-negative integer j.
Then

$$\binom{b}{m} = p^{k+\ell}d_1b_2 \cdot \frac{p^{k+\ell}d_1b_2 - 1}{1} \cdots \frac{p^{k+\ell}d_1b_2 - p^j}{p^j} \cdots \frac{p^{k+\ell}d_1b_2 - m + 1}{m - 1} \cdot \frac{1}{m},$$

which is a multiple of $p^{k+\ell}$ and therefore a multiple of p^{k+1}. It follows that p^{k+1} divides t_m.

Case (ii): $m = p^j$ for some positive integer j.
Then

$$\binom{b}{m} = p^{k+\ell}d_1b_2 \cdot \frac{p^{k+\ell}d_1b_2 - 1}{1} \cdots \frac{p^{k+\ell}d_1b_2 - p^j + 1}{p^j - 1} \cdot \frac{1}{p^j},$$

which is a multiple of $p^{k+\ell-j}$. From this we see that t_m is divisible by

$$p^{k+\ell-j}p^{\ell(p^j-2)} = p^{k+\ell-j+p^j\ell-2\ell} = p^{k+(p^j-1)\ell-j}.$$

It follows that t_m is divisible by p^{k+1} (since $(p^j - 1)\ell - j \geq 1$).

Thus, in both cases, we see that t_m is divisible by p^{k+1}, as claimed.

From (5) we now see that $d_1a_1^{b-1}$ is a multiple of p, which contradicts the fact that a_1 and d_1 are each relatively prime to p.

Consequently, there are no solutions to the equation in the problem statement other than the pairs $(a, 1)$, where a is any positive integer.

2868. [2003 : 399; 2004 : 379] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, we have $c^4 = a^4 + b^4$.

(a) Show that $\triangle ABC$ is acute angled.

(b) Determine the range of $\angle ACB$.

(c)⋆ How can we generalize to $c^n = a^n + b^n$?

Ed: This problem was the subject of a recent article by Guanshen Ren in *The College Mathematics Journal* (Sept 2004, pp. 305–307). The readership may be interested to compare the approach taken by Ren with that of our featured solver, Chip Curtis.

2886. [2003 : 468; 2004 : 519] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

If a, b, c are positive real numbers such that $abc = 1$, prove that

$$ab^2 + bc^2 + ca^2 \ \geq \ ab + bc + ca\,.$$

Editor's Comment: Mea culpa! We goofed! Two of the three featured solutions to this problem, which appeared in [2004 : 519], are flawed.

Vedula N. Murty, Dover, PA, USA, and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON noticed that Solution II is flawed. Using the Cauchy–Schwartz Inequality on the vectors $\left[\sqrt{\frac{b}{c}}, \sqrt{\frac{c}{a}}, \sqrt{\frac{a}{b}}\,\right]$ and $\left[\frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{a}}\right]$ yields $\left(\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right) \geq \left(\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)^2$, from which inequality (1) does not follow.

Yufei Zhao and D. Kipp Johnson, Beaverton, OR, USA indicated that Solution III is flawed, since Muirhead's Theorem does not yield inequality (2) as claimed. Muirhead's Theorem states that if x, y, z are positive and the vector $[p,q,r]$ majorizes the vector $[u,v,w]$, then

$$\sum_{\text{permutation}} x^p y^q z^r \ \geq \ \sum_{\text{permutation}} x^u y^v z^w\,,$$

where there are six permutations in the case of a 3–vector. For this problem, the above inequality becomes

$$\begin{aligned} &2x^3y^3z^0 + 2x^3y^0z^3 + 2x^0y^3z^3 \\ &\qquad \geq \ x^3y^2z^1 + x^3y^1z^2 + x^2y^3z^1 + x^2y^1z^3 + x^1y^3z^2 + x^1y^2z^3\,, \end{aligned}$$

from which inequality (2) does not follow. However, since Solution III has an alternate method of establishing (2) via the AM–GM Inequality, it still represents a valid approach.

2897. [2003 : 518, 2004 : 530] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

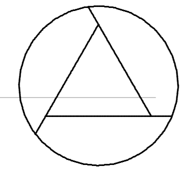
(a) Show that it is possible to divide a circular disc into four parts with the same area by means of three line segments of the same length.

(b) Does there exist a straight edge and compass construction (in the classical sense; that is, with a finite number of steps)?

Comment by Dave Ehren and Stan Wagon, Macalester College, St. Paul, MN, USA.

The featured solution to part (b) of this problem [2004 : 530] attempted to show that no arrangement of lines that satisfies part (a) can be constructed by straight edge and compass. The assertion in the opening statement of the argument is false, we believe, and hence the proof is incorrect.

For suppose the disc has area A. Place an equilateral triangle having area $A/4$ so that it is centred in the disc. This triangle will not touch the circle. Extend each side until it hits the circle, as in the figure. This construction partitions the disc into four parts with the same area, but it does not have a chord, which the proof of (b) says is necessary.



The circumradius of the equilateral triangle is $\sqrt{\pi/(3\sqrt{3})}$ if the original circle has radius 1. Hence, this particular partition of the disc cannot be constructed with straight edge and compass. But there is still a possibility that some other partition could be found.

