SKOLIAD No. 77

Shawn Godin

Please send your solutions to the problems in this edition by *1 October 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 3** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (∗) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

This month's problems are drawn from the county-wide and national mathematics competitions held by the Croatian Mathematical Society in 2003. Thanks to Mr. Željko Hanjš of the Croatian Mathematical Society for making these problems available.

Croatian Mathematical Society City-Level Competition
Junior Level (Grade 1), March 7, 2003

1. A road construction unit is made up of a certain number of workers and a certain amount of equipment. Three units have paved 20 km of a road in 10 days. How many additional units are needed if the remaining 50 km of the road must be paved in 15 days?

2. Let $\triangle ABC$ be an isosceles triangle whose angle at vertex A equals 120°. The line passing through this vertex and perpendicular to one of the adjacent sides of the triangle divides the triangle into two triangles, one of which is obtuse and has an inscribed circle with radius equal to 1. Determine the area of $\triangle ABC$.

3. Calculate the sum

$$\frac{2}{2\cdot 5}+\frac{2}{5\cdot 8}+\cdots+\frac{2}{1997\cdot 2000}+\frac{2}{2000\cdot 2003}\,.$$

4. If the real numbers a, b, c satisfy

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1\,,$$

prove that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} = 0\,.$$

Next we present a solution from the 2003 Fryer Contest in the September 2003 issue ([2003 : 259]).

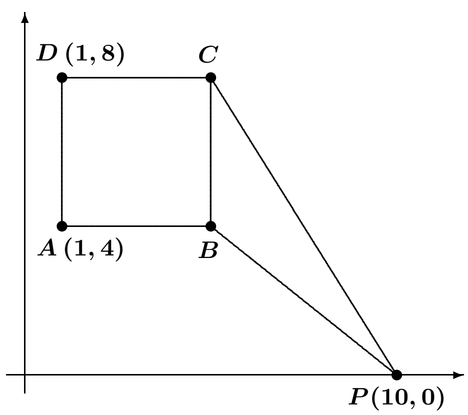
3. In the diagram, $ABCD$ is a square and the coordinates of A and D are as shown.

(a) ($*$) The point P has coordinates $(10,0)$. Show that the area of triangle PCB is 10.

(b) ($*$) Point $E(a,0)$ is on the x–axis such that triangle CBE lies entirely outside square $ABCD$. If the area of the triangle is equal to the area of the square, what is the value of a?

(c) ($*$) Show that there is no point F on the x–axis for which the area of triangle ABF is equal to the area of square $ABCD$.

Extension to #3: ($*$) G is a point on the line passing through the points $M(0,8)$ and $N(3,10)$ such that $\triangle DCG$ lies entirely outside the square. If the area of $\triangle DCG$ is equal to the area of the square, determine the coordinates of G.



Solution by Karthik Natarajan, Grade 5 student, Edgewater Park Public School, Thunder Bay, ON.

(a) We need to calculate the base, which is side CB, and the height of triangle PCB. Since we know the coordinates of $A = (1,4)$ and $D = (1,8)$, side AD must be 4. Since $ABCD$ is a square, $CB = AD = AB = CD = 4$. Because B is to the right of A, we must have $B = (5,4)$. The height of triangle PCB is now $10 - 5 = 5$. Therefore, the area of triangle PCB is $4 \times 5/2 = 10$.

(b) Because triangle CBE lies outside of $ABCD$, the x–coordinate of E is greater than 5. Also, $ABCD$ is a square of area 16. Side $CB = 4$, and because the area of triangle CBE is equal to the area of square $ABCD$, the height of triangle CBE must be 8. We add the height to the x–coordinate of B to get E as $(13,0)$.

(c) In triangle ABF, point F is on the x–axis and side AB has length 4. We know that the distance from AB to the x-axis is 4. Therefore, the height of F from AB is always 4. This means that the area of triangle ABF is always $4 \times 4/2 = 8$. Thus, there cannot be a point F on the x–axis such that the area of triangle ABF is equal to the area of square $ABCD$ (which is 16).

Extension to #3: Side DC has length 4. Since the area of triangle DCG is equal to the area of square $ABCD$, which is 16, the height of G from DC must be 8 (because $4 \times 8/2 = 16$). Now we add the height to the distance from the x–axis to line DC, and we get 16, which is the y–coordinate of G.

I figured out the x–coordinate of G by following a pattern. The point G is on a line passing through the points $(0,8)$ and $(3,10)$. We see that when the x–coordinate goes up by 3, the y–coordinate goes up by 2 on this line. Following this pattern, we get: $(0,8)$, $(3,10)$, $(6,12)$, $(9,14)$, and $(12,16)$. Therefore, the coordinates of G are $(12,16)$.

Lastly, we present some solutions and generalizations to the 2002 W.J. Blundon Mathematics Contest [2003 : 261–262].

1. $(*)$ Five years ago Janet was one sixth of her mother's age. In thirteen years she will be half her mother's age. What is Janet's present age?

Solution by Yufei Zhao, grade 10 student, Don Mills Collegiate Institute, Toronto, ON.

Let Janet's and her mother's present ages be x and y, respectively. Then the conditions given in the problem form a system of equations

$$\begin{aligned} x - 5 &= \tfrac{1}{6}(y - 5)\,, \\ x + 13 &= \tfrac{1}{2}(y + 13)\,. \end{aligned}$$

Solving this simple system, we get the solution $x = 9\frac{1}{2}$ and $y = 32$. Hence, Janet is $9\frac{1}{2}$ years old now.

2. $(*)$ If $a + b + c = 0$, prove that $a^3 + b^3 + c^3 = 3abc$.

Solution by Yufei Zhao, grade 10 student, Don Mills Collegiate Institute, Toronto, ON.

Since $c = -a - b$,

$$\begin{aligned} a^3 + b^3 + c^3 &= a^3 + b^3 + (-a - b)^3 \\ &= a^3 + b^3 - a^3 - 3a^2b - 3ab^2 - b^3 \\ &= 3ab(-a - b) \\ &= 3abc\,. \end{aligned}$$

3. $(*)$ A certain rectangle has area 6 and diagonal of length $2\sqrt{5}$. What is its perimeter?

Solution by Yufei Zhao, grade 10 student, Don Mills Collegiate Institute, Toronto, ON.

Let the two adjacent sides of the rectangle have lengths a and b. Then its area is $ab = 6$, and its diagonal has length $\sqrt{a^2 + b^2} = 2\sqrt{5}$, giving $a^2 + b^2 = 20$. Thus,

$$\begin{aligned} (a + b)^2 = a^2 + b^2 + 2ab = 32 \quad &\Longrightarrow \quad a + b = 4\sqrt{2}\,, \\ (a - b)^2 = a^2 + b^2 - 2ab = 8 \quad &\Longrightarrow \quad a - b = 2\sqrt{2}\,. \end{aligned}$$

Solving the system, we get $a = 3\sqrt{2}$ and $b = \sqrt{2}$. Therefore, the perimeter of the rectangle is $8\sqrt{2}$.

6. Points A and B are on the parabola $y = 2x^2 + 4x - 2$. The origin is the mid-point of the line segment joining A and B. Find the length of this line segment.

Solution and generalization by Yufei Zhao, grade 10 student, Don Mills Collegiate Institute, Toronto, ON.

Consider a half-turn about the origin. This reflection will exchange the points A and B, so that A and B belong on both the original parabola and the image parabola. The equation of the latter is $-y = 2(-x)^2 + 4(-x) - 2$, or equivalently, $y = -2x^2 + 4x + 2$. Thus, we need to solve the system

$$\begin{aligned} y &= 2x^2 + 4x - 2\,, \\ y &= -2x^2 + 4x + 2\,. \end{aligned}$$

By adding the two equations, we get $y = 4x$, and by substituting this back into either equation, we get $x = \pm 1$. Continuing, we see that the two points of intersection are $(1, 4)$ and $(-1, -4)$, which are the points A and B. The distance between them is $2\sqrt{17}$.

In general, if the parabola were given by the equation $y = ax^2 + bx + c$, then, by following the exactly same procedure as above, we find A and B to be $(\pm\sqrt{-c/a}, \pm b\sqrt{-c/a})$ and $|AB| = 2\sqrt{-c(1 + b^2)/a}$. Of course, in order for it to be feasible, $-c/a$ must be non-negative.

9. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

Generalization by Yufei Zhao, grade 10 student, Don Mills Collegiate Institute, Toronto, ON.

In general, if we are given a line $Ax + By + C = 0$ and an ellipse $(x/a)^2 + (y/b)^2 = 1$, then, by applying the transformation $(x, y) \mapsto (ax, yb)$, the line and the ellipse (which is now a circle) become $aAx + bBy + C = 0$ and $x^2 + y^2 = 1$, respectively. If they are tangent to each other, then the new line must be one unit away from the origin. Thus, $\dfrac{|C|}{\sqrt{a^2A^2 + b^2B^2}} = 1$, or equivalently, $a^2A^2 + b^2B^2 = C^2$, and this is the necessary and sufficient condition for the original line and ellipse to be mutually tangent.

That ends this issue of Skoliad, and the winner of the past volume of Mathematical Mayhem is Yufei Zhao. Congratulations Yufei! Keep sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). ~~The Assistant Mayhem Editor is John Grant McLoughlin~~ (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo) and Dan MacKinnon (Ottawa Carleton District School Board).

Mayhem Problems

Please send your solutions to the problems in this edition by **1 October 2004**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M138. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS and Sarah McCurdy, University of New Brunswick, Fredericton, NB.*

Five points are located on a line. When the ten distances between pairs of points are listed from smallest to largest, the list reads:

$$2,\quad 4,\quad 5,\quad 7,\quad 8,\quad k,\quad 13,\quad 15,\quad 17,\quad 19\,.$$

Determine the value of k.

M139. *Proposed by the Mayhem Staff.*

The digits 1, 2, 3, 4, and 5 are each used once to compose a 5–digit number $abcde$, such that the 3–digit number abc is divisible by 4, bcd is divisible by 5, and cde is divisible by 3. Find the 5–digit number $abcde$.

M140. *Proposed by the Mayhem Staff.*

Arthur, Bernie, and Charlie play a game in which the loser has to triple the money of each other player. Three games are played, in which the losers are Arthur, Bernie, and Charlie, in that order. Each player ends with $27. How much money did each person have at the outset?

M141. *Proposed by the Mayhem Staff.*

Create a list of perfect squares in which all of the digits are perfect squares (that is, 0, 1, 4, 9).

M142. *Proposed by Ali Feizmohammadi, University of Toronto, Toronto, ON.*

For every natural number n, define $S(n)$ to be the unique integer m (if it exists) which satisfies the equation

$$n = \lfloor m \rfloor + \left\lfloor \frac{m}{2!} \right\rfloor + \left\lfloor \frac{m}{3!} \right\rfloor + \cdots + \left\lfloor \frac{m}{k!} \right\rfloor + \cdots,$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(a) Find $S(3438)$.

(b) Does there exist a number k such that, for any non-negative integer n, at least one of $S(n+1)$, $S(n+2)$, $\ldots$, $S(n+k)$ exists?

M143. *Proposed by the Mayhem Staff.*

Find the equation(s) of the line(s) through the point $(2, 5)$ for which the y–intercept is a prime number and the x–intercept is an integer.

. . . .

M138. *Proposé by Richard Hoshino, Université Dalhousie, Halifax, NS et Sarah McCurdy, Université du Nouveau Brunswick, Frédéricton, NB.*

On donne cinq points sur une droite. Quand on écrit les dix distances entre les points en ordre croissant, on obtient la liste :

$$2,\quad 4,\quad 5,\quad 7,\quad 8,\quad k,\quad 13,\quad 15,\quad 17,\quad 19.$$

Trouver la valeur de k.

M139. *Proposé par l'Équipe de Mayhem.*

Chacun des chiffres 1, 2, 3, 4 et 5 est utilisé une seule fois pour écrire un nombre de cinq chiffres $abcde$, et ceci de telle sorte que le nombre de 3 chiffres abc soit divisible par 4, que bcd soit divisible par 5, et que cde soit divisible par 3. Trouver ce nombre de 5 chiffres $abcde$.

M140. *Proposé par l'Équipe de Mayhem.*

Arthur, Bernard et Charles jouent un jeu dans lequel le perdant doit tripler l'argent de chacun des autres joueurs. On joue trois parties et tour à tour, Arthur, puis Bernard et enfin Charles, sort perdant. Chaque joueur finit avec \$27 en poche. Combien d'argent chacun d'eux avait-il au départ ?

M141. *Proposé par l'Équipe de Mayhem.*

Trouver une liste de carrés parfaits où tous les chiffres sont des carrés parfaits (c'est-à-dire 0, 1, 4, 9).

M142. *Proposé par Ali Feizmohammadi, Université de Toronto, Toronto, ON.*

Pour chaque nombre naturel n, on définit $S(n)$ comme l'unique entier m (s'il existe) satisfaisant l'équation

$$n = \lfloor m \rfloor + \left\lfloor \frac{m}{2!} \right\rfloor + \left\lfloor \frac{m}{3!} \right\rfloor + \cdots + \left\lfloor \frac{m}{k!} \right\rfloor + \ldots$$

où $\lfloor x \rfloor$ est le plus grand entier plus petit ou égal à x.

(a) Trouver $S(3438)$.

(b) Existe-t-il un nombre k tel que pour tout entier non négatif n, au moins un des $S(n+1)$, $S(n+2)$, ..., $S(n+k)$ existe ?

M143. *Proposé par l'Équipe de Mayhem.*

Trouver l'équation d'une (ou plusieurs) droite passant par le point $(2, 5)$ et dont l'intersection avec l'axe des x est un entier et celle avec l'axe des y est un nombre premier.

Mayhem Solutions

M77. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS.*

Find *all* ordered pairs of integers (a, b) such that the equation $x^2 + |y^2 - 6ay + b| = b - a^2 + 6$ has *exactly* 2001 solutions in positive integers (x, y).

Editor's Note. The problem cannot be solved as stated. The problem will be reworded and reposted in an upcoming issue.

M78. *Proposed by K.R.S. Sastry, Bangalore, India.*

In a right-angled triangle we consider the two vertices at the two acute angles and draw medians from them to the opposite sides. Determine the maximum (acute) angle between these medians.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

We will show that the maximum angle has measure $\arctan\frac{3}{4}$ and this value occurs only for isosceles triangles.

Let us consider the points $A(0, y_0)$, $B(0,0)$, $C(x_0, 0)$ with x_0, $y_0 > 0$. These points determine a right triangle with right angle at B. The median from C to the side AB joins the point $C(x_0, 0)$ to the point $\left(0, \frac{1}{2}y_0\right)$, and the median from A to the side BC joins the point $A(0, y_0)$ to the point $\left(\frac{1}{2}x_0, 0\right)$. The slopes of these medians are $\frac{-y_0}{2x_0}$ and $\frac{-2y_0}{x_0}$, respectively.

Let α be the acute angle between these medians. Then

$$\alpha \;=\; \arctan\frac{-y_0}{2x_0} - \arctan\frac{-2y_0}{x_0}\,.$$

Applying the well-known identity $\arctan x - \arctan y = \arctan\left(\frac{x-y}{1+xy}\right)$, we obtain

$$\alpha \;=\; \arctan\frac{3y_0x_0}{2(x_0^2+y_0^2)}\,.$$

From the AM–GM Inequality, we have $\frac{x_0^2+y_0^2}{2} \geq x_0y_0$. Therefore, $\frac{3x_0y_0}{2(x_0^2+y_0^2)} \leq \frac{3}{4}$. Thus, we obtain

$$\alpha \;=\; \arctan\frac{3y_0x_0}{2(x_0^2+y_0^2)} \;\leq\; \arctan\frac{3}{4}\,.$$

Note that equality holds if and only if $x_0 = y_0$.

Also solved by Andrew Mao, A.B. Lucas Secondary School, London, ON; Yifei Chen, student, West Windsor Plainsboro High School North, Plainsboro, NJ, USA; and Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

M79. *Proposed by the Mayhem Staff.*

Three people play the following game. N marbles are placed in a bowl and the players, in turn, remove 1, 2, or 3 marbles from the bowl. The person who removes the last marble loses. For what values of N can the first and third player work together to force the second player to lose? (Inspired by a recent problem on the Canadian Open Mathematics Challenge.)

Solution by Geneviève Lalonde, Massey, ON.

Clearly, if $N = 2$, 3, 4, it is possible, as the first player can just remove 1, 2, or 3 marbles and force the second player to take the last marble. If $N = 5$, 6, the first and third players cannot force the second player to lose because he can always play so that the sum of his and the first player's play is 4 (which forces the third player to lose if $N = 5$, or the first player to lose if $N = 6$).

For $N > 6$ the first player and the third player can always force the second player to lose by adopting the following strategy: on the first play of the game the first player always removes 1 marble, leaving $N - 1$ marbles. After the second player's first play, there will be k marbles remaining where $N - 4 \le k \le N - 2$. Thus, $3 \le k$.

Now the third and the first players can conspire to leave k' marbles, $k' \equiv 1 \pmod 5$. This is always possible, because the sum of the marbles removed by the third player and the first player can be 2, 3, 4, 5, or 6. Thus, for $k \equiv 0, 1, 2, 3, 4 \pmod 5$, they should play a sum of 4, 5, 6, 2, 3, respectively. Since $k \ge 3$, there are always sufficient marbles to do this.

For each subsequent move, the third and the first players play so that the sum of the three moves (the second player, then the third, then the first) is 5, which is always possible (for example, the third player can always play 1, then the first player plays 4 minus the second player's play). Thus, the second player is always left with a number congruent to $1 \pmod 5$ and will eventually be left with 1, at which time he will lose.

M80. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Compute the number of ways that 4 tires can be rotated so that each tire is relocated. (*Editor's note:* "rotating" a car's tires means changing their position on the car so that they can wear more evenly.)

Combined solution by Andrew Mao, A.B. Lucas Secondary School, London, ON; and Yifei Chen, student, West Windsor Plainsboro High School North, Plainsboro, NJ, USA.

This problem is an example of a derangement problem. The derangements of an ordered set are the permutations in which there are no fixed points, in this case tires. The number of such derangements is

$$4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3}1! + \binom{4}{4}0! \;=\; 9\,.$$

The formula is the result of applying the principle of inclusion/exclusion to the problem. The first term in the formula is simply the total number of ways of arranging 4 symbols. The second term is the correction to ensure that all such permutations with one fixed point are removed. But, since the permutations with two fixed points have now been removed twice, we must make another correction to ensure that we have exactly zero of these. This correction gives rise to our third term. If we continue to correct our formula so that permutations with three and four fixed points are counted zero times, we arrive at the formula above.

