SKOLIAD No. 74

Shawn Godin

Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6, or emailed to

mayhem-editors@cms.math.ca.

We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 June 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 8** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

This issue we present the 2003 Manitoba Mathematical Contest, for students in Senior 4. It was sponsored by the Winnipeg Actuaries Club, the Manitoba Association of Mathematics Teachers, the Canadian Mathematical Society, and the University of Manitoba.

Manitoba Mathematical Contest 2003
Concours mathématique du Manitoba 2003
(Senior 4 / 12ième année)

1. (a) Résoud l'équation $\frac{1}{x} + \frac{1}{x+2} = \frac{1}{x^2+2x}$.

(b) Si a et b sont des nombres réels différents de zéro tels que $9a^2 - 12ab + 4b^2 = 0$, trouve la valeur numérique de a/b.

. . . .

(a) Solve the equation $\frac{1}{x} + \frac{1}{x+2} = \frac{1}{x^2+2x}$.

(b) If a and b are non-zero real numbers such that $9a^2 - 12ab + 4b^2 = 0$, find the numerical value of a/b.

2. (a) Aujourd'hui le fils de Joseph a $\frac{1}{3}$ de l'âge de Joseph. Cinq ans auparavant, il avait $\frac{1}{4}$ de l'âge de Joseph à ce moment. Quel âge a le fils de Joseph maintenant ?

(b) a et b sont des nombres réels différent de zéro. Si l'équation $ax^2 + bx + 8 = 0$ a exactement une solution, trouve la valeur numérique de b^2/a.

(a) Today, Joe's son is $\frac{1}{3}$ of Joe's age. Five years ago he was $\frac{1}{4}$ of Joe's age at that time. How old is Joe's son?

(b) a and b are non-zero real numbers. If the equation $ax^2+bx+8=0$ has exactly one solution, find the numerical value of b^2/a.

3. (a) Un rectangle a une diagonale de longueur 5, et sa longueur est le double de sa largeur. Quelle est son aire ?

(b) Si le périmètre d'un triangle-rectangle isocèle est 8, quelle est son aire ?

. . . .

(a) A rectangle is twice as long as it is wide and has a diagonal of length 5. What is its area?

(b) If the perimeter of an isosceles right-angled triangle is 8, what is its area?

4. (a) Trouve la longueur du diamètre d'un cercle dont laire est triplée lorsque la longueur du rayon est augmentée de 2.

(b) Si a et b sont des nombres réels tels que $3\,(2^a) + 2^b = 7\sqrt{2}$ et $5\,(2^a) - 2^b = 9\sqrt{2}$, trouve a et b.

. . . .

(a) Find the length of the diameter of a circle whose area is tripled when the length of its radius is increased by 2.

(b) If a and b are real numbers such that $3\,(2^a) + 2^b = 7\sqrt{2}$ and $5\,(2^a) - 2^b = 9\sqrt{2}$, find a and b.

5. (a) Si $\sec\theta + 9\cos\theta = 6$, quelle est la valeur numérique de $\sec\theta$?

(b) Le point A se trouve sur la droite dont l'équation est $y = x$. Le point B se trouve sur la droite dont l'équation est $y = -x$. Le segment de droite $\overline{AB}$ est de longueur 2. Prouve que le point au milieu du segment de droite $\overline{AB}$ se trouve sur un cercle de rayon 1 centré à l'origine.

. . . .

(a) If $\sec\theta + 9\cos\theta = 6$, what is the numerical value of $\sec\theta$?

(b) The point A lies on the line whose equation is $y = x$. The point B lies on the line whose equation is $y = -x$. The line segment $\overline{AB}$ has length 2. Prove that the mid-point of the segment $\overline{AB}$ lies on a circle of radius 1 with centre at the origin.

6. (a) Dans ce problème, O est l'origine, A est le point $(3, 1)$ et P est un point dans le premier quadrant qui se trouve sur le graphe de $3x-4y=0$. Si $\angle APO = 45^\circ$, trouve l'aire du triangle AOP.

(b) Si r, s et t sont des nombres réels tels que $r - 2s + 3t \geq 2$ et $2r + s - 3t \geq 1$, prouve que $7r - 4s + 3t \geq 8$.

(a) In this problem, O is the origin, A is the point $(3, 1)$ and P is a point in the first quadrant on the graph of $3x - 4y = 0$. If $\angle APO = 45°$, find the area of triangle AOP.

(b) If r, s, and t are real numbers such that $r - 2s + 3t \geq 2$ and $2r + s - 3t \geq 1$, prove that $7r - 4s + 3t \geq 8$.

7. L'aire d'un triangle-rectangle est 5. La hauteur perpendiculaire à l'hypoténuse est de longueur 2. Trouve les longueurs des trois côtés du triangle.

.·.

A right-angled triangle has area 5. The altitude perpendicular to the hypotenuse has length 2. Find the lengths of the three sides of the triangle.

8. A et B sont des points sur le graphe de l'équation

$$(x^2 + y^2 - 1)\left\{(x-1)^2 + (y-1)^2 - 2\right\} = 0\,.$$

Quelle est la plus grande valeur possible pour la longueur du segment $\overline{AB}$? Prove que ta réponse est juste.

.·.

A and B are points on the graph of the equation

$$(x^2 + y^2 - 1)\left\{(x-1)^2 + (y-1)^2 - 2\right\} = 0\,.$$

What is the largest possible value for the length of the line segment $\overline{AB}$? Prove that your answer is correct.

9. Dans ce problème, O est l'origine ; P est un point sur le graphe de $y = x^2$; les coordonnées de P sont des entiers différents de zéro. Prouve que la longueur du segment de droite $\overline{OP}$ ne peut pas être un entier.

.·.

In this problem O is the origin; P is a point on the graph of $y = x^2$; the coordinates of P are non-zero integers. Prove that the length of the line segment $\overline{OP}$ cannot be an integer.

10. (a) Résoud l'équation $x^4 - 6x^2 + 9 = (x+1)^2$.

(b) Résoud l'équation $x^4 - 7x^2 = 4x - 5$.

.·.

(a) Solve the equation $x^4 - 6x^2 + 9 = (x+1)^2$.

(b) Solve the equation $x^4 - 7x^2 = 4x - 5$.

That ends Skoliad for another year. Continue to send me your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, 2191 Saturn Cres., Orleans, ON, K4A 3T6**. The electronic address is

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The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Problems

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N'oubliez pas d'inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d'étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le *premier juin 2004*. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M119. *Proposé par l'Équipe de Mayhem.*

André, Bernard et Charles jouent à un jeu qui oblige le perdant à tripler l'argent que les autres joueurs possédaient en début de partie. On a joué trois fois et le perdant a été successivement André, puis Bernard et, finalement, Charles. Chaque joueur finit avec **27$** en poche ; combien d'argent avait-il avant de commencer ?

Andrew, Bernard, and Charles play a game in which the loser has to triple the money of each other player. Three games are played, and the successive losers are Andrew, then Bernard, and finally, Charles. Each player ends with $27. How much money did each person have at the outset?

M120. *Proposé par l'Équipe de Mayhem.*

Résoudre l'équation

$$2^{1+3\cos x} - 10 \times 2^{-1+2\cos x} + 2^{2+\cos x} - 1 \;=\; 0$$

par rapport à x, si $0 \leq x < 2\pi$.

. . . .

Solve for x, where $0 \leq x < 2\pi$:

$$2^{1+3\cos x} - 10 \times 2^{-1+2\cos x} + 2^{2+\cos x} - 1 \;=\; 0\,.$$

M121. *Proposé par l'Équipe de Mayhem.*

Utiliser chacun des 9 chiffres 1, 2, 3, 4, 5, 6, 7, 8, 9 une seule fois pour former des nombres premiers dont la somme soit la plus petite possible.

. . . .

Use each of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to form prime numbers whose sum is as small as possible.

M122. *Proposé par l'Équipe de Mayhem.*

Dans une certaine province, les plaques minéralogiques comportent exactement trois lettres suivies de trois chiffres. Il paraît qu'il en coûte $n$$ pour chaque chiffre ($n > 0$) et 10$ pour chaque zéro. De plus, le coût des lettres varie proportionnellement à leur position dans l'alphabet, partant de 1$ pour A, 2$ pour B, jusqu'à 26$ pour Z.

(a) Calculer le coût de production d'un lot complet de plaques, de AAA 000 à ZZZ 999.

(b) Trouver combien de plaques coûteraient exactement 100$.

. . . .

In a certain province, vehicle license plates each have exactly three letters followed by three digits. We are told that to produce such a license plate, it costs $$n$ for each digit $n > 0$ and $10 for each digit 0. For letters, the costs are proportional to the position of the letter in the alphabet, namely, $1 for A, $2 for B, and so on, up to $26 for Z.

(a) Find the cost of producing an entire set of license plates (that is, from AAA 000 to ZZZ 999).

(b) Determine how many plates would cost exactly $100.

M123. *Proposé par l'Équipe de Mayhem.*
Trouver la base dans laquelle 221 est un facteur de 1215.

. . . .

In what base is 221 a factor of 1215?

M124. *Proposé par l'Équipe de Mayhem.*
Sans l'aide d'une table, trouver si 2003 est un nombre premier.

. . . .

Determine (without checking a table) whether 2003 is prime.

M125. *Proposé par l'Équipe de Mayhem.*
Etablir la liste de tous les nombres entiers positifs plus petits que 122003 qui sont des carrés ou des cubes.

. . . .

List all of the positive integers less than 122003 that are both perfect squares and perfect cubes.

Pólya's Paragon

They All Fall Down

Paul Ottaway

Often in mathematics we are faced with problems that are expressed in such a way that we are actually asking infinitely many problems at once. Fermat's Last Theorem, for example, asks us to prove that there are no integer solutions x, y, z to the equation $x^n + y^n = z^n$ for any integer $n > 2$. This is asking us to prove that there are no solutions to $x^3 + y^3 = z^3$ AND no solutions to $x^4 + y^4 = z^4$ AND, etc. It would be nice if there were some way we could relate all these different problems and solve them all in one fell swoop. One way of doing this is to use what is called *Mathematical Induction*. While this does not actually help us prove Fermat's Last Theorem, it will allow us to solve some similarly worded problems.

When we need to prove that a particular statement is true for infinitely many successive natural numbers n, we proceed in following way:

1. Prove that the result is true for some particular natural number n—typically an easy case where n is 0 or 1 (the *base case*).
2. Prove that, if the statement is true for some natural number k, then it must also be true for the value $k + 1$ (the *inductive step*).
3. Conclude that the statement is true for all n greater than or equal to the value used in step 1.

To get an intuitive notion as to why we are allowed to make the conclusion in step 3, we examine what the first two steps tell us. First, we show that our statement is true for $n = 1$. Step 2 then tells us that, since it is true for $n = 1$, it is also true for $n = 2$. But then, by the same logic, since it is true for $n = 2$, it will also be true for $n = 3$. Clearly, this will continue for ever, and we will never find a value of n where the statement will not be true.

As an analogy, think of a string of dominoes standing on end next to each other. Step 1 says 'we cause the first domino to fall over'. Step 2 says 'if a domino falls over, the next in line will also fall over'. Knowing only these two facts we know for certain that every domino must fall over. This is the process of Mathematical Induction.

When this is taught as part of the high-school or early university curriculum, many of the problems involve finding the sum of some sequence of n terms. While these problems are adequate, in my experience, students who are only exposed to such problems learn only the specific pattern of how to solve that type of problem rather than really understanding the principle of Mathematical Induction. I will now show two problems that can be solved using Mathematical Induction that are a little different than what you would see in most textbooks.

Problem 1. An ancient puzzle called the *Tower of Hanoi* consists of three pegs on a stand and n punctured discs of different sizes that are placed in decreasing order of size on one of the pegs. The object of the puzzle is to transfer the pile of discs to another peg, by moving one disc at a time, and without placing any disc on top of a smaller disc. Show that it is possible to solve this puzzle in $2^n - 1$ moves.

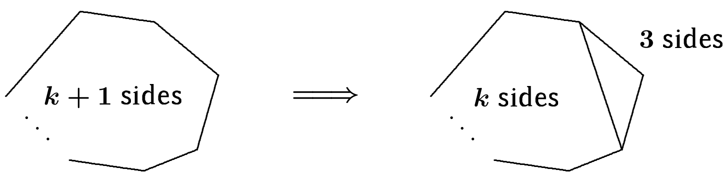
Solution. For a base case, we simply examine the game which begins with only one disc. Clearly, this will only take one move to solve since we can move the disc to the appropriate peg directly. Since $2^1 - 1 = 1$, the statement is true for $n = 1$.

Now we assume that it is possible to solve the puzzle starting with k discs in $2^k - 1$ moves. To solve the puzzle with $k + 1$ discs we first move the first k discs to another peg (taking $2^k - 1$ moves); then we move the last disc to the empty peg (taking 1 move); and finally, we replace the first k discs to rebuild the tower (taking another $2^k - 1$ moves). We have now solved the puzzle and used $(2^k - 1) + (1) + (2^k - 1) = 2 \cdot 2^k - 2 + 1 = 2^{k+1} - 1$ moves. Therefore, we can solve the puzzle with $k + 1$ discs in $2^{k+1} - 1$ moves. By Mathematical Induction, we can conclude that the statement is true for all n.

Problem 2. Prove that the sum of the interior angles of a convex n–gon is exactly $(n - 2) \times 180°$. (I should note that this statement holds for all n–gons, but the proof is much clearer with convex n–gons.)

Solution. Our base case for this problem is $n = 3$, since it is impossible to create a polygon with fewer than 3 sides. For $n = 3$, we have a triangle which has an interior angle sum of 180°. (Exercise: can you prove this rigorously?) We can also check that $(3 - 2) \times 180° = 180°$. Thus, the statement is true for $n = 3$.

Now assume that every convex k–gon has an interior angle sum of $(k-2) \times 180°$. We now consider a convex $(k+1)$–gon. If we draw a line from one vertex to another to form a triangle, we notice that this line divides the $(k+1)$–gon into a k–gon (which has an interior angle sum of $(k - 2) \times 180°$) and a triangle (which has an interior angle sum of 180°).



Now we see that the interior angle sum of the $(k + 1)$–gon is exactly $(k-2) \times 180° + 180° = ((k+1) - 2) \times 180°$, which is what we needed in order to show that the statement is true. Therefore, by Mathematical Induction, we can conclude that every convex n–gon has an interior angle sum of exactly $(n - 2) \times 180°$.

Each of the following problems can be proven with or without Mathematical Induction. Try to find as many solutions as possible to each of them.

PROBLEMS:

1. Prove that $2^{3n} - 1$ is divisible by 7 for all natural numbers n.
2. Prove that the sum of the terms in the n^{th} row of Pascal's Triangle is exactly 2^n.
3. Prove that the product of n consecutive integers is divisible by $n!$.

The Power of Symmetry: An Example Linking Algebra and Geometry

Nicholea Gusita and Michael Silbert

The formula for the area of a triangle is first introduced in the middle grades. An approach that is commonly used is to decompose the triangle in order to construct a parallelogram (Figure 1a), which is then decomposed to construct a rectangle (Figure 1b). The area of the rectangle can be calculated using the formula $A = lw$, which has been previously developed, and the area of the triangle is then determined to be $A = \frac{1}{2}bh$.

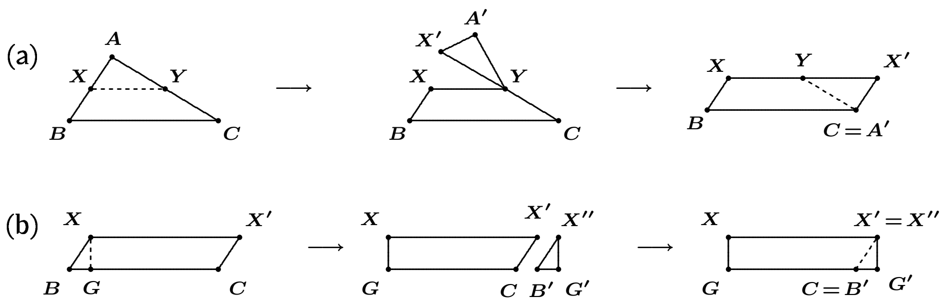


Figure 1

One difficulty students often have in applying the formula $A = \frac{1}{2}bh$ is that they believe the so-called *base* must be oriented horizontally. If the triangle is rotated so that none of the sides is horizontal, some students become confused and unable to apply the formula successfully. The underlying problem often appears to be that the student does not recognize that *any* side can serve as a base. The student fails to appreciate the symmetry.