Editor's Comments: Part (b) seems to have two legitimate interpretations. A narrow interpretation is that the construction used in part (a) cannot be made by ruler and compass. Except for the featured solution, all submitted solutions (including the proposer's) correctly proved this. The more general question—whether an arrangement of three equal segments that divide the disc into four parts of equal area can be constructed using Euclidean tools—remains open.

2914. [2004 : 106, 109] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

On the sides of an acute-angled triangle ABC, similar isosceles triangles DBC, ECA, FAB are constructed externally, such that

$$\begin{aligned}\angle DBC &= \angle DCB = \angle EAC = \angle ECA \\ &= \angle FAB = \angle FBA = \angle BAC\,.\end{aligned}$$

Let M be the mid-point of BC, and let P and Q be the intersections of DE with AC and of DF with AB, respectively.

Prove that $MP : MQ = AB : AC$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Locate N on the ray AC such that $AN = FB$. Then $AFBN$ is a rhombus. Since $AE : AN = AC : AB$, we see that $\triangle AEN \sim \triangle ACB$. Hence, $EN = CD$ and $EN \parallel CD$ [because of equal alternate angles with respect to the transversal CN]. Therefore, $ENDC$ is a parallelogram. Consequently, P is the mid-point of CN. Thus, $MP = \frac{1}{2}BN = \frac{1}{2}AF$. Similarly, $MQ = \frac{1}{2}AE$. Hence, $MP : MQ = AF : AE = AB : AC$.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

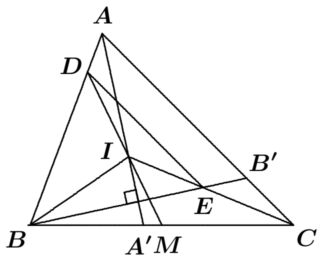
Janous, who used rectangular coordinates, points out that his proof makes no use of the condition that $\triangle ABC$ is acute-angled. The featured argument clearly is valid were $\angle B$ or $\angle C$ not acute. This editor checked that it is valid also when $\angle A$ is obtuse, but the picture becomes cluttered with overlapping triangles. Of course $\angle A$ could not be a right angle, because then points D, E, F would not be defined.

2915. [2004 : 106, 109] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $AB < AC$, let I be its incentre and let M be the mid-point of BC. Suppose that D is the intersection of IM with AB and that E is the intersection of CI with the perpendicular from B to AI.

Prove that $DE \parallel AC$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.



Let $A' = AI \cap BC$ and $B' = BE \cap AC$. Let the lengths of BC, CA, AB, be a, b, c, respectively. Since $BB' \perp AIA'$, we have $AB = AB' = c$. Hence, $B'C = b - c$. Since EC bisects $\angle B'CB$, we have

$$\frac{BE}{EB'} = \frac{BC}{CB'} = \frac{a}{b-c}.$$

Since $BA' : A'C = AB : AC$, we have $BA' = \frac{ac}{b+c}$. By Menelaus' Theorem, we have

$$\frac{BD}{DA} = \left(\frac{BM}{MA'}\right)\left(\frac{A'I}{IA}\right) = \left(\frac{\frac{a}{2}}{\frac{a}{2} - BA'}\right)\left(\frac{BA'}{c}\right)$$
$$= \left(\frac{1}{1 - \frac{2c}{b+c}}\right)\left(\frac{a}{b+c}\right) = \frac{a}{b+c-2c} = \frac{a}{b-c}.$$

Thus, $BE : EB' = BD : DA$, which implies that DE is parallel to AC.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

The solvers showed a wide variety of methods of solving this problem, including barycentric coordinates, Cartesian coodinates, trigonometry, and vectors.

2916. [2003 : 106, 109] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $S = A_1A_2A_3A_4$ be a tetrahedron and let M be the Steiner point; that is, the point M is such that $\sum_{j=1}^{4} A_jM$ is minimized. Assuming that M is an interior point of S, and denoting by A'_j the intersection of A_jM with the opposite face, prove that

$$\sum_{j=1}^{4} A_jM \geq 3\sum_{j=1}^{4} A'_jM.$$

Solution by Michel Bataille, Rouen, France.

The sum $\sum_{j=1}^{4} A_jM$, being extremal at the interior point M, satisfies $\text{grad}\left(\sum_{j=1}^{4} A_jM\right) = \overrightarrow{0}$; that is,

$$\overrightarrow{u_1} + \overrightarrow{u_2} + \overrightarrow{u_3} + \overrightarrow{u_4} = \overrightarrow{0}\,, \tag{1}$$

where $\overrightarrow{u_j}$ denotes the unit vector $\dfrac{\overrightarrow{MA_j}}{MA_j}$ for $j = 1, 2, 3, 4$. [*Ed:* If we let $M = (x, y, z)$ and $A_j = (x_j, y_j, z_j)$, then the terms of our sum have the form $F_j = \sqrt{(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2}$ and $\text{grad}(F_j) = \overrightarrow{u_j}$.]

