SKOLIAD No. 75

Shawn Godin

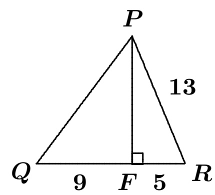
Please send your solutions to the problems from this issue by *1 August, 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 1** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will print solutions to problems marked with an asterisk (*) only if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

Our featured contest this issue is the 2002 Canadian Open Mathematics Challenge. My thanks go to Peter Crippin and Ian VanderBurgh from the Canadian Mathematics Competitions for forwarding the material. We first give the English. The French follows.

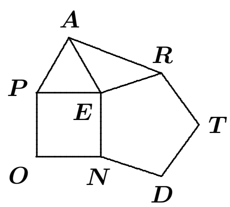
Canadian Open Mathematics Challenge, 2002
Part A

1. In triangle PQR, F is the point on QR such that PF is perpendicular to QR. If $PR = 13$, $RF = 5$, and $FQ = 9$, what is the perimeter of $\triangle PQR$?



2. If $x + y = 4$ and $xy = -12$, what is the value of $x^2 + 5xy + y^2$?

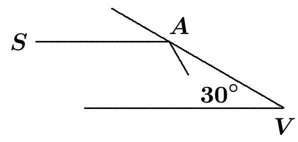
3. A regular pentagon is a five-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram, $TREND$ is a regular pentagon, PEA is an equilateral triangle, and $OPEN$ is a square. Determine the size of $\angle EAR$.



4. In a sequence of numbers, the sum of the first n terms is equal to $5n^2+6n$. What is the sum of the 3^{rd}, 4^{th}, and 5^{th} terms in the original sequence?

5. If m and n are non-negative integers with $m < n$, we define $m\nabla n$ to be the sum of the integers from m to n, including m and n. For example, $5\nabla 8 = 5+6+7+8 = 26$. For every positive integer a, the numerical value of $\dfrac{(2a-1)\nabla(2a+1)}{(a-1)\nabla(a+1)}$ is the same. Determine this value.

6. Two mirrors meet at an angle of $30°$ at the point V. A beam of light, from a source S, travels parallel to one mirror and strikes the other mirror at point A, as shown. After a number of reflections, the beam comes back to S. If SA and AV are both 1 metre in length, determine the total distance travelled by the beam.



7. N is a five-digit positive integer. A six-digit integer P is constructed by placing a 1 at the right-hand end of N. A second six-digit integer Q is constructed by placing a 1 at the left-hand end of N. If P is three times Q, determine the value of N.

8. Suppose that M is an integer with the property that if x is randomly chosen from the set $\{1, 2, 3, \ldots, 999, 1000\}$, the probability that x is a divisor of M is $\frac{1}{100}$. If $M \leq 1000$, determine the maximum possible value of M.

Part B

1. Square $ABCD$ has vertices $A\,(0,0)$, $B\,(0,8)$, $C\,(8,8)$, and $D\,(8,0)$. The points $P\,(0,5)$ and $Q\,(0,3)$ are on side AB, and the point $F\,(8,1)$ is on side CD.

(a) What is the equation of the line through Q parallel to the line through P and F?

(b) If the line from part (a) intersects AD at the point G, what is the equation of the line through F and G?

(c) The centre of the square is the point $H\,(4,4)$. Determine the equation of the line through H perpendicular to FG.

(d) A circle is drawn with centre H that is tangent to the four sides of the square. Does this circle intersect the line through F and G? Justify your answer. (A sketch is not sufficient justification.)

2. (a) Let A and B be digits (that is, A and B are integers between 0 and 9 inclusive). If the product of the three-digit integers $2A5$ and $13B$ is divisible by 36, determine with justification the four possible ordered pairs (A, B).

(b) An integer n is said to be a multiple of 7 if $n = 7k$ for some integer k.

(i) If a and b are integers and $10a + b = 7m$ for some integer m, prove that $a - 2b$ is a multiple of 7.

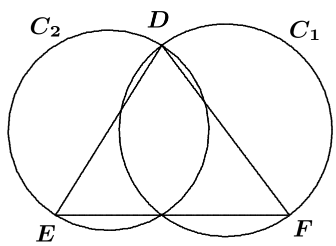
(ii) If c and d are integers and $5c + 4d$ is a multiple of 7, prove that $4c - d$ is also a multiple of 7.

3. There are some marbles in a bowl. Alphonse, Beryl, and Colleen each take turns removing one or two marbles from the bowl, with Alphonse going first, then Beryl, then Colleen, then Alphonse again, and so on. The player who takes the last marble from the bowl is the loser, and the other two players are the winners.

(a) If the game starts with 5 marbles in the bowl, can Beryl and Colleen work together and force Alphonse to lose?

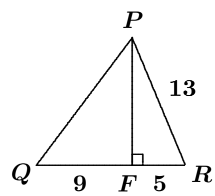
(b) The game is played again, this time starting with N marbles in the bowl. For what values of N can Beryl and Colleen work together and force Alphonse to lose?

4. Triangle DEF is acute. Circle C_1 is drawn with DF as its diameter, and circle C_2 is drawn with DE as its diameter. Points Y and Z are on DF and DE, respectively, so that EY and FZ are altitudes of $\triangle DEF$. EY intersects C_1 at P, and FZ intersects C_2 at Q. EY extended intersects C_1 at R, and FZ extended intersects C_2 at S. Prove that P, Q, R, and S are concyclic points.



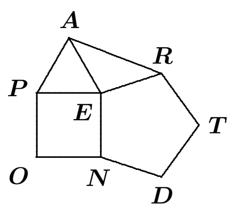
Canadian Open Mathematics Challenge, 2002
Partie A

1. Dans le triangle PQR, F est un point sur QR de manière que PF soit perpendiculaire à QR. Si $PR = 13$, $RF = 5$, et $FQ = 9$, quel est le périmètre du triangle PQR?



2. Si $x + y = 4$ et $xy = -12$, quelle est la valeur de $x^2 + 5xy + y^2$?

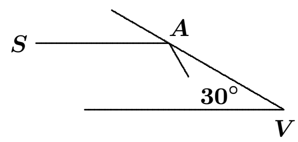
3. Un pentagone régulier est un polygone à cinq côtés dont tous les angles sont égaux et tous les côtés sont égaux. Dans le diagramme, $TREND$ est un pentagone régulier, PEA est un triangle équilatéral et $OPEN$ est un carré. Déterminer la mesure de l'angle $\angle EAR$.



4. Dans une suite de nombres, la somme des n premiers termes est égale à $5n^2 + 6n$. Quelle est la somme des 3^e, 4^e et 5^e termes de la suite donnée ?

5. Si m et n sont des entiers non négatifs et si $m < n$, on définit $m\nabla n$ comme étant la somme des entiers de m à n, incluant m et n. Par exemple, $5\nabla 8 = 5+6+7+8 = 26$. Pour chaque entier strictement positif a, la valeur numérique de $\frac{(2a-1)\nabla(2a+1)}{(a-1)\nabla(a+1)}$ est la même. Déterminer cette valeur.

6. On a placé deux miroirs de manière à former un angle de $30°$ au point V. Le diagramme illustre un rayon lumineux dont la source est au point S, dont le trajet est parallèle à un miroir et qui frappe l'autre miroir au point A. Après un certain nombre de réflexions, le rayon lumineux revient au point S. Si SA et AV ont chacun une longueur de 1 m, déterminer la distance totale parcourue par le rayon lumineux.



7. N est un entier positif de cinq chiffres. On construit un entier positif P de six chiffres en plaçant un 1 à l'extrémité droite de N. On construit un deuxième entier positif de six chiffres, Q, en plaçant un 1 à l'extrémité gauche de N. Sachant que P est égal à trois fois Q, déterminer la valeur de N.

8. On considère un entier positif M qui vérifie la propriété suivante : Si on choisit au hasard un nombre x dans l'ensemble $\{1, 2, 3, \ldots, 999, 1000\}$, la probabilité pour que x soit un diviseur de M est égale à $\frac{1}{100}$. Si $M \leq 1000$, déterminer la plus grande valeur possible de M.

Partie B

1. Un carré $ABCD$ a pour sommets $A\,(0,0)$, $B\,(0,8)$, $C\,(8,8)$, et $D\,(8,0)$. Les points $P\,(0,5)$ et $Q\,(0,3)$ sont sur le côté AB et le point $F\,(8,1)$ est sur le côté CD.

(a) Quelle est l'équation de la droite qui passe par Q et qui est parallèle à la droite passant par P et F ?

(b) Si la droite de la partie (a) coupe AD au point G, quelle est l'équation de la droite qui passe par F et G ?

(c) Le point $H\,(4,4)$ est le centre du carré. Déterminer l'équation de la droite qui passe par H et qui est perpendiculaire à FG.

(d) On construit un cercle de centre H de manière qu'il soit tangent aux quatre côtés du carré. Ce cercle coupe-t-il la droite qui passe par F et G ? Justifier sa réponse. (Un dessin n'est pas considéré comme une justification suffisante.)

2. (a) Soit deux chiffres A et B (A et B sont donc des symboles de 0 à 9 utilisés pour écrire les entiers). Sachant que le produit des deux nombres de trois chiffres, $2A5$ et $13B$, est divisible par 36, déterminer les quatre couples (A, B) possibles. Justifier sa réponse.

(b) On dit qu'un entier n est un multiple de 7 si $n = 7k$ pour un entier quelconque k.

(i) Si a et b sont des entiers tels que $10a + b = 7m$ pour un entier quelconque m, démontrer que $a - 2b$ est un multiple de 7.

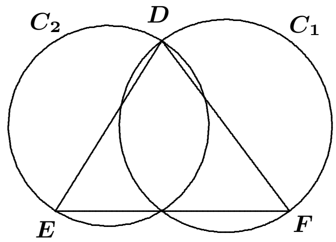
(ii) Si c et d sont des entiers tels que $5c+4d$ est un multiple de 7, démontrer que $4c - d$ est aussi un multiple de 7.

3. Il y a un nombre de billes dans un bol. À tour de rôle, Antoine, Brigitte et Carla enlèvent chacun une ou deux billes du bol. Antoine est premier, Brigitte est deuxième, Carla est troisième, puis Antoine joue de nouveau et ainsi de suite. La personne qui enlève la dernière bille est perdante et les deux autres sont gagnantes.

(a) S'il y a 5 billes dans le bol au début, Brigitte et Carla peuvent-elles travailler ensemble pour s'assurer qu'Antoine perde ?

(b) On recommence avec N billes dans le bol. Pour quelles valeurs de N est-ce que Brigitte et Carla peuvent travailler ensemble pour s'assurer qu'Antoine perde ?

4. Le triangle DEF est acutangle. On a tracé un cercle C_1 de diamètre DF et un cercle C_2 de diamètre DE. Les points respectifs Y et Z sont sur DF et DE de manière que EY et FZ soient des hauteurs du triangle DEF. EY coupe C_1 au point P et FZ coupe C_2 au point Q. Le prolongement de EY coupe C_1 au point R et le prolongement de FZ coupe C_2 au point S. Démontrer que les points P, Q, R et S sont situés sur un même cercle.



Next we give the solutions to the 2001 Invitational Mathematics Challenge that appeared in [2003 : 129].

2001 Invitational Mathematics Challenge
(Grade 10)

1. Thirty years ago, the ages of Xavier, Yolanda, and Zoë were in the ratio $1 : 2 : 5$. Today, the ratio of Xavier's age to Yolanda's age is $6 : 7$. What is Zoë's present age?

Solution by Chenchen Zhang, Lisgar Collegiate Institute, Ottawa, ON.

Let x be Xavier's age thirty years ago. From the ratios given we have $x + 30 = \frac{6}{7}(2x + 30)$. Solving for x, we have $x = 6$. Thirty years ago, Zoë was five times older than Xavier, which gives Zoë's current age as $5 \times 6 + 30 = 60$.

2. (a) Determine the number of integers between 100 and 999, inclusive, that contain exactly two digits that are the same.

(b) Determine the probability that a positive integer less than 1000 contains exactly two digits that are the same.

Solution by Chenchen Zhang, Lisgar Collegiate Institute, Ottawa, ON.

(a) There are three cases to consider. Case 1 includes numbers like 100, 122, 133, etc. (but not 111). There are 9 such numbers in each hundred group, for a total of $9 \times 9 = 81$ numbers. Case 2 includes numbers like 121, 131, etc., of which there are 9 in each hundred group, for a total of $9 \times 9 = 81$ numbers. Case 3 includes numbers like 112, 113, etc., of which there are 9 in each hundred group, for a total of $9 \times 9 = 81$ numbers. All three cases give us a total of 243 numbers.

(b) There are 9 numbers less that 100 that contain two digits the same. Adding this to the previous result, we have $243 + 9 = 252$ positive integers less than 1000 that have exactly two digits the same. The probability that a positive integer less than 1000 contains exactly two digits that are the same is $\dfrac{252}{999} = \dfrac{28}{111}$.

3. Solve the system of equations:

$$\begin{aligned} x + y + z &= 2\,, & (1)\\ x^2 - y^2 - z^2 &= 2\,, & (2)\\ x - 3y^2 + z &= 0\,. & (3) \end{aligned}$$

Solution by Chenchen Zhang, Lisgar Collegiate Institute, Ottawa, ON.

Substituting equation (1) in (3) yields the quadratic $3y^2 + y - 2 = 0$, which, via the quadratic formula, gives $y = -1$ or $y = \frac{2}{3}$. Substituting these values for y into equations (1) and (2) allows us to factor equation (2) and solve for x, giving $x = 2$ or $x = \frac{19}{12}$. Substituting the values for x and y into equation (1) gives $z = 1$ or $z = -\frac{1}{4}$. The two solution sets are $x = 2$, $y = -1$, $z = 1$ and $x = \frac{19}{12}$, $y = \frac{2}{3}$, $z = -\frac{1}{4}$.

4. A flat mirror is perpendicular to the xy–plane and stands on the line $y = x + 4$. A laser beam from the origin strikes the mirror at $P(-1, 3)$ and is reflected to the point Q on the x–axis. Determine the coordinates of the point Q.

Solution by Chenchen Zhang, Lisgar Collegiate Institute, Ottawa, ON.

If we draw a line parallel to the mirror through the origin, it will intersect the line segment $\overline{PQ}$ at a point R. Label the mid-point of the line segment $\overline{OR}$ as G. The line segment $\overline{PG}$ provides an axis of symmetry for the incident and reflected beams. Note that G lies both on the line segment $\overline{OR}$, where $y = x$ (since $\overline{OR}$ is parallel to the mirror $y = x + 4$ and goes through the origin), and on the line segment $\overline{PG}$, where $y = -x + 2$ (since $\overline{PG}$ is perpendicular to the mirror $y = x + 4$ and passes through $P(-1, 3)$). Therefore $G = (1, 1)$. Since $|\overline{OG}| = |\overline{GR}|$, we have $R = (2, 2)$. From the coordinates of P and R, the equation of $\overline{PR}$ is $y = -\frac{1}{3}x + \frac{8}{3}$. Since Q is on the x–axis, we can set $y = 0$ to obtain $Q = (8, 0)$.

2001 Invitational Mathematics Challenge
(Grade 11)

1. Farmer Haas has six containers with capacities of 15, 16, 18, 19, 20, and 31 litres. One of these containers is filled with cream and the other five are filled with either white milk or chocolate milk. Farmer Haas has twice as much white milk as chocolate milk.

(a) What is the volume of the container that is filled with cream?

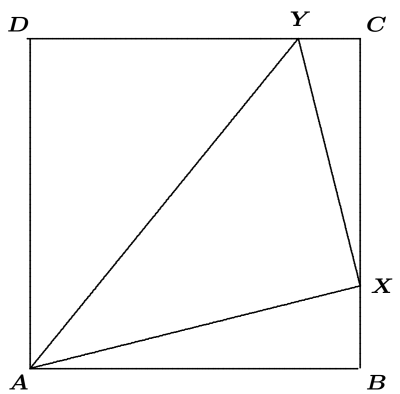
(b) The price of the cream is $\$3$ per litre, the price of the chocolate milk is $\$2$ per litre, and the price of the white milk is $\$1$ per litre. What is the total value of the contents of the six containers?

Solution by Mayhem Staff.

(a) The total volume of all the containers is $119 \equiv 2 \pmod 3$. Thus, since there is twice as much white milk as chocolate milk, the amount of white and chocolate milk together is divisible by 3. If c is the amount of cream, we must have $c \equiv 2 \pmod 3$, and the only possibility is $c = 20$.

(b) Since there are 20 litres of cream, there must be 33 litres of chocolate milk and 66 litres of white milk. Hence, the total value of the contents is $\$3 \times 20 + \$2 \times 33 + \$1 \times 66 = \$192$.

3. Points X and Y are on sides BC and CD of square $ABCD$, as shown below. The lengths of XY, AX, and AY are 3, 4, and 5, respectively. Determine the side length of square $ABCD$.



Solution by Mayhem Staff.

Note that $\angle AXY = 90^\circ$, because $AX^2 + XY^2 = AY^2$. Therefore, $\angle YXC + \angle AXB = 90^\circ$. Also, $\angle XAB + \angle AXB = 90^\circ$. Thus, $\angle XAB = \angle YXC$, and $\triangle ABX \sim \triangle XCY$. Hence, $\frac{AB}{4} = \frac{XC}{3}$; that is, $XC = \frac{3}{4}AB$. Then we have

$$BX \;=\; BC - XC \;=\; AB - \tfrac{3}{4}AB \;=\; \tfrac{1}{4}AB\,.$$

The Pythagorean Theorem applied to $\triangle ABX$ yields $AB^2 + \left(\frac{1}{4}AB\right)^2 = 4^2$, which gives us the length of the side as $\frac{16}{\sqrt{17}}$ units.

Note: We can generalize this to any points X on BC and Y on CD such that $\angle AXY$ is a right angle. We find that

$$AB = \frac{AX^2}{\sqrt{AX^2 + (AX - XY)^2}}\,.$$

4. A flat mirror is perpendicular to the xy–plane and stands along a line L. A laser beam from the origin strikes the mirror at $P(-1,5)$ and is reflected to the point $Q(24,0)$. Determine the equation of the line L.

Solution by Chenchen Zhang, Lisgar Collegiate Institute, Ottawa, ON.

