# SKOLIAD No. 79

Shawn Godin

Please send your solutions to the problems in this edition by *1 March, 2005*.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

This month's questions are taken from the British Columbia Colleges High School Mathematics Contest 2004.

## BC Colleges High School Math Contest 2004
## Junior Preliminary Round
### Wednesday, March 10, 2004

**1**. Of the 2004 students who wrote a mathematics contest, 239 wore Hip jeans and Fast runners, 252 wore Hip jeans but not Fast runners, and 1213 wore Fast runners but not Hip jeans. The number of students who wore neither is:

(a) 298 (b) 299 (c) 300 (d) 301 (e) 302

**2**. Two logs found in a wood pile are identical in every respect. Using a power saw, Kate takes 9 seconds to cut the first log into four smaller logs. Assuming the time it takes her to make each cut remains constant, the time (in seconds) it takes her to cut the second log into five smaller logs is:

(a) 12 (b) 11.5 (c) 11.25 (d) 11.75 (e) none of these

**3**. Using the standard $xy$–coordinate plane, the area, in square units, of a triangle whose vertices have the coordinates $(0, 0)$, $(1, 5)$, and $(7, 3)$ is:

(a) 15 (b) 16 (c) 17.5 (d) 18 (e) none of these

**4**. A box contains red and blue pencils only. If the number of red pencils is two-thirds the number of blue pencils, then the proportion of pencils in the box that are red is:

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{5}$ (e) $\frac{3}{5}$

**5**. Last year a skateboard cost \$100 and a helmet cost \$40. This year the cost of the skateboard increased by 12% and the cost of the helmet increased by 5%. The increase in the combined cost of the skateboard and the helmet is:

(a) 17% (b) 10% (c) 9.5% (d) 8.5% (e) 7.5%

**6**. Two vertical poles, one $10$ metres high and the other $15$ metres high, stand $12$ metres apart. The distance, in metres, between the tops of the poles is:

(a) $16$ (b) $15$ (c) $14$ (d) $13$ (e) $12$

**7**. The number of 4-digit numbers in which the digits sum to greater than $33$ is:

(a) $18$ (b) $13$ (c) $15$ (d) $11$ (e) none of these

**8**. A large square, of perimeter $20$ centimetres, has double the area of a smaller square. The perimeter, in centimetres, of the smaller square is:

(a) $10$ (b) $10\sqrt{2}$ (c) $20\sqrt{2}$ (d) $40$ (e) none of these

**9**. Define the operation $\star$ to mean $A \star B = \frac{A+2B}{3}$. Then the value of $\big[(4\star 7)\star 8\big] - \big[4\star(7\star 8)\big]$ is:

(a) $-\frac{28}{9}$ (b) $-\frac{2}{9}$ (c) $0$ (d) $\frac{8}{9}$ (e) $\frac{29}{9}$

**10**. A box contains $20$ yellow discs, $9$ red discs, and $6$ blue discs. If discs are selected at random, then the smallest number of discs that need to be selected to be assured of selecting at least two discs of each colour is:

(a) $7$ (b) $17$ (c) $23$ (d) $28$ (e) $31$

**11**. If $\frac{a}{d+b+c} = \frac{4}{3}$ and $\frac{a}{b+c} = \frac{3}{5}$, then the value of $\frac{d}{a}$ is:

(a) $\frac{7}{6}$ (b) $\frac{6}{7}$ (c) $-\frac{12}{11}$ (d) $-\frac{11}{12}$ (e) $\frac{15}{11}$

**12**. Lana has a collection of nickels. When she collects them in groups of three, there is one left over; when she piles them in groups of five, there are two left over; and when she puts them in piles of seven, there are three left over. The sum of the digits of the smallest number of nickels that Lana can have is:

(a) $9$ (b) $7$ (c) $10$ (d) $12$ (e) $3$

**13**. The value of $A+B$ that satisfies

$$\left(6^{30}+6^{-30}\right)\left(6^{30}-6^{-30}\right) = 3^A 8^B - 3^{-A} 8^{-B}$$

is:

(a) $30$ (b) $40$ (c) $60$ (d) $80$ (e) $120$

**14**. Let $x = 0.7181818\ldots$, where the digits '18' repeat. When $x$ is expressed as a fraction in lowest terms, then its denominator exceeds its numerator by:

(a) 18 (b) 31 (c) 93 (d) 141 (e) 279

**15**. A student has three different Mathematics books, two different English books, and four different Science books. The number of ways that the books can be arranged on a shelf, if all books of the same subject are kept together, is:

(a) 288 (b) 864 (c) 1260 (d) 1544 (e) 1728

# BC Colleges High School Math Contest 2004
## Senior Preliminary Round
### Wednesday, March 10, 2004

**1**. Same as question #15 on the Junior Prelimary Round above.

**2**. Same as question #13 on the Junior Prelimary Round above.

**3**. A mixing bowl is hemispherical in shape, with a radius of 12 cm. If it contains water to half its depth, then the angle through which it must be tilted before water will begin to pour out is:

(a) $15°$ (b) $30°$ (c) $45°$ (d) $60°$ (e) $75°$

**4**. The hill behind Antonino's house is long and steep. He can walk down it at $4\frac{1}{2}$ km/hr, but he can walk up it at only $1\frac{1}{2}$ km/hr. If it takes him 6 hours to make the round trip, the distance, in kilometres, from his house to the top of the hill is:

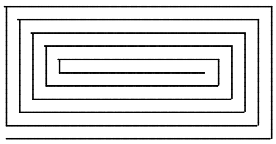
(a) 18 (b) $\frac{27}{2}$ (c) 9 (d) $\frac{27}{4}$ (e) 6

**5**. Same as question #12 on the Junior Prelimary Round above.

**6**. Suppose the line $\ell$ is parallel to the line $y = \frac{3}{4}x + 6$ and four units from it. A possible equation of the line $\ell$ is:

(a) $y = \frac{3}{4}x$ (b) $y = \frac{3}{4}x + 1$ (c) $y = \frac{3}{4}x + 2$
(d) $y = \frac{3}{4}x + 3$ (e) $y = \frac{3}{4}x + \frac{9}{2}$

**7**. You have a rectangular garden twenty metres long and ten metres wide. A one-metre wide path fills up the garden. If you walk along the centre of the path from beginning to end, the number of metres that you walk is:

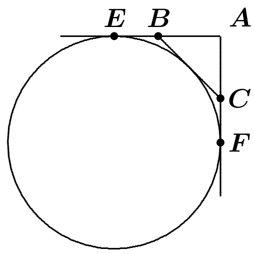


(a) $199\frac{1}{2}$ (b) $200$ (c) $200\frac{1}{2}$ (d) $209\frac{1}{2}$ (e) $210$

**8**. A square grid is made up of a set of parallel lines, 5 cm apart, which are intersected at right angles by another set of parallel lines, also 5 cm apart. If a circular disc of diameter 3 cm is dropped on the square grid, the probability that the disc will not touch a grid line is:

(a) $0.04$ (b) $0.10$ (c) $0.24$ (d) $0.30$ (e) $0.16$

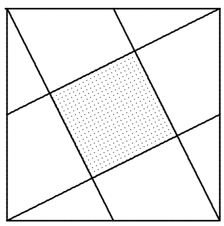
**9**. In the diagram, a circle is tangent to the hypotenuse of the isosceles right triangle $ABC$. Side $AB$ is extended and is tangent to the circle at $E$. Side $AC$ is extended and is tangent to the circle at $F$. If the area of triangle $ABC$ is 9, then the area of the circle is:




(a) $9\pi(3-2\sqrt{2})$ (b) $9\sqrt{2}\pi$ (c) $9\pi(3+2\sqrt{2})$
(d) $18\sqrt{2}\pi$ (e) $36\pi$

**10**. In the diagram at the right, segments join the vertices of a square with area 1 to midpoints of its sides. The area of the shaded quadrilateral is:

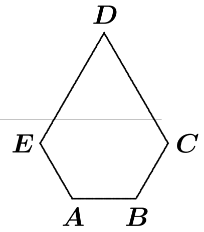
(a) $\frac{1}{2}$ (b) $\frac{2}{5}$ (c) $\frac{1}{3}$
(d) $\frac{1}{4}$ (e) $\frac{1}{5}$



**11**. The longer base of a trapezoid has length 15 and the line segment joining the midpoints of the two diagonals has length 3. The length of the shorter base of the trapezoid is:

(a) $6$ (b) $\frac{15}{2}$ (c) $9$ (d) $10$ (e) $12$

**12**. In the diagram at the right, $\angle A = \angle B = 120°$, $EA = AB = BC = 2$, and $CD = DE = 4$. The area of the pentagon $ABCDE$ is:




(a) $7\sqrt{3}$ (b) $9\sqrt{3}$ (c) $3 + 6\sqrt{3}$
(d) $12$ (e) $6\sqrt{5}$

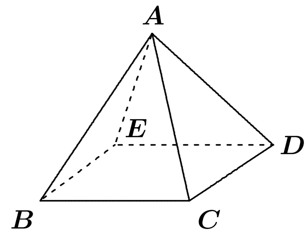
**13**. Each of 600 people have at most twenty \$5 bills; some may have none. Divide the 600 into groups of people with the same number of \$5 bills. The smallest possible maximum size of any of these groups is:

(a) 26 (b) 27 (c) 28 (d) 29 (e) 30

**14**. Consider the positive integers whose first digit is 1 and which have the property that if this digit is transferred to the end of the number, the resulting number is exactly 3 times as large as the original. For example, 139 would be transformed into 391, which is not exactly 3 times as large as 139. If $N$ is the smallest such positive integer, then the remainder when $N$ is divided by 9 is:

(a) 0 (b) 3 (c) 4 (d) 5 (e) 8

**15**. The pyramid $ABCDE$ has a square base, and all four triangular faces are equilateral. The measure of the angle $ABD$ is:




(a) 30° (b) 45° (c) 60°
(d) 75° (e) 90°

# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.



## Mayhem Problems

*Please send your solutions to the problems in this edition by* **1 March 2005**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.*

*The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.*

**M151**. *Proposed by Babis Stergiu, Chalkida, Greece.*

Let $a$, $b$, $c$ be real numbers with $abc = 1$. Prove that

$$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \;\geq\; 2(a^2b + b^2c + c^2a)\,.$$

**M152**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A right-angled triangle has the property that, when a square is drawn externally on each side of the triangle, the six vertices of the squares that are not vertices of the triangle are concyclic. Characterize such triangles.

**M153**. *Proposed by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Two similar triangles $APB$ and $BQC$ are erected externally on a triangle $ABC$. If $R$ is a point such that $PBQR$ is a parallelogram, show that triangles $ARC$ and $APB$ are congruent.

**M154**. *Proposed by the Mayhem Staff.*

Eight rooks are placed randomly on different squares of a chessboard. What is the probability that none of the rooks is under attack by another rook?

**M155**. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine the cubic $x^3 + px + q = 0$, given that

(i) it has one repeated root, and

(ii) $p$ and $q$ are integers such that $q$ is the smallest permissible positive integral multiple of $p$.

**M156**. *Proposed by the Mayhem Staff.*

Solve for $x$ where $0 \leq x < 2\pi$:

$$2^{1+3\cos x} - 10(2)^{-1+2\cos x} + 2^{2+\cos x} - 1 \;=\; 0\,.$$

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**M151**. *Proposé par Babis Stergiu, Chalkida, Grèce.*

Soit $a$, $b$ et $c$ des nombres réels avec $abc = 1$. Montrer que

$$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \;\geq\; 2(a^2b + b^2c + c^2a)\,.$$

**M152**. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Caractériser les triangles rectangles qui ont la propriété suivante : les six sommets extérieurs des carrés construits sur chacun de leurs côtés sont situés sur un même cercle.

**M153**. *Proposé par Yufei Zhao, étudiant, Don Mills Collegiate Institute, Toronto, ON.*

Extérieurement au triangle $ABC$, on construit deux triangles semblables $APB$ et $BQC$. Si $R$ est un point tel que $PBQR$ est un parallélogramme, montrer qu'alors les triangles $ARC$ et $APB$ sont congruents.

**M154**. *Proposé par l'Équipe de Mayhem.*

On place huit tours au hasard sur différentes cases d'un échiquier. Quelle est la probabilité qu'aucune des tours ne soit menacée ?

**M155**. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Déterminer la cubique $x^3 + px + q = 0$, sachant que

(i) elle possède une racine double, et

(ii) $p$ et $q$ sont des entiers tels que $q$ est le plus petit multiple entier positif possible de $p$.

**M156**. *Proposé par l'Équipe de Mayhem.*

Sachant que $0 \leq x < 2\pi$, résoudre l'équation :

$$2^{1+3\cos x} - 10(2)^{-1+2\cos x} + 2^{2+\cos x} - 1 = 0 .$$

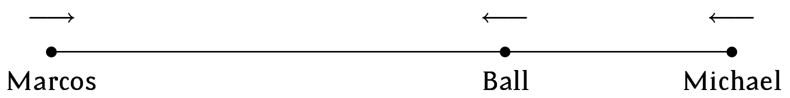
# Problem of the Month

Ian VanderBurgh, University of Waterloo

Sometimes the most interesting and most fun problems are the simplest ones. This type of problem is easy to state, relatively easy to understand (especially after drawing a quick sketch), accessible to virtually anyone, but can be quite confusing to try to solve!

**Problem** (2004 Cayley and Fermat Contests)

A soccer ball rolls at 4 m/s towards Marcos in a direct line from Michael. The ball is 15 m ahead of Michael who is chasing it at 9 m/s. Marcos is 30 m away from the ball and is running towards it at 8 m/s. What is the distance between Michael and Marcos when the ball is touched for the first time by one of them?




Here are three solutions to this problem.

**Solution 1**.

Since the ball is rolling at 4 m/s towards Marcos and Marcos is running towards the ball at 8 m/s, then the distance between Marcos and the ball decreases at a rate of 12 m/s. Since Marcos starts 30 m from the ball, he will reach the ball in 2.5 s.

Since the ball is rolling at 4 m/s away from Michael and Michael is running towards the ball at 9 m/s, then the distance between Michael and the ball decreases at a rate of 5 m/s. Since Michael starts 15 m from the ball, he will reach the ball in 3 s.

Thus, Marcos reaches the ball first. After 2.5 s, the ball has rolled 10 m and Michael has run 22.5 m. Hence, Michael is now 2.5 m away from the ball when Marcos touches it.

**Solution 2**.

Imagine that you were a (very tiny) person standing on top of the soccer ball. Then from your frame of reference, the ball is stationary.

Again from your frame of reference, Marcos starts 30 m away and appears to be running towards you at 12 m/s. Also, Michael starts 15 m away and appears to be running towards you at 5 m/s.

Therefore, Marcos will reach you in 2.5 s and Michael will reach you in 3 s. After 2.5 s, Michael will have closed the gap by 12.5 m, and so will be 2.5 m away from you.

(Of course, this is almost exactly the same solution as Solution 1, but using the fancy terminology of "frame of reference". It would require pretty fancy footwork to stay on top of the rolling ball for that long . . .)

**Solution 3.**

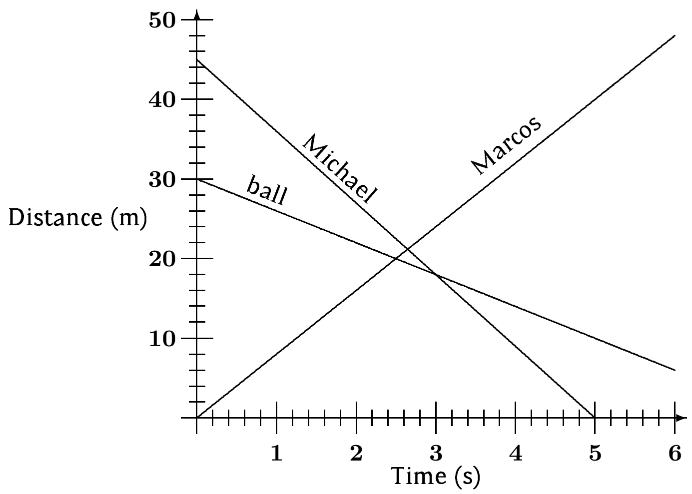
This problem can also be solved using a coordinate axis. Suppose that at time $t = 0$ Marcos is at position $x = 0$, the ball is at position $x = 30$ and Michael is at position $x = 45$.

At a general time $t$, the ball will be at position $x = 30 - 4t$ since it is rolling at 4 m/s towards Marcos. Also, Marcos will be at position $x = 8t$ and Michael will be at position $x = 45 - 9t$.

Marcos meets the ball (if it hasn't been stopped first) when his position is equal to the ball's position; that is, $8t = 30 - 4t$ or $12t = 30$ or $t = 2.5$.

Michael meets the ball (if it hasn't been stopped first) when his position is equal to the ball's position ie. $45 - 9t = 30 - 4t$ or $15 = 5t$ or $t = 3$.

Therefore, Marcos first meets the ball after 2.5 s.




At $t = 2.5$, Marcos and the ball are at position $x = 20$ and Michael is at position $x = 45 - 9(2.5) = 22.5$, which is 2.5 m away.

# THE OLYMPIAD CORNER

## No. 239

R.E. Woodrow

I slipped up and used the same problem set a second time in the May number of the *Corner*. The 2000 Taiwan Mathematical Olympiad appears both in [2004 : 203] and [2003 : 88]. Thanks go to Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON for being the first to notice the repetition.

We begin this number of the *Corner* after the summer break with the problems of the 11th Japanese Mathematical Olympiad (2001), Second Round. Thanks go to Christopher Small, Canadian Team Leader to the 42nd IMO, for collecting them for our use.

## 11th JAPANESE MATHEMATICAL OLYMPIAD

### Second Round

**February 11, 2001**

**1**. An $m \times n$ chessboard is given. Each square is painted black or white in such a way that for every black square, the number of black squares adjacent to it is odd. Prove that the number of black squares is even. (Two squares are *adjacent* if they are different and have a common edge.)

**2**. A positive integer $n$ is written in decimal notation as $a_m a_{m-1} \cdots a_1$; that is,

$$n = 10^{m-1}a_m + 10^{m-2}a_{m-1} + \cdots + a_1 ,$$

where $a_m, a_{m-1}, \ldots, a_1 \in \{0, 1, \ldots, 9\}$ and $a_m \neq 0$. Find all $n$ such that

$$n = (a_m + 1) \times (a_{m-1} + 1) \times \cdots \times (a_1 + 1) .$$

**3**. Three real numbers $a, b, c \geq 0$ satisfy

$$a^2 \leq b^2 + c^2, \qquad b^2 \leq c^2 + a^2, \qquad c^2 \leq a^2 + b^2 .$$

Prove the inequality

$$(a + b + c)(a^2 + b^2 + c^2)(a^3 + b^3 + c^3) \geq 4(a^6 + b^6 + c^6) .$$

When does equality hold?

**4**. Let $p$ be a prime number and $m$ a positive integer. Show that there exists a positive integer $n$ such that the decimal representation of $p^n$ contains a string of $m$ consecutive 0s.

**5**. Two triangles $ABC$ and $PQR$ have the following properties:

(i) $A$ and $P$ are the mid-points of the segments $QR$ and $BC$, respectively.