In this article, exploration of the area of a triangle using only basic algebraic tools leads to Heron's formula and another formula similar to Heron's but not as well known. In the process, it becomes clear that symmetry plays a fundamental role in both the geometry and the algebraic development.

The triangle area formula has three symmetric forms (see Figure 2):

$$A = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c\,. \qquad (1)$$

From (1), we get $2A = ah_a = bh_b = ch_c$, which, when squared, gives

$$4A^2 = a^2h_a^2 = b^2h_b^2 = c^2h_c^2\,. \qquad (2)$$

We still have three distinct symmetric equations that apply to Figure 2. However, if we add the three equations together, we obtain the single combined equation

$$12A^2 \;=\; a^2h_a^2 + b^2h_b^2 + c^2h_c^2\,. \tag{3}$$

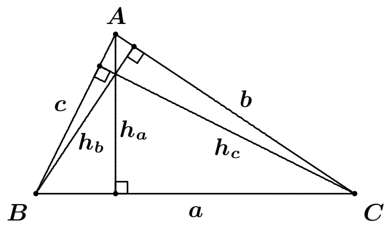


Figure 2

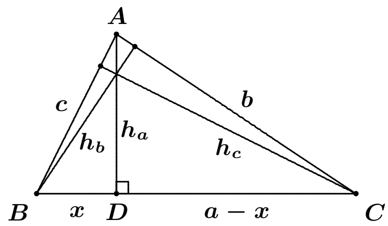


Figure 3

Let the height relative to side BC intersect BC at D (Figure 3), and let $|BD| = x$. Then $|DC| = a - x$. By the Pythagorean Theorem, we have

$$\begin{aligned} h_a^2 + x^2 &= c^2\,, \\ h_a^2 + (a-x)^2 &= b^2\,. \end{aligned} \tag{4}$$

Adding these two equations, expanding, and rearranging the terms yields

$$2h_a^2 \;=\; b^2 + c^2 - a^2 + 2x(a-x)\,. \tag{5}$$

By subtracting the equations in (4) instead of adding them, we obtain, after simplification,

$$x \;=\; \frac{c^2 + a^2 - b^2}{2a}\,. \tag{6}$$

Substituting (6) into (5), we get

$$2h_a^2 \;=\; b^2 + c^2 - a^2 + 2\left(\frac{a^2 - b^2 + c^2}{2a}\right)\left(\frac{a^2 + b^2 - c^2}{2a}\right)\,.$$

This relationship can be simplified as follows:

$$\begin{aligned} 4h_a^2 &= 2b^2 + 2c^2 - 2a^2 + \frac{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}{a^2}\,, \\ 4a^2h_a^2 &= 2a^2b^2 + 2a^2c^2 - 2a^4 + \left(a^2 - [b^2 - c^2]\right)\left(a^2 + [b^2 - c^2]\right) \\ &= 2a^2b^2 + 2a^2c^2 - 2a^4 + a^4 - [b^2 - c^2]^2 \\ &= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\,. \end{aligned}$$

As there are three symmetric expressions for the area of a triangle in terms of base and height, there are likewise three symmetric expressions for the heights h_a, h_b, h_c in terms of the lengths of the sides:

$$\begin{aligned} 4a^2h_a^2 &= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\,, \\ 4b^2h_b^2 &= 2b^2c^2 + 2a^2b^2 + 2a^2c^2 - b^4 - c^4 - a^4\,, \\ 4c^2h_c^2 &= 2a^2c^2 + 2b^2c^2 + 2a^2b^2 - c^4 - a^4 - b^4\,. \end{aligned} \tag{7}$$

Adding the above equations gives us

$$4(a^2h_a^2+b^2h_b^2+c^2h_c^2) = 3\left[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)\right] . \quad (8)$$

Using (3) and (8), we get

$$16A^2 \;=\; 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)\,. \qquad (9)$$

Equation (9) gives us the area A of a triangle in terms of the lengths of the sides. But what about the expression on the right side of (9)? Can it be simplified using only the basic tools of intermediate algebra?

Let us start by considering what happens when a general trinomial is squared: $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$. The expression on the right side of (9) looks like

$$(a^2+b^2+c^2)^2 \;=\; a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2\,,$$

except that the signs of some terms are negative. In attempting to factor the expression on the right side of (9), it might be worthwhile to try factors of the form $(\pm a \pm b \pm c)$. Due to the symmetry in the problem, it would be reasonable to expect the presence of the three factors $(-a+b+c)$, $(a-b+c)$, and $(a+b-c)$. Since the fourth powers in (9) are negative, the fourth factor would have to be $a+b+c$. Multiplying these four factors together confirms that

$$\begin{aligned} &2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)\\ &\quad= (a+b+c)(-a+b+c)(a-b+c)(a+b-c)\,. \end{aligned} \qquad (10)$$

The same result can be obtained in another way. Although the expression on the right side of (9) resists factoring as a perfect square, we can factor it as a difference of squares:

$$\begin{aligned} &2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)\\ &\quad= 4a^2b^2+2b^2c^2+2c^2a^2-2a^2b^2-a^4-b^4-c^4\\ &\quad= 4a^2b^2-[c^4-2b^2c^2-2c^2a^2+a^4+2a^2b^2+b^4]\\ &\quad= 4a^2b^2-[c^4-2(a^2+b^2)c^2+(a^2+b^2)^2]\\ &\quad= (2ab)^2-[c^2-(a^2+b^2)]^2\\ &\quad= [2ab-c^2+a^2+b^2][2ab+c^2-a^2-b^2]\\ &\quad= [(a+b)^2-c^2][c^2-(a-b)^2]\\ &\quad= (a+b+c)(a+b-c)(c+a-b)(c-a+b)\,, \end{aligned}$$

which is the same result as (10).

Substituting (10) into (9) yields

$$\begin{aligned} 16A^2 &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c)\,,\\ A^2 &= \left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right). \end{aligned}$$

We now define the semi-perimeter, s, as $s = \dfrac{a+b+c}{2}$. Then

$$A^2 = s(s-a)(s-b)(s-c);$$

that is,

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Thus, Heron's formula is established.

Not only does $\triangle ABC$ have three sides, a, b, and c, and three heights, h_a, h_b, and h_c, but it also has three medians, m_a, m_b, and m_c (Figure 4). In the development of Heron's formula, the semi-perimeter, s, was introduced. In a parallel fashion, we define the semi-median sum $s_m = \dfrac{m_a + m_b + m_c}{2}$.

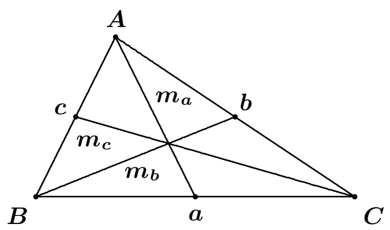


Figure 4

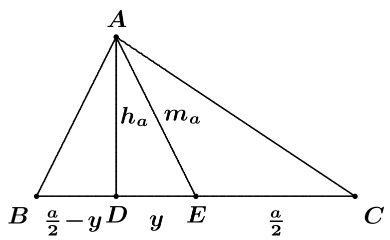


Figure 5

Let the median m_a from A to side BC intersect BC at E, and let $|DE| = y$. Assume that D is closer to B than to C (as in Figure 5). Then $|BD| = (a/2) - y$ and $|EC| = a/2$. In $\triangle ADE$, we have $m_a^2 = h_a^2 + y^2$. Hence, $h_a^2 = m_a^2 - y^2$.

From equation (5), we obtain

$$\begin{aligned} 2h_a^2 &= b^2 + c^2 - a^2 + 2|BD| \cdot |DC| \\ &= b^2 + c^2 - a^2 + 2\left(\frac{a}{2} - y\right)\left(\frac{a}{2} + y\right) \\ &= b^2 + c^2 - \frac{a^2}{2} - 2y^2. \end{aligned} \tag{11}$$

Substituting $h_a^2 = m_a^2 - y^2$ into (11) yields

$$\begin{aligned} 2m_a^2 - 2y^2 &= b^2 + c^2 - \frac{a^2}{2} - 2y^2, \\ 4m_a^2 &= 2(b^2 + c^2) - a^2. \end{aligned}$$

Again, by symmetry, there are three equations:

$$\begin{aligned} 4m_a^2 &= 2(b^2 + c^2) - a^2, \\ 4m_b^2 &= 2(c^2 + a^2) - b^2, \\ 4m_c^2 &= 2(a^2 + b^2) - c^2. \end{aligned} \tag{12}$$

Adding these three equations produces

$$4(m_a^2 + m_b^2 + m_c^2) \;=\; 3(a^2 + b^2 + c^2)\,. \tag{13}$$

Adding together the *squares* of the three equations in (12) produces

$$16(m_a^4 + m_b^4 + m_c^4) \;=\; 9(a^4 + b^4 + c^4)\,. \tag{14}$$

Note that the square of (13) contains all the terms in (14), plus some additional terms. By squaring (13) and subtracting (14) from the result, we get

$$16\left[2(m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2)\right] \;=\; 9[2(a^2b^2 + b^2c^2 + c^2a^2)]\,. \tag{15}$$

We would like to express the area A strictly in terms of the medians (and the semi-median sum). From (9), we know that

$$A^2 \;=\; \frac{1}{16}\left[2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)\right]\,.$$

In this equation, we substitute for $2(a^2b^2 + b^2c^2 + c^2a^2)$ from (15) and for $a^4 + b^4 + c^4$ from (14):

$$\begin{aligned} A^2 &= \frac{1}{16}\cdot\frac{16}{9}\left[2(m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2) - (m_a^4 + m_b^4 + m_c^4)\right] \\ &= \frac{1}{9}\left[2(m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2) - (m_a^4 + m_b^4 + m_c^4)\right]\,. \end{aligned} \tag{16}$$

The *form* of the expression in the square brackets in (16) should look familiar. It is just the left side of (10) with a, b, and c replaced by m_a, m_b, and m_c. Hence, the same algebraic development as before leads to

$$A \;=\; \frac{4}{3}\sqrt{s_m(s_m - m_a)(s_m - m_b)(s_m - m_c)}\,. \tag{17}$$

Thus, the area of a triangle can be expressed in terms of the lengths of its medians.

By considering the area of a triangle as *base* $\times$ *height*, we tend to mask the inherent symmetry. On the other hand, by considering the area of a triangle as *length of side* $\times$ *corresponding height*, we open up the richness of the symmetry in the geometric situation. This leads to the remarkable algebraic symmetry in the Sine and Cosine Laws, Heron's formula, and (17).

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Year End Wrap Up

Shawn Godin

Another year has come to an end. Mayhem has grown over the last year and we have seen, and continue to see, new solvers. To all contributors, both old and new, thank you so much for helping Mayhem to be better. We hope that we will continue to attract new readers and contributors and continue hearing from the old pros.

At this point I would like to thank some people who make Mayhem possible. First, and foremost, I would like to thank Mayhem assistant editor, JOHN GRANT McLOUGHLIN. John has really breathed some life into the problems section, and made the problems more relevant to our high school readers. Some of the new proposed problems are very open-ended, and will allow students to pursue them to greater and greater depths. I look forward to the proposals for 2004.

The other member of the Mayhem Staff is PAUL OTTAWAY, who continues to dedicate his time and talent to help make Mayhem all it is. Pólya's Paragon has given our readers some gems over the last year. Thanks, Paul!

I also want to thank some other people who have been helpful "behind the scenes" over the year: RICHARD HOSHINO, JOSEPH KHOURY, BRUCE SHAWYER, GRAHAM WRIGHT, and LARRY RICE.

All the best of the season to our readers and contributors! ***CRUX with MAYHEM*** is what it is because of our staff and our readers. Please continue to help us make our journal grow and improve. Happy problem solving, we'll see you in 2004.

THE OLYMPIAD CORNER

No. 234

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

To start off this number of the *Corner*, we have another five Klamkin Quickies. Thanks go to Murray S. Klamkin, University of Alberta, Edmonton, AB, for creating them for our use. Try them before looking for his Quickie solutions later in this issue.

FIVE KLAMKIN QUICKIES

December 2003

1. The numbers 2^{2003} and 5^{2003} are written out in base 10 one right after the other to create a single number. How many digits are there in this number? (No log tables or calculators, please!)

2. When an integer-sided cube C is cut into 729 smaller integer-sided cubes, exactly 728 of them are unit cubes. What is the smallest possible volume of C?

3. Points A, B, C, D, E, F are the consecutive vertices of an inscribed hexagon such that $AB = BC$, $CD = DE$, and $EF = FA$. Find the ratio of the area of triangle BDF to the area of the hexagon.

4. Express $(x^n + x^{n-1} + \cdots + 1)^2 - x^n$ as a product of non-constant polynomials.

5. Determine the maximum value of

$$\frac{x}{y(1+z+x)} + \frac{y}{z(1+x+y)} + \frac{z}{x(1+y+z)}$$

where x, y, $z \geq 1$.

As a first Olympiad problem set, we give the Olympiades Académiques de Mathématiques, Session de 2001, Classe de Première. My thanks go to Michel Bataille, Rouen, France for sending them for our use.

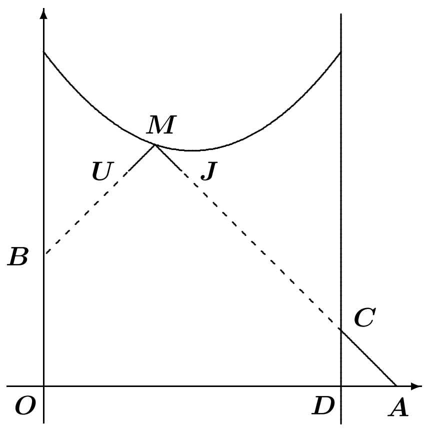
OLYMPIADES ACADÉMIQUES DE MATHÉMATIQUES
Session de 2001
Classe de Première (Durée : 4 heures)

1. Les faces d'un dé en forme de tétraèdre régulier sont numérotées de 1 à 4. Le dé est posé sur une table, face «1» contre cette table. Une étape consiste à faire basculer le dé autour de l'une quelconque des arêtes de sa base. À l'issue de chaque étape, on note le numéro de la face contre la table. On fait la somme s de tous ces nombres après 2001 étapes, en comptant aussi le «1» initial.

(i) Donner la valeur maximale et la valeur minimale que l'on peut ainsi obtenir pour s.

(ii) La somme s peut-elle prendre toutes les valeurs entières entre ces deux valeurs ?

2. Une lampe entourée d'un abat-jour est suspendue entre deux murs distants de 8 mètres à une rampe. La situation est représentée par le schéma à droite.



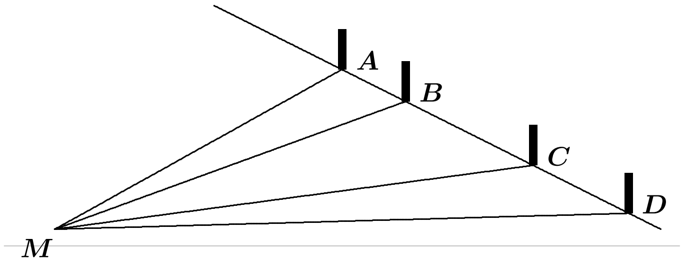
- Les murs ont pour équations $x = 0$, $x = 8$, et la rampe a pour équation $y = \frac{1}{6}(x-4)^2 + \frac{19}{3}$.
- L'abat-jour est symbolisé par un triangle rectangle isocèle UMJ de côtés 1 et $\sqrt{2}$.

(i) Vérifier que les bords de l'abat-jour ne touchent ni la rampe ni les murs lorsque $1 < x < 7$.

(ii) Calculer l'aire du polygone éclairé $OBMCD$ correspondant à $x = 3$.

(iii) Trouver la position de la lampe sur la rampe qui donne un éclairage maximal.