Let α_1, α_2, α_3, α_4 be the positive real numbers such that

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 1 \\ \text{and} \qquad \alpha_1A_1 + \alpha_2A_2 + \alpha_3A_3 + \alpha_4A_4 &= M\,. \end{aligned}$$

From $\sum_{j=1}^{4} \alpha_j\overrightarrow{MA_j} = \overrightarrow{0}$ and (1), we deduce

$$\begin{aligned} (\alpha_1MA_1 - \alpha_4MA_4)\overrightarrow{u_1} + (\alpha_2MA_2 - \alpha_4MA_4)\overrightarrow{u_2} & \\ + (\alpha_3MA_3 - \alpha_4MA_4)\overrightarrow{u_3} &= \overrightarrow{0}\,, \end{aligned}$$

which implies that $\alpha_1MA_1 = \alpha_2MA_2 = \alpha_3MA_3 = \alpha_4MA_4$. Thus, if c denotes this common positive value, we have

$$\sum_{j=1}^{4} MA_j = c\sum_{j=1}^{4}\frac{1}{\alpha_j}\,. \tag{2}$$

Furthermore, $(1-\alpha_j)\overrightarrow{MA'_j} = -\alpha_j\overrightarrow{MA_j}$ for each $j = 1, 2, 3, 4$. For example, considering $j = 1$, the point $\dfrac{M-\alpha_1A_1}{1-\alpha_1} = \dfrac{\alpha_2A_2+\alpha_3A_3+\alpha_4A_4}{1-\alpha_1}$ is on the line A_1M as well as in the plane $A_2A_3A_4$, and therefore, this point is A'_1. Thus, $(1-\alpha_1)A'_1 = M - \alpha_1A_1$, and hence, we obtain $(1-\alpha_1)\overrightarrow{MA'_1} = -\alpha_1\overrightarrow{MA_1}$.

It follows that

$$\sum_{j=1}^{4} MA'_j = c\sum_{j=1}^{4}\frac{1}{1-\alpha_j}\,. \tag{3}$$

From (2) and (3), the requested inequality is equivalent to

$$\sum_{j=1}^{4}\frac{1}{1-\alpha_j} \le \frac{1}{3}\sum_{j=1}^{4}\frac{1}{\alpha_j}\,. \tag{4}$$

Now, with the help of the AM–HM Inequality, we obtain

$$\frac{1}{1-\alpha_1} = \frac{1}{\alpha_2+\alpha_3+\alpha_4} \le \frac{1}{9}\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\frac{1}{\alpha_4}\right)\,.$$

Adding this to the corresponding inequalities for $\frac{1}{1-\alpha_2}$, $\frac{1}{1-\alpha_3}$, $\frac{1}{1-\alpha_4}$, we get

$$\sum_{j=1}^{4} \frac{1}{1-\alpha_j} \leq 3 \times \frac{1}{9}\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\frac{1}{\alpha_4}\right),$$

and (4) follows.

Also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Zhou comments that the result may be extended to an n–simplex $S_n = A_1A_2\cdots A_{n+1}$ with the conclusion $\sum_{i=1}^{n+1} A_iM \geq n\sum_{i=1}^{n+1} A_i'M$.

2917★. [2003 : 107, 109] *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

If x_1, x_2, x_3, x_4, $x_5 \geq 0$ and $x_1+x_2+x_3+x_4+x_5 = 1$, prove or disprove that

$$\frac{x_1}{1+x_2}+\frac{x_2}{1+x_3}+\frac{x_3}{1+x_4}+\frac{x_4}{1+x_5}+\frac{x_5}{1+x_1} \geq \frac{5}{6}.$$

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

The statement is false. If we let $x_1 = .5$,$x_2 = .4$, $x_3 = .05$, $x_4 = .03$, and $x_5 = .02$, then x_1, x_2, x_3, x_4, $x_5 \geq 0$ and $x_1+x_2+x_3+x_4+x_5 = 1$, but

$$\begin{aligned}\frac{x_1}{1+x_2}+\frac{x_2}{1+x_3}+\frac{x_3}{1+x_4}+\frac{x_4}{1+x_5}+\frac{x_5}{1+x_1} \\ = \frac{.5}{1.4}+\frac{.4}{1.05}+\frac{.05}{1.03}+\frac{.03}{1.02}+\frac{.02}{1.5} = .829384\ldots < \frac{5}{6}.\end{aligned}$$

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany.

Specht reports numerical experiments that suggest the minimum is obtained when two successive values x_k and x_{k+1} are zero. Under this assumption the minimum value is approximately 0.8161123672.

2918. [2004 : 107, 110] *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a_1, a_2, $\ldots$, a_{100} be real numbers satisfying:

$$\begin{aligned} a_1 \geq a_2 \geq \cdots \geq a_{100} &\geq 0; \\ a_1^2+a_2^2 &\geq 200; \\ a_3^2+a_4^2+\cdots+a_{100}^2 &\geq 200. \end{aligned}$$

What is the minimum value of $a_1+a_2+\cdots+a_{100}$?

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposers.

First we note that $a_2 > 0$; for otherwise, $a_3 = a_4 = \cdots = a_n = 0$ and the condition $a_3^2 + a_4^2 + \cdots + a_{100}^2 \geq 200$ fails.

Since $a_2 \geq a_j$ for $j = 3, 4, \ldots, 100$, we have

$$a_2(a_3 + a_4 + \cdots + a_{100}) \;\geq\; a_3^2 + a_4^2 + \cdots + a_{100}^2 \;\geq\; 200\,,$$

which implies that $a_3 + a_4 + \cdots + a_{100} \geq 200/a_2$. Using this inequality, the fact that $a_1 \geq a_2$, and the AM–GM Inequality, we obtain

$$\begin{aligned} a_1 + a_2 + \cdots + a_{100} &\geq a_1 + a_2 + \frac{200}{a_2} \;\geq\; 2a_2 + \frac{200}{a_2} \\ &\geq 2\sqrt{(2a_2)\frac{200}{a_2}} \;=\; 40\,. \end{aligned}$$

Inspecting the inequalities above, we find that $a_1 + a_2 + \cdots + a_{100}$ attains its minimum value of 40 if and only if $a_1 = a_2 = a_3 = a_4 = 10$ and $a_5 = a_6 = \cdots = a_{100} = 0$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There was one incomplete solution.