(ii) $QR$ and $BC$ are the bisectors of $\angle BAC$ and $\angle QPR$, respectively.

Prove that $AB + AC = PQ + PR$.

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Next we give the problems of the 14th Mexican Mathematical Olympiad (2000). Thanks again go to Christopher Small, Canadian Team Leader to the 42nd IMO, for collecting them for our use.

## 14th MEXICAN MATHEMATICAL OLYMPIAD
### Day 1
#### November 13, 2000

**1**. Let $A$, $B$, $C$, and $D$ be circles such that (i) $A$ and $B$ are externally tangent at $P$, (ii) $B$ and $C$ are externally tangent at $Q$, (iii) $C$ and $D$ are externally tangent at $R$, and (iv) $D$ and $A$ are externally tangent at $S$. Assume that $A$ and $C$ do not intersect and that $B$ and $D$ do not intersect.

(a) Prove that $P$, $Q$, $R$, and $S$ lie on a circle.

(b) Assume further that $A$ and $C$ have radius $2$, $B$ and $D$ have radius $3$, and the distance between the centres of $A$ and $C$ is $6$. Determine the area of $PQRS$.

**2**. A triangle like the one shown is constructed with the numbers from 1 to 2000 in the first row. Each number in the triangle, except those in the first row, is the sum of the two numbers above it. What number occupies the lowest vertex of the triangle? (Write your final answer as a product of primes.)

1 2 3 4 5
3 5 7 9
8 12 16
20 28
48

**3**. Given a set $A$ of positive integers, a set $A'$ is constructed consisting of all elements of $A$ as well as all positive integers that can be obtained as follows: some elements of $A$ are chosen, without repetition, and for each of them a sign ($+$ or $-$) is chosen; the signed numbers are then added and the result is placed in $A'$. For example, if $A = \{2, 8, 13, 20\}$, then two elements of $A'$ are 8 and 14 (since 8 belongs to $A$ and $14 = 20 + 2 - 8$). From $A'$, a set $A''$ is constructed in the same fashion as $A'$ is constructed from $A$. What is the minimum number of elements that $A$ must have if $A''$ is to contain all integers from 1 to 40 (including 1 and 40)?

## Day 2
**November 14, 2000**

**4**. For positive integers $a$ and $b$, with $b$ not divisible by $5$, a list of numbers is constructed as follows: the first number is $5$, and every number after that is obtained by multiplying the preceding number (in the list) by $a$ and adding $b$ to the product. For example, if $a = 2$ and $b = 4$, then the first three numbers in the list are $5$, $14$, $32$ (since $14 = 5 \cdot 2 + 4$ and $32 = 14 \cdot 2 + 4$). What is the maximum number of primes that may be produced before the first non-prime?

**5**. An $n \times n$ square is divided into unit squares and painted black and white in a checkerboard pattern. The following operation may be performed on the board: choose a sub-rectangle whose side lengths are both odd or both even, but not both $1$, and reverse the colours of the unit squares in this rectangle (that is, black squares become white and white squares become black).

Find all values of $n$ for which it is possible to make all unit squares the same colour by a finite sequence of operations.

**6**. Let $ABC$ be a triangle with $\angle B > 90^\circ$ such that, for some point $H$ on $AC$, we have $AH = BH$, and $BH$ is perpendicular to $BC$. Let $D$ and $E$ be the mid-points of $AB$ and $BC$, respectively. Through $H$ a parallel to $AB$ is drawn, intersecting $DE$ at $F$. Prove that $\angle BCF = \angle ACD$.

Next we present some problems in French from the Midi Finale 2001 and the Maxi Finale 2001 of the Vingt-sixième Olympiade Mathématique Belge. Thanks again go to Christopher Small, Canadian Team Leader to the 42nd IMO for collecting them for our use.

## 26e OLYMPIADE MATHÉMATIQUE BELGE
### Midi Finale
**25 avril 2001**

**1**. Les parallélogrammes $ABCD$ et $AEFG$ sont tels que $E$ appartient à la droite $BC$ et $D$ à la droite $FG$. Comparer les aires de ces deux parallélogrammes. Sont-elles égales ? L'une est-elle toujours plus grande que l'autre ? Si oui, laquelle ?

**2**. Trouver tous les entiers $x$ pour lesquels $\sqrt{x}$ et $\sqrt{x - \sqrt{x}}$ sont eux-mêmes des entiers.

**3**. Déterminer tous les triplets $(x, y, z)$ constitués d'entiers naturels qui satisfont à l'équation

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1\,.$$

**4**. (a) Quelles sont toutes les valeurs prises par le reste (par défaut) de la division de $a^3$ par $7$ lorsque $a$ est un entier naturel quelconque ?

(b) Les entiers naturels $a$, $b$ et $c$ sont tels que $a^3 + b^3 + c^3$ est divisible par $7$. Que vaut alors le reste de la division de $a \cdot b \cdot c$ par $7$ ?

## Maxi Finale

**25 avril 2001**

**1**. Étant donné un rectangle $ABCD$, déterminer deux points $K$ et $L$ respectivement sur $[BC]$ et sur $[CD]$ tels que les triangles $ABK$, $AKL$ et $ADL$ aient la même aire.

**2**. Trouver toutes les solutions du système suivant d'inconnues réeles $x$, $y$, $u$ et $v$.

$$\begin{array}{rclcrcl} x^2+y^2 &=& 1\,, & \quad & xu+yv &=& 1\,, \\ u^2+v^2 &=& 1\,, & & xu-yv &=& \frac{1}{2}\,. \end{array}$$

**3**. Déterminer tous les réels $r$ tels que $-20$, $1$, $10$ et $r$ soient les quatre solutions d'une équation de la forme $p(q(x)) = 0$ dans laquelle $p(x)$ et $q(x)$ sont des trinômes du second degré.

**4**. Les entiers $a_0$, $a_1$, $a_2$, ..., $a_{100}$ satisfont aux conditions suivantes :

$$\begin{array}{rcl} a_1 &>& a_0 \ \geq\ 0\,, \\ a_{k+2} &=& 3a_{k+1} - 2a_k \quad (\text{pour } k = 0, 1, 2, \ldots, 98)\,. \end{array}$$

Comparer les nombres $a_{100}$ et $2^{99}$. Sont-ils égaux ? L'un est-il toujours plus grand que l'autre ? Si oui, lequel ?

We have received the following short solution to problem 2 of the Final (Selection) Round of the Estonian Mathematical Contests 1995–96 [2000 : 6]. A longer solution was given previously [2002 : 75–76].

**2**. Let $a$, $b$, $c$ be the sides of a triangle and $\alpha$, $\beta$, $\gamma$ the opposite angles of the sides, respectively. Prove that if the inradius of the triangle is $r$, then $a\sin\alpha + b\sin\beta + c\sin\gamma \geq 9r$.

*Alternate solution by Vedula N. Murty, Dover, PA, USA.*

Letting $\Delta$ be the area of the triangle, we have the known formulae $\sin\alpha = \frac{2\Delta}{bc}$, $\sin\beta = \frac{2\Delta}{ca}$, $\sin\gamma = \frac{2\Delta}{ab}$, and $r = \frac{\Delta}{s} = \frac{2\Delta}{a+b+c}$. Using these formulae, we find that the inequality to be proved is equivalent to

$$\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right)(a+b+c) \ \geq\ 9\,,$$

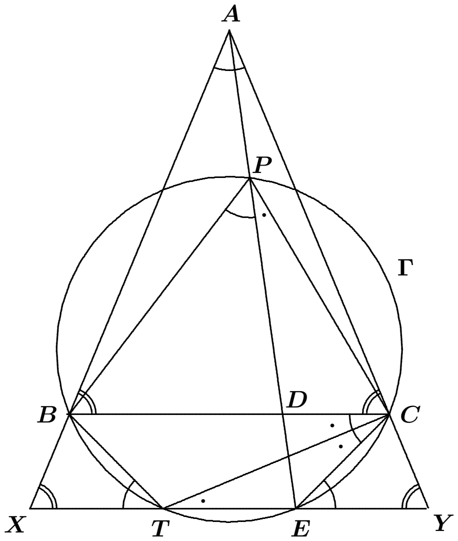
which can be proved by applying the AM–GM Inequality to each factor on the left side.

Now we turn to the March 2002 *Corner* and solutions to problems of the VI$^{th}$ Turkish Mathematical Olympiad, Second Round [2002 : 65].

**1**. On the base of the isosceles triangle $ABC$ ($|AB| = |AC|$) we choose a point $D$ such that $|BD| : |DC| = 2 : 1$, and on $[AD]$ we choose a point $P$ such that $m(\widehat{BAC}) = m(\widehat{BPD})$.

Prove that $m(\widehat{DPC}) = m(\widehat{BAC})/2$.

*Solved by Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Lefkogia, Crete, Greece; Christopher J. Bradley, Bristol, UK; Toshio Seimiya, Kawasaki, Japan; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the solution of Seimiya.*




We denote the circumcircle of $\triangle PBC$ by $\Gamma$. Let $E$ be the second intersection of $PD$ with $\Gamma$. The line through $E$ parallel to $BC$ meets $\Gamma$ at $T$, and it meets $AB$ and $AC$ at $X$ and $Y$, respectively.

Then $XE : EY = BD : DC = 2 : 1$. Hence, we get

$$XE = 2EY\,. \tag{1}$$

Note that $\angle CEY = \angle BCE = \angle BPE = \angle BPD = \angle BAC$ and $\angle CYE = \angle ACB$. It follows that $\triangle ECY \sim \triangle ABC$. Thus,

$$\angle ECY = \angle ABC = \angle ACB = \angle CYE\,.$$

Hence, we have $EC = EY$. Since $\angle BTX = \angle BPE = \angle BPD = \angle BAC$ and $\angle BXT = \angle ABC$, it follows that

$$\angle XBT = \angle ACB = \angle ABC = \angle BXT\,;$$

whence, $TB = TX$.

Since $TE \parallel BC$ and $\angle BTX = \angle CEY$, we get $TB = EC$. Thus, we have $TX = TB = EC = EY$. Using (1), we get $XE = 2TX$. Then $TE = XE - TX = TX = EY = EC$. Therefore,

$$\angle CPE = \angle CTE = \angle ECT = \tfrac{1}{2}\angle CEY = \tfrac{1}{2}\angle BAC .$$

This implies that $\angle DPC = \frac{1}{2}\angle BAC$.

**2**. Prove that

$$(a+3b)(b+4c)(c+2a) \geq 60abc$$

for all real numbers $0 \leq a \leq b \leq c$.

*Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; Pavlos Maragoudakis, Lefkogia, Crete, Greece; Vedula N. Murty, Dover, PA, USA; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the solution by Murty.*

The AM–GM Inequality gives us

$$\frac{a+3b}{4} \geq a^{\frac{1}{4}}b^{\frac{3}{4}}, \qquad \frac{b+4c}{5} \geq b^{\frac{1}{5}}c^{\frac{4}{5}}, \qquad \frac{c+2a}{3} \geq c^{\frac{1}{3}}a^{\frac{2}{3}} .$$

These three inequalities imply that

$$(a+3b)(b+4c)(c+2a) \geq 60a^{\frac{11}{12}}b^{\frac{19}{20}}c^{\frac{17}{15}} .$$

Now, since $0 \leq a \leq b \leq c$, we have

$$c^{\frac{17}{15}} = c^{\frac{1}{12}}c^{\frac{1}{20}}c \geq a^{\frac{1}{12}}b^{\frac{1}{20}}c ,$$

and hence, $60a^{\frac{11}{12}}b^{\frac{19}{20}}c^{\frac{17}{15}} \geq 60abc$. This proves the proposed inequality.

**3**. The points of a circle are coloured by three colours. Prove that there exist infinitely many isosceles triangles with vertices on the circle and of the same colour.

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give Sinefakopoulos' write-up.*

We recall the finite form of Van der Waerden's Theorem: For any positive integers $k$ and $r$, there exists a minimum positive integer $n = W(k,r)$ such that, if we colour $\{1, 2, \ldots, n\}$ with $r$ colours, then there is a monochromatic $k$–term arithmetic progression.

It is known that $W(3,3) = 27$. Therefore, by Van der Waerden's Theorem, if we colour a regular 27–gon with three colours, we must produce a monochromatic isosceles triangle. Infinitely many different regular 27–gons may be inscribed in the circle given in the problem. Therefore, there exist infinitely many monochromatic isosceles triangles with vertices on the circle.

**4**. Determine all positive integers $x$, $n$ satisfying the equation $x^3 + 3367 = 2^n$.

*Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Lefkogia, Crete, Greece; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give Bataille's write-up.*

One solution is $x = 9$, $n = 12$, since $9^3 + 3367 = 4096 = 2^{12}$. We will prove that this is the only solution.

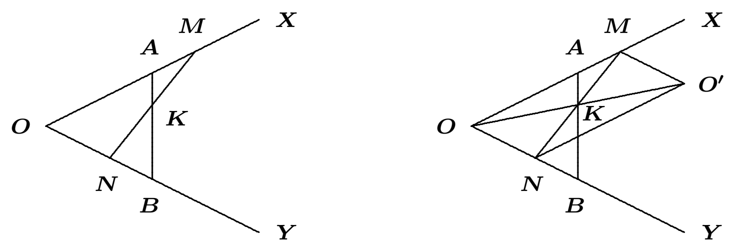
Let $x$, $n$ satisfy the given equation. Since $3367 = 7 \times 13 \times 37$, we have $x^3 \equiv 2^n \pmod 7$. This implies that $n$ is a multiple of 3, since otherwise $2^n$ is congruent to 2 or 4 modulo 7, while the cube of an integer is congruent to 0, 1, or 6 modulo 7. Thus, $n = 3m$ for some positive integer $m$. The equation becomes $(2^m)^3 - x^3 = 3367$; that is,

$$(2^m - x)\big((2^m - x)^2 + 3x \cdot 2^m\big) \;=\; 3367\,. \qquad (1)$$

It follows that $d = 2^m - x$ is a divisor of 3367 such that $d^3 < 3367$. The only possibilities are $d = 1$, $d = 7$, or $d = 13$. The case $d = 1$ transforms (1) into $2^m(2^m - 1) = 2 \times 561$, which clearly has no solution, and the case $d = 13$ transforms (1) into $2^m(2^m - 13) = 2 \times 15$, which also has no solution. That leaves $d = 7$, which yields $2^m(2^m - 7) = 2^4 \times 3^2$. Hence, $m = 4$. As a result, $n = 3m = 12$ and $x = 9$.

**5**. Given the angle $XOY$, variable points $M$ and $N$ are considered on the arms $[OX]$ and $[OY]$, respectively, so that $|OM| + |ON|$ is constant. Determine the geometric locus of the mid-point of $[MN]$.

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; Pavlos Maragoudakis, Lefkogia, Crete, Greece; and Toshio Seimiya, Kawasaki, Japan. We give the solution of Maragoudakis.*




Let $OM + ON = 2a$, and let $A \in OX$ and $B \in OY$ such that $OA = OB = a$. Then $AM = NB$. (See the diagram on the left above.) We will prove that the geometric locus of the mid-point of $MN$ is $[AB]$.

Let $K = AB \cap MN$. Applying the Theorem of Menelaus to $\triangle OMN$ with $AKB$ as transversal, we get

$$\frac{AM}{AO} \cdot \frac{KN}{KM} \cdot \frac{BO}{BN} = 1,$$

which implies that $KN = KM$. Thus, the mid-point of $MN$ belongs to $AB$.

Conversely, if $K \in [AB]$, let $O'$ be such that $K$ is the mid-point of $OO'$. (See the diagram on the right above.) Let $M \in OX$ and $N \in OY$ such that $O'M \parallel OY$ and $O'N \parallel OX$. Then $ONO'M$ is a parallelogram. Applying the Theorem of Menelaus to $\triangle OMN$, we deduce that $AM = BN$. Then $OM + ON = a + AM + a - BN = 2a$. Thus, $K$ belongs to the geometric locus.

Now we turn to solutions to problems of the Turkish Team Selection Examination for the 40th IMO given [2002 : 66–67].

**1**. Let $m \leq n$ be positive integers and $p$ be a prime. Let $p$–expansions of $m$ and $n$ be

$$\begin{aligned} m &= a_0 + a_1 p + \cdots + a_r p^r, \\ n &= b_0 + b_1 p + \cdots + b_s p^s, \end{aligned}$$

respectively, where $a_r$, $b_s \neq 0$, for all $i \in \{0, 1, \ldots, r\}$ and for all $j \in \{0, 1, \ldots, s\}$, we have $0 \leq a_i, b_j \leq p - 1$.

If $a_i \leq b_i$ for all $i \in \{0, 1, \ldots, r\}$, we write $m \prec_p n$. Prove that

$$p \nmid \binom{n}{m} \iff m \prec_p n.$$

*Solved by Pierre Bornsztein, Maisons-Laffitte, France.*

It will be more convenient to write $n = b_0 + b_1 p + \cdots + b_r p^r$, where $b_{s+1} = \cdots = b_r = 0$ if $r > s$. With the usual convention that $\binom{b}{a} = 0$ when $b < a$, we deduce that, if $p$ is a prime and $0 \leq a, b \leq p - 1$, then $\binom{b}{a} \equiv 0 \pmod{p}$ if and only if $b < a$. The desired result is now an immediate consequence of the following Theorem of Lucas (see [1]): If $p$ is a prime, then

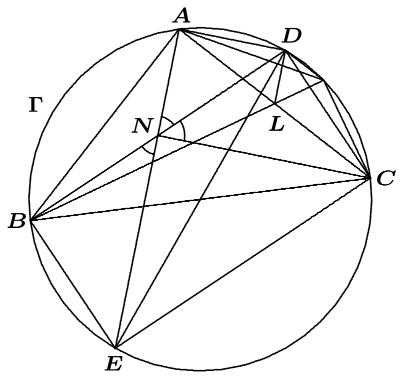
$$\binom{n}{m} \equiv \binom{b_0}{a_0}\binom{b_1}{a_1} \cdots \binom{b_r}{a_r} \pmod{p}.$$

**Reference:**

[1] T. Andreescu, R. Gelca, *Mathematical Olympiad Challenges*, Birkhäuser, 2000, p. 84.

**2**. Let $L$ and $N$ be the mid-points of the diagonals $[AC]$ and $[BD]$ of the cyclic quadrilateral $ABCD$, respectively. If $BD$ is the bisector of the angle $ANC$, then prove that $AC$ is the bisector of the angle $BLD$.

*Solved by Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.*




We denote the circumscribed circle of $ABCD$ by $\Gamma$. Let $E$ be a point on $\Gamma$ such that $CE \parallel BD$. Since $CE \parallel BD$, we have $BE = DC$ and $\angle NBE = \angle NDC$. Also, we are given that $BN = DN$. Therefore, $\triangle BNE \cong \triangle DNC$, which implies that $\angle BNE = \angle DNC = \angle DNA$. Hence, $A$, $N$, and $E$ are collinear.