3. Sur un terrain de jeu sont alignés quatre poteaux, plantés en A, B, C et D dans cet ordre. Ces poteaux délimitent trois buts de largeurs : $AB = 1$, $BC = 2$, $CD = d$, où d est une longueur donnée.



Déterminer l'ensemble des points M du terrain d'où l'on voit les trois buts sous des angles $\angle AMB$, $\angle BMC$ et $\angle CMD$ égaux.

4. Dessinez un cube C (un dessin même approximatif en perspective suffira). Soient A un de ses sommets et B le sommet opposé, c'est-à-dire tel que le milieu du segment $[AB]$ soit le centre du cube.

Considérons un autre cube C' admettant aussi (A, B) comme couple de sommets opposés. Certaines arêtes de C rencontrent des arêtes de C'. Justifiez le fait que, en dehors de A et B, on obtient ainsi six points d'intersection entre une arête de C et une arête de C'.

Placez l'un d'eux sur le dessin et expliquez comment placer alors les cinq autres. V étant le volume de C, quelle est la valeur minimale du volume de la portion d'espace commune aux cubes C et C' ?

As another set of problems for your pleasure over the holiday season, we give the Selected Problems of the Ukrainian Mathematical Olympiad, March 2001. Thanks go to Chris Small, Canadian Team Leader to the XLI IMO, for collecting them.

UKRAINIAN MATHEMATICAL OLYMPIAD
March 2001
Selected Problems

1. (Grade 9) All 5-digit positive integers with digits in increasing order (from left to right) are given. Is it possible to take away one digit from each number so that we obtain all 4-digit positive integers with digits in increasing order?

2. (Grade 9) Let I be the incentre of triangle ABC; let the bisectors of $\angle BAC$ and $\angle ACB$ meet the sides BC and AB at A_1 and C_1, respectively; and let M be an arbitrary point on the segment AC. Lines through M parallel to these angle bisectors meet AA_1, CC_1, AB, and CB at points H, N, P, and Q, respectively. Denote $BC = a$, $AC = b$, $AB = c$, and let d_1, d_2, d_3 be the respective distances from H, I, N to the line PQ. Prove that

$$\frac{d_1}{d_2} + \frac{d_2}{d_3} + \frac{d_3}{d_1} \geq \frac{2ab}{a^2+bc} + \frac{2ca}{c^2+ab} + \frac{2bc}{b^2+ca} .$$

3. (Grade 10) Let a_1, a_2, $\ldots$, a_n be real numbers such that

$$a_1 + a_2 + \cdots + a_n \;\geq\; n^2 \quad \text{and} \quad a_1^2 + a_2^2 + \cdots + a_n^2 \;\leq\; n^3 + 1\,.$$

Prove that $n - 1 \leq a_k \leq n + 1$ for all k.

4. (Grade 10) There are n mathematicians in each of three countries. Each mathematician corresponds with at least $n+1$ foreign mathematicians. Prove that there exist three mathematicians who correspond with each other.

5. (Grade 11) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$ the following equality holds?

$$f(xy) \;=\; \max\{f(x), y\} + \min\{f(y), x\}\;.$$

6. (Grade 11) Positive integers a and n are such that n divides $a^2 + 1$. Prove that there exists a positive integer b such that $n(n^2+1)$ divides b^2+1.

7. (Grade 11) An acute triangle ABC, with $AC \neq BC$, is inscribed in a circle ω. The points A, B, C divide the circle into disjoint arcs $\widehat{AB}$, $\widehat{BC}$, and $\widehat{CA}$. Let M and N be the mid-points of $\widehat{BC}$ and $\widehat{AC}$, respectively, and let K be an arbitrary point of $\widehat{AB}$. Let D be the point of $\widehat{MN}$ such that $CD \parallel NM$. Let O, O_1, O_2 be the incentres of triangles ABC, CAK, CBK, respectively. Let L be the intersection point of the line DO and the circle ω, where $L \neq D$. Prove that the points K, O_1, O_2, L are concyclic.

8. (Grade 11) Let a, b, c and α, β, γ be positive real numbers such that $\alpha + \beta + \gamma = 1$. Prove the inequality

$$\alpha a + \beta b + \gamma c + 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(ab + bc + ca)} \;\leq\; a + b + c\,.$$

Next we give the official solutions by Murray S. Klamkin to the set of "Klamkin Quickies" that opened this *Corner*.

SOLUTIONS TO FIVE KLAMKIN QUICKIES

December 2003

1. Let m and n denote the number of digits in 2^{2003} and 5^{2003} when expressed in base 10. Then

$$10^{m-1} \;<\; 2^{2003} \;<\; 10^m \quad \text{and} \quad 10^{n-1} \;<\; 5^{2003} \;<\; 10^n\,,$$

so that $10^{m+n-2} < 10^{2003} < 10^{m+n}$. Thus, $m + n - 1 = 2003$ and the number of digits is $m + n = 2004$.

2. We must satisfy the Diophantine equation $m^3 = n^3 + 728$, with $n > 1$. Write the equation as

$$(m - n)(m^2 + mn + n^2) = 7 \cdot 8 \cdot 13 .$$

Noting that $(m-n)^2 < m^2+mn+n^2$, we find that $m-n$ can only be 1, 2, 4, 7, or 8. Since $(m^2+mn+n^2)-(m-n)^2 = 3mn$, the only solutions are $(m,n) = (9,1)$, which does not satisfy the condition $n > 1$, and $(12,10)$. Thus, the only possible volume is $12^3 = 1728$.

3. Let the angles subtended at the centre by the sides AB, CD and EF be 2α, 2β, and 2γ, respectively. Then $\alpha + \beta + \gamma = 90^\circ$. Let R be the circumradius. The area of $ABCDEF$ is

$$\begin{aligned}
&[ABCDEF] \\
&\quad= 2(R\sin\alpha)(R\cos\alpha) + 2(R\sin\beta)(R\cos\beta) + 2(R\sin\gamma)(R\cos\gamma) \\
&\quad= R^2(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) ,
\end{aligned}$$

and similarly, the area of BDF is

$$\begin{aligned}
[BDF] &= \tfrac{1}{2}R^2\big(\sin 2(\alpha+\beta) + \sin 2(\beta+\gamma) + \sin 2(\gamma+\alpha)\big) \\
&= \tfrac{1}{2}R^2(\sin 2\gamma + \sin 2\alpha + \sin 2\beta) ,
\end{aligned}$$

since $\alpha + \beta + \gamma = 90^\circ$. Thus, we have $2[BDF] = [ABCDEF]$.

4. Letting $S = x^n + x^{n-1} + \cdots + 1$, the factors are seen to be $(S + x^{n+1})(S - x^n)$ by expanding.

5. Since $y(1 + z + x) \geq y + z + x$, etc., the given sum is ≤ 1, with equality if and only if $x = y = z = 1$.

Now we have readers' solutions to problems of the 15th Balkan Mathematical Olympiad 1998, given in [2001 : 357].

1. Consider the terms of the finite sequence $\left\lfloor \frac{k^2}{1998} \right\rfloor$, $k = 1, 2, \ldots, 1997$, where $\lfloor x \rfloor$ denotes the integral part of x. How many of the terms of this sequence are different?

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Aassila's solution.

Observe that

$$\left\lfloor \frac{998^2}{1998} \right\rfloor = 498 < 499 = \left\lfloor \frac{999^2}{1998} \right\rfloor .$$

We have

$$\frac{(k+1)^2}{1998} - \frac{k^2}{1998} = \frac{2k+1}{1998} \begin{cases} < 1 , & \text{for } k = 1, 2, \ldots, 998, \\ > 1 , & \text{for } k = 999, 1000, \ldots, 1997 . \end{cases}$$

It follows that each integer from 0 to 499 inclusive is a term of the sequence, while each $k > 999$ corresponds to a distinct term of the sequence. Thus, the total number of distinct terms is $500 + (1997 - 999) = 1498$.

2. Let n be an integer, $n \geq 2$, and $0 < a_1 < a_2 < \cdots < a_{2n+1}$ be real numbers. Prove that the following inequality holds:

$$\begin{aligned} &\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} - \cdots - \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n+1}} \\ &\qquad < \sqrt[n]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}}\,. \end{aligned}$$

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Pontoise, France. We give Bornsztein's write-up.

Let $n \geq 2$ be an integer.

Lemma. If $0 \leq a < b \leq c < d$ are real numbers such that $a^n + d^n = b^n + c^n$, then $a + d < b + c$.

Proof. Let $x = b - a$, $y = c - b$, and $z = d - c$. From the Binomial Theorem, we have

$$\begin{aligned} a^n + (a+x+y+z)^n &= a^n + d^n = b^n + c^n \\ &= (a+x)^n + (a+x+y)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \left(x^k + (x+y)^k\right) \\ &< 2a^n + \sum_{k=1}^{n} \binom{n}{k} a^{n-k} (2x+y)^k \\ &= a^n + (a+2x+y)^n\,. \end{aligned}$$

Then $z < x$; that is $a + d < b + c$. The lemma is proved.

Let $0 < a_1 < a_2 < \cdots < a_{2n+1}$ be real numbers. From the lemma, we deduce that

$$\begin{aligned} &\sqrt[n]{a_1} + \sqrt[n]{a_3 - \cdots - a_{2n} + a_{2n+1}} \\ &\qquad < \sqrt[n]{a_2} + \sqrt[n]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}}\,, \\ &\sqrt[n]{a_3} + \sqrt[n]{a_5 - \cdots - a_{2n} + a_{2n+1}} \\ &\qquad < \sqrt[n]{a_4} + \sqrt[n]{a_3 - a_4 + \cdots - a_{2n} + a_{2n+1}}\,, \\ &\qquad \vdots \\ &\sqrt[n]{a_{2n-1}} + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n-1} - a_{2n} + a_{2n+1}}\,. \end{aligned}$$

Summing, we get the desired result.

3. Denote by S the set of all points of $\triangle ABC$ except one interior point T. Show that S can be represented as a union of disjoint (line) segments.

Solved by Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

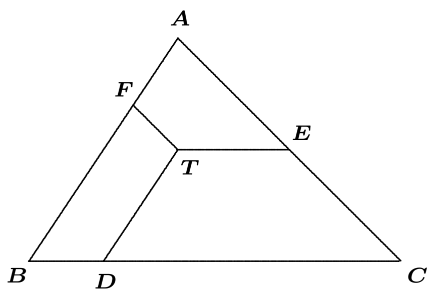
Lemma. If $MNPQ$ is a trapezoid, with MN parallel to PQ, then the set S of all the points interior to, or on the boundary of, $MNPQ$ except the line segment $[MN]$, can be represented as a union of disjoint line segments.

Proof. For any point X on the half-open segment $[QM[$, let Y_X be the point on the half-open segment $[PN[$ such that XY_X is parallel to MN. Then we have

$$S = \bigcup_{X \in [QM[} [XY_X],$$

which proves the lemma.

Let T be an interior point of the triangle ABC. Let ℓ_1, ℓ_2, ℓ_3 be the lines through T parallel to AB, BC, CA, respectively. Let D, E, F be the respective intersections of ℓ_1 with BC, of ℓ_2 with AC, and of ℓ_3 with AB.



To get the conclusion, it suffices to see that S is the disjoint union of the three trapezoids $TFBD - [TD]$, $TDCE - [TE]$, $TEAF - [TF]$, and to use the lemma in these three cases.

4. Prove that the equation $y^2 = x^5 - 4$ has no integer solutions.

Solved by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Pontoise, France. We give Aassila's write-up.

We have $(x^5)^2 = x^{10} \equiv 0$ or $1 \bmod 11$ for all x [by Fermat's Theorem]. Hence, $x^5 \equiv -1$, 0, or $1 \bmod 11$. Thus, $x^5 - 4 \equiv 6$, 7, or $8 \bmod 11$.

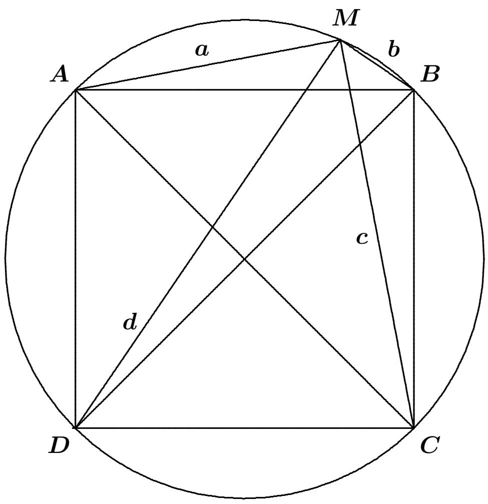
But all squares are congruent to 0, 1, 3, 4, 5, or $9 \bmod 11$. Therefore, the equation has no solutions in integers.

Next we turn to readers' solutions to problems of the 1st Mediterranean Mathematical Olympiad, April 22, 1998, given [2001 : 357–358].

1. [*Greece*]

Let $ABCD$ be a square inscribed in a circle. If M is a point on the arc AB show that $MC \cdot MD > 3\sqrt{3} \cdot MA \cdot MB$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Bradley's answer.



Without loss of generality, let the side of the square be equal to 1. Write $MA = a$, $MB = b$, $MC = c$, $MD = d$.

By Ptolemy's Theorem for $DAMB$ we have

$$b + a\sqrt{2} = d\,,$$

and for $AMBC$ we have

$$a + b\sqrt{2} = c\,.$$

Hence,

$$cd = \sqrt{2}a^2 + \sqrt{2}b^2 + 3ab\,,$$

which, by the AM-GM Inequality, yields

$$cd \geq (3 + 2\sqrt{2})ab > 3\sqrt{3}ab\,.$$

The last inequality is justified by observing that

$$3 + 2\sqrt{2} > 3\sqrt{3} \iff 17 + 12\sqrt{2} > 27 \iff 12\sqrt{2} > 10\,,$$

which is clearly true.

2. [*Croatia*]

(a) Prove that the polynomial $z^{2n} + z^n + 1$, $n \in \mathbb{N}$, is divisible by the polynomial $z^2 + z + 1$ if and only if n is not a multiple of 3.

(b) Find the necessary and sufficient condition that the natural numbers p, q must satisfy for the polynomial $z^p + z^q + 1$ to be divisible by $z^2 + z + 1$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Pontoise, France. We give Bataille's solution.

We begin with the more general question (b) and show that the condition which we seek is $pq \equiv 2 \pmod 3$.

The roots of $z^2 + z + 1$ are ω and ω^2, where $\omega = \exp(2\pi i/3)$. Hence, $z^p + z^q + 1$ is divisible by $z^2 + z + 1$ if and only if $\omega^p + \omega^q + 1 = 0$ and $\omega^{2p} + \omega^{2q} + 1 = 0$. Suppose that these conditions hold. Then

$$\begin{aligned} 0 \;=\; (\omega^p + \omega^q + 1)^2 &= \omega^{2p} + \omega^{2q} + 1 + 2(\omega^{p+q} + \omega^q + \omega^p) \\ &= 2(\omega^{p+q} - 1)\,. \end{aligned}$$

Hence, $\omega^{p+q} = 1$, which implies that $p + q \equiv 0 \pmod 3$. It follows that $\omega^q = \omega^{-p} = \omega^{2p}$. Then $\omega^p + \omega^{2p} + 1 = 0$, which implies that $p \equiv 1$ or $p \equiv 2 \pmod 3$. Since $q \equiv -p \pmod 3$, we get $pq \equiv 2 \pmod 3$.

Conversely, if $pq \equiv 2 \pmod 3$, say $p \equiv 1$ and $q \equiv 2$, then

$$\omega^p + \omega^q + 1 \;=\; \omega^{2p} + \omega^{2q} + 1 \;=\; \omega^2 + \omega + 1 \;=\; 0\,.$$

The conclusion follows.

As for (a), the result just obtained provides the condition $2n^2 \equiv 2$; that is, $n^2 \equiv 1 \pmod 3$. This is clearly equivalent to $n \not\equiv 0 \pmod 3$, which means n is not a multiple of 3.

3. [*Spain*]

In a triangle ABC, I is the incentre and $D \in (BC)$, $E \in (CA)$, $F \in (AB)$ are the points of tangency of the incircle with the sides of the triangle. Let $M \in (BC)$ be the foot of the interior bisector of $\angle BIC$ and $\{P\} = FE \cap AM$. Prove that DP is the interior bisector of the angle $\angle FDE$.