[*Editor:* The reader might like to see how similar ideas are handled in the Mayhem article "Binomial Inversion: Two Proofs and an Application to Derangements", by Heba Hathout [2003 : 275–278].]

M81. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let $a \neq 0$, b, c be integers and $\sin\theta$, $\cos\theta$ be the rational roots of the equation $ax^2 + bx + c = 0$. Show that $a \pm 2c$ are perfect squares.

Solution by Robert Bilinski, Outremont, QC.

To arrive at this result, it is necessary to suppose that the coefficients have had all common factors removed. Thus, $(a, c) = 1$. The two solutions of the quadratic equation $ax^2+bx+c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Setting these two roots to $\cos\theta$ and $\sin\theta$, the identity $\sin^2\theta + \cos^2\theta = 1$ is equivalent to $b^2 - 2ac = a^2$, or $b^2 = a(a + 2c)$.

Then $b^2 - 4ac = a(a - 2c)$. But both sides of this equation must be perfect squares, since the roots $\cos\theta$ and $\sin\theta$ are rational. Since $(a, c) = 1$, it follows that a and $a - 2c$ have no common factors, and each must be a perfect square separately. Also, from the equation $b^2 = a(a + 2c)$, the condition $(a, a + 2c) = 1$ forces $a + 2c$ to be a perfect square, since a and b^2 are perfect squares.

Also solved by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; Andrew Mao, A.B. Lucas Secondary School, London, ON; Yifei Chen, student, West Windsor Plainsboro High School North, Plainsboro, NJ, USA; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

M82. *Proposed by the Mayhem staff.*

In number theory the function $\omega(n)$ is the number of *distinct* primes dividing n. For example, $\omega(12) = 2$ since $12 = 2 \times 2 \times 3$. Prove that for each positive integer n

$$\ln n \;\geq\; \omega(n)\ln 2\,.$$

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{\omega(n)}^{\alpha_{\omega(n)}}$, where p_1, p_2, …, $p_{\omega(n)}$ are distinct primes and α_1, α_2, …, $\alpha_{\omega(n)}$ are positive integers. Since $\alpha_i > 0$ and $p_i \geq 2$, we have

$$\begin{aligned} \ln n \;=\; \ln\left(\prod_{i=1}^{\omega(n)} p_i^{\alpha_i}\right) \;&\geq\; \ln\left(\prod_{i=1}^{\omega(n)} p_i\right) \;\geq\; \ln\left(\prod_{i=1}^{\omega(n)} 2\right) \\ &=\; \ln 2^{\omega(n)} \;=\; \omega(n)\ln 2\,. \end{aligned}$$

Also solved by the Austrian IMO Team 2003; Robert Bilinski, Outremont, QC; Yifei Chen, student, West Windsor Plainsboro High School North, Plainsboro, NJ, USA; and Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

THE OLYMPIAD CORNER

No. 237

R.E. Woodrow

To begin this number, we give Selected Problems from Israel Mathematical Olympiads, 2001. Thanks go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting them for us.

ISRAEL MATHEMATICAL OLYMPIADS 2001
Selected Problems

1. Find all solutions of

$$\begin{aligned} x_1 + x_2 + \cdots + x_{2000} &= 2000\,, \\ x_1^4 + x_2^4 + \cdots + x_{2000}^4 &= x_1^3 + x_2^3 + \cdots + x_{2000}^3\,. \end{aligned}$$

2. Given 2001 real numbers x_1, x_2, $\ldots$, x_{2001} such that $0 \le x_n \le 1$ for each $n = 1, 2, \ldots, 2001$, find the maximum value of

$$\left(\frac{1}{2001}\sum_{n=1}^{2001} x_n^2\right) - \left(\frac{1}{2001}\sum_{n=1}^{2001} x_n\right)^2 .$$

Where is this maximum attained?

3. We are given 2001 lines in the plane, no two of which are parallel and no three of which pass through a common point. These lines partition the plane into some regions (not necessarily finite) bounded by segments of these lines. These segments are called *sides*, and the collection of the regions is called a *map*. Two regions on the map are called *neighbours* if they share a side.

The set of intersection points of the lines is called the set of vertices. Two vertices are called *neighbours* if they are found on the same side.

A *legal colouring of the map* is a colouring of the regions (one colour per region) such that neighbouring regions have different colours.

A *legal colouring of the vertices* is a colouring of the vertices (one colour per vertex) such that neighbouring vertices have different colours.

(i) What is the minimum number of colours required for a legal colouring of the map?

(ii) What is the minimum number of colours required for a legal colouring of the vertices?

4. The lengths of the sides of triangle ABC are 4, 5, 6. For any point D on one of the sides, drop the perpendiculars DP, DQ onto the other two sides (P, Q are on the sides). What is the minimal value of PQ?

5. Triangle ABC in the plane Π is called *good* if it has the following property: For any point D in space, not in the plane Π, it is possible to construct a triangle with sides of length $|CD|$, $|BD|$, $|AD|$. Find all the good triangles.

6. (a) Find a pair of integers (x, y) such that

$$15x^2 + y^2 = 2^{2000}.$$

(b) Does there exist a pair of integers (x, y) with x odd, such that

$$15x^2 + y^2 = 2^{2000}?$$

Next we give the problems of the 21st Brazilian Mathematical Olympiad. Thanks again to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting them for us.

21st BRAZILIAN MATHEMATICAL OLYMPIAD 2001

First Day

1. Let $ABCDE$ be a regular pentagon such that the star $ACEBD$ has area 1. Let P be the point of intersection of AC and BE, and let Q be the point of intersection of BD and CE. Find the area of $APQD$.

2. Prove that there is at least one non-zero digit between the $1\,000\,000^{\text{th}}$ and the $3\,000\,000^{\text{th}}$ decimal digits of $\sqrt{2}$.

3. One must place n pieces on the squares of a 10×10 board such that no four pieces are the vertices of a rectangle with sides parallel to the sides of the board. Find the greatest value of n for which this is possible.

Second Day

4. Planet Zork is spherical and has many towns. For each town there is a corresponding antipodal town (that is, symmetric in relation to the centre of the planet).

There are roads connecting pairs of towns in Zork. If there is a road connecting towns P and Q, then there is also a road connecting towns P' and Q', where P' is the antipode of P and Q' is the antipode of Q. The roads do not cross each other. For any two given towns P and Q, it is possible to travel from P to Q along some sequence of roads.

The prices of Kriptonita in Urghs (the planetary currency) in two towns connected by a road differ by no more than 100 Urghs. Prove that there exist two antipodal towns such that the prices of Kriptonita in these towns differ by no more than 100 Urghs.

5. There are n football teams in Tumbolia. A championship is to be organized in which each team plays against every other exactly once. Every match must take place on a Sunday, and no team can play more than once on the same day.

Find the least positive integer m for which it is possible to set up a championship lasting m Sundays.

6. Given a triangle ABC, explain how to construct with straight-edge and compass a triangle $A'B'C'$ of minimum area such that

$$C' \in AC\,, \quad A' \in AB\,, \quad B' \in BC\,,$$

and

$$\angle B'A'C' = \angle BAC\,, \quad \angle A'C'B' = \angle ACB\,.$$

To round out the problem sets for this number we present the 49$^{\text{th}}$ Mathematical Olympiad of Lithuania. Thanks again go to Chris Small for providing us with this set for our puzzling pleasure.

49$^{\text{th}}$ MATHEMATICAL OLYMPIAD OF LITHUANIA 2000

Forms 9 and 10

1. In a family there are four children of different ages, each age being a positive integer not less than 2 and not greater than 16. A year ago the square of the age of the eldest child was equal to the sum of the squares of the ages of the remaining children. One year from now the sum of the squares of the youngest and the oldest will be equal to the sum of the squares of the other two. How old is each child?

2. A sequence a_1, a_2, a_3, $\ldots$ is defined such that $a_n = n^2 + n + 1$ for all $n \geq 1$. Prove that the product of any two consecutive members of the sequence is itself a member of the given sequence.

3. In the triangle ABC, the point D is the mid-point of the side AB. Point E divides BC in the ratio $BE : EC = 2 : 1$. Given that $\angle ADC = \angle BAE$, determine $\angle BAC$.

4. Find all the triples of positive integers x, y, z with $x \leq y \leq z$ such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

is a positive integer.

Forms 11 and 12

5. For the aircraft pilots K, L, M, who intended to become star fighters, an examination consisting of several tests was organized. In each test, the pilots were ranked first, second, and third, and points were awarded as follows: A points for first, $-B$ points for second, and $-C$ points for third (where A, B, C are positive integers with $A > B > C > 0$). After all the tests, K had accumulated 22 points, while L and M had -9 points each. The test of reaction times was won by L. Who took second place in the running test?

6. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following equation for all real x and y:

$$(x + y)(f(x) - f(y)) \;=\; f(x^2) - f(y^2)\,.$$

Find: (a) one such function; (b) all such functions.

7. A line divides both the area and the perimeter of a triangle into two equal parts. Prove that this line passes through the incentre of the triangle. Does the converse statement always hold?

8. The equation $x^2 + y^2 + z^2 + u^2 = xyzu + 6$ is given. Find:

(a) at least one solution in positive integers;

(b) at least 33 such solutions;

(c) at least 100 such solutions.

Now we turn to readers' submissions for problems of the Russian Mathematical Olympiad 1999, 11^{th} Form given [2001 : 420–421].

1. [*O. Podlipsky*] Do there exist 19 different positive integers that sum to 1999 and such that the sum of the decimal digits of each is the same?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB. We give Bradley's write-up.

Let $S(n)$ be the digital sum of n. This function is known to satisfy

$$S(m+n) \;\equiv\; S(m) + S(n) \pmod 9\ .$$

If s is the common digital sum of the 19 numbers that we wish to sum to 1999, it follows that

$$19s \;\equiv\; 1 + 9 + 9 + 9 \pmod 9\ .$$

Hence, $s \equiv 1 \pmod 9$.

Clearly, s cannot equal 1; thus, the first possibility is $s = 10$. The smallest 19 numbers with digital sum 10, namely 19, 28, 37, 46, 55, 64, 73, 82, 91, 109, 118, 127, 136, 145, 154, 163, 172, 181, and 190, sum to 1990. Since the next integer with digital sum equal to 10 is 208, the next smallest possible sum is $1990 + 208 - 190 = 2008$. Hence, s cannot equal 10.

For $s \geq 19$ the smallest possible 19 numbers add up to much more than 1999; whence it is impossible to obtain the required sum.

2. [*S. Berlov*] A function $f : \mathbb{Q} \to \mathbb{Z}$ is considered. Prove that there exist two rational numbers a and b such that

$$\frac{f(a)+f(b)}{2} \leq f\left(\frac{a+b}{2}\right).$$

Combination of solutions by Pierre Bornsztein, Maisons-Laffitte, France; and Bruce Crofoot, University College of the Cariboo, Kamloops, BC.

Assume the contrary (with the aim of reaching a contradiction). Then, for all distinct rational numbers a and b,

$$f\left(\frac{a+b}{2}\right) < \frac{f(a)+f(b)}{2}. \qquad (1)$$

This condition is not affected by adding a constant to the function f. Therefore, we may assume that $f(-1) \leq 0$ and $f(1) \leq 0$, without loss of generality.

We claim that for all $n \in \mathbb{N} \cup \{0\}$ and $x \in \{-2^{-n}, 0, 2^{-n}\}$, we have $f(x) \leq -n$. To prove this, we use induction on n. For $n = 0$, we must consider $x \in \{-1, 0, 1\}$. We have $f(\pm 1) \leq 0$, and hence, using (1),

$$f(0) = f\left(\frac{1-1}{2}\right) < \frac{f(1)+f(-1)}{2} \leq 0.$$

Thus, the claim is true for $n = 0$.

Now consider any fixed n for which the claim is true. Using (1) and the induction hypothesis, we have

$$f(\pm 2^{-n-1}) = f\left(\frac{0 \pm 2^{-n}}{2}\right) < \frac{f(0)+f(\pm 2^{-n})}{2} \leq \frac{-n-n}{2} = -n.$$

Since f takes only integer values, we must have $f(\pm 2^{-n-1}) \leq -(n+1)$. Then, using (1) again,

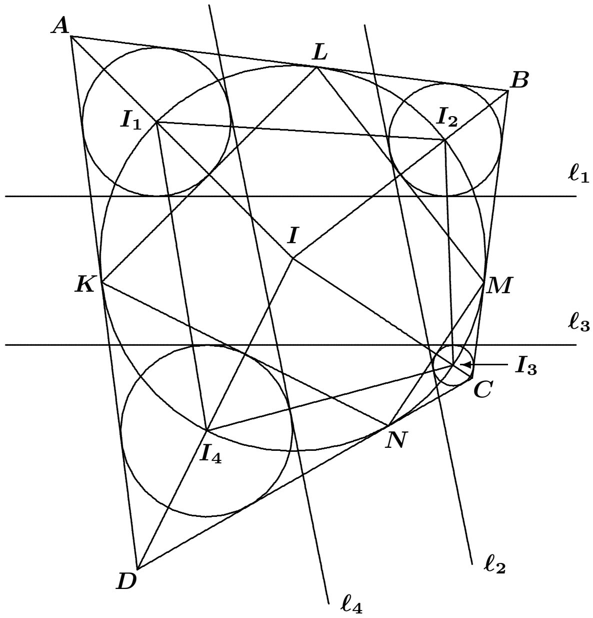
$$f(0) = f\left(\frac{2^{-n-1}-2^{-n-1}}{2}\right) < \frac{f(2^{-n-1})+f(-2^{-n-1})}{2} \leq -(n+1).$$

This completes the induction and proves the claim.

We have shown that $f(0) \leq -n$ for all non-negative integers n. This is impossible. Therefore, there is no function $f : \mathbb{Q} \to \mathbb{Z}$ that satisifies (1) for all distinct rational numbers a and b.

3. [*M. Sonkin*] The incircle of quadrilateral $ABCD$ touches the sides DA, AB, BC, CD in K, L, M, N, respectively. Let S_1, S_2, S_3, S_4 be the incircles of triangles AKL, BLM, CMN, DKN, respectively. Let l_1, l_2, l_3, l_4 be the common external tangents to the pairs S_1 and S_2, S_2 and S_3, S_3 and S_4, S_4 and S_1, different from the sides of quadrilateral $ABCD$. Prove that l_1, l_2, l_3, l_4 intersect in the vertices of a rhombus.

Solution by Christopher J. Bradley, Bristol, UK, modified by the editor.



Let r be the radius of the incircle of quadrilateral $ABCD$. Let

$$\alpha = \tfrac{1}{2}\angle A\,, \quad \beta = \tfrac{1}{2}\angle B\,, \quad \gamma = \tfrac{1}{2}\angle C\,, \quad \delta = \tfrac{1}{2}\angle D\,.$$

Note that $\alpha + \beta + \gamma + \delta = 180^\circ$.

Triangles AKI and ALI are congruent, since they are right-angled, with a common hypotenuse and $KI = LI = r$. Therefore, $AK = AL$ and $\angle IAK = \angle IAL = \alpha$. Since AI is the bisector of $\angle A$ in the isosceles triangle KAL, we have $AI \perp KL$. Then, since $\angle ALI = 90^\circ$, we see that $\angle KLI = \angle IAL = \alpha$ (and also $\angle LKI = \alpha$). Hence, $KL = 2r\cos\alpha$.

Let I_1 be the point at which the incircle of $ABCD$ intersects the line segment AI, and let r_1 be the distance from I_1 to AL (which equals the distance from I_1 to AK). We claim that I_1 and r_1 are the centre and radius of S_1. To prove this, first note that $AI = AI_1 + r$, with $AI = r\csc\alpha$ and $AI_1 = r_1\csc\alpha$. Thus, we have $r\csc\alpha = r_1\csc\alpha + r$; that is,

$$r_1 = r(1 - \sin\alpha)\,. \tag{1}$$

Since the distance from I to KL is $r\sin\angle KLI = r\sin\alpha$, the distance from I_1 to KL is $r - r\sin\alpha = r_1$. We conclude that the point I_1 is the same distance r_1 from each side of $\triangle AKL$, which proves the claim.

Similar considerations apply to the incircles S_2, S_3, and S_4. We denote their centres by I_2, I_3, and I_4, and their radii by r_2, r_3, and r_4, respectively.

Now consider $\triangle II_1I_2$. It is isosceles, with $II_1 = II_2 = r$. Since $\angle I_1II_2 = \angle AIB = 180^\circ - \alpha - \beta$, we have

$$\angle II_1I_2 \;=\; \angle II_2I_1 \;=\; \tfrac{1}{2}(\alpha+\beta)\,, \tag{2}$$

and hence,

$$I_1I_2 \;=\; 2r\cos\left[\tfrac{1}{2}(\alpha+\beta)\right]\,. \tag{3}$$

Let θ be the angle between the lines I_1I_2 and ℓ_2 that contains the point I in its interior. We will prove that $\theta = \gamma + \frac{1}{2}(\alpha+\beta)$.

Case 1. $\beta < \gamma$ (or, equivalently, $r_2 > r_3$, as in the figure).

Then the lines BC and ℓ_2 meet on the extension of BC beyond C. Let the point where they meet be denoted by S, and let T be the point where ℓ_2 intersects I_1I_2. Thus, $\angle I_1TS = \theta$.

From (2), by symmetry, we have $\angle II_2I_3 = \frac{1}{2}(\beta+\gamma)$. This angle is exterior to $\triangle I_2BS$ opposite the interior angles $\angle I_2BS = \beta$ and $\angle I_2SB$. Therefore, $\angle I_2SB = \frac{1}{2}(\beta+\gamma) - \beta = \frac{1}{2}(\gamma-\beta)$. Since SI_2 bisects $\angle TSB$, we also have $\angle I_2ST = \frac{1}{2}(\gamma-\beta)$.

Now θ is an exterior angle of $\triangle TI_2S$ opposite the interior angles $\angle I_2ST = \frac{1}{2}(\gamma-\beta)$ and $\angle TI_2S = \frac{1}{2}(\alpha+\beta) + \frac{1}{2}(\beta+\gamma) = \beta + \frac{1}{2}(\alpha+\gamma)$. Therefore,

$$\theta \;=\; \beta + \tfrac{1}{2}(\alpha+\gamma) + \tfrac{1}{2}(\gamma-\beta) = \gamma + \tfrac{1}{2}(\alpha+\beta)\,.$$

Case 2. $\beta = \gamma$ (or, equivalently, $r_2 = r_3$).

Then the three lines BC, I_2I_3, and ℓ_2 are parallel. Hence,

$$\theta \;=\; \angle I_1I_2I_3 \;=\; \tfrac{1}{2}(\alpha+\beta)+\tfrac{1}{2}(\beta+\gamma) \;=\; \beta+\tfrac{1}{2}(\alpha+\gamma) \;=\; \gamma+\tfrac{1}{2}(\alpha+\beta)\,.$$

Case 3. $\beta > \gamma$ (or, equivalently, $r_2 < r_3$).