The length of $\overline{PO}$ is $\sqrt{26}$ (since $PO^2 = 5^2 + 1^2 = 26$). There must be a point R on $\overline{PQ}$ such that $|\overline{PR}| = |\overline{PO}| = \sqrt{26}$. The line segment $\overline{OR}$ is parallel to the line L (they have the same slope). The equation of $\overline{PQ}$ is $y = -\frac{1}{5}x + 4.8$, since P is $(-1,5)$ and Q is $(24,0)$. The point R will be $\left(x, -\frac{1}{5}x + 4.8\right)$, for some $x > -1$. From the formula for the length of $\overline{PR}$, we have

$$(x+1)^2 + \left(-\tfrac{1}{5}x + 4.8 - 5\right)^2 \;=\; 26\,.$$

Solving for x, we find that $R = (4,4)$. Consequently, $\overline{OR}$ has slope 1. Line L is parallel to $\overline{OR}$ and goes through the point $P\,(-1,5)$; thus, L has the equation $y = x + 6$.

5. Let $f(n) = n^4 + 2n^3 - n^2 + 2n + 1$.

(a) Show that $f(n)$ can be written as the product of two quadratic polynomials with integer coefficients.

(b) Determine all integers n for which $|f(n)|$ is a prime number.

Solution by Mayhem Staff.

(a) If $f(n)$ can be factored into two quadratic polynomials with integer coefficients, it must have the form

$$(n^2+an+1)(n^2+bn+1) \;=\; n^4+(a+b)n^3+(ab+2)n^2+(a+b)n+1$$

or the form

$$(n^2+cn-1)(n^2+dn-1) \;=\; n^4+(c+d)n^3+(cd-2)n^2-(c+d)n+1\,.$$

Since the coefficients of n^3 and n as given in the problem are both positive, we must have the first case above, with

$$\begin{aligned} a+b &= 2\,, \\ ab+2 &= -1\,. \end{aligned}$$

The solutions of this system are $a = -1$, $b = 3$ and $a = 3$, $b = -1$. Thus, the desired product is

$$f(n) = (n^2 - n + 1)(n^2 + 3n + 1)\,.$$

(b) If $|f(n)|$ is to be prime, then we need one of the factors to be ± 1. Thus, we have 4 cases:

Case 1: $n^2 - n + 1 = 1$.
We have $n^2 - n = 0$, which yields $n = 0$ or 1. Substituting gives $f(0) = 1$, which is not prime, and $f(1) = 5$, which is prime.

Case 2: $n^2 - n + 1 = -1$.
We have $n^2 - n + 2 = 0$, which has no integer solutions.

Case 3: $n^2 + 3n + 1 = 1$.
We have $n^2 + 3n = 0$ which yields $n = 0$ or -3. We already know that $n = 0$ does not work; we check that $f(-3) = 13$, which is prime.

Case 4: $n^2 + 3n + 1 = -1$.
We have $n^2 + 3n + 2 = 0$, which yields $n = -1$ or -2. Substituting gives $f(-1) = -3$ and $f(-2) = -7$. Thus, both $|f(-1)|$ and $|f(-2)|$ are prime.

Therefore, $|f(n)|$ is prime for $n \in \{-3, -2, -1, 1\}$.

One last note: The previous editor had received solutions to questions 1 and 3 from the Mandelbrot Competition, Individual test, from Karthik Natarajan, a Grade 5 student, from Edgewater Park Public School, Thunder Bay, Ontario. Both these solutions were correct, but they were misplaced, and we failed to mention Karthik in the November issue. Apologies to Karthik. We hope to continue hearing from him.

That closes another Skoliad. Please continue sending your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo) and Dan MacKinnon (Ottawa Carleton District School Board).

Mayhem Editorial

Shawn Godin

Welcome to another volume of Mayhem! All the best of **2004** to all of our readers and their families. We are looking forward to another great year of problem solving. We have a new face joining the Mayhem team and one team member moving on that we must acknowledge at this time.

Paul Ottaway, who has been writting Pólya's Paragon for the last year and a half, is leaving us to study for his Ph.D. Paul has done a wonderful job, and his contribution to Mayhem will be missed. We are hoping to continue Pólya's Paragon, and are in the process of finding someone to take it over.

New to Mayhem this year is Dan MacKinnon, who is an Educational Technology Integrator with the Ottawa Carleton District School Board in charge of teacher training with regards to technology. Welcome Dan!

Unfortunately, both the December 2003 issue and the current issue are devoid of solutions to the Mayhem Problems. We apologize to our readers for this; we will make up for it in some subsequent issues, which will have more solutions than normal.

I hope to hear from many of our readers with problems and solutions over the next year. Let's make it a great one!

Mayhem Problems

Please send your solutions to the problems in this edition by **1 August 2004***. Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M126. *Proposed by the Mayhem Staff.*

Given that the letters A, B, C, D, E, F represent distinct decimal digits, find the values of these letters so that

$$ABC \times DEF = 232323\,.$$

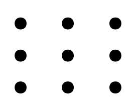
Here, ABC and DEF represent 3-digit numbers.

M127. *Proposed by Ovidiu Gabriel Dinu, Balcesti, Valcea, Romania.*

Prove that if a, $b \in \mathbb{R}$ and $a - b = 1$, then $a^3 - b^3 \geq \frac{1}{4}$.

M128. *Proposed by Lobzang Dorji, Paro, Bhutan.*

(a) Nine dots are uniformly spaced in a 3×3 square array as shown. Verify that 8 non-congruent triangles can be formed using three of the dots as vertices.



(b) Suppose that a 4×4 square array of dots is employed. How many non-congruent triangles could be formed using three of the dots as vertices?

M129. *Proposed by the Mayhem Staff.*

A die is tossed. If the die lands on '1' or '2', then one coin is tossed. If the die lands on '3', then two coins are tossed. Otherwise, three coins are tossed. Given that the resulting coin tosses produced no 'heads', what is the probability that the die landed on '1' or '2'.

M130. *Proposed by the Mayhem Staff.*

Tickets are numbered 1, 2, 3, 4, $\ldots$, N. Exactly half of the tickets have the digit 1 on them.

(a) If N is a three-digit number, determine all possible values of N.

(b) Determine some possible values for N if N is a four-digit number, or a five-digit number, etc.

M131. *Proposed by the Mayhem Staff.*

The triangular array of numbers shown has the following properties:

1. The bottom row contains each of the numbers $1, 2, \ldots, 8$ exactly once.
2. Each number in a row above the bottom row is the sum of the two neighbouring numbers in the row immediately below, if this sum is less than 10; otherwise, 9 is subtracted from this sum.

$$\begin{array}{ccccccccccccccc} &&&&&&&6&&&&&&& \\ &&&&&&7&&8&&&&&& \\ &&&&&3&&4&&4&&&&& \\ &&&&8&&4&&9&&4&&&& \\ &&&7&&1&&3&&6&&7&&& \\ &&4&&3&&7&&5&&1&&6&& \\ &5&&8&&4&&3&&2&&8&&7& \\ 2&&3&&5&&8&&4&&7&&1&&6 \end{array}$$

Is it possible to create a triangular array with the above properties using each number from 1 to 9 exactly four times?

. . . .

M126. *Proposé par l'Équipe de Mayhem.*

Si chaque lettre A, B, C, D, E, F représente un chiffre distinct (parmi les chiffres de 0 à 9), trouver les valeurs de ces lettres de sorte que

$$ABC \times DEF = 232323\,,$$

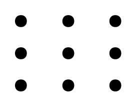
où ABC et DEF sont deux nombres de 3 chiffres.

M127. *Proposé par Ovidiu Gabriel Dinu, Balcesti, Valcea, Roumanie.*

Montrer que si a, $b \in \mathbb{R}$ et $a - b = 1$, alors $a^3 - b^3 \geq \frac{1}{4}$.

M128. *Proposé par Lobzang Dorji, Paro, Bhutan.*

(a) Etant donné une grille formée de 3×3 points uniformément espacés comme dans la figure, vérifier qu'on peut construire 8 triangles non congruents ayant trois de ces points comme sommets.



(b) Combien de triangles non congruents peut-on construire à partir d'une grille de 4×4 en suivant la même procédure que ci-dessus ?

M129. *Proposé par l'Équipe de Mayhem.*

On lance un dé. S'il s'arrête sur le '1' ou le '2', on lance un pièce de monnaie. S'il s'arrête sur le '3', on lance deux pièces de monnaie. Dans les autres cas, on lance trois pièces de monnaie. Quelle est la probabilité pour que le dé se soit arrêté sur le '1' ou le '2', si aucune des pièces de monnaie lancées n'a montré 'face'.

M130. *Proposé par l'Équipe de Mayhem.*

On numérote des billets 1, 2, 3, 4, $\ldots$, N. La moitié d'entre eux comporte le chiffre 1.

(a) Sachant que N est un nombre de trois chiffres, déterminer toutes les valeurs possibles de N.

(b) Trouver certaines valeurs plus grandes que N pourraient avoir, si c'était un nombre de quatre chiffres, de cinq chiffres, etc.

M131. *Proposé par l'Équipe de Mayhem.*

Huit rangées de nombres forment un triangle avec les propriétés suivantes :

1. La base comporte les huit entiers $1, 2, \ldots, 8$, écrits dans n'importe quel ordre.
2. En dessus de la base, chaque nombre est la somme des deux nombres de la ligne inférieure immédiatement voisins, pour autant que cette somme soit plus petite que 10 ; sinon on soustrait 9 de la somme.

```
              6
            7   8
          3   4   4
        8   4   9   4
      7   1   3   6   7
    4   3   7   5   1   6
  5   8   4   3   2   8   7
2   3   5   8   4   7   1   6
```

Est-il possible de former un triangle de nombres avec les deux propriétés ci-dessus en utilisant chaque nombre de 1 à 9 exactement quatre fois ?

Pólya's Paragon

Au Contraire, Mon Ami

Shawn Godin

Sometimes in mathematics, it is easier to do what you don't want to do than what you do want to do! In the September 2003 issue, we looked at proofs by contradiction. The main idea behind these proofs is that, instead of showing that what you want to prove is true, you show that the opposite must be untrue because, if it were true, it would produce a contradiction.

For example, let's look at the second problem that was left for you in September:

Prove that there are infinitely many primes.

Proof: We start by assuming the opposite, namely, that there are finitely many primes. Label these primes $p_1, p_2, \ldots, p_n$. Let's create the number

$$N = p_1 \times p_2 \times \cdots \times p_n + 1 .$$

We know that any integer greater than 1 is either prime or composite. Therefore, we have only two possibilities for N:

Case 1: N is prime.

Clearly N is not one of the primes $p_1, p_2, \ldots, p_n$, since, by its construction, N must be larger than all of these primes. Our original assumption is that we had **all** primes in our list. Thus, we have a contradiction.

Case 2: N is composite.

Then there must be some prime p that divides N. Clearly, p is not one of the primes $p_1, p_2, \ldots, p_n$, since, by the construction of N, each of the primes $p_1, p_2, \ldots, p_n$ leave a remainder of 1 when dividing into N; thus, none of them divides N. Therefore, p is a "new" prime (that is, one that was not in our list), and again we have a contradiction. ■

Our proof shows that, no matter what, we end up with a contradiction. That is, if we have a list of "all" primes, we can always construct a number that will give us a prime that isn't on the list. So, we have proved that we can never list them all; that is, there are not finitely many primes.

You can try this yourself. If you assume that 2 is the only prime, you can create

$$2 + 1 = 3 ,$$

which is prime. Then, adjoining 3 to the list of primes, you get

$$2 \times 3 + 1 = 7 ,$$

which is prime. Repeating this process yields

$$2 \times 3 \times 7 + 1 = 43 ,$$

which is prime, and then

$$2 \times 3 \times 7 \times 43 + 1 = 1807 = 13 \times 139 ,$$

where 13 and 139 are both prime. If you continue this process, you will always get new primes at each step (although the factoring will become very difficult very quickly!)

A proof by contradiction is called an "indirect proof" because we get at the result indirectly, by considering another problem. Very often counting problems can be attacked indirectly as well, rather than looking at all the possibilities. For example, consider the following problem:

In how many ways can parents line up their 6 children for a picture if Sally and John cannot be put beside each other (since they fight)?

If we were to do this directly, we would have to find all the possible ways in which we can place these two children so that they are not side by side (which turns out to be 10 different cases). Alternately, we can look at the "opposite" problem:

In how many ways can parents line up their 6 children for a picture if Sally and John must be put beside each other?

In this case, we can "attach" Sally and John and treat them as one "child". We can do this in 2 ways, since we can attach them as Sally & John or John & Sally. Now we are arranging only 5 "children", and there are $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$ ways to do this. Thus, there are $2 \times 120 = 240$ ways to arrange the children so that Sally and John will be together. Going back to the original problem, we know there are $6! = 720$ ways to line up all 6 children, if we don't care how we do it. Therefore, there must be $720 - 240 = 480$ ways to line them up with Sally and John apart. ■

Thus, the main idea behind the indirect method is to calculate the cases you don't want and remove them from the total. This will save you some work in quite a few problems.

Here are a couple of problems for you to consider.

1. In how many ways can we rearrange the letters in the word MAYHEM so that the A and E are apart and the two M's are together?

2. Thirteen people each pick a number from 1 to 100 and secretly write it on a piece of paper. What is the probability that at least two people picked the same number? (This is a variation of the classic "birthday problem" that we will visit in an upcoming issue.)

THE OLYMPIAD CORNER

No. 235

R.E. Woodrow

Here is the start of another volume of ***Crux Mathematicorum*** and the occasion to thank all those who contributed problems, comments, solutions and generalizations used in the 2003 number of the *Corner*.

Mohammed Aassila	Geoffrey A. Kandall
Miguel Amengual Covas	Murray S. Klamkin
Jean-Claude Andrieux	Matti Lehtinen
Rahul Bamotra	Hongyi Li
Marcus Emmanuel Barnes	Andy Liu
Michel Bataille	Pavlos Maragoudakis
Robert Bilinski	Tobias Reuter
Pierre Bornsztein	Samapti Samapti
Christopher J. Bradley	Heinz-Jürgen Seiffert
Bruce Crofoot	Toshio Seimiya
José Luis Díaz-Barrero	Chris Small
George Evagelopoulos	D.J. Smeenk
J. Chris Fisher	Claude Tardif
Joan Hutchison	Panos E. Tsaoussoglou
Walther Janous	Stan Wagon
Athanasias Kalakos	Edward T.H. Wang

Special thanks also go to Joanne Longworth for her exceptional skills deciphering my handwriting, producing high quality LaTeX materials, and being far more organized, on track, and on schedule than I seem to be.

To rekindle your problem solving for the new year, we first give a set of five Klamkin Quickies. Try them before looking up his "Quickie" solutions later in this number. Thanks to Murray S. Klamkin, University of Alberta, Edmonton, AB for creating them for us.

FIVE KLAMKIN QUICKIES
February 2004

1. Determine all polynomials $P(x)$ such that either $(x+1)P(x) = xP(x-1)$ or else $(x+1)P(x) = xP(x+1)$ for all x.

2. Show that the following polynomial has no real roots:

$$P(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - \cdots - 2nx + 2n + 1\,.$$

3. Determine all integral solutions of $x^3 + y^3 = z^6 + 3$.

4. Determine the maximum value of $\tan x + \tan y$, where

$$(1 + \sin x)(1 + \sin y) \;=\; \cos x \cos y\,.$$

5. Determine whether or not the inequality $(x^2 + y^2)^4 \geq 4xy(x^3 + y^3)^2$ is valid for all real x, y.

Next we give the problems of the 40th IMO Vietnam Team Selection Test written in May 2001. My thanks go to Pham van Thuan for translating them and submitting them for our use.

40th INTERNATIONAL MATHEMATICAL OLYMPIAD

Vietnam Team Selection Test

Hanoi, Vietnam May 8–9, 2001

1. Let a sequence of integers $\{a_n\}$, $n \in \mathbb{N}$, be defined by

$$a_0 \;=\; 1, \quad \text{and} \quad a_n \;=\; a_{n-1} + a_{\lfloor n/3 \rfloor} \quad \text{for every} \quad n \in \mathbb{N}^*\,,$$

where $\lfloor x \rfloor$ denotes the integer part of the real number x. Prove that for each prime number p there exists a natural number k such that a_k is divisible by p.

2. Two circles are given, intersecting each other at two points A and B. Let ℓ be a common tangent of the two circles, and let P and T be the points of tangency. Denote by S the intersection point of the two tangents to the circumcircle of triangle APT at P and T. Let H be the reflection of point B across the line ℓ. Prove that A, S, H are collinear.

3. There are 42 members in a club. Among any 31 of them, there is a pair consisting of a man and a woman who know each other. Prove that there are at least 12 man-woman pairs who know each other.

4. Given positive real numbers x, y, z satisfying the inequality

$$21xy + 2yz + 8zx \;\leq 12\,,$$

determine the minimum value of the function

$$f(x,y,z) \;=\; \frac{1}{x} + \frac{2}{y} + \frac{3}{z}\,.$$

5. Let n be an integer, $n > 1$. Denote by A the set of points $P(x,y,z)$ such that x, y, z are integers satisfying $1 \leq x$, y, $z \leq n$. Colour some of the points in A in such a manner that if point $M(x_0,y_0,z_0)$ is coloured, point $N(x_1,y_1,z_1)$ with $x_1 \leq x_0$, $y_1 \leq y_0$, $z_1 \leq z_0$ is not coloured.

Determine, with proof, the maximum possible number of points that can be coloured.

6. Denote by $\{a_n\}$, $n \in \mathbb{N}^*$, a sequence of positive integers satisfying the following condition

$$0 < a_{n+1} - a_n \leq 2001 \quad \text{for all } n \in \mathbb{N}^* .$$

Prove that there are infinitely many pairs of positive integers (p, q) such that $p < q$ and a_p is a divisor of a_q.

As a second Olympiad set we give the XVII National Mathematical Contest of Italy written at Cesenatico, May 4, 2001. My thanks go to Chris Small, Canadian Team Leader to the 41$^{\text{st}}$ IMO, for collecting them for us.

XVII NATIONAL MATHEMATICAL CONTEST OF ITALY
May 4, 2001

1. In a hexagon with all angles equal, the lengths of four consecutive edges are 5, 3, 6 and 7 (in this order). Find the lengths of the remaining two edges.

2. In a basketball tournament, each team played twice against each other team. Two points were awarded for a win and no points for a loss. (No game could finish in a draw.) A single team won the tournament with 26 points, and exactly two teams finished last with 20 points. How many teams participated in the tournament?

3. Given the equation $x^{2001} = y^x$,

(a) find all solution pairs (x, y) consisting of positive integers with x prime;

(b) find all solution pairs (x, y) consisting of positive integers.

(Recall that $2001 = 3 \cdot 23 \cdot 29$.)

4. Call a positive integer *monotone* if, in its decimal representation, there are at least two digits, all digits are different from zero, and the digits appear in a strictly increasing or strictly decreasing order. (For instance, 127 and 9742 are monotone, whereas 172, 1224, and 7320 are not.)

(a) Compute the sum of all monotone numbers having five digits.