Hess suggested the following generalization of the problem: If $a_1, a_2, \ldots, a_n$ are real numbers, find the minimum value of $a_1 + a_2 + \cdots + a_n$ subject to the conditions

$$\begin{aligned} a_1 \geq a_2 \geq \cdots \geq a_n &\geq 0\,; \\ a_1^2 + a_2^2 + \cdots + a_m^2 &\geq A\,; \\ a_{m+1}^2 + a_{m+2}^2 + \cdots + a_n^2 &\geq B\,. \end{aligned}$$

He then proceeded to discuss some particular cases of the general question.

2919★. [2004 : 107, 110] *Proposed by Ross Cressman, Wilfrid Laurier University, Waterloo, ON.*

Let $n \in \mathbb{N}$ with $n > 1$, and let

$$T_n = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; x_j > 0 \text{ for } j = 1, \ldots, n, \text{ and } \sum_{j=1}^n x_j = 1 \right\}.$$

Let p, q, $r \in T_n$ such that $\sum\limits_{j=1}^n \sqrt{q_j r_j} < \sum\limits_{j=1}^n \sqrt{p_j r_j}$.

Prove or disprove:

(a) $\sum\limits_{j=1}^n \sqrt{q_j\,(r_j + p_j)} \;<\; \sum\limits_{j=1}^n \sqrt{p_j\,(r_j + p_j)}$,

(b) for all $\lambda \in [0,1]$,

$$\sum_{j=1}^n \sqrt{q_j\,(\lambda r_j + (1-\lambda)p_j)} \;<\; \sum_{j=1}^n \sqrt{p_j\,(\lambda r_j + (1-\lambda)p_j)}\,.$$

[Proposer's remarks: (a) is the special case of (b) with $\lambda = \frac{1}{2}$. This question is connected with properties of the Shahshahani metric on T_n, a metric important for population genetics.]

Solution by Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON.

We show that (b) is true when $n = 2$, which implies that (a) is also true when $n = 2$. We then show how to construct examples to prove that (a) and (b) are false when $n \geq 3$.

For p, $q \in \mathbb{R}^n$, let $p \cdot q$ denote the standard inner product in $\mathbb{R}^n$. Let

$$S_n = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; x_j > 0 \text{ for } j = 1, \ldots, n, \text{ and } \sum_{j=1}^{n} x_j^2 = 1 \right\}.$$

Part (b) is equivalent to the proposition below:

Proposition: Let $n \in \mathbb{N}$ with $n > 1$. If p, q, $r \in S_n$ such that $q \cdot r < p \cdot r$, and if $s \in S_n$ with components given by $s_j = \sqrt{\lambda r_j^2 + (1-\lambda) p_j^2}$ for some $\lambda \in [0, 1]$, then $q \cdot s < p \cdot s$.

We first show that the proposition is true for $n = 2$. For convenience, we regard each $p \in \mathbb{R}^n$ both as a vector and as a point. When $n = 2$, we see that p, q, r, s are all on the unit circle and in the interior of the first quadrant. Let $\widehat{pq}$ denote the acute angle between vectors p and q. Then $q \cdot r < p \cdot r$ if and only if $\cos(\widehat{pr}) > \cos(\widehat{qr})$, or equivalently, $\widehat{pr} < \widehat{qr}$, which is true if and only if r is closer to p than to q. Since $\min\{p_j, r_j\} \leq s_j \leq \max\{p_j, r_j\}$ for all j, we see that s is on the arc between p and r. Thus, s is closer to p than to q; hence, $q \cdot s < p \cdot s$.

Now we construct an example to show that the proposition above is false for $n = 3$. Take $p = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$, $r = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and $\lambda = \frac{1}{2}$. Then p, r, and $s = \left(\frac{1}{2}, \frac{\sqrt{5}}{2\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are all on the unit sphere and in the interior of the first octant; that is, they are all in S_3. Note that s is not on the great circle that goes through p and r, since these three points are all on the same latitude with $x_3 = \frac{1}{\sqrt{3}}$.

Consider the circle C_1 on S_3 with centre r passing through the point p. (Note that C_1 need not lie entirely on S_3 since S_3 is only $\frac{1}{8}$ of the unit sphere.) This circle divides S_3 into two regions, one "inside" the circle (think of a polar cap) and one "outside". Similarly, the circle C_2 on S_3 with centre s passing through the point p divides S_3 into two regions. Since p, r, s do not lie on a great circle, C_1 and C_2 are not tangent at p. Thus, there are points on S_3 arbitrarily close to p that are outside C_1 and inside C_2. Clearly, $q \cdot r < p \cdot r$ if and only if q is outside C_1, and $q \cdot s > p \cdot s$ if and only if q is inside C_2. Hence, it suffices to take q to be any such point.

In closing, we remark that this method works for any fixed $\lambda \in (0, 1)$ as well, and can be further extended to any $n > 3$.