Then the lines BC and ℓ_2 meet on the extension of BC beyond B. The angle at which they meet is $\beta - \gamma$ (from Case 1, with β and γ interchanged). An argument similar to that of Case 1, leads to $\theta = \gamma + \frac{1}{2}(\alpha+\beta)$.

In all three cases,

$$\theta \;=\; \gamma + \tfrac{1}{2}(\alpha+\beta) \;=\; \gamma + \tfrac{1}{2}(180^\circ - \gamma - \delta) \;=\; 90^\circ + \tfrac{1}{2}(\gamma - \delta)\,.$$

By symmetry, the angle between I_1I_2 and ℓ_4 which contains I in its interior is $90^\circ + \frac{1}{2}(\delta-\gamma)$. Since the sum of this angle and θ is 180°, the lines ℓ_2 and ℓ_4 are parallel. Similarly, ℓ_1 and ℓ_3 are parallel. Thus, ℓ_1, ℓ_2, ℓ_3, ℓ_4 intersect in the vertices of a parallelogram.

Since θ is one of the two angles between I_1I_2 and ℓ_2 (the other being $180^\circ - \theta$), the perpendicular distance between the lines parallel to ℓ_2 that pass through I_1 and I_2 is $I_1I_2 \sin\theta = I_1I_2 \cos[\frac{1}{2}(\gamma - \delta)]$, where I_1I_2 is given by (3). Therefore, the perpendicular distance between ℓ_2 and ℓ_4 is

$$\begin{aligned} &2r\cos\left[\tfrac{1}{2}(\alpha+\beta)\right]\cos\left[\tfrac{1}{2}(\gamma-\delta)\right] - r_1 - r_2 \\ &\quad = \; r\cos\left[\tfrac{1}{2}(\alpha+\beta+\gamma-\delta)\right] + r\cos\left[\tfrac{1}{2}(\alpha+\beta-\gamma+\delta)\right] - r_1 - r_2 \\ &\quad = \; r\cos\left[\tfrac{1}{2}(180^\circ - 2\delta)\right] + r\cos\left[\tfrac{1}{2}(180^\circ - 2\gamma)\right] - r_1 - r_2 \\ &\quad = \; r\sin\delta + r\sin\gamma - r_1 - r_2 \\ &\quad = \; (r - r_4) + (r - r_3) - r_1 - r_2 \; = \; 2r - (r_1 + r_2 + r_3 + r_4)\,. \end{aligned}$$

This, by symmetry, is equal to the perpendicular distance between ℓ_1 and ℓ_3.

Hence, ℓ_1, ℓ_2, ℓ_3, ℓ_4 intersect in the vertices of a rhombus.

7. [*D. Tereshin*] The plane α passing through the vertex A of tetrahedron $ABCD$ is tangent to the circumsphere of the tetrahedron. Prove that the angles between the lines of intersection of α with the planes ABC, ACD, and ABD are equal if and only if $AB \cdot CD = AC \cdot BD = AD \cdot BC$.

Solution by Christopher J. Bradley, Bristol, UK.

Let K be the centre of the circumsphere of $ABCD$. Without loss of generality, assume that the radius of the sphere is 1. Take A to be the origin of vectors, and denote $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, and $\overrightarrow{AK}$ by b, c, d, and k, respectively. Since $KB = 1$ and $|\mathrm{k}| = 1$, we have

$$1 \;=\; KB^2 \;=\; |\mathrm{k} - \mathrm{b}|^2 \;=\; 1 + |\mathrm{b}|^2 - 2(\mathrm{k}\cdot\mathrm{b})\,.$$

Therefore, $|\mathrm{b}|^2 = 2(\mathrm{k}\cdot\mathrm{b})$. Similarly, $|\mathrm{c}|^2 = 2(\mathrm{k}\cdot\mathrm{c})$ and $|\mathrm{d}|^2 = 2(\mathrm{k}\cdot\mathrm{d})$.

A vector normal to the plane α at A is k, and a vector normal to the plane ABC is $\mathrm{b}\times\mathrm{c}$. Hence, a vector along the line of intersection of α with the plane ABC is

$$\mathrm{k}\times(\mathrm{c}\times\mathrm{b}) \;=\; (\mathrm{k}\cdot\mathrm{b})\mathrm{c} - (\mathrm{k}\cdot\mathrm{c})\mathrm{b} \;=\; \tfrac{1}{2}\left(|\mathrm{b}|^2\mathrm{c} - |\mathrm{c}|^2\mathrm{b}\right)\,.$$

Similarly, a vector along the line of intersection of α with the plane ACD is $\frac{1}{2}\left(|\mathrm{c}|^2\mathrm{d} - |\mathrm{d}|^2\mathrm{c}\right)$, and a vector along the line of intersection of α with the plane ADB is $\frac{1}{2}\left(|\mathrm{d}|^2\mathrm{b} - |\mathrm{b}|^2\mathrm{d}\right)$.

Let h, i, j be scaled versions of these vectors as follows:

$$\begin{aligned} \mathrm{h} &= |\mathrm{d}|^2\left(|\mathrm{b}|^2\mathrm{c} - |\mathrm{c}|^2\mathrm{b}\right)\,, \\ \mathrm{i} &= |\mathrm{b}|^2\left(|\mathrm{c}|^2\mathrm{d} - |\mathrm{d}|^2\mathrm{c}\right)\,, \\ \mathrm{j} &= |\mathrm{c}|^2\left(|\mathrm{d}|^2\mathrm{b} - |\mathrm{b}|^2\mathrm{d}\right)\,. \end{aligned}$$

Then h, i, j are coplanar, since they all lie in α. Also, we have $\mathrm{h}+\mathrm{i}+\mathrm{j} = 0$.

We calculate

$$\begin{aligned} |\mathbf{h}|^2 &= |\mathbf{d}|^4 \Big| |\mathbf{b}|^2\mathbf{c} - |\mathbf{c}|^2\mathbf{b} \Big|^2 \\ &= |\mathbf{d}|^4 \Big(|\mathbf{b}|^4|\mathbf{c}|^2 + |\mathbf{c}|^4|\mathbf{b}|^2 - 2|\mathbf{b}|^2|\mathbf{c}|^2\mathbf{b}\cdot\mathbf{c} \Big) \\ &= |\mathbf{d}|^4|\mathbf{b}|^2|\mathbf{c}|^2 \Big(|\mathbf{b}|^2 + |\mathbf{c}|^2 - 2\mathbf{b}\cdot\mathbf{c} \Big) \\ &= |\mathbf{d}|^4|\mathbf{b}|^2|\mathbf{c}|^2|\mathbf{b}-\mathbf{c}|^2 \;=\; |\mathbf{b}|^2|\mathbf{c}|^2|\mathbf{d}|^2(AD^2BC^2)\,; \end{aligned}$$

that is, $|\mathbf{h}| = |\mathbf{b}||\mathbf{c}||\mathbf{d}|(AD \cdot BC)$. Similarly, $|\mathbf{i}| = |\mathbf{b}||\mathbf{c}||\mathbf{d}|(AB \cdot CD)$ and $|\mathbf{j}| = |\mathbf{b}||\mathbf{c}||\mathbf{d}|(AC \cdot BD)$.

Now we apply the known result that three coplanar vectors $\mathbf{h}$, $\mathbf{i}$, $\mathbf{j}$ such that $\mathbf{h} + \mathbf{i} + \mathbf{j} = \mathbf{0}$ make angles of 120° with one another if and only if $|\mathbf{h}| = |\mathbf{i}| = |\mathbf{j}|$. (One interpretation of this is that three coplanar forces with resultant zero are equally inclined to each other if and only if they are of equal magnitude.) We conclude that our particular vectors $\mathbf{h}$, $\mathbf{i}$, $\mathbf{j}$ make angles of 120° with one another if and only if $AD \cdot BC = AB \cdot CD = AC \cdot BD$, which is the required condition.

Next we turn to the December 2001 number of the *Corner* and solutions by our readers to problems given there. We begin with the Composition de Mathématiques 1999, Classe Terminal S [2001 : 484–485].

2. Résoudre dans $\mathbb{N}$ l'équation en n:

$$(n+3)^n \;=\; \sum_{k=3}^{n+2} k^n\,.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

The only solutions are $n = 2$ and $n = 3$.

Let $f(n) = \sum\limits_{k=3}^{n+2} k^n$ and $g(n) = (n+3)^n$. Then

$$\begin{aligned} f(1) &= 3 \;\neq\; 4 \;=\; g(1)\,, \\ f(2) &= 3^2+4^2 \;=\; 5^2 \;=\; g(2)\,, \\ f(3) &= 3^3+4^3+5^3 \;=\; 216 \;=\; 6^3 \;=\; g(3)\,. \end{aligned}$$

Thus, $n = 2$ and $n = 3$ are solutions. To prove that there are no solutions $n \geq 4$, we will show that for all $n \geq 4$,

$$f(n) \;<\; g(n)\,. \tag{1}$$

Inequality (1) is easily checked for $n = 4$:

$$\begin{aligned} f(4) &= 3^4 + 4^4 + 5^4 + 6^4 = 81 + 256 + 625 + 1296 \\ &= 2258 < 2401 = 7^4 = g(4) . \end{aligned}$$

As an induction hypothesis, suppose that (1) holds for some $n \geq 4$. Then

$$\begin{aligned} f(n+1) &= \sum_{k=3}^{n+3} k^{n+1} = \sum_{k=3}^{n+2} k \cdot k^n + (n+3)^{n+1} \\ &< (n+2) \sum_{k=3}^{n+2} k^n + (n+3)^{n+1} \\ &= (n+2) f(n) + (n+3) g(n) \\ &< (n+2) g(n) + (n+3) g(n) = (n+3)^n (2n+5) . \end{aligned}$$

Since $g(n+1) = (n+4)^{n+1}$, it suffices to show that

$$(n+3)^n (2n+5) < (n+4)^{n+1} . \tag{2}$$

Since $(n+3)^n(2n+5) < (n+3)^n(2n+6) = 2(n+3)^{n+1}$, inequality (2) would be true if

$$\left(\frac{n+4}{n+3}\right)^{n+1} > 2 . \tag{3}$$

But

$$\begin{aligned} \left(\frac{n+4}{n+3}\right)^{n+1} - 2 &= \left(1 + \frac{1}{n+3}\right)^{n+1} - 2 \\ &> 1 + \frac{n+1}{n+3} + \binom{n+1}{2} \frac{1}{(n+3)^2} - 2 \\ &= \frac{n+1}{n+3} + \frac{n(n+1)}{2(n+3)^2} - 1 = \frac{n^2 - 3n - 12}{2(n+3)^2} \\ &> \frac{n^2 - 3n - 18}{2(n+3)^2} = \frac{n-6}{2(n+3)} . \end{aligned}$$

Therefore, inequality (3), and hence (2), holds for all $n \geq 6$.

When $n = 4$, inequality (2) becomes $7^4 \times 13 < 8^5$ or $31213 < 32768$; when $n = 5$, inequality (2) becomes $8^5 \times 15 < 9^6$ or $491520 < 531441$. Thus, (2) holds for all $n \geq 4$, and our induction is complete.

Next we turn to solutions for problems of the 16th Iranian Mathematical Olympiad 1998–1999, First Round, given [2001 : 485–486].

1. Suppose that $a_1 < a_2 < \cdots < a_n$ are real numbers. Prove that:

$$a_1a_2^4+a_2a_3^4+\cdots+a_{n-1}a_n^4+a_na_1^4 \;\geq\; a_2a_1^4+a_3a_2^4+\cdots+a_na_{n-1}^4+a_1a_n^4\,.$$

Solution by Mohammed Aassila, Strasbourg, France.

We prove the result by induction on n. For $n = 2$, we have equality. The case $n = 3$ will be needed below. For $n = 3$, we have to show that

$$a_1a_2^4 + a_2a_3^4 + a_3a_1^4 \;\geq\; a_2a_1^4 + a_3a_2^4 + a_1a_3^4\,.$$

This is true, since

$$\begin{aligned} a_1a_2^4 + a_2a_3^4 + a_3a_1^4 - a_2a_1^4 - a_3a_2^4 - a_1a_3^4 \\ = \;\; & \tfrac{1}{2}(a_2-a_1)(a_3-a_2)(a_3-a_1) \\ & \cdot\left[(a_1+a_2)^2+(a_2+a_3)^2+(a_3+a_1)^2\right] \;\geq\; 0\,. \end{aligned}$$

Assume that the claim is true for $n-1$, and let us prove it for n. By applying the induction hypothesis, we find that it is sufficient to prove that

$$a_{n-1}a_n^4 + a_na_1^4 - a_{n-1}a_1^4 \;\geq\; a_na_{n-1}^4 + a_1a_n^4 - a_1a_{n-1}^4\,,$$

which is the case $n = 3$.

2. Suppose that n is a natural number. The n–tuple $(a_1, a_2, \ldots, a_n)$ is said to be *good*, if $a_1 + a_2 + \cdots + a_n = 2n$ and furthermore, no subset of $\{a_1, \ldots, a_n\}$ has a sum equal to n. Find all good n–tuples.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

First note that the problem is not clear. Since the a_i's are not assumed to be (positive) integers, there exist an infinite number of good n–tuples. Moreover, the notation $\{a_1, \ldots, a_n\}$ is ambiguous because it supposes that the a_i's are pairwise distinct. If this is the case, and if the a_i's are positive integers, there will be no good n–tuples for $n > 1$.

Following [1], we will assume that the a_i's are positive integers and that "for every $k \in \{1, \ldots, n\}$, no k of the n integers add up to n."

Clearly, the only good 1–tuple is (2). The good 2–tuples are $(1, 3)$ and $(3, 1)$, and the good 3–tuples are $(2, 2, 2)$ and all permutations of $(1, 1, 4)$.

We will prove by induction on n that, if $a = (a_1, a_2, \ldots, a_n)$ is a good n–tuple with $a_1 \leq a_2 \leq \cdots \leq a_n$, then either $a = (2, 2, \ldots, 2)$ or $a = (1, 1, \ldots, 1, n+1)$.

The result is true for $n \in \{1, 2, 3\}$. Suppose that the result holds for $n-1$, for some $n \geq 4$. Let $a = (a_1, a_2, \ldots, a_n)$ be a good n–tuple with $a_1 \leq a_2 \leq \cdots \leq a_n$. Since a is good, we have $\sum\limits_{i=1}^{n} a_i = 2n$, which implies that the average of the a_i's is 2. If either $a_1 \geq 2$ or $a_n \leq 2$, then we must have $a_1 = a_2 = \cdots = a_n = 2$. Therefore, we may assume that $a_1 = 1$ and $a_n \geq 3$.

Let $j = \max\{i \mid a_i = 1\}$. Then $1 \leq j < n$. Define the $(n-1)$–tuple a' as follows:

$$\begin{aligned} a'_i &= a_{i+1} \qquad \text{for } i \neq j\,, \\ a'_j &= a_{j+1} - 1\,. \end{aligned}$$

Note that $1 \leq a'_1 \leq a'_2 \leq \cdots \leq a'_{n-1}$ and $\sum\limits_{i=1}^{n-1} a'_i = 2(n-1)$.

Now suppose that a' is not good. Then there exists a subset $E \subseteq \{1, 2, \ldots, n-1\}$, with $E \neq \emptyset$, such that $\sum\limits_{i \in E} a'_i = n-1$. We consider two cases:

Case 1. $j \notin E$.

Define $\overline{E} = \{1\} \cup \{i+1 \mid i \in E\}$. Then

$$\sum_{i \in \overline{E}} a_i \;=\; a_1 + \sum_{i \in E} a_{i+1} \;=\; 1 + \sum_{i \in E} a'_i \;=\; 1 + n - 1 \;=\; n\,.$$

Case 2. $j \in E$.

Define $\overline{E} = \{i+1 \mid i \in E\}$. Then

$$\sum_{i \in \overline{E}} a_i \;=\; \sum_{i \in E} a_{i+1} \;=\; (a'_j + 1) + \sum_{i \in E \setminus \{j\}} a'_i \;=\; 1 + \sum_{i \in E} a'_i \;=\; n\,.$$

In either case we have contradicted the hypothesis that a is good. Therefore, a' is good.

By the induction hypothesis, we have either $a' = (1, 1, \ldots, n)$ or $a' = (2, 2, \ldots, 2)$. But $a' = (2, 2, \ldots, 2)$ is not possible, since this would imply that $a = (1, 3, 2, \ldots, 2)$, which does not have $a_2 \leq a_3$. Therefore, $a' = (1, 1, \ldots, 1, n)$. Then $a = (1, 1, \ldots, 2, n)$ or $a = (1, 1, \ldots, 1, n+1)$. Obviously $(1, 1, \ldots, 2, n)$ is not good; hence $a = (1, 1, \ldots, 1, n+1)$.

Thus, $a = (1, 1, \ldots, 1, n+1)$ or $a = (2, 2, \ldots, 2)$, and the induction is complete.

It follows easily that the good n–tuples are:

(a) those obtained by a permutation of $(1, 1, \ldots, 1, n+1)$;

(b) $(2, 2, \ldots, 2)$, if n is odd.

Reference:

[1] *Mathematical Contests 1998–1999: Olympiad Problems and Solutions from around the World*, edited by T. Andreescu and Z. Feng, MAA.

3. Let I be the incentre of the triangle ABC and AI meet the circumcircle of ABC at point D. Denote the foot of perpendiculars dropped from I on $IE + IF = \frac{1}{2}AD$, find the value of $\angle BAC$.

Solved by Christopher J. Bradley, Bristol, UK. Remark by Toshio Seimiya, Kawasaki, Japan.

This is problem 2280 [1997 : 481; 1998 : 516].

4. Let ABC be a triangle with $BC > CA > AB$. Select points D on BC and E on the extension of AB such that $BD = BE = AC$. The circumcircle of BED intersects AC at point P and BP meets the circumcircle of ABC at point Q. Show that $AQ + CQ = BP$.

Solved by Christopher J. Bradley, Bristol, UK. Remark by Toshio Seimiya, Kawasaki, Japan

This is problem 1881 [1993 : 264; 1994 : 209].

5. Suppose that n is a positive integer and $d_1 < d_2 < d_3 < d_4$ are the four smallest positive integers dividing n. Find all integers n satisfying $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Aassila's write-up.

If n is odd, then $d_1^2 + d_2^2 + d_3^2 + d_4^2 \equiv 1 + 1 + 1 + 1 \equiv 0 \pmod 4$, and we cannot have $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$. Thus, we can assume that 2 divides n. Then $d_1 = 1$ and $d_2 = 2$, and hence,

$$n \;\equiv\; 1 + 0 + d_3^2 + d_4^2 \;\not\equiv\; 0 \pmod 4 \;.$$

Thus, $4 \nmid n$.