(b) Determine the number of final zeroes of the decimal representation of the least common multiple of all monotone numbers (without any restriction on the number of digits).

5. The incircle γ of triangle ABC touches the side AB at T. Let D be the point on γ diametrically opposite to T, and let S be the intersection of the line through C and D with the side AB. Show that $AT = SB$.

6. A square is filled with n^2 lamps, arranged in n rows and n columns. Some of them are alight, the others are out. To each lamp corresponds a push-button that, when pressed, switches all lamps of its row and its column (including the lamp itself). Determine the states from which it is possible to light all the lamps if

(a) $n = 10$; (b) $n = 9$.

As a third Olympiad set we give the problems of the 52nd Polish Mathematical Olympiad, Final Round, April 2–3, 2001. Thanks again go to Chris Small, Canadian Team Leader to the XLI IMO, for collecting them.

52nd POLISH MATHEMATICAL OLYMPIAD
Final Round
April 2 (Day 1), 2001 – Time: 5 hours

1. Show that the inequality

$$\sum_{i=1}^{n} ix_i \le \binom{n}{2} + \sum_{i=1}^{n} x_i^i$$

holds for every integer $n \ge 2$ and all real numbers $x_1, x_2, \ldots, x_n \ge 0$.

2. Consider an arbitrary point P inside the regular tetrahedron with an edge of length 1. Show that the sum of the distances from P to the vertices of the tetrahedron does not exceed 3.

3. The sequence $x_1, x_2, x_3, \ldots$ is defined recursively by

$$x_1 = a, \quad x_2 = b, \quad \text{and} \quad x_{n+2} = x_{n+1} + x_n \quad \text{for } n = 1, 2, 3, \ldots,$$

where a and b are real numbers. A number c will be called a *repeated value* if $x_k = x_l = c$ for at least two distinct indices k and l. Prove that the initial terms a and b can be chosen so that there are more than 2000 repeated values, but it is impossible to choose a and b so that there are infinitely many repeated values.

April 3 (Day 2), 2001 – Time: 5 hours

4. The integers a and b have the property that, for every non-negative integer n, the number $2^n a + b$ is the square of an integer. Show that $a = 0$.

5. Let $ABCD$ be a parallelogram, and let K and L be points lying on the sides BC and CD, respectively, such that $BK \cdot AD = DL \cdot AB$. The segments DK and BL intersect at P. Show that $\angle DAP = \angle BAC$.

6. Let $n_1 < n_2 < \cdots < n_{2000} < 10^{100}$ be positive integers. Prove that the set $\{n_1, n_2, \ldots, n_{2000}\}$ has two non-empty disjoint subsets A and B with equally many elements, equal sums of their elements, and equal sums of the squares of their elements.

SOLUTIONS TO FIVE KLAMKIN QUICKIES

February 2004

1. Setting $x = -1$ in the first relation, we get $P(-2) = 0$. It then follows that $P(-n) = 0$ for all $n \geq 2$. Then $P(x)$ cannot be a polynomial. Setting $P(x) = xQ(x)$ in the second relation, we get $Q(x) = Q(x+1)$, so that $Q(x)$ must be a constant. Therefore, the only solution is $P(x) = kx$.

2. Clearly, $P(x)$ has no negative roots. To show there are no non-negative real roots, we note that for all $x \geq 0$,

$$\begin{aligned}(1+x)P(x) &= x^{2n+1} - x^{2n} + \cdots + x + 2n + 1 \\ &= \frac{x(x^{2n+1}+1)}{x+1} + 2n + 1 \ > \ 0\,.\end{aligned}$$

3. We consider values modulo 7. Cubic residues are 0, -1, and 1. Thus, the right side is either 3 or 4, while the left side can only be 0, 1, 2, -1, or -2. Hence, there are no solutions.

4. Let x and y satisfy the given relation. Letting $\tan x = \sinh u$ and $\tan y = \sinh v$, we have $\sec x = \pm\cosh u$ and $\sec y = \pm\cosh v$. The given relation becomes $(\sinh u \pm \cosh u)(\sinh v \pm \cosh v) = 1$, which implies that $e^{u+v} = 1$ and hence, $u + v = 0$. Then $\tan x + \tan y = \sinh u + \sinh v = 0$.

5. The inequality is clearly valid for $x = y = 1$. Next, we try $x = 1 + \varepsilon$ and $y = 1$ where ε is arbitrarily small. Expanding both sides to order ε^2, the left side is $16 + 64\varepsilon + 128\varepsilon^2$, while the right side is $16 + 64\varepsilon + 132\varepsilon^2$. Thus, the inequality is invalid.

Next we turn to readers' contributions for problems of the 47$^{\text{th}}$ Czech and Slovak Mathematical Olympiad 1998 given [2001 : 360–361].

1. Find all solutions in the real domain of the equation

$$x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor \ = \ 88\,,$$

where $\lfloor a \rfloor$ is the integer part of a real number a; that is, the integer satisfying $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1$. For instance, $\lfloor 3.7 \rfloor = 3$, $\lfloor -3.7 \rfloor = -4$ and $\lfloor 6 \rfloor = 6$.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Maisons-Laffitte, France; Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; George Evagelopoulos, Athens, Greece; Pavlos Maragoudakis, Pireas, Greece. We give the solution by Maragoudakis.

A straightforward calculation shows that $x = \frac{22}{7}$ is a solution. We will prove that this is the only solution.

Let x be a solution and let $n = \lfloor x \rfloor$. Then $n \le x < n+1$.

Case 1. $x > 0$.

Then

$$\begin{aligned} n^2 &\le x \cdot \lfloor x \rfloor < n^2 + n\,, \\ n^2 &\le \lfloor x \cdot \lfloor x \rfloor \rfloor < n^2 + n\,, \\ n^3 &\le x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor < n(n+1)^2\,, \\ n^3 &\le \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor < n(n+1)^2\,, \\ n^4 &\le x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor < n(n+1)^3\,, \\ n^4 &\le 88 < n(n+1)^3\,. \end{aligned}$$

The inequalities in the last line are true only if $n = 3$. Now the equation becomes $x \cdot \lfloor x \cdot \lfloor 3x \rfloor \rfloor = 88$.

Let $k = \lfloor 3x \rfloor$. Then $k \le 3x < k+1$. Thus,

$$\begin{aligned} \frac{k}{3} &\le x < \frac{k+1}{3}\,, \\ \frac{k^2}{3} &\le x \cdot \lfloor 3x \rfloor < \frac{k(k+1)}{3}\,, \\ \frac{k^2}{3} - 1 &< \lfloor x \cdot \lfloor 3x \rfloor \rfloor < \frac{k(k+1)}{3}\,, \\ \frac{k^3 - 3k}{9} &< x \cdot \lfloor x \cdot \lfloor 3x \rfloor \rfloor < \frac{k(k+1)^2}{9}\,, \\ \frac{k^3 - 3k}{9} &< 88 < \frac{k(k+1)^2}{9}\,. \end{aligned}$$

These last inequalities are true only if $k = 9$. Now the equation becomes $x\lfloor 9x \rfloor = 88$.

Let $\ell = \lfloor 9x \rfloor$. Then $\ell \le 9x < \ell + 1$. Hence,

$$\begin{aligned} \frac{\ell}{9} &\le x < \frac{\ell+1}{9}\,, \\ \frac{\ell^2}{9} &\le x \cdot \lfloor 9x \rfloor < \frac{\ell^2 + \ell}{9}\,, \\ \frac{\ell^2}{9} &\le 88 < \frac{\ell^2 + \ell}{9}\,, \\ \ell^2 &\le 792 < \ell^2 + \ell\,. \end{aligned}$$

These inequalities are true only if $\ell = 28$. Finally, from the equation $x\lfloor 9x \rfloor = 88$, we get $x = \dfrac{88}{28} = \dfrac{22}{7}$.

Case 2. $x < 0$.

Then (since $n < 0$) we have

$$\begin{aligned} n^2 &\geq x \cdot \lfloor x \rfloor > n^2 + n\,, \\ n^2 &\geq \lfloor x \cdot \lfloor x \rfloor \rfloor \geq n^2 + n\,, \\ n^3 &\leq x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \leq n(n+1)^2\,, \\ n^3 &\leq \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor \leq n(n+1)^2\,, \\ n^4 &\geq x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor \geq n(n+1)^3\,, \\ n^4 &\geq 88 \geq n(n+1)^3\,. \end{aligned}$$

No integer $n < 0$ satisfies these inequalities.

2. Show that from any fourteen different natural numbers it is possible to choose, for a suitable k ($1 \leq k \leq 7$), two disjoint k–element subsets $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_k\}$ in such a way that the sums

$$A = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} \quad \text{and} \quad B = \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_k}$$

differ by less than 0.001; that is, $|A - B| < 0.001$.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and George Evagelopoulos, Athens, Greece. We present Bornsztein's write-up.

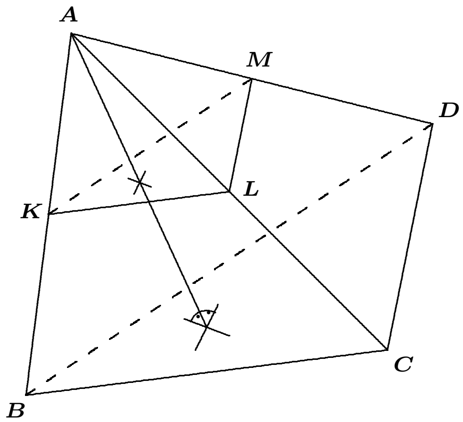
Let E be a set of fourteen different natural numbers. The number of subsets of E having cardinality 7 is $\binom{14}{7} = 3432$. Let X be one of these 7–element subsets, and set $f(X) = \sum_{a \in X} \frac{1}{a}$. We then have

$$f(X) \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{363}{140} < 2.6\,.$$

Thus, each of the 3432 values of $f(X)$, when X varies over all the possible 7–element subsets of E, belongs to one of the 2600 pairwise disjoint intervals of the form $\left(\frac{p}{1000}, \frac{p+1}{1000}\right]$, where $p = 0, 1, \ldots, 2599$. Applying the Pigeon-Hole Principle, we deduce that there exist two different 7–element subsets of E, say X and Y, such that $f(X)$ and $f(Y)$ belong to the same interval.

Let X' and Y' be the sets formed from X and Y by removing any elements common to X and Y, and let $A = f(X')$, $B = f(Y')$. Then X' and Y' are two disjoint subsets of E with the same cardinality, and we have $|A - B| = |f(X') - f(Y')| = |f(X) - f(Y)| < 0.001$.

3. A sphere is inscribed in a given tetrahedron $ABCD$. Its four tangent planes, which are parallel to the faces of the tetrahedron, cut four smaller tetrahedra from the tetrahedron. Prove that the sum of lengths of all their 24 edges is equal to twice the sum of the lengths of the edges of the tetrahedron $ABCD$.



Solution by George Evagelopoulos, Athens, Greece.

Denote by r the radius of the inscribed sphere and by U_A, U_B, U_C, U_D the four heights of the given tetrahedron, labelled according to the vertices from which they emanate. The smaller tetrahedron $AKLM$ (see figure) is homothetic (with centre A) to the whole tetrahedron $ABCD$. Thus, the ratio of the sum of the edge lengths of $AKLM$ to the sum of the edge lengths of $ABCD$ is the same as the ratio of their heights from the common vertex A. This ratio is $(U_A - 2r) : U_A$ (since $2r$ is the distance between the parallel planes KLM and BCD, both of which are tangent to the inscribed sphere). A similar argument applies to the other three small tetrahedra.

Our task is then to show that

$$\frac{U_A - 2r}{U_A} + \frac{U_B - 2r}{U_B} + \frac{U_C - 2r}{U_C} + \frac{U_D - 2r}{U_D} \;=\; 2\,,$$

or equivalently,

$$r\left(\frac{1}{U_A} + \frac{1}{U_B} + \frac{1}{U_C} + \frac{1}{U_D}\right) \;=\; 1\,.$$

We prove this by an argument involving the volume V and surface area S of the tetrahedron $ABCD$. Let S_X denote the area of the face not containing the vertex X. Then $S = S_A + S_B + S_C + S_D$. Furthermore,

$$V \;=\; \tfrac{1}{3}S_AU_A \;=\; \tfrac{1}{3}S_BU_B \;=\; \tfrac{1}{3}S_CU_C \;=\; \tfrac{1}{3}S_DU_D\,,$$

and $V = \frac{1}{3}rS$. Combining these formulas, we obtain

$$r\left(\frac{1}{U_A} + \frac{1}{U_B} + \frac{1}{U_C} + \frac{1}{U_D}\right) \;=\; \frac{3V}{S}\left(\frac{S_A}{3V} + \frac{S_B}{3V} + \frac{S_C}{3V} + \frac{S_D}{3V}\right) \;=\; 1\,,$$

which completes the proof.

4. For each date of the current year (1998) we evaluate the expression

$$\text{day}^{\text{month}} - \text{year}$$

and then find the highest power of 3 dividing it. For instance, for April 21 we obtain $21^4 - 1998 = 192\,483 = 3^3 \cdot 7129$, which is a multiple of 3^3, but not of 3^4. Find all days for which the corresponding power is the greatest.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and George Evagelopoulos, Athens, Greece. We give Bornsztein's write-up.

Let $f(d,m) = d^m - 1998$, where d and m are the day and the month, respectively. Let $\nu(d,m)$ be the exponent of 3 in the prime decomposition of $f(d,m)$. Since 3^3 is the highest power of 3 which divides 1998, we can have $\nu(d,m) \geq 4$ only if $d^m \equiv 0 \bmod 3^3$ and $d^m \not\equiv 0 \bmod 3^4$. To have $\nu(d,m) \geq 4$, we then need $\alpha m = 3$, where α is the exponent of 3 in the prime decomposition of d.

First case. $\alpha = 3$ and $m = 1$.

Since $d \in \{1, 2, \ldots, 31\}$, we have $d \equiv 0 \bmod 3^3$ if and only if $d = 27$. Then, the date is January 27, and $f(27,1) = 27 - 1998 = -1971 = -3^3 \times 73$; whence, $\nu(27,1) = 3$.

Second case. $\alpha = 1$ and $m = 3$.

Then $d \in \{3, 6, 12, 15, 21, 24, 30\}$. Direct computation in these seven cases leads to:

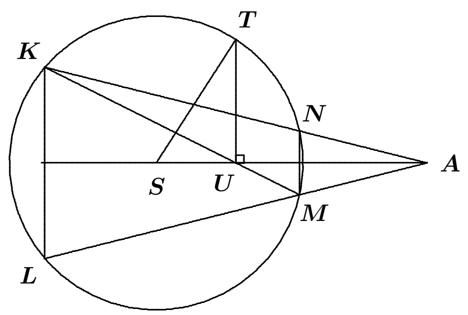
$$\nu(3,3) = \nu(12,3) = \nu(21,3) = \nu(30,03) = 3\,,$$
$$\nu(6,3) = \nu(15,3) = \nu(24,3) = 4\,.$$

Thus, the power of 3 is greatest for March 6, March 15, and March 24.

Conjecture. This exercise was given to the competitors March 24, 1998 (according to the dates given in [2001 : 360]).

5. In the exterior of a circle k a point A is given. Show that the diagonals of all trapezoids which are inscribed into the circle k and whose extended arms intersect at the point A intersect at the same point U.

Solution by George Evagelopoulos, Athens, Greece.



Let S be the centre of the circle k, and let $KLMN$ be an inscribed trapezoid whose extended arms KN and LM intersect at A (see figure). We use the axial symmetry of the trapezoid $KLMN$ with respect to the line AS to conclude that the intersection U of its diagonals must also lie on this line. Let T be one of the endpoints of the chord of the circle k which is perpendicular to SA and passes through U. The power of the point U with respect to the circle k is $|KU| \cdot |MU| = |TU|^2$. Considering the triangle KAM, in which AU bisects $\angle KAM$, we see that

$$|AU|^2 \;=\; |AK| \cdot |AM| - |KU| \cdot |MU| \;=\; |AK| \cdot |AN| - |TU|^2\,.$$

Hence,

$$|AK| \cdot |AN| \;=\; |AU|^2 + |TU|^2 \;=\; |AT|^2\,.$$

Since $|AK| \cdot |AN| = |AT|^2$, the point T is the point of tangency of one of the two tangents to k passing through the point A. These two tangents are independent of the choice of the trapezoid $KLMN$. Hence, the same is true for the point U.

Let us also remark that, by the Theorem of Euclid, the leg ST of the right triangle ATS satisfies $|ST|^2 = |SU| \cdot |SA|$, showing that U and A are images of each other in the inversion with respect to the circle k.

6. Let a, b, c be positive numbers. Show that the triangle with sides a, b, c exists if and only if the system of equations

$$\frac{y}{z} + \frac{z}{y} = \frac{a}{x}\,, \quad \frac{z}{x} + \frac{x}{z} = \frac{b}{y}\,, \quad \frac{x}{y} + \frac{y}{x} = \frac{c}{z}$$

has a solution in the real domain.

Solved by Michel Bataille, Rouen, France; George Evagelopoulos, Athens, Greece; and Pavlos Maragoudakis, Pireas, Greece. We feature Bataille's solution.

Suppose first that the given system has a solution (x,y,z), where x, y, and z are (non-zero) real numbers. Then,

$$\begin{aligned} b+c-a &= y\left(\frac{z}{x}+\frac{x}{z}\right)+z\left(\frac{x}{y}+\frac{y}{x}\right)-x\left(\frac{y}{z}+\frac{z}{y}\right) \\ &= \frac{2yz}{x} = \frac{2(y^2+z^2)}{a} > 0\,. \end{aligned}$$

Thus, $b+c > a$. Similarly, $c+a > b$ and $a+b > c$. Therefore, a, b, c are the sides of some triangle.

Conversely, if a, b, c are the sides of a triangle, let $s = \dfrac{a+b+c}{2}$ and

$$x \;=\; \sqrt{(s-b)(s-c)}\,, \quad y \;=\; \sqrt{(s-c)(s-a)}\,, \quad z \;=\; \sqrt{(s-a)(s-b)}\,.$$

Then (x, y, z) is a solution of the system. Indeed,

$$\frac{y}{z} + \frac{z}{y} \;=\; \frac{\sqrt{s-c}}{\sqrt{s-b}} + \frac{\sqrt{s-b}}{\sqrt{s-c}} \;=\; \frac{s-c+s-b}{x} \;=\; \frac{a}{x}\,,$$

and the other two equations are similarly verified.

