SKOLIAD No. 73

Shawn Godin

Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6, or emailed to

mayhem-editors@cms.math.ca.

We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 April 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 6** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The first item in this issue is the 2003 W.J. Blundon Mathematics Contest. My thanks go out to Don Rideout of Memorial University for forwarding the material to me.

The Twentieth W.J. Blundon Mathematics Contest ($*$)

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

February 19, 2003

1. Solve: $\log_2(9 - 2^x) = 3 - x$.

2. Show that $(\sqrt{5}+2)^{\frac{1}{3}} - (\sqrt{5}-2)^{\frac{1}{3}}$ is rational, and find its value.

3. If $a^3 + b^3 = 4$ and $ab = \frac{2}{3}$, where a and b are real, find $a + b$.

4. Find x, y, and z such that when any one of them is added to the product of the other two, the result is 2.

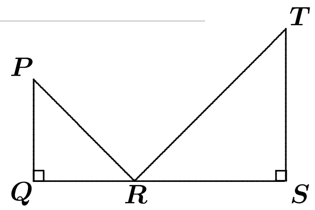
5. If a, b, and c are the three zeros of $P(x) = x^3 - x^2 + x - 2$, find $a + b + c$ and $a^2 + b^2 + c^2$.

6. If $\sin x + \cos x = \sqrt{\dfrac{2+\sqrt{3}}{2}}$, with $0 < x < \frac{\pi}{2}$, find x.

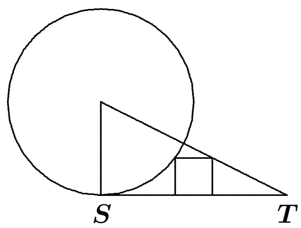
7. Prove that two consecutive odd positive integers cannot have a common factor other than 1.

8. Triangle ABC has vertices $A(3,1)$, $B(5,7)$, and $C(1,y)$. Find all y so that angle C is a right angle.

9. In the diagram to the right, $PQ = 8$, $TS = 12$, and $QS = 20$. Find QR so that $\angle PRT$ is a right angle.



10. A square of side 2 is placed with one side on a tangent to a circle of radius 5 so that the square lies outside the circle, and one vertex of the square lies on the circle. A line is drawn from the centre of the circle through the vertex of the square that is not on the tangent and not on the circle. This line cuts the tangent at a point T. If the tangent meets the circle at S, find the length of the line segment TS.



Next we look at the solutions to the 1997 Mandelbrot Competition, Round 1, individual and team tests from [2003 : 65–67].

The Mandelbrot Competition
Division B Round One Individual Test
November 1997

1. ($*$) What angle less than 180° is formed by the hands of a clock at $2{:}30$ pm? (Express the answer in degrees.) **(1 point)**

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

If the hour hand is pointing at the two and the minute hand at six, the angle between them is $30^\circ \times (6-2) = 120^\circ$. But the hour hand is actually pointing at the mid-point between two and three, which means we have to subtract $30^\circ/2 = 15^\circ$. Therefore, the angle is 105°.

2. ($*$) If $x = \sqrt{\frac{6}{7}}$, then evaluate $\left(x + \frac{1}{x}\right)^2$. **(1 point)**

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

$$\left(x+\frac{1}{x}\right)^2 \;=\; x^2+2+\frac{1}{x^2} \;=\; \frac{6}{7}+2+\frac{7}{6} \;=\; \frac{169}{42}\,.$$

3. (∗) How many possible values are there for the sum $a+b+c$ if a, b, and c are positive integers and $abc = 50$? (**2 points**)

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Without loss of generality, let $a \geq b \geq c$. When $abc = 50$, the only possible values for (a, b, c) are $(50, 1, 1)$, $(25, 2, 1)$, $(10, 5, 1)$, and $(5, 5, 2)$. Since their sums are distinct, there are four possible values for the sum.

4. List the numbers $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[5]{5}$ in order from greatest to least. (**2 points**)

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Comparing the size of $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[5]{5}$ is equivalent to comparing $\left(\sqrt{2}\right)^{30}$, $\left(\sqrt[3]{3}\right)^{30}$, and $\left(\sqrt[5]{5}\right)^{30}$, since they are all positive.

First, note that $\left(\sqrt{2}\right)^{30} = 2^{15} = 8^5$ and $\left(\sqrt[3]{3}\right)^{30} = 3^{10} = 9^5$. Since $9^5 > 8^5$, it follows that $\sqrt[3]{3} > \sqrt{2}$. Similarly, $\left(\sqrt{2}\right)^{30} = 2^{15} = 32^3$ and $\left(\sqrt[5]{5}\right)^{30} = 5^6 = 25^3$. Since $32^3 > 25^3$, it follows that $\sqrt{2} > \sqrt[5]{5}$.

Thus, the order is $\sqrt[3]{3} > \sqrt{2} > \sqrt[5]{5}$.

5. Compute 3^A where $A = \dfrac{(\log 1 - \log 4)(\log 9 - \log 2)}{(\log 1 - \log 9)(\log 8 - \log 4)}$. All logarithms are base three. (**2 points**)

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Simplifying, using the Laws of Logarithms, yields

$$A \;=\; \frac{(0-2\log 2)(2-\log 2)}{(0-2)(\log\frac{8}{4})} \;=\; \frac{-2\log 2(2-\log 2)}{-2\log 2} \;=\; 2-\log 2\,.$$

Therefore,

$$\begin{aligned}
\log 2 &= 2-A\,,\\
2 &= 3^{2-A}\,,\\
2\cdot 3^A &= 3^2\,,\\
3^A &= \frac{9}{2}\,.
\end{aligned}$$

6. Joe and Andy are playing a simple game on a circular board with n spaces. First Joe advances five spaces from the starting space, then Andy advances seven, then Joe advances five, then Andy advances seven, and so on. The first player to land back on the original space wins. If n is a random two-digit number, what is the probability that Joe wins? (**3 points**)

Official solution.

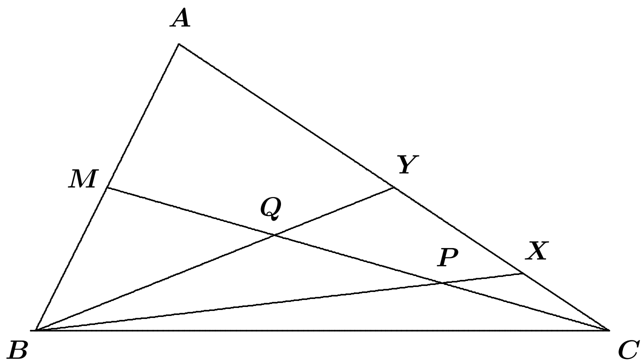
If n is not divisible by either 5 or 7, then Joe will finish when he has advanced $5n$ spaces, while Andy will finish when he has advanced $7n$ spaces. This will require n moves for each player. Since Joe goes first, he will finish first.

If n is divisible by 5 but not 7, Joe will win, since he will finish in only $n/5$ moves. Similarly, if n is divisible by 7 but not 5, then Andy will win, finishing in $n/7$ moves.

Finally, if n is divisible by both 5 and 7, then Andy will win, since he will finish in $n/7$ moves versus Joe's $n/5$.

Hence, Andy will win if and only if n is divisible by 7, or 13 times out of the 90 two-digit numbers. Thus, the probability that Joe wins is $77/90$.

7. In the diagram, M is the mid-point of AB and Y is the mid-point of AC. Hence, Q is a trisection point of CM; we call the other trisection point P and extend BP to meet AC at X. Evaluate $(CX + AY)/XY$. (**3 points**)



Official solution.

Applying Menelaus' Theorem to $\triangle ACM$ as intersected by the line BP, we have

$$\frac{AX}{XC} \cdot \frac{CP}{PM} \cdot \frac{MB}{BA} = 1 .$$

Since $CP/PM = 1/2$ and $MB/BA = 1/2$, we have $AX = 4XC$. Thus, $XC = \frac{1}{5}AC$. Since $YC = \frac{1}{2}AC$, we have $XY = \frac{3}{10}AC$, which yields $(CX + AY)/XY = 7/3$.

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

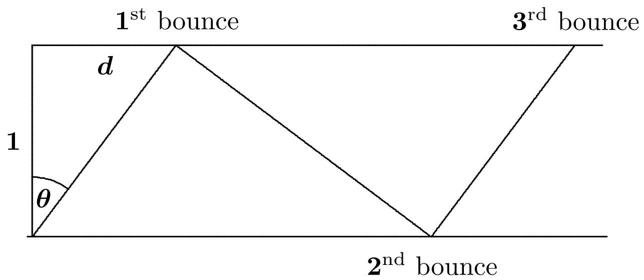
The Mandelbrot Competition
Division B Round One Team Test
November 1997

Facts: The Weighted Power Mean Inequality for two positive variables states that if x_1, x_2, w_1 and w_2 are positive real numbers, and m and n are non-zero integers with $m > n$, then

$$\left(\frac{w_1x_1^m + w_2x_2^m}{w_1 + w_2}\right)^{\frac{1}{m}} \geq \left(\frac{w_1x_1^n + w_2x_2^n}{w_1 + w_2}\right)^{\frac{1}{n}},$$

which is quite a mouthful. The positive variables are x_1 and x_2. The two sides are equal if and only if $x_1 = x_2$. The numbers w_1 and w_2 "weight" the variables in different proportions. Try $w_1 = w_2 = 1$ to see the standard Power Mean Inequality; then compare with $w_1 = 1$ and $w_2 = 2$, which emphasizes x_2. Finally, the non-zero integers m and n vary the powers. For example, use $m = 1$ and $n = -1$ to obtain the Arithmetic Mean–Harmonic Mean Inequality.

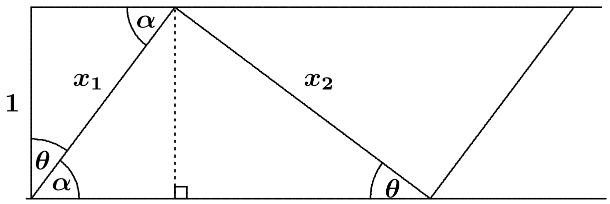
Setup: On the planet Garth a certain laser has the curious property that when reflected off a special mirror it always continues in a direction perpendicular to the original path. Some popular Garthian children's games are based on this phenomenon. The ones described below involve a player situated in the corner of a rectangular mirrored hallway of width one plog (about 7.3 metres), as shown below. The player directs the laser beam an angle of θ away from the left wall, hitting the far wall a distance d from the end of the hall on the first bounce.



Problems: (*Please, no calculus-based solutions.*)

Part i: (4 points) Show that the laser's path length up to the second bounce is $\dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}$.

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.



From the diagram, we get $\cos\theta = \frac{1}{x_1}$, or $x_1 = \frac{1}{\cos\theta}$. Similarly, $\sin\theta = \frac{1}{x_2}$, or $x_2 = \frac{1}{\sin\theta}$. Thus, $x_1 + x_2 = \frac{1}{\sin\theta} + \frac{1}{\cos\theta}$.

Part ii: (4 points) The object of one of the simpler games is to have the shortest path length after two bounces. Use the *standard* Power Mean Inequality with $m = 2$, $n = -1$, $x_1 = \cos\theta$, and $x_2 = \sin\theta$ to prove that the shortest possible path length is $2\sqrt{2}$. Invoke the equality condition to show that we need $\tan\theta = 1$ (that is, $d = 1$) to obtain this minimum.

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Substituting $m = 2$ and $n = -1$ into the standard Power Mean Inequality, we get

$$\sqrt{\frac{\cos^2\theta + \sin^2\theta}{2}} \geq \left(\frac{\frac{1}{\cos\theta} + \frac{1}{\sin\theta}}{2}\right)^{-1}.$$

Since $\cos^2\theta + \sin^2\theta = 1$, this simplifies to

$$\begin{aligned} \sqrt{\frac{1}{2}} &\geq \frac{2}{\frac{1}{\cos\theta} + \frac{1}{\sin\theta}}, \\ \frac{1}{\cos\theta} + \frac{1}{\sin\theta} &\geq \frac{2}{\sqrt{\frac{1}{2}}} = 2\sqrt{2}\,. \end{aligned}$$

Thus, the minimum value for $\frac{1}{\cos\theta} + \frac{1}{\sin\theta}$ is $2\sqrt{2}$. For this to occur, $x_1 = x_2$; that is, $\sin\theta = \cos\theta$, or $\tan\theta = 1$.

Part iii: (4 points) Show that the total path length after three bounces (a more challenging version) is $2\left(\frac{1}{\cos\theta}\right) + \frac{1}{\sin\theta}$. To minimize this, we employ a clever strategy. Begin by finding numbers w_1 and λ_1 such that $w_1\lambda_1^2 = 1$ and $w_1/\lambda_1 = 2$.

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Let x_3 be the length of the third bounce. Then $\cos\theta = \dfrac{1}{x_3}$, or $x_3 = \dfrac{1}{\cos\theta}$. Hence, the total length is $2\left(\dfrac{1}{\cos\theta}\right) + \dfrac{1}{\sin\theta}$.

Solving $w_1\lambda_1^2 = 1$ and $w_1 = 2\lambda_1$ yields $\lambda_1^3 = \frac{1}{2}$ Thus, $\lambda_1 = \frac{1}{\sqrt[3]{2}}$ and $w_1 = \frac{2}{\sqrt[3]{2}}$.

Part iv: (4 points) Now apply the Weighted Power Mean Inequality with $m = 2$ and $n = -1$ as before, using w_1 from **Part iii**, $w_2 = 1$, $x_1 = \lambda_1\cos\theta$, and $x_2 = \sin\theta$, to prove that the minimum path length is $\left(1 + 2^{2/3}\right)^{3/2}$. What value of d should the player aim for?

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

By the Weighted Power Mean Inequality, we get

$$\left(\frac{\frac{2}{\sqrt[3]{2}}\cdot\left(\frac{1}{\sqrt[3]{2}}\right)^2\cdot\cos^2\theta + \sin^2\theta}{\frac{2+\sqrt[3]{2}}{\sqrt[3]{2}}}\right)^{\frac{1}{2}} \geq \left(\frac{\frac{2}{\sqrt[3]{2}}\cdot\frac{\sqrt[3]{2}}{\cos\theta} + \frac{1}{\sin\theta}}{\frac{2+\sqrt[3]{2}}{\sqrt[3]{2}}}\right)^{-1}.$$

Since $\sin^2\theta + \cos^2\theta = 1$, we get

$$\left(\frac{\sqrt[3]{2}}{2+\sqrt[3]{2}}\right)^{\frac{1}{2}} \geq \frac{\frac{2+\sqrt[3]{2}}{\sqrt[3]{2}}}{\dfrac{2}{\cos\theta} + \dfrac{1}{\sin\theta}} = \frac{2^{\frac{2}{3}} + 1}{\dfrac{2}{\cos\theta} + \dfrac{1}{\sin\theta}}.$$

Then

$$\begin{aligned}\frac{2}{\cos\theta} + \frac{1}{\sin\theta} &\geq (2^{\frac{2}{3}} + 1)\cdot(2^{\frac{2}{3}} + 1)^{\frac{1}{2}} \\ &= (2^{\frac{2}{3}} + 1)^{\frac{3}{2}}.\end{aligned}$$

Therefore, the minimum path length is $(2^{\frac{2}{3}} + 1)^{\frac{3}{2}}$. Equality occurs when $\frac{1}{\sqrt[3]{2}}\cos\theta = \sin\theta$; that is, $d = \tan\theta = \frac{1}{\sqrt[3]{2}}$. Thus, the player should aim for $d = \frac{1}{\sqrt[3]{2}}$.

Part v: (5 points) Use this technique to find the shortest path length after five bounces.

Official solution.

The total path length is $3\left(\dfrac{1}{\cos\theta}\right) + 2\left(\dfrac{1}{\sin\theta}\right)$. Thus, we set $m = 2$, $n = -1$, $x_1 = \lambda_1\cos\theta$, and $x_2 = \sin\theta$ as before. However, this time we use $w_2 = 2$ because of the 2 in front of $\left(\dfrac{1}{\sin\theta}\right)$ in the expression for the path length. Incorporating these into the Weighted Power Mean Inequality, we arrive at

$$\left(\frac{w_1\lambda_1^2\cos^2\theta + 2\sin^2\theta}{w_1+2}\right)^{\frac{1}{2}} \geq \left(\frac{w_1\lambda_1^{-1}(\cos\theta)^{-1} + 2(\sin^2\theta)^{-1}}{w_1+2}\right)^{-1}.$$

To guarantee nice simplifications, we need $w_1\lambda_1^2 = 2$. To obtain our expression for path length, we need $w_1\lambda_1^{-1} = 3$. Solving these equations produces $w_1 = \sqrt[3]{18}$ and $\lambda_1 = \sqrt[3]{\frac{2}{3}}$. Using these values in the above inequality yields

$$\sqrt{\frac{2}{\sqrt[3]{18}+2}} \geq \frac{\sqrt[3]{18}+2}{3\left(\frac{1}{\cos\theta}\right) + 2\left(\frac{1}{\sin\theta}\right)}.$$

Multiplying through as usual reveals the desired minimum path length:

$$\text{path length} \geq (2+\sqrt[3]{18})^{3/2}/\sqrt{2},$$

which can be written in the more interesting form

$$\text{path length} \geq \left(2^{\frac{2}{3}} + 3^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

The five-bounce minimum is attained when $\sqrt[3]{\frac{2}{3}}\cos\theta = \sin\theta$. Therefore, we must aim for $d = \sqrt[3]{\frac{2}{3}}$.

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

That ends another Skoliad. Continue to send me your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **~~Mathematical Mayhem, 2191 Saturn Cres., Orleans, ON, K4A~~ 3T6**. The electronic address is

`mayhem-editors@cms.math.ca`

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, 2191 Saturn Crescent, Orleans, Ontario, K4A 3T6** or emailed to

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Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M113. *Proposed by Neven Jurič, Zagreb, Croatia.*

The king is on an open $m \times n$ chessboard. On each of its mn cells the total number of possible moves by the king from that cell is written. Find the sum of all these mn numbers.

. . . .

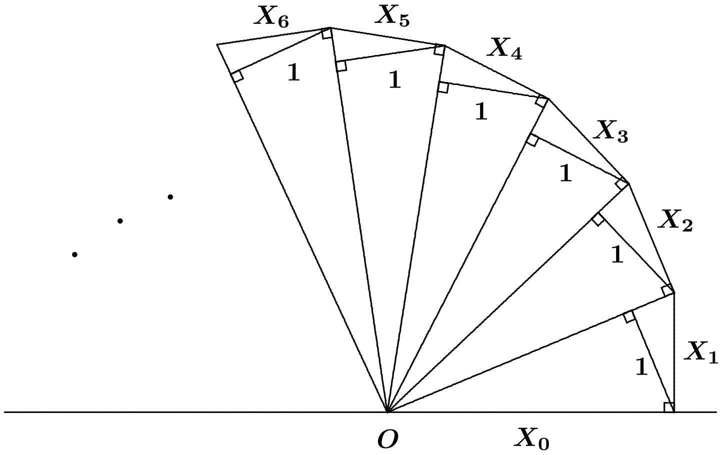
Le roi est sur un échiquier ouvert de dimensions $m \times n$. Sur chacune des mn cases est écrit le nombre total des mouvements possibles du roi à partir de cette case. Trouver la somme totale de tous ces mn nombres.