Hence, $(d_1, d_2, d_3, d_4) = (1, 2, p, q)$ or $(d_1, d_2, d_3, d_4) = (1, 2, p, 2p)$ for some odd primes p, q. In the first case, $n \equiv 3 \pmod 4$, a contradiction. Thus, $n = 5(1 + p^2)$ and $5 \mid n$. Therefore, $p = d_3 = 5$ and $n = 130$.

6. Suppose that $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are two 0/1 sequences. The distance of A from B is defined to be the number of i for which $a_i \neq b_i$ $(1 \leq i \leq n)$ and is denoted by $d(A, B)$.

Suppose that A, B, C are three 0/1 sequences and

$$d(A,B) \;=\; d(A,C) \;=\; d(B,C) \;=\; \delta\,.$$

(a) Prove that δ is an even number.

(b) Prove that there exists a 0/1 sequence D such that

$$d(D,A) \;=\; d(D,B) \;=\; d(D,C) \;=\; \frac{1}{2}\delta\,.$$

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.

We denote by $N(\alpha)$ the number of 1's in any 0/1 sequence α.

If $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are 0/1 sequences, let $X + Y$ be the 0/1 sequence $\{x_1 + y_1,\ x_2 + y_2,\ \ldots,\ x_n + y_n\}$, where addition is performed modulo 2. Clearly, $d(X, Y) = N(X + Y)$. Let k be the number of indices i for which $x_i = y_i = 1$. Then $x_i = 1$, $y_i = 0$ for $N(X) - k$ indices, and $x_i = 0$, $y_i = 1$ for $N(Y) - k$ indices. Therefore, $d(X, Y) = N(X + Y) = N(X) + N(Y) - 2k$.

Suppose now that A, B, C are 0/1 sequences with

$$d(A, B) \;=\; d(A, C) \;=\; d(B, C) \;=\; \delta\,.$$

Let m be the number of indices i such that $a_i + b_i = b_i + c_i = 1$. On the one hand,

$$N\big((A + B) + (B + C)\big) \;=\; N(A + C) \;=\; d(A, C) \;=\; \delta\,.$$

On the other hand,

$$\begin{aligned} N\big((A + B) + (B + C)\big) \;&=\; N(A + B) + N(B + C) - 2m \\ &=\; d(A, B) + d(B, C) - 2m \;=\; 2\delta - 2m\,. \end{aligned}$$

Thus, $\delta = 2(\delta - m)$, and (a) is proved. In addition, we see that $\frac{1}{2}\delta = m$.

Now, consider $D = \{d_1,\ d_2,\ \ldots,\ d_n\}$ defined by the following rule: $d_i = 1$ if $N(a_i, b_i, c_i) \geq 2$, and $d_i = 0$ otherwise (for $1 \leq i \leq n$).

In order to obtain $d(D, B)$, we observe that

- if i is such that $b_i = 0$, then $d_i = 0$ as well, unless $a_i = c_i = 1$;
- if i is such that $b_i = 1$, then $d_i = 1$ as well, unless $a_i = c_i = 0$.

We have $d_i \neq b_i$ exactly when $a_i + b_i = b_i + c_i = 1$. Thus, the number of indices such that $d_i \neq b_i$ is $m = \frac{1}{2}\delta$. This yields $d(D, B) = \frac{1}{2}\delta$. By symmetry, $d(D, C) = \frac{1}{2}\delta = d(D, A)$, and (b) is proved.

Next are solutions to problems of the Second Round of the 16th Iranian Mathematical Olympiad 1998–1999 given [2001 : 486].

1. Define the sequence $\{x_i\}_{i=0}^{\infty}$ by $x_0 = 0$ and,

$$\begin{aligned} x_n \;&=\; x_{n-1} + \frac{3^r - 1}{2}\,, \quad \text{if } n = 3^{r-1}(3k+1)\,, \\ x_n \;&=\; x_{n-1} - \frac{3^r + 1}{2}\,, \quad \text{if } n = 3^{r-1}(3k+2)\,, \end{aligned}$$

where k and r are integers. Prove that every integer occurs exactly once in this sequence.

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bornsztein's solution.

Lemma. If $n = \sum_{i=0}^{p} \alpha_i 3^i$ is the ternary expansion of a non-negative integer n, then

$$x_n \;=\; \sum_{i\in U_n} 3^i - \sum_{i\in T_n} 3^i\,, \tag{1}$$

where U_n and T_n are the sets of values of the index i such that $\alpha_i = 1$ and $\alpha_i = 2$, respectively.

Proof. We use induction on n. Equation (1) is clearly true for $n \in \{0, 1\}$. Let $n \geq 1$ be fixed, and let $n = \sum_{i=0}^{p} \alpha_i 3^i$ be the ternary expansion of n. Suppose that (1) holds for n.

Case1. $\alpha_0 = 0$.

Then $n + 1 = 1 + \sum_{i=1}^{p} \alpha_i 3^i = 3^0(3k + 1)$, for some integer k. Hence, $x_{n+1} = x_n + \dfrac{3^1 - 1}{2} = x_n + 1$. We have

$$\begin{aligned} x_{n+1} &= \left(\sum_{i\in U_n} 3^i - \sum_{i\in T_n} 3^i\right) + 1 \;=\; \left(\sum_{i\in U_n} 3^i + 3^0\right) - \sum_{i\in T_n} 3^i \\ &= \sum_{i\in U_{n+1}} 3^i - \sum_{i\in T_{n+1}} 3^i\,, \end{aligned}$$

which proves the result in this case.

Case 2. $\alpha_0 = 1$.

Then $n + 1 = 2 + \sum_{i=1}^{p} \alpha_i 3^i = 3^0(3k + 2)$, for some integer k. Hence, $x_{n+1} = x_n - \dfrac{3^1 + 1}{2} = x_n - 2$. We have

$$\begin{aligned} x_{n+1} &= \left(\sum_{i\in U_n} 3^i - \sum_{i\in T_n} 3^i\right) - 2 \;=\; \left(\sum_{i\in U_n} 3^i - 3^0\right) - \left(\sum_{i\in T_n} 3^i + 3^0\right) \\ &= \sum_{i\in U_{n+1}} 3^i - \sum_{i\in T_{n+1}} 3^i\,, \end{aligned}$$

which proves the result in this case.

Case 3. $\alpha_0 = 2$.

Define $\alpha_{p+1} = 0$. Let $t = \min\{i \mid \alpha_i \neq 2\}$. Then $1 \leq t \leq p + 1$. We have

$$n \;=\; 2\sum_{i=0}^{t-1} 3^i + \sum_{i\geq t} \alpha_i 3^i \;=\; 3^t - 1 + \sum_{i\geq t} \alpha_i 3^i\,,$$

and

$$\begin{aligned} x_n &= \sum_{i\in U_n} 3^i - \sum_{i\in T_n} 3^i &= \sum_{i\in U_n} 3^i - \sum_{\substack{i\in T_n\\ i>t}} 3^i - \sum_{i=0}^{t-1} 3^i \\ & &= \sum_{i\in U_n} 3^i - \sum_{\substack{i\in T_n\\ i>t}} 3^i - \frac{3^t-1}{2}. \end{aligned}$$

Then $n+1 = 3^t + \sum_{i\geq t} \alpha_i 3^i = 3^t(3k+\alpha_t+1)$, for some integer k. This leads to two subcases:

Subcase 3(a). $\alpha_t = 0$.

Then $n+1 = 3^t(3k+1)$, and hence, $x_{n+1} = x_n + \dfrac{3^{t+1}-1}{2}$. Thus,

$$\begin{aligned} x_{n+1} &= \left(\sum_{i\in U_n} 3^i - \sum_{\substack{i\in T_n\\ i>t}} 3^i - \frac{3^t-1}{2}\right) + \frac{3^{t+1}-1}{2} \\ &= \left(\sum_{i\in U_n} 3^i + 3^t\right) - \sum_{\substack{i\in T_n\\ i>t}} 3^i = \sum_{i\in U_{n+1}} 3^i - \sum_{i\in T_{n+1}} 3^i, \end{aligned}$$

and the result holds in this case.

Subcase 3(b). $\alpha_t = 1$.

Then $n+1 = 3^t(3k+2)$, and hence, $x_{n+1} = x_n - \dfrac{3^{t+1}+1}{2}$. Thus,

$$\begin{aligned} x_{n+1} &= \left(\sum_{i\in U_n} 3^i - \sum_{\substack{i\in T_n\\ i>t}} 3^i - \frac{3^t-1}{2}\right) - \frac{3^{t+1}+1}{2} \\ &= \left(\sum_{i\in U_n} 3^i - 3^t\right) - \left(\sum_{\substack{i\in T_n\\ i>t}} 3^i + 3^t\right) = \sum_{i\in U_{n+1}} 3^i - \sum_{i\in T_{n+1}} 3^i, \end{aligned}$$

and the result holds in this case.

Thus, in every case we have the desired expression for x_{n+1}, which ends the induction and proves the lemma.

Let p be a fixed positive integer, and let $E_p = \{0, 1, \ldots, 3^{p+1}-1\}$ and $F_p = \left\{-\dfrac{3^{p+1}-1}{2}, \ldots, -1, 0, 1, \ldots, \dfrac{3^{p+1}-1}{2}\right\}$. It is well known that every integer $n \in E_p$ can be written in a unique way in the form $n = \sum_{i=0}^{p} \alpha_i 3^i$, where $\alpha_i \in \{0, 1, 2\}$ for each i. This is just the ternary expansion of n. Furthermore, every integer $m \in F_p$ can be written in a unique way in the form $m = \sum_{j=0}^{p} \beta_j 3^j$, where $\beta_j \in \{-1, 0, 1\}$ for each j (see [2000 : 402–403]).

Let $f_p : E_p \longrightarrow F_p$ be the function defined by

$$f_p\left(\sum_{i=0}^{p} \alpha_i 3^i\right) \;=\; \sum_{j=0}^{p} \beta_j 3^j \,,$$

where $\beta_i = \alpha_i$ if $\alpha_i \neq 2$, and $\beta_i = -1$ if $\alpha_i = 2$. The function f_p is clearly injective, and since $|E_p| = 3^{p+1} = |F_p|$, we deduce that f_p is bijective from E_p onto F_p. The lemma implies that $f_p(n) = x_n$ for each $n \in E_p$.

Let k be an integer. There exists a positive integer p such that $k \in F_p$. Then, from above, there exists an integer $n \in E_p$ such that $k = f_p(n) = x_n$. It follows that every integer occurs in the sequence $\{x_n\}$.

Now suppose that there are two non-negative integers m and n such that $x_m = x_n$. There exists a positive integer p such that m and n are both in E_p. Since f_p is injective, we have $m = n$. Thus, an integer cannot occur more than once in the sequence, and we are done.

2. Suppose that $n(r)$ denotes the number of points with integer coordinates on a circle of radius $r > 1$. Prove that,

$$n(r) \;<\; 6\sqrt[3]{\pi r^2}\,.$$

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Aassila's write-up.

If $n \le 8$, then, since $r > 1$ and $6\sqrt[3]{\pi} > 8$, we have $n < 6\sqrt[3]{\pi r^2}$, as required. Now suppose that for some $r > 1$ we have $n > 8$. Let the points with integer coordinates that lie on the circle be P_1, P_2, $\dots$, P_n, in counterclockwise order. Since $\widehat{P_1P_3} + \widehat{P_2P_4} + \cdots + \widehat{P_nP_2} = 4\pi$, one of the arcs $\widehat{P_iP_{i+2}}$ is at most $4\pi/n$. The triangle $P_iP_{i+1}P_{i+2}$ is inscribed in an arc of angle at most $4\pi/n$.

To simplify the notation, write A, B, C in place of P_i, P_{i+1}, P_{i+2}, respectively. Let $\theta = \widehat{AC}$ and $t = \widehat{AB}$. Then $0 < t < \theta \le 4\pi/n$, and

$$\begin{aligned}
[ABC] \;=\; \frac{abc}{4r} \;&=\; \frac{\left(2r\sin\frac{t}{2}\right)\left(2r\sin\frac{\theta}{2}\right)\left(2r\sin\frac{\theta-t}{2}\right)}{4r} \\
&\le\; 2r^2\left(\frac{t}{2}\right)\left(\frac{\theta}{2}\right)\left(\frac{\theta-t}{2}\right) \;=\; \frac{r^2\theta t(\theta-t)}{4} \\
&\le\; \frac{r^2\theta\left(\frac{\theta}{2}\right)^2}{4} \;=\; \frac{r^2\theta^3}{16} \;\le\; \frac{r^2\left(\frac{4\pi}{n}\right)^3}{16} \;=\; \frac{4r^2\pi^3}{n^3}\,.
\end{aligned}$$

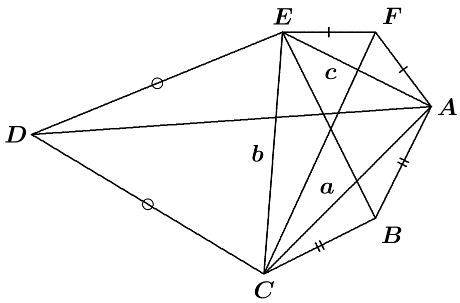
Thanks to Pick's Theorem, we know that $[ABC] \ge \frac{1}{2}$. Therefore,

$$\begin{aligned}
\frac{1}{2} \;&\le\; \frac{4r^2\pi^3}{n^3}\,, \\
n \;&\le\; \sqrt[3]{8r^2\pi^3} \;=\; 2\pi\sqrt[3]{r^2} \;<\; 6\sqrt[3]{\pi r^2}\,.
\end{aligned}$$

3. Suppose that $ABCDEF$ is a convex hexagon with $AB = BC$, $CD = DE$, and $EF = FA$. Prove that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



We put $AC = a$, $CE = b$, and $AE = c$. By the well-known generalization of Ptolemy's Theorem for quadrilateral $ABCE$, we have

$$AC \cdot BE \leq AB \cdot CE + BC \cdot AE = BC(CE + AE);$$

that is, $a \cdot BE \leq BC(b + c)$. Hence,

$$\frac{BC}{BE} \geq \frac{a}{b+c}.$$

Similarly,

$$\frac{DE}{DA} \geq \frac{b}{c+a} \quad \text{and} \quad \frac{FA}{FC} \geq \frac{c}{a+b}.$$

Thus,

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}. \qquad (1)$$

By the AM–GM Inequality, we have

$$\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq 3\sqrt[3]{\frac{1}{b+c} \cdot \frac{1}{c+a} \cdot \frac{1}{a+b}} \qquad (2)$$

and

$$(b+c) + (c+a) + (a+b) \geq 3\sqrt[3]{(b+c)(c+a)(a+b)};$$

that is,

$$a + b + c \geq \frac{3}{2}\sqrt[3]{(b+c)(c+a)(a+b)}; \qquad (3)$$

Multiplying (2) by (3), we get

$$\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b} \;\geq\; \frac{9}{2},$$

which simplifies to

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \;\geq\; \frac{3}{2}. \tag{4}$$

Finally, from (1) and (4),

$$\frac{BC}{BE}+\frac{DE}{DA}+\frac{FA}{FC} \;\geq\; \frac{3}{2}.$$

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying,

$$f\big(f(x)+y\big) \;=\; f(x^2-y)+4f(x)y\,,$$

for all real numbers x, $y \in \mathbb{R}$.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.

The functions $x \mapsto 0$ and $x \mapsto x^2$ are clearly solutions. We now show that there is no other solution.

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f\big(f(x)+y\big) \;=\; f(x^2-y)+4f(x)y \tag{1}$$

for all x, $y \in \mathbb{R}$. Let $a = f(0)$.

Taking $x = 0$ in (1) gives

$$f(a+y) \;=\; f(-y)+4ay \tag{2}$$

for all y. In (2) , we first take $y = 0$ to get $f(a) = a$, then $y = -a$ to get $a = a - 4a^2$. It follows that $a = 0$. Then, from (2), f is an even function. Comparing the results of the substitutions $y = -f(x)$ and $y = x^2$ in (1) easily leads to $4\big(f(x)\big)^2 = 4x^2f(x)$. Thus, $f(x) = 0$ or $f(x) = x^2$.

Assume now that there exists x_0 such that $f(x_0) \neq 0$. Then $x_0 \neq 0$ and $f(x_0) = x_0^2$. Since f is even, we may suppose that $x_0 > 0$. Let x be any non-zero real number. By (1) with $y = -x_0$, we obtain

$$f\big(f(x)-x_0\big) \;=\; f(x^2+x_0)-4f(x)x_0\,.$$

If $f(x) = 0$, then

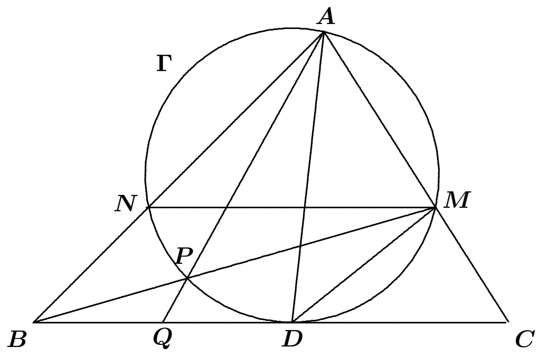
$$f(x^2+x_0) \;=\; f(-x_0) \;=\; f(x_0) \;=\; x_0^2 \;\neq\; 0\,.$$

This implies that $f(x^2 + x_0) = (x^2 + x_0)^2$, and thus, $(x^2 + x_0)^2 = x_0^2$. This is not possible, since $x_0 > 0$ and $x \neq 0$. Therefore, $f(x) = x^2$.

Thus, $f(x) = 0$ for all x or $f(x) = x^2$ for all x.

5. In triangle ABC, the angle bisector of $\angle BAC$ meets BC at point D. Suppose that Γ is the circle which is tangent to BC at D and passes through the point A. Let M be the second point of intersection of Γ and AC and BM meets the circle at P. Prove that AP is a median of triangle ABD.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Amengual's solution.