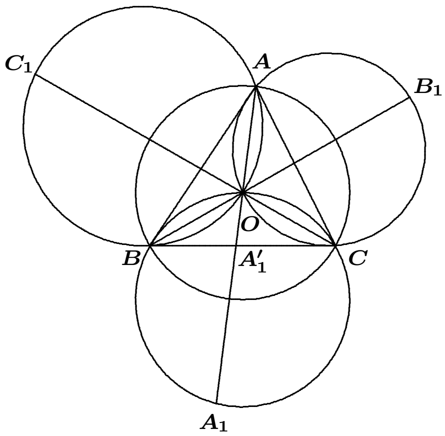
Next we turn to the November 2001 number of the *Corner* and readers' solutions and comments to Selection Questions for the Armenian Team for IMO99 given [2001 : 419–420].

1. Let O be the centre of the circumcircle of the acute triangle ABC. The lines CO, AO, and BO intersect for the second time the circumcircles of the triangles AOB, BOC, and AOC at C_1, A_1, and B_1, respectively.

Prove that

$$\frac{AA_1}{OA_1} + \frac{BB_1}{OB_1} + \frac{CC_1}{OC_1} \;\ge\; 4.5\,.$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's argument.



Let Γ be the circumcircle of $\triangle ABC$, and let R be its radius. Let f be the inversion in Γ. For any point P distinct from O, let $P' = f(P)$. Then $A' = A$, $B' = B$, and $C' = C$. The image of the circumcircle of $\triangle OBC$ is a line containing B' and C' and is therefore the line BC. Since the line through A, O, and A_1 is mapped onto itself by f, we deduce that A_1' is the intersection of this line with BC, and we have

$$OA_1 \cdot OA_1' \;=\; R^2\,. \tag{1}$$

For any points M and N distinct from O, it is well known that

$$M'N' \;=\; \frac{R^2 \cdot MN}{OM \cdot ON}\,.$$

Thus,

$$AA_1 = \frac{R^2 \cdot AA_1'}{OA \cdot OA_1'},$$

and hence, using (1),

$$\frac{AA_1}{OA_1} = \frac{R^2 \cdot AA_1'}{OA \cdot OA_1'} \cdot \frac{OA_1'}{R^2} = \frac{AA_1'}{OA}. \tag{2}$$

Let I and J be the respective orthogonal projections of A and O onto the line BC. Let $x = [OBC]$ and $S = [ABC]$. From Thales' Theorem, we have

$$\frac{OA_1'}{AA_1'} = \frac{OJ}{AI} = \frac{x}{S}.$$

Since O is interior to $\triangle ABC$ (because $\triangle ABC$ is acute), it follows that $OA = AA_1' - OA_1'$, and therefore

$$\frac{OA}{AA_1'} = 1 - \frac{OA_1'}{AA_1'} = \frac{S-x}{S}.$$

Then, using (2),

$$\frac{AA_1}{OA_1} = \frac{S}{S-x}.$$

Similarly,

$$\frac{BB_1}{OB_1} = \frac{S}{S-y} \quad \text{and} \quad \frac{CC_1}{OC_1} = \frac{S}{S-z},$$

where $y = [OAC]$ and $z = [OAB]$. Then

$$\frac{AA_1}{OA_1} + \frac{BB_1}{OB_1} + \frac{CC_1}{OC_1} = \frac{S}{S-x} + \frac{S}{S-y} + \frac{S}{S-z}. \tag{3}$$

We have x, y, $z > 0$ and $S = x + y + z$. From the Cauchy-Schwarz Inequality, we have

$$\left(\frac{S}{S-x} + \frac{S}{S-y} + \frac{S}{S-z}\right)\left((S-x)+(S-y)+(S-z)\right) \geq \left(\sqrt{S}+\sqrt{S}+\sqrt{S}\right)^2;$$

that is, $2S\left(\frac{S}{S-x} + \frac{S}{S-y} + \frac{S}{S-z}\right) \geq 9S$. Thus,

$$\frac{S}{S-x} + \frac{S}{S-y} + \frac{S}{S-z} \geq \frac{9}{2}. \tag{4}$$

The result follows directly from (3) and (4). Note that equality occurs if and only if $\triangle ABC$ is equilateral.

2. Let the escribed circle (opposite $\angle A$) of the triangle ABC ($\angle A$, $\angle B$, $\angle C < 120^\circ$) with centre O be tangent to the sides of the triangle AB, BC and CA at points C_1, A_1 and B_1 respectively. Denote the mid-points of the segments AO, BO, and CO by A_2, B_2, and C_2, respectively.

Prove that lines A_1A_2, B_1B_2, and C_1C_2 intersect at the same point.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

We will use areal coordinates. The following coordinates are known:

$$A\colon (1,0,0)\,; \quad B\colon (0,1,0)\,; \quad C\colon (0,0,1)\,;$$
$$A_1\colon \frac{1}{2a}(0, a-b+c, a+b-c)\,;$$
$$B_1\colon \frac{1}{2b}(b-c-a, 0, a+b+c)\,;$$
$$C_1\colon \frac{1}{2c}(c-a-b, a+b+c, 0)\,;$$
$$O\colon \frac{1}{b+c-a}(-a,b,c)\,.$$

The coordinates of the mid-point of a line segment are the averages of the coordinates of the endpoints. Thus, the coordinates of A_2, B_2, and C_2 are:

$$A_2\colon \frac{1}{2(b+c-a)}(b+c-2a, b, c)\,;$$
$$B_2\colon \frac{1}{2(b+c-a)}(-a, 2b+c-a, c)\,;$$
$$C_2\colon \frac{1}{2(b+c-a)}(-a, b, 2c+b-a)\,.$$

The equation of A_1A_2 is

$$\det\begin{pmatrix} x & y & z \\ b+c-2a & b & c \\ 0 & a-b+c & a+b-c \end{pmatrix} = 0\,,$$

which, on expansion, becomes

$$x(b-c)(a+b+c) - y(b+c-2a)(a+b-c) + z(b+c-2a)(a-b+c) = 0\,.$$

Similarly, B_1B_2 has the equation

$$x(a+b+c)(2b+c-a) + y(a+c)(a+b-c) + z(a-b+c)(2b+c-a) = 0\,,$$

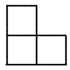
and C_1C_2 has the equation

$$x(a+b+c)(2c+b-a) + y(a+b-c)(2c+b-a) + z(a+b)(a-b+c) = 0\,.$$

The three lines meet at the point with unnormalized areal coordinates

$$
\begin{aligned}
x &= \frac{(b+c)(2a-b-c)}{a+b+c}\,, \\
y &= \frac{(2b+c-a)(c-a)}{a+b-c}\,, \\
z &= \frac{(2c+b-a)(b-a)}{a-b+c}\,.
\end{aligned}
$$

5. Any 9 squares are removed from the 40 white squares of a 9×9 chess-like painted board. Prove that the remaining board is impossible to cover using 24 pieces of the kind as shown in the figure.

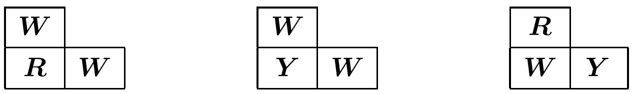


Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

Since the original board has 40 white squares and 41 black squares, the four squares at its corners are black. Label the rows from 1 to 9. Colour the black squares red or yellow according to the parity of their row, as shown in the figure below. After removing the 9 white squares, there are 25 red squares, 16 yellow squares, and $40 - 9 = 31$ white squares.

1	R	W	R	W	R	$\cdots$
2	W	Y	W	Y	W	$\cdots$
3	R	W	R	W	R	$\cdots$
4	W	Y	W	Y	W	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
9	R	W	R	W	R	$\cdots$

Now, suppose that a covering is possible. It uses three types of pieces:



Type R Type Y Type W

Let r, y, w be the numbers of pieces of types R, Y, W, respectively, used for the covering. Then $r + y + w = 24$. Moreover, since there are 25 red squares, we must have $r + w = 25$. Thus, $y = -1$, a contradiction. The conclusion follows.

6. Solve the equation

$$\frac{1}{x^2} + \frac{1}{(4-\sqrt{3}\,x)^2} = 1\,.$$

Solved by Christopher J. Bradley, Clifton College, Bristol, UK; Laura Gu, student, Sir Winston Churchill High School, Calgary, AB; Pavlos Maragoudakis, Pireas, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; Panos E. Tsaoussoglou, Athens, Greece; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Chen and Wang.

For any solution x, there is some number θ such that $\cos\theta = 1/x$ and $\sin\theta = \dfrac{1}{4-\sqrt{3}x}$; hence, there is some θ such that

$$4 \;=\; \sqrt{3}\sec\theta + \csc\theta\,. \tag{1}$$

Conversely, if θ satisfies the above equation, then, letting $x = \sec\theta$, we have $4 - \sqrt{3}x = \csc\theta$, and x is a solution of the given equation.

Equation (1) is equivalent to

$$2\sin\theta\cos\theta \;=\; \tfrac{\sqrt{3}}{2}\sin\theta + \tfrac{1}{2}\cos\theta\,;$$

that is,

$$\sin(2\theta) \;=\; \sin\left(\theta + \frac{\pi}{6}\right)\,.$$

This equation is satisfied either when $2\theta = \theta + \frac{\pi}{6} + 2k\pi$ or when $2\theta = (2k+1)\pi - \left(\theta + \frac{\pi}{6}\right)$, where $k \in \mathbb{Z}$. Thus, the solutions of (1) are given by

$$\theta \;=\; \frac{\pi}{6} + 2k\pi \tag{2}$$

or

$$\theta \;=\; \frac{5\pi}{18} + \frac{2k\pi}{3}\,. \tag{3}$$

Due to the periodicity of the secant function, all values of θ given by (2) yield $x = \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Three more distinct values of x are obtained from the values of θ given by (3). These are $x = \sec\left(\frac{5\pi}{18}\right)$, $x = \sec\left(\frac{17\pi}{18}\right)$, and $x = \sec\left(\frac{29\pi}{18}\right)$.

7. It is known that all members of the infinite sequence $a-b$, a^2-b^2, a^3-b^3, $\ldots$, are natural numbers. Prove that a and b are integers.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Pavlos Maragoudakis, Pireas, Greece. We feature the solution by Maragoudakis.

Let $n_1 = a - b$ and $n_2 = a^2 - b^2$, where n_1, $n_2 \in \mathbb{N}^*$. Then $a = \dfrac{1}{2}\left(n_1 + \dfrac{n_2}{n_1}\right) \in \mathbb{Q}$ and $b = a - n_1 \in \mathbb{Q}$. Let $a = \dfrac{k}{\ell}$ and $b = \dfrac{m}{n}$, where k, ℓ, m, $n \in \mathbb{Z}$, with ℓ, $n > 0$ and $\gcd(k,\ell) = \gcd(m,n) = 1$. Now, $\dfrac{k}{\ell} - \dfrac{m}{n} = n_1$; that is,

$$kn - m\ell \;=\; n_1 n\ell\,.$$

Since $n \mid kn$ and $n \mid n_1 n\ell$, it follows that $n \mid m\ell$. Since $\gcd(m,n) = 1$, we must have $n \mid \ell$. Similarly, $\ell \mid n$. Therefore, $\ell = n$.

Now $a = \dfrac{k}{\ell}$ and $b = \dfrac{m}{\ell}$. We must show that $\ell = 1$. Suppose instead that $\ell \neq 1$. Then there exists a prime $p > 0$ such that $p \mid \ell$. Note that $p \nmid k$, since $\gcd(k, \ell) = 1$. For each $j = 1, 2, \ldots$, we have $a^j - b^j \in \mathbb{N}^*$, by hypothesis, and hence $\ell^j \mid (k^j - m^j)$. Then $p^j \mid (k^j - m^j)$, and therefore $k^j \equiv m^j \pmod{p}$. Thus, for $j = 2, 3, 4, \ldots$,

$$\frac{k^j - m^j}{k - m} = k^{j-1} + k^{j-2}m + \cdots + km^{j-2} + m^{j-1} \equiv jk^{j-1} \pmod{p}\ .$$

Let $m_0 \in \mathbb{N}^*$ be such that p^{m_0} is the largest power of p that divides $k - m$, and choose any integer n_0 such that $n_0 > m_0$ and $p \nmid n_0$. Then, since $p \nmid k$, it follows that $p \nmid n_0 k^{n_0 - 1}$, and thus,

$$p \nmid \frac{k^{n_0} - m^{n_0}}{k - m}\,.$$

Therefore, the largest power of p which divides $k^{n_0} - m^{n_0}$ is p^{m_0}. Since $n_0 > m_0$ we see that $p^{n_0} \nmid (k^{n_0} - m^{n_0})$, a contradiction. Thus, $\ell = 1$, and a and b are integers.

8. Prove that if m and n are natural numbers, such that the number $2^{mn} - 1$ is divisible by the number $(2^m - 1)(2^n - 1)$, then the number $2(3^{mn} - 1)$ is divisible by $(3^m - 1)(3^n - 1)$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

Let m and n be natural numbers such that $2^{mn} - 1$ is divisible by $(2^m - 1)(2^n - 1)$.

Claim. The numbers m and n are coprime.
Proof. Let $d = \gcd(m, n)$. Then $m = dx$ and $n = dy$, where $\gcd(x, y) = 1$. Since d divides both m and n, it follows that $2^d - 1$ divides both $2^m - 1$ and $2^n - 1$; that is, $2^m \equiv 2^n \equiv 1 \pmod{2^d - 1}$. Since $2^n - 1$ divides $\dfrac{2^{mn} - 1}{2^m - 1}$ (by hypothesis), we deduce that $2^d - 1$ divides $\dfrac{2^{mn} - 1}{2^m - 1}$. On the other hand,

$$\frac{2^{mn} - 1}{2^m - 1} = \sum_{k=0}^{n-1} 2^{mk} \equiv \sum_{k=0}^{n-1} 1 = n \pmod{2^d - 1}\ .$$

Therefore, $n \equiv 0 \pmod{2^d - 1}$. In a similar way, we conclude that $m \equiv 0 \pmod{2^d - 1}$. Thus, $dy \equiv dx \equiv 0 \pmod{2^d - 1}$.

Let p, q be positive integers such that

$$dy = p(2^d - 1) \quad \text{and} \quad dx = q(2^d - 1)\,. \tag{1}$$

Then $qdy = pdx$, which leads to $qy = px$. From Gauss' Theorem, we deduce that x divides q; that is, $q = ax$ for some positive integer a. Then, from (1), we have $d = a(2^d - 1)$. It follows that $2^d - 1$ divides d. But an easy induction shows that $2^d - 1 > d$ for $d \geq 2$. It follows that $d = 1$, as claimed. ∎

It is well known ([1], p. 26) that if a, $b \in \mathbb{N}^*$, and if $k > 1$ is an integer, then $\gcd(k^a - 1, k^b - 1) = k^{\gcd(a,b)} - 1$. From this and the claim, we deduce that $\gcd(3^m - 1, 3^n - 1) = 3^1 - 1 = 2$.

At least one of the numbers m and n must be odd, by the claim. Say $m = 2i + 1$. Then $3^m - 1 = 3 \times 9^i - 1 \equiv 2 \pmod 4$. Thus, $3^m - 1 = 2r$, where r is an odd integer and $\gcd(r, 3^n - 1) = 1$. Since each of $3^n - 1$ and $3^m - 1$ is a divisor of $3^{mn} - 1$, we deduce that $3^{mn} - 1$ is divisible by $r(3^n - 1)$. Then $2(3^{mn} - 1)$ is divisible by $2r(3^n - 1) = (3^m - 1)(3^n - 1)$.

Reference.

[1] W. Sierpinski, *Elementary Theory of Numbers*, North-Holland.

9. Find all natural numbers k for which the sequence $x_n = \frac{S(n)}{S(kn)}$, $(n = 1, 2, \ldots)$ will be bounded. Here $S(a)$ denotes the sum of the digits of the natural number a.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that the sequence is bounded if and only if $k = 2^\alpha 5^\beta$, for some non-negative integers α, β.

We will use the following lemma, which is known to regular readers of ***CRUX with MAYHEM*** (see, for example, [2001 : 179–180]): "If n is a natural number, then $S(2n) \le 2S(n) \le 10S(2n)$." From this, we deduce that, for all positive integers n,

$$\frac{1}{2} \le \frac{S(n)}{S(2n)} \le 5\,. \tag{1}$$

Replacing n by $5n$ we get

$$\frac{1}{2} \le \frac{S(5n)}{S(10n)} \le 5\,.$$

Since $S(10n) = S(n)$, it follows that

$$\frac{1}{5} \le \frac{S(n)}{S(5n)} \le 2\,. \tag{2}$$

Let B be the set of $k \in \mathbb{Z}^+$ such that the sequence $\left\{\frac{S(n)}{S(kn)}\right\}$ is bounded. Note that $1 \in B$ (obviously), and also $2 \in B$ and $5 \in B$, from (1) and (2).

Suppose that $k = 2^\alpha a$, where α and a are positive integers and a is odd. Let n be a positive integer. Then

$$\frac{S(n)}{S(kn)} = \frac{S(n)}{S(an)} \cdot \frac{S(an)}{S(2an)} \cdot \frac{S(2an)}{S(2^2an)} \cdots \frac{S(2^{\alpha-1}an)}{S(2^\alpha an)}\,,$$

and, using (1),

$$\frac{1}{2^\alpha} \le \frac{S(an)}{S(2an)} \cdot \frac{S(2an)}{S(2^2an)} \cdots \frac{S(2^{\alpha-1}an)}{S(2^\alpha an)} \le 5^\alpha\,.$$

Thus,

$$\frac{1}{2^\alpha}\frac{S(n)}{S(an)} \le \frac{S(n)}{S(kn)} \le 5^\alpha \frac{S(n)}{S(an)}\,.$$

Since α does not depend on n, it follows that the sequence $\left\{\frac{S(n)}{S(kn)}\right\}$ is bounded if and only if the sequence $\left\{\frac{S(n)}{S(an)}\right\}$ is bounded. That is,

$$2^\alpha a \in B \quad \text{if and only if} \quad a \in B\,. \tag{3}$$

Since $1 \in B$, it follows that $2^\alpha \in B$ for any non-negative integer α.

Next, suppose that $k = 5^\beta b$, where β and b are positive integers, and $b \not\equiv 0 \pmod 5$. Using (2), we can prove in a similar way that

$$5^\beta b \in B \quad \text{if and only if} \quad b \in B\,. \tag{4}$$

Since $2^\alpha \in B$, it follows that $2^\alpha 5^\beta \in B$, where α and β are any non-negative integers.

Now let k be an integer with $k > 1$ and $\gcd(k, 10) = 1$. Then, from the Fermat-Euler Theorem, we have $10^{\varphi(k)} \equiv 1 \pmod k$, where φ denotes Euler's totient function. Set $a = \frac{10^{\varphi(k)} - 1}{k}$. Then, $a \in \mathbb{N}^*$, and $a < 10^{\varphi(k)} - 1$, from which we deduce that $1 + a$ has at most $\varphi(k)$ decimal digits.