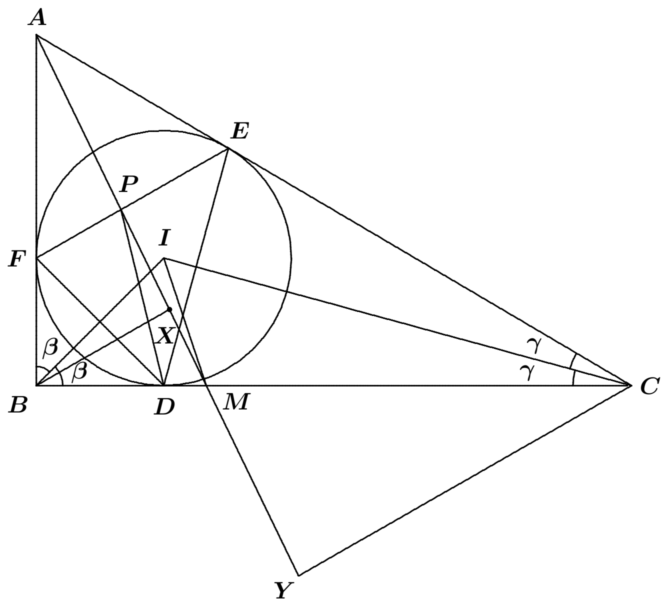
Solved by Christopher J. Bradley, Clifton College, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.

Let X, Y be points on AM such that $BX \parallel CY \parallel FE$. Since $\triangle BMX$ and $\triangle CMY$ are similar, and $\angle BIM = \angle CIM$, we have

$$\frac{CY}{BX} = \frac{CM}{BM} = \frac{CI}{BI}\,.$$

Since $EP \parallel CY$, $FP \parallel BX$ and $AE = AF$, we have

$$\frac{EP}{FP} = \frac{EP}{CY} \cdot \frac{BX}{FP} \cdot \frac{CY}{BX} = \frac{AE}{AC} \cdot \frac{AB}{AF} \cdot \frac{CY}{BX} = \frac{AB}{AC} \cdot \frac{CI}{BI}\,. \qquad (1)$$



Let $\angle ABC = 2\beta$ and $\angle ACB = 2\gamma$. Then

$$\angle ABI = \angle IBC = \beta \quad \text{and} \quad \angle ACI = \angle ICB = \gamma .$$

By the Law of Sines for $\triangle ABC$ and $\triangle IBC$, we get

$$\frac{AB}{AC} = \frac{\sin 2\gamma}{\sin 2\beta} = \frac{2\sin\gamma\cos\gamma}{2\sin\beta\cos\beta} = \frac{\sin\gamma\cos\gamma}{\sin\beta\cos\beta},$$

and

$$\frac{CI}{BI} = \frac{\sin\beta}{\sin\gamma}.$$

Thus, from (1),

$$\frac{EP}{FP} = \frac{\sin\gamma\cos\gamma}{\sin\beta\cos\beta}\cdot\frac{\sin\beta}{\sin\gamma} = \frac{\cos\gamma}{\cos\beta}. \tag{2}$$

Let r be the radius of the incircle. Then

$$DE = 2r\sin\angle DFE = 2r\sin\angle EDC = 2r\sin(90^\circ-\gamma) = 2r\cos\gamma .$$

Similarly, we have $DF = 2r\cos\beta$. Hence,

$$\frac{DE}{DF} = \frac{2r\cos\gamma}{2r\cos\beta} = \frac{\cos\gamma}{\cos\beta}. \tag{3}$$

From (2) and (3) we obtain $\frac{EP}{FP} = \frac{DE}{DF}$.

Therefore, DP is the interior bisector of $\angle FDE$.

Now we turn to readers' solutions to problems of the Final National Selection Competition 1998 for the Greek Team, given [2001 : 358].

1. If x, y, $z > 0$, $k > 2$ and $a = x + ky + kz$, $b = kx + y + kz$, $c = kx + ky + z$, show that

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \geq \frac{3}{2k+1}\,.$$

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Laura Gu, student, Sir Winston Churchill High School, Calgary, AB; Michel Bataille, Rouen, France; and Heinz-Jürgen Seiffert, Berlin, Germany. We give Seiffert's generalization.

More generally, if $x_0, x_1, \ldots, x_n > 0$ $(n \geq 1)$, $k > 1$, and

$$a_i = x_i + k \sum_{\substack{j=0 \\ j \neq i}}^{n} x_j \quad (i = 0, 1, \ldots, n)\,,$$

then

$$\sum_{i=0}^{n} \frac{x_i}{a_i} \geq \frac{n+1}{nk+1}\,.$$

Proof. Let $S = \sum\limits_{i=0}^{n} x_i$. Then, for each i we have $a_i = kS - (k-1)x_i$; that is,

$$x_i = \frac{kS - a_i}{k-1}\,.$$

For all $i, j \in \{0, 1, \ldots, n\}$,

$$\frac{x_i}{a_i} \leq \frac{x_j}{a_j} \iff \frac{kS - a_i}{a_i} \leq \frac{kS - a_j}{a_j} \iff a_j \leq a_i\,.$$

Hence,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\left(\frac{x_i}{a_i} - \frac{x_j}{a_j}\right)(a_j - a_i) \geq 0\,,$$

or, equivalently,

$$\left(\sum_{i=0}^{n}\frac{x_i}{a_i}\right)\left(\sum_{i=0}^{n} a_i\right) \geq (n+1)S\,.$$

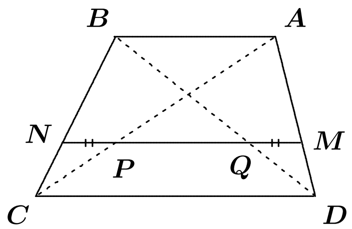
The stated inequality now follows by noting that $\sum\limits_{i=0}^{n} a_i = (nk+1)S$.

To solve the present proposal, take $n = 2$ and rename $x_0 = x$, $x_1 = y$, and $x_2 = z$.

2. Let $ABCD$ be a trapezoid ($AB \parallel CD$) and M, N be points on the lines AD and BC, respectively, such that $MN \parallel AB$. Prove that

$$DC \cdot MA + AB \cdot MD \;=\; MN \cdot AD\,.$$

Solved by Rahul Bamotra, student, Sir Winston Churchill High School, Calgary, AB; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Laura Gu, student, Sir Winston Churchill High School, Calgary, AB; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



Let MN meet AC and BD at the points P and Q, respectively. Since $PM \parallel AB \parallel CD$, we have $PM \parallel CD$. Thus, $\frac{MA}{AD} = \frac{MP}{DC}$; that is,

$$DC \cdot MA \;=\; MP \cdot AD\,. \tag{1}$$

Since $MN \parallel AB$ and $MN \parallel DC$, we have

$$\frac{NP}{BA} \;=\; \frac{CP}{CA} \;=\; \frac{DM}{DA} \;=\; \frac{QM}{BA}\,.$$

Therefore,

$$NP \;=\; QM\,. \tag{2}$$

Since $\frac{QM}{BA} = \frac{DM}{DA}$, then

$$AB \cdot MD \;=\; QM \cdot AD\,.$$

Hence, using (2),

$$AB \cdot MD \;=\; NP \cdot AD\,. \tag{3}$$

From (1) and (3) it follows that

$$\begin{aligned} DC \cdot MA + AB \cdot MD &= MP \cdot AD + NP \cdot AD \\ &= (MP + NP) \cdot AD \;=\; MN \cdot AD\,. \end{aligned}$$

3. Prove that if the number $A = \underbrace{1\,1\,1\,\ldots\,1}_{n \text{ digits}}$ is prime then the number n must be prime. Is the converse true?

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

[**Ed.** This problem also appeared as problem #7 of the Chilean Mathematical Olympiads 1994–95, and a solution appears in [2003 : 294].]

4. (a) A polynomial $P(x)$ with integer coefficients takes the value -2 for seven distinct integer values of x. Prove that it cannot take the value 1996.

(b) Prove that there are irrational numbers x, y such that the number x^y is rational.

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bradley's solution.

(a) Suppose $P(x)$ takes the value -2 for the distinct values x_1, x_2, x_3, x_4, x_5, x_6, x_7. Then, by the Remainder Theorem, we have

$$P(x)+2 \;=\; (x-x_1)(x-x_2)(x-x_3)\cdots(x-x_7)Q(x)\,,$$

where $Q(x)$ is also a polynomial with integer coefficients.

Suppose now that $P(k)=1996$ for some integer k. Then

$$(k-x_1)(k-x_2)\cdots(k-x_7)Q(k) \;=\; 1998 \;=\; 2\times 27\times 37\,.$$

The factors $(k-x_i)$ are distinct, since the x_i are distinct. But the maximum number of distinct factors of 1998 whose product is 1998 is six. [*Ed.* To get the largest number of distinct factors, we could include 1, -1, 3, and -3; the remaining factor 3 would have to be combined with 2, 37, or 3 or -3 in order to maintain distinct factors, leaving us a maximum of six distinct factors. (For example, 1, -1, -3, 9, 2, 37, or 1, -1, 3, -3, 6, 37, etc.).] This contradiction shows that $P(k)\neq 1996$ for any integer k.

[*Ed.* Note that $P(x)$ may take the value 1996 when x is not an integer.]

(b) Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is irrational. If the former, we are done. If the latter, setting $x=\sqrt{2}^{\sqrt{2}}$ and $y=\sqrt{2}$ gives $x^y=2$.

5. Let I be an open interval of width $\frac{1}{n}$, $n\in\mathbb{N}-\{0\}$. Determine the maximum number of irreducible fractions $\frac{a}{b}$ with $1\le b\le n$ that lie in I.

Solved by Pierre Bornsztein, Pontoise, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

We prove that the maximum is $\left\lfloor\frac{n+1}{2}\right\rfloor$, where $\lfloor x\rfloor$ denotes the integer part of x.

For $n\in\mathbb{N}^*$, let $f(n)$ be the maximum number of irreducible fractions $\frac{a}{b}$ with $1\le b\le n$ that may lie in an open interval of width $\frac{1}{n}$, and let $g(n)$ be the maximum number of elements a subset S of $\{1, 2, \ldots, n\}$ can have if no element of S is a multiple of another element of S.

Let I be an open interval of width $\frac{1}{n}$.

Claim 1. Let $\frac{a}{b}$ and $\frac{a'}{b'}$ be distinct, irreducible fractions such that $1 \leq b' \leq b \leq n$ and $b = kb'$ for some $k \in \mathbb{N}^*$. Then $\frac{a}{b}$ and $\frac{a'}{b'}$ cannot both belong to I.

Proof of Claim 1. Suppose that $\frac{a}{b}, \frac{a'}{b'} \in I$, with $\frac{a}{b} \neq \frac{a'}{b'}$. Then

$$\frac{1}{n} > \left|\frac{a}{b} - \frac{a'}{b'}\right| = \frac{|a - ka'|}{b} \geq \frac{1}{b} \geq \frac{1}{n},$$

a contradiction, which proves the claim.

From Claim 1, it follows easily that

$$f(n) \leq g(n). \tag{1}$$

Claim 2. We have $g(n) = \left\lfloor \frac{n+1}{2} \right\rfloor$.

Proof of Claim 2. Let $k = \left\lfloor \frac{n+1}{2} \right\rfloor + 1$, and let $a_1, a_2, \ldots, a_k$ be any k integers satisfying

$$1 \leq a_1 < a_2 < \cdots < a_k \leq n.$$

For each i, let $a_i = 2^{\alpha_i}\beta_i$, where α_i, β_i are non-negative integers, and β_i is odd.

Suppose that n is even, say $n = 2p$. Then $k = p + 1$, and there are exactly p odd integers in $\{1, 2, \ldots, n\}$. Thus, $\beta_i = \beta_j$ for some $i < j$. Since $a_i < a_j$, we must have $\alpha_i < \alpha_j$, from which we deduce that a_i divides a_j.

Suppose that n is odd, say $n = 2p + 1$. Then $k = p + 2$, and there are exactly $p + 1$ odd integers in $\{1, 2, \ldots, n\}$. As above, we deduce that there are integers $i < j$ such that a_i divides a_j.

In either case, for each choice of k distinct integers in $\{1, 2, \ldots, n\}$, there are two of them, say a and b, such that a divides b. It follows that

$$g(n) \leq \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{2}$$

Conversely, consider the set $S = \{p + 1, p + 2, \ldots, n\}$, where $p = \lfloor n/2 \rfloor$. The set S contains $\left\lfloor \frac{n+1}{2} \right\rfloor$ elements, and if a and b are two distinct elements of S, then neither a divides b, nor b divides a. Thus,

$$g(n) \geq \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{3}$$

From (2) and (3), Claim 2 is proved.

From Claim 2 and (1), we deduce that

$$f(n) \ \leq \ \left\lfloor \frac{n+1}{2} \right\rfloor . \tag{4}$$

If $n = 2p+1$, then, for each $r \in \{1,\, 2,\, \dots,\, p+1\}$, the irreducible fraction $\frac{1}{p+r}$ belongs to the open interval $I = \left(\frac{1}{2p+1} - \varepsilon\,,\, \frac{2}{2p+1} - \varepsilon \right)$, where $\varepsilon = \frac{1}{2(2p+1)(p+1)}$. If $n = 2p$, then, for each $r \in \{1,\, 2,\, \dots,\, p\}$, the irreducible fraction $\frac{1}{p+r}$ belongs to the open interval $I = \left(\frac{1}{2p} - \varepsilon\,,\, \frac{1}{p} - \varepsilon \right)$, where $\varepsilon = \frac{1}{2p(p+1)}$. In either case, the open interval I of width $\frac{1}{n}$ contains at least $\left\lfloor \frac{n+1}{2} \right\rfloor$ irreducible fractions $\frac{a}{b}$ with $1 \leq b \leq n$. It follows that

$$f(n) \ \geq \ \left\lfloor \frac{n+1}{2} \right\rfloor . \tag{5}$$

From (4) and (5), we have $f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor$, as claimed.

The next problem set given in the October 2001 *Corner* was the 38th National Mathematical Olympiad of Slovenia 1994 Final Round [2001 : 359]. Some of our readers have better memories and a more careful systematic approach than I seem to have demonstrated. Pierre Bornsztein points out that this problem set was given previously in the *Corner* [1998 : 132] with solutions appearing in [1999 : 208–211] and [1999 : 266–269]. Let me thank those who submitted their solutions to some of the problems in response to the 2001 call:

- Robert Bilinski, Outremont, QC (Grade 4: #3)
- Christopher J. Bradley, Clifton College, Bristol, UK (Grade 3: #1, 3, 4; Grade 4: #1, 2, 3)
- Laura Gu, student, Sir Winston Churchill High School, Calgary, AB (Grade 3: #2)
- Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB (Grade 3: #1; Grade 4: #1)
- Toshio Seimiya, Kawasaki, Japan (Grade 3: #4; Grade 4: #4)

That completes the *Corner* for this issue. Send me your nice solutions and generalizations, as well as contest materials.

BOOK REVIEW

John Grant McLoughlin

Hungarian Problem Book III, Based on the Eötvös Competitions: 1929–1943
Translated and edited by Andy Liu, published by the Mathematical Association of America (Anneli Lax New Mathematical Library Series), 2001
ISBN 0-88385-644-1, paperbound, 163 pages, US$29.95
*Reviewed by **Jozef Širáň**, Slovak University of Technology, Bratislava, Slovakia.*

With origins going back to 1894, the *Eötvös Competition* is the world's oldest mathematics competition for high-school students. It was organized in Hungary on a national scale annually up to the 1943 interruption due to war. Resuming in 1947, it was renamed in 1949 as the *Kürschág Competition*. According to the tradition, solving competition problems challenges the student's ability for independent mathematical thinking but requires no knowledge beyond the standard high-school background.

So far, four volumes of collected competition problems (with solutions) have appeared in Hungarian, covering the periods 1894-1928, 1929-1963, 1964-1987, and 1988-1997. The first Hungarian volume was translated and published in 1963 by the New Mathematical Library in two books entitled *Hungarian Problem Book I* and *Hungarian Problem Book II*.