Let N be the second point of intersection of Γ and AB, and let AP meet BC at Q. Recalling that the angle between a tangent and a chord is equal to the angle subtended by the chord at a point on the circumference on the opposite side of the chord, we get $\angle MDC = \angle CAD = \frac{1}{2}\angle A$. Then

$$\begin{aligned} \angle ADM &= \angle ADC - \angle MDC \\ &= (180^\circ - \angle CAD - \angle DCA) - \angle MDC \\ &= \left(180^\circ - \tfrac{1}{2}\angle A - \angle C\right) - \tfrac{1}{2}\angle A \\ &= 180^\circ - \angle A - \angle C \;=\; \angle B\,. \end{aligned}$$

Also, $\angle ADM = \angle ANM$, since these angles are subtended by the same arc of Γ. Thus, $\angle ANM = \angle B$ (which shows that NM is parallel to BC). Hence, $\angle QPB = \angle APM = \angle ANM = \angle B$. Since we also have $\angle BQP = \angle BQA$, the triangles BPQ and ABQ are similar. Then, from the proportional sides, we get

$$\frac{BQ}{QA} = \frac{QP}{BQ},$$

giving $BQ^2 = QP \cdot QA$. We also have $QP \cdot QA = QD^2$ (the power of the point Q with respect to Γ). Hence $BQ = QD$, establishing AP as a median of $\triangle ABD$.

6. Suppose that ABC is a triangle. If we paint the points of the plane in red and green, prove that either there exist two red points which are one unit apart or three green points forming a triangle equal to ABC.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

With no loss of generality, we may suppose that there exists at least one red point (if not, the conclusion holds trivially) and that the sides of ABC have lengths a, b, c with $a \leq b$ and $a \leq c$.

Assume, for the purpose of contradiction, that there exist neither two red points which are one unit apart nor three green points forming a triangle equal to ABC.

Suppose that there are two red points M and N such that $MN = a$. Let P be such that triangle PMN is equal to triangle ABC. Let Γ_P, Γ_M, and Γ_N be the circles of radius 1 centred at P, M, and N, respectively. Then Γ_M and Γ_N are entirely green. If Γ_P is entirely red, then, since Γ_P has radius 1, there are two red points on Γ_P which are one unit apart—a contradiction. Therefore, there must be a green point on Γ_P, say X. From M and N, using the translation with vector $\overrightarrow{PX}$, we construct green points Y on Γ_M and Z on Γ_N such that XYZ is a green triangle equal to ABC, giving a contradiction.

Thus, there do not exist red points M and N such that $MN = a$.

Now, let Ω be a red point, and let C_Ω be the circle with radius a and centre Ω. From above, C_Ω is entirely green. Let $U \in C_\Omega$. Let $V \in C_\Omega$ be a point such that $UV = a$. (Thus, $\angle U\Omega V = \frac{\pi}{3} \pmod{2\pi}$.) Since U and V both lie on C_Ω, they are both green. Since $a = \min\{a, b, c\}$, there exists a point T outside C_Ω such that the triangle TUV is equal to ABC. Clearly, T must be red (if not, we would have a green triangle equal to ABC).

When we rotate U on C_Ω the set of corresponding points T is a circle Γ with centre Ω and radius $r > a$. Since Γ is entirely red, we may find two red points on it, say M and N, such that $MN = a$, a final contradiction.

Remark. More generally, one can prove ([1]): Given $\{A, B, C, D\}$ any arbitrary configuration of four points in the plane, and given any colouring of the plane with two colours, say red and green, with no two red points at distance 1 from each other, there exists a green configuration which is congruent to $\{A, B, C, D\}$.

Reference

[1] R. Juhász, Ramsey Type Theorems in the Plane, *Journal of Combinatorial Theory*, Series A, **27** (1979), p. 152–160.

Next we turn to solutions for problems of the Third Round of the 16^{th} Iranian Mathematical Olypiad 1998–1999 given [2001 : 487].

3. Suppose that $C_1, \ldots, C_n$ are circles of radius one in the plane such that no two of them are tangent, and the subset of the plane, formed by the union of these circles, is connected. If $S = \{C_i \cap C_j \mid 1 \leq i < j \leq n\}$, prove that $|S| \geq n$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

For each i, denote by $n(C_i)$ the number of elements of S which belong to C_i. Since the union of the circles is connected, we have $n(C_i) > 0$. For each $M \in S$, denote by $n(M)$ the number of circles C_i which contain M. Thus, $n(M) \geq 2$.

Let $M \in S$, and let C_i be any of the given unit circles such that $M \in C_i$. Since there is no tangency, each of the $n(M) - 1$ other circles which contain M must intersect C_i in another point. These points are pairwise distinct, because, for any two given points of the plane, there are at most two unit circles which contain the two of them. Thus, in addition to M, the circle C_i contains at least $n(M) - 1$ other elements of S. It follows that, for each $M \in S$ and each circle C_i such that $M \in C_i$, we have $n(M) \leq n(C_i)$.

Let $N = \sum\limits_{(M,C_i)} \dfrac{1}{n(C_i)}$, where the sum is for the pairs (M, C_i) such that C_i is one of the given unit circles and $M \in S \cap C_i$. We have

$$N \;=\; \sum_{C_i} \left(\sum_{M \in C_i \cap S} \frac{1}{n(C_i)} \right) \;=\; \sum_{C_i} n(C_i) \frac{1}{n(C_i)} \;=\; n\,.$$

On the other hand, since $n(M) \leq n(C_i)$, we have

$$\begin{aligned} N \;&\leq\; \sum_{(M,C_i)} \frac{1}{n(M)} \;=\; \sum_{M \in S} \left(\sum_{C_i : M \in C_i} \frac{1}{n(M)} \right) \\ &=\; \sum_{M \in S} n(M) \frac{1}{n(M)} \;=\; \sum_{M \in S} 1 \;=\; |S|\,. \end{aligned}$$

Thus, $|S| \geq n$.

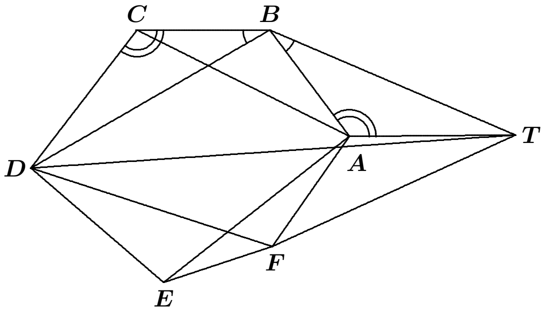
5. Suppose that $ABCDEF$ is a convex hexagon with $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} \;=\; 1\,.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} \;=\; 1\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



Since $\angle B + \angle D + \angle F = 360^\circ$, we get

$$\angle A + \angle C + \angle E = 360^\circ . \tag{1}$$

We construct $\triangle BAT$ directly similar to $\triangle BCD$, as shown in the figure. Then $\angle CBA = \angle DBT$ and $BC/AB = DB/BT$. Therefore, $\triangle BCA \sim \triangle BDT$. Hence,

$$\frac{BC}{CA} = \frac{BD}{DT} . \tag{2}$$

Since $\triangle BCD \sim \triangle BAT$, we have

$$\frac{AB}{BC} = \frac{AT}{CD} . \tag{3}$$

Since $\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$, from (3) we see that $\frac{AT}{CD} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$; that is,

$$\frac{FA}{AT} = \frac{EF}{DE} . \tag{4}$$

From (1), we have $\angle BCD + \angle BAF + \angle FED = 360^\circ$. Since $\angle BAT + \angle BAF + \angle FAT = 360^\circ$, we find that $\angle FAT = \angle FED$. This together with (4) implies that $\triangle FAT \sim \triangle FED$. Then $\triangle FAE \sim \triangle FTD$, which implies that

$$\frac{AE}{EF} = \frac{TD}{DF} . \tag{5}$$

From (2) and (5) we obtain

$$\frac{BC}{CA} \cdot \frac{AE}{EF} = \frac{BD}{DT} \cdot \frac{TD}{DF} = \frac{BD}{DF} ,$$

and hence,

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1 .$$

6. Suppose that $r_1, \ldots, r_n$ are real numbers. Prove that there exists $I \subseteq \{1, 2, \ldots, n\}$ such that I meets $\{i, i+1, i+2\}$ in at least one and at most two elements, for $1 \le i \le n-2$ and

$$\left|\sum_{i \in I} r_i\right| \;\ge\; \frac{1}{6}\sum_{i=1}^{n} |r_i| \,.$$

Solution by Mohammed Aassila, Strasbourg, France.

Let $r = \sum\limits_{i=1}^{n} |r_i|$. For $i = 0, 1, 2$, define

$$s_i \;=\; \sum_{\substack{r_j \ge 0 \\ j \equiv i \pmod 3}} r_j \quad\text{and}\quad t_i \;=\; \sum_{\substack{r_j < 0 \\ j \equiv i \pmod 3}} r_j \,.$$

Then we have $r = s_0 + s_1 + s_2 - t_0 - t_1 - t_2$, and

$$\begin{aligned} 2r \;&=\; (s_0 + s_1) + (s_1 + s_2) + (s_2 + s_0) \\ &\quad - (t_0 + t_1) - (t_1 + t_2) - (t_2 + t_0)\,. \end{aligned}$$

Therefore, there exist i_1 and i_2 with $i_1 \ne i_2$ such that $s_{i_1} + s_{i_2} \ge \frac{1}{3}r$ or $t_{i_1} + t_{i_2} \le -\frac{1}{3}r$. Assume, without loss of generality, that

$$s_{i_1} + s_{i_2} \;\ge\; \tfrac{1}{3}r \quad\text{and}\quad s_{i_1} + s_{i_2} \;\ge\; -(t_{i_1} + t_{i_2})\,.$$

Then $s_{i_1} + s_{i_2} + t_{i_1} + t_{i_2} \ge 0$, and we have

$$(s_{i_1} + s_{i_2} + t_{i_1}) + (s_{i_1} + s_{i_2} + t_{i_2}) \;\ge\; s_{i_1} + s_{i_2} \;\ge\; \tfrac{1}{3}r\,.$$

Hence, one of $s_{i_1} + s_{i_2} + t_{i_1}$ and $s_{i_1} + s_{i_2} + t_{i_2}$ must be at least $\frac{1}{6}r$.

To complete this number of the *Corner* we look at solutions from our readers to problems of the 1999 Chinese Mathematical Olympiad given [2001 : 488–489].

1. In acute triangle $\triangle ABC$, $\angle ACB > \angle ABC$. Point D is on BC such that $\angle ADB$ is obtuse. Let H be the orthocentre of $\triangle ABD$. Suppose point F is inside $\triangle ABC$ and on the circumcircle of $\triangle ABD$. Prove that point F is the orthocentre of $\triangle ABC$ if and only if HD is parallel to CF and H is on the circumcircle of $\triangle ABC$.

Solved by Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's write-up.

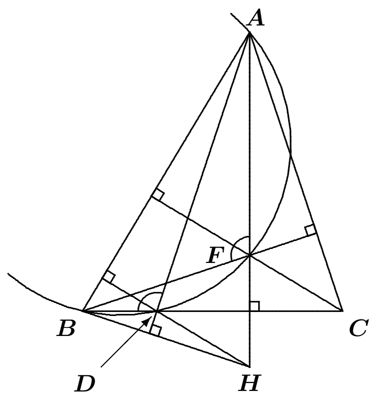


Figure 1

First let us suppose that F is the orthocentre of $\triangle ABC$ (see Figure 1). Then $CF \perp AB$. As H is the orthocentre of $\triangle ABD$, we have $HD \perp AB$. Thus, $HD \parallel CF$.

Since A, B, D, and F are concyclic,

$$\angle AFB \;=\; \angle ADB\,. \tag{1}$$

As F and H are orthocentres of $\triangle ABC$ and $\triangle ABD$, respectively, we have $AF \perp BC$, $BF \perp AC$, $AD \perp BH$, and $BD \perp AH$. It follows that $\angle AFB + \angle ACB = 180^\circ$ and $\angle ADB + \angle AHB = 180^\circ$. Then, using (1), $\angle ACB = \angle AHB$. Therefore A, B, C, and H are concyclic.

Now we consider the converse (see Figure 2). Assume that $HD \parallel CF$ and that A, B, C, and H are concyclic. Since A, B, D, and F are concyclic, $\angle AFB = \angle ADB$. Since A, B, C, and H are concyclic, $\angle ACB = \angle AHB$. As H is the orthocentre of $\triangle ABD$, we have $\angle ADB + \angle AHB = 180^\circ$. Thus, $\angle AFB + \angle ACB = 180^\circ$.

Let G be the reflection of F through AB. Then $\angle AGB = \angle AFB$ and $\angle ABG = \angle ABF$. Thus,

$$\angle AGB + \angle ACB \;=\; \angle AFB + \angle ACB = 180^\circ\,.$$

Hence, A, G, B, and C are concyclic.

Since $CF \parallel HD$ and $HD \perp AB$, we have $CF \perp AB$. Then, since $FG \perp AB$, the points C, F, and G must be collinear. Hence,

$$\angle ACF \;=\; \angle ACG \;=\; \angle ABG \;=\; \angle ABF\,.$$

Since $CF \perp AB$, we have $\angle ACF + \angle BAC = 90^\circ$, which gives us $\angle ABF + \angle BAC = 90^\circ$. Then $BF \perp AC$. Thus, F is the orthocentre of $\triangle ABC$.

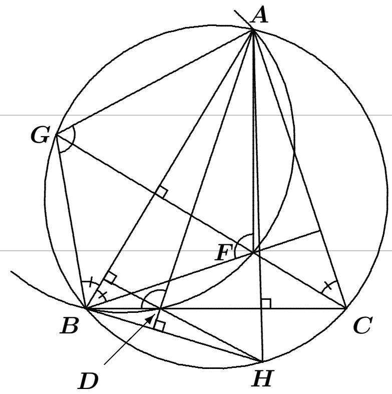


Figure 2

4. Let m be a given integer. Prove that there exist integers a, b, and k such that both a, b are not divisible by 2, $k \geq 0$, and

$$2m = a^{19} + b^{99} + k \cdot 2^{1999} .$$

Solution by Mohammed Aassila, Strasbourg, France.

Let n be a positive integer, and let r be an odd positive integer. For any odd positive integers x and y

$$x^r \equiv y^r \pmod{2^n} \qquad \Longleftrightarrow \qquad x \equiv y \pmod{2^n} ,$$

because

$$x^r - y^r = (x-y)\left(x^{r-1} + x^{r-2}y + \cdots + y^{r-1}\right)$$

and $x^{r-1} + x^{r-2}y + \cdots + y^{r-1}$ is odd. It follows that the set of congruence classes of 1^r, 3^r, 5^r, ..., $(2^n - 1)^r$ modulo 2^n is the same as the set of congruence classes of $1, 3, 5, \ldots, 2^n - 1$, which is the set of all odd congruence classes modulo 2^n.

Taking $r = 19$ and $n = 1999$, we see that there exists an odd number a_0 such that $2m - 1 \equiv a_0^{19} \pmod{2^{1999}}$. Choose $a \equiv a_0 \pmod{2^{1999}}$ to be sufficiently negative so that $2m - 1 - a^{19} > 0$. Then a solution is

$$(a, b, k) = \left(a, 1, \frac{2m - 1 - a^{19}}{2^{1999}}\right) .$$

That completes this number of the *Corner*. We are in Olympiad season. Send me your nice solutions and generalizations as well as Olympiads.

BOOK REVIEW

John Grant McLoughlin

Senior Mathematical Challenge: The UK National Mathematics Contest 1988-1996
by Tony Gardiner, published by Cambridge University Press, 2002
ISBN 0-521-66567-1, softcover, 180 pages, £13.50.
Reviewed by ***Ralph T. Mason****, University of Manitoba, Winnipeg, MB.*

A first resource for high school mathematics contest preparation?

Perhaps the outstanding feature of this collection of high school mathematics contest questions is its versatility. The core of the book is nine years of mathematics contests from the United Kingdom. Each contest provides 25 multiple-choice questions, generally arranged from easier to more difficult. As well as answers, full solutions are provided for all the questions.

Of course, a high school teacher could easily use the mathematics contests as practice opportunities for interested and capable students. The full solutions mean that the teacher need not fear that a question might expose one of the gaps in background we all have to varying degrees. Alternatively, the solutions are well enough written to be readable by students, meaning that a student who has read a solution could lead the discussion of any question and its solutions.

Better still, these contest questions and solutions could be used by any student preparing independently for contests. When a student has solved a question, there is satisfaction and encouragement in reading a comprehensible and explanatory solution. When a question has stymied a student, the student will appreciate that the solutions describe an approach to solving the question, rather than only providing a rationale for the answer. Often, the mathematical foundation of a solution is explicitly stated. (For example, the solution to Question 1 of the 1996 test, which involves expressions with integer powers being odd or even, states "Odd times odd is odd; so any power of an odd number is odd.") As a result, the independent reader can distinguish between mathematical background deficits and difficulties due to applying the mathematics to the particular question.

Many solutions include explicit advice to readers about how to approach questions, backed up by the details a reader might need when looking to expand her/his repertoire of skills. A good example is question 4 from the 1996 test, which technically could be solved inefficiently by using arithmetic. The solution first states, "You must resist the temptation to 'multiply out'. Instead, use elementary algebra to find effortless ways of calculating each option." Then, the solution includes algebraic approaches to calculating each of the question's five choices, not just the right one. A vertical bar in the margin identifies all insertions of advice and background content, making them easy to find or skip over, depending upon the reader's purpose.

The book has two other elements, both of supplemental value. The first element begins with a brief discussion of the qualities of, and the difficulty of finding, books of mathematics and mathematics problems suitable for interested high school students. Two subdivided lists of briefly annotated resources follow. The first list, with 27 entries, is "Books of problems and puzzles on a level similar to those in this book." The second, with 25 entries, is "Books which explore related mathematical content in a readable way." As the author mentions, neither list is exhaustive, providing more of a starting point than a full bibliography. The lists' best features may be their exclusivity: the author has excluded books that are better suited for more mathematically capable audiences.

The second additional element is ten sets each containing ten more multiple-choice questions. These questions have answers provided, but no solutions. At first, I considered this to be a drawback of the book. However, I now recognize that these questions provide the leader of a math contest club with a different opportunity. The 100 questions are not just further practice opportunities; they provide a chance for students to write solutions that could be useful to peers, a chance to build on the good examples the author provides in the 225 solutions provided in the main section of the book.

Generally, the book's questions and references survive relocation to North America quite well. The British monetary pound is now a cents-based system, and measurements such as the earth's circumference or the speed of a train tend to be in metric units. However, there are a scant few questions that could delight North American readers because they are from an unfamiliar context. One question, for instance (Short Paper 2, question 3), asks for the smallest number of coins needed to be able to make every amount from 1p up to £1. North American readers might wonder what difference in coinage results in the answer being one less than the number of coins needed to make any amount of change from 1 cent to \$1.

I consider this book to be a valuable addition to the resources of any teacher preparing high school students for mathematics contests. As well, its potential for the use of a student operating independently makes it an ideal addition to high school libraries. Its best usage, however, one that takes advantage of all of its features, would be by a teacher who is just beginning to build her/his resources for math contest leadership. Such a teacher could rely on this book as a primary resource before building a richer repertoire.