Let q be any positive integer such that $10^{q\varphi(k)} > k - 1$. We have $10^{q\varphi(k)} \equiv 1^q = 1 \pmod k$, and hence, $10^{q\varphi(k)} + (k-1) = kn_q$, for some $n_q \in \mathbb{N}^*$. Then $S(kn_q) = 1 + S(k-1)$. But

$$\begin{aligned} n_q &= 1 + \frac{10^{q\varphi(k)} - 1}{k} \\ &= 1 + \frac{10^{\varphi(k)} - 1}{k}\left(1 + 10^{\varphi(k)} + 10^{2\varphi(k)} + \cdots + 10^{(q-1)\varphi(k)}\right) \\ &= 1 + a + a10^{\varphi(k)} + a10^{2\varphi(k)} + \cdots + a10^{(q-1)\varphi(k)}\,. \end{aligned}$$

Therefore, $S(n_q) = S(1+a) + (q-1)S(a) \ge (q-1)S(a)$. It follows that

$$\frac{S(n_q)}{S(kn_q)} \ge \frac{(q-1)S(a)}{1 + S(k-1)}\,.$$

Then $\frac{S(n_q)}{S(kn_q)} \to +\infty$ as $q \to +\infty$, and therefore, $k \notin B$.

Thus, if k is an integer with $k > 1$ and $\gcd(k, 10) = 1$, then $k \notin B$. It follows, using (3) and (4), that $k \in B$ if and only if $k = 2^\alpha 5^\beta$.

That completes the *Olympiad Corner* for this issue. Send me your contests as well as your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

Mathematical Diamonds
by Ross Honsberger, published by the Math. Association of America, 2003
ISBN 0-88385-332-9, paperbound, 256 pages, US$35.95.
Reviewed by ***Daryl Tingley****, University of New Brunswick, Fredericton, NB.*

In the tradition of Professor Honsberger's other nine books in the Dolciani Mathematical Expositions series from the Mathematical Association of America, *Mathematical Diamonds* is a collection of beautiful mathematical essays.

Although the essays are written at a level that makes them accessible to mathematically talented high school and undergraduate students, the clear exposition and the variety of interesting topics makes the book enjoyable reading for anyone interested in mathematical ideas and thought. It would be especially valuable to an organizer of a mathematics club or problem solving group (at either the high school or undergraduate level).

Each essay carefully explains a beautiful math problem or result, what the author calls a "gem". These gems have been gleaned from many sources. Books and journals, both texts and recreational works, contests, communications between the author and other people (not necessarily professional mathematicians), as well as the author's imagination have all been used as the basis for an essay. Most of the topics covered come from Euclidean and Combinatorial Geometry, Combinatorics, Algebra, and Number Theory.

In the preface, the author states that "There is no attempt to give instruction . . . ". By this he means that he only defines terms or states results he uses if they are directly applicable to the "gem" that follows. The terms used are, for the most part, elementary, and will be understood by most readers. Taking time to explain them would detract from the mathematics. Most results used in the book are well known to the intended audience and the lesser-known results are either proven or referenced.

Perhaps there was no attempt made to give instruction, but the book is certainly instructive. The reader is introduced to many interesting problems and gains insights as to how such problems should be solved. The exposition illustrates that mathematics can be written in a serious but also entertaining way: there is enough rigour to keep a mathematician happy, but the author does not allow himself to get lost in details.

Exploring the Real Numbers
by Frederick W. Stevenson, published by Prentice Hall, 2001
ISBN 0-13-040261-3, hardcover, 365 + xi pages, US$87.00.
Reviewed by ***Amar Sodhi****, Sir Wilfred Grenfell College, Corner Brook, NL.*

One aim of this book, according to the author, is to present "interesting topics which arise from the study of real numbers". Upon reading the Table of Contents, one is led to believe that the author deals with the natural numbers, the integers, the rational numbers, and the real numbers as four discrete topics. However, there is a thread that connects the chapters, namely, the Euclidean Algorithm.

The first chapter deals with basic notions of divisibility in the natural numbers and introduces the Euclidean Algorithm. After proving the Fundamental Theorem of Arithmetic, the author describes the "search for primes" and discusses other special numbers.

Chapter 2 deals mainly with Diophantine equations. A linear Diophantine equation is solved via both the Euclidean Algorithm and modular arithmetic. The equations of Pythagoras and Pell are also discussed. In solving Diophantine equations via the Euclidean Algorithm, the author introduces a tabular method of finding such solutions that is well utilized in the rest of the book. However, the premature introduction of this method distracts, rather than aids, in understanding and enjoying the material, and should be postponed to the continued fraction section of the next chapter.

The highlight of Chapter 3 is a discussion of continued fractions. In Chapter 4 we see how to solve Pell's Equation by the use of such fractions. The author, whether by accident or design, manages to convey that, by exploring real numbers, one can answer problems that pertain to the integers. Other topics discussed in chapter 4 include algebraic numbers, ruler and compass constructions, and a rather frenetic discussion on transcendental numbers.

The last chapter of the book addresses the author's desire to "provide the readers an opportunity to further study [topics in numbers] on their own." This chapter contains twenty-one guided research projects that are accessible to motivated high-school students.

Exploring the Real Numbers is a textbook that is well suited for a university-level course designed for students who like mathematics but are not pursuing a mathematics degree. It is also a book that I would recommend to a mathematically motivated high-school student. A reader of this book will learn some aspects of numbers and number theory together with historical anecdotes. If the reader is armed with paper and pencil, then the reader should enjoy tackling the numerous problems and research projects contained in the text.

On Expanding $4/n$ into Three Egyptian Fractions

John H. Jaroma

A notable conjecture of P. Erdős and E. Straus states that the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

is solvable in positive integers, for $n \in \{4, 5, \dots\}$. The fractions on the right side above, reciprocals, are called "Egyptian" fractions. According to R. Guy [1], the problem has been verified by N. Franceschine for $n < 10^8$. In addition, Guy notes that, when the results of L. Bernstein, R. Obláth, K. Yamamoto, and L. Rosati on this problem are summarized, the conjecture is seen to be true for all n, except possibly where

$$n \equiv 1, 121, 169, 289, 361, 529 \pmod{840}.$$

Also in [1], it is indicated that A. Schinzel has relaxed the condition that x, y, z be positive. It is known that $4/n$ can be expressed as the sum of Egyptian fractions where the algebraic sign of at least one of the Egyptian fractions is negative.

The objective of this note is to present a universal three-term Egyptian fraction decomposition for $4/n$, where $n > 3$. Specifically, we introduce a single formula that allows a proper fraction of the form $4/n$ to be expressed as

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} - \frac{1}{z},$$

where x, y, $z \in \mathbb{Z}^+$.

Without loss of generality, we may assume that n is prime. Otherwise, if $n = 2^k$ for $k \geq 2$, then $4/n$ is equivalent to the Egyptian fraction $1/2^{k-2}$, and we may choose $x = 1/2^{k-2}$, and $y = z$, any positive integer. Furthermore, if $n = rp$, where p is an odd prime, then $4/n = (1/r) \cdot (4/p)$, and a three-term Egyptian fraction decomposition for $4/n$ may be obtained by expanding $4/p$ in terms of Egyptian fractions and then multiplying through by $1/r$.

Theorem. Let n be an odd prime. Then

$$\frac{4}{n} = \frac{1}{\left(\frac{n-1}{2}\right)} + \frac{1}{\left(\frac{n+1}{2}\right)} - \frac{1}{n\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)}.$$

Examples.

1. $n = 109$:

$$\frac{4}{109} = \frac{1}{\left(\frac{109-1}{2}\right)} + \frac{1}{\left(\frac{109+1}{2}\right)} - \frac{1}{109\left(\frac{109-1}{2}\right)\left(\frac{109+1}{2}\right)}$$
$$= \frac{1}{54} + \frac{1}{55} - \frac{1}{323730}.$$

2. $n = 1001$:

$$\frac{4}{1001} = \frac{1}{7\cdot 11}\left(\frac{4}{13}\right) = \frac{1}{77}\left[\frac{1}{\left(\frac{13-1}{2}\right)} + \frac{1}{\left(\frac{13+1}{2}\right)} - \frac{1}{13\left(\frac{13-1}{2}\right)\left(\frac{13+1}{2}\right)}\right]$$
$$= \frac{1}{462} + \frac{1}{539} - \frac{1}{42042}.$$

References.

[1] Richard K. Guy, *Unsolved Problems in Number Theory*, 2nd Ed., Springer Verlag, New York, (1994).

John H. Jaroma
Department of Mathematics & Computer Science
Austin College
Sherman, TX, USA 75090
jjaroma@austincollege.edu

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 September 2004**. *An asterisk* (⋆) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

2887. Correction. *Proposed by Vedula N. Murty, Dover, PA, USA.*

If a, b, c are the sides of $\triangle ABC$ in which at most one angle exceeds $\frac{\pi}{3}$, and if R is its circumradius, prove that

$$a^2 + b^2 + c^2 \;\le\; 6R^2 \sum_{\text{cyclic}} \cos A\,.$$

2888★. Correction. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let a, b, c be the sides of $\triangle ABC$, in which at most one angle exceeds $\frac{\pi}{3}$. Give an algebraic proof of

$$8a^2b^2c^2 + \prod_{\text{cyclic}} \left(b^2 + c^2 - a^2\right) \;\le\; 3abc \sum_{\text{cyclic}} a\left(b^2 + c^2 - a^2\right)\,.$$

2901★. *Proposed by Stanley Rabinowitz, Westford, MA, USA.*

Let I be the incentre of $\triangle ABC$. The circles d, e, f inscribed in $\triangle IAC$, $\triangle IBC$, $\triangle ICA$ touch the sides AB, BC, CA at the points D, E, F, respectively. The line IA is one of the two common internal tangents between the circles d and f. Let ℓ be the other common internal tangent. Prove that ℓ passes through the point E.

[This problem was suggested by experiments using *Geometer's Sketchpad*.]

2902. *Proposed by Stanley Rabinowitz, Westford, MA, USA.*

Let P be a point in the interior of $\triangle ABC$. Let D, E, F be the feet of the perpendiculars from P to BC, CA, AB, respectively. If the three quadrilaterals $AEPF$, $BFPD$, $CDPE$ each have an incircle tangent to all four sides, prove that P is the incentre of $\triangle ABC$.

2903. *Proposed by Stanley Rabinowitz, Westford, MA, USA.*

Three disjoint circles A_1, A_2, and A_3 are given in the plane, none being interior to any other. The common internal tangents to A_j and A_k are α_{jk} and β_{jk}.

If the α_{jk} are concurrent, prove that the β_{jk} are also concurrent.

[This is a known result—but not well-known.]

2904. *Proposed by Mohammed Aassila, Strasbourg, France.*

Suppose that $x_1 > x_2 > \cdots > x_n$ are real numbers. Prove that

$$\sum_{k=1}^{n} x_k^2 \;-\; \sum_{1\le j\le k\le n} \ln(x_j - x_k) \;\ge\; \frac{n(n-1)}{4}(1+2\ln 2) - \frac{1}{2}\sum_{k=1}^{n} k\ln k\,.$$

2905. *Proposed by Titu Zvonaru, Bucharest, Romania.*

The lines joining the vertices to the centroids of the squares constructed externally over the sides of a triangle are concurrent (at Vectin's point). Suppose that Vectin's point is also the symmedian point. Characterize the triangle.

2906. *Proposed by Titu Zvonaru, Bucharest, Romania.*

Suppose that $k \in \mathbb{N}$. Find $\min\limits_{n\in\mathbb{N}} \left(\frac{2}{n} + \frac{n^2}{k}\right)$.

2907. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Rhombus $ABCD$ has incircle Γ with centre O. Circle Γ touches sides AB and AD at M and N, respectively. Suppose that a tangent to Γ meets the segments AM and AN at E and F, respectively, and that EF intersects BC and CD at P and Q, respectively.

Prove that $[AMON]$ is the geometric mean of $[AEF]$ and $[CPQ]$.
($[Z_1Z_1\ldots Z_n]$ denotes the area of the n–gon $Z_1Z_2\ldots Z_n$.)

2908. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

On the sides of triangle ABC, triangles DBC, ECA, FAB, are constructed externally such that $\angle DBC = \angle DCB = \angle ECA = \angle FBA = \phi$ and $\angle CAE = \angle BAF = \theta$, where $\angle BAC + \theta < 180^\circ$, $\angle ABC + \phi < 180^\circ$ and $\angle ACB + \phi < 180^\circ$.

Prove that $[ABDE] = [CAFD]$. ($[Z_1Z_1\cdots Z_n]$ denotes the area of the n–gon $Z_1Z_2\ldots Z_n$.)

2909. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a convex quadrilateral $ABCD$, a line parallel to AD meets segments AB, AC, BD, CD, at E, F, G, H, respectively.

Prove that $[EBCF] : [GBCH] = EF : GH$. ($[Z_1Z_1 \ldots Z_n]$ denotes the area of the n–gon $Z_1Z_2 \ldots Z_n$.)

2910. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In triangle ABC, let D, E, F, be the mid-points of sides BC, CA, AB, respectively. Let M, N, P be points on the segments FD, FB, DC, respectively, such that $FM : FD = FN : FB = DP : DC$.

Prove that AM, EN, FP are concurrent.

2911. *Proposed by Mihály Bencze, Brasov, Romania.*

(a) If z, $w \in \mathbb{C}$ and $|z| = 1$, prove that

$$(n-1)\sum_{k=1}^{n} \left|w + z^k\right| \;\geq\; \sum_{k=1}^{n-1}(n-k)\left|1 - z^k\right| \,.$$

(b) If $x \in \mathbb{R}$, prove that

$$(n-1)\sum_{k=1}^{n} |\cos(kx)| \;\geq\; \sum_{k=1}^{n-1}(n-k)\,|\sin(kx)| \,.$$

2912. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} \log_a \left(\sum_{k=1}^{a^n} (1+k)^{a^{-n}} \right) \;=\; e \,,$$

where $a \geq 2$ is a positive integer.

2913. *Proposed by Mihály Bencze, Brasov, Romania.*

If a, b, $c > 1$ and $\alpha > 0$, prove that

$$a^{\sqrt{\alpha \log_a b}+\sqrt{\alpha \log_a c}} + b^{\sqrt{\alpha \log_b a}+\sqrt{\alpha \log_b c}} + c^{\sqrt{\alpha \log_c a}+\sqrt{\alpha \log_c b}} \;\leq\; \sqrt{abc}\left(a^{\alpha-\frac{1}{2}} + b^{\alpha-\frac{1}{2}} + c^{\alpha-\frac{1}{2}}\right) \,.$$

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2887. Correction. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Si a, b et c sont les côtés d'un triangle ABC dont au plus un des angles excède $\frac{\pi}{3}$, et si R est le rayon du cercle circonscrit, montrer que

$$a^2+b^2+c^2 \;\leq\; 6R^2 \sum_{\text{cyclique}} \cos A\,.$$

2888★. Correction. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit a, b et c les côtés d'un triangle ABC dont au plus un des angles excède $\frac{\pi}{3}$. Donner une démonstration algébrique de

$$8a^2b^2c^2 + \prod_{\text{cyclique}} \left(b^2+c^2-a^2\right) \;\leq\; 3abc \sum_{\text{cyclique}} a\left(b^2+c^2-a^2\right)\,.$$

2901★. *Proposé par Stanley Rabinowitz, Westford, MA, USA.*

Soit I le centre du cercle inscrit du triangle ABC. Les cercles inscrits d, e et f des triangles IAC, IBC et ICA touchent respectivement les côtés AB, BC et CA aux points D, E et F. La droite IA est l'une des deux tangentes internes communes aux deux cercles d et f. Soit ℓ l'autre tangente interne commune. Montrer que ℓ passe par le point E.

[Ce problème a été suggéré à la suite d'expériences menées avec le logiciel *Geometer's Sketchpad*.]

2902. *Proposé par Stanley Rabinowitz, Westford, MA, USA.*

Soit P un point intérieur du triangle ABC. Soit respectivement D, E et F les pieds des perpendiculaires abaissées de P sur BC, CA et AB. Montrer que si chacun des trois quadrilatères $AEPF$, $BFPD$ et $CDPE$ possède un cercle inscrit tangent aux quatres côtés, alors le point P est le centre du cercle inscrit du triangle ABC.

2903. *Proposé par Stanley Rabinowitz, Westford, MA, USA.*

Dans le plan, on donne trois cercles disjoints A_1, A_2 et A_3, aucun n'étant intérieur aux deux autres. Désignons par α_{jk} et β_{jk} les tangentes communes internes à A_j et A_k.

Montrer que si les α_{jk} sont concourantes, alors les β_{jk} le sont aussi.

[Ce résultat mériterait d'être plus connu qu'il semble ne l'être.]

2904. *Proposé par Mohammed Aassila, Strasbourg, France.*

On suppose que $x_1 > x_2 > \cdots > x_n$ sont des nombres réels. Montrer que

$$\sum_{k=1}^{n} x_k^2 - \sum_{1\le j\le k\le n} \ln(x_j - x_k) \ \ge\ \frac{n(n-1)}{4}(1+2\ln 2) - \frac{1}{2}\sum_{k=1}^{n} k\ln k\,.$$

2905. *Proposé par Titu Zvonaru, Bucarest, Roumanie.*

Les droites joignant les sommets aux centres de gravité des carrés extérieurs construits sur les côtés d'un triangle sont concourantes (au point de Vectin). Supposons de plus que le point de Vectin est aussi le point "symmediant" Caractériser le triangle.

2906. *Proposé par Titu Zvonaru, Bucarest, Roumanie.*

Soit k un entier positif. Trouver $\displaystyle\min_{n\in\mathbb{N}}\left(\frac{2}{n}+\frac{n^2}{k}\right)$.

2907. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit O le centre du cercle inscrit Γ du losange $ABCD$. Le cercle Γ touche les côtés AB et AD en M et N respectivement. On suppose qu'une tangente à Γ coupe les segments AM et AN en E et F et que EF intersecte BC et CD en P et Q, respectivement.

Montrer que $[AMON]$ est la moyenne géométrique de $[AEF]$ et $[CPQ]$. ($[Z_1Z_1\ldots Z_n]$ désigne l'aire du n–gone $Z_1Z_2\ldots Z_n$.)
(n–gone : polygone convexe à n côtés.)

2908. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Sur les côtés d'un triangle ABC, on construit les triangles extérieurs DBC, ECA et FAB tels que $\angle DBC = \angle DCB = \angle ECA = \angle FBA = \phi$ et $\angle CAE = \angle BAF = \theta$, où $\angle BAC + \theta < 180°$, $\angle ABC + \phi < 180°$ et $\angle ACB + \phi < 180°$.