Since $AE$ bisects $BD$, triangles $BAE$ and $DAE$ have the same area. Thus,

$$\tfrac{1}{2}AB \cdot BE \sin \angle ABE = \tfrac{1}{2}AD \cdot DE \sin \angle ADE .$$

Since $\angle ABE + \angle ADE = 180°$, we have $\sin \angle ABE = \sin \angle ADE$, and consequently,

$$AB \cdot BE = AD \cdot DE .$$

Since $CE \parallel BD$, we have $BE = CD$ and $DE = BC$. Thus, the above equation becomes

$$AB \cdot CD = AD \cdot BC . \qquad (1)$$

Let $S$ be the point on $\Gamma$ such that $DS \parallel AC$. Then $AD = CS$ and $CD = AS$. It follows from (1) that $AB \cdot AS = CS \cdot BC$. This, together with the relation $\angle BAS + \angle BCS = 180°$, implies that triangles $ABS$ and $CBS$ have the same area. Therefore, $BS$ bisects $AC$. Consequently, $BS$ passes through $L$.

Since $DS \parallel AC$, we have $AD = CS$ and $\angle DAL = \angle SCL$. Hence, $\triangle ALD \cong \triangle CLS$, and therefore,

$$\angle ALD = \angle CLS = \angle ALB .$$

Thus, $AC$ bisects $\angle BLD$.

**3**. Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that the set

$$\left\{ \frac{f(x)}{x} : x \neq 0 \quad \text{and} \quad x \in \mathbb{R} \right\}$$

is finite, and for all $x \in \mathbb{R}$

$$f\big(x - 1 - f(x)\big) \;=\; f(x) - x - 1\,.$$

*Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the solution of Sinefakopoulos.*

We shall show that $f$ must be the identity function on $\mathbb{R}$. Note that the identity satisfies all the given conditions.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the given conditions, and define the set

$$Y \;=\; \{y \mid y = x - f(x)\,, \quad x \in \mathbb{R}\}\,.$$

If $y \in Y$, then $y = x - f(x)$ for some $x \in \mathbb{R}$, and hence,

$$f(y-1) \;=\; f\big(x - f(x) - 1\big) \;=\; f(x) - x - 1 \;=\; y - 1\,. \qquad (1)$$

Then $2y = (y-1) - f(y-1)$, and therefore $2y \in Y$.

Consider any fixed $a \in Y$. An easy induction implies that $2^n a \in Y$ for all $n = 0, 1, 2, \ldots$. From the given conditions on $f$, we know that the following set is finite:

$$\left\{ \frac{f(2^n a - 1)}{2^n a - 1} : n \in \mathbb{N}\,,\, a \neq 2^{-n} \right\}\,.$$

Therefore, we may choose two positive integers $m$ and $n$, with $m \neq n$, $a \neq 2^{-m}$, and $a \neq 2^{-n}$ such that

$$\frac{f(2^n a - 1)}{2^n a - 1} \;=\; \frac{f(2^m a - 1)}{2^m a - 1}\,.$$

Using (1), we have

$$\frac{-2^n a - 1}{2^n a - 1} \;=\; \frac{-2^m a - 1}{2^m a - 1}\,,$$

which forces $2^n a = 2^m a$, and hence $a = 0$. Then $Y = \{0\}$, proving that $f$ is the identity function.

*Comment*: This problem is slightly more general than Mathematics Magazine problem 1491, Vol. 70 (1997), No. 1, p. 68.

**6**. Prove that the plane is not a union of the inner regions of finitely many parabolas. (The outer region of a parabola is the union of the lines not intersecting the parabola. The inner region of a parabola is the set of points of the plane that do not belong to the outer region of the parabola.)

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the solution by Sinefakopoulos.*

Suppose, to the contrary, that the plane is the union of the inner regions of finitely many parabolas, say $P_1$, $P_2$, …, $P_k$. For each $i = 1, 2, \ldots, k$, let $\ell_i$ be the axis of symmetry of the parabola $P_i$. Choose a line $\ell$ which is *not* parallel to any of the lines $\ell_1$, $\ell_2$, …, $\ell_k$. This choice ensures us that if $\ell$ meets a parabola $P_i$ at all, then either it is tangent to $P_i$ or it meets $P_i$ at two distinct points $A_i$ and $B_i$. By renumbering, if necessary, assume that $P_1$, $P_2$, …, $P_m$ ($m \le k$) are the parabolas that meet line $\ell$ at two distinct points. Since every point on $\ell$ must be in the inner region of at least one of the parabolas $P_1$, $P_2$, …, $P_k$, we find that

$$\text{length of } \ell \;\le\; \sum_1^m \|A_iB_i\| \;<\; +\infty\,,$$

an obvious contradiction.

Next we give solutions to problems of the Japanese Mathematical Olympiad 1999, Final Round, given [2002 : 67].

**2**. Let $f(x) = x^3 + 17$. Prove that for each natural number $n$, $n \ge 2$, there is a natural number $x$, for which $f(x)$ is divisible by $3^n$ but not by $3^{n+1}$.

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Pavlos Maragoudakis, Lefkogia, Crete, Greece. We give Maragoudakis' solution.*

First we prove two lemmas.

**Lemma 1**. Let $n \ge 2$. Let $x \in \mathbb{N}$ such that $x^3 + 17 \equiv 0 \pmod{3^n}$ and $x^3 + 17 \not\equiv 0 \pmod{3^{n+1}}$. Then either

$$\begin{aligned} (x + 3^{n-1})^3 + 17 &\equiv 0 \pmod{3^{n+1}} \\ \text{or} \quad (x + 2\cdot 3^{n-1})^3 + 17 &\equiv 0 \pmod{3^{n+1}}\,. \end{aligned}$$

*Proof*: Since $3 \mid (x^3 + 17)$, we see that $x \equiv 1 \pmod 3$. Then $x^2 = 3m + 1$ for some $m \in \mathbb{N}$. Also, $n \ge 2$ gives $2n - 1 \ge n + 1$ and $3n - 3 \ge n + 1$. By assumption, we have $x^3 + 17 = 3^n k$ and $k = 3q + 1$ or $k = 3q + 2$.

If $k = 3q + 1$, then

$$\begin{aligned} (x + 2\cdot 3^{n-1})^3 + 17 &= x^3 + 17 + 2\cdot x^2 \cdot 3^n + 4\cdot x\cdot 3^{2n-1} + 8\cdot 3^{3n-3} \\ &= 3^n(3q + 1 + 6m + 2 + 3y) \\ &= 3^{n+1}(q + 2m + y + 1) \\ &\equiv 0 \pmod{3^{n+1}}\,. \end{aligned}$$

If $k = 3q+2$, then

$$\begin{aligned}(x+3^{n-1})^3+17 &= x^3+17+x^2\cdot 3^n + x\cdot 3^{2n-1}+3^{3n-3}\\ &= 3^n(3q+2+3m+1+3y)\\ &= 3^{n+1}(q+m+y+1)\\ &\equiv 0 \pmod{3^{n+1}}\ .\end{aligned}$$

■

**Lemma 2**. Let $n \geq 2$. Let $x \in \mathbb{N}$ such that $x^3+17 \equiv 0 \pmod{3^{n+1}}$. Then

$$\begin{aligned}(x+3^{n-1})^3+17 &\equiv 0 \pmod{3^n}\\ \text{and}\qquad (x+3^{n-1})^3+17 &\not\equiv 0 \pmod{3^{n+1}}\ .\end{aligned}$$

*Proof*: As in the proof of Lemma 1, we have $x^2 = 3m+1$ for some $m \in \mathbb{N}$. Since $n \geq 2$, we have $2n-1 \geq n+1$ and $3n-3 \geq n+1$. By assumption, $x^3+17 = 3^{n+1}\cdot k$. Hence,

$$\begin{aligned}(x+3^{n-1})^3+17 &= 3^{n+1}\cdot k + 3^n\cdot x^2 + 3^{2n-1}\cdot x + 3^{3n-3}\\ &= 3^n(3k+3m+1+3y)\,,\end{aligned}$$

from which the desired result follows. ■

Now we solve the given problem by constructing a sequence of natural numbers $x_n$ such that $3^n \mid (x_n^3+17)$ and $3^{n+1} \nmid (x_n^3+17)$, for all $n$. We proceed recursively, starting the sequence with $x_2 = 1$. Note that

$$x_2^3+17 = 18 \equiv 0 \pmod{3^2} \quad\text{and}\quad x_2^3+17 \not\equiv 0 \pmod{3^3}\ .$$

Suppose $x_n$ has been defined such that $3^n \mid (x_n^3+17)$ and $3^{n+1} \nmid (x_n^3+17)$. By Lemma 1, we can let $\tilde{x}_{n+1} = x_n + 3^{n-1}$ or $\tilde{x}_{n+1} = x_n + 2\cdot 3^{n-1}$ to obtain $3^{n+1} \mid (\tilde{x}_{n+1}^3+17)$. Then we define

$$x_{n+1} = \begin{cases}\tilde{x}_{n+1} & \text{if } 3^{n+2} \nmid (\tilde{x}_{n+1}^3+17),\\ \tilde{x}_{n+1}+3^n & \text{if } 3^{n+2} \mid (\tilde{x}_{n+1}^3+17).\end{cases}$$

By Lemma 2, we see that $3^{n+1} \mid (x_{n+1}^3+17)$ but $3^{n+2} \nmid (x_{n+1}^3+17)$.

**3**. Let $2n+1$ weights ($n$ is a natural number, $n \geq 1$) satisfy the following condition.

*Condition*: If any one weight is excluded, then the remaining $2n$ weights can be divided into a pair of $n$ weights that balance each other.

Prove that all the weights are equal in this case.

*Solution by Pierre Bornsztein, Maisons-Laffitte, France.*

Let $a_1, a_2, \ldots, a_{2n+1}$ be the weights. For each $i \in \{1, 2, \ldots, 2n+1\}$, when we exclude $a_i$, we may write an equation of the form

$$\sum_{j\in A} a_j - \sum_{j\in B} a_j = 0\,, \tag{1}$$

where $A$ and $B$ are two disjoint subsets of $\{1, 2, \ldots, 2n+1\} - \{i\}$, each of size $n$.

Now consider the $(2n+1) \times (2n+1)$ matrix $M$ whose entries in row $i$ are the coefficients of $a_1, a_2, \ldots, a_{2n+1}$ in (1), for each $i$. The main diagonal of $M$ contains only 0, and the off-diagonal entries of $M$ are 1 or $-1$, with exactly $n$ entries equal to 1 in each row. Let

$$V = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n+1} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

By construction, we have $MV = 0$, and it is easy to see that $MU = 0$. Then $M$ has rank at most $2n$. The desired conclusion will follow from the fact that $M$ has rank equal to $2n$.

Let us consider the matrix $M$ modulo 2 (that is, each entry of $M$ is regarded as an integer modulo 2). Then all of the off-diagonal entries are 1. We perform row reduction by adding the first row of $M$, modulo 2, to each of the other rows, thereby obtaining the matrix

$$\widetilde{M} = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The matrices $M$ and $\widetilde{M}$ must have the same rank over the field of integers modulo 2, and we can see from $\widetilde{M}$ that this rank is at least $2n$.

Thus, $M$ modulo 2 has rank at least $2n$. It follows that there is a minor of $M$ with size $2n \times 2n$ which is non-zero modulo 2, and hence non-zero. Thus, $M$ has rank at least $2n$, and the conclusion follows.

**4**. Prove that

$$f(x) = (x^2+1^2)(x^2+2^2)(x^2+3^2)\cdots(x^2+n^2)+1$$

cannot be expressed as a product of two integral-coefficient polynomials with degree greater than 1.

*Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.*

For the purpose of contradiction, suppose that $f(x) = P(x) \cdot Q(x)$ where $P(x)$ and $Q(x)$ are integral-coefficient polynomials whose degrees are greater than 1.

The complex number $ki$ satisfies the equation $P(ki)Q(ki) = 1$ for $k = \pm 1, \pm 2, \ldots, \pm n$. Since the complex numbers $P(ki)$, $Q(ki)$ are of the form $a + bi$ with $a$, $b \in \mathbb{Z}$, we must have

$$\big(P(ki), Q(ki)\big) \;\in\; \{(1,1)\,,\,(-1,-1)\,,\,(i,-i)\,,\,(-i,i)\}\;. \tag{1}$$

In all cases, $P(ki) = \overline{Q(ki)} = Q(-ki)$. Thus, the polynomial $P(x)-Q(-x)$ has at least $2n$ distinct roots (the complex numbers $\pm i$, $\pm 2i, \ldots, \pm ni$), while its degree is less than $2n$. Therefore, $P(x) - Q(-x)$ is the zero polynomial; that is, $P(x) = Q(-x)$. Hence,

$$\deg P(x) = \deg Q(x) = n\,.$$

Since $f(x)$ is monic (that is, having 1 as the coefficient of the highest power of $x$), we may suppose that $P(x)$ and $Q(x)$ are both monic. Then the polynomial $\big(P(x)\big)^2 - \big(Q(x)\big)^2$ has degree less than $2n$. This polynomial has at least the $2n$ distinct roots $ki$ (for $k = \pm 1$, $\pm 2$, $\ldots$, $\pm n$), because of (1). Hence, $\big(P(x)\big)^2 - \big(Q(x)\big)^2 = 0$. We cannot have $P(x) = -Q(x)$, since $P(x)$ and $Q(x)$ are both monic. We conclude that $P(x) = Q(x)$. Then $f(x) = (P(x))^2$. This implies that $(P(0))^2 = f(0) = (n!)^2 + 1$, which is impossible with $P(0) \in \mathbb{Z}$ and $n \geq 1$. This contradiction establishes the result.

Next we look at readers' solutions to problems given in the April 2002 number of the *Corner* beginning with the Swiss Mathematical Contest (1999) [2002 : 129].

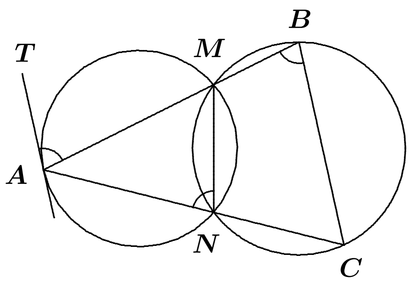
**1**. Two circles intersect each other in points $M$ and $N$. An arbitrary point $A$ of the first circle, which is not $M$ or $N$, is connected with $M$ and $N$, and the straight lines $AM$ and $AN$ intersect the second circle again in the points $B$ and $C$. Prove that the tangent to the first circle at $A$ is parallel to the straight line $BC$.

*Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.*

Let $AT$ be the tangent to the first circle at $A$. Since

$$\angle TAM \;=\; \angle ANM \;=\; \angle MBC\,,$$

we have $\angle TAB = \angle ABC$. Thus, $AT \parallel BC$.



**2**. Is it possible to partition the set $\{1, 2, \ldots, 33\}$ into 11 disjoint subsets, each with three elements, such that in each subset one of the elements is the sum of the other two elements?

*Solved by Michel Bataille, Rouen, France; and Robert Bilinski, Outremont, QC. We give Bataille's write-up.*

No such partition is possible.

Suppose, for the purpose of contradiction, that the set $\{1, 2, \ldots, 33\}$ can be partitioned into the pairwise disjoint sets $A_1, A_2, \ldots, A_{11}$, where $A_i = \{a_i, b_i, c_i\}$ and $a_i + b_i = c_i$, for $i = 1, 2, \ldots, 11$.

Then
$$\sum_{k=1}^{33} k \;=\; \frac{33 \times 34}{2} \;=\; 561\,,$$
which is odd. On the other hand,
$$\sum_{k=1}^{33} k \;=\; \sum_{k=1}^{11} (a_i + b_i + c_i) \;=\; 2\sum_{k=1}^{11} c_i\,,$$
which is even. We have a contradiction.

**3**. Determine all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, satisfying
$$\frac{1}{x} f(-x) + f\left(\frac{1}{x}\right) = x \text{ for all } x \in \mathbb{R} \setminus \{0\}.$$

*Solved by Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's write-up.*

Let $f$ be a solution. Replacing $x$ by $-1/x$ in the given equation, we get
$$-xf\left(\frac{1}{x}\right) + f(-x) \;=\; -\frac{1}{x}\,,$$
and hence
$$-f\left(\frac{1}{x}\right) + \frac{1}{x} f(-x) \;=\; -\frac{1}{x^2}\,.$$
Adding this equation to the given equation gives $\frac{2}{x} f(-x) = x - \frac{1}{x^2}$. Then $f(-x) = \frac{x^2}{2} - \frac{1}{2x}$, and finally, $f(x) = \frac{x^2}{2} + \frac{1}{2x}$ for all $x \in \mathbb{R}\setminus\{0\}$.

Conversely, it is easy to check that the function $x \mapsto f(x) = \frac{x^2}{2} + \frac{1}{2x}$ for all $x \in \mathbb{R}\setminus\{0\}$, is actually a solution. Thus, it is the unique solution to the problem.

**4**. Find all solutions $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ of the system
$$\frac{4x^2}{1 + 4x^2} \;=\; y\,, \qquad \frac{4y^2}{1 + 4y^2} \;=\; z\,, \qquad \frac{4z^2}{1 + 4z^2} \;=\; x\,.$$

*Solved by Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Maisons-Laffitte, France; D.J. Smeenk, Zaltbommel, the Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use Wang's solution.*

Clearly, $x$, $y$, and $z$ must be non-negative, and they are all zero if any one of them is zero. Assume, then, that $x$, $y$, and $z$ are positive.

By the AM–GM Inequality, we have $1 + 4x^2 \geq 2\sqrt{4x^2} = 4x$ and hence, $y \leq \frac{4x^2}{4x} = x$. Similarly, $z \leq \frac{4y^2}{4y} = y$ and $x \leq \frac{4z^2}{z} = z$. It follows that $x = y = z$. Setting $y = x$ in the equation $\frac{4x^2}{1+4x^2} = y$, we obtain $1 + 4x^2 = 4x$; that is, $(2x-1)^2 = 0$, or $x = \frac{1}{2}$.

Thus, there are exactly two solutions, namely $(x,y,z) = (0,0,0)$ and $(x,y,z) = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$.

**5**. Let $ABCD$ be a rectangle, $P$ a point on the line $CD$. Let $M$ and $N$ be the mid-points of $AD$ and $BC$, respectively. $PM$ intersects $AC$ in $Q$. Show that $MN$ is the bisector of the angle $QNP$.

*Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; D.J. Smeenk, Zaltbommel, the Netherlands; and Toshio Seimiya, Kawasaki, Japan. We give the solution of Bornsztein.*

Let $O$ be the centre of the rectangle, and let $R$ be the intersection of the lines $CD$ and $QN$. From Thales' Theorem, we have

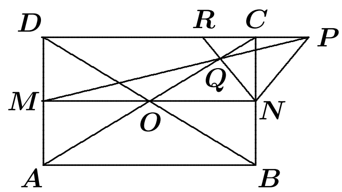
$$\frac{PC}{MO} = \frac{CQ}{OQ} = \frac{CR}{ON}.$$

Since $OM = ON$, we deduce that $PC = CR$. Thus, $C$ is the mid-point of $PR$. Since $CN \perp PR$, we deduce that $N$ belongs to the perpendicular bisector of $PR$. It follows that the triangle $PNR$ is isosceles, and

$$\angle CPN = \angle NRP. \qquad (1)$$

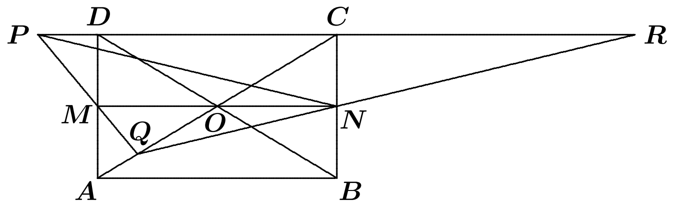
**Case 1**: Points $P$ and $D$ are on opposite sides of $C$ (see figure to the right).