This *Hungarian Problem Book III* covers the Eötvös-Kürschág Competition of the period 1929-1943. The material is divided into six chapters. The first chapter lists the contest problems in chronological order. The 45 problems are then classified by subject into five groups: Combinatorics, Number Theory, Algebra, Geometry - Part I, and Geometry - Part II. Each of these groups forms a separate chapter.

The book is suitable for "beginners" (students interested in challenging problems with no prior experience) as well as "advanced" high school students. Beginners are encouraged to go through Chapters 2 to 6 in order. For example, Chapter 2 (Combinatorics) presents the problems in a restated form, and students are encouraged to solve them in the given order. They can check their success against solutions in Section 2.2. If they find the problems too hard, they are invited to read Section 2.1, which gives detailed discussions. Experienced students can simply try the problems and then compare their work with the (often multiple) solutions offered. The problems are in many cases accompanied with discussions and necessary background material, providing generalizations or making relevant comments.

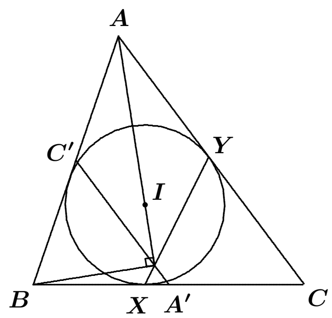
I believe that the book will be a success among high-school students interested in mathematics. Also, high-school mathematics teachers will find it to be a valuable and enjoyable source of problems, ideas, and clever solution methods.

Another Unlikely Concurrence

Jean-Louis Ayme

In section 3.4 of [4], under the heading "An Unlikely Concurrence", Ross Honsberger discusses the following theorem:

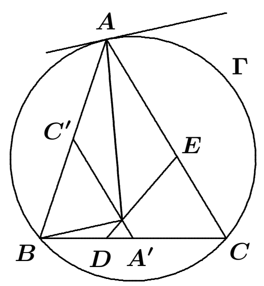
If I is the incentre of $\triangle ABC$, and X and Y the points of contact of the incircle with BC and CA, then the lines AI, XY, and the perpendicular from B to AI are concurrent.



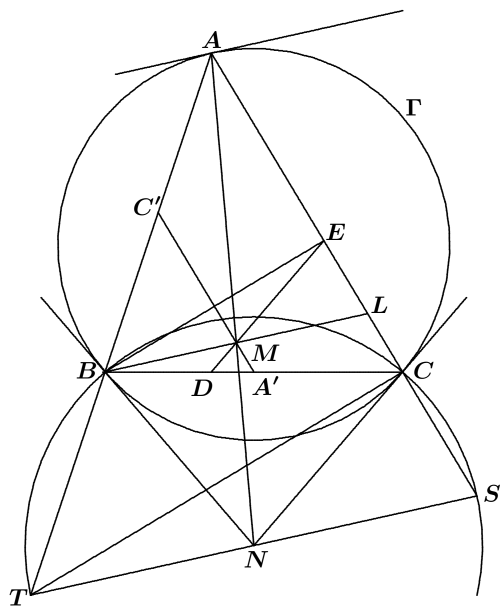
Altshiller Court has this result as an exercise in [1] (p. 118, ex. 43). Earlier, Papelier [6] (p. 19) had stated it for a right triangle. Moreover, long before that (in 1859) Lascases [5] showed that this point also lies on the line that joins the mid-point of BC to the mid-point of BA: more precisely, he proved that the line joining the two mid-points contains the foot of the perpendicular from B to AI. (See [3] p. 327, no. 761.)

The following theorem gives a similarly unlikely concurrence of four lines—a result which I was unable to locate in the literature.

Theorem. Let ABC be a triangle and Γ its circumcircle. Let D and E be the feet of the altitudes from A to BC and B to CA, and let A' and C' be the respective mid-points of BC and BA. Let s_A be the symmedian through A, and let a'_B be the parallel through B to the tangent to Γ at A. Then the four lines s_A, a'_B, DE, and $A'C'$ are concurrent.



Proof. Let a', b', c' be the tangents to Γ at the vertices A, B, C, respectively; let L, M be the respective intersections of the line a'_B with AC and DE; let N be the intersection of b' and c'; let a'_N be the parallel to a' through N; and let S and T be the respective intersections of a'_N with AC and AB. See the figure on the next page.



According to Lemoine, we have $s_A = AN$ (the symmedian through one vertex of a triangle passes through the intersection point of the tangents to the circumcircle at the other two vertices; see [1], p. 248, Theorem 560).

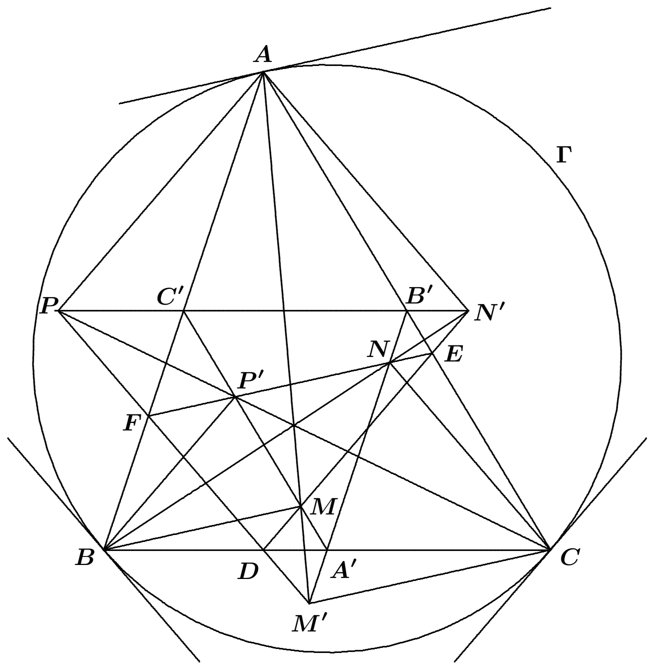
Boutin's Theorem [2] states: If three circles are externally tangent in pairs, the lines joining the contact point of two of the circles to the other two contact points meet the third circle for the second time at the ends of a diameter; furthermore, that diameter is parallel to the line joining the centres of the first two circles. (See [3], p. 553, no. 1256).

Consider now the circle centred at N which passes through B and C, the circle centred at the intersection of a' and b' which passes through A and B, and the circle centred at the intersection of a' and c' which passes through A and C. Boutin's Theorem applied to these circles implies that B, C, S, and T lie on a circle having ST for diameter and N for centre. Consequently, CT is perpendicular to the line ACS. Since $ACS \perp BE$ (by definition), we have $CT \parallel BE$.

A theorem of Carnot (see [3], p. 284) states that the sides of the orthic triangle are parallel to the tangents to the circumcircle of a given triangle at its vertices. Thus, we have $CN \parallel DE$. Also, since $a'_B \parallel a'$ and $a' \parallel a'_N$, we have $a'_B \parallel a'_N$; that is, $BM \parallel TN$. Therefore, the triangles BME and TNC are homothetic. The Theorem of Desargues applied to these triangles (triangles perspective from a line—here the line at infinity—are perspective from a point), implies that A, M, N are collinear.

Since $BL \parallel TS$ and N is the mid-point of TS, M is the mid-point of BL; consequently, M lies on $A'C'$ (since $A'C'$ passes through the mid-points of all cevians through B). Thus, M lies on the four lines as desired.

Of course, we get a second concurrence at M', say, by interchanging the roles of B and C. By cyclically permuting the vertices, we get two more pairs of quadruple-concurrence points, N, N' and P, P', indicated in the complete figure below.



References.

[1] Nathan Altshiller Court, *College Geometry*, Barnes and Noble, 1952.

[2] M.A. Boutin, *Journal de mathématiques élémentaires*, (1890), p. 113.

[3] F.G.-M., *Exercices de géométrie*, 4th edition (1907).

[4] Ross Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, New Mathematical Library 37, The Mathematical Association of America, 1995.

[5] A. Lascases, *Nouvelles annales* **18** (1859), p. 171, No. 477.

[6] G. Papelier, *Exercices de géométrie modernes*, Pôles et polaires, 1927.

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PROBLEMS

Faire parvenir les propositions de problèmes et les solutions à Jim Totten, Département de mathématiques et de statistique, University College of the Cariboo, Kamloops, BC V2C 5N3. Les propositions de problèmes doivent être accompagnées d'une solution ainsi que de références et d'autres indications qui pourraient être utiles à la rédaction. Si vous envoyez une proposition sans solution, vous devez justifier une solution probable en fournissant suffisamment d'information. Un numéro suivi d'une astérisque (⋆) indique que le problème a été proposé sans solution.

Nous sollicitons en particulier des problèmes originaux. Cependant, d'autres problèmes intéressants pourraient être acceptables s'ils ne sont pas trop connus et si leur provenance est précisée. Normalement, si l'auteur d'un problème est connu, il faut demander sa permission avant de proposer un de ses problèmes.

Pour faciliter l'étude de vos propositions, veuillez taper ou écrire à la main (lisiblement) chaque problème sur une feuille distincte de format $8\frac{1}{2}$"×11" ou A4, la signer et la faire parvenir au rédacteur en chef. Les propositions devront lui parvenir au plus tard le **1er juin 2004**. *Vous pouvez aussi les faire parvenir par courriel à* **crux-editors@cms.math.ca**. (*Nous apprécierions de recevoir les problèmes et solutions envoyés par courriel au format* LaTeX). *Les fichiers graphiques doivent être de format « epic » ou « eps » (encapsulated postscript). Les solutions reçues après la date ci-dessus seront prises en compte s'il reste du temps avant la publication. Veuillez prendre note que nous n'acceptons pas les propositions par télécopieur.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

2889. *Proposé par Vedula N. Murty, Dover, PA, USA.*

On suppose que A, B et C sont les angles d'un triangle ABC, et que r et R sont respectivement les rayons du cercle inscrit et du cercle circonscrit. Montrer que

$$4\cos(A)\cos(B)\cos(C) \;\leq\; 2\left(\frac{r}{R}\right)^2 .$$

....

Suppose that A, B, C are the angles of $\triangle ABC$, and that r and R are its inradius and circumradius, respectively. Show that

$$4\cos(A)\cos(B)\cos(C) \;\leq\; 2\left(\frac{r}{R}\right)^2 .$$

2890. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelona, Espagne.*

On suppose que le polynôme $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ possède la factorisation $A(z) = \prod_{k=1}^{n} (z - z_k)$, où les z_k sont des nombres réels positifs.

Montrer que, pour $k = 1, 2, \ldots, n-1$,

$$\left|\frac{a_{n-k}}{C(n,k)}\right|^{\frac{1}{k}} \geq \left|\frac{a_{n-k-1}}{C(n,k+1)}\right|^{\frac{1}{k+1}},$$

où $C(n,k)$ désigne le coefficient binomial $\binom{n}{k}$. Quand y a-t-il égalité ?

. . . .

Suppose that the polynomial $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ can be factored into $A(z) = \prod_{k=1}^{n} (z - z_k)$, where the z_k are positive real numbers.

Prove that, for $k = 1, 2, \ldots, n-1$,

$$\left|\frac{a_{n-k}}{C(n,k)}\right|^{\frac{1}{k}} \geq \left|\frac{a_{n-k-1}}{C(n,k+1)}\right|^{\frac{1}{k+1}},$$

where $C(n,k)$ denotes the binomial coefficient $\binom{n}{k}$. When does equality occur?

2891. *Proposé par Vedula N. Murty, Dover, PA, USA, adapté par les rédacteurs.*

On a demandé à deux correcteurs, Charles et Paul, de travailler sur un manuscript pour en trouver les erreurs. Soit B le nombre d'erreurs dénichées par Charles et Paul, C le nombre d'erreurs dénichées par Charles seulement et P le nombre d'erreurs dénichées par Paul seulement ; finalement, soit N le nombre d'erreurs que tous les deux ont laissé passer.

Montrer que $\sqrt{(B+P)(C+N)(B+C)(P+N)} \geq |BN - CP|$.

. . . .

Two proofreaders, Chris and Pat, were asked to read a manuscript and find the errors. Let B be the number of errors which both Chris and Pat found, C the number of errors found only by Chris, and P the number found only by Pat; lastly, let N be the number of errors found by neither of them.

Prove that $\sqrt{(B+P)(C+N)(B+C)(P+N)} \geq |BN - CP|$.

2892. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

(a) Soit A et B deux matrices 2×2 complexes quelconques. Montrer que, pour tous les nombres complexes α, β et γ,

$$\det(\alpha I + \beta AB + \gamma BA) \;=\; \det(\alpha I + \gamma AB + \beta BA)\,.$$

(Ici, I désigne la matrice identité 2×2 .)

(b)★ Y a-t-il une identité semblable pour des matrices $n \times n$?

[Le proposeur donne une démonstration-machine pour (a). Nous en voulons une démonstration *purement algébrique*.]

. . . .

(a) Let A and B be arbitrary 2×2 matrices over $\mathbb{C}$. For all complex numbers α, β, γ, prove that

$$\det(\alpha I + \beta AB + \gamma BA) \;=\; \det(\alpha I + \gamma AB + \beta BA)\,.$$

(Here, I denotes the 2×2 identity matrix.)

(b)★ Is there a similar identity for $n \times n$ matrices?

[The proposer gives a "Machine" proof for (a). We want a *purely algebraic* proof.]

2893. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Dans [2001 : 45–47], on trouve trois démonstrations de l'inégalité classique

$$1 \le \sum_{\text{cyclique}} \cos(A) \le \frac{3}{2}\,.$$

Dans [2002 : 85–87], on trouve des illustrations (dûes à Klamkin) de l'inégalité de majorisation (dite aussi inégalité de Karamata).

Montrer «l'inégalité classique» ci-dessus à l'aide de l'inégalité de majorisation.

. . . .

In [2001 : 45–47], we find three proofs of the classical inequality

$$1 \le \sum_{\text{cyclic}} \cos(A) \le \frac{3}{2}\,.$$

In [2002 : 85–87], we find Klamkin's illustrations of the Majorization (or Karamata) Inequality.

Prove the above "classical inequality" using the Majorization Inequality.

2894. 2002–077 *Proposé par Vedula N. Murty, Dover, PA, USA.*

On suppose que le triangle ABC est acutangle. Avec la notation habituelle, montrer que

$$4abc \;<\; \left(a^2+b^2+c^2\right)\left(a\cos A+b\cos B+c\cos C\right) \;\leq\; \frac{9}{2}abc\,.$$

. . . .

Suppose that $\triangle ABC$ is acute-angled. With the standard notation, prove that

$$4abc \;<\; \left(a^2+b^2+c^2\right)\left(a\cos A+b\cos B+c\cos C\right) \;\leq\; \frac{9}{2}abc\,.$$

2895. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit A et B deux événements avec les probabilités $P(A)$ et $P(B)$ tels que $0 < P(A) < 1$ et $0 < P(B) < 1$. On pose

$$K \;=\; \frac{2[P(A\cap B)-P(A)P(B)]}{P(A)+P(B)-2P(A)P(B)}\,.$$

Montrer que $|K| < 1$, et donner une interprétation de la valeur $K = 0$.

. . . .

Suppose that A and B are two events with probabilities $P(A)$ and $P(B)$ such that $0 < P(A) < 1$ and $0 < P(B) < 1$. Let

$$K \;=\; \frac{2[P(A\cap B)-P(A)P(B)]}{P(A)+P(B)-2P(A)P(B)}\,.$$

Show that $|K| < 1$, and interpret the value $K = 0$.

2896. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Supposons que $0 < x_0 < x_1$ et que, pour $n = 1, 2, 3, \ldots,$

$$\sqrt{1+x_n}\left(1+\sqrt{x_{n-1}x_{n+1}}\right) \;=\; \sqrt{1+x_{n-1}}\left(1+\sqrt{x_n x_{n+1}}\right)\,.$$

(a) Montrer que la suite $\{x_n\}$ est convergente.

(b) Trouver $\lim\limits_{n\to\infty} x_n$.

. . . .