The Monty Hall Problem and Napier's Number

Fabio Zucca

1. The original Monty Hall problem

The well-known Monty Hall problem is one of the best examples in probability theory of a game where common sense often results in a wrong decision. The aim of this paper is to introduce and analyze some generalizations and determine the best strategy and the probability of winning the prize for these generalizations.

The original Monty Hall problem (named after one of the creators of a popular U.S. game show in the 1970's called "Let's Make A Deal") has the following basic rules: a prize is hidden behind one of three doors. The contestant chooses a door without its being opened. The host opens one of the remaining two doors without revealing the prize. The contestant is then asked if he wants to change his original choice.

If he does not change, then his probability of winning is $1/3$, since he had one chance in three originally of picking the right door. If he does change, then his probability of winning is $2/3$, since there are just these two possibilities and the total sum of the probabilities must equal 1. Often one hears the wrong answer that the probability in both cases is $1/2$, since the prize is behind one of two doors.

2. A generalization of the Monty Hall problem

Consider n indistinguishable unopened boxes ($n \geq 3$), one of them containing a prize, the others being empty. The rules are:

Step 1. The contestant chooses an unopened box.

Step 2. If there is only one unopened box besides the one chosen by the contestant, the chosen box is opened. Otherwise, the host opens at random one of the empty unopened boxes that was not chosen. Then the remaining unopened boxes are shuffled (the contestant does not watch this operation).

Step 3. The host asks if the contestant wants to choose a different unopened box. If yes, we go to Step 1. If no, we go to Step 2.

We say that we have a winning ending if the box chosen by the contestant, when opened, contains the prize.

Represent the strategy of the contestant by an $(n-2)$–dimensional vector $x = (x_1, x_2, \ldots, x_{n-2}) \in \{0,1\}^{n-2}$, where $x_i = 1$ if and only if the contestant changes his choice of box after i boxes have been opened. Denote by $p_{x,n}(i)$ (or simply $p(i)$) the probability of the event "the box chosen by the contestant before carrying out the instruction x_i contains the prize", and denote that box by B_i. Let $p_{x,n}(n-1)$ be the probability that the final box chosen, B_{n-1}, contains the prize. This is the probability of a winning ending.

Proposition 1 The sequence $\{p_{x,n}(i)\}_{i=1}^{n-1}$ satisfies $p_{x,n}(1) = 1/n$ and, for $i = 1, 2, \ldots, n-2$,

$$p_{x,n}(i+1) \;=\; \frac{x_i\big(1-p_{x,n}(i)\big)}{n-i-1} + (1-x_i)p_{x,n}(i)\,. \tag{1}$$

Proof. Denote by β_i the event "the prize is in B_i". Clearly, $p_{x,n}(1) = 1/n$. If $x_i = 1$, then $p(i+1) = p(\beta_{i+1}|\beta_i^c)\cdot p(\beta_i^c) = \dfrac{1-p(i)}{n-i-1}$, and if $x_i = 0$, then $p(i+1) = p(\beta_{i+1}) = p(\beta_i) = p(i)$. Equation (1) follows.

3. The best and worst strategies

In the original game, the best strategy was to change the door and the worst was to keep it. What are the corresponding strategies in the generalized game? To formulate an answer we have to study the sequence $\{p_{x,n}(i)\}_{i=1}^{n-1}$.

Proposition 2 For any given $i = 1, 2, \ldots, n-1$, we have

$$\frac{1}{n} \;\le\; p_{x,n}(i) \;\le\; \frac{i}{n}\,. \tag{2}$$

Proof. The claim follows by induction on i with n fixed. If $i = 1$, then this is trivial. Suppose it is true for some $i \in \{1, 2, \ldots, n-2\}$. Then

$$p(i+1) \;\ge\; \frac{x_i}{n}\frac{n-i}{n-i-1} + \frac{1-x_i}{n} \;\ge\; \frac{1}{n}\,,$$

and

$$\begin{aligned} p(i+1) \;&\le\; \frac{x_i}{n-i-1}\frac{n-1}{n} + (1-x_i)\frac{i}{n} \\ &\le\; \frac{x_i}{n}(i+1) + (1-x_i)\frac{i+1}{n} \;=\; \frac{i+1}{n}\,. \end{aligned}$$

We have used the fact that, for $i = 1, 2, \ldots, n-2$, we have $\dfrac{n-1}{n-i-1} \le i+1$.

In particular, the probability of winning the prize (that is, $p_{x,n}(n-1)$) always lies between $1/n$ and $1-1/n$. We will show that both of these values can be attained by choosing appropriate (unique) strategies. See also [1].

Proposition 3 For any $n \ge 3$ the following equivalences hold:

$$p_{x,n}(n-1) \;=\; \frac{1}{n} \quad\Longleftrightarrow\quad x_i \;=\; 0\,,\ \forall\, i = 1, 2, \ldots, n-2,$$

$$p_{x,n}(n-1) \;=\; \frac{n-1}{n} \quad\Longleftrightarrow\quad x_i \;=\; \begin{cases} 0 & \forall\, i \le n-3, \\ 1 & i = n-2. \end{cases}$$

Proof. Consider the first statement. If $x_i = 0$ for all $i = 1, 2, \ldots, n-2$, then $p_{x,n}(n-1) = 1/n$ follows from (1). We prove the converse by contradiction. Let i_0 be the smallest value with $x_{i_0} = 1$. Then, by equation (1), we have $p(i_0) > 1/n$. Using Proposition 1, we have $p(i) > 1/n$ for any $i \geq i_0$, and this contradicts the assumption that $p(n-1) = 1/n$.

Now consider the second statement. If $\{x_i\}$ satisfies the conditions on the right side, then $p_{x,n}(n-1) = 1 - 1/n$. Conversely, assume that $p(n-1) = (n-1)/n$. If $x_{n-2} = 0$, then $p(n-2) = p(n-1) = (n-1)/n$ by (1), and this contradicts (2). By (1), we have $p(n-2) = 1/n = p(1)$, and this is only possible if $x_{n-3} = 0 = \cdots = x_2 = x_1$.

Thus, if one follows the best strategy for this game, then the probability of a winning ending approaches 1 as the number of boxes grows.

4. Another generalization, its best strategy, and e

We next consider what happens if we change the choice of box whenever possible; that is, $x_i = 1$ for $i = 1, 2, \ldots, n-1$. The probability of a winning ending turns out to be related to Napier's number e, as stated in the next proposition.

Proposition 4 Let $n \geq 3$ and $x_i = 1$, for $i = 1, 2, \ldots, n-1$. Then

$$p_{x,n}(n-1) \;=\; 1 - \sum_{j=0}^{n} \frac{(-1)^j}{j!}\,,$$

and $\lim_{n \to \infty} p_{x,n}(n-1) = 1 - 1/e$.

Proof. Define a function $f_r(x) = (1-x)/r$, for $x \in \mathbb{R}$ and $r > 0$. Denote by q_n the probability $p_{x,n}(n-1)$ corresponding to the chosen strategy. Set $F_n = f_1 \circ f_2 \circ \cdots \circ f_{n-2}$. Proposition 1 implies that $q_n = F_n(1/n)$. By induction on n, it is easy to show that

$$F_n(x) - F_n(y) \;=\; (-1)^n \frac{x-y}{(n-2)!}\,, \qquad \forall\, x, y \in \mathbb{R}.$$

Hence,

$$\begin{aligned} q_{n+1} \;&=\; F_{n+1}\left(\frac{1}{n+1}\right) \;=\; F_n\left(f_{n-1}\left(\frac{1}{n+1}\right)\right) - F_n\left(\frac{1}{n}\right) + F_n\left(\frac{1}{n}\right) \\ &=\; \frac{(-1)^n}{(n-2)!}\left(f_{n-1}\left(\frac{1}{n+1}\right) - \frac{1}{n}\right) + q_n \;=\; \frac{(-1)^n}{(n+1)!} + q_n\,. \end{aligned}$$

Since $q_3 = 2/3$, it follows that

$$q_{n+1} \;=\; \sum_{j=3}^{n} \frac{(-1)^j}{(j+1)!} + \frac{2}{3} \;=\; 1 - \sum_{j=0}^{n+1} \frac{(-1)^j}{j!} \;\longrightarrow\; 1 - \frac{1}{e}$$

It is easy to see that this is the best strategy of the game if we modify Step 3 to the following: if the contestant decides not to change boxes, then the game ends and the chosen box is opened. According to this additional rule, the only strategies that are admitted are those for which $x_i = 1$ for all $i < i_0$ and $x_i = 0$ for all $i \geq i_0$, for some $i_0 \in \{1, 2, \ldots, n-2\}$.

Proposition 5 The best strategy is $x_i = 1$ for all $i = 1, 2, \ldots, n-2$ and the probability of a winning ending is $1 - \sum_{j=0}^{n} (-1)^j / j!$.

Proof. Observe that for any i, we have $p(i) > 1/(n-i)$. Hence, if $x_i = 1$,

$$p(i+1) \;=\; \frac{1-p(i)}{n-i-1} \;=\; p(i) + \frac{1-p(i)(n-i)}{n-i-1} \;>\; p(i)\,.$$

Since the game is over with probability $p(i)$ if any $x_i = 0$, the best strategy is $x = (1, 1, \ldots, 1)$. Proposition 4 now yields the conclusion.

Remark: If the contestant is not allowed to choose a previously chosen box (implying that n is odd), then the probability of a winning ending for the best strategy is "surprisingly" related to the number π. See the paper by Zorzi [2].

[**Editor's remark:** The reader can justify many of the arguments given above by keeping in mind the following principle. When the host eliminates a box, the remaining unchosen boxes "inherit" their probabilities, proportionally divided among these boxes. For example, if the contestant chooses a box and does not change until the last step, then its probability of containing the prize is $1/n$, while the other remaining box at the last step will have probability $(n-1)/n$, since it has inherited all the $1/n$ probabilities from the $n-2$ boxes that the host has eliminated together with its own $1/n$ probability.]

References

[1] V.V. Bapeswara Rao, and M. Bhaskara Rao, "A three-door game show and some of its variants", *The Mathematical Scientist*, **17**(1992), No. 2, 89–94.

[2] A. Zorzi, "Automobili, capre e π", *Archimede*, **LV**(2003), No. 4, 173–178.

Fabio Zucca
Dipartimento di Matematica
Politecnico di Milano
Piazza Leonardo da Vinci 32
20133 Milano, Italy
zucca@mate.polimi.it

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 November 2004**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

2926. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In circle Γ with centre O and radius R, we have three parallel chords A_1A_2, B_1B_2, and C_1C_2. Show that the orthocentres of the eight triangles having vertices A_i, B_j, and C_k ($i, j, k \in \{1, 2\}$) are collinear.

2927$\bigstar$. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \;\geq\; \frac{3(ab+bc+ca)}{a+b+c}\,.$$

2928. *Proposed by Christopher J. Bradley, Bristol, UK.*

Suppose that ABC is an equilateral triangle and that P is a point in the plane of $\triangle ABC$. The perpendicular from P to BC meets AB at X, the perpendicular from P to CA meets BC at Y, and the perpendicular from P to AB meets CA at Z.

1. If P is in the interior of $\triangle ABC$, prove that $[XYZ] \leq [ABC]$.
2. If P lies on the circumcircle of ABC, prove that X, Y, and Z are collinear.

2929. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Suppose that $\triangle ABC$ has $\angle A = 90^\circ$ and $\angle B > \angle C$. Let H be the foot of the perpendicular from A to BC. The point B' lies on BC and is the mirror image of B in the line AH. Suppose that D is the foot of the perpendicular from B' to AC, that E is the foot of the perpendicular from D to BC, that F is the foot of the perpendicular from B to AB', and that G is the foot of the perpendicular from F to BC. Prove that $AH = DE + FG$.

2930. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Suppose that a, b, and c are positive real numbers. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-27\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)^{-2} \geq \frac{1}{3}\left[\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{b}-\frac{1}{c}\right)^2+\left(\frac{1}{c}-\frac{1}{a}\right)^2\right].$$

2931. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given quadrilateral $ABCD$, let P, Q, R, S, M and N be the mid-points of AB, BC, CD, DA, AC and BD, respectively. Suppose that the diagonals AC and BD intersect at E. Let O be the point such that quadrilateral $NEMO$ is a parallelogram.

Prove that $[OPAS] = [OQBP] = [ORCQ] = [OSDR]$ (where $[WXYZ]$ represents the area of quadrilateral $WXYZ$.)

2932. *Proposed by Titu Zvonaru, Bucharest, Romania.*

In $\triangle ABC$, suppose that the points M, N lie on the line segment BC, the point P lies on the line segment CA, and the point Q lies on the line segment AB, such that $MNPQ$ is a square. Suppose further that

$$\frac{AM}{AN} = \frac{AC+\sqrt{2}AB}{AB+\sqrt{2}AC}.$$

Characterize $\triangle ABC$.

2933. *Proposed by Titu Zvonaru, Bucharest, Romania.*

Prove, without the use of a calculator, that $\sin(40^\circ) < \sqrt{\frac{3}{7}}$.

2934. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$ with circumradius R, let AD, BE and CF be the altitudes. Let P be any interior point of the triangle. The line through P parallel to EF intersects the line AC at E_1 and the line AB at F_1. The line through P parallel to FD intersects the line AB at F_2 and the line BC at D_2. The line through P parallel to DE intersects the line BC at D_3 and the line AC at E_3.

Show that

$$E_1F_1\cot A + F_2D_2\cot B + D_3E_3\cot C = 2R.$$

2935. *Proposed by Titu Zvonaru, Bucharest, Romania.*

Suppose that a, b, and c are positive real numbers which satisfy $a^2 + b^2 + c^2 = 1$, and that $n > 1$ is a positive integer. Prove that

$$\frac{a}{1-a^n} + \frac{b}{1-b^n} + \frac{c}{1-c^n} \;\ge\; \frac{(n+1)^{1+\frac{1}{n}}}{n}\,.$$

2936. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Consider $\triangle ABC$ with $\angle ABC = 2\angle ACB$ and $\angle BAC > 90^\circ$. Given that the perpendicular to AC through C meets AB at D, prove that

$$\frac{1}{AB} - \frac{1}{BD} \;=\; \frac{2}{BC}\,.$$

2937. *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that x_1, ..., x_n $(n \ge 2)$ are positive real numbers. Prove that

$$(x_1^2 + \cdots + x_n^2)\left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \;\ge\; \frac{n^2}{2}\,.$$

2938. *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that x_1, ..., x_n, α are positive real numbers. Prove that

(a) $\sqrt[n]{(x_1-\alpha)\cdots(x_n-\alpha)} \;\ge\; \alpha + \sqrt[n]{x_1\cdots x_n}\,;$

(b) $\sqrt[n]{(x_1-\alpha)\cdots(x_n-\alpha)} \;\le\; \alpha + \dfrac{x_1 + \cdots + x_n}{n}\,.$

. . . .

2926. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Dans un cercle Γ de centre O et de rayon R, on a trois cordes parallèles A_1A_2, B_1B_2 et C_1C_2. Montrer que les orthocentres des huit triangles ayant pour sommets A_i, B_j et C_k $(i, j, k \in \{1, 2\})$ sont colinéaires.

2927★. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Supposons que a, b et c sont des nombres réels positifs. Montrer que

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \;\ge\; \frac{3(ab + bc + ca)}{a + b + c}\,.$$

2928. *Proposé par Christopher J. Bradley, Bristol, GB.*

Soit P un point dans le plan d'un triangle équilatéral ABC. La perpendiculaire abaissée de P sur BC coupe AB en X, celle abaissée de P sur CA coupe BC en Y, et celle abaissée de P sur AB coupe CA en Z.

1. Montrer que $[XYZ] \leq [ABC]$ si P est à l'intérieur du triangle ABC.
2. Montrer que X, Y et Z sont colinéaires si P est situé sur le cercle passant par les sommets de ABC.

2929. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit ABC un triangle rectangle d'hypoténuse BC et d'angle en B plus grand que celui en C. Soit H le pied de la perpendiculaire abaissée de A sur BC et B' le symétrique de B par rapport à cette perpendiculaire. Désignons respectivement par D, E, F et G les pieds des perpendiculaires abaissées de B' sur AC, de D sur BC, de B sur AB' et de F sur BC. Montrer que $AH = DE + FG$.

2930. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Supposons que a, b, et c sont des nombres réels positifs. Montrer que

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-27\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)^{-2} \geq \frac{1}{3}\left[\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{b}-\frac{1}{c}\right)^2+\left(\frac{1}{c}-\frac{1}{a}\right)^2\right].$$

2931. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Dans un quadrilatère donné $ABCD$, soit respectivement P, Q, R, S, M et N les points milieux des segments AB, BC, CD, DA, AC et BD. Supposons que les diagonales AC et BD se coupent en E. Soit O le point tel que le quadrilatère $NEMO$ soit un parallélogramme.

Montrer que $[OPAS] = [OQBP] = [ORCQ] = [OSDR]$ (où $[WXYZ]$ représente l'aire du quadrilatère $WXYZ$.)

2932. *Proposé par Titu Zvonaru, Bucarest, Roumanie.*

Dans un triangle ABC, on suppose que les points M, N sont sur le côté BC, que le point P est sur le côté CA et que le point Q est sur le côté AB, de telle sorte que $MNPQ$ soit un carré. On suppose de plus que

$$\frac{AM}{AN} = \frac{AC+\sqrt{2}AB}{AB+\sqrt{2}AC}.$$

Que peut-on dire du triangle ABC?

2933. *Proposé par Titu Zvonaru, Bucarest, Roumanie.*

Sans calculatrice, montrer que $\sin(40°) < \sqrt{\frac{3}{7}}$.

2934. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Dans un triangle ABC dont le rayon du cercle circonscrit est R, soit AD, BE et CF les hauteurs. Soit P un point quelconque à l'intérieur du triangle. La parallèle à EF par P coupe la droite AC en E_1 et la droite AB en F_1. La parallèle à FD par P coupe la droite AB en F_2 et la droite BC en D_2. La parallèle à DE par P coupe la droite BC en D_3 et la droite AC en E_3.