Then $CN$ is the internal bisector of $\angle QNP$. Since $MN \perp CN$, it follows that $MN$ is the external bisector of $\angle QNP$.




**Case 2**: Points $P$ and $D$ are on the same side of $C$ (see figure below).

Then $\angle CPN = \angle PNM$ (since $MN$ and $CP$ are parallel) and $\angle NRP = \angle QNM$ (since $MN$ and $PR$ are parallel). From (1), it follows that $\angle PNM = \angle QNM$; that is, $MN$ is the internal bisector of $\angle QNP$.




Now we turn to problems of the second day of the Swiss Mathematical Olympiad (1999) given [2002 : 130].

**1**. Let $m$ and $n$ be two positive integers such that $m^2 + n^2 - m$ is divisible by $2mn$. Prove that $m$ is the square of an integer.

*Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bornsztein's write-up.*

Let $m$ and $n$ be two positive integers such that $m^2+n^2-m$ is divisible by $2mn$. Then there exists an integer $k$ such that $m^2 + n^2 - m = 2kmn$; that is, $n^2-2kmn+m^2-m = 0$. Thus, $n$ is an integer root of the quadratic polynomial

$$P(x) \;=\; x^2 - 2kmx + m^2 - m\,.$$

The discriminant of $P$ is

$$\Delta \;=\; 4k^2m^2 - 4m^2 + 4m \;=\; 4m(k^2m - m + 1)\,,$$

which must be a perfect square. Since $\gcd(m; k^2m - m + 1) = 1$, it follows that both $m$ and $k^2m - m + 1$ are perfect squares, and we are done.

**2**. A square is partitioned into rectangles whose sides are parallel to the sides of the square. For each rectangle, the ratio of its shorter side to its longer side is determined. Prove that the sum $S$ of these ratios is always at least 1.

*Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Assume that the side of the square has length 1 and that the sides of rectangle $i$ have lengths $a_i$, $b_i$ where $a_i \ge b_i$. Then $1 \ge a_i \ge b_i$ and $\sum a_ib_i = 1$. Therefore, $S = \sum b_i/a_i \ge \sum b_i \ge \sum a_ib_i = 1$.

*Comment by Pierre Bornsztein, Maisons-Laffitte, France.*

This problem appeared in the 1981 Leningrad High School Olympiad (Third Round). A solution may be found in [1987 : 142]. This solution also appears in R. Honsberger, *In Pólya's Footsteps*, MAA, p. 151–152.

**3**. Determine all integers $n \in \mathbb{N}$ such that there exist positive real numbers $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ satisfying

$$\sum_{i=1}^{n} a_i \;=\; 96\,, \qquad \sum_{i=1}^{n} a_i^2 \;=\; 144\,, \qquad \sum_{i=1}^{n} a_i^3 \;=\; 216\,.$$

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Klamkin's comment.*

This was on the 13th Iranian Mathematical Olympiad (1995) given in the *Corner* [1999 : 456]. Comments appeared [2002 : 68].

**4**. Prove that for every polynomial $P(x)$ of degree 10 with integer coefficients there is an (in both directions) infinite arithmetic progression which does not contain $P(k)$ for any integer $k$.

*Solution by Pierre Bornsztein, Maisons-Laffitte, France.*

More generally, we claim that the result holds for any polynomial $P(x)$ with integer coefficients which is *not* of the form $P(x) = mx + p$ for some integers $m$ and $p$ with $m \in \{-1, 1\}$.

**Lemma**. Let $P(x) = \sum\limits_{i=0}^{q} a_i x^i$, with $q \geq 1$, $a_i \in \mathbb{Z}$ for all $i$, and $a_q > 0$. Then there exists an integer $n$ such that $P(x+n) = \sum\limits_{i=0}^{q} b_i x^i$, with $b_i \in \mathbb{Z}$ for all $i$, $b_q = a_q$, and $b_i \geq 3$ for all $i < q$.

*Proof of Lemma*. If $a_i \geq 3$ for all $i < q$, then we just choose $n = 0$ and we are done. Therefore, we can assume that there is some $i < q$ such that $a_i < 3$. Let $t$ be the greatest such $i$. Then $a_i \geq 3$ for $t < i < q$.

Let $n \in \mathbb{Z}$. The polynomial $P(x+n)$ clearly has integer coefficients and degree equal to $q$. Hence, $P(x+n) = \sum\limits_{i=0}^{q} b_i x^i$, where $b_i \in \mathbb{Z}$ for all $i$. Furthermore, $b_q = a_q$, since $x^q$ appears only in the term $a_q(x+n)^q$.

Now consider any fixed integer $j$ such that $t < j < q$. In $P(x+n)$, the power $x^j$ appears in the terms $a_i(x+n)^i$ only for $i \geq j$. Since $a_i > 0$ for $i > t$, we have $b_j \geq a_j \geq 3$.

By similar reasoning, we deduce that $b_t \geq a_t + (t+1)na_{t+1}$. Since $a_{t+1} > 0$, we may choose some $n$ sufficiently large so that $b_t \geq 3$.

Finally, we obtain the desired result by iterating the above process, with $P(x+n)$ in place of $P(x)$, until $b_i \geq 3$ for all $i < q$ (a finite number of iterations). Note that if $Q(x) = P(x+n_1)$, then $Q(x+n_2) = P(x+n_3)$, where $n_3 = n_1 + n_2$. ■

We are now ready to prove the claim. It is obviously true in the case where $P$ is constant. If $P(x) = mx+p$ with $m \notin \{-1, 0, 1\}$, we just choose the arithmetic progression $u_n = mn + p - 1$. From now on, we suppose that $P$ has degree at least 2.

Let $P(x) = \sum_{i=0}^{q} a_i x^i$, with $q \geq 2$ and $a_q > 0$. (The case $a_q < 0$ may be obtained by considering the polynomial $-P(x)$.) For any integer $n$, the sets $\{P(k) \mid k \in \mathbb{Z}\}$ and $\{P(k+n) \mid k \in \mathbb{Z}\}$ are equal. Therefore, by the lemma, it suffices to prove the result when $a_i \geq 3$ for all $i < q$.

Let $S = \sum_{\substack{i=0 \\ i \text{ odd}}}^{q} a_i$. Thus, $S \geq a_1 \geq 3$. Let $k$ be any integer, and let $k = pS + r$ with $r \in \{0, 1, \ldots, S-1\}$. Then

$$P(k) \;=\; \sum_{i=0}^{q} a_i (pS+r)^i \;\equiv\; \sum_{i=0}^{q} a_i r^i \pmod{S} \;=\; P(r)\,.$$

Therefore, the number $P(k)$ is congruent, modulo $S$, to one of the numbers $P(0)$, $P(1)$, $\ldots$, $P(S-1)$. Moreover,

$$\begin{aligned} P(S-1) \;=\; \sum_{i=0}^{q} a_i (S-1)^i \;&\equiv\; \sum_{i=0}^{q} a_i (-1)^i \pmod{S} \;=\; P(1) - 2S \\ &\equiv\; P(1) \pmod{S}\,. \end{aligned}$$

Therefore, the set $\{P(k) \bmod S \mid k \in \mathbb{Z}\}$ contains at most $S-1$ elements.

Consequently, there exists $a \in \{0, 1, \ldots, S-1\}$ such that, for each integer $k$, we have $P(k) \not\equiv a \pmod{S}$. Then none of the terms of the arithmetic progression $u_n = nS + a$ belongs to the set $\{P(k) \mid k \in \mathbb{Z}\}$, and we are done.

**5**. Prove that the product of five consecutive positive integers is never a perfect square.

*Comment by Pierre Bornsztein, Maisons-Laffitte, France.*

This problem was proposed to the jury for the 1985 IMO, but not used. A solution may be found in [1987 : 77]. This solution also appears in R. Honsberger, *From Erdős to Kiev*, MAA, 1995, pp. 207–208. More generally, Erdős and Selfridge have proved that the product of any number ($\geq 2$) of consecutive positive integers is never a perfect (non-trivial) power. See P. Erdős, J.L. Selfridge, *The product of consecutive integers is never a power*, Illinois Journal of Math. (1975) pp. 292–301.

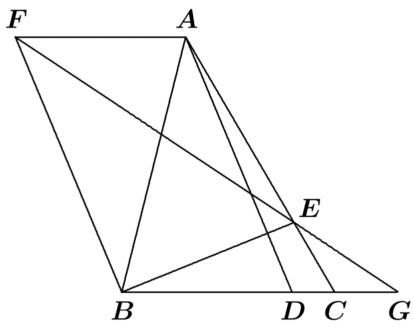
Next we turn to solutions from our readers to problems of the 50th Polish Mathematical Olympiad 1999 given [2002 : 130–131].

**1**. Let $D$ be a point on side $BC$ of triangle $ABC$ such that $AD > BC$. Point $E$ on side $AC$ is defined by the equation $\dfrac{AE}{EC} = \dfrac{BC}{AD - BC}$. Show that $AD > BE$.

*Solution by Toshio Seimiya, Kawasaki, Japan.*

The condition $\frac{AE}{EC} = \frac{BC}{AD - BC}$ is incorrect. The correct condition is $\frac{AE}{EC} = \frac{BD}{AD - BC}$, which is what we shall use in our proof.

Let $F$ be the point such that $AF \parallel BD$ and $BF \parallel AD$. Since $AFBD$ is a parallelogram, we have $FB = AD$ and $FA = BD$.




Let $G$ be the intersection of $EF$ with $BC$. Since $FA \parallel CG$, it follows that

$$\frac{FA}{CG} = \frac{AE}{EC} = \frac{BD}{AD - BC}.$$

Since $FA = BD$, we have

$$\frac{BD}{CG} = \frac{BD}{AD - BC}.$$

Thus, $CG = AD - BC$, which implies that

$$AD = BC + CG = BG.$$

Since $BF = AD$, we have $BF = BG$. Consequently,

$$\angle BEF > \angle BGF = \angle BFG = \angle BFE.$$

Therefore, $BF > BE$. That is, $AD > BE$.

**2**. Given are non-negative integers $a_1 < a_2 < a_3 < \cdots < a_{101}$ smaller than 5050. Show that one can choose four distinct integers $a_k$, $a_l$, $a_m$, $a_n$ so that the number $a_k + a_l - a_m - a_n$ is divisible by 5050.

*Solution by Pierre Bornsztein, Maisons-Laffitte, France.*

There are $\binom{101}{2} = \frac{101\times100}{2} = 5050$ ordered pairs $(a_i, a_j)$ where $i < j$. Suppose that these pairs give distinct sums $a_i + a_j$ modulo 5050. Thus, the sums are all of the residues modulo 5050. Let $S = \sum_{i<j}(a_i + a_j)$. Then $S = 100\sum_{i=1}^{101} a_i$, which is even. On the other hand, since the sums are the residues modulo 5050, we have

$$S \equiv \sum_{i=0}^{5049} i = 5049 \times 2525 \equiv -2525 \pmod{5050},$$

from which we deduce that $S$ is odd—a contradiction.

It follows that there exist two pairs $(k,l) \neq (m,n)$, with $k < l$ and $m < n$, such that $a_k + a_l \equiv a_m + a_n \pmod{5050}$. If $k = m$, then $l \neq n$ and $a_l \equiv a_n \pmod{5050}$, which is impossible, since $0 \leq a_l,\, a_n < 5050$. Thus, $k \neq m$. Similarly, $k \neq n$, $l \neq n$ and $l \neq m$. Then $a_k$, $a_l$, $a_m$, $a_n$ are four distinct numbers such that $a_k + a_l - a_m - a_n \equiv 0 \pmod{5050}$.

**3**. Prove that there exist distinct positive integers $n_1, n_2, n_3, \ldots, n_{50}$ such that $n_1 + S(n_1) = n_2 + S(n_2) = n_3 + S(n_3) = \cdots = n_{50} + S(n_{50})$, where $S(n)$ denotes the sum of the digits of $n$.

*Solution by Pierre Bornsztein, Maisons-Laffitte, France.*

We prove by induction that for each integer $k \geq 1$ there exist integers $n_1 > n_2 > \cdots > n_k > 0$ such that, for $i = 1, 2, \ldots, k$, we have

$$n_i + S(n_i) \;=\; n_1 + S(n_1)\,.$$

The claim is trivial for $k = 1$. Let $k \geq 1$ be a fixed integer, and suppose that we have integers $n_1 > n_2 > \cdots > n_k > 0$ such that

$$n_i + S(n_i) \;=\; n_1 + S(n_1) \qquad \text{for } i = 1, 2, \ldots, k. \tag{1}$$

We first make two remarks:

(R1) $S(n) \equiv n \pmod 9$ for any positive integer $n$.

(R2) $S(n + 10^p) = 1 + S(n)$ for any positive integers $n$ and $p$ such that $10^p > n$.

Using (R2), we deduce that the relations (1) remain valid when $n_i$ is replaced by $n_i + 10^p$ for all $i = 1, 2, \ldots, k$, provided that $p$ is an integer such that $10^p > n_1$. Thus, with no loss of generality, we may suppose that $n_k > 9$ and $S(n_1) \geq 7$.

Let $n \in \{1, 2, \ldots, 9\}$ such that $n \equiv n_1 + 1 \pmod 9$. Note that $S(n) = n$ and $n + S(n) \leq 18$. Using (R1), we obtain

$$n_1 + S(n_1) - \big(n + S(n)\big) \;\equiv\; -11 \pmod 9\ .$$

Thus, for some integer $p$, we have

$$n_1 + S(n_1) - \big(n + S(n)\big) \;=\; -11 + 9p\,. \tag{2}$$

Since $n + S(n) = 2n \leq 18$ and $S(n_1) \geq 7$, it follows that

$$n_1 \;\leq\; n_1 + S(n_1) - \big(n + S(n)\big) + 11 \;=\; 9p\,.$$

But it is easy to prove by induction that, for all integers $q \geq 1$, we have $9q < 10^q$. Thus, $n_1 < 10^p$.

Let $m_i = n_i + 10^{p+1}$ for $i = 1, 2, \ldots, k$, and $m_{k+1} = n + 10^{p+1} - 10$. Since

$$m_k \;=\; n_k + 10^{p+1} \;>\; 9 + 10^{p+1} \;\geq\; n + 10^{p+1} \;>\; m_{k+1}\,,$$

we deduce that $m_1 > m_2 > \cdots > m_k > m_{k+1}$. Using (R2), we have $m_i + S(m_i) = m_1 + S(m_1)$ for $i \leq k$. Moreover,

$$\begin{aligned} m_{k+1} \;=\; n + 10^{p+1} - 10 \;&=\; 99\ldots 90 + n \quad \text{(with } p \text{ digits "9")} \\ &=\; 99\ldots 9n \quad \text{(since } 1 \leq n \leq 9)\,. \end{aligned}$$

Then $S(m_{k+1}) = 9p + n = 9p + S(n)$. It follows that

$$\begin{aligned} m_{k+1} + S(m_{k+1}) &= n + 10^{p+1} - 10 + 9p + S(n) \\ &= n_1 + S(n_1) + 11 + 10^{p+1} - 10 \quad \text{(from (2))} \\ &= n_1 + 10^{p+1} + S(n_1) + 1 \\ &= m_1 + S(m_1)\,, \end{aligned}$$

which ends the proof of the induction step and completes the proof.

**4**. Find all integers $n \geq 2$ for which the system of equations

$$\left\{ \begin{aligned} x_1^2 + x_2^2 + 50 &= 16x_1 + 12x_2 \\ x_2^2 + x_3^2 + 50 &= 16x_2 + 12x_3 \\ x_3^2 + x_4^2 + 50 &= 16x_3 + 12x_4 \\ \cdots\cdots\cdots\cdots &\cdots \cdots\cdots \\ x_{n-1}^2 + x_n^2 + 50 &= 16x_{n-1} + 12x_n \\ x_n^2 + x_1^2 + 50 &= 16x_n + 12x_1 \end{aligned} \right.$$

has a solution in integers $x_1, x_2, x_3, \ldots, x_n$.

*Solution by Pierre Bornsztein, Maisons-Laffitte, France.*

The desired integers are those which are divisible by $3$.

Let $n \geq 2$ be an integer for which the system has a solution in integers $x_1, x_2, \ldots, x_n$. Subscripts are considered modulo $n$.

For all $i$, we have

$$x_i^2 + x_{i+1}^2 + 50 \;=\; 16x_i + 12x_{i+1}\,;$$

that is, $(x_i - 8)^2 + (x_{i+1} - 6)^2 = 50$. Since the decompositions of $50$ as a sum of two squares are $50 = 1 + 49 = 25 + 25$, we deduce that

$$\begin{aligned} & x_i - 8 = \pm 1 \quad \text{and} \quad x_{i+1} - 6 = \pm 7\,, \\ \text{or} \quad & x_i - 8 = \pm 7 \quad \text{and} \quad x_{i+1} - 6 = \pm 1\,, \\ \text{or} \quad & x_i - 8 = \pm 5 \quad \text{and} \quad x_{i+1} - 6 = \pm 5\,. \end{aligned}$$

That is, $(x_i, x_{i+1})$ is one of the pairs $(9,13)$, $(7,-1)$, $(9,-1)$, $(7,13)$, $(15,7)$, $(1,5)$, $(15,5)$, $(1,7)$, $(13,11)$, $(3,1)$, $(13,1)$, $(1,11)$.

Each of the $x_i$'s must be the first entry in one of these pairs and the second entry in another. We deduce that $x_i \in \{1,\,7,\,13\}$ for all $i$. Moveover, if $x_i = 7$ then $x_{i+1} = 13$, $x_{i+2} = 1$, $x_{i+3} = 7$, and so on. It follows that $x_i = x_j$ if and only if $i \equiv j \pmod 3$. Since $x_i = x_{n+i}$, we deduce that $n \equiv 0 \pmod 3$.

Conversely, if $n = 3k$ for some integer $k \geq 1$, we let $x_{3i+1} = 7$, $x_{3i+2} = 13$, and $x_{3i+3} = 1$, for $i = 0,\,1,\,\ldots,\,k-1$. Then the $x_i$'s clearly form a solution of the system in integers.

**6**. In a convex hexagon $ABCDEF$,

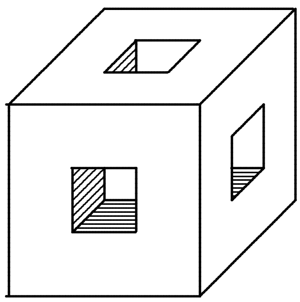
$$\angle A + \angle C + \angle E \;=\; 360^\circ\,, \quad \frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FA} \;=\; 1\,.$$

Prove that $\dfrac{AB}{BF}\cdot\dfrac{FD}{DE}\cdot\dfrac{EC}{CA} = 1$.

*Solved by Toshio Seimiya, Kawasaki, Japan. The problem and solution are very similar to problem 5 of the Third Round of the 16th Iranian Mathematical Olympiad 1998–1999* [2001 : 487; 2004 : 161–162]. *Therefore, we simply refer the interested reader to that solution.*

Next we turn to the solutions for problems of the Chilean Mathematical Olympiad 1996 given [2002 : 131–132].