Suppose that $0 < x_0 < x_1$ and that, for $n = 1, 2, 3, \ldots,$

$$\sqrt{1+x_n}\left(1+\sqrt{x_{n-1}x_{n+1}}\right) \;=\; \sqrt{1+x_{n-1}}\left(1+\sqrt{x_n x_{n+1}}\right)\,.$$

(a) Prove that the sequence $\{x_n\}$ is convergent.

(b) Find $\lim\limits_{n\to\infty} x_n$.

2897. *Proposé par Václav Konečný, Big Rapids, MI, USA.*
(a) Montrer qu'il est possible de partager un disque en quatre parties d'aire égale à l'aide de trois segments de longueur égale.

(b) Existe-t-il une construction avec la règle et le compas (au sens classique, c'est-à-dire en un nombre fini de pas) ?

. . . .

(a) Show that it is possible to divide a circular disc into four parts with the same area by means of three line segments of the same length.

(b) Does there exist a straight edge and compass construction (in the classical sense; that is, with a finite number of steps)?

2898. *Proposé par Mihály Bencze, Brasov, Roumanie.*
Montrer que $\frac{(2^n)!}{1!2!4!\cdots(2^{n-1})!}$ est divisible par $\prod_{k=1}^{n}\left(2^{k-1}+1\right)$.

. . . .

Prove that $\frac{(2^n)!}{1!2!4!\cdots(2^{n-1})!}$ is divisible by $\prod_{k=1}^{n}\left(2^{k-1}+1\right)$.

2899. *Proposé par Hiroshi Kotera, Nara City, Japon.*
Trouver l'aire maximale d'un pentagone $ABCDE$ inscrit dans un cercle de rayon 1 tel que la diagonale AC soit perpendiculaire à la diagonale BD.

. . . .

Find the maximum area of a pentagon $ABCDE$ inscribed in a unit circle such that the diagonal AC is perpendicular to the diagonal BD.

2900★. *Proposé par Stanley Rabinowitz, Westford, MA, USA.*
Dans le triangle ABC, soit I le centre du cercle inscrit, r_1 le rayon du cercle inscrit du triangle IAB, et r_2 le rayon du cercle inscrit du triangle IAC. L'utilisation du logiciel *Geometer's Sketchpad* suggère la conjecture $r_2 < \frac{5}{4}\,r_1$.

(a) Démontrer ou réfuter cette conjecture.

(b) Peut-on remplacer $\frac{5}{4}$ par une constante plus petite ?

. . . .

Let I be the incentre of $\triangle ABC$, r_1 the inradius of $\triangle IAB$ and r_2 the inradius of $\triangle IAC$. Computer experiments using *Geometer's Sketchpad* suggest that $r_2 < \frac{5}{4}r_1$.

(a) Prove or disprove this conjecture.

(b) Can $\frac{5}{4}$ be replaced by a smaller constant?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2790. [2002 : 533] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that $$\sum_{1\le i\le j\le n} \sin^2\left(\frac{(j-i)\pi}{n}\right) \;=\; \frac{n^2}{4}\,.$$

I. *Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina.*

Suppose $n \ge 2$, as the claim does not hold for $n = 1$. It is well known that

$$\sum_{k=1}^{n}\left(\cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right)\right) \;=\; \sum_{i=1}^{n} e^{\frac{2k\pi}{n}i} \;=\; 0\,.$$

Hence, we have

$$\begin{aligned}
&\sum_{1\le i,j\le n}\cos\left(\frac{2(j-i)\pi}{n}\right)\\
&\quad= \sum_{1\le i,j\le n}\left(\cos\left(\frac{2j\pi}{n}\right)\cos\left(\frac{2i\pi}{n}\right)+\sin\left(\frac{2j\pi}{n}\right)\sin\left(\frac{2i\pi}{n}\right)\right)\\
&\quad= \left(\sum_{k=1}^{n}\cos\left(\frac{2k\pi}{n}\right)\right)^2+\left(\sum_{k=1}^{n}\sin\left(\frac{2k\pi}{n}\right)\right)^2\\
&\quad= \left|\sum_{k=1}^{n}\left(\cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right)\right)\right|^2 \;=\; 0\,.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{1\le i\le j\le n}\sin^2\left(\frac{(j-i)\pi}{n}\right) &= \frac{1}{4}\sum_{1\le i,j\le n}\left(1-\cos\left(\frac{2(j-i)\pi}{n}\right)\right)\\
&= \frac{n^2}{4}-\frac{1}{4}\sum_{1\le i,j\le n}\cos\left(\frac{2(j-i)\pi}{n}\right) \;=\; \frac{n^2}{4}\,.
\end{aligned}$$

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA (modified slightly by the editor).*

The claim is clearly false for $n = 1$. We assume that $n \ge 2$. Let S denote the sum to be evaluated, and let $\theta = \frac{\pi}{n}$. Then

$$S \;=\; \sum_{k=1}^{n-1}(n-k)\sin^2(k\theta)\,, \tag{1}$$

Let $m = n - k$. Then $k\theta = n\theta - m\theta = \pi - m\theta$. Hence,

$$S \;=\; \sum_{m=1}^{n-1} m\sin^2(\pi - m\theta) \;=\; \sum_{m=1}^{n-1} m\sin^2(m\theta)\,. \qquad (2)$$

Adding (1) and (2), and noting that $\sin(n\theta) = 0$, we get

$$2S \;=\; \sum_{k=1}^{n-1} n\sin^2(k\theta) \;=\; n\sum_{k=1}^{n}\sin^2(k\theta)\,.$$

Since $n \geq 2$ implies $\sin\theta \neq 0$, we have

$$\begin{aligned} S &= \frac{n}{4}\sum_{k=1}^{n}(1-\cos(2k\theta)) \\ &= \frac{n^2}{4} - \frac{n}{8\sin\theta}\sum_{k=1}^{n}[\sin(2k+1)\theta - \sin(2k-1)\theta] \\ &= \frac{n^2}{4} - \frac{n}{8\sin\theta}[\sin(2n+1)\theta - \sin\theta] \\ &= \frac{n^2}{4} - \frac{n}{8\sin\theta}[\sin(2\pi+\theta) - \sin\theta] \;=\; \frac{n^2}{4}\,. \end{aligned}$$

Also solved by JEAN-CLAUDE ANDRIEUX, Beaune, France; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, Concordia University, Montréal, QC; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; NEVEN JURIČ, Zagreb, Croatia; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; STAN WAGON, Macalester College, St. Paul, MN, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Most of the submitted solutions used complex exponential functions.

Wagon commented that "there are well-known algorithms for determining sums such as these (Gosper's algorithm is the most prominent), and these are built into Mathematica, Maple, etc." He stated that the answer $n^2/4$ came out immediately when he typed the given summation into Mathematica. He further commented that "the generation of such symbolic sums is completely algorithmic, analogous to getting 100 digits of $\sqrt{2}$".

2791. [2002 : 533] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $f : [0,1] \to (0,\infty)$ is a continuous function. Prove that if there exists $\alpha > 0$ such that, for $n \in \mathbb{N}$,

$$\int_0^1 x^\alpha\big(f(x)\big)^n\,dx \;\geq\; \frac{1}{(n+1)\alpha+1} \;\geq\; \int_0^1 \big(f(x)\big)^{n+1}\,dx\,,$$

then α is unique.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Assume that $0 < \alpha < \beta$ and that both α and β satisfy the given inequalities for every $n \in \mathbb{N}$. Choose $N \in \mathbb{N}$ sufficiently large so that $\alpha < \left(\frac{N+1}{N+2}\right)\beta$. Then

$$\begin{aligned}\frac{1}{(N+1)\beta+1} &\geq \int_0^1 \big(f(x)\big)^{N+1}\,dx \geq \int_0^1 x^{\alpha}\big(f(x)\big)^{N+1}\,dx \\ &\geq \frac{1}{(N+2)\alpha+1},\end{aligned}$$

a contradiction.

Also solved by MICHEL BATAILLE, Rouen, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and the proposer.

2792. [2002 : 533, 2003 : 47] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $A_k \in M_n(\mathbb{R})$ $(k = 1, 2, \ldots, m \geq 2)$ for which

$$\sum_{1\leq i<j\leq m} (A_iA_j + A_jA_i) = 0_n .$$

Prove that $$\det\left(\sum_{k=1}^{m} (I_n + A_k)^2 - (m-2)I_n\right) \geq 0 .$$

Solution by Michel Bataille, Rouen, France.

Let B be the matrix $I_n+A_1+A_2+\cdots+A_m$. Using the given condition, we get

$$\left(\sum_{k=1}^{m} A_k\right)^2 = \sum_{k=1}^{m} A_k^2 ,$$

and hence,

$$B^2 = \left(I_n + \sum_{k=1}^{m} A_k\right)^2 = I_n + 2\sum_{k=1}^{m} A_k + \sum_{k=1}^{m} A_k^2 .$$

Let

$$C = \sum_{k=1}^{m} (I_n + A_k)^2 - (m-2)I_n .$$

Then

$$C = 2I_n + 2\sum_{k=1}^{m} A_k + \sum_{k=1}^{m} A_k^2 = I_n + B^2 .$$

Since the entries of B are real numbers, we obtain

$$\begin{aligned}\det(C) &= \det(I_n + B^2) \;=\; \det\big((I_n+iB)(I_n-iB)\big)\\ &= \det\big((I_n+iB)\overline{(I_n+iB)}\big) \;=\; \det(I_n+iB)\det\overline{(I_n+iB)}\\ &= \det(I_n+iB)\overline{\det(I_n+iB)} \;=\; |\det(I_n+iB)|^2 \;\geq\; 0\,,\end{aligned}$$

as desired.

Also solved by CON AMORE PROBLEM GROUP, The Danish University of Education Copenhagen, Denmark; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2793. [2002 : 534] *Proposed by Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.*

Show that the area of the image of the portion of the unit disc which lies in the first quadrant under the mapping $\zeta = \cosh^{-1}(z)$ is a well-known constant.

Solution by Michel Bataille, Rouen, France.

The required area is **Catalan's constant**, $G = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)^2}$.

Let $z = x+iy$ and $\zeta = \xi + i\eta$, where x, y, ξ and η denote real numbers. Let Δ be the portion of the unit disk lying in the first quadrant. The required area is

$$A \;=\; \iint\limits_{\cosh^{-1}(\Delta)} d\xi\,d\eta \;=\; \iint\limits_{\Delta} \left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right| dx\,dy \;=\; \iint\limits_{\Delta} \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right|^{-1} dx\,dy\,,$$

where the change of variables is defined by

$$z \;=\; x+iy \;=\; \cosh\zeta \;=\; \cosh\xi\cos\eta + i\sinh\xi\sin\eta\,.$$

We readily obtain that

$$\begin{aligned}\frac{\partial(x,y)}{\partial(\xi,\eta)} &= \begin{vmatrix} \dfrac{\partial x}{\partial\xi} & \dfrac{\partial x}{\partial\eta} \\ \dfrac{\partial y}{\partial\xi} & \dfrac{\partial y}{\partial\eta}\end{vmatrix} \;=\; \sinh^2\xi\cos^2\eta + \cosh^2\xi\sin^2\eta\\ &= \sinh^2\xi + \sin^2\eta \;=\; \left|\sinh^2\zeta\right| \;=\; \left|z^2-1\right|\,.\end{aligned}$$

Using polar coordinates, we let $z = \rho\cos\theta + i\rho\sin\theta$, for $\theta \in (0,\pi/2)$. Then

$$\begin{aligned}\left|z^2-1\right|^2 &= \left(x^2-y^2-1\right)^2 + 4x^2y^2\\ &= \left(\rho^2\cos(2\theta)-1\right)^2 + \rho^4\sin^2(2\theta)\\ &= \sin^2(2\theta)\left(1+\left(\frac{\rho^2-\cos(2\theta)}{\sin(2\theta)}\right)^2\right)\,.\end{aligned}$$

It follows that

$$\begin{aligned}
A &= \int_0^{\pi/2}\int_0^1 \frac{\dfrac{\rho}{\sin(2\theta)}}{\sqrt{1+\left(\dfrac{\rho^2-\cos(2\theta)}{\sin(2\theta)}\right)^2}}\,d\rho\,d\theta. \\
&= \frac{1}{2}\int_0^{\pi/2}\int_{-\cot(2\theta)}^{\tan\theta} \frac{du}{\sqrt{1+u^2}}\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2}\Bigg(\ln\left(\tan\theta+\sqrt{1+\tan^2\theta}\right) \\
&\qquad\qquad -\ln\left(-\cot(2\theta)+\sqrt{1+\cot^2(2\theta)}\right)\Bigg)\,d\theta \\
&= \frac{1}{2}\left(\int_0^{\pi/2}\ln(1+\sin\theta)\,d\theta - \int_0^{\pi/2}\ln(\sin\theta)\,d\theta\right).
\end{aligned}$$

It is well known that $\int_0^{\pi/2}\ln(\sin\theta)\,d\theta = -\frac{\pi}{2}\ln 2$. Furthermore, we have

$$\begin{aligned}
\int_0^{\pi/2}\ln(1+\sin\theta)\,d\theta &= \int_0^{\pi/2}\ln(1+\cos t)\,dt \\
&= \int_0^{\pi/2}\ln\left(2\cos^2\left(\tfrac{t}{2}\right)\right)dt = \frac{\pi}{2}\ln 2 + 2I\,,
\end{aligned}$$

where $I = \int_0^{\pi/2}\ln\left(\cos\left(\frac{t}{2}\right)\right)dt$. In order to compute I, we introduce the complementary integral $J = \int_0^{\pi/2}\ln\left(\sin\left(\frac{t}{2}\right)\right)dt$. Note that

$$J+I = \int_0^{\pi/2}\ln\left(\tfrac{1}{2}\sin t\right)dt = -\pi\ln 2\,,$$

and

$$\begin{aligned}
J-I = \int_0^{\pi/2}\ln\left(\tan\left(\tfrac{t}{2}\right)\right)dt &= -\int_0^{\pi/2}\frac{t}{\sin t}\,dt \\
&= -2\int_0^1\frac{\arctan x}{x}\,dx = -2G\,,
\end{aligned}$$

where we have used integration by parts, followed by the substitution $t = 2\arctan(x)$, and then the expansion $\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$, which is uniformly convergent for $x\in[0,1]$. It follows that $2I = -\pi\ln 2 + 2G$.

Substituting these results into the above expression for A, we obtain

$$A = \frac{1}{2}\left(\frac{\pi}{2}\ln 2 - \pi\ln 2 + 2G + \frac{\pi}{2}\ln 2\right) = G\,.$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK, ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Two incomplete solutions were received, where the solvers did not identify the name of the constant.

Specht and Zhou made reference to WEB sites with this and other "neat" constants. Loeffler's authority was MATHEMATICA.

2794. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $z_k \in \mathbb{C}^*$ $(k = 1, 2, \ldots, n)$ such that

$$\begin{aligned}|z_1 + z_2 + \cdots + z_n| + |z_2 + z_3 + \cdots + z_n| + \cdots + |z_{n-1} + z_n| + |z_n| \\ = \ |z_1 + 2z_2 + \cdots + nz_n|\ .\end{aligned}$$

Prove that the z_k are collinear.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

By the well-known generalization of the Triangle Inequality, we have, for any complex numbers $w_1, \ldots, w_n$,

$$\sum_{k=1}^{n} |w_k| \ \geq\ \left|\sum_{k=1}^{n} w_k\right|\ .$$

Equality holds if and only if the ratio w_i/w_j is real and positive, for any two non-zero terms w_i and w_j. Assuming $w_n \neq 0$, equality holds if and only if $w_k = c_k w_n$ for $k = 1, 2, \ldots, n-1$, where each c_k is real and non-negative. [*Ed:* Zhou called the inequality above Minkowski's Inequality, and gave the reference [1] below.]

Let $w_k = \sum\limits_{j=k}^{n} z_j$ for $1 \leq k \leq n$. Then the given equation becomes

$$\sum_{k=1}^{n} |w_k| \ =\ \left|\sum_{k=1}^{n} w_k\right|\ .$$

Since $w_n \neq 0$, we have $w_k = c_k w_n$ for $k = 1, 2, \ldots, n-1$, and therefore, $z_k = w_k - w_{k+1} = (c_k - c_{k+1})z_n$. That is, the z_k are collinear.