Montrer que

$$E_1F_1 \cot A + F_2D_2 \cot B + D_3E_3 \cot C \;=\; 2R\,.$$

2935. *Proposé par Titu Zvonaru, Bucarest, Roumanie.*

Supposons que a, b et c sont des nombres réels positifs satisfaisant $a^2 + b^2 + c^2 = 1$, et que $n > 1$ est un entier positif. Montrer que

$$\frac{a}{1-a^n} + \frac{b}{1-b^n} + \frac{c}{1-c^n} \;\geq\; \frac{(n+1)^{1+\frac{1}{n}}}{n}\,.$$

2936. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

On donne un triangle ABC tel que l'angle ABC est le double de l'angle ACB et que l'angle BAC soit supérieur à $90°$. De plus, la perpendiculaire à AC en C coupe AB en D. Montrer que

$$\frac{1}{AB} - \frac{1}{BD} \;=\; \frac{2}{BC}\,.$$

2937. *Proposé par Todor Mitev, Université de Rousse, Rousse, Bulgarie.*

Supposons que $x_1, \ldots, x_n$ $(n \geq 2)$ sont des nombres réels positifs. Montrer que

$$\left(x_1^2 + \cdots + x_n^2\right)\left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \;\geq\; \frac{n^2}{2}\,.$$

2938. *Proposé par Todor Mitev, Université de Rousse, Rousse, Bulgarie.*

Supposons que $x_1, \ldots, x_n$, α sont des nombres réels positifs. Montrer que

(a) $\sqrt[n]{(x_1-\alpha)\cdots(x_n-\alpha)} \;\geq\; \alpha + \sqrt[n]{x_1\cdots x_n}\,;$

(b) $\sqrt[n]{(x_1-\alpha)\cdots(x_n-\alpha)} \;\leq\; \alpha + \dfrac{x_1+\cdots+x_n}{n}\,.$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2778. [2002 : 457; 2003 : 414–415] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $z \neq 1$ is a complex number such that $z^n = 1$ $(n \geq 1)$. Prove that

$$\left|nz - (n+z)\right| \;\leq\; \frac{(n+1)(2n+1)}{6}\,|z-1|^2\,.$$

Editor's Remark. At the end of the featured solution to this problem, there was a conjecture by Walther Janous that the best multiplier of $|z-1|^2$ on the right side of the inequality would be

$$\frac{\sqrt{4n(n-1)\sin^2\left(\frac{\pi}{n}\right)+1}}{4\sin^2\left(\frac{\pi}{n}\right)}\,.$$

We have received a proof of this conjecture.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

We are given that $z = e^{2k\pi i/n}$ for some $k \in \{1, 2, \ldots, n-1\}$. Thus,

$$\begin{aligned} |z-1|^2 &= \left(\cos\frac{2k\pi}{n} - 1\right)^2 + \left(\sin\frac{2k\pi}{n}\right)^2 \\ &= 2\left(1-\cos\frac{2k\pi}{n}\right) \;=\; 4\sin^2\frac{k\pi}{n} \end{aligned}$$

and

$$\begin{aligned} |nz-(n+z)| &= \sqrt{\left((n-1)\cos\frac{2k\pi}{n} - n\right)^2 + \left((n-1)\sin\frac{2k\pi}{n}\right)^2} \\ &= \sqrt{(n-1)^2 + n^2 - 2n(n-1)\cos\frac{2k\pi}{n}} \\ &= \sqrt{1 + 2n(n-1)\left(1-\cos\frac{2k\pi}{n}\right)} \\ &= \sqrt{1+4n(n-1)\sin^2\frac{k\pi}{n}}\,. \end{aligned}$$

Let a be a positive real number. Then $|nz-(n+z)| \leq a|z-1|^2$ if and only if

$$16a^2\sin^4\frac{k\pi}{n} - 4n(n-1)\sin^2\frac{k\pi}{n} - 1 \;\geq\; 0\,.$$

By the quadratic formula, this is true if and only if

$$\sin^2\frac{k\pi}{n} \;\ge\; \frac{n(n-1)+\sqrt{n^2(n-1)^2+4a^2}}{8a^2} \;=\; \frac{1}{2\left(\sqrt{n^2(n-1)^2+4a^2}-n(n-1)\right)}\,.$$

Since $\sin^2\frac{k\pi}{n} \ge \sin^2\frac{\pi}{n}$ for all $k \in \{1, 2, \ldots, n-1\}$, the above inequality is true for all $k \in \{1, 2, \ldots, n-1\}$ if and only if it is true when $k = 1$, in which case

$$\sqrt{n^2(n-1)^2+4a^2}-n(n-1) \;\ge\; \frac{1}{2\sin^2\frac{\pi}{n}}\,,$$

$$a \;\ge\; \frac{\sqrt{4n(n-1)\sin^2\frac{\pi}{n}+1}}{4\sin^2\frac{\pi}{n}}\,.$$

2826. [2003 : 174] *Proposed by Bernardo Recamán Santos, Bogota, Colombia.*

Show that, for every sufficiently large integer n, it is possible to split the integers $1, 2, \ldots, n$ into two disjoint subsets such that the sum of the elements in one set equals the product of the elements in the other.

Essentially the same solution by Pierre Bornsztein, Maisons-Laffitte, France; Christopher Bowen, Halandri, Greece; Con Amore Problem Group, The Danish University of Education, Copenhagen, Denmark; Chip Curtis, Missouri Southern State College, Joplin, MO, USA; Keith Ekblaw, Walla Walla, WA, USA; Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina; Josh Guinn, student, Polk Community College, Winter Haven, FL, USA; James Holton, student, Polk Community College, Winter Haven, FL, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; D. Kipp Johnson, Beaverton, OR, USA; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Kee-Wai Lau, Hong Kong, China; Undine Leopold, student, Ludwigsgymnasium, Koethen, Germany; David Loeffler, student, Trinity College, Cambridge, UK; Gottfried Perz, Pestalozzigymnasium, Graz, Austria; Robert P. Sealy, Mount Allison University, Sackville, NB; Southwest Missouri State University Problem Solving Group, Springfield, MO, USA; Mike Spivey, Samford University, Birmingam, AL, USA; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.

Recall that

$$\sum_{i=1}^{n} i \;=\; \frac{n(n+1)}{2}\,.$$

For odd $n \ge 5$,

$$\frac{n(n+1)}{2} - 1 - \frac{n-1}{2} - (n-1) \;=\; (1)\left(\frac{n-1}{2}\right)(n-1)\,,$$

and for even $n \geq 6$,

$$\frac{n(n+1)}{2} - 1 - \frac{n-2}{2} - n \;=\; (1)\left(\frac{n-2}{2}\right)(n)\,.$$

Several solvers pointed out that a partition exists for $n = 1$ (using an empty product), and $n = 3$; however, no partition is possible for $n = 2$ or $n = 4$.

Holton asks if the partition is unique. The proposer asks if the partition is unique for infinitely many n.

2827. [2003 : 175, 314] Corrected. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a non-negative integer. Determine

$$\sum_{k=0}^{n} \frac{\tanh(2^k)}{1 + 2\sinh^2(2^k)}\,.$$

[*Editor*: The problem is stated above as it was intended by the proposers. As first given in [2003 : 175], it contained *two* errors, only one of which was corrected later [2003 : 314]. The editors wish to extend an apology to the proposers and the readers for spoiling a very nice problem.]

Solution by the proposers.

We will prove that for any real number x,

$$\sum_{k=0}^{n} \frac{\tanh(2^k x)}{1 + 2\sinh^2(2^k x)} \;=\; \tanh(2^{n+1}x) - \tanh(x)\,. \qquad (1)$$

It will then follow, by setting $x = 1$, that the sum given in the problem is $\tanh(2^{n+1}) - \tanh(1)$.

For each $k = 0, 1, 2, \ldots$, we have

$$\begin{aligned}
\frac{\tanh(2^k x)}{1 + 2\sinh^2(2^k x)} &= \frac{\sinh(2^k x)}{\cosh(2^{k+1}x)\cosh(2^k x)} \\
&= \frac{\sinh(2^{k+1}x - 2^k x)}{\cosh(2^{k+1}x)\cosh(2^k x)} \\
&= \frac{\sinh(2^{k+1}x)\cosh(2^k x) - \cosh(2^{k+1}x)\sinh(2^k x)}{\cosh(2^{k+1}x)\cosh(2^k x)} \\
&= \frac{\sinh(2^{k+1}x)}{\cosh(2^{k+1}x)} - \frac{\sinh(2^k x)}{\cosh(2^k x)} \\
&= \tanh(2^{k+1}x) - \tanh(2^k x)\,.
\end{aligned}$$

Consequently, the sum on the left side of (1) telescopes to give the right side.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incorrect solution.

If the finite summation is replaced by a series with k going from 0 to ∞ (as in the version of the problem that was originally published), then the sum is $1 - \tanh(1)$, as we can see by letting $n \to \infty$ in the result above.

2828. [2003 : 175] *Proposed by Achilleas Pavlos Porfyriadis, student, American College of Thessaloniki "Anatolia", Thessaloniki, Greece* (*adapted by the Editors*).

Suppose that f satisfies the functional equation

$$f(x) + 2f\left(\frac{x+2000}{x-1}\right) \;=\; 4011 - x\,.$$

Find the value of $f(2002)$.

Solution by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA; Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina; and Neven Jurič, Zagreb, Croatia.

Let $g(x) = \dfrac{x+2000}{x-1}$, for $x \neq 1$. The given equation becomes

$$f(x) + 2f\big(g(x)\big) \;=\; 4011 - x\,. \qquad (1)$$

It is easy to check that $g(x) \neq 1$ and $g\big(g(x)\big) = x$. Replacing x by $g(x)$ in (1), we get

$$f\big(g(x)\big) + 2f(x) \;=\; 4011 - g(x)\,. \qquad (2)$$

Eliminating $f\big(g(x)\big)$ from (1) and (2), we have

$$3f(x) \;=\; 4011 - 2g(x) + x\,;$$

whence,

$$f(x) \;=\; \frac{4011 - 2g(x) + x}{3} \;=\; \frac{x^2 + 4008x - 8011}{3(x-1)}\,.$$

In particular, $f(2002) = 2003$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO TEAM 2003; DIONNE BAILEY and ELSIE CAMPBELL, Angelo State University, San Angelo, TX, USA; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHRISTOPHER BOWEN, Halandri, Greece; CHRISTOPHER J. BRADLEY, Bristol, UK; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; PAOLO CUSTODI, Fara Novarese, Italy; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, *Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; KEE-WAI LAU, Hong Kong, China; UNDINE LEOPOLD, student, Ludwigsgymnasium, Koethen, Germany; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; GOTTFRIED PERZ, Pestalozzi-gymnasium, Graz, Austria; ROBERT P. SEALY, Mount Allison University, Sackville, NB; BOB SERKEY, Leonia, NJ, USA; MIKE SPIVEY, Samford University, Birmingam, AL, USA; MIHAÏ STOËNESCU, Bischwiller, France; ROBERT VAN DEN HOOGEN, Saint Francis Xavier University, Antigonish, NS; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer. There was one incorrect solution submitted.*

Janous solved the more general problem: Find all functions $f:\mathbb{R}\setminus\{1\}\to\mathbb{R}$ satisfying the functional equation

$$f(x)+af\left(\frac{x+b}{x-1}\right) \;=\; c-x\,,$$

where $|a|\neq 1$ and $b\neq -1$. Using the same approach as in the presented solution, he has shown that the solution is

$$f(x) \;=\; \frac{x^2+\big(a(c-1)-c-1\big)x-a(b+c)+c}{(a^2-1)(x-1)}\,.$$

The original equation is obtained when $a=2$, $b=2000$, and $c=4011$.

2829. [2003 : 175] (Corrected [2003 : 315]) *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Given $\triangle ABC$ with sides a, b, c, prove that

$$\frac{3\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2} \;\geq\; 2\,.$$

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Without loss of generality, assume that $a\leq b\leq c$. The desired inequality is equivalent to

$$3(a^4+b^4+c^4)+(ab+bc+ca)(a^2+b^2+c^2)-2(a^2+b^2+c^2)^2 \;\geq\; 0\,,$$

which is equivalent to

$$\tfrac{1}{4}\Big(a(2a-b-c)^2(a+2b+2c) + (b-c)^2(4b^2+4c^2+12bc+2ab+2ca-9a^2)\Big) \;\geq\; 0\,.$$

The last inequality is clearly true. Equality holds if and only if $a=b=c$. Note that this proof does not require that a, b, and c are the sides of a triangle, as long as they are non-negative and $a^2+b^2+c^2>0$.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The desired inequality is equivalent to

$$3(a^4+b^4+c^4)+(ab+bc+ca)(a^2+b^2+c^2)-2(a^2+b^2+c^2)^2 \ \geq\ 0\,.$$

Since this inequality is symmetric, we can assume that $a \leq b \leq c$. Let $b = a+t$ and $c = a+t+s$, where $s, t \geq 0$. The inequality becomes

$$\begin{aligned} &3\big(a^4+(a+t)^4+(a+t+s)^4\big) \\ &\quad+\big(a(a+t)+(a+t)(a+t+s)+(a+t+s)a\big) \\ &\quad\cdot\big(a^2+(a+t)^2+(a+t+s)^2\big) \\ &\quad-2\big(a^2+(a+t)^2+(a+t+s)^2\big)^2 \ \geq\ 0\,, \end{aligned}$$

which simplifies to

$$5a^2(s^2+st+t^2)+2a(3s^3+7s^2t+3st^2+2t^3)+s^4+5s^3t+5s^2t^2 \ \geq\ 0\,.$$

The last inequality is clearly true.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER J. BRADLEY, Bristol, UK; VASILE CÎRTOAJE, University of Ploiesti, Romania; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; D. KIPP JOHNSON, Beaverton, OR, USA; NEVEN JURIČ, Zagreb, Croatia; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Bucharest, Romania.

Zhou mentions that he had used the same factoring technique as in his solutions to Crux problems 2807 and 2821. Originally, the problem was proposed as

$$\frac{2(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2} \ \geq\ 2\,,$$

which is not true. Several solvers have noticed this fact and provided counter-examples. Alt and Loefler have gone a step further and proved that the expression on the left side of the originally proposed inequality is between $3/2$ and $11/6$. Most solvers have solved the corrected problem.

2830. [2003 : 176] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that $\Gamma(O,R)$ is the circumcircle of $\triangle ABC$. Suppose that side AB is fixed and that C varies on Γ (always on the same side of AB).

Suppose that I_a, I_b, I_c, are the centres of the excircles of $\triangle ABC$ opposite A, B, C, respectively. If Ω is the centre of the circumcircle of $\triangle I_aI_bI_c$, determine the locus of Ω as C varies.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let I be the incentre of $\triangle ABC$. Because external and internal bisectors are perpendicular, we see that AI_a, BI_b, CI_c are the altitudes of $\triangle I_aI_bI_c$, the point I is its orthocentre, and Γ (through the feet of the altitudes) is its nine-point circle. Since I and Ω are the orthocentre and circumcentre of $\triangle I_aI_bI_c$, the line $I\Omega$ is its Euler line and O (the nine-point centre) must be the mid-point of $I\Omega$. As C moves along Γ, we note that $\angle AIB = 90° + C/2$ is fixed. Hence, I moves on a fixed circle Γ' through A and B [whose centre is the mid-point of the arc AB opposite C]. Since the mid-point O of $I\Omega$ is fixed, the point Ω must move on a circle that is the reflection of Γ' in O (whose centre is the mid-point of the arc AB on the same side as C). As C runs from A to B on Γ, the point Ω runs on the arc [interior to Γ] of its circle from the point of Γ diametrically opposed to A to the point opposed to B.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; JOHN G. HEUVER, Grande Prairie, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2831. [2003 : 176] *Proposed by Achilleas Pavlos Porfyriadis, student, American College of Thessaloniki "Anatolia", Thessaloniki, Greece.*

For a convex polygon, prove that it is impossible for two sides without a common vertex to be longer than the longest diagonal.

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Suppose that AB and CD are two sides without a common vertex. We may assume that the diagonals AC and BD intersect at a point E inside the polygon. By the Triangle Inequality,

$$AB + CD \;<\; AE + EB + DE + EC \;=\; AC + BD \;\le\; 2d\,,$$

where d is the length of the longest diagonal. Hence, AB and CD cannot both be greater than d.

II. *Solution by Christopher Bowen, Halandri, Greece.*

Suppose to the contrary that we are given a polygon with n sides and that two of its sides, say AB and CD, have length greater than all the diagonals of the polygon.

Since the sides AB and CD have no vertex in common, the vertices A, B, C, D are distinct and n is at least 4. If AA' is a side of the polygon, then BA' is a diagonal, implying that $BA > BA'$. Thus, in $\triangle ABA'$, we have $\angle A'AB < \angle AA'B$. In particular, $\angle A'AB$, the internal angle at vertex A, must be acute. This holds equally well for internal angles at B, C, and D.

Therefore, the polygon has at least 4 obtuse external angles and cannot be convex.

Also solved by PIERRE BORNSZTEIN, Maisons-Laffitte, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2832★. [2003 : 176] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n be a positive integer, and let

$$a(n) \;=\; \left|\sum_{j=0}^{3n}(-2)^j\left(\binom{6n+2-j}{j+1}+\binom{6n+1-j}{j}\right)\right|\,.$$

Prove that

(a) $a(n)=3$ if and only if $n=1$, and

(b) the sequence $\{a(n)\}_{n=1}^{\infty}$ is strictly increasing.

Editor: There were no solutions submitted for this problem. As a result, problem 2832 remains an open problem which the readers of ***CRUX with MAYHEM*** are encouraged to revisit.

2833★. [2003 : 177] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a be a positive real number, and let $n \geq 2$ be an integer. For each $k=1, 2, \ldots, n$, let x_k be a non-negative real number, λ_k be a positive real number, and let $y_k = \lambda_k x_k + \dfrac{x_{k+1}}{\lambda_{k+1}}$. Here and elsewhere, indices greater than n are to be reduced modulo n.

(a) If $a > 1$, prove that

$$n+\sum_{k=1}^{n} a^{y_k} \;\geq\; 2\sum_{k=1}^{n} a^{x_k} \qquad \text{and} \qquad 3n+\sum_{k=1}^{n} a^{y_k+y_{k+1}} \;\geq\; \sum_{k=1}^{n}\left(1+a^{x_k}\right)^2\,.$$

(b) If $0 < a < 1$, prove that the opposite inequalities hold.

[The proposer has proofs for the cases $n=3$ and $n=4$.]