**1**.



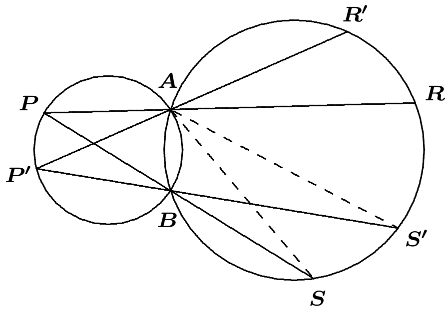
Consider a cube of edge 18 cm. In the centre of three different (and not opposite) faces we bore a square perforation of side 6 cm that goes across the cube as far as the opposite face. We thus obtain the following figure:

Determine the surface area of the resulting solid.

*Solution by Robert Bilinski, Outremont, QC.*

Each outside face of the solid has area $18^2 - 6^2 = 288$ cm$^2$, for a total area of $6 \cdot 288 = 1728$ cm$^2$. The tunnels are made up of 7 removed identical cubes, but they only contribute 6 groups of four squares of side 6 cm to the area of the solid. Thus, the inside area of the solid is $6 \cdot 4 \cdot 6^2 = 864$ cm$^2$. Hence, the solid has total area $2592$ cm$^2$.

**2**. Two circles intersect at $A$ and $B$. $P$ is a point on arc $\widehat{AB}$ on one of the circles. $PA$ and $PB$ intersect the other circle at $R$ and $S$ (see figure). If $P'$ is any point on the same arc as $P$ and if $R'$ and $S'$ are the points in which $P'A$ and $P'B$ intersect the second circle, prove that $\widehat{RS} = \widehat{R'S'}$.




*Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.*

Since $A$, $B$, $P$, $P'$ are concyclic we have

$$\angle APS = \angle APB = \angle AP'B = \angle AP'S' . \qquad (1)$$

Since $A$, $B$, $S$, $S'$ are concyclic we have

$$\angle ASP = \angle ASB = \angle AS'B = \angle AS'P' . \qquad (2)$$

From (1) and (2), we get

$$\angle RAS = \angle APS + \angle ASP = \angle AP'S' + \angle AS'P' = \angle R'AS' .$$

Hence, we have $\widehat{RS} = \widehat{R'S'}$.

**3**. Let $a$, $b$, $c$, $d$ be integers such that $ad \neq bc$.

(a) Prove that it is always possible to write the fraction

$$\frac{1}{(ax+b)(cx+d)}$$

in the form

$$\frac{r}{ax+b} + \frac{s}{cx+d} ,$$

where $r$, $s$ are rational numbers.

(b) Calculate the sum

$$\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{1993\cdot 1996} .$$

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. We give the solution by Díaz-Barrero.*

(a) In order to determine $r$ and $s$, we assume the identity

$$\frac{1}{(ax+b)(cx+d)} = \frac{r}{ax+b} + \frac{s}{cx+d}. \tag{1}$$

Multiplying this by $ax+b$ and taking the limit as $x \to -b/a$ yields

$$r = \lim_{x\to -b/a} \frac{1}{cx+d} = \frac{a}{ad-bc}.$$

Again, multiplying (1) by $cx+d$ and taking the limit as $x \to -d/c$, we get

$$s = \lim_{x\to -d/c} \frac{1}{ax+b} = -\frac{c}{ad-bc}.$$

Thus, we obtain rational numbers $r$ and $s$ which make (1) an identity.

(b) Let $S$ be the given sum. We claim that $S = 665/1996$. In fact, both the first and second factors in the denominators of $S$ are in arithmetic progression; that is, $1, 4, 7, 10, \ldots, 1993$, whose general term is $3n-2$ and $4, 7, 10, 13, \ldots, 1996$, whose general term is $3n+1$. Taking into account part (a), we can write

$$\frac{1}{(3n-2)(3n+1)} = \frac{1}{3}\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right).$$

We will use this for proving that $S$ telescopes to the claimed result. Indeed,

$$\begin{aligned} S &= \frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{1993\cdot 1996} \\ &= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{1}{7} - \frac{1}{10}\right) + \cdots + \frac{1}{3}\left(\frac{1}{1993} - \frac{1}{1996}\right) \\ &= \frac{1}{3}\left(1 - \frac{1}{1996}\right) = \frac{665}{1996}. \end{aligned}$$

That completes the *Corner* for this issue. Send me Olympiad materials and your nice solutions and generalizations to the problems.

# BOOK REVIEW

## John Grant McLoughlin

*The William Lowell Putnam Mathematical Competition 1985-2000: Problems, Solutions, and Commentary*
By Kiran S. Kedlaya, Bjorn Poonen, and Ravi Vakil, published by the Mathematical Association of America, 2002
ISBN 0-88385-807-X, hardcover, 337 pages, US$44.95.
*Reviewed by* ***Richard Hoshino****, Dalhousie University, Halifax, NS.*

This book covers the 192 problems that appeared on the Putnam Competition from 1985 to 2000. The Putnam Competition is an annual mathematics contest written by approximately thirty-five hundred students from the United States and Canada, which features twelve extremely challenging problems. The contest is so difficult that in most years, the median score on the Putnam is zero points, out of a possible 120. The top twenty-five contestants receive cash prizes, and the top five are awarded the distinction of being named Putnam Fellows. Canadian students have enjoyed great success in the Putnam. For example, in the most recent (2003) competition, five of the top sixteen finishers are alumni of the Canadian IMO program.

In addition to problems and detailed multiple solutions to each problem, the book contains background information about the competition, a list of winning individuals and teams (with current information about the career paths of the Putnam Fellows), and other Putnam Competition trivia. The authors do an excellent job of highlighting the connections to other Putnam problems, to past Olympiad problems, and to the undergraduate mathematics curriculum. For students training for the Putnam competition, this book is a must-read. For Putnam coaches, this book is an excellent resource to help prepare your students for the competition. I was particularly impressed with the multiple approaches available to solve each of the problems, and in the process, I learned a great deal of new mathematics from reading this book.

The three authors are among the most prolific and accomplished under-40 mathematicians in the world. They hold professorial appointments in their respective institutions, and have won numerous academic distinctions and fellowships. In total, the authors wrote the Putnam competition twelve times during their undergraduate career, and they were Putnam Fellows eleven times. Many ***CRUX with MAYHEM*** readers will know Ravi Vakil, who represented Canada at the International Mathematical Olympiad three times, and was a Founding Editor of ***Mathematical Mayhem***.

# Rationals Whose Sum Equals the Reciprocal of their Product

Alexander Fink and Bill Sands

For which positive integers $n$ do there exist positive integers $a_1, \ldots, a_{n+1}$ such that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_{n+1}} = \frac{a_{n+1}}{a_1}\,? \tag{1}$$

The second author proposed a special case of this problem for the 2002 IMO, but it was not shortlisted. The full problem was then given to the first author to work on, as part of his 2003 Summer NSERC Undergraduate Research Fellowship project. This note is the result.

We first restate the problem in a more convenient form.

**Problem**: For which positive integers $n$ do there exist positive rationals $b_1, \ldots, b_n$ such that

$$b_1 + b_2 + \cdots + b_n = \frac{1}{b_1 b_2 \cdots b_n}\,? \tag{2}$$

Given a positive integer solution to (1), we obtain a positive rational solution to (2) simply by letting $b_i = a_i/a_{i+1}$ for each integer $i$, $1 \leq i \leq n$. Conversely, given a positive rational solution to (2), we can obtain a positive rational solution to (1) by choosing an arbitrary positive rational $a_1$ and letting $a_{i+1} = a_i/b_i$ for each integer $i$, $1 \leq i \leq n$. From there we can obtain a positive integer solution by multiplying all of the $a_i$'s by a suitable integer. Note that (2) is symmetric in all of the $b_i$'s, and that reordering the $b_i$'s can give distinct solutions to (1).

The case $n = 1$ has the obvious solution $b_1 = 1$. For $n = 2$, it is an old, but not widely known, fact that there is no solution to this problem [2]; that is, *there is no solution in positive integers to*

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} = \frac{a_3}{a_1}\,.$$

We next include a proof for the convenience of the reader.

We rewrite the above equation by putting $A = a_1$, $B = a_2$, $C = a_3$, and clearing denominators, obtaining

$$A^2C + B^2A = C^2B\,. \tag{3}$$

The work of both authors was supported by NSERC Discovery Grant #8306.

(This is the form appearing in [2].) Suppose (3) has a solution in positive integers. We can clearly assume that $A$, $B$, $C$ have no common factor greater than 1.

Let $p$ be a prime dividing both $A$ and $B$ (but not $C$). Write $A = p^a A'$ and $B = p^b B'$, where $a$, $b$, $A'$, $B' \geq 1$ and $(p, A') = (p, B') = 1$. Then (3) becomes

$$p^{2a} A'^2 C + p^{2b+a} A' B'^2 \;=\; p^b B' C^2 \,,$$

where $(p, C) = 1$. Note that if $2b + a \leq 2a$, then $p^{2b+a}$ must divide the right side of this equation, which forces $2b + a \leq b$, an impossibility. Therefore, $2a < 2b + a$, which forces both sides of the equation to be divisible by $p^{2a}$ and by no larger power of $p$. Thus, $2a = b$.

This holds for every prime $p$ dividing both $A$ and $B$. Hence, letting $R = (A, B)$, we can write $A = RS$ and $B = R^2 T$ for positive integers $R$, $S$, $T$ satisfying $(R, C) = (R, S) = (R, T) = (S, T) = 1$. Equation (3) now becomes $R^2 S^2 C + R^5 S T^2 = R^2 T C^2$, which simplifies to

$$S^2 C + R^3 S T^2 \;=\; T C^2 \,. \tag{4}$$

Since $(S, T) = 1$, we get $T \mid C$. Then $TC^2$ and $R^3 S T^2$ are both divisible by $T^2$, forcing $T^2 \mid C$. Writing $C = T^2 D$ and cancelling turns (4) into

$$S^2 D + R^3 S \;=\; T^3 D^2 \,. \tag{5}$$

Since $(D, R) = 1$, we get $D \mid S$, which (as above) forces $D^2 \mid S$. But since $(S, T) = 1$, we also get from (5) that $S \mid D^2$, which means that $S = D^2$. Thus, (5) simplifies to

$$D^3 + R^3 \;=\; T^3 \,.$$

This is the exponent 3 case of Fermat's Last Theorem and is known to have no solution in positive integers (see for instance [3], [4]). □

For $n > 2$, the only reference we have found is [1], where solutions due to Euler are given for the values $n = 3$ and $n = 4$. However, the solution for $n = 4$ contains negative rationals, and Euler's purpose apparently was not to consider equation (2) in general.

Anyway, here is our result.

**Theorem**. The equation

$$b_1 + b_2 + \cdots + b_n \;=\; \frac{1}{b_1 b_2 \cdots b_n}$$

has a positive rational solution for each positive integer $n \geq 3$.

*Proof*: We consider four cases, depending on the congruence class of $n$ modulo 4, and exhibit solutions in each case. Here and below, let b denote $(b_1, b_2, \ldots, b_n)$, although, as we mentioned above, the order of the $b_i$'s is not important.

The following table contains a solution for each of the various cases, where, for example, we write $(1/k)^k$ to denote $k$ copies of $1/k$.

| Case | Solution $b_1, b_2, \ldots, b_n$ | $\sum_{i=1}^{n} b_i$ |
|---|---|---|
| $n \equiv 1 \pmod 4$ | | |
| $n = 4k-3,\ k \geq 2$ | $\left(\frac{1}{k}\right)^k,\ k^{k-2},\ 1^{2k-1}$ | $k^2$ |
| $n \equiv 3 \pmod 4$ | | |
| $n = 3$ | $\frac{4}{3},\ \frac{3}{2},\ \frac{1}{6}$ | $3$ |
| $n = 4k-5,\ k \geq 3$ | $\left(\frac{1}{k}\right)^k,\ k^{k-2},\ 1^{2k-6},\ \frac{1}{2},\ \frac{1}{2},\ 4$ | $k^2$ |
| $n \equiv 0 \pmod 4$ | | |
| $n = 4$ | $\frac{1}{6},\ \frac{1}{3},\ 2,\ 2$ | $\frac{9}{2}$ |
| $n = 4k-4,\ k \geq 3$ | $\left(\frac{1}{k}\right)^k,\ k^{k-2},\ 1^{2k-6},\ \frac{1}{2},\ \frac{1}{2},\ 2,\ 2$ | $k^2$ |
| $n \equiv 2 \pmod 4$ | | |
| $n = 6$ | $\frac{1}{6},\ \frac{1}{3},\ 1,\ 1,\ \frac{3}{2},\ 2$ | $6$ |
| $n = 10$ | $\frac{1}{18},\ \frac{1}{9},\ \frac{1}{3},\ \frac{1}{2},\ 1,\ 1,\ 1,\ 2,\ 3,\ 9$ | $18$ |
| $n = 14$ | $\frac{1}{6},\ \left(\frac{1}{3}\right)^4,\ \left(\frac{1}{2}\right)^3,\ 2^3,\ 3^3$ | $18$ |
| $n = 4k-6,\ k \geq 6$ | $\left(\frac{1}{k}\right)^k,\ k^{k-2},\ 1^{2k-12},\ \left(\frac{1}{3}\right)^3,\ 3,\ \left(\frac{3}{2}\right)^2,\ 2^2$ | $k^2$ |

For example, when $n = 5$, we put $k = 2$ to get the solution $\mathrm{b} = (1/2, 1/2, 1, 1, 1)$, yielding the following solution to (1):

$$\frac{1}{2} + \frac{2}{4} + \frac{4}{4} + \frac{4}{4} + \frac{4}{4} \ = \ \frac{4}{1}\,.$$

By reordering b, we could get other solutions to (1) differing slightly from this one. □

We remark that the number of distinct positive rational solutions to (2) grows arbitrarily large as $n$ increases. (Two solutions that are permutations of each other are not considered distinct.) In particular,

(2) *has at least* $\left\lceil \frac{n-22}{36} \right\rceil$ *distinct positive rational solutions for any* $n$.

Fix $n$. Let $d_i = \lceil (5i-1)/2 \rceil \geq 0$ for $0 \leq i \leq 3$, and write

$$n \ = \ 4(9t + r + d_s) - s + 1$$

for integers $r$, $s$, $t$, where $0 \leq r \leq 8$, $0 \leq s \leq 3$; this is always possible uniquely, since $s$ is determined by the congruence class of $n$ modulo 4, and then $(n + s - 1)/4 - d_s$ can be written uniquely as $9t + r$. If $n$ is sufficiently large (precisely, $n > 22$), then $t \geq 0$.

Suppose that $n > 22$. We claim that (2) has the $t+1$ distinct solutions given by

$$\mathrm{b} \;=\; \left(\left(\frac{1}{k}\right)^{k}, k^{k-2}, 1^{2k-1-20i-5s}, \left(\frac{1}{2}\right)^{8i+2s}, 2^{8i+2s}\right) \tag{6}$$

as $i$ ranges over the integers from 0 to $t$, where $k = 9t + d_s + r + i + 1 \geq 1$. Note that

$$2k-1-20i-5s \;=\; 18(t-i)+(2d_s-5s)+2r+1 \;\geq\; 0-1+0+1 \;=\; 0\,,$$

and the length of b comes out to be

$$4k-3-4i-s \;=\; 36t+4d_s+4r+1-s \;=\; n\,.$$

Therefore, the vector in (6) does indeed have length $n$. It is easily verified that $\sum\limits_{j=1}^{n} b_j \;=\; k^2$ and $\prod\limits_{j=1}^{n} b_j \;=\; \frac{1}{k^2}$. Therefore, (2) holds for each of the solutions in (6). Finally, the number of solutions is

$$\begin{aligned} t+1 \;&=\; \frac{n+35-4r-(4d_s-s)}{36} \\ &\geq\; \left\lceil\frac{n+35-32-(4d_3-3)}{36}\right\rceil \;=\; \left\lceil\frac{n-22}{36}\right\rceil\,. \end{aligned}$$

To end, here is an unsolved problem, suggested by Filip Saidak:

*For which positive integers $n$, if any, does equation* (2) *have infinitely many positive rational solutions?*

**References.**


[1] L. E. Dickson, *History of the Theory of Numbers, Volume II*, Chelsea, 1992, p. 648 (item 187).

[2] H. S. Vandiver, W. F. King, Solution to Problem 101, *Amer. Math. Monthly* **10** (1903), p. 22.

[3] P. Ribenboim, *13 Lectures on Fermat's Last Theorem*, Springer-Verlag, 1979, pp. 39–45.

[4] R. D. Carmichael, *Diophantine Analysis*, Wiley, 1915.



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# PROBLEMS

*Solutions to problems in this issue should arrive no later than* **1 March 2005**. *An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.*

*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.*

*The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.*

**2938**. Correction. *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that $x_1, \ldots, x_n, \alpha$ are positive real numbers. Prove that

(a) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \;\ge\; \alpha + \sqrt[n]{x_1\cdots x_n}$;

(b) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \;\le\; \alpha + \dfrac{x_1+\cdots+x_n}{n}$.

**2951**. *Proposed by Nevena Sybeva, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

Let $M$ and $N$ be interior points of $\triangle ABC$. Define the point $T_A$ to be the point on $BC$ such that light travelling from $M$ to $T_A$, and undergoing perfect reflection at $T_A$, will pass through $N$. Define $T_B$ and $T_C$ similarly.

Prove that if the three possible light paths $MT_AN$, $MT_BN$, $MT_CN$ have equal length, then the lines $AT_A$, $BT_B$, and $CT_C$ are concurrent.

**2952**. *Proposed by C.R. Pranesachar and Prithu Bharti, Indian Institute of Science, Bangalore, India.*

Find a closed form for the real series

$$\sum_{\substack{r\ge 0\\ r\ge -n}} \binom{n+2r}{r} x^r\,, \qquad x \in \left(-\tfrac{1}{4}, \tfrac{1}{4}\right)\,,$$

where $n$ is an integer (positive, negative, or zero).

**2953**. *Proposed by Titu Zvonaru, Bucharest, Romania.*

Let $m$, $n$ be positive integers with $n > 1$, and let $a$, $b$, $c$ be positive real numbers satisfying $a^{m+1}+b^{m+1}+c^{m+1}=1$. Prove that

$$\frac{a}{1-ma^n}+\frac{b}{1-mb^n}+\frac{c}{1-mc^n} \;\ge\; \frac{(m+n)^{1+\frac{m}{n}}}{n}\,.$$

**2954**. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let $\Gamma$ be the circumcircle of $\triangle ABC$. The tangents to $\Gamma$ at $B$ and $C$ intersect at $M$. The line through $M$ parallel to $AB$ intersects $\Gamma$ at $D$ and $E$, and intersects $AC$ at $F$.

Prove that $F$ is the mid-point of $DE$.