Reference

[1] D.O. Shklarsby, N.N. Chentzov, and I.M. Yaglom, The USSR Olympiad Problem Book, Dover, 1993, pp.42–43 and 290–291 (Problem 186 (a) and its solution).

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2795. [2002 : 534] *Proposed by Mihály Bencze, Brasov, Romania.*

A convex polygon with sides a_1, a_2, …, a_n is inscribed in a circle of radius R. Prove that

$$\sum_{k=1}^{n} \sqrt{4R^2 - a_k^2} \;\le\; 2nR\sin\left(\frac{(n-2)\pi}{n}\right) .$$

[*Editor's comments:* The denominator on the right side of the inequality should have been $2n$ instead of n. All solvers made this correction. There is another error, a missing hypothesis, as explained in the featured solution below. Failure to note the need for this hypothesis did not, in itself, disqualify a solution.]

Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.

The stated result is not true. Just consider a case where all sides have lengths close to zero. The problem should have stated that the centre of the circle is not in the exterior of the polygon.

For $k = 1, 2, \ldots, n$, let θ_k be half the angle at the centre of the circle subtended by a_k. Then $a_k = 2R\sin\theta_k$ and $\sqrt{4R^2 - a_k^2} = 2R\cos\theta_k$. Since $\cos\theta$ is concave in the interval $(0, \pi/2)$, we have

$$\sum_{k=1}^{n} \cos\theta_k \;\le\; n\cos\left(\frac{1}{n}\sum_{k=1}^{n}\theta_k\right) \;=\; n\cos\left(\frac{\pi}{n}\right) \;=\; n\sin\left(\frac{(n-2)\pi}{2n}\right) .$$

Then

$$\sum_{k=1}^{n} \sqrt{4R^2 - a_k^2} \;=\; 2R\sum_{k=1}^{n}\cos\theta_k \;\le\; 2nR\sin\left(\frac{(n-2)\pi}{2n}\right) .$$

Equality holds for regular polygons.

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

Specht gives the following argument showing that the inequality as originally printed is true for $n \le 6$ (with the extra hypothesis). First note that

$$\sin\left(\frac{(n-2)\pi}{n}\right) \;=\; \sin\left(\frac{2\pi}{n}\right) \;=\; 2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right) .$$

If $n \leq 6$*, then* $2\sin(\pi/n) \geq 2\sin(\pi/6) = 1$ *and*

$$2nR\sin\left(\frac{(n-2)\pi}{n}\right) \;\geq\; 2nR\cos\left(\frac{\pi}{n}\right) \;\geq\; \sum_{k=1}^{n}\sqrt{4R^2 - a_k^2}\,.$$

The inequality as given originally is false for a regular n*-gon when* $n > 6$*.*

2796★. [2002 : 535] *Proposed by Fernando Castro G., Maturín Estado Monagas, Venezuela.*

Let $\{p_n\}$ be the sequence of prime numbers. Prove that, for each $n \geq 2$, the set $I = \{1, 2, \ldots, n\}$ can be partitioned into two sets A and B, where $A \cup B = I$, in such a way that,

$$1 \;\leq\; \frac{\prod_{i\in A} p_i}{\prod_{j\in B} p_j} \;\leq\; 2\,.$$

Comment: Both Michel Bataille, Rouen, France, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria, observe that the proposer has published Problem 2796 with a solution in *The Mathematical Gazette*, 86 (2002), pp. 264–265.

Li Zhou, Polk Community College, Winter Haven, FL, USA, observes that, from the Prime Number Theorem, we can say that for each $\epsilon > 0$, there is an N such that for any $n \geq 1$, the set $\{N+1, N+2, \ldots, N+2n\}$ can be partitioned into A, B with $|A| = |B| = n$ such that

$$1 < \frac{\prod_{i\in A} p_i}{\prod_{j\in B} p_j} < 1 + \epsilon$$

Also solved by PIERRE BORNSZTEIN, Pontoise, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and, DANIEL MARCOTTE, student, Bishop's University, Lennoxville, QC.

2797. [2002 : 535] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

In $\triangle ABC$, suppose that AD is an altitude. Suppose that perpendiculars from D meet the sides AB and AC at E and F, respectively. Suppose that G and H are points of AB and AC, respectively, such that $DG \parallel AC$ and $DH \parallel AB$. Prove that

(a) EF and GH intersect at A^* on BC.

Defining B^* and C^* similarly, prove that

(b) A^*, B^* and C^* are collinear.

Solution by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

(a) Since triangle ADB has a right angle at D and $DE \perp AB$, we have $\frac{AE}{EB} = \frac{AD^2}{BD^2}$. Similarly, $DF \perp AC$ in $\triangle ADC$, and hence, $\frac{CF}{FA} = \frac{CD^2}{AD^2}$. If we define A^* to be the point where BC meets EF, then Menelaus' Theorem applied to triangle ABC and the line EFA^* yields

$$1 \;=\; \frac{BA^*}{CA^*}\cdot\frac{CF}{FA}\cdot\frac{AE}{EB} \;=\; \frac{BA^*}{CA^*}\cdot\frac{CD^2}{AD^2}\cdot\frac{AD^2}{BD^2}\,.$$

Then

$$\frac{BA^*}{CA^*} \;=\; \frac{BD^2}{CD^2}\,. \tag{1}$$

We prove that A^*, H, and G are collinear by the converse of Menelaus' Theorem applied to triangle ABC. Since $DG \parallel AC$, we have $\frac{AG}{GB} = \frac{CD}{DB}$; since $DH \parallel AB$, we have $\frac{CH}{HA} = \frac{CD}{DB}$. Thus, the product

$$\frac{BA^*}{CA^*}\cdot\frac{CH}{HA}\cdot\frac{AG}{GB} \;=\; \frac{BA^*}{CA^*}\cdot\frac{CD}{DB}\cdot\frac{CD}{DB} \;=\; \frac{BA^*}{CA^*}\cdot\frac{CD^2}{BD^2} \;=\; 1\,,$$

according to (1). This implies that A^*, H, and G are collinear, as desired.

(b) In triangle ABC, let P be the foot of the altitude from B, and let R be the foot of the altitude from C. By arguments used to obtain (1), we see that

$$\frac{AB^*}{CB^*} \;=\; \frac{AP^2}{CP^2} \quad\text{and}\quad \frac{AC^*}{BC^*} \;=\; \frac{AR^2}{BR^2}\,. \tag{2}$$

Ceva's Theorem applied to D, P, and R gives us $\frac{BD}{DC}\cdot\frac{CP}{PA}\cdot\frac{AR}{RB} = 1$. Hence, by (1) and (2),

$$\frac{BA^*}{CA^*}\cdot\frac{CB^*}{B^*A}\cdot\frac{AC^*}{C^*B} \;=\; \left(\frac{BD}{DC}\cdot\frac{CP}{PA}\cdot\frac{AR}{RB}\right)^2 \;=\; 1\,.$$

By the converse to Menelaus' Theorem, we conclude that A^*, B^*, and C^* are collinear.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SATURNINO CAMPO RUIZ, Salamanca, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Editor's generalization:

This problem is a special case of a theorem in projective geometry, in which the altitudes of $\triangle ABC$ in our given problem are replaced by any three concurrent cevians. Of course, the line at infinity could also be replaced by an arbitrary line l, but it is convenient for us to continue to call two lines *parallel* if they meet on l.

> Let P be any point not on a side of triangle ABC, and let D be the point where AP meets BC. Suppose that the parallel to CP through D meets AB at E and the parallel to BP through D meets AC at F, while G and H are points of AB and AC, respectively, such that $DG \parallel AC$ and $DH \parallel AB$. Then EF and GH intersect at a point A^* on BC. Moreover, with B^* and C^* defined similarly, A^*, B^*, and C^* are collinear.

This result is an immediate consequence of an elementary theorem concerning projective mappings between lines. I will first give this projective argument, and then outline other proofs for readers who prefer life in a Euclidean world.

Let us fix D on BC and allow point X to move along AD. For each position of X on AD we define X' on AB to be the point where the parallel to CX through D meets AB, and we define X'' on AC to be the point where the parallel to BX through D meets AC. In this way X' and X'' are projectively related. Note that when X is the point at infinity of AD (that is, the point where AD meets l), $X' = X'' = A$, so that A is fixed by this projectivity. A projectivity relating the points of two distinct lines that fixes their common point must be a perspectivity, which means that the lines $X'X''$ go through a common point (which is the centre of the perspectivity); see, for example, H.S.M. Coxeter, *Projective Geometry*, 2nd ed. (Springer, 1987), p. 35, Theorem 4.22. We apply this theorem to our given triangle by noting that when X is at D, $X' = B$ and $X'' = C$; when X is at P, $X' = E$ and $X'' = F$; when X is at A, $X' = G$ and $X'' = H$. Thus, BC, EF, and GH pass through the centre of the perspectivity (the point we call A^*).

Coordinates provide an easy alternative proof. Since in the projective plane we are free to choose four lines for our coordinate axes, let us place A at the origin, B and C at infinity on the x– and y–axes, and let our line l be $x + y = 1$. Let D be the point at infinity of $y = mx$ (where m is a fixed non-zero number). If our moving point X has coordinates (p, mp), then $X' = \left(\frac{p(1+m)-1}{m}, 0\right)$, $X'' = (0, mp(1+m) - m)$, and $X'X''$ will have slope $-m^2$ (making A^* the point at infinity of $y = -m^2 x$). For yet another approach, one can leave l at infinity and modify the featured solution above with a second application of Ceva's Theorem. The argument involving the line GH remains unchanged, while the argument for EF uses Ceva's Theorem to deduce that EF is parallel to the line joining the point where CP meets AB to the point where BP meets AC.

For the final claim concerning the collinearity of A^*, B^*, and C^*, the proof in the featured solution above goes through in the general setting without change (since the points like G and H were defined without the use of perpendicularity).

2798★. [2002 : 535] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove or disprove the inequality $\displaystyle\sum_{j=1}^{n} \frac{1}{1 - \frac{P}{x_j}} \le \frac{n}{1 - \left(\frac{1}{n}\right)^{n-1}}$, where $\displaystyle\sum_{j=1}^{n} x_j = 1$, $x_j \ge 0$ $(j = 1, 2, \ldots, n)$, and $P = \displaystyle\prod_{j=1}^{n} x_j$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, adapted by the editor.

The proposed result is true for all $n \geq 3$. We will prove a slightly more general theorem.

Theorem. Let $n \geq 3$, $h > 0$ and $k \geq h^{n-1}$. Let

$$W(x_1, x_2, \ldots, x_n) \;=\; \sum_{j=1}^{n} \frac{1}{k - \frac{P}{x_j}}\,,$$

where $\sum\limits_{j=1}^{n} x_j = h$, $x_j \geq 0$ ($j = 1, 2, \ldots, n$), and $P = \prod\limits_{j=1}^{n} x_j$. Then

$$W(x_1, x_2, \ldots, x_n) \;\leq\; \frac{n}{k - \left(\frac{h}{n}\right)^{n-1}}\,.$$

Equality holds if and only if $(x_1, x_2, \ldots, x_n) = \left(\frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n}\right)$.

Proof: The set of admissible values for $(x_1, x_2, \ldots, x_n)$ is a compact set in $\mathbb{R}^n$, and W is continuous on this set. Therefore, W attains a maximum value at some point in the set. We will prove that the maximum cannot occur at any point other than $\left(\frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n}\right)$. Since

$$W\left(\frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n}\right) \;=\; \frac{n}{k - \left(\frac{h}{n}\right)^{n-1}}\,,$$

we will then have the desired result.

We proceed by induction on n. The case $n = 3$ is essentially CRUX Problem #2786 [2002 : 459; 2003 : 476–477], but we need a stronger version of it. We will leave this until the end of the proof. As our induction hypothesis, we assume that the statement in the theorem is true with n replaced by $n - 1$, for some particular $n \geq 4$.

Let $(x_1, x_2, \ldots, x_n)$ be any admissible point other than $\left(\frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n}\right)$. If $x_j = 0$ for more than one value of the index j, then $P/x_j = 0$ for all j, and hence $W(x_1, x_2, \ldots, x_n) = \frac{n}{k} < W\left(\frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n}\right)$. Thus, in this case, W is not maximal at $(x_1, x_2, \ldots, x_n)$.

Now suppose that $x_j = 0$ for at most one j. By re-labelling the points if necessary, we can assume that $x_n \neq 0$ and that $x_1, x_2, \ldots, x_{n-1}$ are not all equal. Then

$$W(x_1, x_2, \ldots, x_n) \;=\; \frac{1}{k - P'} + \frac{1}{x_n} \sum_{j=1}^{n-1} \frac{1}{k' - \dfrac{P'}{x_j}}\,, \tag{1}$$

where $P' = \prod_{j=1}^{n-1} x_j$ and $k' = k/x_n$.

Letting $h' = h - x_n$, we have $\sum_{j=1}^{n-1} x_j = h'$. Consider what happens if we replace $(x_1, x_2, \ldots, x_{n-1})$ by $\left(\frac{h'}{n-1}, \frac{h'}{n-1}, \ldots, \frac{h'}{n-1}\right)$, keeping x_n fixed. This increases the product P' and, consequently, increases the first term on the right side of (1). Also, by the induction hypothesis, the sum on the right side of (1) is increased. Thus,

$$W(x_1, x_2, \ldots, x_n) \;<\; W\left(\frac{h'}{n-1}, \frac{h'}{n-1}, \ldots, \frac{h'}{n-1}, x_n\right) .$$

We conclude that W is not maximal at the point $(x_1, x_2, \ldots, x_n)$.

It remains for us to prove the case $n = 3$. For convenience, let $x = x_1$, $y = x_2$, and $z = x_3$. Without loss of generality, we assume that $h = 1$. (Otherwise, we could scale the variables x, y, z by the factor $1/h$, with a corresponding change in the value of k.) Let (x, y, z) be any admissible point other than $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. By re-labelling the points if necessary, we can assume that $z \leq \frac{1}{2}$ and $x \neq y$. Now

$$\begin{aligned}
W(x,y,z) &= \frac{1}{k-xy} + \frac{1}{k-yz} + \frac{1}{k-zx} \\
&= \frac{3k^2 - 2k(xy+yz+zx) + xyz(x+y+z)}{k^3 - k^2(xy+yz+zx) + kxyz(x+y+z) - x^2y^2z^2} \\
&= \frac{3k^2 - 2k(xy+yz+zx) + xyz}{k^3 - k^2(xy+yz+zx) + kxyz - x^2y^2z^2} \\
&= \frac{3}{k} + \frac{1}{k} \cdot \frac{k^2(xy+yz+zx) - 2kxyz + 3x^2y^2z^2}{k^3 - k^2(xy+yz+zx) + kxyz - x^2y^2z^2} \\
&= \frac{3}{k} + \frac{kz(1-z) + xy(k-2z) + \frac{3}{k}x^2y^2z^2}{k^3 - k^2z(1-z) - kxy(k-z) - x^2y^2z^2} .
\end{aligned}$$

Replacing (x, y) by $\left(\frac{1}{2}(x+y), \frac{1}{2}(x+y)\right)$ causes the product xy to increase (since $x \neq y$), which increases the numerator and decreases the denominator in the second term on the right side above. Therefore, $W(x,y,z) < W\left(\frac{x+y}{2}, \frac{x+y}{2}, z\right)$. We conclude that W is not maximal at the point (x, y, z). Therefore, W must attain its maximum value at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and only at this point.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina. There was one incorrect solution.

A generalization of Problems 2798 and 2799 has been proposed by Guersenzvaig and independently by Mihály Bencze, Brasov, Romania. See the notes to Problem 2799 below.

2799★. [2002 : 536] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove or disprove the inequality

$$\sum_{\substack{i,\,j\,\in\,\{1,\,2,\,\ldots\,,n\}\\ 1\le i<j\le n}} \frac{1}{1-x_ix_j} \;\le\; \binom{n}{2}\frac{1}{1-\frac{1}{n^2}}\,,$$

where $\sum\limits_{j=1}^{n} x_j = 1$, $x_j \ge 0$.