2920. [2004 : 108, 110] *Proposed by Simon Marshall, student, Onslow College, Wellington, New Zealand.*

Let a, b, and c be positive real numbers. Prove that

$$a^4+b^4+c^4+2\left(a^2b^2+b^2c^2+c^2a^2\right) \;\geq\; 3\left(a^3b+b^3c+c^3a\right) .$$

Composite of almost identical solutions by Vasile Cîrtoaje, University of Ploiesti, Romania; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Note that the given inequality is equivalent to

$$(a^2+b^2+c^2)^2-3(a^3b+b^3c+c^3a) \;\geq\; 0 .$$

Let $E=(a^2+b^2+c^2)^2-3(a^3b+b^3c+c^3a)$. Let

$$a \;=\; x\,, \qquad b \;=\; x+p\,, \qquad \text{and} \qquad c \;=\; x+q\,.$$

Then, after some simplifications, we find that

$$\begin{aligned} E \;=\; E(x) \;&=\; (p^2-pq+q^2)x^2-(p^3-5p^2q+4pq^2+q^3)x \\ &\qquad +(p^4-3p^3q+2p^2q^2+q^4)\,. \end{aligned}$$

Regarding $E(x)$ as a quadratic in x, its discriminant is

$$\begin{aligned} \Delta \;&=\; (p^3-5p^2q+4pq^2+q^3)^2 \\ &\qquad -4(p^2-pq+q^2)(p^4-3p^3q+2p^2q^2+q^4) \\ &=\; -3(p^6-2p^5q-3p^4q^2+6p^3q^3+2p^2q^4-4pq^5+q^6) \\ &=\; -3(p^3-p^2q-2pq^2+q^3)^2\,. \end{aligned}$$

Since $\Delta \leq 0$ and $p^2-pq+q^2 \geq 0$, it follows that $E(x) \geq 0$ for all real x.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. There were six incorrect solutions and one incomplete solution. Four of the incorrect solutions made the convenient assumption that $a \leq b \leq c$, which is technically incorrect, since the right side of the given inequality possesses only cyclic symmetry rather than complete symmetry. Three solutions gave (without proof) the wrong statement that equality holds if and only if $a=b=c$, even though the given problem did not ask for the determination of the equality cases.

It is clear from the featured solution that the assumption that a, b, and c are positive is superfluous. But, strangely enough, no one pointed this out explicitly.

The proposer gave a more complicated and sophisticated proof using complex numbers and primitive roots of unity. Using this approach, he was able to determine that the equality holds if and only if either $a=b=c$ or, for some constant k, we have $a=k(1+z+\overline{z})$, $b=k(1+wz+w^2\overline{z})$, and $c=k(1+w^2z+w\overline{z})$ (in any cyclic order), where $w=e^{2\pi i/3}$, $z=\frac{1}{\sqrt{7}}\cos\theta$, and $\theta=-\frac{1}{3}\operatorname{Arg}(2+6w)=-\frac{1}{3}\operatorname{Arg}(-1+3\sqrt{3}i)$. If we take $k=1$ and use the approximate value $\theta\approx 33.63^\circ$, we see that $(a,b,c)\approx(1.629, 1.048, 0.323)$ or any cyclic permutation thereof.

Mihály Bencze, Brasov, Romania, sent in a comment indicating that in one of the 1997 issues of Octogon Mathematical Magazine, he proved the following more general inequality: If $x_k > 0$ *for* $k = 1, 2, \ldots, n$ *and* $\alpha, \beta > 0$*, then*

$$\alpha \sum_{k=1}^{n} x_k^4 + \beta \sum_{k=1}^{n} x_k^2 x_{k+1}^2 \ \geq \ (\alpha + \beta) \sum_{k=1}^{n} x_k^3 x_{k+1} ,$$

where $x_{k+1} = x_1$. *The current problem is the special case when* $n = 3$*,* $\alpha = 1$*, and* $\beta = 2$.

Cîrtoaje and another solver remarked that the given inequality follows from the identity

$$\begin{aligned} 4(a^2+b^2+c^2-ab-bc-ca)((a+b+c)^2 - 3(a^3b+b^3c+c^3a)) \\ = \ (A-5B+4C)^2 + 3(A-B-2C+2D)^2 , \end{aligned}$$

where $A = a^3+b^3+c^3$*,* $B = a^2b+b^2c+c^2a$*,* $C = ab^2+bc^2+ca^2$*, and* $D = 3abc$*. Clearly, verifying this identity by hand is a formidable task. The editor turned to MAPLE for help and found the identity to be correct. Unfortunately, the solvers did not explain how they arrived at this amazing identity.*

Cîrtoaje mentioned that the given inequality has appeared before as problem #22694 on page 287 of Gazeta Matematica 7–8, 1992 (in Romanian), in the following (equivalent) form: If a*,* b*, and* c *are real numbers, then*

$$a^2(a-b)(a-2b) + b^2(b-c)(b-2c) + c^2(c-a)(c-2a) \ \geq \ 0 .$$

However, he stated that a solution was not published there.

2921. [2004 : 108, 111] *Proposed by Barry R. Monson, University of New Brunswick, Fredericton, NB; and J. Chris Fisher, University of Regina, Regina, SK.*

These days, with *Cinderella*™ and the *Lénárt sphere*™ at hand, one can do actual spherical constructions, using a spherical ruler to draw the complete great circle through points A and B, and spherical compasses to draw the circle with centre A and radius BC ($\leq \frac{\pi}{2}$, say, on a unit sphere).