**2955**. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $n$ be a positive integer. For each positive integer $k$, let $f_k$ be the $k^{\text{th}}$ Fibonacci number; that is, $f_1 = 1$, $f_2 = 1$, and $f_{k+2} = f_{k+1} + f_k$ for all $k \geq 1$. Prove that

$$\left(\sum_{k=1}^{n} f_{k+1}^2\right)\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) \;\geq\; n^2\,.$$

**2956**. *Proposed by David Loeffler, student, Trinity College, Cambridge, UK.*

Let $A$, $B$, $C$ be the angles of a triangle. Prove that

$$\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \;<\; 2$$

if and only if

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) \;<\; 2\,.$$

**2957**. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let $ABC$ and $A'B'C'$ be two triangles having $BC = a$, $B'C' = s - a$, etc., where $s = \frac{1}{2}(a + b + c)$. Prove that the triangles are isosceles if and only if $\tan\left(\frac{B}{2}\right)$ is the geometric mean of $\tan\left(\frac{A'}{2}\right)$ and $\tan\left(\frac{B'}{2}\right)$.

**2958**. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $ABC$ be a right triangle with right angle at $A$. Let $ACDE$, $BAFG$, and $CBIJ$ be squares mounted externally on the sides of $\triangle ABC$. Let $H$ be the intersection of the interior angle bisector of angle $A$ (extended) with the line segment $EF$, and let $A'$ be the point outside the square $CBIJ$ such that $\triangle A'JI$ is directly congruent to $\triangle ABC$.

Show that $A'DHG$ is a cyclic quadrilateral.

**2959**. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Given a non-isosceles triangle $ABC$, prove that there exists a unique inscribed equilateral triangle $PQR$ of minimal area, with $P$, $Q$, $R$ on $BC$, $CA$, and $AB$, respectively. Construct it by straightedge and compass.

**2960**. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $m_a$, $w_a$, and $h_a$ be the lengths of the median, the angle-bisector, and the altitude, respectively, from the right-angled vertex $A$ of triangle $ABC$ to the hypotenuse. Suppose that the sides $a$ and $c$ are fixed in length, while the length of side $b$ varies subject to $a > b \geq c$.

Evaluate $\lim\limits_{b \to c} \dfrac{m_a - h_a}{w_a - h_a}$.

**2961**. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $ABC$ and $A'B'C'$ be two right triangles with right angles at $A$ and $A'$. If $w_a$ and $w_{a'}$ are the interior angle bisectors of angles $A$ and $A'$, respectively, prove that $aw_a a'w_{a'} \geq bcb'c'$, with equality if and only if both $ABC$ and $A'B'C'$ are isosceles.

**2962**. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $ABC$ and $A'B'C'$ be two triangles satisfying $a \geq b \geq c$ and $a' \geq b' \geq c'$. If $h_a$, $h_{a'}$ are the altitudes from the vertices $A$, $A'$, respectively, to the opposite sides, prove that

(i) $bb' + cc' \geq ah_{a'} + a'h_a$, (ii) $bc' + b'c \geq ah_{a'} + a'h_a$.

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**2938**. Correction. *Proposé par Todor Mitev, Université de Rousse, Rousse, Bulgarie.*

Supposons que $x_1, \ldots, x_n, \alpha$ sont des nombres réels positifs. Montrer que

(a) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \;\geq\; \alpha + \sqrt[n]{x_1 \cdots x_n}$;

(b) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \;\leq\; \alpha + \dfrac{x_1 + \cdots + x_n}{n}$.

**2951**. *Proposé par Nevena Sybeva, Académie des Sciences de Bulgarie, Sofia, Bulgarie.*

Soit $M$ et $N$ deux points intérieurs du triangle $ABC$. Désignons par $T_A$ le point sur $BC$ tel qu'un rayon lumineux issu de $M$ en direction de $T_A$ va passer par $N$ après une réflexion parfaite en $T_A$. On définit $T_B$ et $T_C$ de manière analogue.

Montrer que si les trois trajectoires possibles pour un rayon lumineux issu de $M$ et aboutissant en $N$ en passant par réflexion en $T_A$, $T_B$ ou $T_C$ sont de longueur égale, alors les droites $AT_A$, $BT_B$ et $CT_C$ sont concourantes.

**2952**. *Proposé par C.R. Pranesachar et Prithu Bharti, Institut Indien des Sciences, Bangalore, Inde.*

Trouver une formule pour exprimer la somme de la série réelle

$$\sum_{\substack{r \geq 0 \\ r \geq -n}} \binom{n+2r}{r} x^r , \qquad x \in \left(-\tfrac{1}{4}, \tfrac{1}{4}\right) ,$$

où $n$ est un entier (positif, negatif ou nul).

**2953**. *Proposé par Titu Zvonaru, Bucarest, Roumanie.*

Soit $m$ et $n$ deux entiers positifs avec $n > 1$, et soit $a$, $b$, $c$ des nombres réels positifs satisfaisant $a^{m+1} + b^{m+1} + c^{m+1} = 1$. Montrer que

$$\frac{a}{1 - ma^n} + \frac{b}{1 - mb^n} + \frac{c}{1 - mc^n} \;\geq\; \frac{(m+n)^{1+\frac{m}{n}}}{n} .$$

**2954**. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit $\Gamma$ le cercle circonscrit au triangle $ABC$ et $M$ le point d'intersection des tangentes à $\Gamma$ en $B$ et en $C$. La parallèle à $AB$ passant par $M$ coupe $\Gamma$ en $D$ et en $E$ et le côté $AC$ en $F$.

Montrer que $F$ est le point milieu de $DE$.

**2955**. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $n$ un entier positif. Pour chaque entier positif $k$, soit $f_k$ le $k^{\text{e}}$ nombre de Fibonacci ; c'est-à-dire, $f_1 = 1$, $f_2 = 1$, et $f_{k+2} = f_{k+1} + f_k$ pour tout $k \geq 1$. Montrer que

$$\left(\sum_{k=1}^{n} f_{k+1}^2\right)\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) \;\geq\; n^2 .$$

**2956**. *Proposé par David Loeffler, étudiant, Trinity College, Cambridge, GB.*

Soit $A$, $B$ et $C$ les angles d'un triangle. Montrer que

$$\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \;<\; 2$$

si et seulement si

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) \;<\; 2 .$$

**2957**. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Soit $ABC$ et $A'B'C'$ deux triangles avec $BC = a$, $B'C' = s - a$, etc., où $s = \frac{1}{2}(a + b + c)$. Montrer que les triangles sont isocèles si et seulement si $\tan\left(\frac{B}{2}\right)$ est la moyenne géométrique de $\tan\left(\frac{A'}{2}\right)$ et $\tan\left(\frac{B'}{2}\right)$.

**2958**. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit $ABC$ un triangle rectangle d'angle droit en $A$. On désigne par $ACDE$, $BAFG$ et $CBIJ$ les carrés construits extérieurement sur les côtés du triangle $ABC$. Soit $H$ l'intersection de la bissectrice de l'angle $A$ avec la droite $EF$, et soit $A'$ le point extérieur au carré $CBIJ$ tel que le triangle $A'JI$ soit directement congruent au triangle $ABC$.

Montrer que $A'DHG$ est un quadrilatère cyclique.

**2959**. *Proposé par Peter Y. Woo, Biola University, La Mirada, CA, USA.*

On donne un triangle non isocèle $ABC$. Montrer qu'il existe un unique triangle équilatéral inscrit $PQR$ d'aire minimale, avec respectivement $P$, $Q$ et $R$ sur $BC$, $CA$ et $AB$. Trouver une construction de ce triangle avec la règle et le compas.

**2960**. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Dans un triangle rectangle $ABC$ d'angle droit au sommet $A$, on considère respectivement les longueurs $m_a$, $w_a$ et $h_a$ de la médiane, de la bissectrice et de la hauteur abaissée de $A$ sur l'hypoténuse. On suppose que les côtés $a$ et $c$ sont de longueur fixe, alors que la longueur du côté $b$ varie tout en respectant la condition $a > b \geq c$.

Calculer $\lim\limits_{b \to c} \dfrac{m_a - h_a}{w_a - h_a}$.

**2961**. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit $ABC$ et $A'B'C'$ deux triangles rectangles avec l'angle droit en $A$ et $A'$. Si $w_a$ et $w_{a'}$ sont respectivement les bissectrices intérieures des angles $A$ et $A'$, montrer que $aw_a a'w_{a'} \geq bcb'c'$, avec l'égalité si et seulement si $ABC$ et $A'B'C'$ sont tous les deux isocèles.

**2962**. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit $ABC$ et $A'B'C'$ deux triangles satisfaisant $a \geq b \geq c$ et $a' \geq b' \geq c'$. Si $h_a$ et $h_{a'}$ sont les hauteurs abaissées des sommets $A$ et $A'$ sur les côtés opposés respectifs, montrer que

(i) $bb' + cc' \geq ah_{a'} + a'h_a$, (ii) $bc' + b'c \geq ah_{a'} + a'h_a$.

# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

We apologize for misspelling the name of JAMES HOLETON in the list of solvers of 2826, and for omitting the name of ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina from the list of solvers of 2843, and the name of LI ZHOU, Polk Community College, Winter Haven, FL, USA from the list of solvers of 2845.

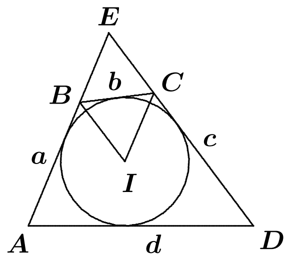
**2849**. [2003 : 242] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In a convex quadrilateral $ABCD$, we have $\angle ABC = \angle BCD = 120^\circ$. Suppose that $AB^2 + BC^2 + CD^2 = AD^2$.

Prove that $ABCD$ has an inscribed circle.

*Solution by Michel Bataille, Rouen, France; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Mitko Kunchev, Baba Tonka School of Mathematics, Rousse, Bulgaria; David Loeffler, student, Trinity College, Cambridge, UK; Andrei Simion, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. Smeenk, Zaltbommel, the Netherlands; and the proposer.*

Let $a = AB$, $b = BC$, $c = CD$, $d = DA$, and let $E$ be the point of intersection of the lines $AB$ and $CD$. Since $\angle ABC = \angle BCD = 120^\circ$, we have $\angle EBC = \angle ECB = 60^\circ$, and therefore, $BE = CE = BC = b$.




Applying the Law of Cosines to $\triangle AED$, we obtain

$$\begin{aligned} d^2 &= (a+b)^2 + (b+c)^2 - (a+b)(b+c) \\ &= a^2 + b^2 + c^2 + ab + bc - ac\,. \end{aligned}$$

Using the given condition $d^2 = a^2 + b^2 + c^2$, we get $ab + bc - ac = 0$. But then $d^2 = a^2 + b^2 + c^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ac = (a + c - b)^2$, so that $d = a + c - b$ or $d = b - a - c$. The latter is impossible, because $d^2 > b^2$. Consequently, $a + c = d + b$, which is a well-known necessary and sufficient condition for a convex quadrilateral to have an inscribed circle.

*Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOHN G. HEUVER, Grande Prairie, AB; NEVEN JURIČ, Zagreb, Croatia; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania. There were also two incorrect solutions submitted.*

**2850**. [2003 : 242] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find all integral solutions of

$$x^2 - 4xy + 6y^2 - 2x - 20y = 29 .$$

*Composite of essentially the same solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain; Con Amore Problem Group, The Danish University of Education, Copenhagen, Denmark; Douglass L. Grant, University College of Cape Breton, Sydney, NS; Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; D. Kipp Johnson, Beaverton, OR, USA; David Loeffler, student, Trinity College, Cambridge, UK; Digby Smith, Mount Royal College, Calgary, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

The given equation can be written as

$$(x - 2y - 1)^2 + 2(y - 6)^2 \;=\; 102 .$$

Thus, $x - 2y - 1$ is even. Setting $x - 2y - 1 = 2u$ and $y - 6 = v$, we then have $2u^2 + v^2 = 51$ for some integers $u$ and $v$. Clearly, $|u| \leq 5$. By setting $u = 0$, $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, and $\pm 5$, we find that $v$ is an integer only when $u = \pm 1$ or $\pm 5$. These values yield eight pairs: $(u, v) = (\pm 1, \pm 7)$ or $(\pm 5, \pm 1)$. Simple substitutions then give eight solution pairs $(x, y)$ for the given equation:

$$(29, 13) ,\ (25, 13) ,\ (1, -1) ,\ (-3, -1) ,\ (25, 7) ,\ (5, 7) ,\ (21, 5) ,\ (1, 5) .$$

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); DIONNE T. BAILEY, ELSIE M. CAMPBEII, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; PAOLO CUSTODI, Fara Novarese, Italy; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; NEVEN JURIČ, Zagreb, Croatia; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB SERKEY, Leonia, NJ, USA; MIKE SPIVEY, Samford University, Birmingham, AL, USA; MIHAÏ STOËNESCU, Bischwiller, France; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; ROGER ZARNOWSKI, Angelo State University, TX, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer. There were four incorrect or incomplete solutions.*

*As pointed out by Con Amore Problem Group, Curtis, and Konečný, the given equation represents an ellipse $E$, on which there can be only a finite number of lattice points. The solutions to the problem simply identify all eight lattice points on $E$.*

**2851**★. [2003 : 315] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $m$, $n$, and $N$ be non-negative integers such that $m+n \geq 2N+1$. Let $K = m+n-N-1$. Prove that

$$\sum_{j=0}^{\infty}(-1)^j \frac{N+1}{N+1+j}\binom{N}{j}\left[\binom{K-j}{m}+\binom{K-j}{n}\right] = \frac{\binom{m+n}{m}}{\binom{2N+1}{N}}.$$

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

In [1], Walther Janous (the proposer of the current problem) and the editors of the "Problems and Solutions" column of the *American Mathematical Monthly* used the so-called "snake-oil" method to establish the identity

$$\sum_{j=0}^{N} \frac{(-1)^j}{N+1+j}\binom{N}{j}\left[\binom{K-j}{m}+\binom{K-j}{n}\right] = \binom{m+n}{n}\sum_{j=0}^{N}\frac{(-1)^j}{N+1+j}\binom{N}{j}.$$

In the editorial comment that followed their composite solution, it was also proved, by the so-called WZ method, that

$$\sum_{j=0}^{N}\frac{(-1)^j}{N+1+j}\binom{N}{j} = \frac{1}{N+1}\binom{2N+1}{N}^{-1}.$$

Putting these two identities together solves the problem.

**Reference**

[1] Solution to Problem 10868 (proposed by M.N. Deshpande and A.V. Kharashikar), American Math. Monthly 110 (2003), pp. 241–243.

*Also solved by G.P. HENDERSON, Garden Hill, Ontario.*

**2852**. [2003 : 316] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, we have $AB < AC$. The internal bisector of $\angle BAC$ meets $BC$ at $D$. Let $P$ be an interior point of the line segment $AD$, and let $E$ and $F$ be the intersections of $BP$ and $CP$ with $AC$ and $AB$, respectively.

Prove that $\frac{PE}{PF} < \frac{AC}{AB}$.

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

**Lemma**. Let the internal bisector of $\angle BAC$ meet $BC$ at $D$. For any interior point $P$ of the segment $AD$ let $E$ and $F$ be the intersections of $BP$ and $CP$ with $AC$ and $AB$, respectively. Then $AB < AC$ implies $BE < CF$; that is, the shorter cevian goes to the longer side.

*Proof.* Denote by $a$, $b$, $c$ the lengths of the sides $BC$, $CA$, $AB$, as usual. Let $BF/FA = t > 0$. By Ceva's Theorem,

$$\frac{BF}{FA}\cdot\frac{AE}{EC}\cdot\frac{CD}{DB} \;=\; 1\,.$$

Thus, $FA = c/(1+t)$ and $AE = bc/(c+bt)$. By the Law of Cosines,

$$BE^2 \;=\; AB^2+AE^2-2AB\cdot AE\cos A \;=\; c^2+\left(\frac{bc}{c+bt}\right)^2-\frac{c(b^2+c^2-a^2)}{c+bt}\,,$$

and

$$CF^2 \;=\; AC^2+FA^2-2AC\cdot FA\cos A \;=\; b^2+\left(\frac{c}{1+t}\right)^2-\frac{b^2+c^2-a^2}{1+t}\,.$$

Therefore,

$$\begin{aligned}
&(1+t)^2(c+bt)^2(CF^2-BE^2)\\
&\qquad= (b^2-c^2)(1+t)^2(c+bt)^2+c^2[(c+bt)^2-b^2(1+t)^2]\\
&\qquad\qquad -(b^2+c^2-a^2)(1+t)(c+bt)(b-c)t\\
&\qquad> (b-c)\big[(b+c)(1+t)^2(c+bt)^2-c^2(c+b+2bt)\\
&\qquad\qquad -(b^2+c^2)(1+t)(c+bt)t\big]\\
&\qquad= (b-c)\big[\big((1+t)(c+bt)^3-c^3-2bc^2t\big)\\
&\qquad\qquad +\big((1+t)(c+bt)bc(1+t^2)-bc^2\big)\big] \;>\; 0\,,
\end{aligned}$$

which proves the lemma. ∎

Turning to the problem, suppose that the lines through $D$ which are parallel to $BE$ and $CF$ intersect $AC$ and $AB$ at $G$ and $H$, respectively. Then

$$\frac{PE}{DG} \;=\; \frac{PA}{DA} \;=\; \frac{PF}{DH}\,.$$

Also,

$$\frac{DG}{BE} \;=\; \frac{CD}{CB}\,,\quad \frac{DH}{CF} \;=\; \frac{DB}{CB}\,,\quad\text{and}\quad \frac{CD}{DB} \;=\; \frac{AC}{AB}$$

(where the last equality holds because $AD$ bisects $\angle BAC$). Hence,

$$\frac{PE}{PF} \;=\; \frac{DG}{DH} \;=\; \frac{CD}{DB}\cdot\frac{BE}{CF} \;=\; \frac{AC}{AB}\cdot\frac{BE}{CF}\,.$$

Since $BE < CF$ from the lemma, we conclude that $PE/PF < AC/AB$, as desired.

II. *Editor's comments, and a 1944 solution by L.M. Kelly.*

The lemma was also used in solutions submitted by Janous, Seimiya, and (implicitly) Paragiou. It generalizes the Steiner-Lehmus theorem, a result that deals with the case where $P$ is the incentre: *The shorter angle bisector goes to the longer side.* According to Coxeter and Greitzer (*Geometry Revisited*, Mathematical Association of America (1967), page 14), that theorem always excites interest; papers on it have appeared with some regularity since 1842. One would therefore expect our lemma to have been discovered long ago. A cursory computer search for references to the Steiner-Lehmus theorem in the Mathematical Association of America journals turned up problem E613 of the *American Mathematical Monthly* (**51**:3 (March 1944), p. 162, and **51**:10 (December 1944), pp. 590–591). Both the problem and the solution came from L.M. Kelly. To solve his problem he used the following:

**Lemma** (Kelly). If the internal cevians $BE$ and $CF$ of triangle $ABC$ are such that $\angle CBE > \angle BCF$ and $\angle ABE > \angle ACF$, then $BE < CF$.