M114. *Proposed by Seyamack Jafari, Bandar Imam, Khozestan, Iran.*

In the spiral below prove that

$$X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 \;=\; X_0^2 \cdot X_1^2 \cdot X_2^2 \cdots X_n^2 \,,$$

where the height of each triangle indicated in the diagram is 1 unit.



Dans la spirale ci-dessus, montrer que

$$X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 \;=\; X_0^2 \cdot X_1^2 \cdot X_2^2 \cdots X_n^2 \,,$$

où la hauteur de chaque triangle indiqué dans le diagram est 1 unité.

M115. *Proposed by the Mayhem Staff.*

The twenty-third term of an arithmetic sequence is three times the value of the fifth term. Find the ratio of the twenty-third term to the first term of the sequence. Express the ratio in the form $p : q$ where p and q are integers.

Le vingt-troisième terme d'une progression arithmétique est trois fois la valeur du cinquième terme. Trouver le rapport du vingt-troisième terme au premier terme de la progression. Exprimer le rapport sous la forme $p : q$ où p et q sont des entiers.

M116. *Proposed by the Mayhem Staff.*

A polynomial $f(x)$ satisfies the condition that $f(5 - x) = f(5 + x)$ for all real numbers x. If $f(x) = 0$ has 4 distinct real roots, find the sum of these roots.

Soit $f(x)$ un polynôme satisfaisant la condition $f(5 - x) = f(5 + x)$ pour tous les nombres réels x. Si $f(x) = 0$ possède 4 racines réelles distinctes, trouver la somme de ces racines.

M117. *Proposed by the Mayhem Staff.*

A person cashes a cheque at the bank. By mistake, the teller pays the number of cents as dollars and the number of dollars as cents. The person spends $\$3.50$ before noticing the mistake, then, on counting the money, finds that the remaining money is exactly double the amount of the cheque. For what amount was the cheque made out?

. . . .

Une personne encaisse un chèque à la banque. Par erreur, le caissier lui remet le nombre de cents en dollars et le nombre de dollars en cents. La personne dépense $\$3.50$ avant de s'apercevoir de l'erreur, puis, comptant son argent, trouve que le reste est exactement le double du montant du chèque. Quel est ce montant ?

M118. *Proposed by Andrew Mao, Grade 12 student, A.B. Lucas Secondary School, London, ON.*

You are given a sheet of paper of size 2003×2004. You are allowed to cut it either horizontally or vertically (that is, parallel to an edge). You wish to obtain 2003×2004 unit squares. You are not allowed to fold or stack pieces of the paper. Determine the minimum number of cuts required.

. . . .

On vous donne une feuille de papier de format 2003×2004. Déterminer le nombre minimal de coupes nécessaires pour obtenir 2003×2004 carrés, si seules les coupes parallèles aux bords sont permises et s'il n'est pas permit de plier le papier ni d'empiler des morceaux.

Mayhem Solutions

M63. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS.*

Let ABC be a right-angled triangle with BC as its hypotenuse. From vertex A, construct altitude AD and internal angle bisector AE (so D and E are on side BC). We are given that $AD = 28$ and $AE = 35$. Determine the area of triangle ABC.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Without loss of generality we may assume that E is between D and C. Since $AD \perp BC$, we see that $\triangle ABD$ and $\triangle ADE$ are right-angled. Applying the Theorem of Pythagoras to $\triangle ADE$ yields $DE = 21$. Then

$$\angle BAD \;=\; \angle BAE - \angle DAE \;=\; 45^\circ - \arcsin \tfrac{3}{5}\,, \qquad (1)$$

since $\angle BAE = \frac{1}{2}\angle BAC = 45^\circ$. Since $\triangle DAB$ and $\triangle ACB$ are similar, we have $\angle BAD = \angle ACB$. Thus,

$$\angle ACB = 45^\circ - \arcsin\tfrac{3}{5}\,. \tag{2}$$

Then,

$$\begin{aligned}\sin(\angle ACB) &= \sin\left(45^\circ - \arcsin\tfrac{3}{5}\right)\\ &= \sin(45^\circ)\cos(\arcsin\tfrac{3}{5}) - \sin(\arcsin\tfrac{3}{5})\cos(45^\circ)\\ &= \tfrac{\sqrt{2}}{2}\cos\left(\arccos\tfrac{4}{5}\right) - \tfrac{3}{5}\tfrac{\sqrt{2}}{2} \;=\; \tfrac{1}{10}\sqrt{2}\,. \end{aligned} \tag{3}$$

Using (2), we get $\angle AEC = 180^\circ - 45^\circ - \angle ACB = 90^\circ + \arcsin\frac{3}{5}$; whence

$$\sin(\angle AEC) \;=\; \cos\left(\arcsin\tfrac{3}{5}\right) \;=\cos\left(\arccos\tfrac{4}{5}\right) \;=\; \tfrac{4}{5}\,. \tag{4}$$

Applying (3), (4), and the Law of Sines to $\triangle AEC$, we have

$$AC \;=\; 140\sqrt{2}\,. \tag{5}$$

Using (1), we obtain $\angle ABC = 90^\circ - \angle BAD = 45^\circ + \arcsin\frac{3}{5}$. Then

$$\begin{aligned}\sin(\angle ABC) &= \sin\left(45^\circ + \arcsin\frac{3}{5}\right)\\ &= \sin(45^\circ)\cos(\arcsin\tfrac{3}{5}) + \sin(\arcsin\tfrac{3}{5})\cos(45^\circ)\\ &= \tfrac{\sqrt{2}}{2}\cos\left(\arccos\tfrac{4}{5}\right) + \tfrac{3}{5}\tfrac{\sqrt{2}}{2} \;=\; \tfrac{7}{10}\sqrt{2}\,.\end{aligned}$$

Hence,

$$AB \;=\; \frac{28}{\sin(\angle ABC)} \;=\; 20\sqrt{2}\,. \tag{6}$$

Finally, using equations (5) and (6), we get

$$\text{Area of }\triangle ABC \;=\; \frac{AB\cdot AC}{2} \;=\; 2800\,.$$

Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.

M64. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

For real numbers x, let $f(x) = \lfloor x\lceil x\rceil\rfloor - \lceil x\lfloor x\rfloor\rceil$ (where $\lfloor x\rfloor$ is the largest integer smaller than or equal to x and $\lceil x\rceil$ is the smallest integer greater than or equal to x).

(a) Show that $f(x) \geq 0$ for all $x \geq 0$, and determine when equality holds.

(b) What is the situation if $x < 0$?

Solution by the proposer.

(a) Clearly, $f(x) = 0$ if $x \in \mathbb{N}$. Suppose $x = k + d$, where $k \in \mathbb{N} \cup \{0\}$ and $0 < d < 1$. Then

$$\begin{aligned} \lfloor x\lceil x\rceil\rfloor &= \lfloor (k+d)(k+1)\rfloor = \lfloor k^2+k+dk+d\rfloor \geq k^2+k\,, && (1)\\ \lceil x\lfloor x\rfloor\rceil &= \lceil (k+d)k\rceil = \lceil k^2+dk\rceil \leq \lceil k^2+k\rceil = k^2+k\,. && (2)\end{aligned}$$

From (1) and (2), we have $f(x) \geq 0$.

Now we look for the cases where equality occurs. We have already noted that $f(x) = 0$ when $x \in \mathbb{N}$. Suppose $x = k + d$, where $k \in \mathbb{N} \cup \{0\}$ and $0 < d < 1$. If $f(x) = 0$, then we must have equality in (1). Hence, $dk + d < 1$, or $d < \dfrac{1}{k+1}$. Then $dk < \dfrac{k}{k+1} < 1$, which implies that $\lceil k^2 + dk\rceil = k^2 + 1 < k^2 + k$. If $k \geq 2$, this is a contradiction. Hence, $k = 0$ or $k = 1$.

When $k = 0$, we have $x = d$. Thus, $\lceil x\lfloor x\rfloor\rceil = 0$ and $\lfloor x\lceil x\rceil\rfloor = \lfloor x\rfloor = 0$; whence, $f(x) = 0$.

When $k = 1$, we have $x = 1 + d$. Thus, $\lfloor x\rfloor = 1$ and $\lceil x\rceil = 2$. Hence, $\lceil x\lfloor x\rfloor\rceil = \lceil x\rceil = 2$, while

$$\lfloor x\lceil x\rceil\rfloor = \lfloor 2x\rfloor = \lfloor 2+2d\rfloor = \begin{cases} 2 & \text{if } 0 < d < \frac{1}{2},\\ 3 & \text{if } \frac{1}{2} \leq d < 1.\end{cases}$$

Hence, we have $f(x) = 0$ if $0 < d < \frac{1}{2}$ and $f(x) \neq 0$ if $\frac{1}{2} \leq d < 1$.

To summarize, $f(x) = 0$ if and only if $x \in \mathbb{N}$ or $x \in \left[0, \frac{3}{2}\right)$.

(b) For $x < 0$, we use the fact that $\lfloor -t\rfloor = -\lceil t\rceil$ for all $t \in \mathbb{R}$. Let $x = -y$ where $y > 0$. Then

$$\begin{aligned} \lfloor x\lceil x\rceil\rfloor &= \lfloor -y\lceil -y\rceil\rfloor = \lfloor (y(-\lfloor y\rfloor)\rfloor = \lfloor y\lfloor y\rfloor\rfloor\,, && (3)\\ \lceil x\lfloor x\rfloor\rceil &= \lceil -y\lfloor -y\rfloor\rceil = \lceil -y(-\lceil y\rceil)\rceil = \lceil y\lceil y\rceil\rceil\,. && (4)\end{aligned}$$

From (3) and (4), we see that $f(x) \leq 0$. Clearly, equality holds if $x \in \mathbb{Z}$. Suppose $y = k + d$, where $k \in \mathbb{N} \cup \{0\}$ and $0 < d < 1$. Then

$$\lfloor y\lfloor y\rfloor\rfloor = \lfloor (k+d)k\rfloor \leq \lfloor k^2+k\rfloor = k^2+k$$

and

$$\begin{aligned} \lceil y\lceil y\rceil\rceil &= \lceil (k+d)(k+1)\rceil = k(k+1) + \lceil (k+1)d\rceil\\ &\geq k(k+1)+1 > k^2+k\,.\end{aligned}$$

Hence, equality can not hold.

To summarize, $f(x) \leq 0$ for all $x < 0$, with equality if and only if x is a negative integer.

M65. *Proposed by the Mayhem Staff.*

I have a number of unit cubes and I arrange them all to form a larger solid cube. I then paint some of the faces of the large cube. When the cube is disassembled it is discovered that 1000 of the cubes have no paint on them. How many faces of the big cube were painted?

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let us notice that, if a large solid cube is formed by n^3 unit cubes, then $(n-2)^3$ of them are interior and the remaining ones form the faces of the large solid cube. Since some of the faces are painted and there are 1000 cubes with no paint, we see that $n \geq 11$. On the other hand, if $n > 12$, we would have at least 1331 interior cubes, all of which would be unpainted. Thus, $n = 11$ or $n = 12$.

For $n = 11$, there are 331 painted unit cubes, provided we paint 3 faces that have a vertex in common. Then the number of cubes which have no paint on them is $1331 - 331 = 1000$.

For $n = 12$, the cube has 1000 interior cubes. Then, all the faces of the big cube were painted.

Therefore, the number of painted faces was either 3 or 6.

M66. *Proposé par Václav Konečný, Big Rapids, MI, USA.*

Trouver des entiers positifs N en base 10 tels que $N!$ en base 6 finisse exactement par 99 zéros.

Solution de Robert Bilinski, Outremont, QC.

Un nombre en base 6 va finir avec 99 zéros si le premier coefficient de son expansion en base 6 qui n'est pas zéro est celui de 6^{99}. Autrement dit, $N!$ doit être un multiple de qui n'est pas un multiple de 6^{99} pour avoir exactement 99 zéros. Puisque l'on accumule plus lentement les 3 que les 2, cela arrivera quand $N!$ contiendra 99 fois le facteur 3. En faisant une étude sommaire, on voit que $3!$ a 1 facteur 3, $6!$ en a 2, $\ldots$, $27!$ en a 13, $54!$ en a 26, $81!$ en a 40, $162!$ en a 80, et ainsi de suite. On arrête avec $204!$ qui a 99 facteurs 3 (et au moins 102 facteurs 2). Ainsi, les nombres qui ont la propriété voulu sont 204, 205 et 206.

Résolu aussi par Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine.

M67. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Find the interval containing r so that three consecutive terms of the geometric sequence a, ar, ar^2, $\ldots$ are the sides of a triangle.

Solution by Geneviève Lalonde, Massey, ON.

Since lengths are positive, we know that a and r must be positive. We have two cases:

Case 1. $r \geq 1$.

In this case ar^2 is the largest. Thus, in order to have a triangle, we must have

$$\begin{aligned} ar^2 &< a + ar\,, \\ r^2 - r - 1 &< 0\,, \\ r &< \frac{1+\sqrt{5}}{2}\,. \end{aligned}$$

This, together with our initial assumption, means $1 \le r < \frac{1}{2}(1+\sqrt{5})$.

Case 2. $0 < r < 1$.

In this case a is the largest. Thus, in order to have a triangle, we must have

$$\begin{aligned} ar^2 + ar &> a\,, \\ r^2 + r - 1 &> 0\,, \\ \frac{-1+\sqrt{5}}{2} &< r\,. \end{aligned}$$

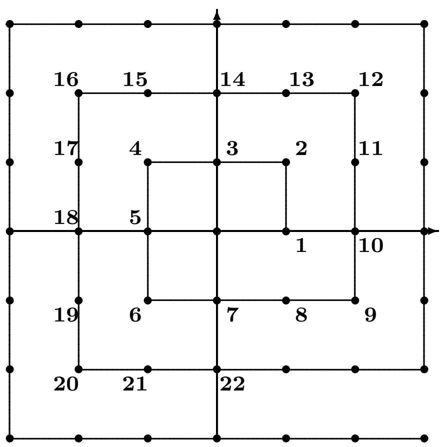
This, together with our initial assumption, means $\frac{1}{2}(-1+\sqrt{5}) < r < 1$.

Putting these two cases together, we get $\frac{1}{2}(-1+\sqrt{5}) < r < \frac{1}{2}(1+\sqrt{5})$.

M68. *Proposed by the Mayhem Staff.*

You go for a spiralling walk on the Cartesian plane. Starting at $(0,0)$, your first five steps are to the points $(1,0)$, $(1,1)$, $(0,1)$, $(-1,1)$ and $(-1,0)$. What point do you arrive at on your 2002^{th} step?

Solution by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.



Considering the points on the negative y-axis, we see that the 7^{th} step is $(0,-1)$ and the 22^{nd} step is $(0,-2)$. The n^{th} point on the negative y-axis is $(0,-n)$, which is step number $(2n+1)^2 - (n+1) = 4n^2 + 3n$.

Let us see if the 2002^{nd} step is on the negative y-axis. We solve:

$$\begin{aligned} 4n^2 + 3n &= 2002\,, \\ 4n^2 + 3n - 2002 &= 0\,, \\ (n-22)(4n+91) &= 0\,. \end{aligned}$$

Thus, we see that the 22^{nd} point on the y-axis, namely $(0, -22)$, is the 2002^{nd} step.

Pólya's Paragon

Apollonius' Problem

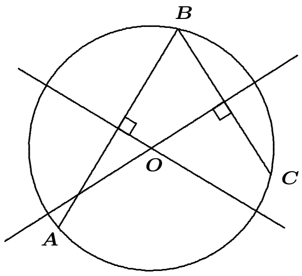
Paul Ottaway

The ancient Greeks are famous for their knowledge of geometry. Among them was a mathematician and scientist named Apollonius, who studied conic sections and astronomy. He lived about 2200 years ago, but the specific years of his birth and death are not known. He posed an interesting question dealing with circles, which has since been aptly named "Apollonius' Problem".

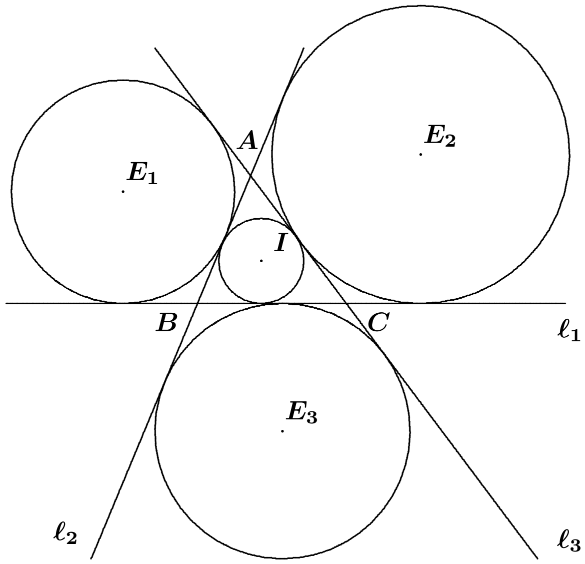
We begin with three objects, each of which may be either a line, a point or a circle. The goal is to construct a circle which is tangent to all three objects. In this case, tangent to a point means passing through the point.

Clearly, this goal is many problems all posed at the same time. In fact, there are ten separate cases that need to be checked in order to verify that such a circle can be constructed. In some cases, there are many circles which satisfy the given criteria. It has been shown that all cases are constructible using only a straightedge and compass. We will solve some of the easiest cases here.

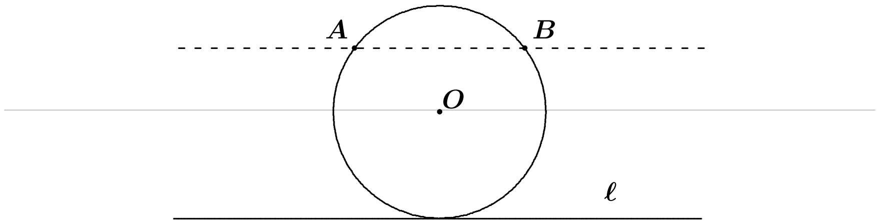
First, we examine the case where we are given three points. If the points all lie on a common line, then no circle can pass through all of them simultaneously. Otherwise, we can think of the three points as being the three vertices of a triangle. Now our goal is to construct a circle which circumscribes this triangle. Let the points be labelled A, B, and C, respectively. Then draw the line segments $\overline{AB}$ and $\overline{BC}$. Construct the perpendicular bisectors of these line segments, and label their intersection as O. Since O is on the perpendicular bisector of $\overline{AB}$, we know that O is equidistant from A and B. Likewise, we find that O is equidistant from B and C. Therefore, O is an equal distance R from all 3 points A, B, and C. We can draw the circle centred at O with radius R. The point O is called the *circumcentre* of the triangle ABC and the circle constructed is called the *circumcircle*. This is the only circle which passes through all three vertices of $\triangle ABC$.