Give a simple spherical construction for the vertices of a regular icosahedron inscribed in the sphere.

Solution by the proposers.

The perpendiculars, mid-points, angle bisectors, and circumcircle that we need here can all be constructed by mimicking planar constructions. First draw (the great circle) AX through two non-antipodal points A and X; then erect the perpendiculars at A and X, meeting at Z, say (one of the two poles for AX). Suppose that AZ, which already has been drawn, meets AX again at B (antipodal to A). At Z erect the perpendicular to AZ, meeting AXB at antipodes Y and W. (The sphere has now been subdivided into congruent octants.)

Find the mid-point K of the quarter great circle AZ, and construct the circle μ with centre K and radius $KZ = KA$. Next construct the circle ν through Z, B, Y, and suppose ν meets μ at V (as well as at Z). Finally, suppose ZV meets AXB at antipodes P (closest to A) and Q.

Claim. A and P are neighbouring vertices of an inscribed icosahedron.

Proof of the claim: Look at where the planes supporting the various circles of the construction meet the 'equatorial plane' $AYBW$. The sphere itself traces the equator λ through these points, a circle whose centre M is the mid-point of the Euclidean line segment AB. But the planes determined by μ and ν trace lines (in the equatorial plane) meeting at a point D, so that $\triangle ABD$ is an isosceles right triangle, with DA tangent to λ at A. It is now easy to see that $APBQ$ is a golden rectangle; in fact, that is the content of ***CRUX with MAYHEM*** problem 2813 [2003 : 47; 2004 : 63–64], namely *If M is the mid-point of side AB of the square $ABCD$, while P (inside the square) and Q are the intersection points of the line MD with the circle centred at M whose radius is MA $(= MB)$, then $APBQ$ is a golden rectangle— $PB : PA = (\sqrt{5}+1)/2$.* Therefore, A, P, B, and Q are four of the twelve vertices of a regular icosahedron. (See Coxeter, *Introduction to Geometry*, section 11.2.) The others are similarly located on the great circle bisectors of $\angle AZP$ and $\angle PZB$.

No other solutions were submitted, but Zhou offered the comment that the construction is quite simple if one wants to work in a plane: construct the angle $\cos^{-1}(1/\sqrt{5})$ [by means of a $(1,\ 2,\ \sqrt{5})$–right triangle such as $\triangle DAM$ in the above solution]. Since this angle is the central angle subtended by adjacent vertices of the icosahedron, one only has to transfer to the sphere the chord of the unit circle which subtends that angle. This is essentially the way Euclid constructed the Platonic solids. Note that the original problem restricts the construction to the sphere's surface; it is only for the proof that we are permitted to embed the sphere in Euclidean space. Perhaps readers are happy with life in the Euclidean plane and find it hard to imagine what it would be like living on a spherical surface. In Bieberbach's Theorie der geometrischen Konstructionen, *there is a short section on spherical constructions. It provides general arguments using stereographic projection to show that the spherical possibilities more or less amount to the Euclidean possibilities.*

2922. [2003 : 108, 111] *Proposed by Michel Bataille, Rouen, France.*

Suppose that n is a non-negative integer. Find a closed expression for

$$\sum_{k=0}^{n}(-1)^k 2^k\binom{n}{k}\binom{2n-k}{n}.$$

Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela.

Let $\langle x^k\rangle f(x)$ denote the coefficient of x^k in the polynomial $f(x)$. Clearly, $\langle x^{n-k}\rangle f(x) = \langle x^n\rangle x^k f(x)$ and $\sum_k \langle x^k\rangle f(x) y^k = f(y)$. Now we have

$$\langle x^k\rangle(1-2x)^n \;=\; (-2)^k\binom{n}{k} \;=\; (-1)^k 2^k\binom{n}{k},$$

and

$$\binom{2n-k}{n} \;=\; \binom{2n-k}{n-k} \;=\; \langle y^{n-k}\rangle(1+y)^{2n-k} \;=\; \langle y^n\rangle y^k(1+y)^{2n-k}.$$

Hence,

$$\sum_{k=0}^{n}(-1)^k 2^k\binom{n}{k}\binom{2n-k}{n}$$

$$\begin{aligned}
&= \sum_{k=0}^{n}\langle x^k\rangle(1-2x)^n\langle y^n\rangle y^k(1+y)^{2n-k}\\
&= \langle y^n\rangle(1+y)^{2n}\sum_{k=0}^{n}\langle x^k\rangle(1-2x)^n\left(\frac{y}{1+y}\right)^k\\
&= \langle y^n\rangle(1+y)^{2n}\left(1-\frac{2y}{1+y}\right)^n \;=\; \langle y^n\rangle(1+y)^n(1-y)^n\\
&= \langle y^n\rangle(1-y^2)^n \;=\; \begin{cases}0 & \text{if } n \text{ is odd,}\\ (-1)^{\frac{n}{2}}\binom{n}{n/2} & \text{if } n \text{ is even.}\end{cases}
\end{aligned}$$

Also solved by CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Janous remarks there is an earlier treatment of this sum, "Unfortunately I have no reference in English, but only one in Russian: Yu. P. Vrazaev, Problems in Algebra and Analysis, *Kiev, 1949. [Mimeographed lecture notes]".*

2923. [2004 : 108, 111] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x, $y \geq 0$ (x, $y \in \mathbb{R}$) and $x^2 + y^3 \geq x^3 + y^4$. Prove that $x^3 + y^3 \leq 2$.