Montrer que $[ABDE] = [CAFD]$. ($[Z_1Z_1\ldots Z_n]$ désigne l'aire du n–gone $Z_1Z_2\ldots Z_n$.)
(n–gone : polygone convexe à n côtés.)

2909. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Dans un quadrilatère convexe $ABCD$, on trace une parallèle à AD coupant respectivement les segments AB, AC, BD et CD, en E, F, G et H.

Montrer que $[EBCF] : [GBCH] = EF : GH$. ($[Z_1Z_1\ldots Z_n]$ désigne l'aire du n–gone $Z_1Z_2\ldots Z_n$.)
(n–gone : polygone convexe à n côtés.)

2910. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit respectivement D, E et F les points milieu des côtés BC, CA et AB d'un triangle ABC. Soit de plus M, N et P des points sur les segments FD, FB et DC tels que $FM : FD = FN : FB = DP : DC$.

Montrer que AM, EN, FP sont concourants.

2911. *Proposé par Mihály Bencze, Brasov, Roumanie.*

(a) Si z, $w \in \mathbb{C}$ et $|z| = 1$, montrer que

$$(n-1)\sum_{k=1}^{n}\left|w+z^k\right| \;\geq\; \sum_{k=1}^{n-1}(n-k)\left|1-z^k\right| \,.$$

(b) Si $x \in \mathbb{R}$, montrer que

$$(n-1)\sum_{k=1}^{n}|\cos(kx)| \;\geq\; \sum_{k=1}^{n-1}(n-k)\,|\sin(kx)| \,.$$

2912. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}\log_a\left(\sum_{k=1}^{a^n}(1+k)^{a^{-n}}\right) \;=\; e\,,$$

où a est un entier plus grand ou égal à 2.

2913. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Si a, b, $c > 1$ et $\alpha > 0$, montrer que

$$a^{\sqrt{\alpha\log_a b}+\sqrt{\alpha\log_a c}} + b^{\sqrt{\alpha\log_b a}+\sqrt{\alpha\log_b c}} + c^{\sqrt{\alpha\log_c a}+\sqrt{\alpha\log_c b}} \;\leq\; \sqrt{abc}\left(a^{\alpha-\frac{1}{2}}+b^{\alpha-\frac{1}{2}}+c^{\alpha-\frac{1}{2}}\right) \,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologise for omitting the name of JOHN G. HEUVER, Grande Prairie, AB from the list of solvers of 2751.

2724★. [2002 : 174] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle and h_a, h_b, h_c, respectively, the corresponding altitudes. Prove that the maximum range of validity of the inequality

$$\left(\frac{h_a^t+h_b^t+h_c^t}{3}\right)^{1/t} \;\le\; \frac{\sqrt{3}}{2}\left(\frac{a^t+b^t+c^t}{3}\right)^{1/t},$$

where $t \neq 0$, is $\dfrac{-\ln 4}{\ln 4-\ln 3} \;<\; t \;<\; \dfrac{\ln 4}{\ln 4-\ln 3}$.

Partial Solution by Vasile Cartoaje, University of Ploiesti, Romania.

In his partial solution of this problem ([2003 : 180]), Murray S. Klamkin proved that the above inequality is equivalent to

$$\sum_{\text{cyclic}} \sin^t A \;\ge\; \left(\frac{2}{\sqrt{3}}\right)^t \sum_{\text{cyclic}} \sin^t B \sin^t C \tag{1}$$

and is valid for

$$0 \;<\; t < \frac{\ln 9-\ln 4}{\ln 4-\ln 3} \;\approx\; 2.818\,.$$

What follows is a more complete, but yet partial, solution. We will show that the inequality (1) is valid for

$$0 \;<\; t < \frac{\ln 3}{\ln 4-\ln 3} \;\approx\; 3.818\,.$$

Following Klamkin, we start with an inequality of A. Makowski and J. Berkes given in [1]:

$$\sum_{\text{cyclic}} \sin^t A \;\le\; 3\left(\frac{\sqrt{3}}{2}\right)^t.$$

The range of validity of this inequality is $0 < t \le \dfrac{\ln 9-\ln 4}{\ln 4-\ln 3}$. Replacing t by mt, where $m = \dfrac{\ln 9-\ln 4}{\ln 3} \approx 0.738$, we obtain

$$\sum_{\text{cyclic}} \sin^{mt} A \;\le\; 3\left(\frac{\sqrt{3}}{2}\right)^{mt},$$

valid for $0 < t < \frac{\ln 3}{\ln 4 - \ln 3}$. Hence, for any t in this range,

$$\left(\frac{2}{\sqrt{3}}\right)^t \left(\frac{1}{3}\sum_{\text{cyclic}} \sin^{mt} A\right)^{\frac{2}{m}} \;\le\; \left(\frac{1}{3}\sum_{\text{cyclic}} \sin^{mt} A\right)^{\frac{1}{m}} \;\le\; \frac{1}{3}\sum_{\text{cyclic}} \sin^t A\,,$$

where the last step follows by the Power-Mean Inequality, since $m < 1$. The lemma below implies that

$$\left(\frac{1}{3}\sum_{\text{cyclic}} \sin^{mt} A\right)^{\frac{2}{m}} \;\ge\; \frac{1}{3}\sum_{\text{cyclic}} \sin^t B \sin^t C\,.$$

Using this result in the previous inequality, we obtain (1).

Equality holds in (1) in the case where $t = \frac{\ln 3}{\ln 4 - \ln 3}$ and the triangle ABC is degenerate with angles 90°, 90°, and 0°. For this same case, in the step where we applied the Power-Mean Inequality above, the inequality is strict. Therefore, this solution cannot be extended to the entire proposed range of t.

Lemma. If $x \ge 0$, $y \ge 0$, $z \ge 0$, and $m = \frac{\ln 9 - \ln 4}{\ln 3} \approx 0.738$, then

$$\left(\frac{x^m + y^m + z^m}{3}\right)^{\frac{2}{m}} \;\ge\; \frac{xy + yz + zx}{3}\,, \tag{2}$$

with equality in the cases: (a) $x = y = z$; (b) $x = 0$, $y = z$; (c) $y = 0$, $z = x$; and (d) $z = 0$, $x = y$.

Proof. Assume, without loss of generality, that $x = \min\{x, y, z\}$. We first prove the inequality in the case $y = z = t$ and $t \ge x$; that is, we prove

$$\left(\frac{x^m + 2t^m}{3}\right)^{\frac{2}{m}} \;\ge\; \frac{2tx + t^2}{3}\,. \tag{3}$$

Since (3) is homogeneous, we may choose $t = 1$, which implies $x \le 1$. Setting $t = 1$ in (3) and taking logarithms, we get

$$2\ln\frac{x^m + 2}{3} \;\ge\; m\ln\frac{2x + 1}{3}\,.$$

To prove this inequality, we consider the function

$$f(x) \;=\; 2\ln\frac{x^m + 2}{3} - m\ln\frac{2x + 1}{3}\,.$$

We will show that $f(x) \ge 0$ for $0 \le x \le 1$.

The derivative

$$f'(x) \;=\; \frac{2mx^{m-1}}{x^m + 2} - \frac{2m}{2x + 1} \;=\; \frac{2m(x - 2x^{1-m} + 1)}{x^{1-m}(x^m + 2)(2x + 1)}$$

has, for $x \in (0,1]$, the same sign as $g(x) = x - 2x^{1-m} + 1$. We find that $g'(x) = 1 - \frac{2(1-m)}{x^m}$. Therefore, $g'(x) < 0$ for $x \in (0, x_1)$ and $g'(x) > 0$ for $x \in (x_1, 1)$, where $x_1 = (2-2m)^{1/m}$. Then, the function $g(x)$ is strictly decreasing for $x \in [0, x_1]$, and strictly increasing for $x \in [x_1, 1]$. Since $g(0) = 1 > 0$ and $g(1) = 0$, it follows that there exists $x_2 \in (0, x_1)$ such that $g(x_2) = 0$, $g(x) > 0$ for $x \in [0, x_2)$, and $g(x) < 0$ for $x \in (x_2, 1]$. Hence, $f'(x_2) = 0$, $f'(x) > 0$ for $x \in (0, x_2)$, and $f'(x) < 0$ for $x \in (x_2, 1)$. Therefore, the function $f(x)$ is strictly increasing for $x \in [0, x_2]$ and strictly decreasing for $x \in [x_2, 1]$. It follows that $f(x) \geq \min\{f(0),\ f(1)\} = 0$, establishing the desired result.

Now, for $t = \left(\frac{y^m + z^m}{2}\right)^{1/m}$, the inequality (3) becomes

$$\left(\frac{x^m + y^m + z^m}{3}\right)^{\frac{2}{m}} \;\geq\; \frac{2tx + t^2}{3}\,.$$

To prove (2), we have to show that

$$2tx + t^2 \;\geq\; x(y+z) + yz\,;$$

that is,

$$t^2 \;\geq\; x(y+z-2t) + yz\,. \tag{4}$$

By the Power-Mean Inequality, since $\frac{1}{2} < m < 1$, we have

$$\left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2 \;\leq\; t \;\leq\; \frac{y+z}{2}\,.$$

Thus, $y+z-2t \geq 0$. Since $x = \min\{x,\, y,\, z\}$, we have $x \leq \sqrt{yz}$. Therefore, to prove (4), it suffices to show that

$$t^2 \;\geq\; \sqrt{yz}(y+z-2t) + yz\,.$$

This inequality is equivalent to each of the following inequalities, the last of which is clearly true.

$$\left(t+\sqrt{yz}\right)^2 \;\geq\; \sqrt{yz}\left(\sqrt{y}+\sqrt{z}\right)^2\,,$$

$$t+\sqrt{yz} \;\geq\; \sqrt[4]{yz}\left(\sqrt{y}+\sqrt{z}\right)\,,$$

$$\left[t - \left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2\right] + \left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2 + \sqrt{yz} - \sqrt[4]{yz}\left(\sqrt{y}+\sqrt{z}\right) \;\geq\; 0\,,$$

$$\left[t - \left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2\right] + \left(\frac{\sqrt{y}+\sqrt{z}}{2} - \sqrt[4]{yz}\right)^2 \;\geq\; 0\,.$$

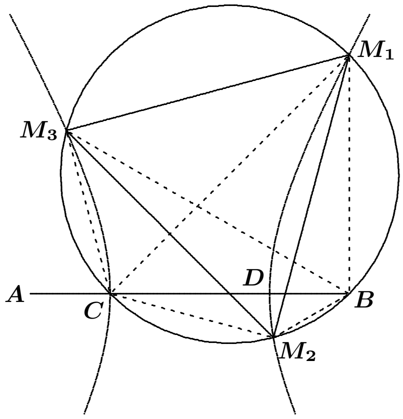
References.

[1] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović & P.M. Vasić, *Geometric Inequalities*, Groningen, 1969

2759. [2002 : 331, 2003 : 341] *Proposed by Michel Bataille, Rouen, France.*

On the line segment AB, let C, D be such that $\frac{AC}{CB} = \frac{BD}{DA} = \frac{1}{3}$. Distinct points M_1, M_2, M_3 lie on a circle passing through B and C and are such that $\angle M_1BC = 2\angle M_1CB$, $\angle M_2BC = 2\angle M_2CB$, and $\angle M_3AD = 2\angle M_3DA$. Show that $\triangle M_1M_2M_3$ is equilateral.

Alternative solution by Shailesh Shirali, Rishi Valley School, India, modified by the editor.



Impose a coordinatization with $A = (-2, 0)$, $B = (2, 0)$, $C = (-1, 0)$, and $D = (1, 0)$. Consider the locus of a point P such that P does not lie on the line AB and

$$2\angle PCB = \angle PBC \qquad \text{or} \qquad 2\angle PCB = \angle PBC + 180^\circ . \qquad (1)$$

Writing $P = (x, y)$, we have

$$\tan\angle PCB = \frac{|y|}{x+1} \qquad \text{and} \qquad \tan\angle PBC = -\frac{|y|}{x-2} .$$

If P satisfies (1), then $\tan(2\angle PCB) = \tan\angle PBC$. Using the double-angle tangent formula, we get the equation

$$-\frac{|y|}{x-2} = \frac{\frac{2|y|}{x+1}}{1-\left(\frac{|y|}{x+1}\right)^2} .$$

Simplifying, and noting that $y \neq 0$ (since P cannot lie on the line AB), we see that the locus of P is the hyperbola $y^2 = 3(x^2 - 1)$, excluding the points C and D (where $y = 0$). For points P on the right branch of this hyperbola (where $x > 0$), we have $2\angle PCB = \angle PBC$; for points on the left branch (where $x < 0$), we have $2\angle PCB = \angle PBC + 180^\circ$.

Under the reflection $x \mapsto -x$, the hyperbola $y^2 = 3(x^2 - 1)$ is invariant, and the points B and C are mapped to A and D, respectively. It follows that the same locus results from the following condition in place of (1):

$$2\angle PDA = \angle PAD \qquad \text{or} \qquad 2\angle PDA = \angle PAD + 180^\circ .$$

Therefore, points P on the right branch of the hyperbola must satisfy both $2\angle PCB = \angle PBC$ and $2\angle PDA = \angle PAD + 180^\circ$, and on the left branch they must satisfy both $2\angle PDA = \angle PAD$ and $2\angle PCB = \angle PBC + 180^\circ$.

We conclude that the points M_1 and M_2 in the problem are on the right branch of the hyperbola, while the point M_3 lies on the left branch. Moreover, we must have

$$2\angle M_3CB = \angle M_3BC + 180^\circ . \tag{2}$$

Now, since $\angle M_1BC = 2\angle M_1CB$ and $\angle M_2BC = 2\angle M_2CB$, we have

$$\begin{aligned} \angle M_1BM_2 &= \angle M_1BC + \angle M_2BC \\ &= 2\angle M_1CB + 2\angle M_2CB = 2\angle M_1CM_2 . \end{aligned}$$

Since we also have $\angle M_1BM_2 + \angle M_1CM_2 = 180^\circ$ (the sum of opposite angles in the cyclic quadrilateral M_1BM_2C), we see that $\angle M_1BM_2 = 120^\circ$. Then, using the cyclic quadrilateral $M_1BM_2M_3$, we get $\angle M_1M_3M_2 = 60^\circ$.

Similarly, using (2), we have

$$\begin{aligned} \angle M_2BM_3 &= \angle M_2BC + \angle M_3BC \\ &= 2\angle M_2CB + 2\angle M_3CB - 180^\circ = 2\angle M_2CM_3 - 180^\circ \end{aligned}$$

This equation, in combination with $\angle M_2BM_3 + \angle M_2CM_3 = 180^\circ$, yields $\angle M_2CM_3 = 120^\circ$, and therefore, $\angle M_2M_1M_3 = 60^\circ$.

It follows that $\triangle M_1M_2M_3$ is equilateral.

2801. [2003 : 43] *Proposé par Heinz-Jürgen Seiffert, Berlin, Allemagne.*

Supposons que le triangle ABC n'aie pas d'angle obtus et soit a, b, et c ses côtés et R le rayon du cercle circonscrit. Montrer que

$$\left(\frac{2A}{\pi}\right)^{\frac{1}{a}} \left(\frac{2B}{\pi}\right)^{\frac{1}{b}} \left(\frac{2C}{\pi}\right)^{\frac{1}{c}} \le \left(\frac{2}{3}\right)^{\frac{\sqrt{3}}{R}} .$$

Quand l'égalité a-t-elle lieu ?

I. *Solution de Jacques Choné, Nancy, France.*

Comme A, B, C sont dans $(0, \pi/2]$ et que

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} ,$$

l'inégalité à démontrer est équivalente à

$$\frac{1}{2\sin A}\ln\frac{\pi}{2A}+\frac{1}{2\sin B}\ln\frac{\pi}{2B}+\frac{1}{2\sin C}\ln\frac{\pi}{2C} \;\geq\; \sqrt{3}\ln\frac{3}{2}\,.$$

Soit f la fonction définie sur $(0,\pi/2]$ par $f(x)=u(x)v(x)$, où

$$u(x)=\frac{1}{2\sin x}\qquad\text{et}\qquad v(x)=\ln\frac{\pi}{2x}=\ln\frac{\pi}{2}-\ln x\,.$$

Donc $f''=uv''+2u'v'+u''v$. On démontre facilement que, sur $(0,\pi/2]$, les fonctions u, v, u'', v'' sont positives et les fonctions u', v' négatives. Donc la fonction f'' est positive et la fonction f convexe. D'après l'inégalité de Jensen,

$$f(A)+f(B)+f(C) \;\geq\; 3f\left(\frac{A+B+C}{3}\right) \;=\; 3f\left(\frac{\pi}{3}\right) \;=\; \sqrt{3}\ln\frac{3}{2}\,,$$

avec égalité si et seulement si $A=B=C=\pi/3$.

On en déduit l'inégalité désirée avec égalité si et seulement si le triangle est équilatéral.

II. *Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

The following proof works for an arbitrary triangle, with no restriction on the angles.

By the AM–HM Inequality,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \;\geq\; \frac{9}{a+b+c}\,.$$

It is well known that $a+b+c\leq 3\sqrt{3}R$. Hence,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \;\geq\; \frac{\sqrt{3}}{R}\,. \tag{1}$$

Also by the AM–HM Inequality,

$$\frac{1}{3}\left(\frac{\pi}{2A}+\frac{\pi}{2B}+\frac{\pi}{2C}\right) \;\geq\; \frac{3}{\frac{2}{\pi}(A+B+C)} \;=\; \frac{3}{2}\,. \tag{2}$$

Letting $f(x)=\ln\frac{\pi}{2x}$, we have $f''(x)=\frac{1}{x^2}>0$. Therefore f is convex. By Jensen's Inequality,

$$\frac{1}{3}\left(\ln\frac{\pi}{2A}+\ln\frac{\pi}{2B}+\ln\frac{\pi}{2C}\right) \;\geq\; \ln\left[\frac{1}{3}\left(\frac{\pi}{2A}+\frac{\pi}{2B}+\frac{\pi}{2C}\right)\right]\,.$$

Using (2) and noting that $\ln x$ is strictly increasing, we have

$$\frac{1}{3}\left(\ln\frac{\pi}{2A}+\ln\frac{\pi}{2B}+\ln\frac{\pi}{2C}\right) \;\geq\; \ln\frac{3}{2}\,. \tag{3}$$

Without loss of generality, we may suppose that $a \leq b \leq c$, which implies that $A \leq B \leq C$. Then $\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c}$ and $\ln\frac{\pi}{2A} \geq \ln\frac{\pi}{2B} \geq \ln\frac{\pi}{2B}$. By Chebyshev's Inequality,

$$\frac{1}{a}\ln\frac{\pi}{2A}+\frac{1}{b}\ln\frac{\pi}{2B}+\frac{1}{c}\ln\frac{\pi}{2C} \;\geq\; \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{\ln\frac{\pi}{2A}+\ln\frac{\pi}{2B}+\ln\frac{\pi}{2C}}{3}\right).$$

Then, using (1) and (3),

$$\frac{1}{a}\ln\frac{\pi}{2A}+\frac{1}{b}\ln\frac{\pi}{2B}+\frac{1}{c}\ln\frac{\pi}{2C} \;\geq\; \frac{\sqrt{3}}{R}\ln\frac{3}{2}.$$

By exponentiating and taking reciprocals, we obtain the desired inequality.