[*Editor's Remark*. The condition $i<j$ replaced the original $i \le j$ in a footnote in [2003 : 47].]

Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina, modified by the editors.

We will prove the given inequality for $n \ge 2$. To begin, we denote the sum on the left side by $F(x_1,x_2,\ldots,x_n)$, which defines a continuous function F whose domain is a compact subset of $\mathbb{R}^n$, say S. Thus, F has a maximum value in S, say M. Then $2M$ is the maximum value over S of

$$2F(x_1,\ldots,x_n) \;=\; \sum_{i=1}^{n}\Bigg(\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n} \frac{1}{1-x_ix_j}\Bigg)\,.$$

Moreover, by considering, for each $i=1,\ \ldots,\ n$ and each $x\in[0,1]$, the compact subset $S_i(x)$ of S formed by the points $(y_1,\ldots,y_n)\in S$ such that $y_i = x$, it is clear that

$$2M \;\le\; \max_{(x_1,\ldots,x_n)\in S}\sum_{i=1}^{n}\Bigg(\max_{(y_1,\ldots,y_n)\in S_i(x_i)}\sum_{\substack{j\,=\,1\\ j\,\neq\, i}}^{n}\frac{1}{1-x_iy_j}\Bigg)\,.$$

Therefore, since F is a symmetric function, in order to complete the proof it will be sufficient to prove the following two facts.

(i) For each $x\in[0,1]$ and each $(y_1,\ldots,y_{n-1},x)\in S_n(x)$, the maximum value of the function $F_{n-1,x}$ defined by

$$F_{n-1,x}(y_1,\ldots,y_{n-1}) \;=\; \sum_{j=1}^{n-1}\frac{1}{1-xy_j}$$

is given by

$$F_{n-1,x}\left(\frac{1-x}{n-1},\ldots,\frac{1-x}{n-1}\right) \;=\; \frac{n-1}{1-\dfrac{x(1-x)}{n-1}}\,.$$

(ii) The maximum value of the function G_n defined on S by

$$G_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{n-1}{1 - \dfrac{x_i(1-x_i)}{n-1}}$$

is given by

$$G_n\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) = 2\binom{n}{2} \frac{1}{1 - \frac{1}{n^2}}$$

We will argue inductively beginning with (i). Cases $x = 0$ and $x = 1$ are clearly true. Therefore, we may assume that $x(1-x) \neq 0$. Case $n = 2$ is also clear because $y_1 = 1 - x$. Thus, suppose that $n \geq 3$, and assume (i) holds for $n - 1$.

Let us notice in the first place that $0 \leq x(1-x) \leq \frac{1}{4}$ for $0 \leq x \leq 1$. Hence, for $n \geq 3$, we have

$$\frac{n-2}{1 - \dfrac{x(1-x)}{n-2}} \leq \frac{n-2}{1 - \dfrac{1}{4(n-2)}} < \frac{n-1}{1} \leq \frac{n-1}{1 - \dfrac{x(1-x)}{n-1}} .$$

Suppose $y_j = 0$ for some j, $1 \leq j \leq n-1$. Without loss of generality, we may assume $y_{n-1} = 0$ because of the symmetry of $F_{n-1,x}$. Observing that

$$1 + F_{n-2,x}(y_1, \ldots, y_{n-2}) = F_{n-1,x}(y_1, \ldots, y_{n-2}, 0)$$

and using the induction hypothesis, we get

$$\begin{aligned} F_{n-1,x}(y_1, \ldots, y_{n-2}, 0) &\leq \frac{n-2}{1 - \dfrac{x(1-x)}{n-2}} \\ &< F_{n-1,x}\left(\frac{1-x}{n-1}, \ldots, \frac{1-x}{n-1}\right) , \end{aligned}$$

which proves that all the coordinates of any maximum point of $F_{n-1,x}$ are non-zero.

Consider the Lagrange function L defined for positive y_1, $\ldots$, y_{n-1}, and $\lambda \in \mathbb{R}$ by

$$L(y_1, \ldots, y_{n-1}, \lambda) = \sum_{j=1}^{n-1} \frac{1}{1 - xy_j} - \lambda(1 - x - y_1 - \cdots - y_{n-1}) .$$

Looking for the critical points of L, we set

$$\frac{\partial L}{\partial y_1} = \cdots = \frac{\partial L}{\partial y_{n-1}} = \frac{\partial L}{\partial \lambda} = 0$$

to obtain
$$\frac{x}{(1-xy_1)^2} = \cdots = \frac{x}{(1-xy_{n-1})^2};$$
whence, $y_1 = \cdots = y_{n-1} = \dfrac{1-x}{n-1}$, which means that (i) holds for n. Thus, the proof of (i) is complete.

Case $n = 2$ of (ii) is true because $x_i(1-x_i) \le \frac{1}{4}$; whence,
$$G_2(x_1,x_2) \le 2\frac{1}{1-\frac{1}{4}} = 2\binom{2}{2}\frac{1}{1-\frac{1}{4}} = G_2\left(\frac{1}{2},\frac{1}{2}\right).$$

Now suppose $n \ge 3$, and assume (ii) holds for $n-1$. We first consider the case that $x_i = 0$ for some i, $1 \le i \le n$. From the symmetry of G_n, we can assume $x_n = 0$. Since for $n \ge 3$, we have
$$(n-1) + 2\binom{n-1}{2}\frac{1}{1-\dfrac{1}{(n-1)^2}} < 2\binom{n}{2}\frac{1}{1-\dfrac{1}{n^2}},$$
using the induction hypothesis, we derive
$$\begin{aligned} G_n(x_1,\ldots,x_{n-1},0) &= n-1+G_{n-1}(x_1,\ldots,x_{n-1}) \\ &\le n-1+2\binom{n-1}{2}\frac{1}{1-\dfrac{1}{(n-1)^2}} \\ &< G_n\left(\frac{1}{n},\cdots,\frac{1}{n}\right), \end{aligned}$$
from which it follows that all the coordinates of any maximum point of G_n are non-zero.

Consider the Lagrange function L defined for positive $x_1, \ldots, x_n$, and $\lambda \in \mathbb{R}$ by
$$L(x_1,\ldots,x_n,\lambda) = \sum_{i=1}^{n}\frac{n-1}{1-\dfrac{x_i(1-x_i)}{n-1}} - \lambda(1-x_1-\cdots-x_n).$$
To complete the proof it will suffice to show that $(\frac{1}{n}, \ldots, \frac{1}{n})$ is the only critical point of L. From the necessary condition
$$\frac{\partial L}{\partial x_1} = \cdots = \frac{\partial L}{\partial x_n} = \frac{\partial L}{\partial \lambda} = 0,$$
after canceling $(n-1)^2$ from each term, we obtain
$$\frac{2x_1-1}{(n-1-x_1+x_1^2)^2} = \cdots = \frac{2x_n-1}{(n-1-x_n+x_n^2)^2}.$$

Hence, by noting that the function f defined for $x \in (0,1)$ by

$$f(x) = \frac{2x-1}{(n-1-x+x^2)^2}$$

is strictly increasing (because $f'(x) = \frac{2(n-2+3x(1-x))}{(n-1-x-x^2)^3} > 0$), we see at once that $x_1 = \cdots = x_n = \frac{1}{n}$, which means that (ii) holds for n. Thus, the proof of (ii) is complete.

There was one incomplete solution.

The author also adds the following remarks.

(a) Let m be the minimum value of the given sum. From cases $x = 0$ and $x = 1$ of (i), it follows that the minimum value of $F_{n-1,x}$ is equal to $F_{n-1,x}(1,0,\ldots,0) = n-1$; whence, $m = \binom{n}{2}$ because

$$2m = \min_{(x_1,\ldots,x_n)\in S} \sum_{i=1}^{n} \left(\min_{(y_1,\ldots,y_n)\in S_i(x_i)} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{1-x_i y_j} \right).$$

[Ed. We can also see this by observing that the original sum has $\binom{n}{2}$ terms, each of which is at least one.]

(b) By the same iterative argument, we can prove by induction over k the following generalization of # 2799 (case $k = 2$) and # 2798 (case $k = n-1$):

Let n, k be arbitrary integers with $2 \le k \le n$. Then

$$\binom{n}{k} \le \sum_{\substack{1 \le x_1 < \cdots < x_k \le n \\ i_j \in \{1,2,\ldots,n\} \\ j=1,\ldots,k}} \frac{1}{1 - \prod_{j=1}^{k} x_{i_j}} \le \binom{n}{k} \frac{1}{1-\frac{1}{n^k}},$$

where $\sum_{j=1}^{n} x_j = 1$, and $x_i \ge 0$ for $i = 1, 2, \ldots, n$.

2800. [2002 : 536] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $z_k \in \mathbb{C}^*$ ($k = 1, 2, \ldots, n$) such that $|z_k| = |M|$ (constant) and

$$\left|\sum_{k=1}^{n} z_k\right| = \left|\sum_{k=1}^{n} z_k - z_1\right| = \left|\sum_{k=1}^{n} z_k - z_2\right| = \cdots = \left|\sum_{k=1}^{n} z_k - z_n\right|$$

($k \in \{1, 2, \ldots, n\}$). Prove that $\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right) = \frac{n}{2}$.

I. *Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let $s = \sum_{k=1}^{n} z_k$. Taking the complex conjugate, we have $\overline{s} = \sum_{k=1}^{n} \overline{z_k}$. Then, for $i = 1, 2, \ldots, n$,

$$\begin{aligned} s \cdot \overline{s} &= \left|\sum_{k=1}^{n} z_k\right|^2 = \left|\sum_{k=1}^{n} z_k - z_i\right|^2 = (s - z_i) \cdot \overline{(s - z_i)} \\ &= (s - z_i) \cdot (\overline{s} - \overline{z_i}) = s \cdot \overline{s} - z_i \cdot \overline{s} - s \cdot \overline{z_i} + |M|^2 . \end{aligned}$$

Hence, $z_i \cdot \overline{s} + s \cdot \overline{z_i} = |M|^2$. Summing this equation over $i = 1, 2, \ldots, n$ gives $2s \cdot \overline{s} = n|M|^2$. Finally,

$$\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right) = s \cdot \left(\sum_{k=1}^{n} \frac{\overline{z_k}}{z_k \cdot \overline{z_k}}\right) = \frac{s \cdot \overline{s}}{|M|^2} = \frac{n}{2} .$$

II. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

Let $s = \sum_{k=1}^{n} z_k$. From the given information, we see that in the complex plane the points z_k all lie on a common circle Γ centred at s and passing through the origin. Furthermore, since $|z_k| = |M|$, the points z_k also lie on the circle centred at 0 with radius $|M|$. Hence, each point z_k is at one or the other of the points of intersection of these two circles.

Since the problem is invariant under multiplication of all the points z_k by a complex constant, we can assume without loss of generality that $|M| = 1$ and that s is real and positive. Then n must be even, and $n/2$ of the points z_k lie at each of the points $e^{i\theta}$ and $e^{-i\theta}$, for some constant θ. Therefore, $s = \frac{n}{2}(e^{i\theta} + e^{-i\theta}) = n\cos\theta$. Since the mapping $z \mapsto \frac{1}{z}$ just interchanges the two sets of points, we also have $\sum\limits_{k=1}^{n} \frac{1}{z_k} = n\cos\theta$.

The triangle with vertices 0, z_1, and s is isosceles ($s - 0 = |s - z_1|$). From this triangle, we have $1 = |z_1| = 2s\cos\theta$. Hence, $1 = 2n\cos^2\theta$. Thus,

$$\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right) = (n\cos\theta)^2 = \frac{n}{2}(2n\cos^2\theta) = \frac{n}{2} .$$

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

YEAR END FINALE

WOW! A whole year has gone by already! Time sure flies when you are having fun!

Many of you have been kind enough to send kudos in my direction for the enjoyment you get from ***CRUX with MAYHEM***, but these kudos need to be spread among the many people who share in putting together each issue. I will try to mention each of them here, at the risk of inadvertently missing some, so that you can more fully appreciate how many people make a contribution before you get the finished product.

First of all, I want to thank Bruce Shawyer, the previous Editor-in-Chief, for convincing me to replace him as Editor-in-Chief. I find that I can hardly wait to get to work each day to see what kind of interesting material has arrived. Thanks, Bruce!

In the actual day-to-day production of ***CRUX with MAYHEM***, there is one person who stands above the rest. That person is Bruce Crofoot, my Associate Editor. Besides putting in vast amounts of time scrutinizing drafts, Bruce's meticulous attention to detail picks up an incredible number of typos (and plain mistakes!) before we ever go to the proof readers.

There are many other people that I wish to thank most sincerely for particular contributions. These include ILIYA BLUSKOV, CHRIS FISHER, EDWARD WANG, RICK BREWSTER, BRUCE CROFOOT, and BRUCE SHAWYER (yes, he never really left!) for their regular and timely service in assessing the solutions; BRUCE GILLIGAN, for ensuring that ***CRUX with MAYHEM*** has quality articles; JOHN GRANT McLOUGHLIN, for ensuring that we have book reviews that are appropriate; ROBERT WOODROW for overseeing the *Olympiad Corner*; SHAWN GODIN for doing likewise with the *Skoliad*; and RICHARD GUY for being there whenever needed.

You will have noticed that we are now posing all of our Mayhem Problems and the CRUX Problems in French, as well as English. The task of doing all the translations has fallen on the shoulders of JEAN-MARC TERRIER and HIDEMITSU SAYEKI. During the year Hidemitsu has had to step down due to health reasons; his replacement is MARTIN GOLDSTEIN. I want thank all three of them for their efforts, and wish all the best to Hidemitsu.

The editors of the ***MATHEMATICAL MAYHEM*** section, our journal-within-a-journal, namely, SHAWN GODIN, JOHN GRANT McLOUGHLIN, PAUL OTTAWAY, and LARRY RICE, have done a great job producing a student-oriented section. Their biggest difficulty is finding interesting problems at the appropriate level. They want your help!

I also want to thank all the proofreaders. MOHAMMED AASSILA, DAVID FELDMAN, BRUCE KADANOFF, and ROGER COROAS assist the editors with this task. The quality of the work of all these people is a vital part of what makes ***CRUX with MAYHEM*** what it is. Thank you one and all.

I also want to thank the University College of the Cariboo and my colleagues in the Department of Mathematics and Statistics for their continued understanding and support, and for believing that my work on this journal is important enough to reduce my teaching load sufficiently to allow me to do it. Thanks, in particular, to CAROL COSTACHE, secretary to our department, who reduces the amount of time I need to spend on the more mundane tasks associated with the journal, which allows me to do my teaching properly.

Also, the LaTeX expertise of JOANNE LONGWORTH at the University of Calgary, SHAUNA GAMMON at Memorial University of Newfoundland, JUNE ALEONG and TAO GONG at Wilfrid Laurier University, the ***MAYHEM*** staff, and all others who produce material, is much appreciated.

Thanks to GRAHAM WRIGHT, the Managing Editor, who keeps me on the right track, and to the U of T Press, and TAMI EHRLICH in particular, who continue to print a fine product.

The online version of ***CRUX with MAYHEM*** continues to grow. We recommend it highly to you. Thanks are due to JUDI BORWEIN and the rest of the team at SFU who are responsible for this.

Last but not least, I send my thanks to you, the readers. Without you, ***CRUX with MAYHEM*** would not be what it is. We receive over 150 problem proposals each year, and we publish 100 of these in each volume. Of course, we receive hundreds of solutions, as you will see in the index that follows. Every year, we receive solutions from new readers. This is very gratifying. We hope that these new solvers will become regular solvers and proposers of new problems. Please ensure that your name and address is on EVERY problem or proposal, and that each starts on a fresh sheet of paper. Otherwise, there may be filing errors, resulting in a submitted solution or proposal being lost.

We need your ARTICLES, PROPOSALS, and SOLUTIONS to keep ***CRUX with MAYHEM*** alive and well. It is all about continued renewal! More and more proposals and solutions are arriving by email. The editor is able to process files sent in LaTeX, in WORD, in WordPerfect, and in PDF formats, as well as (although less desirable) image files.

I wish everyone the compliments of the season, and a very happy, peaceful and prosperous year 2004.

Jim Totten

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