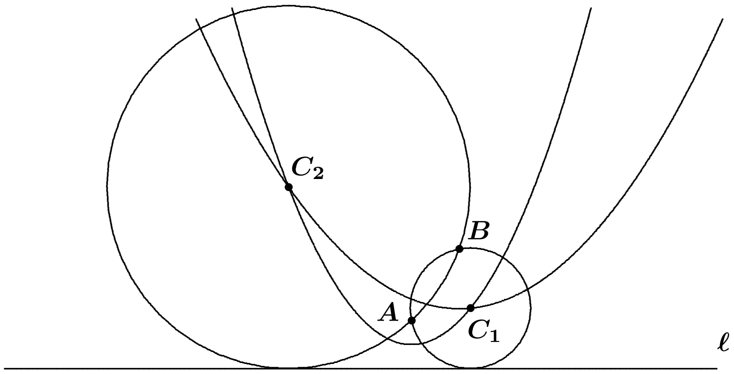
Now, let us consider the case where we are given three lines. It is interesting to note that the number of solutions this time depends on the lines that are given. If all three lines are parallel (and distinct), then there are no solutions. If two of the three lines are parallel (and distinct), then we find exactly two solutions. Finally, if no two of the lines are parallel, then there are exactly four circles which can be drawn such that each is tangent to all three lines. Label the intersections of these lines A, B and C. The centres of these circles are called the *incentre* I and the *excentres* E_1, E_2, and E_3 of triangle ABC.



A slightly more interesting case is when our three objects are one line and two points. If the points are on opposite sides of the line, then there are no solutions, since the circle would have to cross the line to pass through both points. On the other hand, if both points are on the same side of the line, there are still two cases to check. If the two points define a line which is parallel to the given line, then there is exactly one solution, as shown in the following diagram



Otherwise, we can show that there are two circles passing through the two points and tangent to the line. First, we must recall that a parabola is a locus of points which are equidistant from a line (the *directrix*) and a point (the *focus*). We consider the two parabolas that can be constructed by using the given line as the directrix and one of the two points as a focus. These two parabolas intersect at two points which are equidistant from both the given points and the given line. The intersection points are the centres of circles which pass through the given points and are tangent to the given line. (Note that if the two given points define a line parallel to the given line, then the parabolas will intersect at only one point.)



PROBLEMS:

1. Solve the case where there are two lines and one point.
2. For the cases discussed in this article, find a way to construct the solution using only a straightedge and compass. (*Hint:* Make sure you can construct perpendicular bisectors and angle bisectors first!)
3. For the remaining cases of Apollonius' Problem, determine how many circles there are which meet the given requirements.
4. Fill in the details on how to find the incentres and excentres of a triangle.

THE OLYMPIAD CORNER

No. 233

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

As a first Olympiad this issue we give the 14^{th} Nordic Mathematical Contest written March 30, 2000. My thanks go to Matti Lehtinen, who sent the contest for use in the *Corner*.

14^{th} NORDIC MATHEMATICAL CONTEST

March 30, 2000 (Time Allowed: 4 hours)

1. In how many ways can the number 2000 be written as a sum of three positive, not necessarily different, integers? (Sums like $1+2+3$ and $3+1+2$, etc. are considered to be the same.)

2. The persons P_1, P_2, $\dots$, P_{n-1}, P_n sit around a table, in this order, and each one has a number of coins. At the start, P_1 has one coin more than P_2, P_2 has one coin more than P_3, etc., up to P_{n-1}, who has one coin more than P_n. Now P_1 gives one coin to P_2, who in turn gives two coins to P_3, etc., up to P_n, who gives n coins to P_1. The process continues in the same way: P_1 gives $n+1$ coins to P_2, P_2 gives $n+2$ coins to P_3, and so on. The transactions go on until someone has not enough coins to give away one coin more than he just received. At the moment when the process comes to an end in this manner, it turns out that there are two neighbours at the table one of whom has exactly five times as many coins as the other. Determine the number of persons and the number of coins circulating around the table.

3. In the triangle ABC, the bisector of angle B meets AC at D, and the bisector of angle C meets AB at E. The bisectors intersect at O, and $OD = OE$. Prove that either $\triangle ABC$ is isosceles or $\angle BAC = 60^\circ$.

4. The real-valued function f is defined for $0 \leq x \leq 1$, and satisfies $f(0) = 0$, $f(1) = 1$, and

$$\frac{1}{2} \leq \frac{f(z) - f(y)}{f(y) - f(x)} \leq 2\,,$$

for all $0 \leq x < y < z \leq 1$ with $z - y = y - x$. Prove that

$$\frac{1}{7} \leq f\left(\frac{1}{3}\right) \leq \frac{4}{7}\,.$$

A second set for your puzzling pleasure is the Finnish High School Mathematics Competition, written January 28, 2000. Thanks again go to Matti Lehtinen for collecting them for our use.

FINNISH HIGH SCHOOL MATH COMPETITION
Final Round
January 28, 2000 (Time Allowed: 3 hours)

1. Two circles touch each other externally at A. A common tangent touches one circle at B and the other at C ($B \neq C$). The segments BD and CE are diameters of the circles. Prove that D, A, and C are collinear.

2. Prove that the integer part of $(3+\sqrt{5})^n$ is odd for every positive integer n.

3. Determine all positive integers n such that $n! > \sqrt{n^n}$.

4. There are seven points in the plane, no three of which are collinear. Every point is joined to every other by either a blue or a red line segment. Prove that there are at least four monochromatic triangles in the figure.

5. Irja and Valtteri alternate in flipping a coin. Irja starts. They each have a counter, and at the start the counters are in opposite corners of a square. If the person whose turn it is to flip gets heads, that person's counter is moved to the opposite corner; if that person gets tails, his/her counter is moved to an adjacent corner, where the direction of movement is counterclockwise for Irja and clockwise for Valtteri. The winner is the one whose counter is moved to the corner occupied by the opponent's counter. What is the probability that Irja wins?

We turn to readers' solutions to some of the problems of the Swedish Mathematical Competition 1997, Final Round, given in [2001 : 293].

1. Let AC be a diameter of a circle. Assume that AB is tangent to the circle at the point A and that the segment BC intersects the circle at D. Show that if $|AC| = 1$, $|AB| = a$, and $|CD| = b$, then

$$\frac{1}{a^2+\frac{1}{2}} < \frac{b}{a} < \frac{1}{a^2}\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give Bataille's write-up.

Let $\theta = \angle ACD$. Note that $0 < \theta < 90^\circ$. We have $\tan\theta = \dfrac{|AB|}{|AC|} = a$ and $\cos\theta = \dfrac{|CD|}{|AC|} = b$, since $\angle CAB = \angle CDA = 90^\circ$.

Now, $\frac{b}{a} < \frac{1}{a^2}$ holds, since this is equivalent to $ab < 1$, or $\sin\theta < 1$, which is true. Also, $\frac{1}{a^2+1/2} < \frac{b}{a}$ holds, being successively equivalent to

$$\begin{aligned} 2a^2 - 2\frac{a}{b} + 1 &> 0\,, \\ 2(\tan\theta)^2 - 2\left(\frac{\tan\theta}{\cos\theta}\right) + 1 &> 0\,, \\ 2(\sin\theta)^2 - 2\sin\theta + (\cos\theta)^2 &> 0\,, \\ (\sin\theta - 1)^2 &> 0\,, \end{aligned}$$

which is true as well.

2. The bisector of the angle B in the triangle ABC intersects the side AC at the point D. Let E be a point on side AB such that $3\angle ACE = 2\angle BCE$. The segments BD and CE intersect at the point P. One knows that $|ED| = |DC| = |CP|$. Find the angles of the triangle.

Solved by Christopher J. Bradley, Clifton College, Bristol, UK; Pavlos Maragoudakis, Pireas, Greece; and Toshio Seimiya, Kawasaki, Japan. We present Seimiya's solution.

We set $\angle ABD = \angle DBC = \beta$ and $3\angle ACE = 2\angle BCE = 6\alpha$. Then $\angle ACE = 2\alpha$ and $\angle BCE = 3\alpha$. Since $CP = CD$, we see that $\angle CPD = \angle CDP$. Since

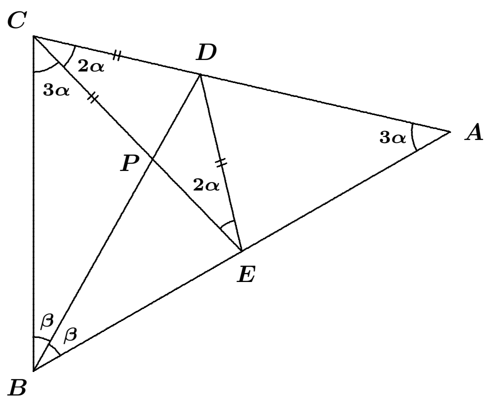
$$\angle CPD \;=\; \angle PBC + \angle PCB \;=\; \beta + 3\alpha\,,$$

and since

$$\angle CDP \;=\; \angle CDB \;=\; \angle A + \angle ABD \;=\; \angle A + \beta\,,$$

we must have $\beta + 3\alpha = \angle A + \beta$, which implies that $\angle A = 3\alpha$. Therefore,

$$\angle BEC \;=\; \angle A + \angle ACE \;=\; 3\alpha + 2\alpha \;=\; 5\alpha\,.$$



Since $DE = DC$, we have $\angle DEC = \angle DCE = 2\alpha$. Therefore,

$$\angle BED \;=\; \angle BEC + \angle DEC \;=\; 5\alpha + 2\alpha = 7\alpha\,.$$

By the Law of Sines for $\triangle BDE$ and $\triangle BDC$, we have

$$\frac{\sin BED}{BD} \;=\; \frac{\sin EBD}{DE} \;=\; \frac{\sin CBD}{DC} \;=\; \frac{\sin BCD}{BD}\,.$$

Thus, $\sin BED = \sin BCD$. Hence, either $\angle BED = \angle BCD$ or $\angle BED + \angle BCD = 180^\circ$. Since $\angle BED = 7\alpha$ and $\angle BCD = 5\alpha$, we see that $\angle BED \neq \angle BCD$. Thus, $\angle BED + \angle BCD = 180^\circ$. This further implies that B, C, D, and E are concyclic. Therefore, $\angle EBD = \angle ECD$; that is, $\beta = 2\alpha$. By examining the interior angles of $\triangle ABC$, we have $3\alpha + 2\beta + 5\alpha = 180^\circ$. Consequently, we have

$$3\alpha + 4\alpha + 5\alpha \;=\; 180^\circ\,,$$

from which we get $\alpha = 15^\circ$.

Hence, $\angle A = 3\alpha = 45^\circ$, $\angle B = 2\beta = 4\alpha = 60^\circ$, and $\angle C = 5\alpha = 75^\circ$. (It is easy to verify that this triangle satisfies the required conditions.)

3. Let the sum of the two integers A and B be odd. Show that any integer can be written in the form $x^2 - y^2 + Ax + By$, where x and y are integers.

Solved by Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We present Bornsztein's solution.

For integers x, y, let $f(x,y) = x^2 - y^2 + Ax + By$. If α is any integer, we have:

$$f(y+\alpha, y) \;=\; (A + B + 2\alpha)y + \alpha^2 + A\alpha\,.$$

Since $A + B$ is odd, we may choose $\alpha = \alpha_0$ such that $A + B + 2\alpha_0 = 1$. Let $k = \alpha_0^2 + A\alpha_0$. Note that k does not depend on y. Thus, $f(y+\alpha_0, y) = y + k$ takes every integer value when y ranges over $\mathbb{Z}$. The conclusion follows.

4. A and B are playing a game consisting of two parts:

• A and B make one throw each with a die. If the outcome is x and y, respectively, a list is created consisting of all two-digit integers $10a + b$, with a, $b \in \{1, 2, 3, 4, 5, 6\}$ such that $10a + b \leq 10x + y$.

For instance, if $x = 2$ and $y = 3$ the list is:

$$11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 21,\ 22,\ 23.$$

• The players now reduce the number of integers in the list by replacing a pair of the integers in the list by the non-negative difference of the chosen integers. If A, for instance, chooses 14 and 21 in the above example, these two integers are removed and replaced by the integer 7. The new list becomes:

$$7,\ 11,\ 12,\ 13,\ 15,\ 16,\ 22,\ 23.$$

In the next move B may choose, for instance, 7 and 23, reducing the number of integers by one, and leaving the list

$$11,\ 12,\ 13,\ 15,\ 16,\ 16,\ 22.$$

The game is over when the list has been reduced to only one integer.

If the integer in the final list has the same parity as the outcome of A's throw, then A is the winner. What is the probability that A wins the game?

Solved by Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB. We give an edited version of his solution.

Let a_1, a_2, $\ldots$, a_n be the list of numbers. First notice that replacing two of the numbers by their difference does not alter the parity of the sum $a_1 + a_2 + \cdots + a_n$. Thus, the parity of the number remaining at the end of the game does not depend on the way in which the list is reduced.

Now, the problem is to determine for which odd tosses of A (1, 3, or 5) the sum $a_1 + \cdots + a_n$ is odd, and for which even tosses of A (2, 4, or 6) the sum is even.

Let $S_{(x,y)}$ be the list obtained when A tosses x and B tosses y. When $x = 1$, the possible lists are

$$\begin{aligned}
S_{(1,1)} &: \ 11 \\
S_{(1,2)} &: \ 11,\ 12 \\
S_{(1,3)} &: \ 11,\ 12,\ 13 \\
S_{(1,4)} &: \ 11,\ 12,\ 13,\ 14 \\
S_{(1,5)} &: \ 11,\ 12,\ 13,\ 14,\ 15 \\
S_{(1,6)} &: \ 11,\ 12,\ 13,\ 14,\ 15,\ 16\,.
\end{aligned}$$

Notice that $S_{(1,1)}$, $S_{(1,2)}$, $S_{(1,5)}$, and $S_{(1,6)}$ have odd sums. Similarly, when $x = 3$, we see that $S_{(3,1)}$, $S_{(3,2)}$, $S_{(3,5)}$, and $S_{(3,6)}$ have odd sums, and when $x = 5$, the sum is odd for four of the six tosses by B. Altogether, A wins 12 times out of 18 possibilities when x is odd.

When x is an even number, since there is an odd number of odd numbers $10a + b$ with $a < x$, the sum of the list entries is even only when B tosses 1, 2, 5, or 6. It follows that A also wins in 12 of the 18 possibilities when x is even.

Thus, A wins 24 of the 36 possible games, a probability of $2/3$.

5. Let $s(m)$ denote the sum of the digits of the integer m. Prove that for any integer n, with $n > 1$ and $n \neq 10$, there is a unique integer $f(n) \geq 2$ such that $s(k) + s(f(n) - k) = n$ for all integers k satisfying $0 < k < f(n)$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Pontoise, France. We give Bataille's write-up.

First suppose that $2 \leq n \leq 9$ and that $f(n) \geq 2$ satisfies the given condition. If $f(n) > n$, then we have $s(n) + s(f(n) - n) = n$, which is a contradiction, since $s(f(n) - n) > 0$ and $s(n) = n$. Thus, $f(n) \leq n$;

whence, $2 \leq f(n) \leq 9$. From $s(1) + s(f(n) - 1) = n$, we now deduce that $1 + f(n) - 1 = n$, or $f(n) = n$. Conversely, $s(k) + s(n-k) = k + n - k = n$ if $0 < k < n \leq 9$.

If $n = 10$, it is readily checked that $f(n) = 10$ or $f(n) = 19$ are two possible solutions, so that the uniqueness fails in this case.

From now on, we assume $n \geq 11$. By the Euclidean Algorithm, we have $n = 9q + r$, where q is a positive integer and $r \in \{0, 1, 2, \ldots, 8\}$. We prove that the integer $f(n) = r \cdot 10^q + (10^q - 1)$ satisfies the condition of the problem.

Indeed, we can express $f(n)$ as

$$f(n) \; = \; r \cdot 10^q + 9 \cdot 10^{q-1} + \cdots + 9 \cdot 10 + 9 \,,$$

and, if $0 < k < f(n)$, we can express k as

$$k \; = \; a_q \cdot 10^q + a_{q-1} \cdot 10^{q-1} + \cdots + a_1 \cdot 10 + a_0 \,,$$

for some integers a_j such that $0 \leq a_q \leq r$ and $0 \leq a_j \leq 9$ for $j < q$. Hence,

$$\begin{aligned} f(n) - k \; = \; & (r - a_q)10^q + (9 - a_{q-1})10^{q-1} + \cdots \\ & \cdots + (9 - a_1)10 + (9 - a_0) \,, \end{aligned}$$

and thus, $s(f(n) - k) = r + 9q - (a_q + a_{q-1} + \cdots + a_0) = n - s(k)$, as desired.

It remains to show the uniqueness of $f(n)$. We first show that any suitable $f(n)$ must satisfy $f(n) \geq 11$. If we had $f(n) \leq 10$, then, using $s(1) + s(f(n) - 1) = n$, we would get $f(n) = n$, a contradiction, since $n \geq 11$ and $f(n) \leq 10$.

Next we show that $s(f(n)) = n$. From the given condition with $k = 9$ and with $k = 10$, we get $s(f(n) - 9) = n - 9$ and $s(f(n) - 10) = n - 1$. It follows that $n - 9 = s(f(n) - 10 + 1) = s(f(n) - 10) + 1 - 9m$, where m denotes the number of 9s at the right end of the integer $f(n) - 10$. (For example, $m = 1$ if $f(n) - 10 = 19129$). Hence,

$$n - 9 = (n - 1) + 1 - 9m = n - 9m \,.$$

Then $m = 1$. Thus, $f(n) - 10$ has a 9 as its rightmost digit and, consequently, so does $f(n)$. From this observation, we deduce that

$$s(f(n)) = s(f(n) - 1) + 1 = s(f(n) - 1) + s(1) = n \,.$$

If $f_1(n)$ and $f_2(n)$ were two suitable integers with $f_1(n) < f_2(n)$, then $s(f_1(n)) + s(f_2(n) - f_1(n)) = n$ yields $s(f_2(n) - f_1(n)) = 0$ (since $s(f_1(n)) = n$), a contradiction. The uniqueness of $f(n)$ follows.

Next we turn to readers' solutions to some of the problems of the Ukrainian Mathematical Olympiad 1998, Selected Problems, April 1998, given in [2001 : 294–295].

1. (9th grade) Prove the inequality

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \;\geq\; 3$$

for positive real numbers a, b, c with $abc = 1$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Rahul Bamotra, student, Sir Winston Churchill High School, Calgary, AB; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; George Evagelopoulos, Athens, Greece; Murray S. Klamkin, University of Alberta, Edmonton, AB; Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB; Pavlos Maragoudakis, Pireas, Greece; Toshio Seimiya, Kawasaki, Japan; D.J. Smeenk, Zaltbommel, the Netherlands; and Panos E. Tsaoussoglou, Athens, Greece. We give Bataille's solution.

Since $abc = 1$, we have

$$\frac{1+ab}{1+a}=\frac{1+\frac{1}{c}}{1+a} \;=\; \frac{1+c}{c(1+a)}\,.$$

It follows that

$$\begin{aligned}\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \;&=\; \frac{1}{c}\cdot\frac{1+c}{1+a}+\frac{1}{a}\cdot\frac{1+a}{1+b}+\frac{1}{b}\cdot\frac{1+b}{1+c}\\ &\geq\; 3\sqrt[3]{\frac{1}{c}\cdot\frac{1+c}{1+a}\cdot\frac{1}{a}\cdot\frac{1+a}{1+b}\cdot\frac{1}{b}\cdot\frac{1+b}{1+c}} \;=\; 3\,,\end{aligned}$$

using the AM-GM Inequality, and recalling that $abc = 1$.