I. *Solution by Chip Curtis, Missouri Southern State College, Joplin, MO, USA.*

We proceed by proving the following string of inequalities:

$$x^3 + y^3 \;\leq\; x^2 + y^2 \;\leq\; x + y \;\leq\; 2\,.$$

Writing

$$x^3 + y^3 \;=\; \langle x^{3/2}, y^2\rangle \cdot \langle x^{3/2}, y\rangle\,,$$

the Cauchy–Schwarz Inequality, the given inequality, and the AM–GM Inequality imply that

$$\begin{aligned}
x^3 + y^3 \;&\leq\; \sqrt{x^3 + y^4}\cdot\sqrt{x^3 + y^2}\\
&\leq\; \sqrt{x^2 + y^3}\cdot\sqrt{x^3 + y^2} \;\leq\; \tfrac{1}{2}(x^2 + y^3 + x^3 + y^2)\,,
\end{aligned}$$

from which it follows that $x^3 + y^3 \leq x^2 + y^2$.

Similarly,

$$\begin{aligned} x^2+y^2 &= \langle x^{3/2}, y^{3/2}\rangle \cdot \langle x^{1/2}, y^{1/2}\rangle \ \le\ \sqrt{x^3+y^3}\cdot\sqrt{x+y} \\ &\le\ \sqrt{x^2+y^2}\cdot\sqrt{x+y}\ \le\ \tfrac{1}{2}(x^2+y^2+x+y)\,, \end{aligned}$$

implying that $x^2+y^2 \le x+y$.

Finally,

$$\begin{aligned} x+y &= \langle x, y\rangle \cdot \langle 1,1\rangle \ \le\ \sqrt{x^2+y^2}\cdot\sqrt{2} \\ &\le\ \sqrt{x+y}\cdot\sqrt{2}\ \le\ \tfrac{1}{2}(x+y+2)\,, \end{aligned}$$

implying that $x+y \le 2$. Therefore, $x^3+y^3 \le 2$.

II. *Generalization by Li Zhou, Polk Community College, Winter Haven, FL, USA, modified slightly by the editor.*

We prove the more general result that if x, $y > 0$ and if r is a real number such that $x^{r-1}+y^r \ge x^r+y^{r+1}$, then $x^r+y^r \le x^{r-3}+y^{r-3}$. The current problem is the special case when $r=3$, since the cases when $x=0$ or $y=0$ are clearly trivial.

Let $S = x^{r-2}(1-x)+y^{r-1}(1-y)$ and $T = x^{r-1}(1-x)+y^r(1-y)$. Then $T \ge 0$ from the assumption. Since

$$\begin{aligned} S-T &= x^{r-2}-2x^{r-1}+x^r+y^{r+1}-2y^r+y^{r-1} \\ &= x^{r-2}(1-x)^2+y^{r-1}(1-y)^2\ \ge\ 0\,, \end{aligned}$$

we have $S \ge T \ge 0$. Furthermore, by straightforward algebra, we find that

$$\begin{aligned} & x^{r-3}+y^{r-3}-x^r-y^r-x^{r-3}(1-x)^2-y^{r-3}(1-y^2)^2 \\ &\quad= -x^r-y^r+2x^{r-2}-x^{r-1}+2y^{r-1}-y^{r+1} \\ &\quad= 2x^{r-2}(1-x)+2y^{r-1}(1-y)+x^{r-1}-x^r+y^r-y^{r+1} \\ &\quad= 2S+T\,. \end{aligned}$$

Hence,

$$\begin{aligned} & x^{r-3}+y^{r-3}-x^r-y^r \\ &\qquad= 2S+T+x^{r-3}(1-x)^2+y^{r-3}(1-y^2)^2\ \ge\ 0\,, \end{aligned}$$

from which $x^r+y^r \le x^{r-3}+y^{r-3}$ follows immediately.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; and the proposer. There was one incorrect solution.

2924. [2003 : 108, 111] *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that $x_1, \ldots, x_n$ $(n \geq 3)$ are positive real numbers satisfying

$$\frac{1}{1+x_2^2x_3\cdots x_n}+\frac{1}{1+x_1x_3^2\cdots x_n}+\cdots+\frac{1}{1+x_1^2x_2\cdots x_{n-1}} \;\geq\; \alpha\,,$$

for some $\alpha > 0$. Prove that

$$\frac{x_1}{x_2}+\frac{x_2}{x_3}+\cdots+\frac{x_n}{x_1} \;\geq\; \frac{n\alpha}{n-\alpha}x_1x_2\cdots x_n\,.$$

Solution Li Zhou, Polk Community College, Winter Haven, FL, USA.