*Kelly's Proof.* Select a point $Q$ on the segment $AE$ so that $\angle QBE = \angle ACF$. Let $CF$ meet $BE$ at $P$ and $BQ$ at $R$. Since triangle $QBC$ has a greater angle at $B$ than at $C$, $QC > QB$. Since triangles $QBE$ and $QCR$ are similar, it follows that $BE < CR < CF$, as desired. ∎

Zhou's Lemma is the special case of Kelly's where $AD$ is the bisector of $\angle BAC$. To see this, reflect $C$ in $AD$ to a point $C'$ on the extension of segment $AB$ beyond $B$, and note that

$$\angle CBE = \angle DBP > \angle DC'P = \angle DCP = \angle BCF,$$

and

$$\angle ABE = \angle ABP > \angle AC'P = \angle ACP = \angle ACF.$$

In problem E613, Kelly showed that when $AB < AC$, the medians $BE$ and $CF$ satisfy the conditions of his lemma; from that he easily deduced that a triangle cannot have two equal symmedians without being isosceles.

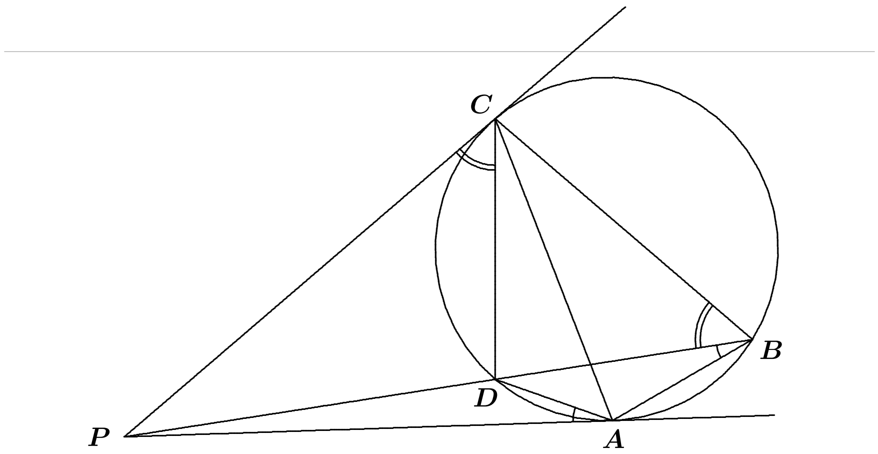
*Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete solution (which stated Zhou's lemma without an accompanying proof or reference).*

**2853**. [2003 : 316] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, we have $AC = 2AB$. The tangents at $A$ and $C$ to the circumcircle of $\triangle ABC$ meet at $P$.

Prove that the line $BP$ bisects the arc $BAC$ (of the circumcircle).

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*




Suppose that $BP$ intersects the arc $BAC$ at the point $D$. Since $\angle CBD = \angle PCD$, the triangles $PBC$ and $PCD$ are similar. Likewise, $\angle ABD = \angle DAP$, implying that the triangles $PAB$ and $PDA$ are similar. Hence,

$$\frac{PC}{PD} = \frac{BC}{CD} \quad \text{and} \quad \frac{AB}{DA} = \frac{PA}{PD}.$$

Since $PA = PC$, it follows that

$$\frac{AB}{DA} = \frac{BC}{CD};$$

that is, $AB \cdot CD = DA \cdot BC$. Using Ptolemy's Theorem, we have

$$AC \cdot BD = AB \cdot CD + DA \cdot BC = 2AB \cdot CD.$$

Since $AC = 2AB$, we obtain $BD = CD$.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Bucharest, Romania; and the proposer.*

**2854**. [2003 : 316] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $M$ and $N$ are the mid-points of the sides $AB$ and $CD$ of quadrilateral $ABCD$, respectively.

Prove that $AN^2 + DM^2 + BC^2 = BN^2 + CM^2 + AD^2$.

I. *Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

By applying the Law of Cosines to $\triangle AND$, we get

$$\begin{aligned} AN^2 &= AD^2 + ND^2 - 2(AD)(ND)\cos\angle ADN \\ &= AD^2 + \tfrac{1}{4}CD^2 - (AD)(CD)\cos\angle ADC\,. \end{aligned}$$

By applying the Law of Cosines to $\triangle ACD$, we get

$$AC^2 \;=\; AD^2 + CD^2 - 2(AD)(CD)\cos\angle ADC\,.$$

We combine these results to obtain

$$AN^2 \;=\; \tfrac{1}{2}(AD^2 + AC^2) - \tfrac{1}{4}CD^2\,. \qquad (1)$$

Similarly,

$$\begin{aligned} DM^2 &= \tfrac{1}{2}(AD^2 + BD^2) - \tfrac{1}{4}AB^2\,, \\ BN^2 &= \tfrac{1}{2}(BC^2 + BD^2) - \tfrac{1}{4}CD^2\,, \\ CM^2 &= \tfrac{1}{2}(AC^2 + BC^2) - \tfrac{1}{4}AB^2\,. \end{aligned}$$

Therefore, $AN^2 + DM^2 - BN^2 - CM^2 = AD^2 - BC^2$, which gives the desired result.

II. *Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.*

Take any four points (vectors) $A$, $B$, $C$, and $D$ in $n$-dimensional space. (The plane is a special case.) Define $M = \frac{1}{2}(A+B)$ and $N = \frac{1}{2}(C+D)$ (the mid-points of the lines joining $A$ to $B$, and $C$ to $D$, respectively). Then

$$\begin{aligned} & AN^2 + DM^2 + BC^2 - BN^2 - CM^2 - AD^2 \\ &\quad= |A-N|^2 + |D-M|^2 + |B-C|^2 \\ &\qquad\quad - |B-N|^2 - |C-M|^2 - |A-D|^2 \\ &\quad= -2A\cdot N - 2D\cdot M - 2B\cdot C + 2B\cdot N + 2C\cdot M + 2A\cdot D \\ &\quad= -A\cdot(C+D) - D\cdot(A+B) - 2B\cdot C \\ &\qquad\quad + B\cdot(C+D) + C\cdot(A+B) + 2A\cdot D \;=\; 0\,, \end{aligned}$$

and thus, $AN^2 + DM^2 + BC^2 = BN^2 + CM^2 + AD^2$.

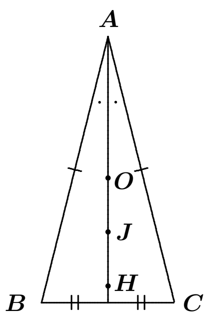
*Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; VEDULA N. MURTY, Dover, PA, USA; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; MARCELO RUFINO DE OLIVEIRA, Belém, Brazil; BOB SERKEY, Leonia, NJ, USA; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESÚS VILLAR RUBIO, Santander,* *Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student (Grade 10), Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

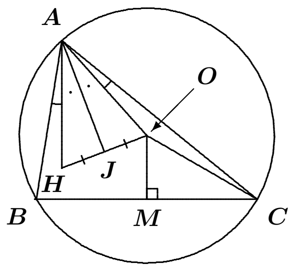
*Most solvers used vectors in some manner more or less like Solution II above. A shortcut is possible in Solution I: equation (1) and the similar equations that follow it are instances of a standard theorem about the median of a triangle. Several solvers used this theorem.*

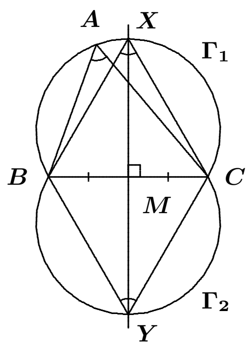
**2855**. [2003 : 316] *Proposed by Antreas P. Hatzipolakis and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given two points $B$ and $C$, find the locus of the point $A$ such that the centre of the nine-point circle of $\triangle ABC$ lies on the interior bisector of $\angle CAB$.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*




Denote the centre of the nine-point circle of $\triangle ABC$ by $J$, and let $O$ and $H$ denote the circumcentre and the orthocentre of $\triangle ABC$, respectively. Then $J$ is the mid-point of $OH$.

If $AB = AC$ (see the first figure), then $O$ and $H$ lie on the interior bisector of $\angle CAB$, and then so does $J$.

Now suppose that $AB \neq AC$. If $\angle CAB \geq 90^\circ$, we can easily verify that $J$ does not lie on the interior bisector of $\angle CAB$. Therefore, we must have $\angle CAB < 90^\circ$ (see the second figure). Since $AH$ and $AO$ are isogonal conjugates with respect to $\angle BAC$, it follows that $AJ$ is the interior bisector of $\angle HAO$. Since $J$ is the mid-point of $HO$, we have $AH = AO$.

Let $M$ be the mid-point of $BC$. Then $OM \perp BC$. It is well known that $AH = 2OM$. Hence, $OC = AO = AH = 2OM$. Thus, $\angle MOC = 60^\circ$. Therefore, $\angle BAC = \angle MOC = 60^\circ$.

Let $\triangle XBC$ and $\triangle YBC$ be equilateral triangles (see the third figure). Denote arc $BXC$ of the circumcircle of $\triangle XBC$ by $\Gamma_1$ and arc $BYC$ of the circumcircle of $\triangle YBC$ by $\Gamma_2$. Since $\angle BAC = \angle BXC = \angle BYC = 60^\circ$, the point $A$ lies either on $\Gamma_1$ or $\Gamma_2$.

We conclude that the locus is the figure consisting of the perpendicular bisector of $BC$ and $\Gamma_1 \cup \Gamma_2$ (excluding $B$, $C$, and the mid-point $M$ of $BC$).

II. *Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

It is known that the centre $X(5)$ of the nine-point circle has trilinear coordinates $\cos(B-C)$, $\cos(C-A)$, $\cos(A-B)$. (See Clark Kimberling's *Encyclopaedia of Triangle Centers*: `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.) Since the trilinears are proportional to the directed distance to the sides $BC$, $CA$, $AB$, and the distance from $X(5)$ to the sides $AB$ and $AC$ are, by assumption, equal (angle bisector property), we are left with the condition:

$$\cos(C-A) \;=\; \cos(A-B)\,.$$

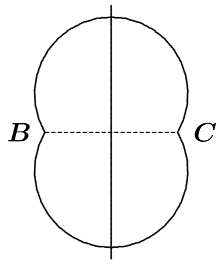
This is equivalent to $C-A=\pm(A-B)$. (There are no other possibilities, since $A$, $B$, $C<\pi$.) If $C-A=A-B$, we get $3A=A+B+C=\pi$, and hence, $A=\pi/3$. If $C-A=-(A-B)$, then $B=C$.

Thus, the desired loci of $A$ are

(i) an arc with $\angle CAB=60^\circ$ constructed over segment $BC$ on one side;

(ii) the same arc constructed over $BC$ on the other side; and

(iii) the perpendicular bisector of segment $BC$.




*Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers. Two incomplete solutions were received, where the solvers found the arcs, but not the straight line. Most solvers used trilinears.*

**2856**. [2003 : 317] *Proposed by Óscar Ciaurri, Universidad de La Rioja, Logroño, Spain.*

Let $a_k=\dfrac{q^k-1}{q-1}$, where $q$ is a real number, $q\neq 1$. For integers $n\geq 0$ and $k\geq 1$, define $C_{n,k}$ as follows: $C_{n,1}=1$, $C_{0,k}=0$ for $k\geq 2$, and $C_{n,k}=\sum\limits_{j=0}^{n-1}\dfrac{a_{k-1}^j}{a_k^{j+1}}C_{j,k-1}$ for $n\geq 1$ and $k\geq 2$.

Show that

$$C_{n,k} \;=\; -(q-1)^{k-1}\sum_{i=1}^{k}\left(\frac{q^i-1}{q^k-1}\right)^n \frac{q^i\big\langle\, q^{-k},q\big\rangle_i}{\big\langle\, q,q\big\rangle_{i-1}\big\langle\, q,q\big\rangle_k}\,,$$

where $\big\langle\, a,q\big\rangle_0=1$ and $\big\langle\, a,q\big\rangle_i=(1-a)(1-aq)\cdots(1-aq^{i-1})$ for $i\geq 1$.

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Note first that $C_{n,k}$ can be defined by the alternative recurrence

$$C_{n,k} = \frac{a_{k-1}^{n-1}}{a_k^n} C_{n-1,k-1} + C_{n-1,k} \tag{1}$$

for $n \geq 1$ and $k \geq 2$. For clarity, we let

$$B_{n,k} = -(q-1)^{k-1} \sum_{i=1}^{k} \left( \frac{q^i - 1}{q^k - 1} \right)^n \frac{q^i \{ q^{-k}, q \}_i}{\{ q, q \}_{i-1} \{ q, q \}_k},$$

and show $B_{n,k}$ satisfies both (1) and the boundary conditions.

Clearly, for $n \geq 0$,

$$B_{n,1} = -\frac{q(1-q^{-1})}{1-q} = 1.$$

Assume next $k \geq 2$. Substituting $z = q$ and $z = q^2$ into the well-known $q$–Binomial Theorem (see [1])

$$\prod_{i=1}^{k} \left( 1 - \frac{z}{q^i} \right) = \sum_{i=0}^{k} \frac{\{ q^{-k}, q \}_i}{\{ q, q \}_i} z^i,$$

we get

$$\sum_{i=0}^{k} \frac{q^i \{ q^{-k}, q \}_i}{\{ q, q \}_i} = 0 \qquad \text{and} \qquad \sum_{i=0}^{k} \frac{q^{2i} \{ q^{-k}, q \}_i}{\{ q, q \}_i} = 0.$$

Subtracting the two equations, we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{k} \frac{q^i (1 - q^i) \{ q^{-k}, q \}_i}{\{ q, q \}_i} \\ &= \sum_{i=1}^{k} \frac{q^i \{ q^{-k}, q \}_i}{\{ q, q \}_{i-1}} = -\frac{\{ q, q \}_k}{(q-1)^{k-1}} B_{0,k}. \end{aligned}$$

Thus, $B_{0,k} = 0$.

Finally, for $n \geq 1$ and $k \geq 2$, we have

$$\begin{aligned}
&\frac{a_{k-1}^{n-1}}{a_k^n} B_{n-1,k-1} + B_{n-1,k} \\
&= -\frac{(q-1)^{k-1}}{(q^k-1)^n} \sum_{i=1}^{k-1} \frac{(q^i-1)^{n-1} q^i \left\{ q^{-k+1}, q \right\}_i}{\left\{ q, q \right\}_{i-1} \left\{ q, q \right\}_{k-1}} \\
&\quad - \frac{(q-1)^{k-1}}{(q^k-1)^{n-1}} \sum_{i=1}^{k} \frac{(q^i-1)^{n-1} q^i \left\{ q^{-k}, q \right\}_i}{\left\{ q, q \right\}_{i-1} \left\{ q, q \right\}_k} \\
&= -\frac{(q-1)^{k-1}}{(q^k-1)^n \left\{ q, q \right\}_k} \left( \sum_{i=1}^{k-1} \frac{(q^i-1)^{n-1} q^i (1-q^k)}{\left\{ q, q \right\}_{i-1}} \right. \\
&\quad \left. \cdot \left[ \left\{ q^{-k+1}, q \right\}_i - \left\{ q^{-k}, q \right\}_i \right] + \frac{(q^k-1)^n q^k \left\{ q^{-k}, q \right\}_k}{\left\{ q, q \right\}_{k-1}} \right) \\
&= -\frac{(q-1)^{k-1}}{(q^k-1)^n \left\{ q, q \right\}_k} \sum_{i=1}^{k} \frac{(q^i-1)^n q^i \left\{ q^{-k}, q \right\}_i}{\left\{ q, q \right\}_{i-1}} \\
&= B_{n,k} \,,
\end{aligned}$$

completing the proof.

**Reference.**

[1] Eric W. Weisstein, "$q$–Binomial Theorem", from MathWorld – A Wolfram Web Resource:
`http://mathworld.wolfram.com/q-BinomialTheorem.html`

*Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and the proposer.*

*Guersenzvaig notes that we must require that $q \neq 0$ and $q \neq -1$. He further notes that the result holds for an arbitrary field $\mathbb{F}$, where $q$ is any non-zero element which is not a root of unity in $\mathbb{F}$.*

**2857**. [2003 : 317] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let $O$ be an interior point of $\triangle ABC$, and let $D$, $E$, $F$, be the intersections of $AO$, $BO$, $CO$ with $BC$, $CA$, $AB$, respectively.

Suppose that $P$ and $Q$ are points on the line segments $BE$ and $CF$, respectively, such that $\frac{BP}{PE} = \frac{CQ}{QF} = \frac{DO}{OA}$.

Prove that $PF \parallel QE$.

I. *Solution by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let the parallel to $BC$ through $O$ intersect $AC$ at $R$ and $AB$ at $S$. The equal ratios

$$\frac{BP}{PE} = \frac{DO}{OA} = \frac{BS}{SA}$$

imply that $PS \parallel CA$. In the same way, $QR \parallel BA$ (using $CQ/QF$). Because corresponding sides are parallel, we have $\triangle OPS \sim \triangle OER$ and $\triangle OFS \sim \triangle OQR$. Moreover, the side $OS$ is common to triangles $OPS$ and $OFS$, while the side $OR$ is common to triangles $OER$ and $OQR$. Consequently, the quadrangles $OFSP$ and $OQRE$ are similar and, moreover, have corresponding sides parallel. It follows that their corresponding parts, $\triangle PSF$ and $\triangle ERQ$, are similar. These two triangles have two pairs of corresponding sides parallel, namely $PS \parallel ER$ and $SF \parallel RQ$; hence, their third sides $PF$ and $EQ$ must also be parallel, as desired.

*Editor's comment.* As an alternative to using similar quadrangles, one can use the dilative rotation that takes $OR$ to $OS$ (defined to be the product of the half-turn about $O$ and the dilatation whose ratio is $OS : OR$). Since this transformation takes each line to a parallel line, it takes $RQ$ to $SF$ and $RE$ to $SP$. Thus, $QE$ must be parallel to its image $FP$.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, shortened by the editor.*

We will use the given equal ratios in the form $\frac{EP}{EB} = \frac{FQ}{FC} = \frac{AO}{AD}$. Let $\frac{CD}{DB} = \frac{\lambda}{1-\lambda}$ (so that $CD/CB = \lambda$ and $BD/BC = 1 - \lambda$). By Menelaus' Theorem applied to triangle $COD$ and line $AFB$,

$$\frac{FO}{FC} = \frac{AO}{AD} \cdot \frac{BD}{BC} = \frac{FQ}{FC} \cdot (1-\lambda)\,;$$

thus, $FO/FQ = 1 - \lambda$, or

$$\frac{FO}{OQ} = \frac{1-\lambda}{\lambda}\,.$$

Similarly, Menelaus' Theorem applied to triangle $BDO$ and line $AEC$ yields

$$\frac{EO}{EB} = \frac{AO}{AD} \cdot \frac{CD}{CB} = \frac{EP}{EB} \cdot \lambda\,;$$

thus, $EO/EP = \lambda$, or

$$\frac{EO}{OP} = \frac{\lambda}{1-\lambda}\,.$$

It follows that $\frac{FO}{OQ} = \frac{PO}{OE}$ and, therefore, that $FP$ is parallel to $QE$.

*Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

**2858**. [2003 : 318] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $P$ is an interior point of $\triangle ABC$, and that $D$, $E$, $F$ are the intersections of $AP$, $BP$, $CP$ with $BC$, $CA$, $AB$, respectively. Suppose that

$$\frac{AE+AF}{BC} = \frac{BF+BD}{CA} = \frac{CD+CE}{AB}.$$

Characterize the point $P$.