Ed: Equality occurs when $\frac{1+ab}{1+a} = \frac{1+bc}{1+b} = \frac{1+ac}{1+c}$, which implies $b + ab + ab^2 - a - bc = 1$ and two other similar equations. Adding these three equations, we obtain $ab^2 + bc^2 + ca^2 = 3$. Hence,

$$ab^2+bc^2+ca^2 \;=\; 3\sqrt[3]{ab^2\cdot bc^2\cdot ca^2}\,.$$

Thus, $ab^2 = bc^2 = ca^2 = 1$, and finally $a = b = c = 1$. The converse is obvious. We conclude that equality occurs if and only if $a = b = c = 1$.

Klamkin notes that, more generally, if $P = a_1a_2\cdots a_n = 1$, then

$$\frac{1+\frac{P}{a_n}}{1+a_1}+\frac{1+\frac{P}{a_1}}{1+a_2}+\cdots+\frac{1+\frac{P}{a_{n-1}}}{1+a_n} \;\geq\; n\,.$$

Here, when n is odd, there is equality if and only if all the a_i are 1. For $n = 2m$, we have equality if and only if $a_1 = a_3 = \cdots = a_{2m-1}$ and $a_2 = a_4 = \cdots = a_{2m} = \frac{1}{a_1}$.

2. (9th grade) A convex polygon with 2000 vertices in a plane is given. Prove that we may mark 1998 points of the plane so that any triangle with vertices which are vertices of the polygon has exactly one marked point as an internal point.

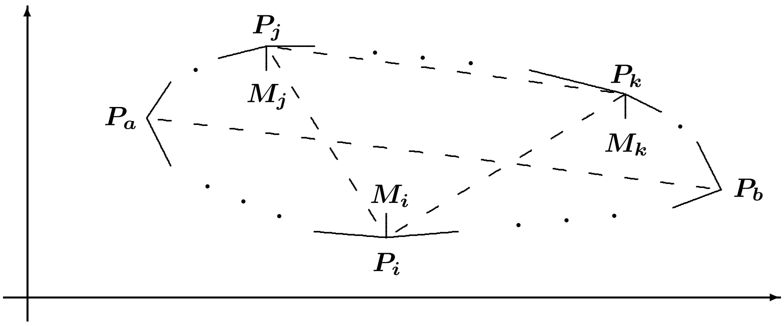
Solved by Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We use Bornsztein's solution.

More generally, for any integer $n \geq 3$, we prove the result with the numbers n and $n-2$ instead of 2000 and 1998, respectively.

Let $\mathcal{K}$ be a convex n–gon in the plane. Since $\mathcal{K}$ has a finite number of vertices, the number of lines that pass through at least two of these vertices is finite. Thus, we may consider an orthonormal coordinate system $\mathcal{R}$ such that, in $\mathcal{R}$, the x–coordinates of the vertices of $\mathcal{K}$ are pairwise distinct. Let $P_i(x_i, y_i)$, for $i = 1, 2, \ldots, n$, be the vertices of $\mathcal{K}$, ordered so that $x_1 < x_2 < \cdots < x_n$.

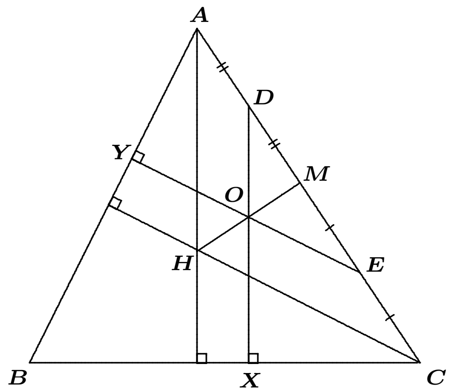
For $i \neq j$, let Δ_{ij} be the line (P_iP_j). Denote by $d(P_k, \Delta_{ij})$ the distance from P_k to Δ_{ij}. Since there is a finite number of lines Δ_{ij}, we may consider the minimum of the set of distances $d(P_k, \Delta_{ij})$, for i, j, k pairwise distinct. Let d denote this minimum. Since the vertices P_i are in convex position, we have $d > 0$.

For $i = 2, \ldots, n-1$, exactly one of the two points with coordinates $\left(x_i, y_i \pm \frac{d}{2}\right)$ is an interior point of $\mathcal{K}$. Denote this point by M_i. The construction of the points M_i ensures that M_i is an exterior point of any of the triangles $P_aP_bP_c$ where $i \notin \{a, b, c\}$. Moreover, M_i is an interior point of $P_iP_jP_k$ if and only if $j < i < k$ or $k < i < j$. Thus, each one of the triangles $P_aP_bP_c$ contains exactly one point M_i. We mark these points, and the conclusion follows.



3. (10th grade) Let M be an internal point on the side AC of a triangle ABC, and let O be the intersection point of perpendiculars from the midpoints of AM and MC to lines BC and AB, respectively. Find the location of M such that the length of segment OM is minimal.

Solved by Christopher J. Bradley, Clifton College, Bristol, UK; George Evagelopoulos, Athens, Greece; and Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.



Let D and E be the mid-points of AM and CM, respectively. Let X and Y be the feet of the perpendiculars from D to BC and from E to AB, respectively. Let H be the orthocentre of $\triangle ABC$.

Since $AH \perp BC$ and $DX \perp BC$, we have $AH \parallel DX$. Because D is the mid-point of AM, we see that DX passes through the mid-point of HM. Similarly, EY passes through the mid-point of HM. Therefore, the intersection of DX and EY is the mid-point of HM. Thus, O is the mid-point of HM. Hence, $OM = \frac{1}{2}HM$. We conclude that OM is minimal when HM is minimal.

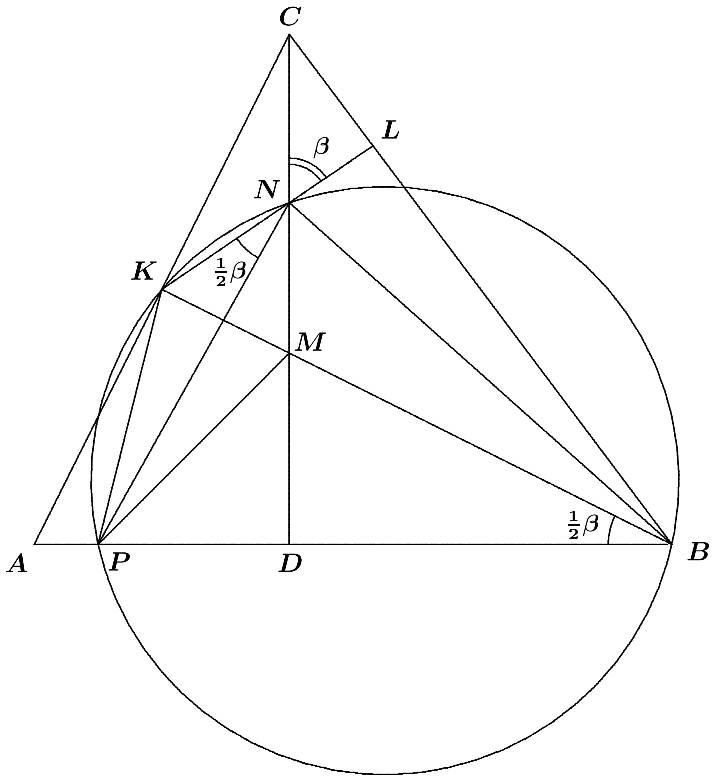
Let P be the foot of the altitude from B to AC. Then P is also the foot of the perpendicular from H to AC. If $\angle BAC$ and $\angle BCA$ are acute, then P is an internal point of the side AC, and the length of HM may be minimized by taking $M = P$. If $\angle BAC \geq 90^\circ$ or $\angle BCA \geq 90^\circ$, then P is not an internal point of AC, and there is no choice for M that minimizes the length of HM.

Thus, the length of OM is minimal when M is the foot of the altitude from B to AC, provided that $\angle BAC$ and $\angle BCA$ are acute. Otherwise, there is no minimal length.

[*Ed.* Although the figure above shows an acute-angled triangle, in which case O is an interior point of $\triangle ABC$, the argument in the proof still holds if $\angle ABC$ is obtuse.]

4. (11th grade) A triangle ABC is given. Altitude CD intersects the bisector BK of $\angle ABC$, and the altitude KL of BKC, at the points M and N, respectively. The circumscribed circle of BKN intersects segment AB at the point $P \neq B$. Prove that triangle KPM is isosceles.

Solved by Christopher J. Bradley, Clifton College, Bristol, UK; George Evagelopoulos, Athens, Greece; Toshio Seimiya, Kawasaki, Japan; and D.J. Smeenk, Zaltbommel, the Netherlands. We give the solution by Smeenk, adapted by the editor.



Quadrilateral $PBNK$ can be inscribed in a circle. Therefore,

$$\angle KNP \;=\; \angle KBP = \tfrac{1}{2}\beta\,, \tag{1}$$

where $\beta \;=\; \angle ABC$. We also observe that $\angle NCL \;=\; 90^\circ \;-\; \beta$, since $CD \perp AB$. This implies that $\angle CNL = \beta$, since $KL \perp BC$. Then

$$\angle KNM \;=\; \angle CNL \;=\; \beta\,. \tag{2}$$

From (1) and (2), we conclude that $\angle KNP = \angle MNP = \frac{1}{2}\beta$. Thus, NP is the bisector of $\angle KNM$. Furthermore, since $\angle KMN$ is an exterior angle of $\triangle MBC$, we have

$$\angle KMN \;=\; \angle MBC + \angle MCB \;=\; \tfrac{1}{2}\beta + (90^\circ - \beta) \;=\; 90^\circ - \tfrac{1}{2}\beta\,.$$

It follows that $NP \perp KM$. This can happen only when $PK = PM$.

5. (11th grade) For real numbers x, y, $z \in (0,1]$, prove the inequality

$$\frac{x}{1+y+zx} + \frac{y}{1+z+xy} + \frac{z}{1+x+yz} \;\leq\; \frac{3}{x+y+z}\,.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; George Evagelopoulos, Athens, Greece; Laura Gu, student, Sir Winston Churchill High School, Calgary, AB; Murray S. Klamkin, University of Alberta, Edmonton, AB; Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB; Pavlos Maragoudakis, Pireas, Greece; Panos E. Tsaoussoglou, Athens, Greece. We use the solution of Amengual Covas.

We will prove the stronger inequality

$$\frac{x}{1+y+zx} \;+\; \frac{y}{1+z+xy} \;+\; \frac{z}{1+x+yz} \;\leq\; 1\,.$$

Since $1+xy = (1-x)(1-y)+x+y$, we have

$$\begin{aligned} 1+z+xy &= (1-x)(1-y)+x+y+z \\ &\geq x+y+z\,. \end{aligned}$$

Analogously, we also have

$$\begin{aligned} 1+x+yz &= (1-y)(1-z)+x+y+z \;\geq\; x+y+z \\ \text{and} \quad 1+y+zx &= (1-z)(1-x)+x+y+z \;\geq\; x+y+z\,. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{x}{1+y+zx} + \frac{y}{1+z+xy} + \frac{z}{1+x+yz} \\ &\quad\leq\; \frac{x}{x+y+z} + \frac{y}{x+y+z} + \frac{z}{x+y+z} \;=\; 1\,, \end{aligned}$$

as claimed.

6. (11th grade) The function $f(x)$ is defined on $[0,1]$ and has values in $[0,1]$. It is known that $\lambda \in (0,1)$ exists such that $f(\lambda) \neq 0$ and $f(\lambda) \neq \lambda$. Also

$$f(f(x)+y) \;=\; f(x)+f(y)$$

for all x and y from the range of definition of the equality.

(a) Give an example of such a function.

(b) Prove that for any $x \in [0,1]$,

$$\underbrace{f(f(\ldots f}_{19}(x)\ldots)) \;=\; \underbrace{f(f(\ldots f}_{98}(x)\ldots))\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; and George Evagelopoulos, Athens, Greece. We use the solution by Bataille.

(a) The function f defined by

$$f(x) \;=\; \begin{cases} 0 & \text{for } x = 0 \text{ or } x \in \left(\frac{1}{2}, 1\right), \\ 1 & \text{otherwise,} \end{cases}$$

provides an example. Indeed, we can take $1/4$ for the required λ. Then $f(f(x)+y) = f(x)+f(y)$ is clearly satisfied when $f(x) = 0$. If $f(x) = 1$, we must have $y = 0$ (since $f(x) + y$ must be in $[0,1]$); whence,

$$f(f(x)+y) \;=\; f(f(x)) \;=\; f(1) \;=\; 1 \;=\; f(x)+f(0) \;=\; f(x)+f(y)\,.$$

(b) In the general case, let $f(0) = a \in [0,1]$. Then,

$$f(f(x)) \;=\; f(f(x)+0) \;=\; f(x)+a$$

for all $x \in [0,1]$. Taking $x = 0$, we have $f(a) = 2a$. Consider now any positive integer n such that $na \leq 1$ and $f(na) = (n+1)a$. (We have just shown that one such integer is $n = 1$.) Then $(n+1)a \leq 1$ (since f has values in $[0,1]$), and

$$f((n+1)a) \;=\; f(f(na)) \;=\; f(na)+a \;=\; (n+1)a+a \;=\; (n+2)a\,.$$

By induction, we have $na \leq 1$ for all positive integers n, which implies that $a = 0$. Consequently, $f(f(x)) = f(x)$ for all $x \in [0,1]$. In other words, we have $f^2 = f$, which, by a simple induction, leads to $f^m = f$ for all positive integers m. (Here f^2 means $f \circ f$, etc.)

Thus, $f^{19} = f^{98}$ $(= f)$, as required.

7. (11th grade) Two spheres with distinct radii are externally tangent at point P. Line segments AB and CD are given such that the first sphere touches them at the points A and C, and the second sphere touches them at the points B and D. Let M and N be the orthogonal projections of the mid-points of segments AC and BD on the line joining the centres of the given spheres. Prove that $PM = PN$.

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; and George Evagelopoulos, Athens, Greece. We give Bataille's solution.

Let S (centre O, radius R) and S' (centre O', radius R') be the given spheres. Denote by I, J, K, L the respective mid-points of segments AB, BD, DC, AC, so that M and N are the respective orthogonal projections of L and J onto OO'. Let $\overrightarrow{U} = \overrightarrow{OO'}/\|\overrightarrow{OO'}\|$, a unit vector in the direction of $\overrightarrow{OO'}$. For any point Z, we denote by $\overrightarrow{Z}$ the vector $\overrightarrow{PZ}$.

Now, $\overrightarrow{N}$ points in the same direction as $\overrightarrow{U}$ and $\overrightarrow{M}$ points in the opposite direction. Therefore, $PN = \overrightarrow{N} \cdot \overrightarrow{U}$ and $PM = -\overrightarrow{M} \cdot \overrightarrow{U}$. Note also that

$\overrightarrow{N} \cdot \overrightarrow{U} = \overrightarrow{J} \cdot \overrightarrow{U}$ and $\overrightarrow{M} \cdot \overrightarrow{U} = \overrightarrow{L} \cdot \overrightarrow{U}$, since M and N are the orthogonal projections of L and J onto OO'. Thus,

$$PN - PM \;=\; \overrightarrow{J} \cdot \overrightarrow{U} + \overrightarrow{L} \cdot \overrightarrow{U} \;=\; \left(\overrightarrow{J} + \overrightarrow{L}\right) \cdot \overrightarrow{U}\,.$$

We have

$$\overrightarrow{J} + \overrightarrow{L} \;=\; (\overrightarrow{B} + \overrightarrow{D}) + (\overrightarrow{A} + \overrightarrow{C}) \;=\; (\overrightarrow{A} + \overrightarrow{B}) + (\overrightarrow{C} + \overrightarrow{D}) \;=\; \overrightarrow{I} + \overrightarrow{K}\,.$$

Hence, $PN - PM = \left(\overrightarrow{I} + \overrightarrow{K}\right) \cdot \overrightarrow{U}$.

But

$$\begin{aligned} IO^2 - IO'^2 &= (IO^2 - R^2) - (IO'^2 - R'^2) + R^2 - R'^2 \\ &= IA^2 - IB^2 + R^2 - R'^2 = R^2 - R'^2 = PO^2 - PO'^2\,. \end{aligned}$$

Thus, I belongs to the plane orthogonal to OO' through P (which is the radical plane of S, S'). It follows that $\overrightarrow{U} \cdot \overrightarrow{I} = 0$. Similarly, $\overrightarrow{U} \cdot \overrightarrow{K} = 0$, and we get $PN - PM = 0$, as required.

8. (11th grade) Let $x_1, x_2, \ldots, x_n, \ldots$ be the sequence of real numbers such that

$$x_1 \;=\; 1\,, \quad x_{n+1} \;=\; \frac{n^2}{x_n} + \frac{x_n}{n^2} + 2\,, \quad n \;\geq\; 1\,.$$

Prove that

(a) $x_{n+1} \geq x_n$ for all $n \geq 4$;

(b) $[x_n] = n$ for all $n \geq 4$ ($[a]$ denotes the whole part of a).

Solved by Michel Bataille, Rouen, France; and George Evagelopoulos, Athens, Greece. We use Evagelopoulos' solution.

We can easily compute the first few terms:

$$x_1 \;=\; 1\,, \quad x_2 \;=\; 4\,, \quad x_3 \;=\; 4\,, \quad x_4 \;=\; \frac{169}{36}\,.$$

By induction on $n \geq 4$, we shall prove the inequalities

$$n + \frac{2}{n} \;<\; x_n \;<\; n + 1\,. \tag{1}$$

The initial case, $n = 4$, is easy to check. For the induction step, we must show that (1) implies

$$n + 1 + \frac{2}{n+1} \;<\; x_{n+1} \;<\; n + 2\,. \tag{2}$$

Let

$$f_n(x) \;=\; \frac{n^2}{x} + \frac{x}{n^2} + 2\,.$$

Then

$$x_{n+1} \;=\; f_n(x_n)\,, \quad n \;\geq\; 1\,. \tag{3}$$

The function $f_n(x)$ has derivative $f_n'(x) = (x^2 - n^4)/(n^2x^2)$. Hence, it is decreasing on $(0, n^2)$. Using (1) and (3), we have

$$f_n(n+1) \;<\; f_n(x_n) \;=\; x_{n+1} \;<\; f_n\left(n + \frac{2}{n}\right)\,. \tag{4}$$

Also

$$\begin{aligned} f_n(n+1) \;&=\; \frac{n^2}{n+1} + \frac{n+1}{n^2} + 2 \\ &=\; n + 1 + \frac{1}{n+1} + \frac{n+1}{n^2} \;>\; n + 1 + \frac{2}{n+1}\,, \end{aligned} \tag{5}$$

and

$$f_n\left(n + \frac{2}{n}\right) \;=\; f_n\left(\frac{n^2+2}{n}\right) \;=\; \frac{n^3}{n^2+2} + \frac{n^2+2}{n^3} + 2\,.$$

Note that

$$f_n\left(n + \frac{2}{n}\right) \;<\; n + 2 \tag{6}$$

for $n \geq 3$, as a consequence of

$$\frac{n^3}{n^2+2} + \frac{n^2+2}{n^3} + 2 - (n+2) \;=\; \frac{-\left[(n^2-2)^2 - 8\right]}{n^3(n^2+2)} \;<\; 0\,.$$

From (5), (4), and (6), we obtain

$$n + 1 + \frac{2}{n+1} \;<\; f_n(n+1) \;<\; x_{n+1} \;<\; f_n\left(n + \frac{2}{n}\right) \;<\; n + 2\,,$$

and thus (2) is proved.