In all of the inequalities above, equality holds if and only if $a = b = c$ (that is, $A = B = C$). Thus, equality holds for equilateral triangles only.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Maisons-Laffitte, France; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Three solutions were incorrect or incomplete.

2802. [2003 : 44] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Four positive integers, a, b, c, d, are said to have property $\mathcal{PS}$ if all of $bc + cd + db$, $ac + cd + da$, $ab + bd + da$, and $ab + bc + ca$ are $\mathcal{P}$erfect $\mathcal{S}$quares.

Suppose that the positive integers m, p, q, and r satisfy $p \leq q \leq r$ and $pq + qr + rp = m^2$. Let $s = p + q + r + 2m$.

Prove that p, q, r, and s have property $\mathcal{PS}$.

Solution by Ateneo Problem-Solving Group, Ateneo de Manila University, The Philippines.

We have

$$\begin{aligned} pq + qs + sp &= pq + (p+q)(p+q+r+2m) \\ &= (pq+qr+rp) + (p+q)^2 + 2m(p+q) \\ &= m^2 + 2m(p+q) + (p+q)^2 \;=\; (m+p+q)^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} pr + rs + sp &= (m+p+r)^2 \\ \text{and}\quad qr + rs + sq &= (m+q+r)^2. \end{aligned}$$

Since $pq + qr + rp = m^2$, it follows that p, q, r, and s have property $\mathcal{PS}$.

Also solved by DIONNE T. BAILEY and ELSIE M. CAMPBELL, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; ROBERT BILINSKI, Outremont, QC; CHRISTOPHER BOWEN, Halandri, Greece; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; YIFEI CHEN, student, West Windsor Plainsboro High School North, Plainsboro, NJ, USA; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; NEVEN JURIČ, Zagreb, Croatia; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; GUSTAVO KRIMKER, Universidad, CAECE, Buenos Aires, Argentina; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; JOSEPH LING, University of Calgary, Calgary, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; BRADLEY LUDERMAN, Angelo State University, San Angelo, TX, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; MIHAÏ STOËNESCU, Bischwiller, France; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; STEFFEN WEBER, student, Martin-Luther-Universität, Halle, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

All the submitted solutions are essentially the same as the one featured above.

Diminnie, Hess, Johnson, Klamkin, Stoënescu, and Zvonaru all pointed out that the condition "$p \leq q \leq r$" is superfluous.

2803. [2003 : 44] *Proposé par I.C. Draghicescu, Bucarest, Roumanie.*

Soit $x_1, x_2, \ldots, x_n$ $(n > 2)$ des nombres réels tels que la somme de $n-1$ d'entre eux est plus grande que le nombre restant. On pose $s = \sum\limits_{k=1}^{n} x_k$. Montrer que

$$\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k} \;\geq\; \frac{s}{n-2}\,.$$

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

D'après les hypothèses, pour tout k, on a $s - 2x_k > 0$. De l'inégalité de Cauchy-Schwarz, on déduit alors que :

$$\left(\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k}\right)\left(\sum_{k=1}^{n}(s - 2x_k)\right) \geq \left(\sum_{k=1}^{n} x_k\right)^2 \;=\; s^2\,.$$

Or, $0 < \sum\limits_{k=1}^{n}(s - 2x_k) = ns - 2s$, d'oú

$$\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k} \;\geq\; \frac{s}{n-2}\,.$$

En outre résolu par ARKADY ALT, San Jose, CA, USA ; MICHEL BATAILLE, Rouen, France ; MIHÁLY BENCZE, Brasov, Roumanie ; YIFEI CHEN, étudiant, West Windsor Plainsboro High School North, Plainsboro, NJ, USA ; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA ; OVIDIU FURDUI, étudiant, Western Michigan University, Kalamazoo, *MI, USA ; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentine ; JOHN G. HEUVER, Grande Prairie, AB ; JOE HOWARD, Portales, NM, USA ; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Autriche ; D. KIPP JOHNSON, Beaverton, OR, USA ; MURRAY S. KLAMKIN, Université de l'Alberta, Edmonton, AB ; KEE-WAI LAU, Hong Kong, Chine ; DAVID LOEFFLER, étudiant, Trinity College, Cambridge, GB ; ECKARD SPECHT, Université Otto-von-Guericke, Magdeburg, Allemagne ; PANOS E. TSAOUSSOGLOU, Athènes, Grèce ; STEFFEN WEBER, étudiant, Martin-Luther-Universität, Halle, Allemagne ; PETER Y. WOO, Biola University, La Mirada, CA, USA ; LI ZHOU, Polk Community College, Winter Haven, FL, USA ; TITU ZVONARU, Bucarest, Roumanie ; et le proposeur.*

En plus de la solution en utilisant l'inégalité de Cauchy-Schwarz, plusieurs solutionneurs ont employé l'inégalité de Jensen.

Il y avaient plusieurs généralisations :

1. *Alt : Soit $x_1, \ldots, x_n$ des nombres réels et soit $y_1, \ldots, y_n$ des nombres réels positifs ; posons n, m, k soient entiers non négatifs, $n > 2$, m est pair, et k est impair, $m > k$, alors on a*
$$\sum_{i=1}^{n} \frac{x_i^m}{y_i^k} \geq \frac{1}{n^{m-k-1}} \cdot \frac{\left(\sum_{i=1}^n x_i\right)^m}{\left(\sum_{i=1}^n y_i\right)^k}$$
2. *Bencze : Soit x_i et $\lambda_{i,j}$ $(i = 1, 2, \ldots, n ; j = 1, 2, \ldots, m)$ des nombres réels positifs, alors on a*
$$\sum_{i=1}^{n} \frac{x_i^{m+1}}{\lambda_{i,1}\lambda_{i,2}\cdots\lambda_{i,m}} \geq \frac{\left(\sum_{i=1}^n x_i\right)^{m+1}}{\left(\sum_{i=1}^n \lambda_{i,1}\right)\left(\sum_{i=1}^n \lambda_{i,2}\right)\cdots\left(\sum_{i=1}^n \lambda_{i,m}\right)}$$
3. *Klamkin : Supposons que $s > rx_j$, $j = 1, 2, \ldots, n$, oú $n > r$, on a*
$$\sum_{k=1}^{n} \frac{x_k^{2m}}{s - rx_k} \geq \frac{s^{2m-1}}{(n-r)n^{2m-2}} .$$

2804. [2003 : 44] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands*

Given three non-concentric circles $\Gamma_j\ (M_j, R_j)$, let μ_j denote the power of a point P with respect to Γ_j.

Determine the locus of P if $2\mu_2 = \mu_1 + \mu_3$.

Solution by Michel Bataille, Rouen, France.

We discard the condition "non-concentric" for greater generality.

The condition $2\mu_2 = \mu_1 + \mu_3$ is successively equivalent to

$$2\left(PM_2^2 - R_2^2\right) = PM_1^2 - R_1^2 + PM_3^2 - R_3^2 ,$$
$$2PM_2^2 - \left(\overrightarrow{PM_2} + \overrightarrow{M_2M_1}\right)^2 - \left(\overrightarrow{PM_2} + \overrightarrow{M_2M_3}\right)^2 = 2R_2^2 - R_1^2 - R_3^2 ,$$
$$\overrightarrow{M_2P} \cdot \overrightarrow{M_2I} = k , \qquad (1)$$

where $k = \frac{1}{4}\left(2R_2^2 - R_1^2 - R_3^2 + M_2M_1^2 + M_2M_3^2\right)$ and I denotes the mid-point of M_1M_3.

Suppose first that $M_2 \neq I$. From (1), we deduce that the locus of P is the line perpendicular to M_2I through the point Q of M_2I defined by $\overrightarrow{M_2Q} = \frac{k}{M_2I^2}\overrightarrow{M_2I}$. If Γ_1, Γ_2 and Γ_3 have a radical centre C, then this point C belongs to the locus (since it has the same power with respect to the three circles), and the locus is the line perpendicular to M_2I through C [see Figure 1, where the locus of P is the dotted line].

In case $M_2 = I$, equation (1) reduces to $k = 0$, where now

$$k \;=\; \frac{1}{4}\left(2R_2^2 - R_1^2 - R_3^2 + 2M_2M_1^2\right) \;.$$

If $R_1^2 + R_3^2 \;=\; 2\left(M_2M_1^2 + R_2^2\right)$, then the locus of P is the whole plane; if $R_1^2 + R_3^2 \;\neq\; 2\left(M_2M_1^2 + R_2^2\right)$, then the locus is empty. [See Figure 2, where R_1, R_2, R_3, M_1M_2 have been taken from the auxiliary figure on the left according to $R_1 = AM$, $R_3 = A'M$, $M_1M_2 = AN$, $R_2 = NB$.]

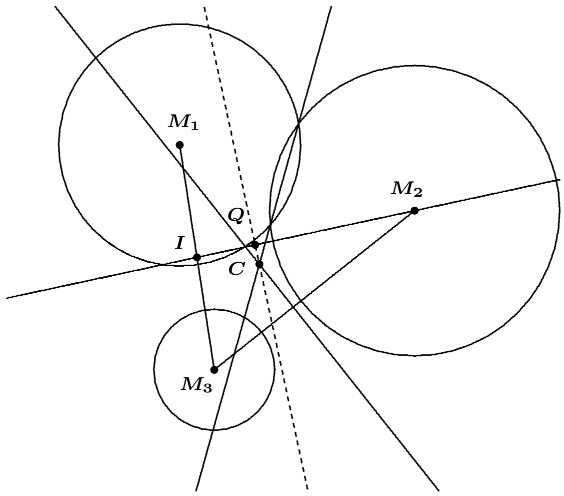


Figure 1

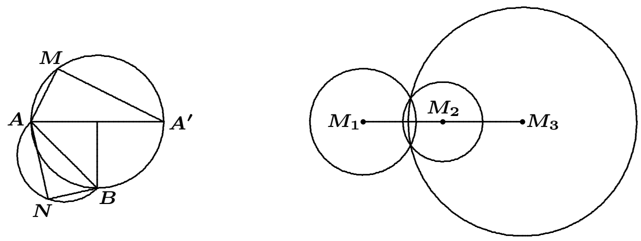


Figure 2

Also solved by CHRISTOPHER BOWEN, Halandri, Greece; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOHN G. HEUVER, Grande Prairie, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Woo also indicated that the locus is constructible with straightedge and compass, or with straightedge alone.

2805. [2003 : 45] *Proposed by Mihály Bencze, Brasov, Romania.*

Let k be a fixed positive integer. For all positive integers n, prove that there exist positive integers a_1, a_2, ..., a_n, such that $(n, a_n) = 1$ and

$$\sum_{j=1}^{n} \frac{j^k}{a_j} = 1 .$$

I. *Essentially the same solution by Michel Bataille, Rouen, France; Christopher Bowen, Halandri, Greece; Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.*

We use induction on n. For $n = 1$, we take $a_1 = 1$.

Suppose that, for some $n \geq 1$, positive integers a_1, a_2, ..., a_n have been found such that $(n, a_n) = 1$ and $\sum\limits_{j=1}^{n} \frac{j^k}{a_j} = 1$.

Let $b_{n+1} = (n+1)^k + 1$, and for each $j = 1, 2, \ldots, n$, let $b_j = a_j b_{n+1}$. Then clearly $(n + 1, b_{n+1}) = 1$ and

$$\sum_{j=1}^{n+1} \frac{j^k}{b_j} = \frac{1}{b_{n+1}} \left(\sum_{j=1}^{n} \frac{j^k}{a_j} + (n+1)^k \right) = \frac{1 + (n+1)^k}{b_{n+1}} = 1 .$$

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $a_n = n^k + n - 1$ and for $j = 1, 2, \ldots, n - 1$, let $a_j = a_n j^k$. Then clearly, $(n, a_n) = 1$ and

$$\sum_{j=1}^{n} \frac{j^k}{a_j} = \frac{n-1}{a_n} + \frac{n^k}{a_n} = 1 .$$

III. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

Evidently, the case $n = 1$ is trivial. For $n > 1$, let $a_n = n^k + 1$, and for $j = 1, 2, \ldots, n - 1$, let $a_j = (n - 1)a_n j^k$. Then $(n, a_n) = 1$ and

$$\sum_{j=1}^{n} \frac{j^k}{a_j} = \frac{n-1}{(n-1)a_n} + \frac{n^k}{a_n} = \frac{1 + n^k}{a_n} = 1 .$$

Loeffler considered the related problem by adding the conditions that $(j, a_j) = 1$ for all $j = 1, 2, \ldots, n - 1$, and he gave a proof for this much harder problem.

It is interesting to compare the values of the a_i's for the different solutions featured above. Bowen computed these values for $n = 1, 2, 3, 4$ when $k = 3$ (Solution I). When $n = 2$, all three solutions yield $(a_1, a_2) = (9, 9)$. For $n = 3$ and $n = 4$, the values of the a_i's obtained are given in the table below (for $k = 3$). It is apparent that Solution II given by Janous would yield much smaller values for the a_i's in general.

Solution	$n = 3$	$n = 4$
I	$(a_1, a_2, a_3) = (252, 252, 28)$	$(a_1, a_2, a_3, a_4) = (16380, 16380, 1820, 65)$
II	$(a_1, a_2, a_3) = (29, 232, 29)$	$(a_1, a_2, a_3, a_4) = (67, 536, 1809, 67)$
III	$(a_1, a_2, a_3) = (56, 448, 28)$	$(a_1, a_2, a_3, a_4) = (195, 1560, 5265, 65)$

2806. [2003 : 45] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that x, y, $z > 0$, $\alpha \in \mathbb{R}$ and $x^\alpha + y^\alpha + z^\alpha = 1$. Prove that

(a) $x^2+y^2+z^2 \;\geq\; x^{\alpha+2}+y^{\alpha+2}+z^{\alpha+2}+2x^2y^2z^2\left(x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}\right)$,

(b) $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2} \;\geq\; x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}+\dfrac{2\left(x^{\alpha+1}+y^{\alpha+1}+z^{\alpha+1}\right)}{xyz}$.

Solution by Arkady Alt, San Jose, CA, USA; David Loeffler, student, Trinity College, Cambridge, UK; and Panos E. Tsaoussoglou, Athens, Greece.

(a) We will show that the proposed inequality is true for $\alpha > 0$ and the reverse inequality holds for $\alpha < 0$. For $\alpha = 0$, the given condition $x^\alpha + y^\alpha + z^\alpha = 1$ is never satisfied.

Let D be the difference between the left side and the right side of the given inequality. Using $x^\alpha + y^\alpha + z^\alpha = 1$, we obtain

$$\begin{aligned} D &= (x^2+y^2+z^2) - x^{\alpha+2} - y^{\alpha+2} - z^{\alpha+2} \\ &\qquad - 2x^\alpha y^2z^2 - 2y^\alpha z^2x^2 - 2z^\alpha x^2y^2 \\ &= (x^2+y^2+z^2)(x^\alpha+y^\alpha+z^\alpha) - x^{\alpha+2} - y^{\alpha+2} - z^{\alpha+2} \\ &\qquad - 2x^\alpha y^2z^2 - 2y^\alpha z^2x^2 - 2z^\alpha x^2y^2 \\ &= x^\alpha(y^2+z^2) - 2x^\alpha y^2z^2 + y^\alpha(z^2+x^2) \\ &\qquad - 2y^\alpha z^2x^2 + z^\alpha(x^2+y^2) - 2z^\alpha x^2y^2 \\ &= x^\alpha[y^2(1-z^2)+z^2(1-y^2)] + y^\alpha[z^2(1-x^2)+x^2(1-z^2)] \\ &\qquad + z^\alpha[x^2(1-y^2)+y^2(1-x^2)]\,. \end{aligned}$$

Now, if $\alpha > 0$, then the condition $x^\alpha + y^\alpha + z^\alpha = 1$ implies that $x < 1$, $y < 1$, and $z < 1$, so that $D \geq 0$ and the inequality holds. If $\alpha < 0$, then $x > 1$, $y > 1$, and $z > 1$ and the reverse inequality holds.

(b) The desired inequality follows easily by applying the condition $x^\alpha + y^\alpha + z^\alpha = 1$ and the AM–GM Inequality:

$$\begin{aligned} \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} &= \left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\cdot(x^\alpha+y^\alpha+z^\alpha) \\ &= x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}+x^\alpha\left(\frac{1}{y^2}+\frac{1}{z^2}\right) \\ &\qquad + y^\alpha\left(\frac{1}{z^2}+\frac{1}{x^2}\right)+z^\alpha\left(\frac{1}{x^2}+\frac{1}{y^2}\right) \\ &\geq x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2} \\ &\qquad + x^\alpha\left(\frac{2}{yz}\right)+y^\alpha\left(\frac{2}{zx}\right)+z^\alpha\left(\frac{2}{xy}\right) \\ &= x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}+\frac{2(x^{\alpha+1}+y^{\alpha+1}+z^{\alpha+1})}{xyz}\,. \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France: OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; BEN HARWOOD, student, Northern Kentucky University, KY, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania. Part (b) only was solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2807. [2003 : 45] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In $\triangle ABC$, denote its area by $[ABC]$ (and its semi-perimeter by s). Show that

$$\min\left\{\frac{2s^4-\left(a^4+b^4+c^4\right)}{[ABC]^2}\right\} \;=\; 38\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Without loss of generality, assume that $a \ge b \ge c$. Then

$$\begin{aligned}
8&\left(2s^4-(a^4+b^4+c^4)-38[ABC]^2\right)\\
&= (a+b+c)^4-8(a^4+b^4+c^4)\\
&\qquad -19(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)\\
&= c(a+b-2c)^2(4a+4b+3c)\\
&\qquad +(a-b)^2(12a^2+12b^2+28ab-19c^2)\\
&\ge 0\,.
\end{aligned}$$

We get equality when $a=b=c$. Note that this proof does not require that a, b, and c are the sides of a triangle, as long as they are non-negative.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France: CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; NEVEN JURIČ, Zagreb, Croatia; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; STEFFEN WEBER, student, Martin-Luther-Universität, Halle, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer. There were also two incorrect solutions submitted.