*A combination of similar solutions by Francisco Bellot Rosado, I.B. Emilio Ferrari, and María Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain; Shailesh Shirali, Rishi Valley School, India; and Andrei Simion, student, Cooper Union for Advancement of Science and Art, New York, NY, USA.*

We show that the given condition is satisfied if and only if $P$ is the Nagel point of triangle $ABC$. As usual, we denote the side lengths of the triangle by $a$, $b$, $c$, and its semiperimeter $(a+b+c)/2$ by $s$.

For equality of the three ratios

$$\frac{AE+AF}{BC}, \quad \frac{BF+BD}{CA}, \quad \frac{CD+CE}{AB},$$

each of them must equal

$$\frac{AE+AF+BF+BD+CD+CE}{BC+CA+AB} = \frac{BC+CA+AB}{BC+CA+AB} = 1.$$

Consequently,

$$AE+AF = a, \quad BF+BD = b, \quad CD+CE = c. \tag{1}$$

Moreover, by definition

$$BD+DC = a, \quad CE+EA = b, \quad AF+FB = c. \tag{2}$$

We shall assume (without loss of generality) that $a \le b \le c$, and let $BD = x$. Then the other relevant segments satisfy

$$DC = a-x, \quad CE = c-a+x, \quad EA = a+b-c-x,$$
$$AF = c-b+x, \quad FB = b-x.$$

Since the three cevians are concurrent at $P$, Ceva's Theorem implies

$$x\cdot(c-a+x)\cdot(c-b+x) = (a-x)\cdot(a+b-c-x)\cdot(b-x). \tag{3}$$

Since all factors in equation (3) are positive when $0 < x < a$, the product on the left increases with $x$ while the product on the right decreases with $x$; therefore, the equation must have exactly one real root. To find that root, we note that the equations in (1) and (2) would be satisfied by $BF = CE$, $AF = CD$, and $AE = BD$. Each of these equalities is equivalent to

$2x = a + b - c$, or $x = s - c$. One easily checks that, indeed, $x = s - c$ is a solution (and therefore, *the* solution) of (3). Our segments are therefore

$$BF = CE = s - a\,, \quad AF = CD = s - b\,, \quad AE = BD = s - c\,.$$

These equalities tell us that $D$, $E$, $F$ are the points of tangency of the excircles with the sides of triangle, which makes $P$ the Nagel point of triangle $ABC$.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK (partial solution); CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

*Two solvers submitted solutions that were correct in every way except for the terminology. For their benefit, and for the benefit of readers who have not yet memorized Clark Kimberling's list of triangle centres [Kimberling, C.,* Encyclopedia of Triangle Centers *($X(8)$ = Nagel Point):* `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X8`], *the Nagel point is defined to be the intersection point of the lines joining each vertex to the point where the opposite side touches the opposite excircle. Equivalently, the Nagel point is the isotomic conjugate of the Gergonne point. Moreover, each of the points $D$, $E$, $F$ bisects the distance around the perimeter from the opposite vertex. More details can be found on the* Mathworld *web page,* `http://mathworld.wolfram.com/`, *or in books dealing with the geometry of the triangle.*

**2859★**. [2003 : 318] *Proposed by Mohammed Aassila, Strasbourg, France.*

Prove that $\displaystyle\sum_{\text{cyclic}} \frac{ab}{c(c+a)} \;\geq\; \sum_{\text{cyclic}} \frac{a}{c+a}$, where $a$, $b$, $c$ represent the three sides of a triangle.

*Solution by Pierre Bornsztein, Maisons-Laffitte, France; and Nick Skombris and Babis Stergiu, Chalkida, Greece.*

Let $x = a/b$, $y = b/c$ and $z = c/a$. Then $xyz = 1$ and the inequality becomes

$$\frac{x-1}{y+1} + \frac{y-1}{z+1} + \frac{z-1}{x+1} \;\geq\; 0\,,$$

which is equivalent to

$$x^2 + y^2 + z^2 - x - y - z + xy^2 + yz^2 + zx^2 - 3 \;\geq\; 0\,. \qquad (1)$$

Applying the AM–QM and AM–GM inequalities, we obtain

$$x^2 + y^2 + z^2 \;\geq\; \frac{1}{3}(x+y+z)^2 \;\geq\; \sqrt[3]{xyz}(x+y+z) \;=\; x+y+z \quad (2)$$

and

$$xy^2 + yz^2 + zx^2 \;\geq\; 3\sqrt[3]{x^3y^3z^3} \;=\; 3\,. \qquad (3)$$

Adding (2) and (3) gives (1). Equality holds when $x = y = z$. Therefore, the desired inequality is true, with equality when $a = b = c$.

*Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); VASILE CIRTOAJE, University of Ploiesti, Romania; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.*

*Most solvers noticed that the condition that $a$, $b$, and $c$ are the sides of a triangle is unnecessarily restrictive; the inequality is true for any three positive real numbers $a$, $b$, and $c$, as seen from the presented solution. Specht and Stergiu indicated that this problem was one of the problems of the 1999 Moldavian Mathematical Olympiad. Specht gave the following web address as a reference:* `http://www.olsedim.com/olympiad/99.html`

**2860**. [2003 : 318] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

In $\triangle ABC$ and $\triangle A'B'C'$, the lengths of the sides satisfy $a \geq b \geq c$ and $a' \geq b' \geq c'$. Let $h_a$ and $h_{a'}$ denote the lengths of the altitudes to the opposite sides from $A$ and $A'$, respectively. Prove that

(a) $bb' + cc' \geq ah_{a'} + a'h_a$ ;

(b) $bc' + b'c \geq ah_{a'} + a'h_a$ .

I. *Solution by Michel Bataille, Rouen, France.*

Note that $A$ is the largest angle. Thus, the foot of the altitude from $A$ certainly falls between $B$ and $C$. It follows that $\sin B = \frac{h_a}{c}$ and $\sin C = \frac{h_a}{b}$. Similarly, $\sin B' = \frac{h_{a'}}{c'}$ and $\sin C' = \frac{h_{a'}}{b'}$. We have

$$\begin{aligned} ah_{a'} + a'h_a &= ab'\sin C' + a'c\sin B \\ &= b'\sin C'(b\cos C + c\cos B) \\ &\qquad + c\sin B(b'\cos C' + c'\cos B') \\ &= bb'\sin C'\cos C + b'c\sin C'\cos B \\ &\qquad + b'c\sin B\cos C' + cc'\sin B\cos B' . \qquad (1) \end{aligned}$$

(a) From (1),

$$\begin{aligned} ah_{a'} + a'h_a &= bb'\sin C'\cos C + cc'\sin B'\cos B \\ &\qquad + bb'\sin C\cos C' + cc'\sin B\cos B' \\ &= bb'\sin(C + C') + cc'\sin(B + B') \;\leq\; bb' + cc' . \end{aligned}$$

(b) From (1),

$$\begin{aligned} ah_{a'} + a'h_a &= bc'\sin B'\cos C + b'c\sin C'\cos B \\ &\qquad + b'c\sin B\cos C' + bc'\sin C\cos B' \\ &= bc'\sin(B' + C) + b'c\sin(B + C') \;\leq\; bc' + b'c . \end{aligned}$$

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

(a) Scale $\triangle ABC$ by $a'$ to get $\triangle PQS$ with $PQ = a'c$, $QS = a'a$, and $SP = a'b$. Scale $\triangle A'B'C'$ by $a$ to get $\triangle RSQ$ with $RS = ac'$, $QR = ab'$, and $R$ and $P$ on opposite sides of $SQ$. Applying Ptolemy's Inequality to $PQRS$, we get

$$aa'bb' + aa'cc' \;=\; SP \cdot QR + PQ \cdot RS \;\geq\; SQ \cdot PR \;\geq\; aa'(ah_{a'} + a'h_a)\,.$$

Equality holds if and only if $PQRS$ is cyclic and $PR \perp SQ$.

(b) We obtain this part similarly by modifying the above proof so that $RS = ab'$ and $QC = ac'$.

*Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; and the proposer.*

*Klamkin observes that $b'(b-c) \geq c'(b-c)$, which is equivalent to $bb'+cc' \geq bc'+b'c$. Thus, it is sufficient to prove part (b).*

**2861**. [2003 : 319] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

The circle $\Gamma(P,r)$ intersects the side $AB$ of $\triangle ABC$ at $A_3$ and $B_3$, the side $BC$ at $B_1$ and $C_1$, and the side $CA$ at $C_2$ and $A_2$.

Given that $|A_3B_3| : |B_1C_1| : |C_2A_2| = |AB| : |BC| : |CA|$, determine the locus of $P$.

*Partial solution by the proposer.*

Denote the lengths of $B_1C_1$, $C_2A_2$, and $A_3B_3$ by $2\lambda a$, $2\lambda b$, and $2\lambda c$, respectively, and let $x$, $y$, and $z$ be the distances from $P$ to the sides of the triangle (that is, to the mid-points of these three segments). Since $P$ is the centre of a circle through the end-points of these segments,

$$x^2 + \lambda^2 a^2 \;=\; y^2 + \lambda^2 b^2 \;=\; z^2 + \lambda^2 c^2\,.$$

Eliminating $\lambda^2$, we find

$$x^2(b^2-c^2) + y^2(c^2-a^2) + z^2(a^2-b^2) \;=\; 0\,,$$

which we recognize as the equation of a conic in trilinear coordinates. Indeed, it is the conic through the circumcentre (where $\lambda = 1/2$), and the four tritangent centres (that is, the incentre and the excentres, where $\lambda = 0$). Moreover, since the conic passes through the tritangent centres, it must be a rectangular hyperbola (because each degenerate conic in the pencil determined by $I$, $I_A$, $I_B$, $I_C$ consists of a perpendicular pair of lines).

*The only submitted solution came from the proposer. That is a pity—this seems to be a nice problem. Smeenk's argument establishes only that if $P$ is a point of the locus, then it lies on the conic. To determine the locus, of course, one must also say which points of the conic are appropriate. It is clear that to be able to solve for $\lambda$ we must have the magnitudes of the* *distances $x$, $y$, $z$ in an order opposite to the corresponding side lengths $a$, $b$, $c$; more precisely, if we label the triangle so that $a \geq b \geq c$, then for $\lambda$ to exist we must have $x \leq y \leq z$. For an equilateral triangle this implies that the locus consists of four points—the four tritangent centres; the fifth point $O$ will coincide with $I$. For an isosceles triangle, say with $b = c$, the hyperbola degenerates into the two bisectors of angle $A$. In this case the locus consists of the segment between the two tritangent centres on one bisector, and of the complement of that segment on the other (depending on whether the circumcentre lies between $I$ and $I_A$ or not). For a scalene triangle it is not obvious to this editor how $\lambda$ is related to the locus, nor is it clear how much of the hyperbola belongs to the locus. Is it not the case that one branch of the hyperbola contains one excentre while the other branch contains the other excentres along with the incentre and the circumcentre?*

**2862**. [2003 : 319] *Proposed by Mihály Bencze, Brasov, Romania.*

The sequence $\{x_n\}$ is defined by $\left(1+\dfrac{1}{n}\right)^{n+x_n} = e$.

(a) Prove that $\{x_n\}$ is convergent, and determine its limit.

(b)$\star$ Determine the asymptotic expansion of the sequence.

*Solution by Chip Curtis, Missouri Southern State College, Joplin, MO, USA.*

We solve the given equation for $x_n$ as follows:

$$\begin{aligned} (n+x_n)\log\left(1+\frac{1}{n}\right) &= 1\,, \\ x_n \log\left(1+\frac{1}{n}\right) &= 1 - n\log\left(1+\frac{1}{n}\right)\,, \\ x_n &= \frac{1-n\log\left(1+\frac{1}{n}\right)}{\log\left(1+\frac{1}{n}\right)}\,. \end{aligned}$$

Using a change of variables and three applications of L'Hôpital's Rule, we obtain

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{1-n\log\left(1+\frac{1}{n}\right)}{\log\left(1+\frac{1}{n}\right)} = \lim_{t\to 0+} \frac{1-\frac{1}{t}\log(1+t)}{\log(1+t)} \\ &= \lim_{t\to 0+} \frac{-\dfrac{1}{t(1+t)} + \dfrac{1}{t^2}\log(1+t)}{\dfrac{1}{1+t}} \\ &= \lim_{t\to 0+} \frac{-t+(1+t)\log(1+t)}{t^2} \\ &= \lim_{t\to 0+} \frac{\log(1+t)}{2t} = \lim_{t\to 0+} \frac{1}{2(1+t)} = \frac{1}{2}\,. \end{aligned}$$

In the discussion following equation (7.59) on p. 352 of [1], it is pointed out that

$$\frac{z}{\log(1+z)} = -\sum_{k\geq 0}(-z)^k\sigma_k(k-1)\,,$$

where $\sigma_k(x)$ is the $k^{\text{th}}$ *Stirling polynomial* ([1], p. 271). Since

$$x_n = \frac{1}{\log\left(1+\frac{1}{n}\right)} - n\,,$$

we have

$$x_n = -n\sum_{k\geq 0}\left(-\frac{1}{n}\right)^k\sigma_k(k-1) - n = \sum_{k\geq 1}\frac{(-1)^{k-1}}{n^{k-1}}\sigma_k(k-1)\,.$$

Again from [1], p. 271, we have the expressions for $\sigma_k(x)$ shown on the left below. Corresponding values of $\sigma_k(k-1)$ are shown on the right.

$$\begin{aligned}
\sigma_0(x) &= \frac{1}{x}\,, & \sigma_0(-1) &= -1\,,\\
\sigma_1(x) &= \frac{1}{2}\,, & \sigma_1(0) &= \frac{1}{2}\,,\\
\sigma_2(x) &= \frac{3x-1}{24}\,, & \sigma_2(1) &= \frac{1}{12}\,,\\
\sigma_3(x) &= \frac{x^2-x}{48}\,, & \sigma_3(2) &= \frac{1}{24}\,,\\
\sigma_4(x) &= \frac{15x^3-30x^2+5x+2}{5760}\,, & \sigma_4(3) &= \frac{19}{720}\,,\\
&\ \vdots & &\ \vdots
\end{aligned}$$

Hence,

$$x_n = \frac{1}{2} - \frac{1}{12n} + \frac{1}{24n^2} - \frac{19}{720n^3} + \cdots\,.$$

**Reference.**

[1] Ronald Graham, Donald Knuth, and Oren Patashnik, *Concrete Mathematics*, Second Edition, Addison-Wesley, 1994.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, EMILIO FERNÁNDEZ, and M. CARMEN MÍNGUEZ, Logroño, Spain; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; VEDULA N. MURTY, Dover, PA, USA; SHAILESH SHIRALI, Rishi Valley School, India; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; MIKE SPIVEY, Samford University, Birmingam, AL, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. PIERRE BORNSZTEIN, Maisons-Laffitte, France solved part* (a).

*Several solvers used the Maclaurin series expansion for* $\ln(1+x)$ *to obtain, for example,*

$$x_n \;=\; \frac{1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right)}{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots} \;=\; \frac{\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots}{1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots}\,,$$

*from which the limit of* $\frac{1}{2}$ *is clear. Moreover, the asymptotic expansion can be discovered as follows:*

$$\begin{aligned} x_n &= \frac{1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{3n^3}{-}\frac{1}{4n^4} + \cdots\right)}{1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots} \\ &= \frac{1}{2} + \frac{-\frac{1}{12n} + \frac{1}{12n^2} - \frac{3}{40n^3} + \cdots}{1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots} \\ &= \frac{1}{2} - \frac{1}{12n} + \frac{\frac{1}{24n^2} - \frac{17}{360n^3} + \cdots}{1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots} \\ &\;\vdots \end{aligned}$$

*Benito, Ciaurri, Fernández, and Mínguez, provided a solution to the problem using the* Rey Pastor *numbers and polynomials.*

**2863**. [2003 : 319] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $a$, $b$, $c$ are complex numbers such that $|a| = |b| = |c|$. Prove that

$$\left|\frac{ab}{a^2 - b^2}\right| + \left|\frac{bc}{b^2 - c^2}\right| + \left|\frac{ca}{c^2 - a^2}\right| \;\geq\; \sqrt{3}\,.$$

*Solution by CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA, modified slightly by the editor.*

[*Ed:* Clearly, we must assume that $a^2$, $b^2$, and $c^2$ are all distinct and hence, $abc \neq 0$.]

Set $a = re^{i\alpha}$, $b = re^{i\beta}$, and $c = re^{i\gamma}$. Then

$$\left|\frac{ab}{a^2 - b^2}\right| \;=\; \frac{1}{\left|\frac{a}{b} - \frac{b}{a}\right|} \;=\; \frac{1}{\left|e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)}\right|} \;=\; \frac{1}{2}|\csc(\alpha-\beta)|\,.$$

By symmetry, the given inequality is then equivalent to

$$|\csc(\alpha-\beta)| + |\csc(\beta-\gamma)| + |\csc(\gamma-\alpha)| \;\geq\; 2\sqrt{3}\,. \qquad (1)$$

By the AM–GM Inequality, we have

$$\begin{aligned} &|\csc(\alpha-\beta)| + |\csc(\beta-\gamma)| + |\csc(\gamma-\alpha)| \\ &\qquad\geq\; 3|\csc(\alpha-\beta)\csc(\beta-\gamma)\csc(\gamma-\alpha)|^{1/3} \\ &\qquad=\; 3|\csc(\alpha-\beta)\csc(\beta-\gamma)\csc(\pi-(\alpha-\gamma))|^{1/3} \\ &\qquad=\; 3|\csc A\csc B\csc C|^{1/3}\,, \qquad (2) \end{aligned}$$

where $A = \alpha - \beta$, $B = \beta - \gamma$, and $C = \pi - (A + B)$.

Consider the function $f(t) = \ln|\sin t|$. Since $f''(t) = -\csc^2 t < 0$, we see that $f$ is convex. Hence, by Jensen's Inequality, we have

$$\begin{aligned}
f(A) + f(B) + f(C) &\leq 3f\left(\frac{A+B+C}{3}\right) = 3\ln\left(\sin\frac{\pi}{3}\right) \\
\ln|\sin A \sin B \sin C| &\leq 3\ln\left(\frac{\sqrt{3}}{2}\right) \\
|\sin A \sin B \sin C| &\leq \left(\frac{\sqrt{3}}{2}\right)^3 \\
|\csc A \csc B \csc C|^{1/3} &\geq \frac{2}{\sqrt{3}}\,. \qquad (3)
\end{aligned}$$

Substituting (3) into (2), we see that (1) follows immediately.

*Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; ÓSCAR CIAURRI, Universidad de La Rioja, Logroño, Spain and JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the propser.*

*Bataille also showed that equality holds if and only if the triangle with $a$, $b$, $c$ as vertices is either equilateral or isosceles with an angle of $2\pi/3$. Janous established the stronger inequality that for non-zero complex numbers $a$, $b$, and $c$, we have*

$$\left|\frac{a^2-b^2}{ab}\right|^2 + \left|\frac{b^2-c^2}{bc}\right|^2 + \left|\frac{c^2-a^2}{ca}\right|^2 \leq 9\,.$$

*It is easy to show, via the Power–Mean Inequality and the AM–HM Inequality, that this result would imply the inequality in the proposed problem.*