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

(a) If u, $v > 1$, we have $(u-1)(v-1) > 0$, or

$$1+uv \;>\; u+v\,. \tag{1}$$

Hence,

$$n+\sum_{k=1}^{n} a^{y_k} \;=\; \sum_{k=1}^{n}\left(1+a^{\lambda_k x_k} a^{\frac{x_{k+1}}{\lambda_{k+1}}}\right) \;>\; \sum_{k=1}^{n} a^{\lambda_k x_k}+\sum_{k=1}^{n} a^{\frac{x_k}{\lambda_k}}\,.$$

Meanwhile, by the AM–GM Inequality,

$$a^{\lambda_k x_k}+a^{\frac{x_k}{\lambda_k}} \;\geq\; 2a^{\frac{1}{2}x_k\left(\lambda_k+\frac{1}{\lambda_k}\right)} \;\geq\; 2a^{x_k}\,. \tag{2}$$

Combining these results, we obtain

$$n+\sum_{k=1}^{n} a^{y_k} \;>\; 2\sum_{k=1}^{n} a^{x_k}\,.$$

Substituting the inequality (1) into itself yields

$$2+uvw \;>\; 1+uv+w \;>\; u+v+w\,. \tag{3}$$

Also,

$$\begin{aligned} y_k+y_{k+1} &= \lambda_k x_k+\left(\lambda_{k+1}+\frac{1}{\lambda_{k+1}}\right)x_{k+1}+\frac{x_{k+2}}{\lambda_{k+2}} \\ &\geq \lambda_k x_k+2x_{k+1}+\frac{x_{k+2}}{\lambda_{k+2}}\,. \end{aligned}$$

Denoting these last three terms by u, v, and w, respectively, in (3), we obtain

$$3n+\sum_{k=1}^{n} a^{y_k+y_{k+1}} \;\geq\; n+\sum_{k=1}^{n} a^{\lambda_k x_k}+\sum_{k=1}^{n} a^{2x_k}+\sum_{k=1}^{n} a^{\frac{x_k}{\lambda_k}}\,,$$

and using (2), we obtain

$$3n+\sum_{k=1}^{n} a^{y_k+y_{k+1}} \;\geq\; n+2\sum_{k=1}^{n} a^{x_k}+\sum_{k=1}^{n} a^{2x_k} \;=\; \sum_{k=1}^{n}(1+a^{x_k})^2\,.$$

(b) The claim is not true. Let $x_k=\lambda_k=1$ for all k. Then the first inequality becomes

$$n+na^2 \;\geq\; 2na \qquad\Longleftrightarrow\qquad n(1-a)^2 \;\geq\; 0\,,$$

which is clearly true for all a. Similarly, the second inequality becomes

$$3n+na^4 \;\geq\; n(1+a)^2 \qquad\Longleftrightarrow\qquad (a-1)^2(a^2+2a+2) \;\geq\; 0\,,$$

which is also true for all a. Thus, the reverse inequalities do not hold for $0<a<1$.

It is not clear whether the inequalities hold in their original forms for $0 < a < 1$; all parts of the proofs go through unchanged except (2), at which point we would need to show that

$$f(t) + f(u) \;\geq\; 2f\left(\frac{t+u}{2}\right),$$

where $f(x) = e^{-e^x}$ at $t = \log_e\big(\lambda_k x_k(-\log_e a)\big)$, $u = \log_e\left(\frac{x_k}{\lambda_k}(-\log_e a)\right)$. By checking the second derivative of f, we see that this does indeed hold as long as t and u are always positive; that is, if all the values $\lambda_k x_k$ and $\frac{x_k}{\lambda_k}$ are at least as large as $\frac{1}{-\log_e a}$. Otherwise, this line of argument fails, but the inequality may still be true.

Also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA.

2834. [2003 : 177] *Proposed by Michel Bataille, Rouen, France.*

Let $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for integers $n > 2$. Then define

$$g_n \;=\; f_{n+6} + 3f_{n+2} + 3f_{n-2} + f_{n-6}$$

for integers $n > 6$. Find $\gcd\{g_{f_{6666}}, g_{f_{666}}\}$.

Solution by Chip Curtis, Missouri Southern State College, Joplin, MO, USA; and Mike Spivey, Samford University, Birmingam, AL, USA.

Repeated application of the recurrence relation for the Fibonacci sequence $\{f_i\}$ gives

$$\begin{aligned} f_{n-2} &= 3f_{n-5} + 2f_{n-6}, \\ f_n &= 8f_{n-5} + 5f_{n-6}, \\ f_{n+2} &= 21f_{n-5} + 13f_{n-6}, \\ f_{n+6} &= 144f_{n-5} + 89f_{n-6}, \end{aligned}$$

so that

$$g_n \;=\; 216f_{n-5} + 135f_{n-6} \;=\; 27(8f_{n-5} + 5f_{n-6}) \;=\; 27f_n\,.$$

The following property of the Fibonacci sequence is well-known:

$$\gcd\{f_m, f_n\} \;=\; f_{\gcd\{m,n\}}\,.$$

(See, for example, Kenneth H. Rossen, editor, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, 2000, p. 143). Applying this property and using $\gcd\{6666, 666\} = 6$, $f_6 = 8$, and $f_8 = 21$, we obtain

$$\begin{aligned} \gcd\{g_{f_{6666}}, g_{f_{666}}\} &= \gcd\{27f_{f_{6666}}, 27f_{f_{666}}\} \;=\; 27\gcd\{f_{f_{6666}}, f_{f_{666}}\} \\ &= 27f_{\gcd\{f_{6666}, f_{666}\}} \;=\; 27f_{f_{\gcd\{6666,666\}}} \\ &= 27f_{f_6} \;=\; 27f_8 \;=\; 27(21) \;=\; 567\,. \end{aligned}$$

Also solved by CHRISTOPHER BOWEN, Halandri, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina (2 solutions); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were also two incorrect solutions submitted.

2835. [2003 : 178] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

For non-negative real numbers x and y, not both equal to 0, prove that

$$\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y} \;\geq\; \frac{5}{8}\,.$$

I. *Composite of essentially identical solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Pierre Bornsztein, Maisons-Laffitte, France; Chip Curtis, Missouri Southern State College, Joplin, MO, USA; Charles Diminnie, Elsie Campbell, and Dionne Bailey, Angelo State University, San Angelo, TX, USA; Kee-Wai Lau, Hong Kong, China; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

If $y=0$, then we have $1\geq\frac{5}{8}$. Suppose that $y\neq 0$ and let $u=\sqrt{\frac{x}{y}}$. Then the given inequality becomes

$$\frac{u^8+1}{(u^2+1)^4}+\frac{u}{u^2+1} \;\geq\; \frac{5}{8}\,.$$

This can be rewritten as:

$$\begin{aligned} 8(u^8+1)+8u(u^2+1)^3-5(u^2+1)^4 &\;\geq\; 0\,,\\ \text{or}\quad (u-1)^2(3u^6+14u^5+5u^4+20u^3+5u^2+14u+3) &\;\geq\; 0\,, \end{aligned}$$

which is clearly true. Equality holds if and only if $u=1$; that is, if and only if $x=y$.

II. *Composite of solutions by Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA (modified slightly by the editor).*

By homogeneity, we may suppose $x+y=1$. Then

$$x^2+y^2 \;=\; (x+y)^2-2xy \;=\; 1-2xy\,,$$

and

$$x^4+y^4 \;=\; (x^2+y^2)^2-2x^2y^2 \;=\; 1-4xy+2x^2y^2\,.$$

Hence, it suffices to show that $1-4xy+2x^2y^2+\sqrt{xy}\geq\frac{5}{8}$.

Let $t = \sqrt{xy}$. Then $0 \le t \le \frac{1}{2}$, and the inequality above becomes

$$\begin{aligned} 2t^4 - 4t^2 + t + \tfrac{3}{8} &\ge 0\,, \\ 16t^4 - 32t^2 + 8t + 3 &\ge 0\,, \\ \text{or} \quad (4t^2-3)(4t^2-1) + 8t(1-2t) &\ge 0\,, \end{aligned}$$

which is true. Equality holds if and only if $t = \frac{1}{2}$; that is, if and only if $x = y$.

III. *Generalization by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Since $\sqrt{xy} \ge \dfrac{2xy}{x+y}$, the given inequality follows from the stronger inequality

$$\frac{x^4+y^4}{(x+y)^4} + \frac{2xy}{(x+y)^2} \ge \frac{5}{8}\,. \tag{1}$$

Without loss of generality, assume $x \ne 0$. Let $y = tx$. Then (1) becomes

$$\frac{t^4+1}{(t+1)^4} + \frac{2t}{(t+1)^2} \ge \frac{5}{8}\,. \tag{2}$$

Straightforward computations show that (2) is equivalent to

$$\begin{aligned} 3t^4 - 4t^3 + 2t^2 - 4t + 3 &\ge 0\,, \\ \text{or} \quad (t-1)^2(3t^2+2t+3) &\ge 0\,, \end{aligned}$$

which is clearly true, with equality if and only if $t = 1$. Therefore, equality holds in (1) if and only if $x = y$. It then follows that equality holds in the original inequality if and only if $x = y$.

Also solved by ARKADY ALT, San Jose, CA, USA; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHRISTOPHER J. BRADLEY, Bristol, UK; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina (2 solutions); RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; D. KIPP JOHNSON, Beaverton, OR, USA; NEVEN JURIČ, Zagreb, Croatia; VEDULA N. MURTY, Dover, PA, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; MIKE SPIVEY, Samford University, Birmingham, AL, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer. There were also two partly incorrect solutions.

Alt considered and solved several related problems. In particular, he proved that for $n \in \mathbb{N}$, $n \ge 2$, the inequality

$$\frac{x^n+y^n}{(x+y)^n} + \frac{\sqrt{xy}}{x+y} \ge \frac{1}{2} + \frac{1}{2^{n-1}}$$

holds if and only if $2 \le n \le 7$. (Note that the given inequality is the special case when $n = 2$.) Janous also extended the result to some inequalities involving the Power Mean.

2836. [2003 : 178] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Suppose that $\triangle ABC$ is equilateral and that P is an interior point. The lines AP, BP, CP intersect the opposite sides at D, E, F, respectively. Suppose that $PD = PE = PF$. Determine the locus of P.

Solution by David Loeffler, student, Trinity College, Cambridge, UK, minimally modified by the editor.

We shall solve the following more general problem: *what is the locus of P such that $PD = PE$?*

Suppose, for convenience, that $AB = BC = CA = 1$. Let x, y, z be the lengths of the perpendiculars from P to BC, CA, AB, respectively. Then $z/y = BD/CD$. Letting M be the mid-point of BC, it follows that

$$MD \;=\; \left|\frac{1}{2} - BD\right| \;=\; \left|\frac{1}{2} - \frac{y}{y+z}\right| \;=\; \frac{1}{2}\left|\frac{y-z}{y+z}\right| .$$

Since the altitude in $\triangle ABC$ is $\frac{\sqrt{3}}{2}$, we have

$$AD^2 \;=\; \left(\frac{\sqrt{3}}{2}\right)^2 + MD^2 \;=\; \frac{3}{4} + \frac{1}{4}\left(\frac{y-z}{y+z}\right)^2 \;=\; \frac{y^2+yz+z^2}{(y+z)^2} .$$

Then, since $\dfrac{PD}{AD} = \dfrac{x}{\sqrt{3}/2}$, we have

$$PD^2 \;=\; \frac{4x^2}{3}AD^2 \;=\; \frac{4x^2(y^2+yz+z^2)}{3(y+z)^2} .$$

Applying the same argument to PE, we see that $PD = PE$ if and only if

$$\frac{x^2\left(y^2+yz+z^2\right)}{(y+z)^2} \;=\; \frac{y^2\left(x^2+xz+z^2\right)}{(z+x)^2} .$$

This is equivalent to

$$x^2(x+z)^2(y^2+yz+z^2) - y^2(y+z)^2(x^2+xz+z^2) \;=\; 0 .$$

Factoring gives

$$(x-y)(x+y+z)(x^2y^2+zx^2y+z^2x^2+zxy^2+z^2xy+z^3x+z^2y^2+z^3y) \;=\; 0 .$$

Since the second and third terms of the factored polynomial are positive, it follows that $PD = PE$ if and only if $x = y$. Thus, the locus is the altitude through C.

It follows that the solution to the original problem is the intersection of the three altitudes; that is, the orthocentre of the equilateral triangle ABC [*Ed.* in this case, also the centroid, the circumcentre, and the incentre as stated by various solvers].

The original problem also solved by MICHEL BATAILLE, Rouen, France; ROBERT BILINSKI, Outremont, QC; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; LI ZHOU, Polk Community College, Winter Haven, FL, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Loeffler also found the locus of a point P such that $AD = BE$. We leave this to the readers to discover. He also expressed interest in the case of a scalene triangle, but stated that he had not been able to make any progress on that case.

Janous suggests the following extension (for which he has no solution at the moment):

> Let ABC be an equilateral triangle. For an interior point P, let the Cevians AP, BP, CP, intersect the opposite sides at D, E, F, respectively. For $\triangle DEF$, determine the three loci of P such that
>
> 1. P is the centroid;
> 2. P is the orthocentre;
> 3. P is the incentre,

2837. [2003 : 178] *Proposed by Christopher J. Bradley, Bristol, UK.*

Suppose that Γ is a circle and that I, J, and K are three distinct points in the plane of Γ, but not on Γ. Let A be any point on Γ. Points B, C, D, E, F, and G on Γ are defined by the conditions that chords AB and DE intersect at I, chords BC and EF intersect at J, and chords CD and FG intersect at K. (A tangent is to be regarded as a chord with its point of contact defined to be a pair of coincident points.)

Is it possible to select the positions of I, J, and K so that G coincides with A for all points A lying on Γ? (Justification required!)

I. *Identical solutions by Peter Y. Woo, Biola University, La Mirada, CA, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

The answer is *YES*. Select I, J, and K to be three collinear points in the plane of the given circle Γ (or, more generally, of a given conic Γ), but not on Γ. For any point A on Γ, the figure $ABCDEF$ is a hexagon inscribed in a conic. Hence, we may apply Pascal's Theorem to it. By definition AB meets DE at I while BC meets EF at J. Moreover, K must be the point where CD intersects IJ, since we have taken K to lie on the line IJ and defined it to lie on CD. Therefore, by Pascal's Theorem (which states that the intersections of the opposite sides of $ABCDEF$ are collinear), we conclude that AF must also pass through K. Since G was defined to be the point where FK meets Γ again, we conclude that G coincides with A for any choice of A.

II. *Solution by Michel Bataille, Rouen, France.*

Editor's comment. Pascal's Theorem forces I, J, and K to be collinear. The theorem continues to hold for positions of A that coincide with any of the other five vertices; for example, should AI be tangent to Γ, then $B = A$.

In fact, the theorem remains valid for as many as three pairs of identical vertices among the six, so long as the three points I, J, and K are well defined. Bataille, on the other hand, interpreted the problem differently: with Γ, I, J, and K fixed, then for any point A on Γ, point B is defined to be the point where AI meets Γ again, point C is where BJ meets Γ again, and so forth. Instead of "AB and DE intersect at I", read "AB and DE pass through I." These were indeed Bradley's words in his original statement of the problem (which the editor changed), so Bataille's interpretation captures the true spirit of the proposal, even though the problem's current wording suggests that opposite sides of the resulting hexagon must intersect in three well-defined points. Here, then, is Bataille's solution.

G will coincide with A for all A on Γ if and only if either I, J, K are collinear or $\triangle IJK$ is a self-polar triangle with respect to Γ (that is, each vertex is the pole of the opposite side). In the latter case we have $A = D$, $B = E$, and $C = F$. To prove this, we identify points with complex numbers and, without loss of generality, suppose that Γ is the unit circle. For A on Γ and M not on Γ, we denote by $\mathrm{M}(A)$ the point of Γ where the line MA intersects Γ again (and $\mathrm{M}(A) = A$ if it is tangent to Γ at A). An easy calculation yields

$$\mathrm{M}(A) \;=\; \frac{A - M}{A\overline{M} - 1}\,,$$

where $\overline{M}$ is the conjugate of the complex number M.

We readily obtain

$$D \;=\; \mathrm{K} \circ \mathrm{J} \circ \mathrm{I}(A) \;=\; \frac{AU - V}{A\overline{V} - \overline{U}}\,,$$

where $U = 1 - \overline{I}J - \overline{J}K + K\overline{I}$ and $V = I - J + K - I\overline{J}K$, and

$$G \;=\; (\mathrm{K} \circ \mathrm{J} \circ \mathrm{I})^2(A) \;=\; \frac{A(U^2 - V\overline{V}) + V\overline{U} - UV}{A(U\overline{V} - \overline{U}\overline{V}) + \overline{U}^2 - V\overline{V}}\,.$$

It follows that $G = A$ for all A on Γ if and only if

$$U^2 - V\overline{V} \;=\; \overline{U}^2 - V\overline{V} \quad\text{and}\quad V\overline{U} - UV \;=\; U\overline{V} - \overline{U}\overline{V} \;=\; 0\,.$$

This reduces to $(U = \overline{U})$ or $(U \neq \overline{U}$ and $V = 0$ and $U + \overline{U} = 0)$. Condition $U = \overline{U}$ implies $I(\overline{J} - \overline{K}) - \overline{I}(J - K) + J\overline{K} - \overline{J}K = 0$, which means that I lies on the line through J and K.

On the other hand, conditions $V = 0 = U + \overline{U}$ yield

$$I - J + K \;=\; I\overline{J}K \tag{1}$$

and

$$a - b + c = 0\,, \tag{2}$$

where we denote by a, b, c the real numbers $\overline{J}K + \overline{K}J - 2$, $\overline{K}I + \overline{I}K - 2$, and $\overline{I}J + \overline{J}I - 2$, respectively.

Since (1) may also be written as $aI - bJ + cK = 0$, we see that (1) and (2) are equivalent to $a + c = b$ and $a(I - J) + c(K - J) = 0$. The latter leads to $a = c = 0$, because I, J, K are not collinear. Finally, the second case occurs if and only if $a = b = c = 0$. This means that $\triangle IJK$ is self-polar since Z is on the polar of M with respect to Γ if and only if $\overline{M}Z + M\overline{Z} = 2$, as it is readily checked.

Notes.

1. For a self-polar triangle IJK, we actually have $\mathbf{K} \circ \mathbf{J} \circ \mathbf{I}(A) = A$ for all A on Γ; thus, $D = A = G$, $E = B$, $F = C$.
2. The result can be extended to a conic Γ (through a projectivity).

Also solved by DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and the proposer.

Bradley and Loeffler both gave explicit positions for I, J, and K that are easily seen to solve the problem directly, without invoking Pascal's Theorem. With the points represented by complex numbers as in solution II, Bradley chose the points 0, r, and $1/r$ (in the order I, J, K, with r real and $0 \neq r \neq 1$), while Loeffler chose r, 0, and $-r$ (r complex and $0 \neq r \neq 1$).

2838★. [2003 : 238] *Proposed by Mohammed Aassila, Strasbourg, France.*

Let P be a real polynomial with integer coefficients such that there is an infinite subsequence of the sequence $\{P(k)\}_{k=1}^{\infty}$ with the property that the subsequence has only finitely many prime divisors.

Prove that P is of the form $P(x) = (ax + b)^n$.

Editor: There were no solutions submitted for this problem. As a result, problem 2838 remains an open problem which the readers of ***CRUX with MAYHEM*** are encouraged to revisit.