Janous has proposed the following generalization: Let x, y, and z be non-negative real numbers. Prove or disprove that the following inequality is valid for all natural numbers $n \ge 1$:

$$2(x+y+z)^{4n}-(y+z)^{4n}-(z+x)^{4n}-(x+y)^{4n} \;\ge\; \frac{2\cdot 3^{4n}-3\cdot 2^{4n}}{3^n}\cdot[xyz(x+y+z)]^n\,.$$

He has proved the claim for $n=1$, 2, and 3. The original inequality can be obtained from Janous' inequality by the substitution $y+z=a$, $z+x=b$, $x+y=c$, for $n=1$. Perhaps our readers can try their hand at this generalization.

2808. [2003 : 46] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In $\triangle ABC$, we have $b < c$ and $a\left(3b^2+c^2-a^2\right) = 2b\left(c^2-b^2\right)$. Determine the ratio $a:b:c$.

Solution by Michel Bataille, Rouen, France.

Let $\lambda = \frac{a}{b}$ and $\mu = \frac{c}{b}$. Dividing both sides of the given relation by b^3, we obtain

$$\lambda(3+\mu^2-\lambda^2) \;=\; 2(\mu^2-1)\,,$$

or

$$(\lambda-2)(\mu-\lambda-1)(\mu+\lambda+1) \;=\; 0\,.$$

Now, $\mu+\lambda+1 > 0$, and $\mu = \lambda+1$ implies that $a+b = c$, which is impossible for a (non-degenerate) triangle. Thus, we must have $\lambda = 2$. Furthermore, $1 < \mu < 3$, because $b < c$ and $c < a+b$. It follows that $a:b:c = 2:1:\mu$ for some μ such that $1 < \mu < 3$.

Conversely, if $a:b:c = 2:1:\mu$, where $1 < \mu < 3$, then $b < c < 3b$ and $a = 2b$. Hence, $|c-b| < a < b+c$, so that ABC is a triangle, and the relation

$$a(3b^2+c^2-a^2) \;=\; 2b(c^2-b^2)$$

holds, as it is readily seen.

Also solved by ATENEO PROBLEM-SOLVING GROUP, Ateneo de Manila University, The Philippines; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK: JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; NEVEN JURIČ, Zagreb, Croatia; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania. There were also three incorrect solutions submitted.

2809. [2003 : 46] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $k \geq 2$ is a fixed integer. For each non-negative integer n, let x_n denote the leftmost digit of n^k.

Prove that the number $0.x_0x_1x_2\ldots x_n\ldots$ is irrational.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

As is well known, a number is rational if and only if its decimal expansion is eventually periodic. We shall show that this sequence of digits contains arbitrarily long blocks of the digit 1; this clearly implies that it cannot be periodic unless it eventually consists entirely of 1's, which is evidently not the case.

Clearly, the first digit of n^k will be 1 if, for some positive integer r,

$$10^{r/k} \;\le\; n < 2^{1/k} \times 10^{r/k} \,.$$

The number of consecutive integers n that satisfy these inequalities, for fixed r, is at least $(2^{1/k} - 1)10^{r/k} - 1$, which tends to ∞ as r increases. This gives us arbitrarily long blocks of 1's, and the result follows. (It may be shown similarly that there are arbitrarily long blocks of every other digit.)

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER BOWEN, Halandri, Greece; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; IGNACIO LOFEUDO, Universidad CAECE, Buenos Aires, Argentina; TREY SMITH, Angelo State University, San Angelo, TX, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Loeffler comments "It seems overwhelmingly likely that this number is transcendental. Anyone fancy trying to prove this?"

Walther Janous comments that the non-periodicity of the leftmost digits of n^k has been known since Euler's time. However, the rightmost digits are periodic. Indeed, the m^{th} digit from the right is periodic. What about the m^{th} digit from the left?

2810. [2003 : 46] *Proposed by I.C. Draghicescu, Bucharest, Romania.*

Suppose that a, b and $x_1, x_2, \ldots, x_n$ $(n \ge 2)$ are positive real numbers. Let $s = \sum\limits_{k=1}^{n} x_k$. Prove that

$$\prod_{k=1}^{n} \left(a + \frac{b}{x_k}\right) \;\ge\; \left(a + \frac{nb}{s}\right)^n \,.$$

I. *Composite of virtually identical solutions by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Paul Bracken, University of Texas, Edinburg, TX, USA; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; Joe Howard, Portales, NM, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Murray S. Klamkin, University of Alberta, Edmonton, AB; Kee-Wai Lau, Hong Kong, China; David Loeffler, student, Trinity College, Cambridge, UK; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let $f(x) = \ln\left(a + \frac{b}{x}\right)$. Then

$$f'(x) \;=\; \frac{-b}{ax^2 + bx} \qquad \text{and} \qquad f''(x) \;=\; \frac{b(2ax+b)}{(ax^2+bx)^2} \,.$$

Since $f''(x) > 0$ for all $x > 0$, we have, by Jensen's Inequality,

$$\sum_{k=1}^{n} f(x_k) \;\ge\; nf\left(\frac{s}{n}\right) \,,$$

or

$$\sum_{k=1}^{n} \ln\left(a+\frac{b}{x_k}\right) \;\ge\; n\ln\left(a+\frac{nb}{s}\right)\,.$$

The given inequality then follows by exponentiation.

II. *Solution by Titu Zvonaru, Bucharest, Romania.*

We use the following known inequality, valid for non-negative real numbers $a_1, a_2, \ldots, a_n$:

$$(1+a_1)(1+a_2)\cdots(1+a_n) \;\ge\; (1+\sqrt[n]{a_1a_2\cdots a_n})^n\,. \qquad (1)$$

We have

$$\begin{aligned}\prod_{k=1}^{n}\left(a+\frac{b}{x_k}\right) &= a^n\prod_{k=1}^{n}\left(1+\frac{b}{ax_k}\right) \;\ge\; a^n\left(1+\left(\prod_{k=1}^{n}\frac{b}{ax_k}\right)^{\frac{1}{n}}\right)^n \\ &= a^n\left(1+\frac{b}{a}\left(\prod_{k=1}^{n}x_k\right)^{-\frac{1}{n}}\right)^n \;=\; \left(a+b\left(\prod_{k=1}^{n}x_k\right)^{-\frac{1}{n}}\right)^n \\ &\ge \left(a+\frac{b}{s/n}\right)^n \;=\; \left(a+\frac{nb}{s}\right)^n\,,\end{aligned}$$

where the last line follows by the AM–GM Inequality.

[*Ed*: The inequality (1) is on page 61 of *Inequalities* by G.H. Hardy, J.E. Littlewood, and G. Pólya. Zvonaru calls this the Huygen's Inequality, and another solver, Chip Curtis, pointed out that it can be found in *The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics* by D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom.]

III. *Solution by Steffen Weber, student, Martin-Luther-Universität, Halle, Germany.*

Note first that the proposed inequality is equivalent to

$$\prod_{k=1}^{n}\left(c+\frac{1}{x_k}\right) \;\ge\; \left(c+\frac{n}{s}\right)^n\,,$$

where $c = a/b$. This in turn is equivalent to

$$\prod_{k=1}^{n}\left(\frac{cs+\frac{s}{x_k}}{n+cs}\right) \;\ge\; 1\,,$$

or

$$\prod_{k=1}^{n}\left(1+\frac{-n+\frac{s}{x_k}}{n+cs}\right) \;\ge\; 1\,. \qquad (1)$$

To prove (1), we will use Bernoulli's Inequality, which states that, for $\lambda_k \geq -1$ $(k = 1, 2, \ldots, n)$,

$$\prod_{k=1}^{n} (1 + \lambda_k) \;\geq\; 1 + \sum_{k=1}^{n} \lambda_k \,.$$

For each $k = 1, 2, \ldots, n$, we have $-n + \frac{s}{x_k} \geq -n - cs$, which implies that

$$\frac{-n + \dfrac{s}{x_k}}{n + cs} \;\geq\; -1 \,.$$

Applying Bernoulli's Inequality, we have

$$\begin{aligned} \prod_{k=1}^{n} \left(1 + \frac{-n + \frac{s}{x_k}}{n + cs} \right) \;&\geq\; 1 + \sum_{k=1}^{n} \frac{-n + \frac{s}{x_k}}{n + cs} \\ &=\; 1 + \frac{1}{n + cs} \left(-n^2 + \sum_{k=1}^{n} \frac{s}{x_k} \right) \,. \end{aligned} \tag{2}$$

By the AM–HM Inequality, we have

$$\frac{s}{n} \;\geq\; \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{x_k} \right)^{-1} \,,$$

and hence, $\sum\limits_{k=1}^{n} \dfrac{s}{x_k} \geq n^2$. Using this result in (2), we obtain (1).

Also solved by ARKADY ALT, San Jose, CA, USA; MIHÁLY BENCZE, Brasov, Romania; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; JOHN G. HEUVER, Grande Prairie, AB; D. KIPP JOHNSON, Beaverton, OR, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania (a second solution); and the proposer.

Using exactly the same argument as in I above, Janous obtained the slightly stronger result that

$$\sum_{k=1}^{n} \left(a + \frac{b}{x_k^{\alpha}} \right) \;\geq\; \left(a + \frac{n^{\alpha} b}{s^{\alpha}} \right)^{n}$$

for all $\alpha > 0$. Bencze cited four results from his earlier papers which are all generalizations of the given inequality. Interested readers might want to contact him directly for these results.

2811. [2003 : 46] *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy, for all real x,

$$f\left(x^3 + x\right) \ \le\ x \ \le\ f^3(x) + f(x)\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Note that the function $x \to x^3 + x$ is an increasing bijection. We prove that, more generally, for any increasing bijection $g : \mathbb{R} \to \mathbb{R}$, the function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(g(x)) \ \le\ x \ \le\ g(f(x))$$

for all $x \in \mathbb{R}$ if and only if $f = g^{-1}$.

Indeed, $f = g^{-1}$ certainly satisfies the functional inequalities (as equalities, in fact). Conversely, if f satisfies the functional inequalities, then

$$f(x) \ = \ f(g(g^{-1}(x))) \ \le\ g^{-1}(x) \ \le\ g^{-1}(g(f(x))) \ = \ f(x)\,.$$

Thus, $f = g^{-1}$ for all $x \in \mathbb{R}$.

In our special case, with $g(x) = x^3 + x$, we have

$$f(x) \ = \ g^{-1}(x) \ = \ \sqrt[3]{\frac{x}{2} + \sqrt{\frac{x^2}{4} + \frac{1}{27}}} + \sqrt[3]{\frac{x}{2} - \sqrt{\frac{x^2}{4} + \frac{1}{27}}}$$

by Cardano's Formula.

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER BOWEN, Halandri, Greece; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; NOVALD MIDTTUN, Royal Norwegian Naval Academy, Bergen, Norway; TITU ZVONARU, Bucharest, Romania; and the proposer.

2812. [2003 : 46] *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all injective functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$(2a + b)f(ax + b) \ \ge\ a\,f^2\left(\frac{1}{x}\right) \ +\ b\,f\left(\frac{1}{x}\right) \ +\ a$$

for all positive real x, where a, $b \in \mathbb{R}$, $a > 0$, $a^2 + 4b > 0$ and $2a + b > 0$.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

It is readily seen that the condition is equivalent to

$$f\left(\frac{a}{x} + b\right) \ \ge\ f(x) + \frac{a}{2a + b}(f(x) - 1)^2$$

for all positive real x.

Let us define $g(x) = \frac{a}{x} + b$. The equation $g(x) = x$ is equivalent to the quadratic $x^2 - bx - a = 0$, which has the roots $x = \frac{b \pm \sqrt{b^2 + 4a}}{2}$. These are real and distinct if $b^2 + 4a > 0$, which is evidently true, since $a > 0$; and they always differ in sign. Let the positive and negative roots be denoted by x_+ and x_-, respectively. Then

$$f(x_+) \;\geq\; f(x_+) + \frac{a}{2a+b}(f(x_+) - 1)^2\,,$$

which implies that $f(x_+) = 1$, since otherwise $\frac{a}{2a+b}(f(x_+) - 1)^2 > 0$.

If we were allowed to assume that the inequality in the question is true for all $x \in \mathbb{R}$, then, by the same reasoning, we would have $f(x_-) = 1$ as well—a contradiction, since f is injective. However, since this not one of the assumptions of the problem, we cannot use this result.

[*Ed*: In fact, the proposer stated the problem with the inequality holding for all non-zero $x \in \mathbb{R}$. The editorial staff incorrectly typeset the problem. The solver is to be commended for anticipating the original problem (and solution), and for continuing with the solution below.]

What happens if we compose g repeatedly with itself?

We know that the fixed points are $x_\pm$; let us study whether or not these are stable. It is well known that a fixed point x is stable if $|g'(x)| < 1$. We have $g'(x) = -a/x^2$. Three cases now arise.

Case (a): $b = 0$. In this case we have $x_\pm = \pm\sqrt{a}$, and both fixed points are neutral (neither attractive nor repulsive). This is not surprising, since we then have $g^2(x) = x$ for all x. Now, since

$$f(x) \;=\; f(g^2(x)) \;\geq\; f(g(x)) \;\geq\; f(x) + \frac{a}{2a+b}(f(x) - 1)^2\,,$$

it follows that $f(x) = 1$ for all $x > 0$, which is a contradiction, since f is injective. Hence, no such functions exist.

Case (b): $b > 0$. In this case, since $x_+ + x_- = b > 0$ and $x_+x_- = -a$, we see that we must have $|x_+| > \sqrt{a} > |x_-|$. It follows that $|g'(x_+)| < 1$ and $|g'(x_-)| > 1$; hence, x_+ is attractive and x_- is repulsive.

Case(c): $b < 0$. In this case, similarly, we see that x_+ is repulsive and x_- is attractive.

Thus, for $b \neq 0$, no iterate of g can be the identity, as it must still have one attractive and one repulsive fixed point. However, since g^k is a Möbius transformation, g^k cannot have more that two fixed points; but we know that it fixes $x_\pm$. Hence, for any $u \neq x_\pm$, the doubly infinite sequence $g^k(x)$ is not periodic.

Let us partition the reals into equivalence classes, where u and v are related if $u = g^k(v)$ for some $k \in \mathbb{Z}$; thus, the equivalence classes are the orbits under g. Evidently, this gives us the singletons $\{x_+\}$ and $\{x_-\}$ and $\mathfrak{c}$ countable sequences of the form $\{g^k(u_0) : k \in \mathbb{Z}\}$ (where $\mathfrak{c}$ is the cardinality of $\mathbb{R}$). We can define $f(x)$ independently on each equivalence class, so long as we ensure that it is injective.

By the Well-Ordering Principle, there exists a well-ordering $\prec$ on the set of equivalence classes, with $\{x_+\}$ and $\{x_-\}$ as the first two elements. Suppose α is an equivalence class, and we have defined f on all classes $\beta \prec \alpha$ in such a way that f is injective and satisfies the given inequality for the domain on which it has so far been defined, with $f(x_+) = 1$.

Pick an arbitrary $u_0 \in \alpha$. We have so far defined $f(v)$ on a set of cardinality $< \mathfrak{c}$, so we may select some $y_0 > 1$ which is not equal to any of the values so far assigned for f, and set $f(u_0) = y_0$. We may also select a $y_1 > y_0 + \frac{a}{2a+b}(y_0 - 1)^2$ and declare it to be $f(u_1)$, where $u_1 = g(u_0)$. Similarly, we may define $f(u_k)$ for all $k > 0$, all the while preserving the injectivity.

For u_{-1}, we note that there is some $z_0 > 1$ such that

$$z_0 + \frac{a}{2a+b}(z_0 - 1)^2 \ < \ y_0 \,,$$

and since there are uncountably many reals in $(1, z_0)$, we may pick one of them to be $f(u_{-1})$. We may repeat the process to obtain $f(u_k)$ for all $k < 0$ in such a way that f remains injective. Then we have successfully defined f on the equivalence class α.

Hence, by transfinite induction, we can define f for all $x \in \mathbb{R}$.

Therefore, there are a large infinity of functions (at least $\mathfrak{c}$ of them) satisfying the given requirements. (We note in passing that they, in fact, satisfy the given inequality for all $x \neq x_-$.)

2813. [2003 : 47] *Proposed by Barry R. Monson, University of New Brunswick, Fredericton, NB and J. Chris Fisher, University of Regina, Regina, SK.*

Suppose that M is the mid-point of side AB of the square $ABCD$. Let P and Q be the points of intersection of the line MD with the circle, centre M, radius MA ($= MB$), where P is inside the square $ABCD$ and Q is outside.

Prove that rectangle $APBQ$ is a golden rectangle; that is,

$$PB : PA = (\sqrt{5} + 1) : 2 \,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Without loss of generality, we may assume that $MA = 1$. Then, we observe that $AD = 2$, $MB = MP = MA = 1$, $MD = \sqrt{5}$, and $\cos\angle AMP = \cos\angle AMD = \frac{1}{\sqrt{5}}$. By the Law of Cosines, we have

$$PA^2 \ = \ PM^2 + AM^2 - 2PM \cdot AM \cos\angle AMP \ = \ \frac{2(\sqrt{5}-1)}{\sqrt{5}} \,,$$

and

$$PB^2 = PM^2 + BM^2 - 2PM \cdot BM \cos(180^\circ - \angle AMP) = \frac{2(\sqrt{5}+1)}{\sqrt{5}}.$$

Hence,

$$\left(\frac{PB}{PA}\right)^2 = \frac{\sqrt{5}+1}{\sqrt{5}-1} = \frac{(\sqrt{5}+1)^2}{4},$$

which gives $PB/PA = (\sqrt{5}+1)/2$.

Also solved by ATENEO PROBLEM-SOLVING GROUP, Ateneo de Manila University, The Philippines; MICHEL BATAILLE, Rouen, France: FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MIHÁLY BENCZE, Brasov, Romania; ROBERT BILINSKI, Outremont, QC; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER BOWEN, Halandri, Greece; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; YIFEI CHEN, student, West Windsor Plainsboro High School North, Plainsboro, NJ, USA; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; PAOLO CUSTODY, Fara Novarese, Italy; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; NEVEN JURIČ, Zagreb, Croatia; GEOFFREY A. KANDALL, Hamden, CT, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina (two solutions); MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; Mª JESÚS VILLAR RUBIO, Santander, Spain; BOB SERKEY, Leonia, NJ, USA; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; MIHAÏ STOËNESCU, Bischwiller, France; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania (two solutions); and the proposers.