Therefore, (1) is true for all $n \geq 4$. Hence, $[x_n] = n$ for all $n \geq 4$, and part (b) is proved.

Now we consider part (a). We have

$$x_{n+1} - x_n \;=\; \frac{n^2}{x_n} + \frac{x_n}{n^2} + 2 - x_n \;=\; \frac{(n^2 + x_n)^2 - (nx_n)^2}{n^2x_n}\,.$$

For $n \geq 4$, we have, from (1), $x_n < n + 1 < \dfrac{n^2}{n-1}$, which implies

$$nx_n \;<\; x_n + n^2\,.$$

Hence, $x_{n+1} > x_n$, for $n \geq 4$. [*Ed.* From above we also have $x_{n+1} \geq x_n$ for $n = 1$, 2, and 3, which means that $x_{n+1} \geq x_n$ for all $n \geq 1$.]

Now we turn to readers' solutions to some of the problems of the Vietnamese Mathematical Olympiad 1998, Category A, Day 1 and Day 2, given [2001 : 295–296].

1. Let $a \geq 1$ be a real number. Define a sequence $\{x_n\}$ ($n = 1, 2, \ldots$) of real numbers by

$$x_1 = a\,, \quad x_{n+1} = 1 + \ln\left(\frac{x_n^2}{1 + \ln x_n}\right)\,.$$

Prove that the sequence $\{x_n\}$ has a finite limit, and determine it.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

We prove that the limit of the sequence $\{x_n\}$ is 1.

Let $f(x) = 1 + \ln\left(\dfrac{x^2}{1 + \ln x}\right)$, defined on $[1, \infty)$. We easily compute $f'(x) = \dfrac{1 + 2\ln x}{x(1 + \ln x)} > 0$ for $x \geq 1$. Thus, f is increasing on $[1, \infty)$.

Let $g(x) = f(x) - x$ on $[1, \infty)$. Then, we have $g'(x) = \dfrac{h(x)}{x(1 + \ln x)}$, where $h(x) = 1 + 2\ln x - x - x\ln x$. We have

$$h'(x) = -1 - \ln x - 1 + \frac{2}{x} = \frac{2 - 2x - x\ln x}{x} \leq 0\,,$$

with equality if and only if $x = 1$. Therefore, h is decreasing on $[1, \infty)$. Since $h(1) = 0$, we deduce that $h(x) \leq 0$ for $x \geq 1$, with equality if and only if $x = 1$.

It follows that $g'(x) \leq 0$ for $x \geq 1$, with equality if and only if $x = 1$. Then g is decreasing on $[1, \infty)$. Since $g(1) = 0$, we must have $g(x) \leq 0$ for $x \geq 1$, with equality if and only if $x = 1$. That is, $f(x) \leq x$, with equality if and only if $x = 1$.

Since $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}^*$, an easy induction leads to the conclusion that the sequence $\{x_n\}$ is non-increasing and has a lower bound of 1. It follows that $\{x_n\}$ is convergent. Since f is continuous, the limit of $\{x_n\}$ is a fixed point of f. The unique fixed point is 1 (since $f(x) = x$ if and only if $x = 1$). Therefore, the limit is 1.

2. Let $ABCD$ be a tetrahedron and AA_1, BB_1, CC_1, DD_1 be diameters of the circumsphere of $ABCD$. Let A_0, B_0, C_0 and D_0 be the centroids of the triangles BCD, CDA, DAB and ABC, respectively. Prove that

(a) the lines A_0A_1, B_0B_1, C_0C_1 and D_0D_1 have a common point, which is denoted by F;

(b) the line passing through F and the mid-point of an edge is perpendicular to its opposite edge.

Solved by Michel Bataille, Rouen, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bradley's write-up.

Take the origin at the circumcentre O of $ABCD$ and set $\overrightarrow{a} = \overrightarrow{OA}$. Then $\overrightarrow{OA_1} = -\overrightarrow{a}$, since AA_1 is a diameter. Define $\overrightarrow{b}$, $\overrightarrow{c}$, and $\overrightarrow{d}$ similarly. If A_0 is the centroid of $\triangle BCD$, then $\overrightarrow{OA_0} = \frac{1}{3}(\overrightarrow{b} + \overrightarrow{c} + \overrightarrow{d})$.

The line A_0A_1 has vector equation

$$\overrightarrow{r} = \tfrac{1}{3}(\overrightarrow{b} + \overrightarrow{c} + \overrightarrow{d}) + t(-\overrightarrow{a} - \tfrac{1}{3}\overrightarrow{b} - \tfrac{1}{3}\overrightarrow{c} - \tfrac{1}{3}\overrightarrow{d}).$$

For $t = -\frac{1}{2}$, we have $\overrightarrow{r} = \frac{1}{2}(\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} + \overrightarrow{d})$. By symmetry, this point lies on B_0B_1, C_0C_1, and D_0D_1 also. Thus, we may use this point as the point F to establish part (a).

Let L be the mid-point of AB. Then, we have $\overrightarrow{OL} = \frac{1}{2}(\overrightarrow{a} + \overrightarrow{b})$ and $\overrightarrow{LF} = \frac{1}{2}(\overrightarrow{c} + \overrightarrow{d})$. Now

$$\overrightarrow{LF} \cdot \overrightarrow{CD} = \frac{1}{2}(\overrightarrow{c} + \overrightarrow{d}) \cdot (\overrightarrow{d} - \overrightarrow{c}) = \frac{1}{2}\|\overrightarrow{d}\|^2 - \frac{1}{2}\|\overrightarrow{c}\|^2 = 0,$$

because $\|\overrightarrow{a}\| = \|\overrightarrow{b}\| = \|\overrightarrow{c}\| = \|\overrightarrow{d}\| = R$, the radius of the circumcircle.

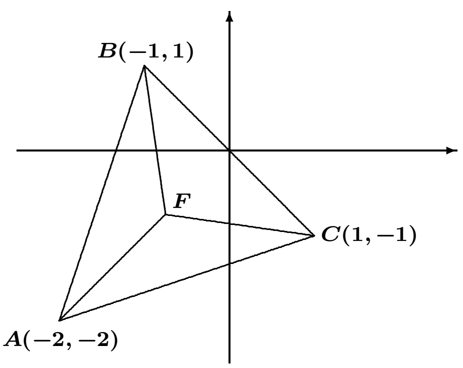
5. Determine the smallest possible value of the following expression:

$$F(x,y) = \sqrt{(x+1)^2 + (y-1)^2} + \sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x+2)^2 + (y+2)^2}$$

where x, y are real numbers.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Daniel Reisz, Vincelles, France. We give Bradley's write-up.

This is the Fermat-Steiner problem for the triangle with vertices at $B(-1,1)$, $C(1,-1)$, $A(-2,-2)$. The Fermat point is the point F such that $\angle BFA = \angle BFC = \angle CFA = 120°$.



We have $BC = \sqrt{8}$ and $AB = AC = \sqrt{10}$. Note that F lies on the line $y = x$, because $\angle BFA = \angle CFA$. Suppose F has coordinates $(-c, -c)$. Then

$$FB^2 \;=\; FC^2 \;=\; (c-1)^2 + (c+1)^2 \;=\; 2c^2 + 2\,.$$

By the Law of Cosines, we have

$$BC^2 \;=\; FB^2 + FC^2 + (FB)(FC) \;=\; 3FC^2 \;=\; 6c^2 + 6\,.$$

Therefore, $6c^2 + 6 = 8$, and hence, $c = 1/\sqrt{3}$. Then

$$FB \;=\; FC \;=\; \sqrt{2c^2+2} \;=\; 2\frac{\sqrt{2}}{\sqrt{3}}\,,$$
$$FA \;=\; \sqrt{(-2+c)^2 + (-2+c)^2} \;=\; \sqrt{2}\left(2 - \frac{1}{\sqrt{3}}\right)\,,$$

and finally,

$$\min(FA + FB + FC) \;=\; 2\sqrt{2} - \sqrt{\frac{2}{3}} + \frac{4\sqrt{2}}{\sqrt{3}} \;=\; 2\sqrt{2} + \sqrt{6}\,.$$

To complete this number of the *Corner*, we turn to readers' solutions to problems of the Vietnamese Mathematical Olympiad 1998, Category B, Day 1 and Day 2, given [2001 : 296–297].

1. Let a be a real number. Define a sequence $\{x_n\}$ $(n = 1, 2, \ldots)$ of real numbers by

$$x_1 \;=\; a\,, \quad x_{n+1} \;=\; \frac{x_n(x_n^2+3)}{3x_n^2+1}$$

for $n \geq 1$. Prove that the sequence has a finite limit, and determine it.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

We first observe that if $a = 0$, then the sequence is constant, with $x_n = 0$ for all $n \geq 1$. Thus, it has limit 0.

Let us now assume that $a \neq 0$. Define $f(x) = \dfrac{x^3+3x}{3x^2+1}$ for $x \in \mathbb{R}$. Note that $x_{n+1} = f(x_n)$ for all $n \geq 1$. Since, for all real numbers x, we have

$$x^3 + 3x - 3x^2 - 1 \;=\; (x-1)^3\,,$$

we deduce that

$$x \geq 1 \implies f(x) \geq 1\,, \tag{1}$$
$$\text{and}\quad x \leq 1 \implies f(x) \leq 1\,. \tag{2}$$

In the same way,

$$\begin{aligned} x \ \geq \ -1 \quad &\Longrightarrow \quad f(x) \ \geq \ -1\,, & (3)\\ \text{and} \quad x \ \leq \ -1 \quad &\Longrightarrow \quad f(x) \ \leq \ -1\,. & (4) \end{aligned}$$

Moreover, $f(x) - x = \dfrac{2x(1-x)(1+x)}{3x^2+1}$. From this we can conclude that

$$f(x) \ = \ x \quad \Longleftrightarrow \quad x \in \{-1, 0, 1\}\,. \qquad (5)$$

Also,

$$\begin{aligned} x \ \leq \ -1 \quad &\Longrightarrow \quad f(x) \ \geq \ x\,, & (6)\\ -1 \ \leq \ x \ \leq \ 0 \quad &\Longrightarrow \quad f(x) \ \leq \ x\,, & (7)\\ 0 \ \leq \ x \ \leq \ 1 \quad &\Longrightarrow \quad f(x) \ \geq \ x\,, & (8)\\ \text{and} \quad 1 \ \leq \ x \quad &\Longrightarrow \quad f(x) \ \leq \ x\,. & (9) \end{aligned}$$

Now we distinguish two cases.

Case 1. $a > 0$.
An easy induction shows that $x_n \geq 0$ for all $n \geq 1$.

If $0 < a \leq 1$, then another induction, using (2) and (8), shows that $x_n \leq x_{n+1} \leq 1$ for all $n \geq 1$. Thus, the sequence $\{x_n\}$ is non-decreasing and has a finite upper bound. Hence, the sequence is convergent, and its limit is a fixed point $\alpha \in (0, 1]$ of the function f. Then, from (5), the limit of $\{x_n\}$ is 1.

If $1 \leq a$, then, as above, using (1) and (9), we show that the sequence $\{x_n\}$ is non-increasing and takes values in $[1, +\infty)$. Hence, the sequence is convergent, and its limit is a fixed point $\beta \in [1, +\infty)$ of the function f. Then, from (5), the limit of $\{x_n\}$ is 1.

Case 2. $a < 0$.
If $-1 \leq a < 0$, then, from (3) and (7), the sequence is convergent, and (from (5)) its limit is -1.

If $a \leq -1$, then, using (4) and (6), the sequence is convergent, and (from (5)) its limit is -1.

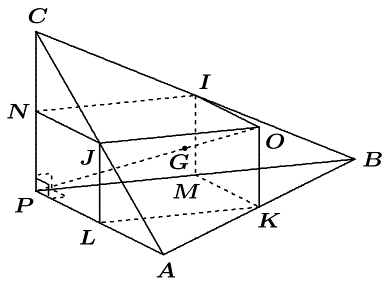
Conclusion. The sequence $\{x_n\}$ is convergent to the limit ℓ, where

$$\ell = \begin{cases} -1 & \text{if } a < 0,\\ 0 & \text{if } a = 0,\\ 1 & \text{if } a > 0. \end{cases}$$

2. Let P be a point lying on a given sphere. Three mutually perpendicular rays from P intersect the sphere at points A, B, and C. Prove that for all such triads of rays from P, the plane of the triangle ABC passes through a fixed point and determine the largest possible value of the area of the triangle ABC.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the solution by Bataille.

Let O be the centre of the sphere, and let I, J, K, L, M, N be the respective mid-points of BC, CA, AB, PA, PB, PC. As the centre of the circumcircle of $\triangle BPC$, the point I is the orthogonal projection of O onto the plane (BPC). Thus, OI is parallel to PA. Similarly, $OJ \parallel PB$ and $OK \parallel PC$. It follows that the points O, I, N, J, L, P, M, K are the vertices of a right-angled parallelepiped.



Let G be the centroid of $\triangle ABC$. We have

$$\overrightarrow{PO} \;=\; \overrightarrow{PL}+\overrightarrow{LK}+\overrightarrow{KO} \;=\; \frac{1}{2}\overrightarrow{PA}+\frac{1}{2}\overrightarrow{PB}+\frac{1}{2}\overrightarrow{PC} \;=\; \frac{1}{2}(3\overrightarrow{PG})\,,$$

and hence, $\overrightarrow{PG}=\frac{2}{3}\overrightarrow{PO}$. Then G is fixed, since P and O are fixed.

Since $PABC$ is a right-angled tetrahedron, the area $[ABC]$ of the triangle ABC satisfies the well-known Pythagoras-like relation

$$[ABC]^2 \;=\; [APC]^2+[BPC]^2+[BPA]^2\,.$$

Thus, $[ABC]^2=\frac{1}{4}(PA^2PC^2+PC^2PB^2+PB^2PA^2)$. Then

$$[ABC]^2 \;\le\; \tfrac{1}{4}(PA^4+PB^4+PC^4)$$

(since $xy+yz+zx\le x^2+y^2+z^2$ for positive numbers x, y, z).

On the other hand,

$$\begin{aligned}PA^4+PB^4+PC^4 &= (PA^2+PB^2+PC^2)^2\\ &\qquad -2(PA^2PC^2+PC^2PB^2+PB^2PA^2)\,,\end{aligned}$$

and

$$\begin{aligned}PA^2+PB^2+PC^2 &= (2PL)^2+(2PM)^2+(2PN)^2\\ &= 4(PL^2+LK^2+KO^2) \;=\; 4PO^2 \;=\; 4r^2\,,\end{aligned}$$

where r denotes the radius of the given sphere.

Thus, $[ABC]^2 \leq \frac{1}{4}((4r^2)^2 - 2(4[ABC]^2))$; whence, $[ABC] \leq \dfrac{2r^2}{\sqrt{3}}$. Furthermore, the value $\dfrac{2r^2}{\sqrt{3}}$ is attained when $PA = PB = PC = \dfrac{2r}{\sqrt{3}}$; that is, when P, A, B, C are four vertices of a cube inscribed in the sphere.

In conclusion, the maximal value of $[ABC]$ is $\dfrac{2r^2}{\sqrt{3}}$.

3. Let a, b be integers. Define a sequence $\{a_n\}$ $(n = 0, 1, 2, \ldots)$ of integers by

$$a_0 = a, a_1 = b, a_2 = 2b - a + 2, a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$$

for $n \geq 0$.

(a) Find the general term of the sequence.

(b) Determine all integers a, b, for which a_n is a perfect square for all $n \geq 1998$.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

(a) The characteristic equation is $X^3 - 3X^2 + 3X - 1 = 0$, which is equivalent to $(X-1)^3 = 0$. Thus, there exist three real numbers α, β, γ such that, for $n \geq 0$, we have $a_n = \alpha n^2 + \beta n + \gamma$.

Using $a_0 = a$, $a_1 = b$, $a_2 = 2b - a + 2$, we easily deduce that $\alpha = 1$, $\beta = b - a - 1$, $\gamma = a$. Thus, $a_n = n^2 + (b-a-1)n + a$, for all $n \geq 0$.

(b) **Lemma**. Let $P \in \mathbb{Z}[x]$, with $\deg P = 2$ and the leading coefficient of P equal to 1. Let $n \in \mathbb{Z}$. If, for all integers $k \geq n$ the number $P(k)$ is the square of an integer, then there exists $Q \in \mathbb{Z}[x]$ such that $P = Q^2$.

Proof. Let $P(x) = x^2 + \alpha x + \beta$, where $\alpha, \beta \in \mathbb{Z}$.

Case 1. α is odd.

$$P(x) = \left(x + \frac{\alpha-1}{2}\right)^2 + x + \beta - \left(\frac{\alpha-1}{2}\right)^2,$$

and $\left(x + \frac{\alpha-1}{2} + 1\right)^2 = \left(x + \frac{\alpha-1}{2}\right)^2 + 2x + \alpha$. For sufficiently large k,

$$\left(k + \frac{\alpha-1}{2}\right)^2 < P(k) < \left(k + \frac{\alpha-1}{2} + 1\right)^2.$$

Then $P(k)$ lies strictly between the squares of two consecutive integers, and therefore cannot be a perfect square itself for k sufficiently large.

Case 2. α is even.

$$P(x) = \left(x + \frac{\alpha}{2}\right)^2 + \beta - \left(\frac{\alpha}{2}\right)^2.$$

For sufficiently large k, we have

$$\left(k+\frac{\alpha}{2}-1\right)^2 \;<\; P(k) \;<\; \left(k+\frac{\alpha}{2}+1\right)^2 .$$

Then, in order for $P(k)$ to be the square of an integer, we must have $P(k) = \left(k+\frac{\alpha}{2}\right)^2$. It follows that $\beta - \left(\frac{\alpha}{2}\right)^2 = 0$. Then, for all $x \in \mathbb{R}$, we have $P(x) = \left(x+\frac{\alpha}{2}\right)^2$, and the lemma is proved.

It is clear that, conversely, if $P = Q^2$ where $Q \in \mathbb{Z}[x]$, then $P(k)$ is a square for all integers k.

Since $a_n = P(n)$, where P is a quadratic polynomial with leading coefficient equal to 1, it follows from the lemma that a_n is a perfect square for all $n \geq 1998$ if and only if $(b-a-1)^2 - 4a = 0$. This is true if and only if $(a,b) = \left(n^2, (n \pm 1)^2\right)$, for some integer n.

Remark. The lemma can be generalized to any polynomial with integer coefficients (not necessary quadratic, or with leading coefficient equal to 1). (See Pólya, Szegő, Billingheimer, *Problems and Theorems in Analysis*, Vol. II, Springer.)

4. Let x_1, x_2, $\ldots$, x_n $(n \geq 2)$ be real positive numbers satisfying

$$\frac{1}{x_1+1998} + \frac{1}{x_2+1998} + \cdots + \frac{1}{x_n+1998} = \frac{1}{1998} .$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \;\geq\; 1998 .$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Heinz-Jürgen Seiffert, Berlin, Germany. We give Bataille's response.

Let $y_j = x_j/1998$ $(j = 1, 2, \ldots, n)$. Then the hypothesis reads

$$\frac{1}{1+y_1} + \frac{1}{1+y_2} + \cdots + \frac{1}{1+y_n} \;=\; 1 .$$

Now, this implies $y_1 y_2 \cdots y_n \geq (n-1)^n$ (a known result: see, for example, [2000 : 167–168] where it is Jimmy Chui's Problem of the Month). Expressing this in terms of the x_j's, the required result follows immediately.

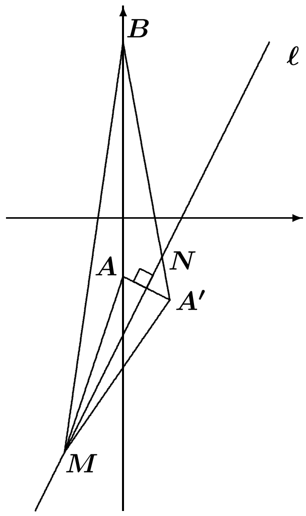
5. Determine the smallest possible value of the following expression

$$\sqrt{x^2+(y+1)^2} \;+\; \sqrt{x^2+(y-3)^2}$$

where x, y are real numbers such that $2x - y = 2$.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Laura Gu, student, Sir Winston Churchill High School, Calgary, AB; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Maragoudakis.

Let ℓ be the line with equation $2x - y = 2$. Let $A = (0, -1)$ and $B = (0, 3)$. Let A' be the reflection of A across the line ℓ; then $A' = \left(\frac{4}{5}, -\frac{7}{5}\right)$. Let N be the point of intersection of $A'B$ and ℓ; then $N = \left(\frac{2}{3}, -\frac{2}{3}\right)$.



Let $M = (x, y)$ be an arbitrary point on the line ℓ. By symmetry, $MA = MA'$. Thus,

$$\sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-3)^2} = MA + MB = MA' + MB\,.$$

By the Triangle Inequality, we have

$$MA' + MB \ \geq \ A'B \ = \ 2\sqrt{5}\,,$$

with equality if and only if $M = N$.

Finally,

$$\sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-3)^2} \ \geq \ 2\sqrt{5}\,.$$

Equality holds only if $(x, y) = \left(\frac{2}{3}, -\frac{2}{3}\right)$.

6. Prove that for each positive odd integer n there is exactly one polynomial $P(x)$ of degree n with real coefficients satisfying

$$P\left(x - \frac{1}{x}\right) \ = \ x^n - \frac{1}{x^n}$$

for all real $x \neq 0$.

Determine if the above assertion holds for positive even integers n.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the solution of Bradley.

The proof is by induction. The case $n = 1$ is trivial: simply choose $P(x) = P_1(x) = x$, and this polynomial is unique. Now suppose that, for $k = 1, 3, \ldots, 2n-1$, we have a unique polynomial $P_k(x)$ of degree k with real coefficients such that

$$P_k\left(x - \frac{1}{x}\right) \ = \ x^k - \frac{1}{x^k}\,.$$

Using the Binomial Theorem, we have

$$\left(x-\frac{1}{x}\right)^{2n+1} = x^{2n+1}-\frac{1}{x^{2n+1}}-\binom{2n+1}{1}\left(x^{2n-1}-\frac{1}{x^{2n-1}}\right) + \cdots + (-1)^n\binom{2n+1}{n}\left(x-\frac{1}{x}\right) .$$

It follows that

$$\begin{aligned} x^{2n+1}-\frac{1}{x^{2n+1}} &= \left(x-\frac{1}{x}\right)^{2n+1} + \binom{2n+1}{1}P_{2n-1}\left(x-\frac{1}{x}\right) \\ &\qquad - \cdots - (-1)^n\binom{2n+1}{n}P_1\left(x-\frac{1}{x}\right) \\ &= P_{2n+1}\left(x-\frac{1}{x}\right) . \end{aligned}$$

We have constructed the unique polynomial explicitly by means of a recursive definition.

The assertion does not hold for even n. For $n = 2$ we would require

$$Q_2\left(x-\frac{1}{x}\right) = x^2-\frac{1}{x^2} ,$$

and Q_2 would have to be of degree 2. Thus, we would require

$$a\left(x-\frac{1}{x}\right)^2 + b\left(x-\frac{1}{x}\right) + c = x^2-\frac{1}{x^2}$$

for all $x \neq 0$. Setting $x = 1$ requires $c = 0$. Then, for $x \neq \pm 1$, we have

$$a\left(x-\frac{1}{x}\right) + b = x+\frac{1}{x} .$$

That is, $ax^2 - a + bx = x^2 + 1$. This would require $b = 0$, $a = 1$, and $a = -1$, which is an obvious contradiction.

That completes this number of the *Corner*. Send me your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

USA & International Mathematical Olympiads 2001
Edited by Titu Andreescu and Zuming Feng, published by The Mathematical Association of America, 2002
ISBN# 0-88385-089-6, softcover, 122 pages, US$16.50
Reviewed by ***Richard Hoshino****, Dalhousie University, Halifax, Nova Scotia.*

This book provides a detailed analysis of 21 contest problems at the Olympiad level: the six problems on the USA Mathematical Olympiad (USAMO), the nine problems on the USA IMO Team Selection Test, and the six problems from the 2001 IMO competition.

Each problem is carefully analyzed and studied, and multiple solutions are presented for each problem. The treatment given to the six IMO problems is particularly impressive. With the exception of one problem, there are at least six solutions presented for each question, with the majority of the solutions provided by the students who took part in the 2001 IMO contest. The work of four students (Reid Barton and Gabriel Carroll from the USA, Zhiqiang Zhang and Liang Xiao of China) is prominent throughout the text. Their insights are brilliant as they managed to solve these IMO problems in ways that were unknown to the mathematicians who created these problems for the competition. These four students were the only ones who scored a perfect 42 out of 42 on the IMO competition. Much will be learned by students who carefully look at these ingenious solutions and attempt to understand the mathematics involved in each proof.

In an interesting connection to ***Crux Mathematicorum***, the fourth problem of the 2001 IMO contest was created by Bill Sands. Most readers familiar with this magazine will recall that Bill was the Editor-in-Chief of ***Crux Mathematicorum*** for many years.

This is a well-written book with multiple solutions to a wide variety of difficult problems. This book will especially benefit young readers who are interested in developing their repertoire for Olympiad-level contests with the hope and ambition of representing Canada at the IMO.

MISTEAKS . . . and how to find them before the teacher does . . .
by Barry Cipra, published by A.K. Peters, Natick, MA, USA, Third Edition, 2000
ISBN 1-56881-122-5, alk. paper, 70 + xv pages, US$5.95.
Reviewed by **Bruce Shawyer**, *Memorial University of Newfoundland, St. John's, NL.*

In the Introduction, the author states "Everybody makes mistakes. Young or old, smart or dumb, student or teacher, we all make 'em". Unfortunately, the main text of the book begins with a mistake, and I am far from convinced that the author meant it as a deliberate mistake. He states:

A definite integral measures the area beneath a curve,

and makes use of this definitive statement in his attempt to help students find mistakes that they may have made in calculating areas.

Now, as every mathematician knows, that statement is not correct. However, in the very next line, the author (sort of) corrects this by stating:

In particular, a *positive* function must have a *positive* integral.

If the book had not begun in this unfortunate way, I would have been able to give it a strong recommendation to students (and to teachers), but I am very fearful because that error in the very first paragraph, unlike the errors illustrated throughout the book, is never corrected.

The book is mostly about the errors that teachers of elementary calculus see over and over again. In this respect, there is a lot of very important ground covered. Not all of the material is about the calculus. A lot of the thinking is about sheer common sense. Impressing that on students is so, so important. I am reminded of the claim that an answer must be correct "because the computer/calculator said so" (see chapter 9). To this, the answer is, of course, GIGO (garbage in — garbage out).

In terms of presentation, the book suffers from the lack of a good TEXnical editor. Back to the start of the main text again. Line four appears as:

$$\int_{-2}^{1} (x^2+1)\,dx \;=\; (\tfrac{1}{3}x^3+x)|_{-2}^{-1}$$

which surely would look much better typeset in the way we write it by hand:

$$\int_{-2}^{1} \left(x^2+1\right)\,dx \;=\; \left(\frac{1}{3}x^3+x\right)\Big|_{-2}^{-1}$$

This is endemic throughout the book.

In conclusion, this book should be used with great caution. (I hope that there are no misteaks in this review!)

On log-trigonometric functions

C.-S. Lin

In our previous article [5], some elementary finite sine and cosine series were applied to produce sums for some finite series of integers, by way of differentiation. Naturally, one might wonder what to expect from using integration instead. In this article, we express log-trigonometric functions in terms of infinite cosine series. These are useful in many applications, such as computing log-trigonometric integrals.

Proposition 1 For $x \neq 0$, $\pm\pi$, $\pm 2\pi$, $\ldots$,

$$\ln|\sin x| \;=\; -\ln 2 - \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k}\,. \tag{1}$$

For $x \neq \pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\ldots$,

$$\ln|\cos x| \;=\; -\ln 2 - \sum_{k=1}^{\infty} (-1)^k \frac{\cos(2kx)}{k}\,. \tag{2}$$

Proof. Observe that (2) follows from (1) by replacing x in (1) by $x + \frac{\pi}{2}$. We will prove (1). For this it is enough to consider x such that $0 < x < \frac{\pi}{2}$, because of the symmetry and periodicity of the sine and cosine functions.

Let us recall a familiar finite sine series from the Lemma in [5]. For any integer $n \geq 1$, and for $x \neq 0$, $\pm 2\pi$, $\pm 4\pi$, $\ldots$,

$$\sum_{k=1}^{n} \sin(kx) \;=\; \frac{\cos\left(\frac{1}{2}x\right) - \cos\left((n+\frac{1}{2})x\right)}{2\sin\left(\frac{1}{2}x\right)}\,. \tag{3}$$

Rewrite (3) as follows:

$$\cos\left((n+\tfrac{1}{2})x\right)\csc\left(\tfrac{1}{2}x\right) \;=\; \cot\left(\tfrac{1}{2}x\right) - 2\sum_{k=1}^{n} \sin(kx)\,.$$

Letting $x = 2t$, where $t \neq 0$, $\pm\pi$, $\pm 2\pi$, $\ldots$, we have

$$\cos\left((2n+1)t\right)\csc t \;=\; \cot t - 2\sum_{k=1}^{n} \sin(2kt)\,. \tag{4}$$

For $0 < x < \frac{\pi}{2}$, we integrate the left side of (4) by parts from x to $\frac{\pi}{2}$, with $u = \csc t$ and $dv = \cos\left((2n+1)t\right)\,dt$:

$$\begin{aligned}
&\int_x^{\frac{\pi}{2}} \cos\big((2n+1)t\big)\csc t\,dt \\
&= \frac{1}{2n+1}\left(\Big[\sin\big((2n+1)t\big)\csc t\Big]_x^{\frac{\pi}{2}} - \int_x^{\frac{\pi}{2}} \sin\big((2n+1)t\big)\,\tfrac{d}{dt}\csc t\,dt\right) \\
&= \frac{1}{2n+1}\left((-1)^n - \sin\big((2n+1)x\big)\csc x - \int_x^{\frac{\pi}{2}} \sin\big((2n+1)t\big)\,\tfrac{d}{dt}\csc t\,dt\right).
\end{aligned}$$

Since $\big|\sin\big((2n+1)x\big)\big| \le 1$ for all x, we have

$$\begin{aligned}
\left|\int_x^{\frac{\pi}{2}} \cos\big((2n+1)t\big)\csc t\,dt\right| &\le \frac{1}{2n+1}\left(1 + \csc x + \int_x^{\frac{\pi}{2}} \big|\tfrac{d}{dt}\csc t\big|\,dt\right) \\
&= \frac{1}{2n+1}\left(1 + \csc x - \int_x^{\frac{\pi}{2}} \tfrac{d}{dt}\csc t\,dt\right) \\
&= \frac{1}{2n+1}\left(1 + \csc x - \big[\csc t\big]_x^{\frac{\pi}{2}}\right) = \frac{2\csc x}{2n+1},
\end{aligned}$$

which approaches 0 as $n \to \infty$.

Now define $h(n) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$. Thus, $h(n)$ is the sum of the first n terms of the alternating harmonic series, which converges by the Alternating Series Test. Integrating the right side of (4) from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ yields

$$\left[\ln(\sin t) + \sum_{k=1}^{n} \frac{\cos(2kt)}{k}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \begin{cases} \frac{1}{2}\ln 2 - h(n) + \frac{1}{2}h\big(\frac{n-1}{2}\big) & \text{if } n \text{ is odd,} \\ \frac{1}{2}\ln 2 - h(n) + \frac{1}{2}h\big(\frac{n}{2}\big) & \text{if } n \text{ is even.} \end{cases}$$

This must approach 0 as $n \to \infty$. Therefore,

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}. \tag{5}$$

For $0 < x < \frac{\pi}{2}$, the integral of the right side of (4) from x to $\frac{\pi}{2}$ is

$$\left[\ln(\sin t) + \sum_{k=1}^{n} \frac{\cos(2kt)}{k}\right]_x^{\frac{\pi}{2}} = -\ln(\sin x) - h(n) - \sum_{k=1}^{n} \frac{\cos(2kx)}{k},$$

which must approach 0 as $n \to \infty$. Thus, using (5), we obtain (1).

Remarks.

1. Formula (3) may be obtained by summing the geometric series $\sum_{k=1}^{n} e^{ikx}$ to get $\sum_{k=1}^{n} e^{ikx} = \dfrac{e^{i(n+1)x} - e^{ix}}{e^{ix} - 1}$, and then taking imaginary parts.

2. Integrating term-by-term in the series in (1) and (2), we can evaluate the following improper integrals:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx &= \int_0^{\frac{\pi}{2}} \ln(\cos x)\,dx \;=\; -\frac{\pi}{2}\ln 2\,, \\ \int_0^{\pi} \ln(\sin x)\,dx &= -\pi\ln 2\,. \end{aligned}$$

While these results are correct, the term-by-term integration of the series in (1) and (2) is not easily justified. (Convergence of the series is not uniform on the relevant intervals.) The seond formula above is used to prove Jensen's formula in elementary Complex Analysis. The evaluation of this improper integral relies heavily on contour integration in [1, p. 159], and is done using the symmetric method in [6, p. 275]. The integral also appears in [2].

3. Trigonometric series for $\ln(\tan x)$, $\ln(\cot x)$, $\ln(\sec x)$, and $\ln(\csc x)$ can be derived easily from (1) and (2). We leave the details to the reader.

4. It is the author's belief that our proof of (5) is new. The normal way of getting $\ln 2$ is by Mercator's log series [3, p. 367],

$$\ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } -1 < x \le 1\,.$$

5. The Fourier series (1) is proved differently in [4, Theorem 216].

References

[1] L.V. Ahlfors, *Complex Analysis*, 2nd ed., McGraw-Hill, New York, 1966.

[2] A.K. Arora, S.K. Goel and D.M. Rodriguez, *Special integration techniques for trigonometric integrals*, Amer. Math. Monthly, **95**(1988), 126–130.

[3] J.H. Eves, *An Introduction To The History Of Mathematics*, Saunders Colleg. Pub., 1990.

[4] E. Landau, *Elementary Number Theory*, Chelsea Publishing Co., New York, 1966.

[5] C.-S. Lin, *Summation of finite series of integers*, Crux Math. **27**(2001), 510–513.

[6] M.R. Spiegel, *Advanced Calculus*, Schaum's Outline Series, McGraw-Hill Book Co., 1963.

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PROBLEMS

Problem proposals and solutions should be sent to Jim Totten, Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, BC, Canada, V2C 5N3. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 May 2004**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX.) *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

2843. Correction. *Proposed by Bektemirov Baurjan, student, Aktobe, Kazakstan.*

Suppose that $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 4+\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}$ for positive real x, y, z. Prove that

$$(1-x)(1-y)(1-z) \;\leq\; \frac{1}{64}\,.$$

On suppose que $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 4+\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}$ pour des réels positifs x, y, z. Montrer que

$$(1-x)(1-y)(1-z) \;\leq\; \frac{1}{64}\,.$$

2876. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given $\triangle ABC$ with incentre I and circumcentre O, suppose that M is the mid-point of BC, and that $IO \perp AM$.

Prove that $\frac{2}{BC} \;=\; \frac{1}{AB}+\frac{1}{AC}$.

. . . .

Dans un triangle ABC avec I comme centre du cercle inscrit et O comme centre du cercle circonscrit, on suppose que M est le milieu du côté BC et que IO est perpendiculaire à AM.

Montrer que $\frac{2}{BC} \;=\; \frac{1}{AB}+\frac{1}{AC}$.

2877. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let O be an interior point of $\triangle ABC$ such that $AB+BO = AC+CO$. Suppose that P is a variable point on the side BC, and that Q and R are points on AB and AC, respectively, such that $PQ \parallel CO$ and $PR \parallel BO$.

Prove that the perimeter of quadrilateral $AQPR$ is constant.

. . . .

Soit O un point intérieur du $\triangle ABC$ tel que $AB+BO = AC+CO$. On suppose que P est un point variable sur le côté BC et que Q et R sont des points sur AB et AC, respectivement, de sorte que PQ et PR soient respectivement parallèles à CO et BO.

Montrer que le périmètre du quadrilatère $AQPR$ est constant.

2878. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given an acute triangle ABC with orthocentre H and circumcentre O, suppose that the perpendicular bisector of AH meets AB and AC at D and E, respectively.

Prove that A is an excentre of $\triangle ODE$.

. . . .

Dans un triangle acutangle ABC, soit H l'orthocentre et O le centre du cercle circonscrit. On suppose que la médiatrice de AH coupe respectivement AB et AC en D et E.

Montrer que A est le centre d'un cercle exinscrit du triangle ODE.

2879. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Let D, E, F be arbitrary points on the sides BC, CA, AB, respectively, of triangle ABC. By the theorem of Miquel, the circles AEF, BFD, CDE are concurrent at O, say.

Let P be an arbitrary point in the plane of $\triangle ABC$, and let A', B', C' be the second points of intersection of PA, PB, PC with the circles AEF, BFD, CDE, respectively.

Prove that O, P, A', B', and C' are concyclic.

. . . .

On donne respectivement les points D, E et F sur les côtés BC, CA et AB d'un triangle ABC. D'après le théorème de Miquel, les cercles AEF, BFD et CDE se coupent en un point, disons O.

Soit P un point arbitraire dans le plan du triangle ABC, et soit respectivement A', B' et C' les seconds points d'intersection de PA, PB et PC avec les cercles AEF, BFD et CDE.

Montrer que O, P, A', B' et C' sont sur un même cercle.

2880. *Proposed by Mihály Bencze, Brasov, Romania.*

1. If x, y, $z > 1$, prove that

   (a) $\left(\log_{yz} x^4yz\right)\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right) \;>\; 25$,

   (b)★ $\left(\log_{yz} x^4yz\right)\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right) \;\geq\; 27$.

2.★ If $x_k > 1$ ($k = 1, 2, \ldots, n$) and $\alpha \geq -1$, prove that

$$\prod_{k=1}^{n} \log_{b_k} b_k x_k^{\alpha+1} \;\geq\; \left(\frac{n+\alpha}{n-1}\right)^n ,$$

where $b_k = x_1 \cdots x_{k-1} x_{k+1} \cdots x_n$.

. . . .

1. Si x, y, $z > 1$, montrer que

   (a) $\left(\log_{yz} x^4yz\right)\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right) \;>\; 25$,

   (b)★ $\left(\log_{yz} x^4yz\right)\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right) \;\geq\; 27$.

2.★ Si $x_k > 1$ ($k = 1, 2, \ldots, n$) et $\alpha \geq -1$, montrer que

$$\prod_{k=1}^{n} \log_{b_k} b_k x_k^{\alpha+1} \;\geq\; \left(\frac{n+\alpha}{n-1}\right)^n ,$$

øù $b_k = x_1 \cdots x_{k-1} x_{k+1} \cdots x_n$.

2881. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

A set of four non-negative integers a, b, c, d are said to have the property $\mathcal{P}$ if all of $bc+cd+db$, $ac+cd+da$, $ab+bd+da$, $ab+bc+ca$ are perfect squares.

The sequence $\{u_n\}$ is defined by $u_1=0$, $u_2=1$, $u_3=1$, $u_4=4$ and, for $n\geq 1$,

$$u_{n+4} \;=\; 2u_{n+3}+2u_{n+2}+2u_{n+1}-u_n\,.$$

Prove that the set $\{u_n,\,u_{n+1},\,u_{n+2},\,u_{n+3}\}$ has the property $\mathcal{P}$ for all $n\geq 1$.

. . . .

On dit qu'un ensemble de quatres nombres entiers non négatifs a, b, c et d possède la propriété $\mathcal{P}$ si les nombres $bc+cd+db$, $ac+cd+da$, $ab+bd+da$ et $ab+bc+ca$ sont tous des carrés parfaits.

On définit la suite $\{u_n\}$ par $u_1=0$, $u_2=1$, $u_3=1$, $u_4=4$ et, pour $n\geq 1$, par

$$u_{n+4} \;=\; 2u_{n+3}+2u_{n+2}+2u_{n+1}-u_n\,.$$

Montrer que l'ensemble $\{u_n,\,u_{n+1},\,u_{n+2},\,u_{n+3}\}$ possède la propriété $\mathcal{P}$ pour tout $n\geq 1$.

2882. *Proposed by Mihály Bencze, Brasov, Romania.*

If $x\in\left(0,\frac{\pi}{2}\right)$, $0\leq a\leq b$, and $0\leq c\leq 1$, prove that

$$\left(\frac{c+\cos x}{c+1}\right)^b \;<\; \left(\frac{\sin x}{x}\right)^a\,.$$

. . . .

Si $x\in\left(0,\frac{\pi}{2}\right)$, $0\leq a\leq b$ et $0\leq c\leq 1$, montrer que

$$\left(\frac{c+\cos x}{c+1}\right)^b \;<\; \left(\frac{\sin x}{x}\right)^a\,.$$

2883. *Proposed by Šefket Arslanagić and Faruk Zejnulaki, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x, y, $z\in[0,1)$ and that $x+y+z=1$. Prove that

$$\sqrt{\frac{xy}{z+xy}}+\sqrt{\frac{yz}{x+yz}}+\sqrt{\frac{zx}{y+zx}} \;\leq\; \frac{3}{2}\,.$$

. . . .

Supposons que x, y, $z\in[0,1)$ et que $x+y+z=1$. Montrer que

$$\sqrt{\frac{xy}{z+xy}}+\sqrt{\frac{yz}{x+yz}}+\sqrt{\frac{zx}{y+zx}} \;\leq\; \frac{3}{2}\,.$$

2884. *Proposed by Niels Bejlegaard, Copenhagen, Denmark.*

Suppose that a, b, c are the sides of a non-obtuse triangle. Give a geometric proof and hence, a geometric interpretation of the inequality

$$a+b+c \;\geq\; \sum_{\text{cyclic}} \sqrt{a^2+b^2-c^2}\,.$$

. . . .

On suppose que a, b, c sont les côtés d'un triangle non-obtus. Donner une démonstration géométrique et donc une interprétation géométrique de l'inégalité

$$a+b+c \;\geq\; \sum_{\text{cyclique}} \sqrt{a^2+b^2-c^2}.$$

2885. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let O and I be the circumcentre and the incentre, respectively, of triangle ABC. Denote the cevians through O by AA', BB', and CC', and those through I by AD, BE, and CF. The sides of the triangle are a, b, and c.

1. If $\dfrac{AA'}{a}=\dfrac{BB'}{b}=\dfrac{CC'}{c}$, prove that $\triangle ABC$ is equilateral.
2. If $\dfrac{AD}{a}=\dfrac{BE}{b}=\dfrac{CF}{c}$, prove that $\triangle ABC$ is equilateral.
3. Give an answer to Sastry's question [1998 : 280]: For an internal point P and its corresponding cevians AD, BE, CF, with $\dfrac{AD}{a}=\dfrac{BE}{b}=\dfrac{CF}{c}$, prove or disprove that $\triangle ABC$ is equilateral.

. . . .

Soit a, b et c les côtés d' un triangle ABC, I le centre du cercle inscrit et O celui du cercle circonscrit. Désignons par AA', BB' et CC' les céviennes passant par O, et par AD, BE et CF celles passant par I.

1. Si $\dfrac{AA'}{a}=\dfrac{BB'}{b}=\dfrac{CC'}{c}$, montrer que le triangle ABC est équilatéral.
2. Si $\dfrac{AD}{a}=\dfrac{BE}{b}=\dfrac{CF}{c}$, montrer que le triangle ABC est équilatéral.
3. Donner une réponse à la question de Sastry [1998 : 280] : Pour un point intérieur P et ses céviennes correspondantes AD, BE et CF, telles que $\dfrac{AD}{a}=\dfrac{BE}{b}=\dfrac{CF}{c}$, démontrer ou réfuter l'assertion : le triangle ABC est équilatéral.

2886. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*
If a, b, c are positive real numbers such that $abc = 1$, prove that

$$ab^2 + bc^2 + ca^2 \ \geq \ ab + bc + ca \,.$$

. . . .

Si a, b, c sont des nombres réels positifs tels que $abc = 1$, montrer que

$$ab^2 + bc^2 + ca^2 \ \geq \ ab + bc + ca \,.$$

2887. *Proposed by Vedula N. Murty, Dover, PA, USA.*
If a, b, c are the sides of $\triangle ABC$ and R is its circumradius, prove that

$$a^2 + b^2 + c^2 \ \leq \ 6R^2 \sum_{\text{cyclic}} \cos A \,.$$

. . . .

Si a, b et c sont les côtés d'un triangle ABC et si R est le rayon du cercle circonscrit, montrer que

$$a^2 + b^2 + c^2 \ \leq \ 6R^2 \sum_{\text{cyclique}} \cos A \,.$$

2888★. *Proposed by Vedula N. Murty, Dover, PA, USA.*
Let a, b, c be the sides of $\triangle ABC$. Give an algebraic proof of

$$8a^2b^2c^2 + \prod_{\text{cyclic}} \left(b^2 + c^2 - a^2\right) \ \leq \ 3abc \sum_{\text{cyclic}} a\left(b^2 + c^2 - a^2\right) \,.$$

. . . .

Soit a, b et c les côtés d'un triangle ABC. Donner une démonstration algébrique de

$$8a^2b^2c^2 + \prod_{\text{cyclique}} \left(b^2 + c^2 - a^2\right) \ \leq \ 3abc \sum_{\text{cyclique}} a\left(b^2 + c^2 - a^2\right) \,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologise for omitting the name of NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina from the list of solvers of 2739.

2701★. [2002 : 52] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Do there exist infinitely many triplets $(n, n+1, n+2)$ of adjacent natural numbers such that all of them are sums of two positive perfect squares?

(Examples are $(232, 233, 234)$, $(520, 521, 522)$ and $(808, 809, 810)$.)

Compare the 2000 Putnam problem A2 [2001 : 3]

Comment : Following the published solution [2003 : 48–49], the Editor remarked "Li Zhou asks the question : Are there infinitely many different similarity classes of Pythagorean triples the hypotenuses of which appear in the set $\{2n^2+1\}_{n\in\mathbb{N}}$?"

Zhou has answered his own question as follows :

The published solution has in fact answered this question. For $a \geq 2$, the numbers $p = (a^2-1)^2 - a^2(a-2)^2$, $q = 2a(a^2-1)(a-2)$ and $r = 2a^2(a-1)^2+1$ satisfy $p^2+q^2=r^2$. Since $q/p \longrightarrow \infty$ as $a \longrightarrow \infty$, there are infinitely many dissimilar triples (p, q, r).

2780. [2002 : 457] *Proposed by Mihály Bencze, Brasov, Romania.*

Solve in $\mathbb{C}$:

$$|x|^2yz = x^4\,, \qquad |y|^2zx = y^4\,, \qquad |z|^2xy = z^4\,.$$

Solution by Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.

If any one of x, y, or z is zero, then the other two are also zero. Thus, we may assume that $xyz \neq 0$. Let $x = |x|e^{i\theta}$, $y = |y|e^{i\phi}$, and $z = |z|e^{i\psi}$. Dividing $|x|^2yz = x^4$ by $|x|^2$, we obtain $yz = |x|^2e^{4i\theta}$, which implies

$$|y||z| = |x|^2\,, \qquad (1)$$
$$\phi+\psi = 4\theta + 2k\pi\,, \qquad (2)$$

for some arbitrary integer k. By symmetry, we have

$$|z||x| = |y|^2\,, \qquad (3)$$
$$\psi+\theta = 4\phi + 2l\pi\,, \qquad (4)$$
$$|x||y| = |z|^2\,, \qquad (5)$$
$$\theta+\phi = 4\psi + 2m\pi\,, \qquad (6)$$

where l and m are arbitrary integers.

Dividing (1) by (3) yields $|x|^3 = |y|^3$. Thus, $|x| = |y|$. Similarly, dividing (3) by (5) yields $|y| = |z|$. Hence, $|x| = |y| = |z| = r$ for some $r > 0$. Solving (2), (4), and (6), we easily find that $\theta = -(3k + l + m)\pi/5$, $\phi = -(k + 3l + m)\pi/5$, and $\psi = -(k + l + 3m)\pi/5$. Clearly, we can drop the minus signs. Hence, besides $(0,0,0)$, all the solutions are given by $(x,y,z) = (re^{(3k+l+m)\pi i/5}, re^{(k+3l+m)\pi i/5}, re^{(k+l+3m)\pi i/5})$, for any integers k, l, m and any $r > 0$.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; ROBERT BILINSKI, Outremont, QC; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; NEVEN JURIČ, Zagreb, Croatia; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, student, New York University, NY, USA; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were three incorrect or incomplete solutions.

Both Díaz-Barrero and the proposer gave the solution set as :

$$S = \{(x,y,z) | x = ru\,, y = ruv\,, z = r^2u^3/v\,, r \ge 0\,, \\ u\,, v \in \mathbb{C} \text{ with } u^{10} = 1 \text{ and } v^5 = 1\}$$

together with all the triples obtained by permuting x, y, and z.

Bataille, Loeffler, Woo, and Zhou listed all the solutions for fixed $r > 0$. In particular, by setting $a = 3k + l + m$, $b = k + 3l + m$, $c = k + l + 3m$ and restricting the values of a, b, c to be in $\{0, 1, 2, \ldots, 9\}$, Zhou obtained the following 14 unordered triples for (a,b,c), where $a \le b \le c$: $(0,0,0)$, $(5,5,5)$, $(1,1,3)$, $(2,2,6)$, $(3,3,9)$, $(2,4,4)$, $(6,6,8)$, $(1,7,7)$, $(4,8,8)$, $(7,9,9)$, $(0,2,8)$, $(0,4,6)$, $(1,5,9)$, $(3,5,7)$. By permuting the components in each triple and by simple counting, we easily see that in addition to the trivial solution $(0,0,0)$, there are, for each fixed $r > 0$, exactly 50 solutions.

2781. [2002 : 458] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(xf(y)) = (1-y)f(xy) + x^2y^2f(y)$$

for all x, $y \in \mathbb{R}$.

Solution by Michel Bataille, Rouen, France.

It is easy to verify that the zero-function, $x \mapsto 0$, and the function $x \mapsto x - x^2$ are solutions. We show that the equation has no other solutions.

Let f satisfy the equation

$$f(x(f(y)) = (1-y)f(xy) + x^2y^2f(y) \qquad (1)$$

for all $x, y \in \mathbb{R}$. Setting $(x,y) = (0,-1)$, $(1,1)$, and $\big(1, f(1)\big)$, in succession, in equation (1), we get $f(0) = 0$, $f\big(f(1)\big) = f(1)$, and $[f(1)]^2\big(f(1)-1\big) = 0$, respectively, and we conclude that $f(1) = 1$ or $f(1) = 0$.

If $f(1) = 1$, then setting $y = 1$ in equation (1) gives $f(x) = x^2$ for all $x \in \mathbb{R}$. However, it is easy to check that the function $x \mapsto x^2$ is not a solution of the given functional equation.

Therefore, we have $f(1) = 0$. Thus, $f(0) = f(1) = 0$.

If there exists $a \in \mathbb{R} \setminus \{0,1\}$ such that $f(a) = 0$, then set $y = a$ in equation (1) to get $0 = (1-a)f(xa)$ for all $x \in \mathbb{R}$. It follows that $f(xa) = 0$ for all $x \in \mathbb{R}$, so that $f(t) = f\big((t/a)a\big) = 0$ for all $t \in \mathbb{R}$. This gives the solution $x \mapsto 0$.

If $f(y) \neq 0$ for all $y \in \mathbb{R} \setminus \{0,1\}$, then set $x = 1/y$ in equation (1) to get

$$f\left(\frac{f(y)}{y}\right) \;=\; f(y)\,. \tag{2}$$

Setting $x = 1$ in (1) gives

$$f\left(f(y)\right) \;=\; (1-y)f(y) + y^2 f(y)\,. \tag{3}$$

Substituting $f(y)/y$ for y in equation (3) yields

$$f\left(f\left(\frac{f(y)}{y}\right)\right) \;=\; \left(1 - \frac{f(y)}{y}\right) f\left(\frac{f(y)}{y}\right) + \left(\frac{f(y)}{y}\right)^2 f\left(\frac{f(y)}{y}\right)\,. \tag{4}$$

Using (2), equation (4) can be simplified to

$$f\left(f(y)\right) \;=\; \left(1 - \frac{f(y)}{y}\right) f(y) + \left(\frac{f(y)}{y}\right)^2 f(y)\,. \tag{5}$$

Eliminating $f\left(f(y)\right)$ from equations (3) and (5), we get a quadratic equation for $f(y)$:

$$[f(y)]^2 - yf(y) + y^3 - y^4 \;=\; 0\,.$$

Thus, $f(y) = y^2$ or $f(y) = y - y^2$. Then $f(y^2) = y^4$ or $f(y^2) = y^2 - y^4$. If $f(y) = y^2$, then equation (3) gives $f(y^2) = (1-y)y^2 + y^4$. Comparing the last equality with $f(y^2) = y^4$ and $f(y^2) = y^2 - y^4$, we conclude that $f(y) = y^2$ can only occur when $y = 1/2$, in which case $f(y) = y - y^2$ as well. Consequently, $f(y) = y - y^2$ for all $y \in \mathbb{R} \setminus \{0,1\}$, and even for all $y \in \mathbb{R}$, because $f(0) = f(1) = 0$. This completes the proof.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; PIERRE BORNSZTEIN, Pontoise, France; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were two incorrect solutions submitted.

2782. [2002 : 458] *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that p is an odd prime number. Prove that

$$\frac{p+1}{2} + p!\sum_{k=1}^{p} \frac{1}{\binom{p}{k}} \equiv 0 \pmod{p} .$$

Solution by David Loeffler, student, Trinity College, Cambridge, UK.
Evidently,

$$\begin{aligned} \frac{p!}{\binom{p}{k}} &= (p-k)!\,k! = 1 \times 2 \times \cdots \times (p-k) \times k! \\ &\equiv (-1)^{p-k}(p-1)(p-2)\cdots(k+1)(k)k! \\ &= (-1)^{p-k}(p-1)!\,k \equiv (-1)^k k \pmod{p} , \end{aligned}$$

since $(p-1)! \equiv -1 \pmod{p}$ by Wilson's Theorem. Hence,

$$\begin{aligned} \sum_{k=1}^{p} \frac{p!}{\binom{p}{k}} &\equiv \sum_{k=1}^{p}(-1)^k k \equiv \sum_{j=1}^{\frac{p-1}{2}}\left((-1)^{2j}(2j) + (-1)^{2j-1}(2j-1)\right) \\ &= \sum_{j=1}^{\frac{p-1}{2}} 1 = \frac{p-1}{2} , \end{aligned}$$

and the result clearly follows.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PAUL BRACKEN, Concordia University, Montréal, QC; IAN JUNE L. GARCES and WINFER C. TABARES, Ateneo de Manila University, The Philippines; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. All submitted solutions made use of Wilson's Theorem and were all similar to the one featured above.

2783. [2002 : 458] *Proposed by Radu Miculescu, University of Bucharest, Bucharest, Romania.*

Let $\{x_n\}$ and $\{y_n\}$ $(n \geq 1)$ be sequences of real numbers satisfying

$$x_{n+1} = 2x_n - y_n \qquad \text{and} \qquad y_{n+1} = 2y_n - x_n$$

for each $n \geq 1$. Suppose that $x_0 \leq y_0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f'' \geq 0$. Prove that the sequence $\{z_n\}$ $(n \geq 1)$ is increasing, where

$$z_n = f(x_n) + f(y_n) .$$

Solution by Michel Bataille, Rouen, France.

Let n be any positive integer. Note that

$$\frac{x_n + y_{n+1}}{2} = y_n \quad \text{and} \quad \frac{y_n + x_{n+1}}{2} = x_n\,.$$

Since $f'' \geq 0$, f is a convex function. Therefore,

$$f(y_n) \leq \frac{1}{2}f(x_n) + \frac{1}{2}f(y_{n+1}) \quad \text{and} \quad f(x_n) \leq \frac{1}{2}f(y_n) + \frac{1}{2}f(x_{n+1})\,.$$

By addition, $z_n \leq \frac{1}{2}z_n + \frac{1}{2}z_{n+1}$, or $z_n \leq z_{n+1}$. Thus, $\{z_n\}$ is increasing.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; DANIEL REISZ, Vincelles, France; JOEL SCHLOSBERG, student, New York University, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2784. [2002 : 459] *Proposed by Radu Miculescu, University of Bucharest, Bucharest, Romania.*

Let $\{x_n\}$ and $\{y_n\}$ ($n \geq 0$) be sequences of real numbers such that (a) $\{x_n\}$ is bounded, (b) $\lim_{n\to\infty} y_n = \alpha$, and (c) $\lim_{n\to\infty}(x_n - y_n x_{n+1}) = \beta$, where $\alpha \in (-1,0) \cup (0,1)$ and $\beta \in \mathbb{R}$.

Show that $\lim_{n\to\infty} x_n = \dfrac{\beta}{1-\alpha}$.

Solution by David Loeffler, student, Trinity College, Cambridge, UK, modified by the editor.

The result, in fact, holds for all $\alpha \neq \pm 1$.

For any real α and β, the hypotheses (a)–(c) in the problem imply that $\lim_{n\to\infty}(x_n - \alpha x_{n+1}) = \beta$. Indeed, we have

$$\begin{aligned}
\lim_{n\to\infty}(x_n - \alpha x_{n+1}) &= \lim_{n\to\infty}(x_n - y_n x_{n+1} + y_n x_{n+1} - \alpha x_{n+1}) \\
&= \lim_{n\to\infty}(x_n - y_n x_{n+1}) + \lim_{n\to\infty}\left(x_{n+1}(y_n - \alpha)\right) \\
&= \beta + 0 = \beta\,,
\end{aligned}$$

where the last limit is zero because $\lim_{n\to\infty}(y_n - \alpha) = 0$ and $\{x_n\}$ is bounded.

We can take $\beta = 0$ without loss of generality. To see this, note that the general case can be reduced to this special case by letting $u_n = x_n - \dfrac{\beta}{1-\alpha}$. Then $\{u_n\}$ is bounded, and

$$\begin{aligned}
\lim_{n\to\infty}(u_n - \alpha u_{n+1}) &= \lim_{n\to\infty}(x_n - \alpha x_{n+1}) - \frac{\beta}{1-\alpha} + \frac{\alpha\beta}{1-\alpha} \\
&= \beta - \beta = 0\,.
\end{aligned}$$

From now on, we assume that $\beta = 0$. We will prove that if $\alpha \neq \pm 1$, then $\lim\limits_{n\to\infty} x_n = 0$.

Let M be a positive real number such that $|x_n| < M$ for all n. Let $\epsilon > 0$ be given.

Case 1 : $|\alpha| < 1$. Since $\lim\limits_{n\to\infty}(x_n - \alpha x_{n+1}) = 0$, there is some $N \in \mathbb{N}$ such that $|x_n - \alpha x_{n+1}| < \frac{\epsilon}{2}(1 - |\alpha|)$ for all $n \geq N$. Then, for any $n \geq N$,

$$|x_n| \leq |x_n - \alpha x_{n+1}| + |\alpha x_{n+1}| < \frac{\epsilon}{2}(1 - |\alpha|) + |\alpha| \cdot |x_{n+1}| .$$

Now an easy induction shows that for all $r \in \mathbb{N}$,

$$|x_n| < \frac{\epsilon}{2}(1 - |\alpha|) \sum_{j=0}^{r-1} |\alpha|^j + |\alpha|^r \cdot |x_{n+r}| .$$

Hence,

$$|x_n| < \frac{\epsilon}{2}(1 - |\alpha|) \sum_{j=0}^{\infty} |\alpha|^j + |\alpha|^r M = \frac{\epsilon}{2} + |\alpha|^r M .$$

Choosing r large enough so that $|\alpha|^r < \frac{\epsilon}{2M}$, we have $|x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Case 2 : $|\alpha| > 1$. Note that

$$\lim_{n\to\infty}\left(x_{n+1} - \frac{x_n}{\alpha}\right) = -\frac{1}{\alpha}\lim_{n\to\infty}(x_n - \alpha x_{n+1}) = 0 .$$

Let $N \in \mathbb{N}$ be large enough so that $\left|x_{n+1} - \frac{x_n}{\alpha}\right| < \frac{\epsilon}{2}\left(1 - \frac{1}{|\alpha|}\right)$ for $n \geq N$. Then, for any $n \geq N$,

$$|x_{n+1}| \leq \left|x_{n+1} - \frac{x_n}{\alpha}\right| + \left|\frac{x_n}{\alpha}\right| < \frac{\epsilon}{2}\left(1 - \frac{1}{|\alpha|}\right) + \frac{|x_n|}{|\alpha|} .$$

By induction, for all $r \in \mathbb{N}$,

$$|x_{n+r}| < \frac{\epsilon}{2}\left(1 - \frac{1}{|\alpha|}\right)\sum_{j=0}^{r-1}\frac{1}{|\alpha|^j} + \frac{|x_n|}{|\alpha|^r} .$$

Hence,

$$|x_{n+r}| < \frac{\epsilon}{2}\left(1 - \frac{1}{|\alpha|}\right)\sum_{j=0}^{\infty}\frac{1}{|\alpha|^j} + \frac{M}{|\alpha|^r} = \frac{\epsilon}{2} + \frac{M}{|\alpha|^r} .$$

Let $m \in \mathbb{N}$ be large enough so that $\frac{1}{|\alpha|^m} < \frac{\epsilon}{2M}$. Then, for all $r \geq m$, we have $|x_{n+r}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus, $|x_n| < \epsilon$ for all $n \geq N + m$.

In both cases, we have shown that $|x_n| < \epsilon$ for all n sufficiently large. This proves that $\lim\limits_{n\to\infty} x_n = 0$.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; PIERRE BORNSZTEIN, Pontoise, France; KEITH EKBLAW, Walla Walla, WA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were 4 incorrect solutions.

Ekblaw gave examples to show that the proposed result is false when $\alpha = \pm 1$. He did not specify a particular sequence $\{y_n\}$, but we can take $y_n = \alpha$ for all n. For $\alpha = -1$, he considered the sequence $\{x_n\}$ that alternates between 2 and 3. Then $x_n - \alpha x_{n+1} = 5$ for all n, and condition (c) in the problem is satisfied with $\beta = 5$. Condition (a) is satisfied also. But $\lim\limits_{n\to\infty} x_n$ does not exist. For $\alpha = 1$, he considered a sequence $\{x_n\}$ that steps between 0 and 1 with steps that decrease in size. An example of such a sequence (slightly different than the example he gave) is defined by

$$x_{2^k+j} = \begin{cases} \dfrac{j}{2^{k-1}} & \text{for } j = 0, 1, \ldots, 2^{k-1}, \\ 2 - \dfrac{j}{2^{k-1}} & \text{for } j = 2^{k-1}+1, \ldots, 2^k - 1, \end{cases}$$

for $k = 2, 3, \ldots$. Here condition (c) is satisfied with $\beta = 0$, but $\lim\limits_{n\to\infty} x_n$ does not exist.

The result and the featured proof above are valid for complex sequences and complex numbers α and β, provided that $|\alpha| \neq 1$. When $|\alpha| = 1$ and $\alpha \neq 1$, a counterexample is the sequence $\{x_n\}$ defined by $x_n = 1/\alpha^n = \bar{\alpha}^n$.

2785. [2002 : 459] *Proposed by Christopher Bowen, Halandri, Greece.*

Solve the Diophantine equation

$$(x-y-z)(x-y+z)(x+y-z) = 8xyz,$$

where x, y, z are relatively prime.

Composite of similar solutions by the SMSU Problem Solving Group, Southwest Missouri University, Springfield, MO, USA; and by Li Zhou, Polk Community College, Winter Haven, FL, USA.

The given equation is equivalent to

$$\begin{aligned} 0 &= (x-y-z)(x-y+z)(x+y-z) - 8xyz \\ &= (x+y+z)(x^2+y^2+z^2-2xy-2yz-2zx). \end{aligned}$$

Solving for z in terms of x and y, we have $z = -x-y$ or $z = x \pm 2\sqrt{xy} + y$. It is clear that in either case, x and y must be relatively prime in order for x, y, and z to be relatively prime. Thus, the first case leads to solutions of the form $(m, n, -m-n)$, where m and n are relatively prime integers.

In the second case, we must have $x = km^2$, $y = kn^2$ where m and n are relatively prime and $k = \pm 1$. Hence, $z = k(m+n)^2$ and we are led to the solutions $\pm(m^2, n^2, (m+n)^2)$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; and the proposer. There were three incomplete and one incorrect solution.

2786. [2002 : 459] *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove or disprove the inequality

$$3 \leq \frac{1}{1-xy} + \frac{1}{1-yz} + \frac{1}{1-zx} \leq \frac{27}{8},$$

where $x+y+z=1$ and x, y, $z \geq 0$.

I. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

The left-hand inequality is obvious, since each term must be at least 1. Equality occurs if and only if $xy = yz = zx = 0$; that is, if two of x, y, z are 0 (and the other is 1).

For the right-hand inequality, multiply by $(1-xy)(1-yz)(1-zx)$ and simplify to get the equivalent form

$$3 - 11(xy+yz+zx) + 19xyz(x+y+z) - 27x^2y^2z^2 \geq 0.$$

Note that $xyz \leq (x+y+z)^3/27 = 1/27$, by the AM–GM Inequality. Thus, $27x^2y^2z^2 \leq xyz$. Applying the last inequality and $x+y+z=1$, we obtain

$$\begin{aligned} 3 - 11(xy+yz+zx) + 19xyz(x+y+z) - 27x^2y^2z^2 \\ \geq \; 3 - 11(xy+yz+zx) + 19xyz - xyz \\ = \; 3 - 11(xy+yz+zx) + 18xyz. \end{aligned}$$

Now, it suffices to prove the inequality

$$3 - 11(xy+yz+zx) + 18xyz \geq 0.$$

The expression on the left-hand side is linear in each of the variables and symmetric, so that the methods of Sato's article [2001 : 529] apply; that is, the extreme values of this expression must occur among the values at the points $(1,0,0)$, $(1/2,1/2,0)$ and $(1/3,1/3,1/3)$. Since these values are 3, $1/4$, and 0, respectively, we can deduce that the expression is always non-negative, as desired.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB; Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA; and Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

For the left-hand inequality, the solution is the same as above.

For the right-hand inequality, we use the elementary symmetric functions $\sigma_1 = x+y+z$, $\sigma_2 = yz+zx+xy$ and $\sigma_3 = xyz$ and rewrite the inequality as

$$3\sigma_1^6 - 11\sigma_1^4\sigma_2 + 19\sigma_1^3\sigma_3 - 27\sigma_3^2 \geq 0,$$

and then as

$$3\sigma_1^3(\sigma_1^3 - 4\sigma_1\sigma_2 + 9\sigma_3) + \sigma_1^3(\sigma_1\sigma_2 - 9\sigma_3) + \sigma_3(\sigma_1^3 - 27\sigma_3) \geq 0.$$

The validity of the last inequality follows from the three known inequalities : $\sigma_1^3 - 4\sigma_1\sigma_2 + 9\sigma_3 = \sum_c x(x-y)(x-z) \geq 0$, a special case of Schur's Inequality ; $\sigma_1\sigma_2 - 9\sigma_3 \geq 0$, Cauchy's Inequality ; and $\sigma_1^3 - 27\sigma_3 \geq 0$, the AM–GM Inequality. Equality holds if and only if $x = y = z$.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain ; PIERRE BORNSZTEIN, Pontoise, France ; CHRISTOPHER BOWEN, Halandri, Greece ; PAUL BRACKEN, Concordia University, Montréal, QC ; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK ; VASILE CARTOAJE, University of Ploiesti, Romania ; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina ; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (*two solutions*) ; *D. KIPP JOHNSON, Beaverton, OR, USA ; KEE-WAI LAU, Hong Kong, China ; PANOS E. TSAOUSSOGLOU, Athens, Greece ; LI ZHOU, Polk Community College, Winter Haven, FL, USA ; and TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania. There were four incorrect or incomplete solutions submitted.*

Cartoaje established the following more general result : Let x, y, z be non-negative real numbers such that $x+y+z=1$, and let r be a real number such that $r<4$. Then

$$\min\left\{3, \frac{27}{9-r}\right\} \leq \frac{1}{1-rxy} + \frac{1}{1-ryz} + \frac{1}{1-rzx} \leq \max\left\{3, \frac{27}{9-r}, \frac{12-2r}{4-r}\right\} .$$

2787. [2002 : 460] *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove or disprove the inequality

$$\frac{27}{8} \leq \frac{1}{1-\left(\frac{x+y}{2}\right)^2} + \frac{1}{1-\left(\frac{y+z}{2}\right)^2} + \frac{1}{1-\left(\frac{z+x}{2}\right)^2} \leq \frac{11}{3},$$

where $x+y+z=1$ and x, y, $z \geq 0$.

I. *Solution by Rey Barcelon, Ian June L. Garces and Winfer C. Tabares, Ateneo de Manila University, The Philippines.*

Using the given condition $x+y+z=1$ and a partial fraction decomposition, we obtain

$$\frac{1}{1-\left(\frac{x+y}{2}\right)^2} = \frac{4}{4-(1-z)^2} = \frac{4}{(1+z)(3-z)} = \frac{1}{1+z} + \frac{1}{3-z} .$$

Hence, the desired inequality is equivalent to

$$\frac{27}{8} \leq \left(\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}\right) + \left(\frac{1}{3-x} + \frac{1}{3-y} + \frac{1}{3-z}\right) \leq \frac{11}{3} .$$

We establish it by proving the two inequalities :

$$\frac{9}{4} \leq \left(\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}\right) \leq \frac{5}{2} \qquad (1)$$

and

$$\frac{9}{8} \leq \left(\frac{1}{3-x} + \frac{1}{3-y} + \frac{1}{3-z}\right) \leq \frac{7}{6} . \qquad (2)$$

The left-hand inequality of (1) follows by the AM–HM Inequality :

$$\frac{3}{\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}} \;\le\; \frac{(1+x)+(1+y)+(1+z)}{3}\,.$$

The proof of the left-hand inequality of (2) is similar. Equality holds in the left-hand inequalities of (1) and (2) if and only if $x=y=z$.

To prove the right-hand inequalities of (1) and (2), we note that

$$\frac{1}{1+x} \;=\; \frac{1}{2}\left(1+\frac{1-x}{1+x}\right) \;\le\; \frac{1}{2}(1+1-x) \;=\; 1-\frac{x}{2}$$

and

$$\frac{1}{3-x} \;=\; \frac{1}{6}\left(3-\frac{3(1-x)}{3-x}\right) \;\le\; \frac{1}{6}\big[3-(1-x)\big] \;=\; \frac{1}{3}+\frac{x}{6}\,.$$

Adding similar inequalities for y and z gives the results. Equality holds in the right-hand inequalities of (1) and (2) if and only if two of x, y, z are 0 are 0 (and the other is 1).

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

As known ([1], [2]) by the Majorization Inequality, if $F(x)$ is convex, $x+y+z=1$, and x, y, $z\ge 0$, then

$$F(1)+(n-1)F(0)\ge F(x)+F(y)+F(z)\ge 3F\left(\frac{1}{3}\right)\,,$$

because $(1,0,0)\succ(x,y,z)\succ\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. We can extend this to the following class of cyclic inequalities : Let $t_i=x_i+x_{i+1}+\cdots+x_{i+r-1}$, for $i=1,2,\ldots,n$, where $x_{i+n}=x_i$, $x_i\ge 0$, and $x_1+x_2+\cdots+x_n=1$. Then

$$rF(1)+(n-r)F(0)\ge F(t_1)+F(t_2)+\cdots+F(t_n)\ge nF\left(\frac{r}{n}\right)\,,$$

because $(1,1,\ldots,1,0,0,\ldots,0)\succ(t_1,t_2,\ldots,t_n)\succ\left(\frac{r}{n},\frac{r}{n},\ldots,\frac{r}{n}\right)$. Now, let $F(x)=1/(a-x^p)^s$, where $a>1$, $p>1$, and $s>0$. Since $F''(x)\ge 0$, then $F(x)$ is convex, so that

$$\frac{r}{(a-1)^s}+\frac{n-r}{a^s} \;\ge\; \sum_{i=1}^{n}\frac{1}{(a-t_i^p)^s} \;\ge\; \frac{n}{a-\left(\frac{r}{n}\right)^p}\,,$$

The proposed inequality corresponds to the special case of $r=2$, $a=4$, $p=2$, $s=1$, and $n=3$.

[1] A.W. Marshall, I. Olkin, *Inequalities : Theory of Majorization and its Applications*, Academic Press, N.Y., 1979.

[2] M.S. Klamkin, *On a "Problem of the Month"*, Crux Mathematicorum with Mathematical Mayhem, 2(2002), 86–87.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, Concordia University, Montréal, QC; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania. There were four incorrect or incomplete solutions submitted.

2788. [2002 : 460] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Let D be the differential operator $\frac{d}{dx}$. Find the value, at $x = \frac{\pi}{2}$, of

$$(xD + n + 1)\ D^n \left(\frac{\sin^3 x}{x}\right).$$

Composite of solutions by Christopher J. Bradley, Clifton College, Bristol, UK; and the proposer.

By Leibniz' Rule, we have

$$D^{n+1}(xy) \;=\; xD^{n+1}y + (n+1)D^n y \;=\; (xD + n + 1)D^n y$$

for any infinitely differentiable function y. It follows that

$$(xD + n + 1)\, D^n\!\left(\frac{\sin^3 x}{x}\right) \;=\; D^{n+1}(\sin^3 x)\,.$$

On the other hand, we also have

$$\begin{aligned} D^{n+1}(\sin^3 x) &= \frac{1}{4}D^{n+1}(3\sin x - \sin 3x) \\ &= \frac{1}{4}\left[3\sin\left(x + \frac{(n+1)\pi}{2}\right) - 3^{n+1}\sin\left(3x + \frac{(n+1)\pi}{2}\right)\right]. \end{aligned}$$

At $x = \frac{\pi}{2}$, we obtain the value

$$\frac{3}{4}\sin\left(\frac{(n+2)\pi}{2}\right) - \frac{3^{n+1}}{4}\sin\left(\frac{(n+4)\pi}{2}\right) \;=\; -\frac{3}{4}(1+3^n)\sin\left(\frac{n\pi}{2}\right).$$

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; ROBERT BILINSKI, Outremont, QC; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There were one incomplete and two incorrect solutions.

2789. [2002 : 532] *Proposed by Mihály Bencze, Brasov, Romania.*

Let A, $B \in M_n(\mathbb{C})$ such that $A + B = I_n$ and, for some $k \in \mathbb{N}^*$, $A^{2k+1} = A^{2k}$. Prove that $I_n + A^k B$ is invertible, and find its inverse.

Solution by Con Amore Problem Group, The Danish University of Education, Copenhagen, Denmark.

We have

$$\begin{aligned}(I_n + A^k B)(I_n - A^k B) &= I_n - A^k B + A^k B - A^k B A^k B \\ &= I_n - A^k (I_n - A) A^k B \\ &= I_n - (A^k - A^{k+1}) A^k B \\ &= I_n - (A^{2k} - A^{2k+1}) B \\ &= I_n - O_n B \\ &= I_n\,,\end{aligned}$$

so that $(I_n + A^k B)(I_n - A^k B) = I_n$. Consequently, $I_n + A^k B$ is invertible and $I_n - A^k B$ is its inverse.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, NS; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NEVEN JURIČ, Zagreb, Croatia; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NF; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution submitted.