The given condition implies that $n > \alpha$. Let

$$t_k \;=\; \left(\frac{x_k}{x_{k+1}}\right)\left(\frac{1}{x_1x_2\ldots x_n}\right)\,,$$

where indices are taken modulo n. Since $f(t) = \dfrac{t}{t+1}$ is a concave function on $(0,\infty)$, we have

$$\begin{aligned}\frac{n(t_1+t_2+\cdots+t_n)}{t_1+t_2+\cdots+t_n+n} &= nf\left(\frac{t_1+t_2+\cdots+t_n}{n}\right)\\ &\geq f(t_1)+f(t_2)+\cdots+f(t_n)\\ &= \frac{1}{1+t_1^{-1}}+\frac{1}{1+t_2^{-1}}+\cdots+\frac{1}{1+t_n^{-1}} \;\geq\; \alpha\,,\end{aligned}$$

which is equivalent to

$$t_1+t_2+\cdots t_n \;\geq\; \frac{n\alpha}{n-\alpha}\,.$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

2925. [2004 : 109, 111] *Proposed by Michel Bataille, Rouen, France.*

Let n be an integer with $n \geq 3$. Determine the zeroes of the function

$$f_n(x) \;=\; \sum_{k=1}^{n-1}\frac{\sin(k\pi/n)}{\sin\big((k\pi/n)-x\big)}\,.$$

Solution by the proposer.

It is quite easy to check that the zeroes of $\sin x$ are not zeroes of $f_n(x)$. Therefore, we can impose the restriction $\sin x \neq 0$. Then we may write

$$(\sin x) f_n(x) \;=\; \sum_{k=1}^{n-1} \frac{1}{\cot x - \cot(k\pi/n)} \;=\; \frac{P'_{n-1}(\cot x)}{P_{n-1}(\cot x)}\,,$$

where

$$P_{n-1}(z) \;=\; \prod_{k=1}^{n-1} \Big(z - \cot(k\pi/n) \Big)\,.$$

Thus, $P_{n-1}(z)$ is a polynomial of degree $n-1$, and its zeroes are the numbers $\cot(k\pi/n)$ for $k \in \{1, 2, \ldots, n-1\}$. Observe that these are also the roots of the equation $\left(\frac{z+i}{z-i}\right)^n = 1$. It follows that

$$P_{n-1}(z) \;=\; \frac{1}{2in}\Big((z+i)^n - (z-i)^n \Big)\,.$$

Then $P'_{n-1}(z) = (n-1)P_{n-2}(z)$. The zeroes of $P'_{n-1}(z)$ are the zeroes of $P_{n-2}(z)$, namely, the numbers $\cot\left(\frac{k\pi}{n-1}\right)$ for $k \in \{1, 2, \ldots, n-2\}$. These numbers are not zeroes of $P_{n-1}(z)$. Therefore, they are the zeroes of $P'_{n-1}(z)/P_{n-1}(z)$.

We can now conclude that the zeroes of $f_n(x)$ are the values of x such that $\cot x = \cot\left(\frac{k\pi}{n-1}\right)$ for $k \in \{1, 2, \ldots, n-2\}$. These are the numbers $\frac{m\pi}{n-1}$, where m is any integer not divisible by $n-1$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incorrect solution.

Janous and Zhou used the identity

$$f_n(x) \;=\; \frac{n \sin\big((n-1)x\big)}{\sin(nx)}\,,$$

which is valid as long as x is not a zero of $\sin(nx)$. Since the zeroes of the right side are quite evident, this identity provides an immediate solution to the problem.

Zhou supplied a proof of the above identity. An elegant proof may be obtained by going a little further with the ideas in the solution above. We leave this as an exercise for the reader.

2926. [2004 : 172, 174] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In circle Γ with centre O and radius R, we have three parallel chords A_1A_2, B_1B_2, and C_1C_2. Show that the orthocentres of the eight triangles having vertices A_i, B_j, and C_k ($i, j, k \in \{1, 2\}$) are collinear.

Composite of similar solutions by Christopher J. Bradley, Bristol, UK; D. Kipp Johnson, Beaverton, OR, USA; Doug Newman, Lancaster, CA, USA; Joel Schlosberg, Bayside, NY, USA; Toshio Seimiya, Kawasaki, Japan; Peter Y. Woo, Biola University, La Mirada, CA, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; Li Zhou, Polk Community College, Winter Haven, FL, USA; Titu Zvonaru, Comănești, Romania; and the proposer.

Choose coordinates so that the end-points of the given chords are

$$\begin{array}{ll} A_1(-\cos\alpha,\,\sin\alpha)\,, & A_2(\cos\alpha,\,\sin\alpha)\,, \\ B_1(-\cos\beta,\,\sin\beta)\,, & B_2(\cos\beta,\,\sin\beta)\,, \\ C_1(-\cos\gamma,\,\sin\gamma)\,, & C_2(\cos\gamma,\,\sin\gamma)\,. \end{array}$$

Then the eight orthocentres are

$$(\pm\cos\alpha\pm\cos\beta\pm\cos\gamma,\,\sin\alpha+\sin\beta+\sin\gamma)\,.$$

(This is a known result; see, for example, L.-S. Hahn, *Complex Numbers and Geometry*, Math. Association of America, 1994, p. 71. It is, however, more easily proved than looked up: If P, Q, and R are three vectors from the origin O to points on a circle centred at O, then the centroid M of the triangle formed by the tips of these vectors is at $(P+Q+R)/3$, and the points O, H, M on the Euler line satisfy $OH = 3OM$.)

Our proof is completed by observing that all eight of the orthocentres lie on the line $y = \sin\alpha+\sin\beta+\sin\gamma$, which is parallel to the given chords.

Also solved by MICHEL BATAILLE, Rouen, France; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany.