SKOLIAD No. 71

Shawn Godin

Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6, or emailed to

mayhem-editors@cms.math.ca.

We are looking for solutions especially from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 March 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 5** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The first item this issue comes from the 2003 first annual Fryer Contest. This and two other contests, the Galois and the Hypatia, were introduced this year for students in grades 9, 10, and 11, respectively, by the Canadian Mathematics Competitions. My thanks go out to Ian VanderBurgh and Peter Crippin of The University of Waterloo for forwarding the material to me. We especially invite students in grade 10 (or equivalent) or earlier to send in solutions.

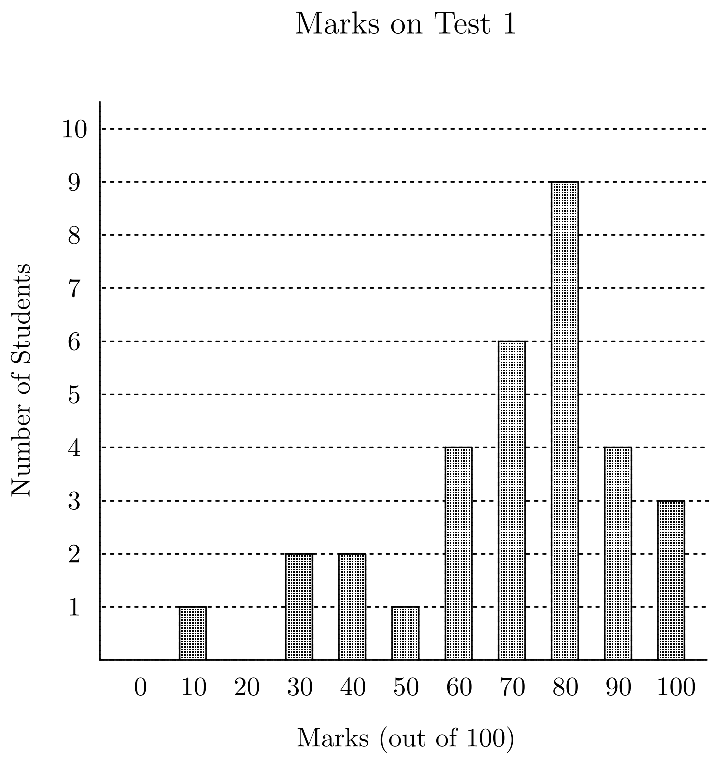
2003 Fryer Contest
April 16, 2003

1. (a) ($*$) The marks of 32 mathematics students on Test 1 are all multiples of 10 and are shown on the bar graph. What was the average (mean) mark of the 32 students in the class?

(b) ($*$) After his first 6 tests, Paul has an average of 86. What will his average be if he scores 100 on his next test?

(c) ($*$) Later in the year, Mary realizes that she needs a mark of 100 on the next test in order to achieve an average of 90 for all her tests. However, if she gets a mark of 70 on the next test, her average will be 87. After she writes the next test, how many tests will she have written?

Extension to #1: ($*$) Mary's teacher records the final marks of the 32 students. The teacher calculates that, for the entire class, the median mark is 80. The teacher also calculates that the difference between the highest and lowest marks is 40 and calculates that the average mark for the entire class is 58. Show that the teacher has made a calculation error.



(a) (∗) Les notes de 32 élèves, lors de leur première épreuve de mathématiques, sont toutes des multiples de 10. Elles sont indiquées dans le diagramme à bâtons. Quelle est la moyenne des notes des 32 élèves de la classe ?

(b) (∗) Après 6 épreuves, Paul a une moyenne de 86. Quelle sera sa moyenne s'il obtient une note de 100 lors de la prochaine épreuve ?

(c) (∗) Plus tard dans l'année, Marie se rend compte qu'elle a besoin d'une note de 100 lors de la prochaine épreuve pour que sa moyenne, dans toutes les épreuves, soit égale à 90. Or si elle obtient une note de 70 dans la prochaine épreuve, sa moyenne sera égale à 87. Lorsqu'elle aura terminé la prochaine épreuve, combien d'épreuves aura-t-elle écrites ?

Prolongement du Problème 1 *:* (∗) L'enseignante de Marie inscrit la note finale des 32 élèves. L'enseignante calcule la médiane de la classe et obtient une note de 80. Elle calcule aussi l'étendue des notes, soit la différence entre la note la plus haute et la note la plus basse, et obtient 40. Elle calcule enfin la moyenne de la classe et obtient 58. Démontrer que l'enseignante a commis une erreur.

2. In a game, Xavier and Yolanda take turns calling out whole numbers. The first number called must be a whole number between and including 1 and 9. Each number called after the first must be a whole number which is 1 to 10 greater than the previous number called.

(a) ($*$) The first time the game is played, the person who calls the number 15 is the winner. Explain why Xavier has a winning strategy if he goes first and calls 4.

(b) ($*$) The second time the game is played, the person who calls the number 50 is the winner. If Xavier goes first, how does he guarantee that he will win?

Extension to #2: ($*$) In the game described in b), the target number was 50. For what different values of the target number is it guaranteed that Yolanda will have a winning strategy if Xavier goes first?

. . . .

Xavier et Yvonne participent à un jeu dans lequel chacun, tour à tour, annonce un numéro qu'il ou elle a choisi. Le premier numéro doit être un entier de 1 à 9. Chaque numéro subséquent doit être un entier qui est de 1 à 10 de plus que le numéro précédent.

(a) ($*$) Lors de la première partie, la personne qui annoncera le numéro 15 sera déclarée gagnante. Expliquer que Xavier a une stratégie gagnante s'il joue premier en annonçant le numéro 4.

(b) ($*$) Lors de la deuxième partie, la personne qui annoncera le numéro 50 sera déclarée gagnante. Si Xavier joue premier, comment peut-il s'assurer de gagner ?

Prolongement du Problème 2 : ($*$) Dans la partie b), le nombre-cible était 50. Quelles sont les valeurs du nombre-cible qui peuvent assurer à Yvonne une stratégie gagnante si Xavier joue premier ?

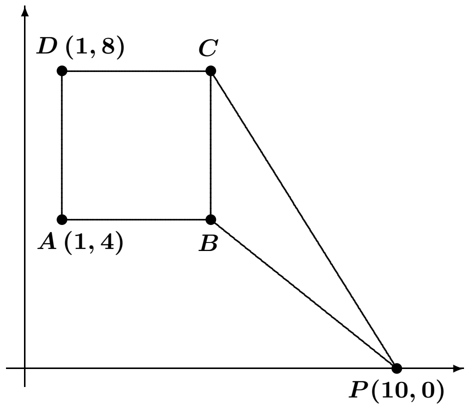
3. In the diagram, $ABCD$ is a square and the coordinates of A and D are as shown.

(a) ($*$) The point P has coordinates $(10, 0)$. Show that the area of triangle PCB is 10.

(b) ($*$) Point $E(a, 0)$ is on the x–axis such that triangle CBE lies entirely outside square $ABCD$. If the area of the triangle is equal to the area of the square, what is the value of a?

(c) ($*$) Show that there is no point F on the x–axis for which the area of triangle ABF is equal to the area of square $ABCD$.

Extension to #3: ($*$) G is a point on the line passing through the points $M(0, 8)$ and $N(3, 10)$ such that $\triangle DCG$ lies entirely outside the square. If the area of $\triangle DCG$ is equal to the area of the square, determine the coordinates of G.



$ABCD$ est un carré et les coordonnées de A et de D sont indiquées.

(a) ($*$) Le point P a pour coordonnées $(10, 0)$. Montrer que le triangle PCB a une aire de 10.

(b) ($*$) Soit un point $E(a, 0)$, sur l'axe des abscisses, de manière que le triangle CBE soit situé complètement à l'extérieur du carré $ABCD$. Si l'aire du triangle est égale à l'aire du carré, quelle est la valeur de a ?

(c) ($*$) Démontrer qu'il n'existe aucun point F, sur l'axe des abscisses, pour lequel l'aire du triangle ABF est égale à l'aire du carré $ABCD$.

Prolongement du Problème 3 : ($*$) Soit G un point sur la droite qui passe par les points $M(0, 8)$ et $N(3, 10)$, de manière que le triangle DCG soit situé complètement à l'extérieur du carré. Déterminer les coordonnées de G, sachant que l'aire du triangle est égale à l'aire du carré.

4. For the set of numbers {1, 10, 100} we can obtain 7 distinct numbers as totals of one or more elements of the set. These totals are 1, 10, 100, $1 + 10 = 11$, $1 + 100 = 101$, $10 + 100 = 110$, and $1 + 10 + 100 = 111$. The *power-sum* of this set is the sum of these totals, in this case, 444.

(a) ($*$) How many distinct numbers may be obtained as a sum of one or more different numbers from the set {1, 10, 100, 1000} ? Calculate the power-sum for this set.

(b) ($*$) Determine the power-sum of the set

$$\{1,\ 10,\ 100,\ 1000,\ 10\,000,\ 100\,000,\ 1\,000\,000\}.$$

Extension to #4: ($*$) Consider the set {1, 2, 3, 6, 12, 24, 48, 96}. How many different totals are now possible if a total is defined as the sum of 1 or more elements of a set?

Étant donné l'ensemble $\{1, 10, 100\}$, on peut obtenir 7 totaux distincts en additionnant un nombre ou plus de cet ensemble. Ces totaux sont 1, 10, 100, $1+10 = 11$, $1+100 = 101$, $10+100 = 110$, et $1+10+100 = 111$. La *somme-puissance* de cet ensemble est la somme de ces totaux. Elle est égale à 444.

(a) ($*$) Étant donné l'ensemble $\{1, 10, 100, 1000\}$, combien peut-on obtenir de totaux distincts en additionnant un nombre ou plus de cet ensemble ? Calculer la somme-puissance de cet ensemble.

(b) ($*$) Déterminer la somme-puissance de l'ensemble

$$\{1, 10, 100, 1000, 10\,000, 100\,000, 1\,000\,000\}.$$

Prolongement du Problème 4 *:* ($*$) Soit l'ensemble $\{1, 2, 3, 6, 12, 24, 48, 96\}$. Combien peut-on obtenir de totaux distincts en additionnant un nombre ou plus de cet ensemble ?

Our second item this month is the 2002 W.J. Blundon Mathematics Contest. My thanks go out to Don Rideout of Memorial University for forwarding the material to me.

The Nineteenth W.J. Blundon Mathematics Contest

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

February 20, 2002

1. ($*$) Five years ago Janet was one sixth of her mother's age. In thirteen years she will be half her mother's age. What is Janet's present age?

2. ($*$) If $a + b + c = 0$, prove that $a^3 + b^3 + c^3 = 3abc$.

3. ($*$) A certain rectangle has area 6 and diagonal of length $2\sqrt{5}$. What is its perimeter?

4. Find all positive numbers x such that $x^{x\sqrt{x}} = (x\sqrt{x}\,)^x$.

5. Rationalize the denominator: $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}}$.

6. Points A and B are on the parabola $y = 2x^2 + 4x - 2$. The origin is the mid-point of the line segment joining A and B. Find the length of this line segment.

7. If $\log_{125} 2 = a$ and $\log_9 25 = b$, find $\log_8 9$ in terms of a and b.

8. Point P lies in the first quadrant on the line $y = 2x$. Point Q is a point on the line $y = 3x$ such that PQ has length 5 and is perpendicular to the line $y = 2x$. Find the point P.

9. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

10. In $\triangle ABC$, we have $\angle ACB = 120$ degrees, $AC = 6$ and $BC = 2$. The internal bisector of $\angle ACB$ meets the side AB at the point D. Determine the length of the line segment CD.

Next, we present the solutions to the 2nd Junior Balkan Mathematical Olympiad (1998) that appeared in the December 2002 issue ([2002 : 522]).

1. (Yugoslavia) Prove that the number

$$\underbrace{11\ldots111}_{1997}\underbrace{22\ldots222}_{1998}5$$

is a perfect square.

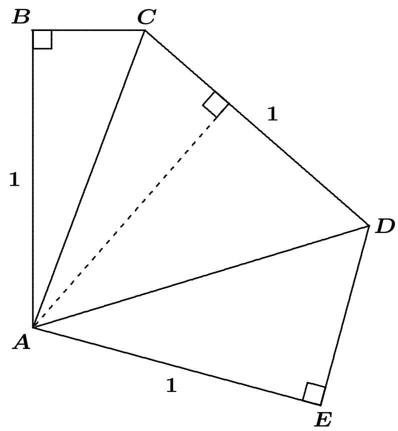
Official Solution.

$$\begin{aligned}
\underbrace{11\ldots111}_{1997}\underbrace{22\ldots222}_{1998}5 &= \underbrace{11\ldots111}_{1997}\times 10^{1999} + \underbrace{22\ldots222}_{1998}\times 10 + 5 \\
&= \frac{10^{1997}-1}{9}\cdot 10^{1999} + 2\cdot\frac{10^{1998}-1}{9}\cdot 10 + 5 \\
&= \frac{1}{9}\left(10^{3996} - 10^{1999} + 2\cdot 10^{1999} - 20 + 45\right) \\
&= \frac{1}{9}\left(10^{3996} + 2\cdot 5\cdot 10^{1998} + 25\right) \\
&= \frac{1}{9}\left(10^{1998} + 5\right)^2 \\
&= (\underbrace{33\ldots33}_{1997}5)^2\,.
\end{aligned}$$

2. (Greece) Let us consider a convex pentagon $ABCDE$, with $AB = AE = CD = 1$, $\angle ABC = \angle DEA = 90^\circ$ and $BC + DE = 1$. Find the area of the pentagon.

Official Solution.

Draw the diagonals AC and AD. Since $\angle B = \angle E = 90°$ and $AB = AE$, we can create a triangle from the triangles ABC and AED having altitude $AB = AE = 1$ and base $BC + DE = 1$. The area of this new triangle is $\frac{1}{2}$. This newly created triangle is then congruent to $\triangle ACD$; hence, the total area of the pentagon is 1.



3. (Albania) Find all pairs of positive integers (x, y) that satisfy the following equation: $x^y = y^{x-y}$.

Official Solution.

If (x, y) is a solution of the given equation and $x \neq 1$, then $x > y$. Otherwise, we would have $y^{x-y} \leq 1$, while $x^y > 1$. Obviously, $(1, 1)$ is a solution of the given equation, and this is the only solution with $y = 1$. Let $x > y \geq 2$. Then we obtain the equation

$$\left(\frac{x}{y}\right)^y = y^{x-2y} .$$

Since $\frac{x}{y} > 1$, we get $x - 2y > 0$ and $\frac{x}{y} > 2$. Also, $\frac{x}{y} \in \mathbb{N}$. Our equation above can be written as

$$\frac{x}{y} = y^{\frac{x}{y}-2} .$$

Since $y^{\frac{x}{y}-2} \geq 2^{\frac{x}{y}-2}$, we obtain $\frac{x}{y} \leq 4$. We conclude that $2 < \frac{x}{y} \leq 4$.

- If $\frac{x}{y} = 3$, then $y = 3$, $x = 9$.
- If $\frac{x}{y} = 4$, then $y = 2$, $x = 8$.

Finally, the set of solutions of the given equation is:

$$\{(1, 1), (8, 2), (9, 3)\} .$$

4. (Bulgaria) Using only three digits can one write 16 three-digit numbers, such that no two of them are giving the same remainder divided by 16?

Official Solution.

Let us suppose that we can write down 16 such numbers. It is clear that 8 of these numbers must be even and the rest must be odd. Therefore, the digits cannot be all even or all odd. Let us examine the case where the given digits are two even and one odd. (The case with two odd digits and one even digit is similar.) Let e_1, e_2, and o denote the two even digits and the odd digit, respectively. We can write exactly 9 odd three-digit numbers with the given digits:

$$e_1e_1o\,,\,e_1e_2o\,,\,e_1oo\,,\,e_2e_1o\,,\,e_2e_2o\,,\,e_2oo\,,\,oe_1o\,,\,oe_2o\,,\,ooo\,.$$

If we rewrite these as a_1o, a_2o, $\ldots$, a_9o, where a_1, a_2, $\ldots$, a_9 are two-digit numbers formed by the first two digits of the numbers in the first list, then $a_io - a_jo$ is divisible by 16 if and only if $a_i - a_j$ is divisible by 8. But there are only three two-digit odd numbers among a_1, a_2, $\ldots$, a_9, whereas four are required to obtain four different odd remainders when divided by 8. Therefore, we cannot write down 16 such three-digit numbers.

Finally, we present the answers to the 2002 British Columbia Colleges Junior High School Mathematics Contest, Preliminary Round that appeared in the December 2002 issue ([2002 : 523–525]).

1. b	**2**. b	**3**. b	**4**. c	**5**. a	**6**. e	**7**. b	**8**. c
9. e	**10**. d	**11**. d	**12**. e	**13**. a	**14**. c	**15**. e	

That brings us to the end of another issue of Skoliad. Continue sending in contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7**. The electronic address is

mayhem-editors@cms.math.ca

The Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, 2191 Saturn Crescent, Orleans, Ontario, K4A 3T6** or emailed to

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Please include in all correspondence your name, school, grade, city, province or state, and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 March 2004*. Solutions received after this time will be considered only if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Hidemitsu Saeki of the University of Montreal for translations of the problems.

M101. *Proposed by the Mayhem Staff.*

Find the smallest value of k such that $k!$ ends with 100 zeros. [*Note:* $k! = k(k-1)(k-2)\cdots(3)(2)(1)$.]

. . . .

Trouver la plus petite valeur de k telle que $k!$ finisse avec 100 zéros. [*Note :* $k! = k(k-1)(k-2)\cdots(3)(2)(1)$.]

M102. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Suppose that $ABCD$ is a parallelogram and that G_A, G_B, G_C, and G_D are the centroids of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$, respectively.

Prove that:

1. $G_AG_BG_CG_D$ is a parallelogram;
2. $[G_AG_BG_CG_D] = \frac{1}{9}[ABCD]$, where $[ABCD]$ is the area of $ABCD$.

. . . .

Dans un parallélogramme $ABCD$ on suppose que G_A, G_B, G_C et G_D sont les centres de gravité respectifs des triangles BCD, ACD, ABD et ABC.

Montrer que :

1. $G_AG_BG_CG_D$ est un parallélogramme ;
2. $[G_AG_BG_CG_D] = \frac{1}{9}[ABCD]$, où $[ABCD]$ désigne l'aire de $ABCD$.

M103. *Proposed by the Mayhem Staff.*

Solve for n:

$$100^{1/n} \times 100^{2/n} \times 100^{3/n} \times \cdots \times 100^{2003/n} = 1000\,.$$

. . . .

Résoudre par rapport à n :

$$100^{1/n} \times 100^{2/n} \times 100^{3/n} \times \cdots \times 100^{2003/n} = 1000\,.$$

M104. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Suppose that $ABCD$ is a parallelogram and that O_A, O_B, O_C, and O_D are the circumcentres of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$, respectively.

Prove that:

1. $O_AO_BO_CO_D$ is a parallelogram;
2. parallelograms $ABCD$ and $O_AO_BO_CO_D$ are similar;
3. AO_BCO_D is a parallelogram;
4. O_ABO_CD is a parallelogram;
5. parallelograms AO_BCO_D and O_ABO_CD are similar.

Dans un parallélogramme $ABCD$ on suppose respectivement que O_A, O_B, O_C and O_D sont les centres des cercles circonscrits des triangles BCD, ACD, ABD and ABC.

Montrer que :

1. $O_AO_BO_CO_D$ est un parallélogramme ;
2. les parallélogrammes $ABCD$ et $O_AO_BO_CO_D$ sont semblables ;
3. AO_BCO_D est un parallélogramme ;
4. O_ABO_CD est un parallélogramme ;
5. les parallélogrammes AO_BCO_D et O_ABO_CD sont semblables.

M105. *Proposed by Andrew Critch, Clarenville High School, Clarenville, NL.*

Suppose that the roots of $P(x) = x^3 - 2kx^2 - 3x^2 + hx - 4$ are distinct, and that $P(k) = P(k+1) = 0$. Determine the value of h.

. . . .

On suppose que les racines de $P(x) = x^3 - 2kx^2 - 3x^2 + hx - 4$ sont distinctes, et que $P(k) = P(k+1) = 0$. Trouver la valeur de h.

M106. *Proposed by the Mayhem Staff.*

A 4 by 4 square has an area of 16 square units and a perimeter of 16 units. That is, the area and perimeter are numerically equivalent (ignoring units of measurement). Are there any other rectangles with integral dimensions that share this property? If possible, show that you have found all such examples.

. . . .

Un carré de 4 par 4 a une aire de 16 unités carrées et un périmètre de 16 unités. Autrement dit, l'aire et le périmètre sont numériquement équivalents (si on laisse tomber les unités). Y a-t-il d'autres rectangles de dimensions entières possédant cette propriété ? Si possible, montrez que vous les avez tous trouvés.

Mayhem Solutions

M51. *Proposed by the Mayhem Staff.*

You have a deck with cards numbered 1 through 25. You perform the following operations on the deck:

- you place the top card on the bottom of the deck.
- you place the new top card on the bottom of the deck.

- you flip the new top card face up on the table.

You continue this process until all cards are face up on the table. Find the order of the cards in the deck if, when the process is performed, the cards get laid out on the table in the order $1, 2, 3, \ldots, 25$.

Solution by Robert Bilinski, Outremont, QC.

First, let $a, b, c, d, e, f, \ldots, y$, be the sequence of 25 cards, in their original positions. In the following table, we put the first few steps of the stated algorithm.

Step	Undistributed cards	Found sequence
0	$a, b, c, d, e, f, \ldots, y$	
1	$d, e, f, g, \ldots, y, a, b$	c
2	$g, h, \ldots, y, a, b, d, e$	c, f

We thus obtain the following sequence:

$$c,\ f,\ i,\ l,\ o,\ r,\ u,\ x,\ b,\ g,\ k,\ p,\ t,\ y,\ e,\ m,\ s,\ a,\ j,\ v,\ h,\ w,\ q,\ d,\ n,$$

which equals $1, 2, 3, \ldots, 25$.

Thus, the original order of the cards was:

$$\begin{array}{l} 18,\ 9,\ 1,\ 24,\ 15,\ 2,\ 10,\ 21,\ 3,\ 19, \\ \quad 11,\ 4,\ 16,\ 25,\ 5,\ 12,\ 23,\ 6,\ 17,\ 13,\ 7,\ 20,\ 22,\ 8,\ 14. \end{array}$$

Also solved by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

M52. *Proposé par J. Walter Lynch, Athens, GA, USA.*

On a deux pièces de monnaie. L'une est une pièce d'un dollar normale et l'autre une fausse pièce d'un dollar, avec deux faces. On jette au hasard chacune des pièces dans deux tiroirs différents. Quelqu'un entre dans la chambre et ouvre un des tiroirs et aperçoît une pièce, côté face. Quelle est la probabilité que cette pièce soit celle à deux faces ?

Solution de Robert Bilinski, Outremont, QC.

Si on regarde les 2 pièces de monnaie, il y a au total 3 faces. Sur ces trois faces, il y en a deux qui appartiennent à la pièce truquée. Ainsi, par la définition de probabilité, on a :

$$P(\text{pièce truquée}) = \frac{\#\text{ cas favorable}}{\#\text{ total}} = \frac{2}{3}.$$

Solutioné aussi par Alexandre Ortan, écolier, École Joseph-François-Perrault, Montréal, QC.

M53. *Proposed by the Mayhem Staff.*

A circular path is surrounded by 17 stepping stones numbered 0, 1, 2, ..., 16. Sally starts on stone 0 and moves 1 step to stone 1, then 4 steps to stone 5, then 9 steps to step 14 and continues in the following pattern until at last she moves 2002^2 steps and stops (to rest). What stone is Sally standing on while she rests?

Solution by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

Let us first find how many steps Sally has made:

$$1^2 + 2^2 + 3^2 + \cdots + 2002^2 \;=\; \frac{2002 \times 2003 \times 4005}{6} \;=\; 2\,676\,679\,005$$

steps. Since her walk is cyclical, the residue of the number of steps on division by 17 will give us the position where Sally ends her walk.

Since $2\,676\,679\,005 = 17 \times 157\,451\,706 + 3$, Sophie ended up on the third step.

Also solved by Robert Bilinski, Outremont, QC.

M54. *Proposed by Gary Tupper, Pedagoguery Software Inc., Terrace, BC.*

An ellipse with major axis AB and foci F and F' is inscribed in a circle with diameter AB and centre C. P is a point on the ellipse and D is a point on the circle so that radius CD bisects FP. Show that line DP is tangent to the ellipse.

Solution by Andrei Ismail, student, 11^{th} Grade, Vasile Alecsandri National College, Galati, Romania.

Let the ellipse have equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then its foci are

$$F(\sqrt{a^2 - b^2}, 0) \qquad \text{and} \qquad F'(-\sqrt{a^2 - b^2}, 0)\,.$$

Because point P is located on the ellipse, its coordinates can be expressed as $P(a\cos t, b\sin t)$. Also, the points A and B have coordinates $A(-a, 0)$ and $B(a, 0)$. The circle with diameter AB has the equation $x^2 + y^2 = a^2$. Let us consider M, the mid-point of FP. It has coordinates

$$M\left(\frac{a\cos t + \sqrt{a^2 - b^2}}{2}, \frac{b\sin t}{2}\right)\,.$$

The condition that CD bisects FP is equivalent to the statement that D is the intersection of the line CM with the circle $x^2 + y^2 = a^2$. Now, it is easy to see that D has coordinates

$$D\left(a \cdot \frac{a\cos t + \sqrt{a^2 - b^2}}{\sqrt{(a\cos t + \sqrt{a^2 - b^2})^2 + (b\sin t)^2}}, a \cdot \frac{b\sin t}{\sqrt{(a\cos t + \sqrt{a^2 - b^2})^2 + (b\sin t)^2}}\right)\,.$$

Without loss of generality, we assume that P is in the half-plane $y > 0$ (and therefore, so is D). Simplifying the denominators yields

$$D\left(a \cdot \frac{a\cos t + \sqrt{a^2 - b^2}}{A},\ a \cdot \frac{b \sin t}{A}\right),$$

where $A = a + \cos t\sqrt{a^2 - b^2}$.

All we have to show now is that the line DP is tangent to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is well-known that the slope of the tangent to the ellipse at the point $P(a\cos t, b\sin t)$ is $-\frac{b\cos t}{a\sin t}$. Thus, it will suffice to show that:

$$\frac{a \cdot \dfrac{b\sin t}{A} - b\sin t}{a \cdot \dfrac{a\cos t + \sqrt{a^2 - b^2}}{A} - a\cos t} = -\frac{b\cos t}{a \sin t},$$

or, equivalently,

$$\frac{\dfrac{a\cos t + \sqrt{a^2 - b^2}}{A} - \cos t}{\dfrac{a\sin t}{A} - \sin t} = -\frac{\sin t}{\cos t}. \tag{1}$$

Now, the left-hand side of (1) can be simplified as

$$\begin{aligned}
\frac{\dfrac{a\cos t + \sqrt{a^2 - b^2}}{A} - \cos t}{\dfrac{a\sin t}{A} - \sin t} &= \frac{a\cos t + \sqrt{a^2 - b^2} - A\cos t}{a \sin t - A \sin t} \\
&= \frac{a\cos t + \sqrt{a^2 - b^2} - a\cos t - \cos^2 t \cdot \sqrt{a^2 - b^2}}{a\sin t - a\sin t - \cos t \sin t \cdot \sqrt{a^2 - b^2}} \\
&= \frac{1 - \cos^2 t}{-\cos t \sin t} \\
&= -\frac{\sin t}{\cos t},
\end{aligned}$$

which is the right-hand side of (1). Therefore, DP is indeed tangent to the ellipse.

Also solved by D.J. Smeenk, Zaltbommel, the Netherlands.

M55. *Proposé par l'équipe de Mayhem*. Trouver la somme des 2002 premiers termes de la suite suivante

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5,\ \ldots.$$

Solution by Robert Bilinski, Outremont, QC.

Il faut en premier déterminer avec la répétition de quel nombre s'arrête la suite lorsqu'elle est à la position 2002. On remarque que chaque nombre est répété sa valeur de fois. On cherche le plus grand n tel que $\sum\limits_{k=1}^{n} k \leq 2002$ ou bien que $\frac{n(n+1)}{2} \leq 2002$ qui équivaut à $n^2+n-4004 \leq 0$. On obtient $n = 62$ qui donne $\frac{n(n+1)}{2} = 1953$. Ce qui veut dire que dans les 2002 premiers nombres, les nombres de 1 à 62 apparaissent "au complet" et qu'il y aura ensuite 49 fois la valeur 63. La somme des 2002 premiers nombres a donc la valeur

$$\sum_{k=1}^{62} k^2 + 49 \cdot 63 \;=\; \frac{62 \cdot 63 \cdot 125}{6} + 49 \cdot 63 \;=\; 84\,462\,.$$

Solutioné aussi par Alexandre Ortan, écolier, École Joseph-François-Perrault, Montréal, QC.

M56. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Prove the identity

$$\left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 \;=\; 4 \cos A \cos B \cos C\,,$$

where the sums are cyclic and $A + B + C = \pi$.

Solution by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.

We claim that, if $A + B + C = \pi$, then

$$\cos^2 A \;=\; 1 - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C\,. \qquad (1)$$

In fact, since $B + C = \pi - A$, we have

$$\begin{aligned}
\cos^2 A &= \cos^2(B+C) = (\cos B \cos C - \sin B \sin C)^2 \\
&= \cos^2 B \cos^2 C + (1 - \cos^2 B)(1 - \cos^2 C) \\
&\qquad\qquad -2 \sin B \sin C \cos B \cos C \\
&= 1 - \cos^2 B - \cos^2 C \\
&\qquad\qquad +2 \cos B \cos C\,(\cos B \cos C - \sin B \sin C) \\
&= 1 - \cos^2 B - \cos^2 C + 2 \cos B \cos C \cos(B+C) \\
&= 1 - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C\,,
\end{aligned}$$

as claimed.

Now we prove the identity proposed. Taking into account (1) and the fact that $\cos A = \sin B \sin C - \cos B \cos C$ (cyclic), we have

$$\begin{aligned}\left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 &= \sum \sin^2 A - \left(1 + \sum \cos^2 A\right) \\ &= \sum (\sin^2 A - 1) + 2 - \sum \cos^2 A \\ &= 2\left(1 - \sum \cos^2 A\right) \\ &= 4 \cos A \cos B \cos C\,,\end{aligned}$$

where again all the above sums are cyclic.

Also solved by Robert Bilinski, Outremont, QC.

This month, Andrei Ismail wins a copy of GrafEq from Pedagoguery Software. Congratulations Andrei! Keep sending your problems and solutions.

Pólya's Paragon

Paul Ottaway

Last September, I began this column with a short discussion about how mathematics is beautiful and how this beauty can arise from the variety of ways of proving mathematical statements. I will start this new school year with a cute proof that not only are numbers beautiful, but they are also interesting!

Theorem 1. All positive integers are interesting.

Proof. First, we suppose for a contradiction that there exists at least one uninteresting positive integer. Then, among all such integers, one of them must be the smallest. Now this integer is very interesting indeed, since it is the smallest one with no other interesting properties! This contradicts our assumption. We must conclude that all positive integers are interesting. □

We have not really proved anything here, because the term 'interesting' has not been defined. Nevertheless, this is a wonderful example of how a "proof by contradiction" works. Basically, you assume the negation of what you are trying to prove and show that this assumption leads to a contradiction. This shows that the assumption is false, which then implies that the original statement is true. More generally, this is known as an "indirect proof".

The next theorem I would like to consider involves a number which I find quite interesting. The proof is one of the most famous examples of proof by contradiction.

Theorem. $\sqrt{2}$ is irrational.

Proof. Assume for a contradiction that $\sqrt{2}$ is rational. Then, we can write $\sqrt{2}$ as a/b where a, b are integers with no common divisor greater than 1 and $b \neq 0$ (this is the definition of a rational number). Now we do a little algebra:

$$\begin{aligned} \sqrt{2} &= \frac{a}{b} \\ 2 &= \frac{a^2}{b^2} \\ 2b^2 &= a^2 \end{aligned}$$

This shows that a^2 is even. Now, since an odd number squared is again odd, we must conclude that a is even; that is, $a = 2n$ for some other integer n. Substituting for a above, we get the following:

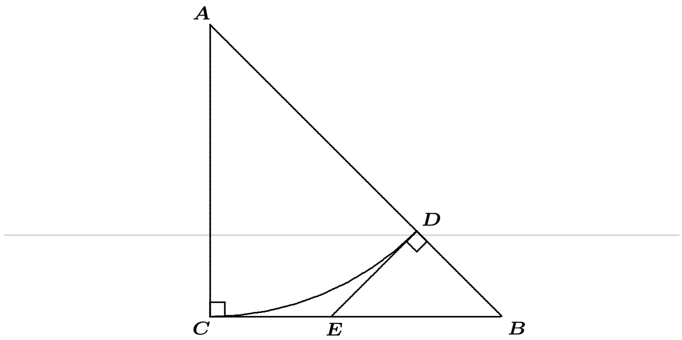
$$\begin{aligned} 2b^2 &= (2n)^2 \\ 2b^2 &= 4n^2 \\ b^2 &= 2n^2 \end{aligned}$$

This means that b^2 is also an even number which, by the same argument as above, shows that b is even. We have now shown that a and b are both even and hence, both divisible by 2. This contradicts the assumption that a and b have no common divisor greater than 1. Therefore, without any further work we must conclude that $\sqrt{2}$ is not a rational number. This, of course, implies that it must be irrational, as required. □

The fact that $\sqrt{2}$ is irrational was known to the Greeks many years before mathematicians used variables to represent unknown quantities. We might wonder how they ever went about proving such a result without expressing the results algebraically. Here is an alternate geometric proof that does not need variables. It is again a proof by contradiction.

Proof. We know by the Pythagorean Theorem that we can construct a right angled triangle with side lengths 1, 1 and $\sqrt{2}$. If we assume that $\sqrt{2}$ is rational, then we can find a similar triangle that has all integer side lengths (you can multiply all the sides by the denominator b if $\sqrt{2} = a/b$). In particular, we choose the smallest such triangle ABC with integer lengths which is similar to the original.

Now we make the following constructions: Draw a circle centred at A with radius equal to one of the shorter sides. This intersects the hypotenuse at D. Construct a line from D perpendicular to the hypotenuse which meets BC at E.



We need only make arguments about which sides in the diagram must have integer lengths. First, $AD = AC$; thus, AD is an integer. Then we know that $DB = AB - AD$ is also an integer, since it is the difference of two integers. Since $\angle DBE = 45°$ and $\angle EDB = 90°$, we know $\angle DEB = 45°$, which means that $DB = DE$. Therefore, DE has integral length as well. Also, DE and CE are tangents to the circle centred at A; hence, they are the same length. Thus, CE has integral length. Finally, we can now see that $EB = CB - CE$ is an integer, since its length can be expressed as the difference of two integers. Hence, $\triangle BDE$ is similar to $\triangle ACB$. We have shown that $\triangle BDE$ has integer side lengths. This contradicts our assumption that $\triangle ABC$ is the smallest such triangle. We must therefore conclude that $\sqrt{2}$ is irrational. □

Clearly, this is not quite as elegant as the first proof. Nevertheless, it is nice to know that such things can be proven without the algebraic tools that we have come to depend on.

Problems for you to try:

There are many other problems that can be solved using an indirect proof. Here are three more problems whose proofs lend themselves to this technique.

1. Prove that if r is an irrational number, and if a and b are rational numbers with $b \neq 0$, then the expression $a + br$ represents an irrational number.

2. Prove that there are infinitely many primes. This is another famous use of indirect proof. [*Hint:* Begin by assuming that there are only finitely many. What can you say about the number you get when you multiply them all together and add 1?]

3. Prove that e is an irrational number. (Note: this is quite hard. You may use the fact that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.)

Binomial Inversion: Two Proofs and an Application to Derangements

Heba Hathout

There is an old problem that goes by many names but generally runs something like this:

> A group of n men enter a restaurant and check their hats. The hat-checker is absent-minded and distributes the hats back to the men at random when they leave. What is the probability, P_n, that no man gets his own hat back, and how does P_n behave as n approaches infinity?

The answer is that

$$P_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} = \sum_{j=0}^{n} (-1)^j \frac{1}{j!}, \qquad (1)$$

which approaches $1/e$ as n approaches infinity.

This problem is usually tackled using the inclusion-exclusion principle. It can also be solved by developing a recursive relationship among P_n, P_{n-1}, and P_{n-2}. This paper introduces a different approach, using a technique called *binomial inversion*.

If a sequence of numbers b_0, b_1, b_2, ... is defined in terms of another sequence of numbers a_0, a_1, a_2, ... by the formula

$$b_k = \sum_{i=0}^{k} \binom{k}{i} a_i, \qquad (2)$$

then this relationship can be inverted and the numbers a_i retrieved by the following formula, called the *Binomial Inversion Formula*:

$$a_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} b_i. \qquad (3)$$

We will first use the formula (3) to solve our problem. Then we will derive the formula by two different methods, the first using infinite series and the second using linear algebra.

Solution using Binomial Inversion

The set of all possible permutations of the hats can be divided into subsets as follows: a subset S_0 comprised of the permutations where none of the men gets his own hat back, a subset S_1 consisting of the permutations where just one of the men gets his own hat, and so on, up to a subset S_n consisting of permutations where all of the men get their own hats.

Consider the subset S_2, for example. If the two men who get the correct hats are man # 1 and man # 2, then the number of possible arrangements is D_{n-2}, the number of derangements for the other $n-2$ hats. Since there are $\binom{n}{2}$ possibilities for the pair of men who get their own hats, the number of permutations in the set S_2 is $|S_2| = \binom{n}{2}D_{n-2}$.

Applying the same logic to each subset S_i, we obtain $|S_i| = \binom{n}{i}D_{n-i}$. The total number of permutations of the hats is

$$\begin{aligned} n! &= |S_0| + |S_1| + |S_2| + \cdots + |S_n| \\ &= \sum_{i=0}^{n}\binom{n}{i}D_{n-i} = \sum_{j=0}^{n}\binom{n}{n-j}D_j = \sum_{j=0}^{n}\binom{n}{j}D_j\,, \end{aligned}$$

where we have changed the index of summation from i to $j = n - i$ and then used the fact that $\binom{n}{n-j} = \dfrac{n!}{j!(n-j)!} = \binom{n}{j}$.

Now we use binomial inversion (with $a_n = D_n$ and $b_n = n!$):

$$\begin{aligned} D_n &= \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}i! = \sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!(n-i)!}\,i! \\ &= n!\sum_{i=0}^{n}(-1)^{n-i}\frac{1}{(n-i)!} = n!\sum_{j=0}^{n}(-1)^j\frac{1}{j!}\,. \end{aligned}$$

Finally, since $P_n = D_n/n!$, we obtain (1).

Binomial Inversion using Infinite Series

We will derive the inversion formula (3) from formula (2) using infinite series. However, since the issue of convergence of the series will be ignored, the argument that we will give here is not completely rigorous.

We introduce exponential generating functions as follows:

$$A(x) = \sum_{k=0}^{\infty}\frac{a_k}{k!}\,x^k\,, \qquad B(x) = \sum_{k=0}^{\infty}\frac{b_k}{k!}\,x^k\,.$$

Substituting for b_k from (2), we get

$$B(x) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}a_i\right)\frac{1}{k!}\,x^k = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{a_i}{i!(k-i)!}\,x^k\,.$$

Interchanging the order of summation and simplifying,

$$\begin{aligned} B(x) &= \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \frac{a_i}{i!(k-i)!}\, x^k \;=\; \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \left(\frac{a_i x^i}{i!}\right)\left(\frac{x^{k-i}}{(k-i)!}\right) \\ &= \sum_{i=0}^{\infty} \frac{a_i x^i}{i!}\left(\sum_{k=i}^{\infty} \frac{x^{k-i}}{(k-i)!}\right) \;=\; \left[\sum_{i=0}^{\infty} \frac{a_i}{i!} x^i\right]\left[\sum_{j=0}^{\infty} \frac{x^j}{j!}\right]. \end{aligned}$$

We recognize the sums in the last expression above as $A(x)$ and e^x, respectively. Therefore, we can now write $B(x) = A(x)e^x$, from which we get $A(x) = e^{-x}B(x)$. Then

$$A(x) = \left[\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}\right]\left[\sum_{i=0}^{\infty} \frac{b_i}{i!} x^i\right] = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^j b_i}{j!i!}\, x^{i+j}.$$

We can re-index (letting $k = i + j$) to get

$$A(x) = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{(-1)^{k-i} b_i}{(k-i)!i!}\, x^k = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} \binom{k}{i}(-1)^{k-i} b_i\right)\frac{1}{k!}\, x^k.$$

The coefficient of x_k in this formula must be the same as the coefficient $a_k/k!$ in the initial formula defining $A(x)$. Thus, we obtain (3).

Binomial Inversion using Linear Algebra

We now assume that the sequences related by the formula (2) are *finite* sequences $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$. (This case is all that we needed for our problem of the hats.) Then (2) can be recast into matrix form as

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

On the right-hand side is an $n+1$ by $n+1$ matrix T whose rows are the first $n+1$ rows of Pascal's triangle, filled in with zeroes. The entry in row i and column j is $T(i,j) = \binom{i}{j}$, where i and j go from 0 to n (rather than the conventional numbering starting at 1). The inversion formula (3) states that the matrix T has an inverse X whose entries are $X(i,j) = \binom{i}{j}(-1)^{i-j}$. This is what we will prove.

Let $M = TX$. The goal is to show that $M = I$, the identity matrix. We must show that the diagonal elements of M are 1 and the off-diagonal elements are 0.

The entry in row a and column b of M is

$$M(a,b) \;=\; \sum_{i=0}^{n} T(a,i)X(i,b) \;=\; \sum_{i=0}^{n} \binom{a}{i}\binom{i}{b}(-1)^{i-b}\,. \tag{4}$$

If $i > a$, we have $\binom{a}{i} = 0$, while if $i < b$, then $\binom{i}{b} = 0$. Thus, some terms in the above sum are zero, in general. In particular, we see that $M(a,b) = 0$ when $a < b$.

The diagonal entries of M occur when $a = b$. In this case, the only non-zero term in the sum in (4) is for $i = a$. We have

$$M(a,a) \;=\; \binom{a}{a}\binom{a}{a}(-1)^{a-a} \;=\; 1\,.$$

Thus, the diagonal elements of M are 1, as desired.

Now consider $a > b$. Then (4) becomes

$$M(a,b) \;=\; \sum_{i=b}^{a} \binom{a}{i}\binom{i}{b}(-1)^{i-b} \;=\; \sum_{i=b}^{a} \frac{a!}{(a-i)!(i-b)!b!}(-1)^{i-b}\,,$$

which is the same as

$$\frac{b!}{a!}M(a,b) \;=\; \sum_{i=b}^{a} \frac{1}{(a-i)!(i-b)!}(-1)^{i-b}\,.$$

Now, we let $j = i - b$ (and hence $i = j + b$). Then our equation is

$$\frac{b!}{a!}M(a,b) \;=\; \sum_{j=0}^{a-b} \frac{1}{(a-b-j)!j!}(-1)^{j}\,.$$

Finally, we let $m = a - b$ and multiply both sides by $m!$ to get

$$m!\frac{b!}{a!}M(a,b) \;=\; \sum_{j=0}^{m} \frac{m!}{(m-j)!j!}(-1)^{j} \;=\; \sum_{j=0}^{m} \binom{m}{j}(-1)^{j}\,.$$

The right-hand side equals 0 by a well-known combinatorics identity, because it represents $(x+y)^m$, when $x = 1$ and $y = -1$. Therefore, $M(a,b)$ is zero, and the proof is complete.

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THE OLYMPIAD CORNER

No. 231

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Last number we gave ten problems from the shortlist for the 2000 International Mathematical Olympiad. We start this number with the remaining 11 problems of the shortlist. My thanks go to Andy Liu, Canadian Team Leader to the IMO in Seoul, for collecting them.

2000 INTERNATIONAL MATHEMATICAL OLYMPIAD

Shortlisted Problems

11. (*Argentina*) Let $ABCD$ be a convex quadrilateral with AB not parallel to CD, and let Y be the point of intersection of the perpendicular bisectors of AB and CD. If X is a point inside $ABCD$ such that $\angle ADX = \angle BCX < 90^\circ$ and $\angle DAX = \angle CBX < 90^\circ$, prove that $\angle AYB = 2\angle ADX$.

12. (*Belarus*) Find all pairs of functions f and g from the set of real numbers to itself such that $f(x+g(y)) = xf(y) - yf(x) + g(x)$ for all real numbers x and y.

13. (*India*) Let O be the circumcentre and H the orthocentre of an acute triangle ABC. Prove that there exist points D, E, and F on sides BC, CA, and AB, respectively, such that $OD + DH = OE + EH = OF + FH$ and the lines AD, BE, and CF are concurrent.

14. (*Iran*) Ten gangsters are standing on a flat surface. The distances between them are all distinct. Simultaneously each of them shoots at the one among the other nine who is the nearest. At least how many gangsters will be shot at?

15. (*Ireland*) A non-empty set A of real numbers is called a B_3–set if the conditions a_1, a_2, a_3, a_4, a_5, $a_6 \in A$ and $a_1 + a_2 + a_3 = a_4 + a_5 + a_6$ imply that the sequences (a_1, a_2, a_3) and (a_4, a_5, a_6) are identical up to a permutation. For a set X of real numbers, let $D(X)$ denote the difference set $\{|x - y| : x, y \in X\}$. Prove that if $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ and $B = \{0 = b_0 < b_1 < b_2 < \cdots\}$ are infinite sequences of real numbers with $D(A) = D(B)$, and if A is a B_3–set, then $A = B$.

16. (*The Netherlands*) In the plane we are given two circles intersecting at X and Y. Prove that there exist four points such that for every circle touching the two given circles at A and B, and meeting the line XY at C and D, each of the lines AC, AD, BC, and BD passes through one of those four points.

17. (*Russia*) For a polynomial P with distinct real coefficients, let $M(P)$ be the set of all polynomials that can be obtained from P by permuting its coefficients. Find all integers n for which there exists a polynomial P of degree 2000 with distinct real coefficients such that $P(n) = 0$ and we can get from any $Q \in M(P)$ a polynomial Q' such that $Q'(n) = 0$ by interchanging at most one pair of coefficients of Q.

18. (*Russia*) Let $A_1A_2 \ldots A_n$ be a convex polygon, $n \geq 4$. Prove that $A_1A_2 \ldots A_n$ is cyclic if and only if each vertex A_i can be assigned a pair (b_i, c_i) of real numbers so that $A_iA_j = b_jc_i - b_ic_j$ for all i and j with $1 \leq i < j \leq n$.

19. (*United Kingdom*) Let a, b, and c be positive integers such that $c > 2b > 4a$. Prove that there exists a real number λ such that the three numbers λa, λb, and λc all have their fractional parts in the interval $\left(\frac{1}{3}, \frac{2}{3}\right]$.

20. (*United Kingdom*) A function F is defined from the set of non-negative integers to itself such that, for every non-negative integer n, $F(4n) = F(2n) + F(n)$, $F(4n + 2) = F(4n) + 1$, and $F(2n + 1) = F(2n) + 1$. Prove that, for each positive integer m, the number of integers n with $0 \leq n < 2^m$ and $F(4n) = F(3n)$ is $F(2^{m+1})$.

21. (*United Kingdom*) The tangents at B and A to the circumcircle of an acute triangle ABC meet the tangent at C at T and U, respectively. The lines AT and BC meet at P, and Q is the mid-point of AP; the lines BU and CA meet at R, and S is the mid-point of BR.

(a) Prove that $\angle ABQ = \angle BAS$.

(b) Determine, in terms of ratios of side lengths, the triangles for which this angle is a maximum.

Next we turn to solutions by our readers to problems of the Vietnamese Mathematical Competition 1997 [2001 : 167].

1. In a plane, let there be given a circle with centre O, with radius R and a point P inside the circle, $OP = d < R$. Among all convex quadrilaterals $ABCD$, inscribed in the circle such that their diagonals AC and BD cut each other orthogonally at P, determine the ones which have the greatest perimeter and the ones which have the smallest perimeter. Calculate these perimeters in terms of R and d.

Solved by Mohammed Aassila, Strasbourg, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Aassila's solution, adapted by the editors.

Let $ABCD$ be a quadrilateral satisfying the given conditions, and let p denote its perimeter. Then

$$\begin{aligned} p^2 &= (AB+BC+CD+DA)^2 \\ &= AB^2+CD^2+BC^2+DA^2+2(AB\cdot CD+AD\cdot BC) \\ &\qquad +2(AB\cdot AD+CB\cdot CD)+2(BA\cdot BC+DA\cdot DC)\,. \end{aligned}$$

Now

$$AB^2+CD^2 \;=\; BC^2+DA^2 \;=\; 4R^2\,. \tag{1}$$

Ptolemy's Theorem gives us

$$AB\cdot CD+AD\cdot BC \;=\; AC\cdot BD\,.$$

Also, with some work one obtains $AC^2+BD^2 = 8R^2-4d^2$. Hence,

$$2AC\cdot BD \;=\; (AC+BD)^2-8R^2+4d^2\,. \tag{2}$$

Thus,

$$2(AB\cdot CD+AD\cdot BC) \;=\; (AC+BD)^2-8R^2+4d^2\,. \tag{3}$$

Furthermore,

$$2(AB\cdot AD+CB\cdot CD) \;=\; 4R\cdot AC\,, \tag{4}$$

and

$$2(BA\cdot BC+DA\cdot DC) \;=\; 4R\cdot BD\,. \tag{5}$$

Using (1), (3), (4), and (5) in our expression for p^2, we get

$$p^2 \;=\; (AC+BD)^2+4R(AC+BD)+4d^2\,.$$

Consequently, the maximum (respectively, minimum) of p corresponds to the maximum (respectively, minimum) of $AC+BD$, which, in view of (2), corresponds to the maximum (respectively, minimum) of $AC\cdot BD$. Noting that

$$2AC\cdot BD \;=\; 8R^2-4d^2-(AC-BD)^2\,,$$

we conclude that the maximum (respectively, minimum) of p corresponds to the minimum (respectively, maximum) of $|AC-BD|$. It follows that p is maximized when $AC = BD$, and p is minimized when $AC = 2R$ and $BD = 2\sqrt{R^2-d^2}$ (the maximum and minimum possible lengths for a chord through P). Hence,

$$p^2_{\max} \;=\; 16R^2-4d^2+8R\sqrt{4R^2-2d^2}\,,$$

and

$$p^2_{\min} \;=\; 16R^2+16R\sqrt{R^2-d^2}\,.$$

Editor's note: We can then obtain the following expressions for $p_{\max}$ and $p_{\min}$:

$$p_{\max} = 2\left(\sqrt{2}R + \sqrt{2R^2 - d^2}\right) = \left(\sqrt{\sqrt{2}R - d} + \sqrt{\sqrt{2}R + d}\right)^2,$$
$$p_{\min} = 2\sqrt{2R}\left(\sqrt{R+d} + \sqrt{R-d}\right).$$

2. Let there be given a whole number $n > 1$, not divisible by 1997. Consider two sequences of numbers $\{a_i\}$ and $\{b_j\}$ defined by:

$$a_i = i + \frac{ni}{1997} \quad (i = 1, 2, 3, \ldots, 1996),$$
$$b_j = j + \frac{1997j}{n} \quad (j = 1, 2, 3, \ldots, n-1).$$

By arranging the numbers of these two sequences in increasing order, we get the sequence $c_1 \leq c_2 \leq \cdots \leq c_{1995+n}$.

Prove that $c_{k+1} - c_k < 2$ for every $k = 1, 2, \ldots, 1994 + n$.

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Pontoise, France. We give the solution of Bornsztein.

First note that $\{a_i\}$ and $\{b_j\}$ are two increasing arithmetical sequences, with difference $\alpha = 1 + \frac{n}{1997}$ and $\beta = 1 + \frac{1997}{n}$, respectively.

Let $i \in \{1, \ldots, 1996\}$ and $j \in \{1, \ldots, n-1\}$, and suppose that $a_i = b_j$. Then $ni = 1997j$. Since 1997 is prime and $n \not\equiv 0 \pmod{1997}$, we deduce that $\gcd(n, 1997) = 1$. From Gauss's Theorem, we then have $i \equiv 0 \pmod{1997}$, which is impossible. It follows that $a_i \neq b_j$.

First Case. $n < 1997$.

We easily see that

$$\alpha < 2 < \beta, \tag{1}$$

$$\text{and} \quad a_1 < b_1, \tag{2}$$

which implies that $c_1 = a_1$. Moreover, $\frac{a_{1996}}{b_{n-1}} = \frac{1996n}{1997(n-1)} > 1$. Then,

$$b_{n-1} < a_{1996}, \tag{3}$$

which implies that $c_{1995+n} = a_{1996}$.

Lemma. For every $j \in \{1, \ldots, n-2\}$, there exists $i \in \{1, \ldots, 1996\}$ such that $b_j < a_i < b_{j+1}$.

Proof. Suppose, for the purpose of contradiction, that there exists $j \in \{1, \ldots, n-2\}$ such that the interval $[b_j, b_{j+1}]$ does not contain any of the a_i's. Let p be the greatest index such that $a_p < b_j$ (such a p does exist, since $a_1 < b_1 \le b_j$). Then $p < 1996$ (since $b_j < b_{n-1} < a_{1996}$), and $a_p < b_j < b_{j+1} < a_{p+1}$. It follows that $\alpha = a_{p+1} - a_p > b_{j+1} - b_j = \beta$, which contradicts (1). Thus, the lemma is proved.

It follows from the lemma that, for every $k \in \{1, \ldots, 1994 + n\}$, we are in one of the three following cases:

(a) $c_k = a_i$ and $c_{k+1} = a_{i+1}$ for some i. Then $c_{k+1} - c_k = \alpha < 2$.

(b) $c_k = a_i$ and $c_{k+1} = b_j$ for some $i < 1996$ (from (3)) and some $j \le n-1$. Then $b_j < a_{i+1}$ and $c_{k+1} - c_k < a_{i+1} - a_i = \alpha < 2$.

(c) $c_k = b_j$ and $c_{k+1} = a_i$ for some $i > 1$ (from (2)) and some $j \le n-1$. Then $a_{i-1} < b_j$ and $c_{k+1} - c_k < a_i - a_{i-1} = \alpha < 2$.

In each case, we have $c_{k+1} - c_k < 2$, as desired.

Second Case. $n > 1997$.

This case is essentially the same as the first case: simply interchange n with 1997 and a with b (and also α with β).

3. How many functions $f : \mathbb{N}^* \to \mathbb{N}^*$ are there that simultaneously satisfy the two following conditions:

(i) $f(1) = 1$,

(ii) $f(n) \cdot f(n+2) = \left(f(n+1)\right)^2 + 1997$ for all $n \in \mathbb{N}^*$?

($\mathbb{N}^*$ denotes the set of all positive integers.)

Solved by Mohammed Aassila, Strasbourg, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Athanasias Kalakos, Athens, Greece. We give the solution and comments of Kalakos.

This is, essentially, one of the problems of the 3rd Balkan Mathematical Olympiad (1986). It was proposed by Bulgaria. Here we give it as a lemma with the same proof (slightly modified by the editors) that was given after the competition by the teams that had participated.

Lemma. A sequence is defined by $a_1 = a$, $a_2 = b$, and

$$a_{n+2} = \frac{a_{n+1}^2 + c}{a_n}, \quad n = 1, 2, \ldots,$$

where a, b, c are real numbers and $c > 0$. Then all a_n $(n \ge 1)$ are integers if and only if a, b, and $\dfrac{a^2 + b^2 + c}{ab}$ are integers.

Proof: If $a = 0$, then a_3 is not defined. Thus, $a \ne 0$. Similarly, if $b = 0$, then a_4 is not defined. Thus, $b \ne 0$. It follows inductively that $a_n \ne 0$, for all $n \ge 1$. More precisely, every term exists and is non-zero.

By the recurrence we find, for all $n \ge 2$,

$$a_{n+2}a_n - a_{n+1}^2 = c = a_{n+1}a_{n-1} - a_n^2,$$

and hence,

$$\frac{a_{n+2}+a_n}{a_{n+1}} = \frac{a_{n+1}+a_{n-1}}{a_n}\,.$$

Therefore, for all $n \geq 1$, we have

$$\frac{a_{n+2}+a_n}{a_{n+1}} = \frac{a_3+a_1}{a_2} = \frac{\frac{b^2+c}{a}+a}{b} = \frac{a^2+b^2+c}{ab}\,,$$

and hence,

$$a_{n+2} = \frac{a^2+b^2+c}{ab}\cdot a_{n+1} - a_n\,.$$

If a, b, $\frac{a^2+b^2+c}{ab}$ are integers, then an easy induction shows that a_n is an integer for all $n \geq 1$.

Conversely, assume that $a_n \in \mathbb{Z}$, for all $n \geq 1$. Then a_1, $a_2 \in \mathbb{Z}$ implies that a, $b \in \mathbb{Z}$. Moreover, we have $c \in \mathbb{Z}$, since $c = aa_3 - b^2$. Therefore, $\frac{a^2+b^2+c}{ab}$ is rational. Write $\frac{a^2+b^2+c}{ab} = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$ and $\gcd(p,q) = 1$. For $s \in \mathbb{N}^*$, we will prove inductively on s the following proposition $P(s)$: $q^s \mid a_n$, for all $n \geq s+1$.

For all $k \geq 1$, the recurrence $a_{k+2} = \frac{p}{q}\cdot a_{k+1} - a_k$ gives $\frac{pa_{k+1}}{q} = a_{k+2} + a_k$, and hence, $q \mid (pa_{k+1})$. Since $\gcd(p,q) = 1$, it follows that $q \mid a_{k+1}$. Thus, $q \mid a_n$ for all $n \geq 2$, and $P(1)$ is true.

Suppose that $P(s)$ holds for some $s \geq 1$. Then $q^s \mid a_n$ for all $n \geq s+1$. Consider any $k \geq s+1$. Since $a_{k+2} = \frac{p}{q}a_{k+1} - a_k$, we have

$$\frac{a_{k+2}+a_k}{q^s} = \frac{pa_{k+1}}{q^{s+1}}\,.$$

By the induction hypothesis, $q^s \mid a_k$ and $q^s \mid a_{k+2}$; hence, $q^s \mid (a_{k+2}+a_k)$. Therefore, $q^{s+1} \mid (pa_{k+1})$. Since $\gcd(p, q^{s+1}) = 1$, we obtain $q^{s+1} \mid a_{k+1}$. Thus, $q^{s+1} \mid a_n$, for all $n \geq s+2$. This proves $P(s+1)$ and completes the induction.

Now let $s \geq 1$ be arbitrary. We have $c = a_{n+2}a_n - a_{n+1}^2$. For $n = s+1$, this yields $c = a_{s+3}a_{s+1} - a_{s+2}^2$. Using $P(s)$,

$$\begin{aligned} q^s \mid a_{s+1}\,,\quad q^s \mid a_{s+3}\,,\quad q^s \mid a_{s+2} &\implies q^{2s} \mid (a_{s+3}a_{s+1} - a_{s+2}^2) \\ &\implies q^{2s} \mid c\,. \end{aligned}$$

Therefore,

$$q^{2s} \leq c \quad \text{for all } s \geq 1\,. \tag{1}$$

If we suppose $q > 1$, then $\lim\limits_{s \to +\infty} q^{2s} = +\infty$, which contradicts (1). Thus, $q = 1$ and $\dfrac{a^2 + b^2 + c}{ab}$ is an integer. The lemma is proved.

We turn now to the initial problem.

Suppose $f : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ is any function such that $f(1) = 1$ and

$$f(n+2)f(n) \;=\; (f(n+1))^2 + 1997\,, \qquad \text{for all } n \in \mathbb{N}^*\,.$$

Let $b = f(2)$. Since $f(n)$ is an integer for all n, we have (from the lemma) that $b \in \mathbb{Z}$ and $\dfrac{1^2 + b^2 + 1997}{1 \cdot b} \in \mathbb{Z}$. Thus, $\dfrac{b^2 + 1998}{b} \in \mathbb{Z}$. Then $\dfrac{1998}{b} \in \mathbb{Z}$, and therefore, $b \mid 1998$. Thus, $f(2) = b$ is a positive divisor of 1998.

Conversely, let b be a positive divisor of 1998. Define $f : \mathbb{N}^* \longrightarrow \mathbb{R}$ by $f(1) = 1$, $f(2) = b$, and $f(n+2) \cdot f(n) = (f(n+1))^2 + 1997$. Since $f(1) \neq 0$ and $f(2) \neq 0$, each $f(n)$ exists and is non-zero (as in the proof of the lemma). Now $b \in \mathbb{Z}$, and

$$\frac{1^2 + b^2 + 1997}{1 \cdot b} \;=\; \frac{b^2 + 1998}{b} \;=\; b + \frac{1998}{b}$$

is an integer, since $b \mid 1998$. By the lemma, $f(n)$ is an integer for all $n \in \mathbb{N}^*$. Thus, $f : \mathbb{N}^* \longrightarrow \mathbb{Z}^*$. An easy induction shows that $f(n) > 0$ for every n, since $f(1) > 0$ and $f(2) > 0$. Thus, $f : \mathbb{N}^* \longrightarrow \mathbb{N}^*$, and we obtain an admissible sequence.

The above discussion reveals that the number of functions that satisfy both conditions (i) and (ii) in the problem is the same as the number of positive divisors of 1998. Since $1998 = 2 \cdot 3^3 \cdot 37$, the number of positive divisors of 1998 is $(1+1) \cdot (3+1) \cdot (1+1) = 16$. (It is known that the number of divisors of $p_1^{a_1} \cdots p_r^{a_r}$ is $(a_1 + 1) \cdots (a_r + 1)$.)

4. (a) Find all polynomials of least degree, with rational coefficients, such that

$$f(\sqrt[3]{3} + \sqrt[3]{9}) \;=\; 3 + \sqrt[3]{3}\,.$$

(b) Does there exist a polynomial with integer coefficients such that

$$f(\sqrt[3]{3} + \sqrt[3]{9}) \;=\; 3 + \sqrt[3]{3}\,?$$

Solved by Pierre Bornsztein, Pontoise, France.

(a) Let $a = 3^{1/3}$.

Lemma. If α, β, $\gamma \in \mathbb{Q}$ such that $\alpha a^2 + \beta a + \gamma = 0$, then $\alpha = \beta = \gamma = 0$.

Proof. Let α, β, $\gamma \in \mathbb{Q}$ such that

$$\alpha a^2 + \beta a + \gamma \;=\; 0\,. \tag{1}$$

Suppose that $\alpha \neq 0$. Since a is a root of the quadratic $\alpha x^2 + \beta x + \gamma = 0$, we must have $a = \dfrac{-\beta \pm \sqrt{\Delta}}{2\alpha}$, where $\Delta = \beta^2 - 4\alpha\gamma \geq 0$. Note that $\sqrt{\Delta} \notin \mathbb{Q}$ (since $a \notin \mathbb{Q}$). Then

$$-24\alpha^3 \;=\; \beta^3 + 3\beta\Delta \pm \sqrt{\Delta}(3\beta^2 + \Delta)\,.$$

It follows that $3\beta^2 + \Delta = 0$, which leads to $\beta = \Delta = 0$. Then we get $\alpha = 0$, a contradiction. Therefore, $\alpha = 0$, and equation (1) becomes $\beta a + \gamma = 0$. Since $a \notin \mathbb{Q}$, we deduce that $\beta = \gamma = 0$, and the lemma is proved.

Let $f \in \mathbb{Q}[x]$ such that $f(a + a^2) = a + 3$. If f has degree 1, then $f(x) = \alpha x + \beta$ for some α, $\beta \in \mathbb{Q}$. Then, $\alpha(a + a^2) + \beta = a + 3$. Then the lemma implies that $\alpha = 0$ and $\alpha = 1$, which is clearly impossible. Therefore, f cannot have degree 1. If f has degree 2, then $f(x) = \alpha x^2 + \beta x + \gamma$ for some α, β, $\gamma \in \mathbb{Q}$. Then, from the lemma, $f(a + a^2) = a + 3$ is equivalent to

$$\begin{aligned} \alpha + \beta \quad\;\; &= 0 \\ 3\alpha + \beta \quad\;\; &= 1 \\ 6\alpha \qquad + \gamma &= 3\,. \end{aligned}$$

The unique solution of this system is $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, and $\gamma = 0$. It follows that there is a unique polynomial f of least degree having rational coefficients such that $f(\sqrt[3]{3} + \sqrt[3]{9}) = 3 + \sqrt[3]{3}$, namely $f(x) = \frac{1}{2}x^2 - \frac{1}{2}x$.

(b) The answer is no.

Suppose, for the purpose of contradiction, that there exists $P \in \mathbb{Z}[x]$ such that $P(a + a^2) = a + 3$. Let $P(x) = \sum_{i=0}^{n} \alpha_i x^i$, where $\alpha_i \in \mathbb{Z}$ for each i. We must have $n \geq 3$, in view of our solution to (a). Note that $(a + a^2)^3 = 12 + 9(a + a^2)$. Then

$$P(a + a^2) \;=\; \sum_{i=0}^{2} \alpha_i (a + a^2)^i + \big(12 + 9(a + a^2)\big) \sum_{i=3}^{n} \alpha_i (a + a^2)^{i-3}\,.$$

It follows that the polynomial

$$Q(x) \;=\; \sum_{i=0}^{2} \alpha_i x^i + (12 + 9x) \sum_{i=3}^{n} \alpha_i x^{i-3}$$

satisfies $Q(a + a^2) = P(a + a^2) = a + 3$, where $Q \in \mathbb{Z}[x]$ with $\deg Q(x) = \deg P(x) - 2$. Now we can apply the same reasoning to Q in place of P. An easy induction leads to a polynomial R of degree at most 2, with integer coefficients, which satisfies $R(a + a^2) = a + 3$. From (a) we must have $R(x) = \frac{1}{2}x^2 - \frac{1}{2}x$, which does not have integer coefficients. This is a contradiction. The conclusion follows.

5. Prove that, for every positive integer n, there exists a positive integer k such that $19^k - 97$ is divisible by 2^n.

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Maragoudakis.

We will define by induction a sequence of positive integers $\{k_n\}_{n=1}^{\infty}$ such that $2^n \mid (19^{k_n} - 97)$, for all $n \in \mathbb{N}^*$.

Since $19^8 - 97 \equiv 0 \pmod{64}$, we let $k_1 = k_2 = \cdots = k_6 = 8$. For $n \geq 6$, suppose that $k_n \in \mathbb{N}^*$ satisfies $2^n \mid (19^{k_n} - 97)$. Define

$$k_{n+1} = \begin{cases} k_n & \text{if } 2^{n+1} \mid (19^{k_n} - 97), \\ k_n(2^{n-5} + 1) & \text{if } 2^{n+1} \nmid (19^{k_n} - 97). \end{cases}$$

We will prove that $2^{n+1} \mid (19^{k_{n+1}} - 97)$. This is obvious if k_{n+1} is defined by the first case in the formula above. In the second case, since $2^n \mid (19^{k_n} - 97)$ and $2^{n+1} \nmid (19^{k_n} - 97)$, we must have $19^{k_n} - 97 = 2^n(2m+1)$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned} 19^{k_{n+1}} - 97 &= 19^{k_{n+1}} - 19^{k_n} + 19^{k_n} - 97 \\ &= \left(19^{k_n \cdot 2^{n-5}} - 1\right) 19^{k_n} + 2^n(2m+1)\,. \end{aligned}$$

Now, in order to prove that $2^{n+1} \mid (19^{k_{n+1}} - 97)$, it is enough to prove that $19^{k_n \cdot 2^{n-5}} - 1 = 2^n(2x+1)$, for some $x \in \mathbb{N}$. We start by factoring:

$$\begin{aligned} 19^{k_n \cdot 2^{n-5}} - 1 &= (19^{k_n} - 1) \cdot (19^{k_n} + 1) \cdot (19^{2k_n} + 1) \cdot \\ &\quad \cdot \left(19^{2^2 k_n} + 1\right) \cdots \left(19^{2^{n-6} k_n} + 1\right)\,. \end{aligned}$$

Since $2^n \mid (19^{k_n} - 97)$, where $n \geq 6$, we get that $32 \mid (19^{k_n} - 1)$ and $64 \nmid (19^{k_n} - 1)$. Also, for $v = 1, 2, \ldots$, we have $19^{vk_n} + 1 \equiv 0 \pmod 2$ and (since k_n is even) $19^{vk_n} + 1 \equiv 2 \pmod 4$. Therefore,

$$19^{k_n \cdot 2^{n-5}} - 1 = 2^5 \cdot \underbrace{2 \cdot 2 \cdots 2}_{n-5 \text{ factors}} \cdot (2x+1) = 2^n(2x+1)\,,$$

for some $x \in \mathbb{N}$.

We have proved that $2^{n+1} \mid (19^{k_{n+1}} - 97)$. The induction is complete.

We next turn to solutions by readers to problems of the Turkey Team Selection Examination for the 38th IMO 1997 [2001 : 168–169].

1. In a triangle ABC which has a right angle at A, let H denote the foot of the altitude belonging to the hypotenuse. Show that the sum of the radii of the incircles of the triangles ABC, ABH, and AHC is equal to $|AH|$.

Solved by Jean-Claude Andrieux, Beaune, France; Mohammed Aassila, Strasbourg, France; Houda Anoun, Bordeaux, France; Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece. We give the solution of Maragoudakis.

Triangles HBA, HAC, ABC are similar. Let r_1, r_2, r be the radii of the incircles of these triangles, respectively. Then

$$\frac{r_1}{AB} = \frac{r_2}{AC} = \frac{r}{BC} = \frac{r_1+r_2+r}{AB+AC+BC}\,.$$

Thus,

$$r_1+r_2+r = \frac{r(AB+AC+BC)}{BC} = \frac{2[ABC]}{BC} = \frac{BC\cdot AH}{BC} = AH\,.$$

2. The sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are defined through $a_1 = \alpha$, $b_1 = \beta$, and $a_{n+1} = \alpha a_n - \beta b_n$, $b_{n+1} = \beta a_n + \alpha b_n$ for all $n \geq 1$. How many pairs (α, β) of real numbers are there such that

$$a_{1997} = b_1 \quad\text{and}\quad b_{1997} = a_1\,?$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Athanasias Kalakos, Athens, Greece. We give the solution by Bataille.

There are 1999 such pairs, namely $(0,0)$ and the pairs $(\cos\theta_k, \sin\theta_k)$ where

$$\theta_k = \frac{\pi}{3996} + \frac{2k\pi}{1998} \quad (k = 0, 1, \dots, 1997)\,.$$

To prove this result, we remark that, for all $n \geq 1$, the complex number $a_{n+1} + ib_{n+1}$ is given by

$$a_{n+1} + ib_{n+1} = (\alpha + i\beta)(a_n + ib_n)\,,$$

so that $a_n + ib_n = (\alpha + i\beta)^n$ (since $a_1 + ib_1 = \alpha + i\beta$). We will have $a_{1997} = b_1$ and $b_{1997} = a_1$ if and only if $(\alpha + i\beta)^{1997} = i(\alpha - i\beta)$. Letting $z = \alpha + i\beta$, we have the equation $z^{1997} = i\bar{z}$, to be solved for $z \in \mathbb{C}$. An obvious solution is $z = 0$. Any non-zero solution z is necessarily of modulus 1, in which case $\bar{z} = 1/z$ and we have $z^{1998} = i$. Since the solutions of $z^{1998} = i$ are the 1998 complex numbers $\exp\left(i\left(\frac{\pi}{3996} + \frac{2k\pi}{1998}\right)\right)$ with $k = 0, 1, \dots, 1997$, we have the announced result.

4. The edge AE of a convex pentagon $ABCDE$ whose vertices lie on the unit circle passes through the centre of this circle. If $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DE| = d$ and $ab = cd = \frac{1}{4}$, compute $|AC| + |CE|$ in terms of a, b, c, d.

Solved by Athanasias Kalakos, Athens, Greece; and Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's write-up.

Let $|AC| = x$, $|CE| = y$, $|AD| = p$, $|BE| = q$. The angles ABE, ACE, ADE are each $90°$, so that $a^2 + q^2 = x^2 + y^2 = p^2 + d^2 = 4$. By Ptolemy's Theorem, $dx + 2c = py$; whence, $d^2x^2 + x + 4c^2 = p^2y^2$. Therefore,

$$\begin{aligned} x &= (4 - d^2)y^2 - 4c^2 - d^2x^2 = 4y^2 - 4c^2 - d^2(x^2 + y^2) \\ &= 4y^2 - 4c^2 - 4d^2 . \end{aligned}$$

Analogously, the relation $ay + 2b = qx$ leads to $y = 4x^2 - 4a^2 - 4b^2$.

Consequently, $x + y = 16 - 4(a^2 + b^2 + c^2 + d^2)$.

5. Prove that, for each prime number $p \geq 7$, there exists a positive integer n and integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ which are not divisible by p, such that

$$\begin{aligned} x_1^2 + y_1^2 &\equiv x_2^2 \pmod{p} , \\ x_2^2 + y_2^2 &\equiv x_3^2 \pmod{p} , \\ &\vdots \\ x_{n-1}^2 + y_{n-1}^2 &\equiv x_n^2 \pmod{p} , \\ x_n^2 + y_n^2 &\equiv x_1^2 \pmod{p} . \end{aligned}$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bradley's solution (modified by the editors).

We consider two cases:

$$p \equiv 1 \pmod{4} \qquad \text{or} \qquad p \equiv 3 \pmod{4} .$$

(a) $p \equiv 3 \pmod{4}$. Then $p = 4k + 3$ for some integer k (where $k > 0$ since $p \geq 7$), and we observe that

$$1^2 + k^2 \equiv (k + 2)^2 \pmod{p} . \tag{1}$$

Setting $x_1 = 1$, $y_1 = k$, and $x_2 = k + 2$, we have $x_1^2 + y_1^2 \equiv x_2^2 \pmod{p}$. Suppose now that we are given $x_i^2 + y_i^2 \equiv x_{i+1}^2 \pmod{p}$ for some $i \geq 1$. We will construct integers y_{i+1} and x_{i+2} such that $x_{i+1}^2 + y_{i+1}^2 \equiv x_{i+2}^2 \pmod{p}$. We first multiply (1) by x_{i+1}^2 to yield

$$x_{i+1}^2 + k^2x_{i+1}^2 \equiv (k + 2)x_{i+1}^2 \pmod{p} .$$

Then we choose $y_{i+1} \equiv kx_{i+1} \pmod{p}$ and $x_{i+2} \equiv (k + 2)x_{i+1} \pmod{p}$.

Since, for any prime p, there are a finite number of quadratic residues, eventually we will have $x_j \equiv x_i \pmod{p}$ for some $j > i$. We can then re-label x_i as x_1 and y_i as y_1, and begin the process there.

For example, if $p = 13$, then $p = 4k + 1$ for $k = 3$. We start with

$$1^2 + 9^2 \equiv 11^2 \pmod{13} ,$$

and proceed to get the following circuit:

$$\begin{aligned} 11^2 + 8^2 &\equiv 4^2 \pmod{13} , \\ 4^2 + 10^2 &\equiv 5^2 \pmod{13} , \\ 5^2 + 6^2 &\equiv 3^2 \pmod{13} , \\ 3^2 + 1^2 &\equiv 7^2 \pmod{13} , \\ 7^2 + 11^2 &\equiv 12^2 \equiv 1^2 \pmod{13} . \end{aligned}$$

(b) $p \equiv 1 \pmod 4$. Then $p = 4k + 1$ for some integer k (where $k > 1$ since $p \geq 7$), and we observe that

$$1^2 + (3k)^2 \equiv (3k + 2)^2 \pmod{p} . \qquad (2)$$

Our process is similar to part (a), only this time we multiply (2) by x_{i+1}^2 and choose $y_{i+1} \equiv 3kx_{i+1} \pmod{p}$ and $x_{i+2} \equiv (3k + 2)x_{i+1} \pmod{p}$.

6. Given an integer $n \geq 2$, find the minimal value of

$$\frac{x_1^5}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^5}{x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n^5}{x_1 + x_2 + \cdots + x_{n-1}}$$

subject to $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, where x_1, x_2, ..., x_n are positive real numbers.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Bornsztein.

More generally, we will find the minimal value of $\sum_{i=1}^{n} \left(\frac{x_i^p}{S - x_i^r} \right)$ subject to $x_1^s + x_2^s + \cdots + x_n^s = 1$, where $S = x_1^r + x_2^r + \cdots + x_n^r$ and p, r, s are positive real numbers such that $r \leq s \leq \frac{p}{2}$. The given problem is the special case $p = 5$, $r = 1$, $s = 2$.

Applying the Cauchy-Schwarz Inequality, we have

$$\left(\sum_{i=1}^{n} (S - x_i^r) \right) \left(\sum_{i=1}^{n} \frac{x_i^p}{S - x_i^r} \right) \geq \left(\sum_{i=1}^{n} x_i^{p/2} \right)^2 ,$$

with equality if and only if there exists a positive real number λ such that $S - x_i^r = \lambda x_i^{p/2}$.

By the Power-Mean Inequality (since $p/2 \geq s$),

$$\left(\sum_{i=1}^{n} x_i^{p/2}\right)^2 \;=\; n^2\left(\sum_{i=1}^{n} \frac{x_i^{p/2}}{n}\right)^2 \;\geq\; n^2\left(\sum_{i=1}^{n} \frac{x_i^s}{n}\right)^{p/s} \;=\; \frac{n^2}{n^{p/s}}\,,$$

with equality if and only if $p = 2s$ or $x_1 = x_2 = \cdots = x_n = \frac{1}{n^{1/s}}$. Also,

$$\begin{aligned}\sum_{i=1}^{n}(S - x_i^r) \;&=\; (n-1)\sum_{i=1}^{n} x_i^r \;=\; n(n-1)\sum_{i=1}^{n}\frac{x_i^r}{n} \\ &\leq\; n(n-1)\left(\sum_{i=1}^{n}\frac{x_i^s}{n}\right)^{r/s} \;=\; \frac{n(n-1)}{n^{r/s}}\,,\end{aligned}$$

by the Power-Mean Inequality (since $s \geq r$). Here, equality occurs if and only if $r = s$ or $x_1 = x_2 = \cdots = x_n = \frac{1}{n^{1/s}}$. Since $\sum_{i=1}^{n}(S - x_i^r) > 0$, we have

$$\sum_{i=1}^{n}\frac{x_i^p}{S - x_i^r} \;\geq\; \frac{\left(\sum_{i=1}^{n} x_i^{p/2}\right)^2}{\sum_{i=1}^{n}(S - x_i^r)} \;\geq\; \frac{\dfrac{n^2}{n^{p/s}}}{\dfrac{n(n-1)}{n^{r/s}}} \;=\; \frac{n^{(r+s-p)/s}}{n-1}\,.$$

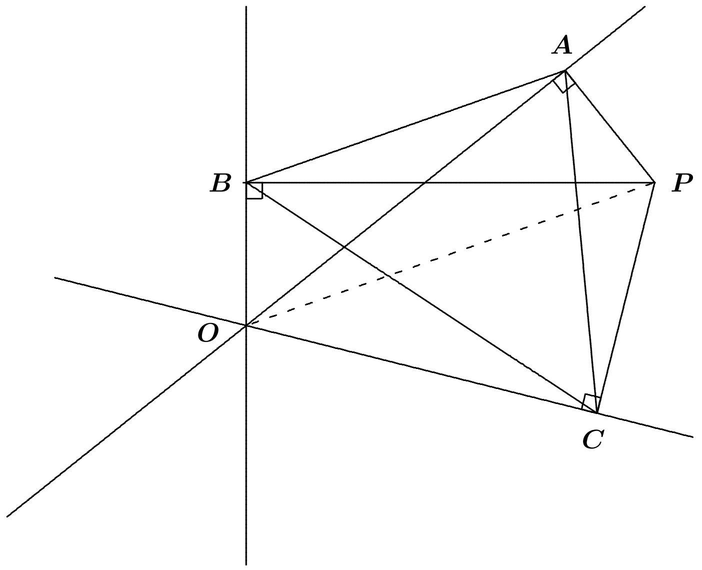
Equality occurs when $x_1 = x_2 = \cdots = x_n = \frac{1}{n^{1/s}}$. Therefore, the expression on the right side above is the minimal value of the sum on the left side, subject to $x_1^s + x_2^s + \cdots + x_n^s = 1$.

Setting $p = 5$, $r = 1$, and $s = 2$, we find that the minimal value in the given problem is $\dfrac{1}{n(n-1)}$.

We now turn to readers' solutions of problems of the Chilean Mathematical Olympiads 1994–95 [2001 : 169–170].

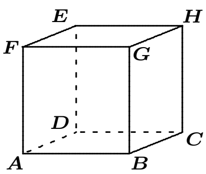
1. Given three straight lines in a plane, that concur at point O, consider the three consecutive angles between them (which, naturally, add up to $180°$). Let P be a point in the plane not on any of these lines, and let A, B, C be the feet of the perpendiculars drawn from P to the three lines. Show that the internal angles of $\triangle ABC$ are equal to those between the given lines.

Solved by Christopher J. Bradley, Clifton College, Bristol, UK; and Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's write-up.



The points O, C, P, A, B lie on the circle having $\overline{OP}$ as diameter. Therefore, $\angle ABC = \angle AOC$ and $\angle ACB = \angle AOB$. As a consequence, $\angle BAC$ is equal to the third consecutive angle at O.

2.



$ABCDEFGH$ is a cube of edge 2. Let M be the mid-point of $\overline{BC}$ and N the mid-point of $\overline{EF}$. Compute the area of the quadrilateral $AMHN$.

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Geoffrey A. Kandall, Hamden, CT, USA. We give the solution of Bilinski.

Since clearly $\overrightarrow{NH} = \overrightarrow{AM}$, the four points A, M, H, and N are coplanar. Since $\triangle ENH$, $\triangle FNA$, $\triangle BMA$, and $\triangle CMH$ are right triangles whose sides have lengths 1 and 2, we see that these triangles are congruent and $AM = MH = HN = NA$. Quadrilateral $NHMA$ is thus a rhombus.

Let us find the length of its diagonals AH and NM.

By applying the Pythagorean Theorem in right triangle $\triangle EHC$, we have $EC = \sqrt{EH^2 + HC^2} = 2\sqrt{2}$. Noticing that $NM = EC$, we get $NM = 2\sqrt{2}$. By the same reasoning, we see that $AC = 2\sqrt{2}$. Applying the Pythagorean Theorem in $\triangle ACH$, we get $AH = \sqrt{AC^2 + HC^2} = 2\sqrt{3}$.

Now, the area of a rhombus is half the product of the lengths of its diagonals. Hence, we obtain $2\sqrt{6}$ as the area.

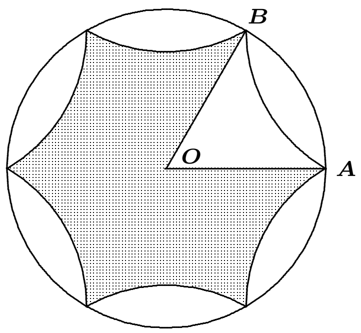
3. Given a trapezoid $ABCD$, where $\overline{AB}$ and $\overline{DC}$ are parallel, and $\overline{AD} = \overline{DC} = \overline{AB}/2$, determine $\angle ACB$.

Solved by Robert Bilinski, Outremont, QC; Christopher J. Bradley, Clifton College, Bristol, UK; Athanasias Kalakos, Athens, Greece; Geoffrey A. Kandall, Hamden, CT, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Bilinski.

Let M be the mid-point of AB, and let E be the point of intersection of AD and BC. Since $DC = AB/2$ and $DC \| AB$, we see by Thales that D is the mid-point of AE and C is the mid-point of BE. Thus, M, D, and C are the mid-points of the sides of $\triangle ABE$. By the Mid-point Theorem, $AMCD$ is a parallelogram and $MC = AD = AB/2$.

In $\triangle ABC$, MC is a median which is half the length of the side AB. Hence, $\triangle ABC$ has a right angle at C. That is, $\angle ACB = 90°$.

4. In a circle of radius 1 are drawn six equal arcs of circles, radius 1, cutting the original circle as in the figure. Calculate the shaded area.



Solved by Robert Bilinski, Outremont, QC; Christopher J. Bradley, Clifton College, Bristol, UK; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use Bilinski's solution.

Since both arcs passing through A and B are of radius 1, they are symmetric about the line AB. Hence, the area enclosed by these two arcs is cut in half by the line segment AB. If line segments are drawn through all vertices of the hexagonal star, we get a regular hexagon inscribed in the circle. From this we can easily calculate the area of the total white border, for it will be twice the area between the circle and the hexagon.

The hexagon's area is six times the area of an equilateral triangle of side 1, namely $6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$. The circle has area $\pi r^2 = \pi$. Hence, the hexagonal star has area $\pi - 2\left(\pi - \frac{3\sqrt{3}}{2}\right) = 3\sqrt{3} - \pi$.

Thus, since the shaded area has only $5/6$ the area of the hexagonal star, its area is $\frac{5}{6}(3\sqrt{3} - \pi)$.

5. In right triangle ABC the altitude $h_c = \overline{CD}$ is drawn to the hypotenuse $\overline{AB}$. Let P, P_1, P_2 be the radii of the circles inscribed in the triangles ABC, ADC, BCD, respectively. Show that $P + P_1 + P_2 = h_c$.

Solution and observation from Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Geoffrey A. Kandall, Hamden, CT, USA.

This is the same as question 1 of the Turkey Team Selection Test given in the same number of the *Corner*. See the solution given above (p. 287).

6. Consider the product of all the positive multiples of 6 that are less than 1000. Find the number of zeroes with which this product ends.

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Kandall.

The multiples under consideration are the numbers $6n$ $(1 \le n \le 166)$. Their product is $P = 2^{166} \cdot 3^{166} \cdot (166!)$. The prime factorization of P contains exactly 40 5's, since

$$\sum_{n=1}^{\infty} \left\lfloor \frac{166}{5^n} \right\rfloor = \left\lfloor \frac{166}{5} \right\rfloor + \left\lfloor \frac{166}{25} \right\rfloor + \left\lfloor \frac{166}{125} \right\rfloor = 33 + 6 + 1 = 40\,.$$

Therefore, P is divisible by 10^{40}, but not by 10^{41}; that is, P ends with 40 zeroes.

7. Let x be an integer of the form

$$x = \underbrace{1\,1\,1\,\ldots\,1}_{n}\,.$$

Show that, if x is a prime, then n is a prime.

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Athanasias Kalakos, Athens, Greece. We use Bradley's write-up.

Note that

$$x = \underbrace{1\,1\,1\cdots1}_{n \text{ times}} = 1 + 10 + \cdots + 10^{n-1} = \frac{10^n - 1}{10 - 1} = \frac{1}{9}(10^n - 1)\,.$$

Now, if n is composite, say $n = n_1 n_2$ (where n_1, $n_2 > 1$), then

$$x = \frac{10^n - 1}{9} = \left(\frac{10^{n_1} - 1}{9}\right)(1 + 10^{n_1} + 10^{2n_1} + \cdots + 10^{(n_2-1)n_1})\,.$$

Since $\dfrac{10^{n_1} - 1}{9}$ is an integer greater than 1, then x is composite.

Hence if x is prime, n must be prime.

8. Let x be a number such that

$$x + \frac{1}{x} = -1 .$$

Compute

$$x^{1994} + \frac{-1}{x^{1994}} .$$

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Athanasias Kalakos, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's solution.

Let x be a number such that $x + \frac{1}{x} = -1$. Then $x = e^{2i\pi/3} = j$ or $x = \bar{j}$. Since $1/j = \bar{j}$ and $j^2 = \bar{j}$ and $j^3 = 1$, we deduce that

- If $x = j$, then $x^{1994} - \frac{1}{x^{1994}} = j^2 - \frac{1}{j^2} = \bar{j} - j = -i\sqrt{3}$.
- If $x = \bar{j}$, then $x^{1994} - \frac{1}{x^{1994}} = \overline{-i\sqrt{3}} = i\sqrt{3}$.

Then

$$x^{1994} - \frac{1}{x^{1994}} = \pm i\sqrt{3} .$$

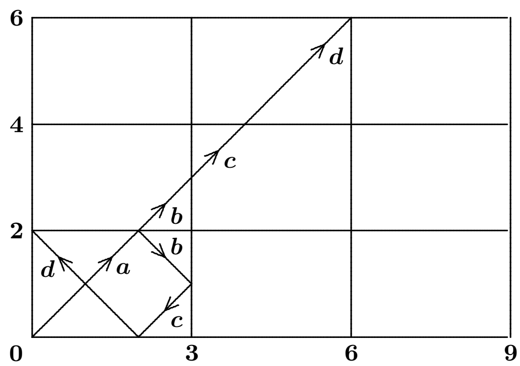
9. Let $ABCD$ be an $m \times n$ rectangle, with $m, n \in \mathbb{N}$. Consider a ray of light that starts from A, is reflected at an angle of 45° on another side of the rectangle, and goes on reflecting in this way.

(a) Show that the ray will finally hit a vertex.

(b) Suppose m and n have no common factor greater than 1. Determine the number of reflections undergone by the ray before it hits a vertex.

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We use Bradley's explanation.

(a) Extend the rectangle in all directions to form a Cartesian grid. (The example in the figure below has $m = 3$ and $n = 2$.)



The actual path is mirrored by a straight line with equation $y = x$. The ray of light will eventually hit the vertex corresponding to the vertex (mn, mn) in the extension.

(b) The ray crosses $m - 1$ lines in one direction and $n - 1$ in the other (in the extension). Thus, the number of reflections is $m + n - 2$.

10. Let a be a natural number. Show that the equation

$$x^2 - y^2 = a^3$$

always has integer solutions for x and y.

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Geoffrey A. Kandall, Hamden, CT, USA; Athanasias Kalakos, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use the solution of Kalakos.

Let $x = \frac{a(a+1)}{2}$, $y = \frac{a(1-a)}{2}$. Clearly, $x, y \in \mathbb{Z}$. Moreover,

$$\begin{aligned} x^2 - y^2 &= (x-y)(x+y) = \frac{a^2+a-a+a^2}{2} \cdot \frac{a^2+a+a-a^2}{2} \\ &= a^2 \cdot a = a^3 . \end{aligned}$$

Next we move to solutions to problems of the May 2001 number of the *Corner* and the 28th Austrian Mathematics Olympiad 1997 [2001 : 231–232].

1. Let a be a fixed whole number.

Determine all solutions x, y, z in whole numbers to the system of equations

$$\begin{aligned} 5x &+ (a+2)y &+ (a+2)z &= a , \\ (2a+4)x &+ (a^2+3)y &+ (2a+2)z &= 3a-1 , \\ (2a+4)x &+ (2a+2)y &+ (a^2+3)z &= a+1 . \end{aligned}$$

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Pavlos Maragoudakis, Pireas, Greece; Vedula N. Murty, Dover, PA, USA. We give Bradley's write-up.

The second and third equations give

$$(a-1)^2(y-z) = 2(a-1) .$$

When $a = 1$, the original system reduces to two independent equations $5x + 3y + 3z = 1$ and $6x + 4y + 4z = 2$, from which we obtain a one-parameter set of whole-number solutions $x = -1$, $y = 1 + t$, $z = 1 - t$, where $t \in Z$.

When $a \neq 1$, we have $y - z = \dfrac{2}{a-1}$. If y, z are whole numbers, then so is their difference, which means that $a - 1 \mid 2$, restricting the possibilities to $a = -1$, $a = 0$, $a = 2$, $a = 3$.

Case 1. $a = -1$. The equations become

$$\begin{aligned} 5x + y + z &= -1\,, \\ 2x + 4y &= -4\,, \\ 2x + 4z &= 0\,, \end{aligned}$$

giving a solution $x = 0$, $y = -1$, $z = 0$.

Case 2. $a = 0$. The equations become

$$\begin{aligned} 5x + 2y + 2z &= 0\,, \\ 4x + 3y + 2z &= -1\,, \\ 4x + 2y + 3z &= 1\,, \end{aligned}$$

giving a solution $x = 0$, $y = -1$, $z = 1$.

Case 3. $a = 2$. The equations become

$$\begin{aligned} 5x + 4y + 4z &= 2\,, \\ 8x + 7y + 6z &= 5\,, \\ 8x + 6y + 7z &= 3\,, \end{aligned}$$

giving a solution $x = -6$, $y = 5$, $z = 3$.

Case 4. $a = 3$. The equations include

$$5x + 5y + 5z = 3\,,$$

which evidently has no whole-number solutions, since $5 \nmid 3$.

2. Let K be a positive whole number. The sequence $\{a_n : n \geq 1\}$ is defined by $a_1 = 1$ and a_n is the n^{th} natural number greater than a_{n-1} which is congruent to n modulo K.

(a) Determine an explicit formula for a_n.

(b) What is the result if $K = 2$?

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Athanasias Kalakos, Athens, Greece; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the write-up of Bornsztein.

(a) Let n be a positive integer.

Since $a_n \equiv n \bmod K$, the first integer which is greater than a_n and congruent to $n+1$ modulo K is $a_n + 1$. Thus, the $(n+1)^{\text{th}}$ natural number greater than a_n which is congruent to n modulo K is $a_{n+1} = a_n + 1 + nK$. Summing these relations, we get that, for every integer $n \geq 1$,

$$a_n = a_1 + n - 1 + K\sum_{i=1}^{n-1} i = n + \frac{n(n-1)}{2}K .$$

(b) For $K = 2$, we immediately have $a_n = n^2$ for $n \geq 1$.

4. Determine all quadruples (a, b, c, d) of real numbers satisfying the equation

$$\begin{aligned} 256a^3b^3c^3d^3 &= (a^6+b^2+c^2+d^2)(a^2+b^6+c^2+d^2) \\ &\quad \times(a^2+b^2+c^6+d^2)(a^2+b^2+c^2+d^6) . \end{aligned}$$

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bornsztein's write-up.

Let (a, b, c, d) be a quadruple of real numbers satisfying the equation. If one of the numbers is zero, then all are 0. From now on, we suppose that none of the four numbers is 0. Since the right-hand side is positive there must be an even number of negative reals amongst a, b, c, d. Then (a, b, c, d) is a solution if and only if $(|a|, |b|, |c|, |d|)$ is a solution. Thus, we may suppose that a, b, c, d are positive.

From the AM–GM Inequality,

$$a^6 + b^2 + c^2 + d^2 \geq 4(a^6b^2c^2d^2)^{1/4} ,$$

and similarly for the other three factors on the right-hand side of the equation. Thus,

$$\begin{aligned} &(a^6+b^2+c^2+d^2)(a^2+b^6+c^2+d^2) \\ &\quad\times(a^2+b^2+c^6+d^2)(a^2+b^2+c^2+d^6) \\ &\geq 256(a^6b^2c^2d^2)^{1/4}(a^2b^6c^2d^2)^{1/4} \times (a^2b^2c^6d^2)^{1/4}(a^2b^2c^2d^6)^{1/4} \\ &= 256a^3b^3c^3d^3 , \end{aligned}$$

which indicates that the given equation is the equality case of the AM/GM Inequality. Therefore,

$$a^6 = b^2 = c^2 = d^2 = a^2 = b^6 = c^6 = d^6 ;$$

that is, $a = b = c = d = 1$.

Then the solutions are $(0, 0, 0, 0)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ where $\varepsilon_i = \pm 1$ for $i = 1, 2, 3, 4$ and $\prod_{i=1}^{4} \varepsilon_i = 1$.

5. We define the following operation which will be applied to a row of bars being situated side-by-side on positions $1, \ldots, N$:

Each bar situated at an odd-numbered position is left as is, while each bar at an even-numbered position is replaced by two bars. After that, all bars will be put side-by-side in such a way that all bars form a new row and are situated (side-by-side) on positions $1, \ldots, M$.

From an initial number $a_0 > 0$ of bars there originates (by successive application of the above-defined operation) a sequence, $\{a_n : n \geq 0\}$ of natural numbers, where a_n is the number of bars after having applied the operation n times.

(a) Prove that for all $n > 0$ we have $a_n \neq 1997$.

(b) Determine the natural numbers that can only occur as a_0 or a_1.

Solved by Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

(a) Let n be a non-negative integer. When the operation is applied to a row of a_n bars, the total number of bars increases by $\frac{a_n}{2}$ if a_n is even, and by $\frac{a_n - 1}{2}$ if a_n is odd. Thus, for every $n \geq 0$,

$$a_{n+1} = \begin{cases} \dfrac{3a_n}{2} & \text{if } a_n \text{ is even,} \\ \dfrac{3a_n - 1}{2} & \text{if } a_n \text{ is odd.} \end{cases}$$

Let p be a natural number with $p \equiv 2 \bmod 3$. Let $n \geq 0$ be an integer. Suppose $a_{n+1} = p$. If a_n is even, then $3a_n = 2a_{n+1} = 2p \equiv 1 \pmod 3$, a contradiction, while if a_n is odd, then $3a_n = 2a_{n+1} + 1 = 2p + 1 \equiv 2 \pmod 3$, a contradiction. Thus, $a_{n+1} \neq p$. Since $1997 \equiv 2 \pmod 3$, part (a) is proved.

(b) We have seen that if $p \equiv 2 \pmod 3$, then p can only occur in the sequence as a_0.

Case 1. $p = 9k$, with $k \in \mathbb{N}^*$.
For $a_0 = 4k$, we have $a_1 = 6k$ and $a_2 = 9k$. Thus, p can occur in the sequence as a_n with $n \geq 2$.

Case 2. $p = 9k + 1$, with $k \in \mathbb{N}$.
For $a_0 = 4k + 1$, we have $a_1 = 6k + 1$ and $a_2 = 9k + 1$. Thus, p can occur in the sequence as a_n with $n \geq 2$.

Case 3. $p = 9k + 3$, with $k \in \mathbb{N}$.
Suppose that there exists an integer $n \geq 1$ such that $a_{n+1} = p$. If a_n is even, then $3a_n = 2p = 2(9k + 3)$, and hence, $a_n = 6k + 2 \equiv 2 \pmod 3$ with $n > 0$, a contradiction. If a_n is odd, then $3a_n = 2p + 1 = 2(9k + 3) + 1 \equiv 1 \pmod 3$, a contradiction. Therefore, p cannot occur in the sequence as a_n with $n \geq 2$.

Case 4. $p = 9k + 4$, with $k \in \mathbb{N}$.
For $a_0 = 4k + 2$, we have $a_1 = 6k + 3$ and $a_2 = 9k + 4$. Thus, p can occur in the sequence as a_n with $n \geq 2$.

Case 5. $p = 9k + 6$, with $k \in \mathbb{N}$.
For $a_0 = 4k + 3$, we have $a_1 = 6k + 4$ and $a_2 = 9k + 6$. Thus, p can occur in the sequence as a_n with $n \geq 2$.

Case 6. $p = 9k + 7$, with $k \in \mathbb{N}$.
Suppose that there exists an integer $n \geq 1$ such that $a_{n+1} = p$. If a_n is even, then $3a_n = 2p = 2(9k + 7) \equiv 2 \pmod 3$, a contradiction. If a_n is odd, then $3a_n = 2p + 1 = 2(9k + 7) + 1$, and hence, $a_n = 6k + 5 \equiv 2 \pmod 3$ with $n > 0$, a contradiction. Therefore, p cannot occur in the sequence as a_n with $n \geq 2$.

It follows that the natural numbers that can only occur as a_0 or a_1 are those congruent to 2, 3, 5, 7, or 8 $\pmod 9$.

6. Let n be a fixed natural number. Determine all polynomials $x^2 + ax + b$, where $a^2 \geq 4b$, such that $x^2 + ax + b$ divides $x^{2n} + ax^n + b$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Athanasias Kalakos, Athens, Greece; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bataille's write-up.

If $n = 1$, all polynomials $x^2 + ax + b$ are solutions. We will suppose $n > 1$ from now on. Since $a^2 \geq 4b$, there exist real numbers x_1, x_2 (not necessarily distinct) such that $x^2 + ax + b = (x - x_1)(x - x_2)$. It follows that $x^{2n} + ax^n + b = (x^n - x_1)(x^n - x_2)$.

Now, if $x^2 + ax + b$ divides $x^{2n} + ax^n + b$, then x_1, x_2 are roots of $x^{2n} + ax^n + b$, so that $x_1^n = x_1$ or $x_1^n = x_2$, and $x_2^n = x_2$ or $x_2^n = x_1$. Therefore, x_1 and x_2 must belong to $\{-1, 0, 1\}$.

Now we check the possible cases:

- If $x_1 = x_2 = 0$, then $a = b = 0$ and $x^2 + ax + b = x^2$ divides $x^{2n} + ax^n + b = x^{2n}$.

- If $x_1 = 0$, $x_2 = -1$, then $a = 1$, $b = 0$ and $x^2 + ax + b = x(x + 1)$ divides $x^n(x^n + 1)$ only if n is odd.

- If $x_1 = 0$, $x_2 = 1$, then $x^2 + ax + b = x(x - 1)$ divides $x^n(x^n - 1)$.

- If $x_1 = -1$, $x_2 = -1$, then $x^2 + ax + b = (x + 1)^2$ divides $(x^n + 1)^2$ only if n is odd.

- If $x_1 = 1$, $x_2 = 1$, then $x^2 + ax + b = (x - 1)^2$ divides $(x^n - 1)^2$.

- If $x_1 = -1$, $x_2 = 1$, then $x^2 + ax + b = x^2 - 1$ divides $x^{2n} - 1$.

In conclusion, for n odd ($n > 1$), the solutions are x^2, $x(x+1)$, $x(x-1)$, $(x+1)^2$, $(x-1)^2$, x^2-1; and for n even, the solutions are x^2, $x(x-1)$, $(x-1)^2$, x^2-1.

Editor's comment: Klamkin points out that if the condition $a^2 \geq 4b$ is eliminated, the zeros x_1, x_2 can be complex cube roots of unity, allowing another possibility, x^2+x+1, provided n is not a multiple of 3.

Next we move to readers' solutions for problems of the Íslenzka Staerðfræðikeppni Framhaldsskólanema 1995–1996 [2001 : 232–233].

1. Calculate the area of the region in the plane determined by the inequality

$$|x|+|y|+|x+y| \;\leq\; 2\,.$$

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the solution of Diaz-Barrero.

Let $\mathcal{A} = \{(x,y) \in \mathbb{R}^2 : |x|+|y|+|x+y| \leq 2\}$ be the given region. We claim that $\mathcal{A}$ has area 3. To prove this, we start by recalling that for all $a \in \mathbb{R}$, $|a| = |-a|$. It follows that $\mathcal{A}$ is symmetric with respect to reflection through the origin. Hence, it suffices to investigate the given inequality only when $y \geq 0$.

In the first quadrant, where $x \geq 0$ and $y \geq 0$, we have $x+y \geq 0$, and the inequality becomes $x+y+x+y \leq 2$. Therefore, the part of $\mathcal{A}$ in this quadrant is the triangle

$$AOB \;=\; \{(x,y) \in \mathbb{R}^2 : x \geq 0\,, y \geq 0\,, x+y \leq 1\}\,.$$

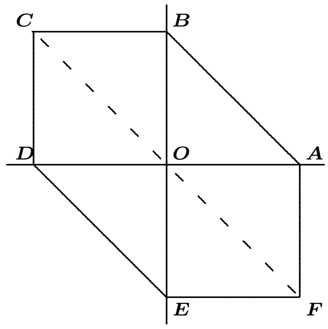
In the second quadrant, where $x \leq 0$ and $y \geq 0$, we have two possibilities: (a) $x+y \geq 0$ or (b) $x+y \leq 0$. In case (a), the inequality becomes $-x+y+x+y \leq 2$, and we have the triangle

$$BOC \;=\; \{(x,y) \in \mathbb{R}^2 : x \leq 0\,, 0 \leq y \leq 1\}\,,$$

In case (b), the inequality becomes $-x+y-(x+y) \leq 2$, determining the triangle

$$COD \;=\; \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 0\,, y \geq 0\}\,.$$

These triangles can be seen in the figure on the next page. Reflecting them through the origin, we see that $\mathcal{A}$ is the hexagon $ABCDEF$, with area 3.



2. Suppose that a, b, and c are the three roots of the polynomial $p(x) = x^3 - 19x^2 + 26x - 2$. Calculate

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\,.$$

Solved by Rahul Banotra, student, Sir Winston Churchill High School and Samapti Samapti, Western Canada High School, Calgary, AB; Marcus Emmanuel Barnes, student, York University; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Elizabeth Park, Western Canada High School, Calgary, AB. We give the solution of Banotra and Samapti.

If a, b, c are the roots of the polynomial, then

$$(x-a)(x-b)(x-c) \;=\; x^3 - 19x^2 + 26x - 2\,. \tag{1}$$

Expanding yields:

$$\begin{aligned}
&(x-a)(x-b)(x-c)\\
&\quad= (x^2 - (a+b)x + ab)(x-c)\\
&\quad= x^3 - (a+b)x^2 + abx - cx^2 + (a+b)cx - abc\\
&\quad= x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc\,.
\end{aligned} \tag{2}$$

From (1) and (2) we get

$$\begin{aligned}
a+b+c &= 19\,,\\
ab+bc+ac &= 26\,,\\
abc &= 2\,.
\end{aligned}$$

Thus, we have $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc+ac+ab}{abc} = \frac{26}{2} = 13$.

3. A collection of 52 integers is given. Show that amongst these numbers it is possible to find two such that 100 divides either their sum or their difference.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Klamkin.

This problem appeared as a Russian Olympiad problem quite some time ago. Unfortunately, I do not have a reference.

Each of the numbers can be expressed in the form $100a + b$, where $-49 \le b \le 50$. Since there are only 51 possible values for $|b|$, at least two of them must be the same. If the corresponding numbers have opposite signs associated with b, then the sum of the numbers is divisible by 100; if they have the same signs, then the difference is divisible by 100.

The result is true more generally if we replace 100 by n and 52 by $\lfloor n/2 \rfloor + 2$.

4. (i) Show that the sum of the digits of every integer multiple of 99, from $1 \cdot 99$ up to and including $100 \cdot 99$, is 18.

(ii) Show that the sum of the digits of every integer multiple of the number $10^n - 1$, from $1 \cdot (10^n - 1)$ up to and including $10^n \cdot (10^n - 1)$, is $n \cdot 9$.

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Samapti Samapti, Western Canada High School, Calgary, AB. We give the solution of Klamkin.

(i) $99n = 100n - n$. Thus, if n is a single-digit number, the digits resulting from subtracting n from the 3–digit number $n00$ are $n - 1$, 9, and $10 - n$, for a sum of 18. If n is a 2–digit number ab with $b = 0$, the digits of $100n - n$ are $a - 1$, 9, $10 - a$, and 0, which still sum to 18; if $b \neq 0$, then the digits of $ab00 - ab$ are a, $b - 1$, $10 - a - 1$, and $10 - b$, again summing to 18. (In a similar way it follows that the sum of the digits of *every* multiple of 99 is 18).

(ii) Proceeding in a similar way as in (i), it follows that the sum of the digits of the number $99\ldots9m$, where there are n 9's, is obtained by subtracting m from $m00\ldots0$. For example, if m is the 2–digit number ab not ending in 0, then the successive digits are a, $b - 1$, 9, 9, $\ldots$, 9, $10 - a - 1$, and $10 - b$. Hence, the sum of the digits is $9n$.

5. The sequence $\{a_n\}$ is defined by $a_1 = 1$ and, for $n \geq 1$,

$$a_{n+1} = \frac{a_n}{1 + na_n}.$$

Find a_{1996}.

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Murray S. Klamkin, University of Alberta, Edmonton, AB; Samapti Samapti, Western Canada High School, Calgary, AB; Heinz-Jürgen Seiffert, Berlin, Germany; D.J. Smeenk, Zaltbommel, the Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Bataille.

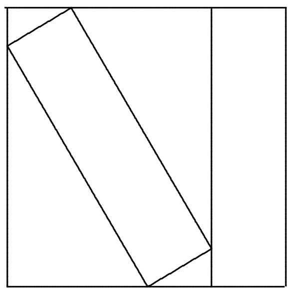
By an immediate induction, $a_n > 0$ for all n.

Now, $\dfrac{1}{a_{n+1}} = \dfrac{1 + na_n}{a_n} = \dfrac{1}{a_n} + n$ for all n. It follows that

$$\frac{1}{a_{1996}} - \frac{1}{a_1} = \sum_{k=1}^{1995}\left(\frac{1}{a_{k+1}} - \frac{1}{a_k}\right) = \sum_{k=1}^{1995} k = \frac{1995 \times 1996}{2}.$$

Since $a_1 = 1$, we deduce that $\dfrac{1}{a_{1996}} = 1 + \dfrac{1995 \times 1996}{2} = 1991011$. Therefore, $a_{1996} = \dfrac{1}{1991011}$.

6. In a square bookcase two identical books are placed as shown in the figure. Suppose the height of the bookcase is 1. How thick are the books?



Solved by Rahul Banotra, student, Sir Winston Churchill High School, Calgary, AB; Robert Bilinski, Outremont, QC; Christopher J. Bradley, Clifton College, Bristol, UK; Murray S. Klamkin, University of Alberta, Edmonton, AB; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bilinski's solution.

The solution is illustrated by a diagram at the end.

Due to the abundance of right angles and complementary angles, we have two pairs of congruent triangles: $\triangle ABI \cong \triangle FGJ$, $\triangle CJB \cong \triangle HIG$. All four of these triangles are similar, since they have corresponding angles equal.

Since $GI = BJ = 1$ (the book height), we have $HI = CJ = \cos\theta$ and $HG = BC = \sin\theta$. Since $AH = CF = 1$, we quickly conclude that $AI = JF = 1 - \cos\theta$.

In $\triangle ABI$, we have $\sin\theta = \dfrac{1-\cos\theta}{BI}$, which gives $BI = \dfrac{1-\cos\theta}{\sin\theta}$, the width of the book ($BI = GJ = CD = FE$). In a similar fashion, we find that $AB = \dfrac{\cos\theta(1-\cos\theta)}{\sin\theta}$.

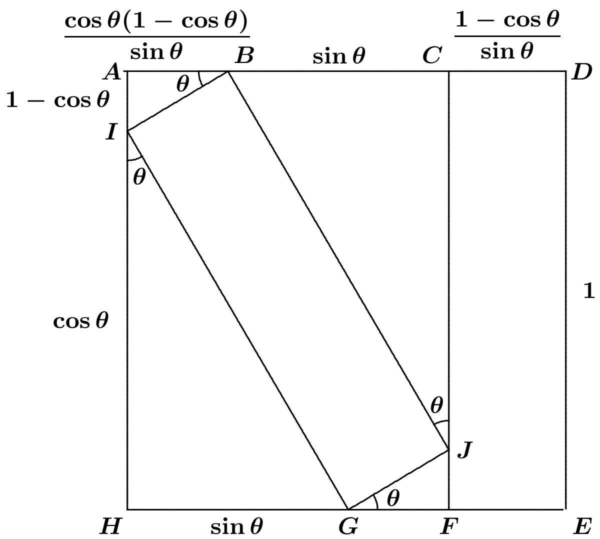
Since

$$1 \;=\; AD \;=\; AB + BC + CD \;=\; \frac{\cos\theta(1-\cos\theta)}{\sin\theta} + \sin\theta + \frac{1-\cos\theta}{\sin\theta}\,,$$

we get

$$\begin{aligned} \frac{\cos\theta - \cos^2\theta + \sin^2\theta + 1 - \cos\theta}{\sin\theta} &= 1\,, \\ \frac{2\sin^2\theta}{\sin\theta} &= 1\,, \\ \sin\theta &= \frac{1}{2}\,. \end{aligned}$$

Hence, $\theta = 30^\circ$, and the book width is $BI = \dfrac{1-\cos 30^\circ}{\sin 30^\circ} = 2 - \sqrt{3}$.



That concludes this issue of the *Corner*. Please keep sending me Olympiad contests and your nice solutions.

BOOK REVIEWS

John Grant McLoughlin

Inverse problems - Activities for Undergraduates
by Charles W. Groetsch, published by The Mathematical Association of America, 1999
ISBN# 0-88385-716-2, softcover, 222 + xii pages, US$26.00
Reviewed by ***Edward Vrscay****, University of Waterloo, Waterloo, Ontario.*

In the preface, the author begins by writing, "This is not a textbook. Nor is it a survey of elementary inverse problems. It is a personal miscellany of activities related to inverse problems that is meant to enrich, and perhaps enliven, the teaching of mathematics in the first two undergraduate years."

Indeed, this "miscellany", written by a specialist in inverse and ill-posed problems, is a delightful gem that could be used at various places and levels of undergraduate school. As the author points out in his first chapter, *Introduction to Inverse Problems*, undergraduate training in mathematics and science is generally dominated by *direct* problems—those in which a student is given enough information to carry out a well-defined procedure that yields a unique solution. In other words, the student is given a "model" (process) along with a "cause" (input) and is then expected to find the "effect" (output). But what about the other side of the coin? For example, if we have a model K and an effect y, can we find the cause x? Or if we know the cause x and effect y, can we come up with the model K? These are *inverse* problems. Not surprisingly, inverse problems constitute a significant chapter in the history of science, engineering and mathematics.

As a very simple example, consider the linear interpolation problem, viewed by the author as the oldest problem in mathematics. The direct problem is to calculate the values of a linear function. The inverse problem is to determine a linear function from some data points (x_i, y_i).

The author briefly discusses a number of classic inverse problems from history, including "Archimedes' Bath" (technically, "nondestructive evaluation"), the two-body Kepler problem and, more recently, computed tomography (Radon transform). These very readable vignettes should impress upon the reader the importance of inverse problems in the history of mathematics and science, which should be reason enough to include them in undergraduate education. However, there is a more important point made by the author in the section entitled *Why Teach Inverse Problems*, namely, that teaching inverse problems nourishes a habit of "inverse thinking" in students. Looking only at the direct problem is to miss much of the whole picture. Teaching only direct methods is, in fact, intellectually limiting. (This reviewer agrees wholeheartedly with the author's sentiments.)

The next four chapters of the book are devoted to inverse problems in precalculus, calculus, differential equations and linear algebra, respectively. Each chapter has six "modules" that consist of: (i) an introduction for the instructor (giving the course level of the problem(s) of concern, goals, mathematical and scientific background required of the student, and necessary hardware/software), (ii) Activities (exercises and computations), and (iii) Notes and Further Reading. Answers and advice for selected problems are found in Appendix A. Appendix B contains source codes for MATLAB routines that can be used for some of the computations in the book. (These source codes can also be downloaded from the author's website.)

For example, the module entitled *Shape Up!* in Chapter 2, (*Inverse Problems in Calculus*), examines direct and inverse problems for the following drainage scenario: A vessel is formed by revolving a curve $x = f(y)$ about the y-axis. For a given initial water depth $y > 0$, let $T(y)$ be the time necessary for the vessel to drain through an orifice of cross-sectional area a at its base. This module examines the relationship between the "drain-time" function T and f. (One will need Torricelli's Law, discussed in the earlier module *A Little Squirt*.) The direct problem is to find T given f: A number of functions f are suggested in the Activities section. Students are also asked to ascertain some mathematical properties of T. The inverse problem is to find f given T. This includes the classic *clepsydra* or water-clock problem (finding f so that $T(y)$ is linear in y).

There are also opportunities to explore more realistic aspects of this inverse problem, for example, when the drain-time function T is not given in closed form, but rather in the form of data points (for example, stopwatch experiments). The student is asked to explore some approximation schemes (numerical integration and differentiation) using the MATLAB programs provided. The accuracy of these schemes is to be examined along with the effects of noise in the data.

The chapter on inverse problems in linear algebra includes problems in elementary tomography, gravimetry and inverse vibration. The mathematical methods used there include projections onto hyperplanes, matrix and generalized inverses, and eigenvalue analysis. For example, the module entitled "L'ART Pour L'Art" examines the *algebraic reconstruction technique* (ART) algorithm, which uses successive orthogonal projections to approximate solutions of linear equations. A simple tomography problem is then considered.

You might be able to find some of the inverse problems that are treated in this book scattered throughout some standard first- and second-year calculus and linear algebra texts. However, you probably won't find these problems treated with the care and possibly the depth that they receive in this book. Nor will you find anywhere else such a comprehensive collection of "inverse thinking" under one cover. This book would serve as a valuable resource not only in Year 1 and 2 Calculus and Linear Algebra courses but also in courses on differential equations and Newtonian mechanics.

My only complaint about this book is concerned with the numbering of sections in Appendix A, *Selected Answers and Advice*: Section $A.m.n$ in the Appendix corresponds to module $(m-1).n$ in the main text. I find that I must almost always go back to the Table of Contents to return from a solution to the appropriate module in the main text. I would hope that future editions of the book would at least add the starting page of the module to the subtitles of the Appendix; for example *Slip Sliding Away* (p. 96). However, this is but a minor blemish in what is a unique and highly recommended book, clearly a labour of love by an expert in the field.

Challenging Brainteasers
by Bernardo Recamán Santos, published by Sterling Publishing Co., Inc., 1997
ISBN 0-8069-2877-8, softcover, 93 pages, US$9.95
Reviewed by **Sandy Graham**, *University of Waterloo, Waterloo, Ontario.*

This book is a compilation of mathematical games and puzzles which the author has either created or collected over the years. He divides the problems into four categories: Arithmetical Puzzles, Geometrical Puzzles, Logical Puzzles, and Algebraic Puzzles. The puzzles range from simple algebraic word problems to some relatively difficult combinatorial problems.

Each section has 14 or 15 questions. Many of the problems are probably familiar to readers of such articles. In the foreword, the author explains that most of the problems are ones that he has collected, and only a few are original. The solutions provided for the problems range from simple answers to more complete discussions of the problem. In some cases, the author suggests variations on the problem, or the problems introduce unusual mathematical terms such as "exotic numbers". The book is probably suitable as a resource for high school or senior elementary school teachers. Teachers can use the problems in their classroom as enrichment, or as a catalyst to start discussing other mathematical concepts.

Although educators can always find a place on their bookshelf for this type of compilation, *Challenging Brainteasers* is not an extraordinary collection of problems and solutions. The solutions section could have been expanded significantly. Ideally it would include more problem-solving techniques, more variations for the problems, and more complete solutions rather than just answers. As a set of classic math problems this book is adequate, but if you are looking for more you will not find it here.

An Elementary Proof of the Inequality: variance $\leq (M - \overline{x})(\overline{x} - m)$

Vedula N. Murty

Introduction.

Whenever I proved an inequality using calculus my teacher always asked me, "See if you can prove it without using calculus". Bhatia and Davis [1] gave a new bound for the variance, and their proof used calculus. The object of this note is to present an elementary proof of their result without using calculus.

In this paper we assume that $x_1, \ldots, x_n$ are real numbers. Recall that the variance of these numbers is defined to be

$$\sigma^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \overline{x})^2, \quad \text{where} \quad \overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j .$$

Let $M = \max\{x_1, x_2, \ldots, x_n\}$ and $m = \min\{x_1, x_2, \ldots, x_n\}$, and set $R = M - m$. The classical bound on the variance, known to students of statistics, is

$$\sigma^2 \leq \frac{R^2}{4} . \tag{1}$$

In [1] Bhatia and Davis proved

$$\sigma^2 \leq (M - \overline{x})(\overline{x} - m) . \tag{2}$$

It is easy to show that the right hand side of (2) is less than or equal to the right hand side of (1) (as we will do following the proof below). This shows that the new bound is sharper than the classical bound.

An Elementary Proof.

Given the population $x_1, x_2, \ldots, x_n$, note that its variance, mean, maximum, and minimum are invariant under a permutation of the x_j's. We, therefore, establish (2) assuming

$$x_1 \leq x_2 \leq \ldots \leq x_n . \tag{3}$$

Now (3) implies $M = x_n$ and $m = x_1$. Thus, we will prove the inequality

$$\sigma^2 \leq (x_n - \overline{x})(\overline{x} - x_1) . \tag{4}$$

Recall that $n\sigma^2 = \sum_{j=1}^{n} x_j^2 - n\overline{x}^2$. Inequality (4) is established by showing

$$n(x_n - \overline{x})(\overline{x} - x_1) - n\sigma^2 \geq 0 . \tag{5}$$

The left hand side of (5) can be written as

$$\begin{aligned} n\overline{x}(x_n + x_1) - nx_1x_n - \sum_{j=1}^{n} x_j^2 &= \left(\sum_{j=1}^{n} x_j\right)(x_n + x_1) - \sum_{j=1}^{n} x_1x_n - \sum_{j=1}^{n} x_j^2 \\ &= \sum_{j=1}^{n}(x_jx_n + x_jx_1 - x_1x_n - x_j^2) \\ &= \sum_{j=1}^{n}(x_n - x_j)(x_j - x_1)\,. \end{aligned}$$

The last expression is clearly greater than or equal to zero, since we have assumed the x's are non-decreasing. This completes the proof of (4) and hence of (2) as well. Note that we have equality in this inequality if and only if all sample values are equal to either x_1 or x_n; that is, if and only if there are at most two sample values.

To prove that the bound is sharper than $R^2/4$, we observe

$$\begin{aligned} &\tfrac{1}{4}(x_n - x_1)^2 - (x_n - \overline{x})(\overline{x} - x_1) \\ &\quad = \overline{x}^2 - \overline{x}(x_1 + x_n) + x_1x_n + \tfrac{1}{4}(x_n - x_1)^2 \\ &\quad = \overline{x}^2 - \overline{x}(x_1 + x_n) + \tfrac{1}{4}(x_n + x_1)^2 \\ &\quad = \left(\overline{x} - \frac{x_1 + x_n}{2}\right)^2 \ \geq\ 0\,, \end{aligned}$$

which proves the claim.

For other useful elementary proofs of statistical inequalities, see [2], which received the George Pólya award.

Acknowledgements: The author wishes to express his gratitude to Dr. Clifford H. Wagner who made several stylistic suggestions and carefully checked the algebra, and to Professor Warren Page who also checked the algebra. The author also thanks the referee and the editor for an improved presentation.

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[1] Rajendra Bhatia and Chandler Davis, *A better bound on the variance*, The American Mathematical Monthly, **107** (2000), No 4, pp. 353–357.

[2] Warren Page and V.N. Murty, *Nearness relations among measures of central tendency and dispersion: Part 1 and Part 2*, The Two Year College Mathematics Journal **13** (1982), No 5, and **14** (1983), No 1.

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A Simple Irreducibility Criterion for $f(X^2)$

Natalio H. Guersenzvaig

Let k be any field, and let $f(X)$ be an arbitrary polynomial of $k[X]$ which is irreducible in $k[X]$. A well-known result of Wahlen-Capelli (see [1], p. 212) establishes necessary and sufficient conditions for the irreducibility of $f(g(X))$ in $k[X]$, where $g(X)$ is any polynomial of $k[X]$. The proof of this result is not elementary because it uses the theory of field extensions.

In this short article we establish, with a very elementary proof, necessary and sufficient conditions for the reducibility of $f(X^2)$ in $Z[X]$, where Z denotes an arbitrary unique factorization domain. As an immediate consequence we obtain a simple sufficient condition for the irreducibility of $f(X^2)$ in $Z[X]$.

Theorem 1. *Let $f(X)$ be any non-zero polynomial in $Z[X]$. The following statements are equivalent.*

(i) *$f(X^2)$ is reducible in $Z[X]$.*

(ii) *$f(X)$ is reducible in $Z[X]$ or there exist polynomials $G(X)$, $H(X)$ in $Z[X]$ and a unit u of Z (that is, an invertible element of $Z \setminus \{0\}$) such that*

$$uf(X) = G^2(X) - XH^2(X). \qquad (\star)$$

Proof. We first suppose that (ii) is true. It is clear that $f(X^2)$ is reducible in $Z[X]$ if $f(X)$ is. Then suppose that $f(X)$ is irreducible in $Z[X]$. Thus, $(\star)$ is true with $H(X) \neq 0$. As a consequence, (i) follows, because $G(X^2)$ and $XH(X^2)$ have degrees of distinct parity and

$$uf(X^2) = \big(G(X^2) - XH(X^2)\big)\big(G(X^2) + XH(X^2)\big).$$

Now suppose that (i) is true. Assume $f(X)$ is irreducible in $Z[X]$ (otherwise we are done). Then $f(X^2) = g(X)h(X)$, where $g(X)$, $h(X) \in Z[X]$ are not units of Z. Collecting even powers in $g(X)$ and $h(X)$, we obtain

$$g(X) = G(X^2) + XL(X^2), \qquad h(X) = H(X^2) + XT(X^2), \qquad (1)$$

for some polynomials G, L, H, and T in $Z[X]$. Hence,

$$\begin{aligned} f(X^2) = {} & G(X^2)H(X^2) + X^2L(X^2)T(X^2) \\ & + XG(X^2)T(X^2) + XL(X^2)H(X^2). \end{aligned} \qquad (2)$$

for some polynomials G, L, H, and T in $Z[X]$.

We claim that $L(X)T(X) \neq 0$. We prove this by contradiction. Suppose for example that $L(X) = 0$ (the case $T(X) = 0$ is analogous). Thus, we have

$$f(X^2) - G(X^2)H(X^2) \;=\; XG(X^2)T(X^2)\,. \tag{3}$$

Both sides of this equality are zero because, otherwise, they have degrees of different parity. Thus, $T(X) = 0$; whence, $f(X^2) = G(X^2)H(X^2)$; that is, $f(X) = G(X)H(X)$, which contradicts the assumption that $f(X)$ is irreducible in $Z[X]$.

It can be assumed that the greatest common divisor of $G(X)$ and $L(X)$, say $D(X)$, is equal to 1, because, otherwise, we consider the factorization $f(X^2) = g^*(X)h^*(X)$, with $h^*(X) = D(X^2)h(X)$ and

$$g^*(X) \;=\; \frac{g(X)}{D(X^2)} \;=\; \frac{G(X^2)}{D(X^2)} + X\frac{L(X^2)}{D(X^2)}\,,$$

where such a condition is satisfied. Note that in order to replace $g(X)$ by $g^*(X)$, we need to know that $g^*(X)$ is not a unit of Z. If it were a unit, then $L(X^2) = 0$ from (1) which implies $L(X) = 0$. But this leads to a contradiction, as was shown in the preceding paragraph.

Now, from (2), via the same argument used in (3), we get

$$G(X)T(X) + L(X)H(X) \;=\; 0\,, \tag{4}$$

and

$$f(X) \;=\; G(X)H(X) + XL(X)T(X)\,.$$

As a consequence,

$$L(X)f(X) \;=\; G(X)L(X)H(X) + XL^2(X)T(X)\,.$$

By using (4) this becomes

$$L(X)f(X) \;=\; -T(X)\big(G^2(X) - XL^2(X)\big)\,;$$

whence, $L(X)$ is a divisor of $T(X)$ because $G(X)$ and $L(X)$ are coprime polynomials. Thus,

$$f(X) \;=\; M(X)\big(G^2(X) - XL^2(X)\big)$$

for some $M(X) \in Z[X]$. But we have assumed that $f(X)$ is irreducible in $Z[X]$. Therefore, $M(X)$ is a unit of Z, and $(\star)$ follows. □

Corollary 1. *Let $f(X)$ be any polynomial of $Z[X]$ which is irreducible in $Z[X]$. Assume that $f(X)$ has leading coefficient A and constant term C. In addition suppose that uA is not a square in Z for each unit u of Z or that AC is not a square in Z. Then*

$$f(X^2) \text{ is irreducible in } Z[X].$$

Remark. If $f(X)$ is detected as irreducible via the well-known Eisenstein's Criterion (see [2, pp. 267-268]), which also works in $Z[X]$ (*mutatis mutandis*), it follows immediately that $f(X^m)$ is irreducible in $Z[X]$ for any positive integer m. However, our result works in cases where Eisenstein's Criterion is inapplicable. As an example of this, we consider the polynomial $f(X) = 3X^2 + 2X + 4 \in \mathbb{Z}[X]$, which is certainly irreducible in $\mathbb{Z}[X]$. Using Corollary 1, we note that $AC = 12$ and ± 3 are not squares in $\mathbb{Z}$. From either of these two facts we have that $f\left(X^{2^m}\right) = 3X^{2^m} + 2X^{2^{m-1}} + 4$ is irreducible in $\mathbb{Z}[X]$ for any positive integer m.

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[1] P.M. Cohn, *Algebra*, Vol. 2, John Wiley & Sons, 1977.

[2] W.K. Nicholson, *Introduction to Abstract Algebra*, Wiley, 1999.

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PROBLEMS

Problem proposals and solutions should be sent to Jim Totten, Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, BC, Canada, V2C 5N3. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 March 2004**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX.) *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Hidemitsu Saeki of the University of Montreal for translations of the problems.

2827. Correction. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a non-negative integer. Determine

$$\sum_{k=0}^{n} \frac{\tanh\left(2^k\right)}{2+2\sinh^2\left(2^k\right)}\,.$$

....

Soit n un entier non négatif. Calculer

$$\sum_{k=0}^{n} \frac{\tanh\left(2^k\right)}{2+2\sinh^2\left(2^k\right)}\,.$$

2829. Correction. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Given $\triangle ABC$ with sides a, b, c, prove that

$$\frac{3\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2} \;\geq\; 2\,.$$

. . . .

Montrer que, dans un triangle ABC de côtés a, b, c,

$$\frac{3\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2} \;\geq\; 2\,.$$

2839. Correction. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Suppose that x, y, and z are real numbers. Prove that

$$\left(x^3+y^3+z^3\right)^2+3(xyz)^2 \;\geq\; 4\left(y^3z^3+z^3x^3+x^3y^3\right)\,.$$

Determine the cases of equality.

. . . .

Si x, y et z sont des nombres réels, montrer que

$$\left(x^3+y^3+z^3\right)^2+3(xyz)^2 \;\geq\; 4\left(y^3z^3+z^3x^3+x^3y^3\right)\,.$$

Déterminer les cas où il y a égalité.

2851★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let m, n, and N be non-negative integers such that $m+n \geq 2N+1$. Let $K=m+n-N-1$. Prove that

$$\sum_{j=0}^{\infty}(-1)^j\frac{N+1}{N+1+j}\binom{N}{j}\left[\binom{K-j}{m}+\binom{K-j}{n}\right] \;=\; \frac{\binom{m+n}{m}}{\binom{2N+1}{N}}\,.$$

. . . .

Soit m, n, et N des nombres entiers non négatifs tels que $m+n \geq 2N+1$, et soit $K=m+n-N-1$. Montrer que

$$\sum_{j=0}^{\infty}(-1)^j\frac{N+1}{N+1+j}\binom{N}{j}\left[\binom{K-j}{m}+\binom{K-j}{n}\right] \;=\; \frac{\binom{m+n}{m}}{\binom{2N+1}{N}}\,.$$

2852. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, we have $AB < AC$. The internal bisector of $\angle BAC$ meets BC at D. Let P be an interior point of the line segment AD, and let E and F be the intersections of BP and CP with AC and AB, respectively.

Prove that $\frac{PE}{PF} < \frac{AC}{AB}$.

. . . .

Dans un triangle ABC, on a $AB < AC$. La bissectrice intérieure de l'angle BAC coupe BC en D. Soit P un point intérieur du segment AD, et soit E et F les intersections respectives de BP et CP avec AC et AB.

Montrer que $\frac{PE}{PF} < \frac{AC}{AB}$.

2853. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, we have $AC = 2AB$. The tangents at A and C to the circumcircle of $\triangle ABC$ meet at P.

Prove that the line BP bisects the arc BAC (of the circumcircle).

. . . .

Dans un triangle ABC, on a $AC = 2AB$. Les tangentes en A et en C au cercle circonscrit du triangle ABC se coupent en P.

Montrer que la droite BP divise l'arc BAC (du cercle circonscrit) en deux parties égales.

2854. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that M and N are the mid-points of the sides AB and CD of quadrilateral $ABCD$, respectively.

Prove that $AN^2 + DM^2 + BC^2 = BN^2 + CM^2 + AD^2$.

. . . .

Supposons que M et N sont les points milieu respectifs des côtes AB et CD d'un quadrilatère $ABCD$.

Montrer que $AN^2 + DM^2 + BC^2 = BN^2 + CM^2 + AD^2$.

2855. *Proposed by Antreas P. Hatzipolakis and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given two points B and C, find the locus of the point A such that the centre of the nine-point circle of $\triangle ABC$ lies on the interior bisector of $\angle CAB$.

. . . .

Etant donné deux points B et C, trouver le lieu du point A tel que le centre du cercle des 9-points du triangle ABC soit situé sur la bissectrice intérieure de l'angle CAB.

2856. *Proposed by Óscar Ciaurri, Universidad de La Rioja, Logroño, Spain.*

Let $a_k = \dfrac{q^k - 1}{q - 1}$, where q is a real number, $q \neq 1$. For integers $n \geq 0$ and $k \geq 1$, define $C_{n,k}$ as follows: $C_{n,1} = 1$, $C_{0,k} = 0$ for $k \geq 2$, and $C_{n,k} = \sum\limits_{j=0}^{n-1} \dfrac{a_{k-1}^j}{a_k^{j+1}} C_{j,k-1}$ for $n \geq 1$ and $k \geq 2$.

Show that

$$C_{n,k} \;=\; -(q-1)^{k-1} \sum_{i=1}^{k} \left(\frac{q^i - 1}{q^k - 1} \right)^n \frac{q^i \langle q^{-k}, q \rangle_i}{\langle q, q \rangle_{i-1} \langle q, q \rangle_k},$$

where $\langle a, q \rangle_0 = 1$ and $\langle a, q \rangle_i = (1-a)(1-aq)\cdots(1-aq^{i-1})$ for $i \geq 1$.

. . . .

Soit q un nombre réel différent de 1 et $a_k = \dfrac{q^k - 1}{q - 1}$. Pour des entiers $n \geq 0$ et $k \geq 1$, on définit $C_{n,k}$ par $C_{n,1} = 1$, par $C_{0,k} = 0$ si $k \geq 2$, et par $C_{n,k} = \sum\limits_{j=0}^{n-1} \dfrac{a_{k-1}^j}{a_k^{j+1}} C_{j,k-1}$ si $n \geq 1$ et $k \geq 2$.

Montrer que

$$C_{n,k} \;=\; -(q-1)^{k-1} \sum_{i=1}^{k} \left(\frac{q^i - 1}{q^k - 1} \right)^n \frac{q^i \langle q^{-k}, q \rangle_i}{\langle q, q \rangle_{i-1} \langle q, q \rangle_k},$$

où $\langle a, q \rangle_0 = 1$ et $\langle a, q \rangle_i = (1-a)(1-aq)\cdots(1-aq^{i-1})$ si $i \geq 1$.

2857. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let O be an interior point of $\triangle ABC$, and let D, E, F, be the intersections of AO, BO, CO with BC, CA, AB, respectively.

Suppose that P and Q are points on the line segments BE and CF, respectively, such that $\dfrac{BP}{PE} = \dfrac{CQ}{QF} = \dfrac{DO}{OA}$.

Prove that $PF \parallel QE$.

. . . .

Soit O un point intérieur du triangle ABC, et soit D, E et F, les intersections de AO, BO et CO avec BC, CA et AB, respectivement.

On suppose que P et Q sont des points sur les segments BE et CF, respectivement, tels que $\dfrac{BP}{PE} = \dfrac{CQ}{QF} = \dfrac{DO}{OA}$.

Montrer que $PF \parallel QE$.

2858. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that P is an interior point of $\triangle ABC$, and that D, E, F are the intersections of AP, BP, CP with BC, CA, AB, respectively. Suppose that

$$\frac{AE+AF}{BC} = \frac{BF+BD}{CA} = \frac{CD+CE}{AB}.$$

Characterize the point P.

. . . .

Soit P un point intérieur d'un triangle ABC et soit D, E et F les intersections respectives de AP, BP et CP avec BC, CA et AB. Supposons que

$$\frac{AE+AF}{BC} = \frac{BF+BD}{CA} = \frac{CD+CE}{AB}.$$

Caractériser le point P.

2859★. *Proposed by Mohammed Aassila, Strasbourg, France.*

Prove that $\displaystyle\sum_{\text{cyclic}} \frac{ab}{c(c+a)} \geq \sum_{\text{cyclic}} \frac{a}{c+a}$, where a, b, c represent the three sides of a triangle.

. . . .

Montrer que $\displaystyle\sum_{\text{cyclic}} \frac{ab}{c(c+a)} \geq \sum_{\text{cyclic}} \frac{a}{c+a}$, où a, b, c représentent les trois côtés d'un triangle.

2860. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

In $\triangle ABC$ and $\triangle A'B'C'$, the lengths of the sides satisfy $a \geq b \geq c$ and $a' \geq b' \geq c'$. Let h_a and $h_{a'}$ denote the lengths of the altitudes to the opposite sides from A and A', respectively. Prove that

(a) $bb' + cc' \geq ah_{a'} + a'h_a$;

(b) $bc' + b'c \geq ah_{a'} + a'h_a$.

. . . .

Dans les triangles ABC et $A'B'C'$, les longueurs des côtés satisfont $a \geq b \geq c$ et $a' \geq b' \geq c'$. Soit h_a et $h_{a'}$ la longueur des hauteurs issues des sommets A et A'. Montrer que

(a) $bb' + cc' \geq ah_{a'} + a'h_a$;

(b) $bc' + b'c \geq ah_{a'} + a'h_a$.

2861. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

The circle $\Gamma(P, r)$ intersects the side AB of $\triangle ABC$ at A_3 and B_3, the side BC at B_1 and C_1, and the side CA at C_2 and A_2.

Given that $|A_3B_3| : |B_1C_1| : |C_2A_2| = |AB| : |BC| : |CA|$, determine the locus of P.

. . . .

Le cercle $\Gamma(P, r)$ coupe le côté AB du triangle ABC en A_3 et B_3, le côté BC en B_1 et C_1, et le côté CA en C_2 et A_2.

En supposant que $|A_3B_3| : |B_1C_1| : |C_2A_2| = |AB| : |BC| : |CA|$, déterminer le lieu de P.

2862. *Proposed by Mihály Bencze, Brasov, Romania.*

The sequence $\{x_n\}$ is defined by $\left(1+\frac{1}{n}\right)^{n+x_n} = e$.

(a) Prove that $\{x_n\}$ is convergent, and determine its limit.

(b)⋆ Determine the asymptotic expansion of the sequence.

. . . .

La suite $\{x_n\}$ est définie par $\left(1+\frac{1}{n}\right)^{n+x_n} = e$.

(a) Montrer que $\{x_n\}$ converge et trouver sa limite.

(b)⋆ Trouver le développement asymptotique de la suite.

2863. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that a, b, c are complex numbers such that $|a| = |b| = |c|$. Prove that

$$\left|\frac{ab}{a^2-b^2}\right| + \left|\frac{bc}{b^2-c^2}\right| + \left|\frac{ca}{c^2-a^2}\right| \geq \sqrt{3}\,.$$

. . . .

On suppose que a, b et c sont des nombres complexes tels que $|a| = |b| = |c|$. Montrer que

$$\left|\frac{ab}{a^2-b^2}\right| + \left|\frac{bc}{b^2-c^2}\right| + \left|\frac{ca}{c^2-a^2}\right| \geq \sqrt{3}\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologise for omitting the name of ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA from the list of solvers of 2717 and 2718, and the name of MICHEL BATAILLE, Rouen, France from the list of solvers of 2729.

2289★. [1997 : 501] *Proposed by Clark Kimberling, Evansville, IN, USA.*

Use any sequence, $\{c_k\}$, of 0's and 1's to define a *repetition-resistant sequence* $s = \{s_k\}$ inductively as follows :

1. $s_1 = c_1$, $s_2 = 1 - s_1$;
2. for $n \geq 2$, let

$$\begin{aligned} L &= \max\{i \geq 1 \ : \ (s_{m-i+2}, \ldots, s_m, s_{m+1}) \\ &\qquad\qquad = (s_{n-i+2}, \ldots, s_n, 0) \quad \text{for some } m < n\}, \\ L' &= \max\{i \geq 1 \ : \ (s_{m-i+2}, \ldots, s_m, s_{m+1}) \\ &\qquad\qquad = (s_{n-i+2}, \ldots, s_n, 1) \quad \text{for some } m < n\}. \end{aligned}$$

(so that L is the maximal length of the tail-sequence of $(s_1, s_2, \ldots, s_n, 0)$ that already occurs in $(s_1, s_2, \ldots, s_n)$, and similarly for L'), and

$$s_{n+1} = \begin{cases} 0 & \text{if } \ L < L', \\ 1 & \text{if } \ L > L', \\ c_n & \text{if } \ L = L'. \end{cases}$$

(For example, if $c_i = 0$ for all i, then

$$s = (0,1,0,0,0,1,1,1,0,1,0,1,1,1,0,0,1,0,0,$$
$$1,1,1,1,0,1,1,0,0,0,0,0,1,0,1,0,\ldots)\,)$$

Prove or disprove that s contains every binary word.

Solution by Alejandro Dau, Universidad de Buenos Aires, Buenos Aires, Argentina, modified slightly by the editors.

We will prove that every binary word occurs infinitely many times in s. For any symbol $\sigma \in \{0, 1\}$, we write $\overline{\sigma}$ for the opposite symbol.

We first show that the words of length 1 appear infinitely many times; that is, the symbols 0 and 1 each appear infinitely often in s. Each symbol appears at least once, because, by definition, s starts with 01 or 10. Suppose the symbol σ appears just finitely many times, the last time being at s_k.

Then $s_{k+i} = \overline{\sigma}$ for all $i \geq 1$. In particular, $s_{2k+2} = \overline{\sigma}$. But this contradicts the definition of s, because the string $s_{k+2} \dots s_{2k+2} = \overline{\sigma}^{k+1}$ already occurs in $s_1 \dots s_{2k+1}$ as $s_{k+1} \dots s_{2k+1}$, whereas the word $\overline{\sigma}^k \sigma$ does not occur as a substring of $s_1 \dots s_{2k+1}$. We conclude that each word of length 1 appears infinitely often in s.

Now suppose there is some word of length greater than 1 that appears only finitely often. Then there is a shortest length $n \geq 2$ for such words. Choose some word of length n that appears only finitely often, say $t = d_1 \dots d_n$, and let $k \geq 0$ be the number of times that t appears. The word $d_1 \dots d_{n-1}$ appears infinitely often, because its length is less than n. Letting $t^* = d_1 \dots d_{n-1}\overline{d_n}$, we see that t^* must appear infinitely often, since t appears only finitely often.

Suppose $k = 0$. Then the word t does not appear at all in s. Let the first two appearances of t^* end at positions i and j, respectively. Thus, $s_{i-n+1} \dots s_i = t^*$ (the first appearance) and $s_{j-n+1} \dots s_j = t^*$ (the second appearance). The fact that $s_j = \overline{d_n}$ contradicts the definition of s, because the word t^* already occurs in $s_1 \dots s_{j-1}$ as $s_{i-n+1} \dots s_i$, whereas the word t does not occur in $s_1 \dots s_{j-1}$.

Suppose $k \geq 1$. Then the word t appears at least once in s. Let the last appearance of t end at position p. Thus, $s_{p-n+1} \dots s_p = t$, and this is the last appearance of t in s. Consequently, there are no words in s of length $p+1$ that end with the string t. On the other hand, there must be some word of length $p+1$ ending in t^* that occurs more than once in s, because there are infinitely many appearances of t^* in s and only finitely many words of length $p+1$. Thus, there must exist two strings $s_{i-p} \dots s_i$ and $s_{j-p} \dots s_j$, with $i < j$, that represent the same word w ending in t^*. We have $s_j = \overline{d_n}$. But the definition of s requires that $s_j = d_n$, because the tail-sequence w ending in t^* already occurs in $s_1 \dots s_{j-1}$ as $s_{i-p} \dots s_i$, whereas there are no words in s of length $p+1$ ending in t. We have a contradiction.

A contradiction comes from assuming that a word appears in s only a finite number of times. Therefore, every word appears infinitely many times.

2664. [2001 : 403] *Remark : In his solution of a generalization of this problem, Walther Janous proved in* [2002 : 410] *the following result :*

Let a, b, c be non-negative reals and let $r \in \mathbb{R}$ with $r \geq 2$.

Then

$$a^r(b+c) + b^r(c+a) + c^r(a+b) \;\geq\; \frac{2}{3^{\frac{r-1}{2}}}(ab+bc+ca)^{\frac{r+1}{2}} . \qquad (1)$$

At the end of his solution he made the remark :

Clearly equality holds in our theorem if $r = 1$. This leads to the natural question : What happens for $r \in (1,2)$?

The following is a solution to this question.

Solution by Vasile Cirtoaje, University of Ploiesti, Romania.

We will prove using two different methods that (1) also holds for all $r \in (1,2)$.

Method I. We will prove (1) for $r \geq 1$ by showing that the function

$$f(x) = \frac{1}{x+1} \ln \left[\frac{a^x(b+c) + b^x(c+a) + c^x(a+b)}{6} \right]$$

is increasing on $[1,\infty)$; that is, by showing that $f'(x) \geq 0$ on $[1,\infty)$. The required inequality then follows from $f(r) \geq f(1)$.

The inequality $f'(x) \geq 0$ is equivalent to $A \geq C$, where

$$A = a^{(b+c)a^x(x+1)} b^{(c+a)b^x(x+1)} c^{(a+b)c^x(x+1)},$$

$$C = \left[\frac{a^x(b+c) + b^x(c+a) + c^x(a+b)}{6} \right]^{a^x(b+c)+b^x(c+a)+c^x(a+b)}.$$

We will show this by proving that $A \geq B \geq C$, where

$$B = a^{(b+c)a^x x + a(b^x+c^x)} b^{(c+a)b^x x + b(c^x+a^x)} c^{(a+b)c^x x + c(a^x+b^x)}.$$

The inequality $A \geq B$ is equivalent to $\ln A \geq \ln B$. To prove the latter true we must show that $g(x) \geq 0$ for all $x \geq 1$, where

$$\begin{aligned} g(x) &= [(b+c)a^x - a(b^x+c^x)] \ln a + [(c+a)b^x - b(c^x+a^x)] \ln b \\ &\quad + [(a+b)c^x - c(a^x+b^x)] \ln c. \end{aligned}$$

We have

$$\begin{aligned} g'(x) &= (b+c)a^x \ln^2 a + (c+a)b^x \ln^2 b + (a+b)c^x \ln^2 c \\ &\quad - (a^x b + ab^x) \ln a \ln b - (b^x c + bc^x) \ln b \ln c \\ &\quad - (c^x a + ca^x) \ln c \ln a \\ &= ab(\ln a - \ln b)(a^{x-1} \ln a - b^{x-1} \ln b) \\ &\quad + bc(\ln b - \ln c)(b^{x-1} \ln b - c^{x-1} \ln c) \\ &\quad + ca(\ln c - \ln a)(c^{x-1} \ln c - a^{x-1} \ln a). \end{aligned}$$

For $x \geq 1$, it easily follows that $g'(x) \geq 0$. Consequently, $g(x)$ is increasing on $[1,\infty)$; that is, $g(x) \geq g(1) = 0$.

To prove the inequality $B \geq C$, we rewrite this inequality as follows :

$$\begin{aligned} & x_1^{x_1} x_2^{x_2} x_3^{x_3} x_4^{x_4} x_5^{x_5} x_6^{x_6} \\ &\quad \geq \left(\frac{x_1+x_2+x_3+x_4+x_5+x_6}{6} \right)^{x_1+x_2+x_3+x_4+x_5+x_6}, \end{aligned}$$

where $x_1 = a^x b$, $x_2 = ab^x$, $x_3 = b^x c$, $x_4 = bc^x$, $x_5 = c^x a$, $x_6 = ca^x$. This known inequality follows from Jensen's Inequality applied to the convex function $h(x) = x \ln x$.

Method II. Since (1) is homogeneous, we can reformulate the problem (without loss of generality) as follows :

If a, b, c are positive integers such that $ab + bc + ca = 3$ and $r \in (1, 2)$, then

$$a^r(b+c) + b^r(c+a) + c^r(a+b) \;\geq\; 6\,. \qquad (2)$$

From $ab+bc+ca = 3$, we may write $a(b+c) = 3 - bc$, $b(c+a) = 3 - ca$, $c(a+b) = 3 - ab$, and (2) becomes

$$a^{r-1}(3-bc) + b^{r-1}(3-ca) + c^{r-1}(3-ab) \;\geq\; 6\,,$$

or

$$a^{r-1} + b^{r-1} + c^{r-1} \;\geq\; a^{r-1}b^{r-1}c^{r-1} \cdot \frac{(ab)^{2-r} + (bc)^{2-r} + (ca)^{2-r}}{3} + 2\,.$$

Since $0 < 2 - r < 1$, the function $f(x) = x^{2-r}$ is concave. Thus, by Jensen's Inequality, we have

$$\frac{(ab)^{2-r} + (bc)^{2-r} + (ca)^{2-r}}{3} \;\leq\; \left(\frac{ab+bc+ca}{3}\right)^{2-r} \;=\; 1\,,$$

and it suffices to prove that

$$a^{r-1} + b^{r-1} + c^{r-1} \;\geq\; a^{r-1}b^{r-1}c^{r-1} + 2\,. \qquad (3)$$

Because the inequality is symmetice, we may assume that $a \geq b \geq c$, without any loss of generality. Now we write (3) as

$$a^{r-1} + b^{r-1} - 2 \;\geq\; (a^{r-1}b^{r-1} - 1)\left(\frac{3-ab}{a+b}\right)^{r-1}\,.$$

Setting $x = \sqrt{ab}$, it follows from the AM-GM Inequality that $a + b \geq 2x$ and $a^{r-1} + b^{r-1} \geq 2x^{r-1}$. Since $a \geq b \geq c$ and $ab + bc + ca = 3$ imply $1 \leq x < \sqrt{3}$, it suffices to prove that

$$2(x^{r-1} - 1) \;\geq\; (x^{2r-2} - 1)\left(\frac{3-x^2}{2x}\right)^{r-1}\,.$$

Divinding by the non-negative factor $x^{r-1} - 1$, this last inequality becomes

$$2 \;\geq\; (x^{r-1} + 1)\left(\frac{3-x^2}{2x}\right)^{r-1}\,,$$

or

$$2 \;\geq\; \left(\frac{3-x^2}{2}\right)^{r-1} + \left(\frac{3-x^2}{2x}\right)^{r-1}\,.$$

This inequality is true because $1 \geq (3 - x^2)/2 \geq (3 - x^2)/(2x)$ for $x \geq 1$.

2682. [2001 : 461] and [2002 : 478] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The sequence of functions, $\{J(n) = J(n, w)\}$, $n = 0, 1, \ldots$, is defined as follows :

$$J(0) = a\,, \qquad J(1) = w + b\,,$$
$$J(n+1) = \frac{J(n)\ (J(n)\ (w\,J(n) - 1) - J(n-1))}{J(n-1)\ (w\,J(n) + 1) + J(n)} \qquad \text{for } n > 0.$$

(a) Show that, if $a = 0$, then the sequence consists of polynomials.

(b) Show that there exists a pair (a, b) of non-zero integers such that all the $J(n)$ are polynomials with integer coefficients.

Solution by Wangwei, student, and Jun-hua Huang, the Middle School attached to Hunan Normal University, Changsha, China.

The given recurrence is equivalent to

$$J(n+1)J(n-1)wJ(n) + J(n+1)J(n-1) + J(n+1)J(n) = w\big(J(n)\big)^3 - \big(J(n)\big)^2 - J(n)J(n-1)\,,$$

or $$wJ(n)\left(\big(J(n)\big)^2 - J(n+1)J(n-1)\right) = \big(J(n+1) + J(n)\big)\big(J(n) + J(n-1)\big)\,.$$

Setting $a_n = wJ(n)$, we obtain

$$a_n(a_n^2 - a_{n+1}a_{n-1}) = (a_{n+1} + a_n)(a_n + a_{n-1})\,. \qquad (1)$$

(a) Assume that $a = 0$. Let $x = wJ(1) = w(w + b)$.

Lemma : Let $\{b_n\}$ be the sequence defined by $b_0 = 0$, $b_1 = x$, and for all $n \geq 1$,

$$b_{n+1} = (x-2)b_n - b_{n-1} + x\,. \qquad (2)$$

Then, for all $n \geq 1$,

$$b_n^2 - b_{n+1}b_{n-1} = xb_n\,. \qquad (3)$$

Proof : The proof is by induction on n. When $n = 1$, equation (3) is true, since $b_0 = 0$ and $b_1 = x$. Now, suppose equation (3) is true for $n = k \geq 1$. Then

$$\begin{aligned} b_{k+1}^2 - b_{k+2}b_k &= \big((x-2)b_k + x - b_{k-1}\big)b_{k+1} \\ &\qquad - \big((x-2)b_{k+1} + x - b_k\big)b_k \\ &= (x-2)b_kb_{k+1} + xb_{k+1} - b_{k-1}b_{k+1} \\ &\qquad - (x-2)b_kb_{k+1} - xb_k + b_k^2 \\ &= xb_{k+1} + b_k^2 - b_{k-1}b_{k+1} - xb_k = xb_{k+1}\,. \end{aligned}$$

Therefore, the lemma holds by induction.

From (2) and (3), one can verify that

$$b_n(b_n^2 - b_{n+1}b_{n-1}) \;=\; (b_{n+1}+b_n)(b_n+b_{n-1})\,.$$

Comparing this to equation (1) and noting that

$$a_0 = wJ(0) = wa = 0 \qquad \text{and} \qquad a_1 = wJ(1) = w(w+b) = x\,,$$

we conclude that $a_n = b_n$ for all $n \geq 0$. Hence,

$$a_{n+1} \;=\; (w(w+b)-2)a_n - a_{n-1} + w(w+b)\,.$$

It is easy to show by induction that a_n is a polynomial in w. Therefore, $J(n)$ is a polynomial in w, since w is a factor of a_n for all n.

(b) Taking $(a,b) = (1,-1)$, we have $J(0) = 1$ and $J(1) = w-1$. Then $a_0 = w$ and $a_1 = w(w-1)$. Let $\{c_n\}$ be the sequence defined by $c_0 = w$, $c_1 = w(w-1)$, and for all $n \geq 1$,

$$c_{n+1} \;=\; (w-2)c_n - c_{n-1} + w\,.$$

Using the same inductive proof as in the lemma above we can establish

$$c_n^2 - c_{n+1}c_{n-1} \;=\; wc_n\,,$$

for all $n \geq 1$ (the only difference in the proofs is in the initial step). Therefore, $a_n = c_n$ for all $n \geq 0$, which means that, for all $n \geq 1$, we have

$$a_{n+1} \;=\; (w-2)a_n + w - a_{n-1}\,.$$

It is easy to show that a_n is a polynomial with integer coefficients for all $n \geq 0$. Since w is a factor of a_n for all $n \geq 0$, we see that $J(n) = J(n,w)$ is a polynomial with integer coefficients for all $n \geq 1$.

2741. [2002 : 245] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

In the plane are given an ellipse with its two foci M and N, and two points, A and B, on it, so that $AB \parallel MN$.

With only an unmarked straight-edge, construct a diameter of the circle $ABNM$.

I. *Solution by Michel Bataille, Rouen, France.*

We shall construct the diameter along the minor axis l of the ellipse (which is the perpendicular bisector of both MN and AB). If AM and BN meet the ellipse again at C and D, respectively, l passes through the intersection points of AN with BM, of DM with CN, and of AM with BN (at most one of which could fail to exist due to parallel defining lines). Note that by symmetry, l passes through the centre of the circle $ABNM$ whose diameter we are to construct.

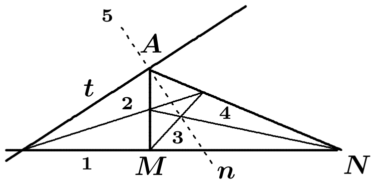


Figure 1

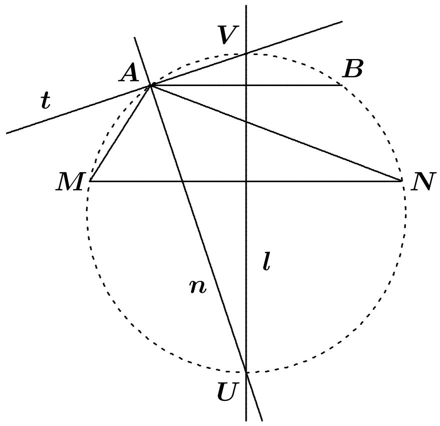


Figure 2

Using only a straight-edge, we now construct the tangent t and normal n to the conic at A. The tangent comes from an application of Pascal's Theorem (as described in problem 2740 [2003 : 246]), while n is the harmonic conjugate of t with respect to AM and AN, whose construction is shown in Figure 1, where the lines are numbered in the order of their appearance in the construction. Let t and n meet l at V and U, respectively. (See Figure 2.) The desired diameter is UV. Indeed, since n is the internal angle bisector of $\angle MAN$, it meets the circle at the mid-point of the arc MN not containing A. This point is also on l. Furthermore t, being perpendicular to n at A, meets the circle at the point diametrically opposite U. This point is V, since it is also on l.

II. *Solution by Václav Konečný, Big Rapids, MI, USA.*

We construct the diameter NI through the focus N. To do this we make use of the following straight-edge constructions.

(a) *Construct the tangent to a given conic at a given point.* This construction was described in problem 2740.

(b) *Construct the line passing through a given point and parallel to a given segment whose mid-point is also given.* This construction was used in problem 2695 [2002 : 553–554]; it can also be found in *A Survey of Geometry* by Howard Eves, p. 175, where there is a discussion of straight-edge constructions. In Figure 3 we are given segment XY with mid-point Z and a point P not on the line XY, and we construct the line through P parallel to XY.

(c) *Construct the line through a focus that is perpendicular to a chord through that focus, given the conic, focus, and corresponding directrix.* This is problem #2 of the J.I.R. McKnight Problems Contest 1982, [1998 : 232]. [Begin by extending the given chord BN beyond N to the point R where it meets the directrix. In Figure 4, RS is the directrix and SN is the desired perpendicular to BN. Perpendicularity is a consequence of the projective property that conjugate lines through a focus are perpendicular. Alternatively, Konečný verified that $BN \perp NS$ using coordinates.]

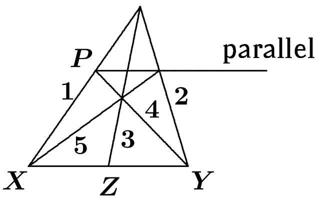


Figure 3

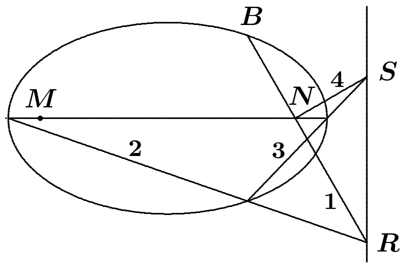


Figure 4

Here is the construction. (See Figure 5.)

1. Draw MN (the major axis), then construct [as in solution I] the minor axis l of the ellipse (the line that is perpendicular to the major axis MN at the centre O of the ellipse).
2. Using (b), construct the line through the focus N parallel to the minor axis (whose mid-point is O). Denote by N' one of the points where this line meets the conic.

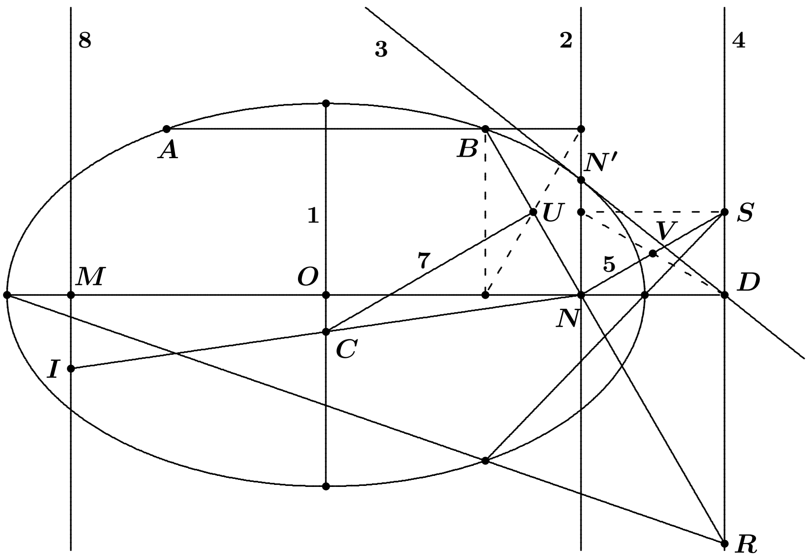


Figure 5

3. By (a), construct the tangent at N'. Call D the point where this tangent intersects the extension of the major axis. Then D lies on the directrix, since it is conjugate to N.

4. By (b), construct the line through D parallel to the minor axis. This line is the directrix corresponding to N.

5. By (c), draw the perpendicular NS to BN at N, where S is on the directrix.

6. We need to construct the mid-points U of BN and V of NS. Since O is the mid-point of the major axis, we can construct the parallel to the major axis through S. Since O is the mid-point of the minor axis, we can construct the parallel to the minor axis through B. These lines complete a pair of rectangles, one with corners B and N whose diagonals intersect at U, the other with corners N and S whose diagonals intersect at V.

7. Since V is the mid-point of NS, we can construct the parallel to NS through U. This is the perpendicular bisector of NB. It therefore passes through the centre of circle $ABNM$, which must be the point C where it intersects the minor axis.

8. Draw the parallel through M to the minor axis, and let I be the point where this line intersects NC. By symmetry, NI is a diameter of circle $ABNM$.

Also solved by VÁCLAV KONEČNÝ, Big Rapids, MI, USA (a second solution); PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

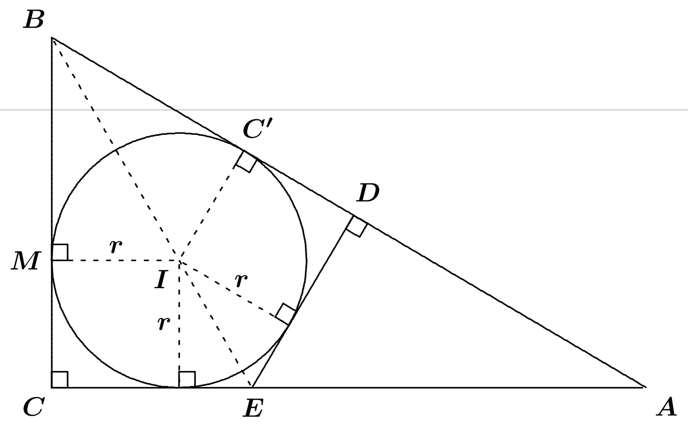
2745. [2002 : 246] *Proposed by K.R.S. Sastry, Bangalore, India.*

Let ABC be a primitive Pythagorean triangle (that is, the gcd of the sides is 1) in which $\angle ACB$ is the right angle. Let D be a point in AB and E a point in AC such that DE is perpendicular to AB and also tangent to the incircle of $\triangle ABC$.

Prove that BE has rational length if and only if the length of AB is the square of an integer.

Combination of solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let a, b, and c be the lengths of the sides BC, CA, and AB, respectively. We show that the problem is correct under the extra assumption that b is even.



First, we show that BE is the bisector of $\angle ABC$. Let C' and M be the points of contact of the incircle of $\triangle ABC$ and the lines AB and BC. Then $BM = BC'$. If r is the radius of the incircle, then

$$BC \;=\; BM + MC \;=\; BM + r \;=\; BC' + r \;=\; BC' + C'D \;=\; BD\,.$$

Similarly, $EC = ED$. Hence, BE is the bisector of $\angle ABC$.

It is well-known that

$$BE \;=\; \frac{2\sqrt{ac\,s(s-b)}}{a+c}\,,$$

where s is the semiperimeter of $\triangle ABC$. We have

$$\begin{aligned} BE \;&=\; \frac{\sqrt{ac(a+b+c)(a-b+c)}}{a+c} \;=\; \frac{\sqrt{ac[(a+c)^2-b^2]}}{a+c} \\ &=\; \frac{\sqrt{ac[(a+c)^2-(c^2-a^2)]}}{a+c} \;=\; \frac{\sqrt{2a^2c(a+c)}}{a+c} \;=\; a\sqrt{\frac{2c}{a+c}}\,. \end{aligned}$$

If b is even, then $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, where m and n are positive integers with $m > n$ and $\gcd(m,n) = 1$, so that $BE = \frac{(m^2-n^2)\sqrt{c}}{m}$ is rational if and only if c is a square. If b is odd, the problem statement is incorrect. For example, if $a = 24$ and $b = 7$, then $c = 25$, and $BE = \frac{120\sqrt{2}}{7}$ is not rational.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2751. [2002 : 328] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

On each side of $\triangle ABC$, draw squares outwards to create six new points, D, E, F, G, H, and I. Characterise those triangles such that the points D, E, F, G, H, and I are concyclic.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

We assume that D, E, F, G, H, and I are concyclic and labelled so that the squares are $ABHI$, $BCDE$, and $CAFG$. Since F, G, H, and I are concyclic, we have $\angle HIG = \angle HFG$. Since $\angle HIA = \angle AFG = 90^\circ$, we get $\angle AIG = \angle AFH$. If these angles are not zero, we have $\triangle AIG \sim \triangle AFH$ (because $\angle IAG = \angle FAH$). This implies $AI : AG = AF : AH$; that is, $AB : \sqrt{2}AC = AC : \sqrt{2}AB$, which then implies that $AB = AC$. On the other hand, if $\angle AIG = \angle AFH = 0$, then I, A, G are collinear, as are F, A, H. In this case, $\angle BAC = 45^\circ$.

Similarly, since D, E, F, G concyclic, we have either $AC = BC$ or $\angle ACB = 45^\circ$. Combining these conditions we get the following four cases.

Case 1. $AB = AC$ and $AC = BC$. Thus, $\triangle ABC$ is equilateral.

Case 2. $AB = AC$ and $\angle ACB = 45^\circ$. Then $\triangle ABC$ is an isosceles right triangle with right angle at A.

Case 3. $\angle BAC = 45^\circ$ and $AC = BC$. Then, as in Case 2, $\triangle ABC$ is an isosceles right triangle with right angle at C.

Case 4. $\angle BAC = 45^\circ$ and $\angle ACB = 45^\circ$. Then $\triangle ABC$ is an isosceles right triangle with right angle at B.

We conclude that $\triangle ABC$ is either equilateral or it is an isosceles right triangle. The converse, if $\triangle ABC$ is either an equilateral triangle or an isosceles right triangle then D, E, F, G, H, and I are concyclic, is straightforward.

Editor's comment : Seimiya provided the details for the converse, but we leave them for the reader in order to make room for an alternative solution.

II. *Combination of solutions by Christopher J. Bradley, Clifton College, Bristol, UK and D.J. Smeenk, Zaltbommel, the Netherlands.*

If D, E, F, G, H, I are concyclic, then the centre of circle $DEFGHI$ must lie on the perpendicular bisector of DE, and hence, on the perpendicular bisector of BC. Using the same argument for FG (with AC), we conclude that the centre must be the circumcentre O of $\triangle ABC$. Thus, the proposed problem is equivalent to characterizing those triangles such that

$$OD = OF = OH . \qquad (1)$$

Let M be the mid-point of BC and M' the mid-point of DE. Let R be the radius of the circumscribed circle of $\triangle ABC$. We have

$$\begin{aligned} OD^2 &= OM'^2 + M'D^2 \;=\; (OM + MM')^2 + M'D^2 \\ &= (R\cos A + 2R\sin A)^2 + (R\sin A)^2 \\ &= R^2(3 + 2\sin 2A - 2\cos 2A)\,, \end{aligned}$$

With similar calculations for OF and OH, we can rewrite (1) as

$$\sin 2A - \cos 2A \;=\; \sin 2B - \cos 2B \;=\; \sin 2C - \cos 2C\,. \qquad (2)$$

One possibility for equality is $\angle A = \angle B = \angle C$.

Suppose that $\angle A \neq \angle B$. Since $\sin 2A - \cos 2A = \sin 2B - \cos 2B$ can be written as $\sin(2A - 45^\circ) = \sin(2B - 45^\circ)$, we conclude that $(2A - 45^\circ) + (2B - 45^\circ) = 180^\circ$. In this case $C = 45^\circ$. Now consider angles A and C. Either $A = C\,(= 45^\circ)$ or $A \neq C$, in which case the same argument as above using (2) gives us $B = 45^\circ$. In either case, $\triangle ABC$ is an isosceles right triangle.

We conclude that the remote vertices of the squares are concyclic if and only if $\triangle ABC$ is either an equilateral triangle or an isosceles right triangle.

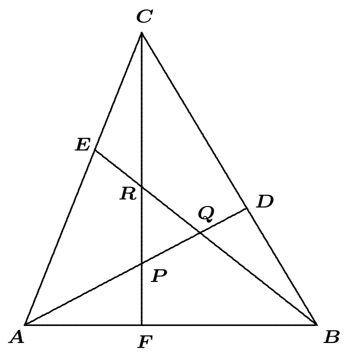
Also solved by MICHEL BATAILLE, Rouen, France; ROBERT BILINSKI, Outremont, QC; IAN JUNE L. GARCES and WINFER C. TABARES, Manila, The Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; BRUCE SHAWYER, Memorial University of Newfoundland, St. John's, NL; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Loeffler, using an argument as in Solution II, explored what would happen if the squares were constructed inwards. He easily found that there would be no non-trivial solutions. With one inward and two outward squares there are various degenerate triangles, but also the isosceles right triangle with an inward square on the hypotenuse (in which case E coincides with I, and D coincides with F). Finally, with two squares constructed inward and one outward there are again various trivial solutions along with a solution triangle having angles 30°, 75°, and 75°.

2752. [2002 : 329] *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A generalization of Putnam 2001, question A4.

Suppose that $\dfrac{AF}{FB} = i$, $\dfrac{BD}{DC} = g$ and $\dfrac{CE}{EA} = h$. Determine the area of $\triangle PQR$ as a proportion of the area of $\triangle ABC$.



Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The content of this problem is a very famous one ; indeed, its answer is given by the following :

Routh's Theorem. *With the notation of the problem, we have :*

$$\frac{[PQR]}{[ABC]} = \frac{(ghi-1)^2}{(gh+g+1)(hi+h+1)(ig+i+1)}$$

and

$$\frac{[DEF]}{[ABC]} = \frac{ghi+1}{(g+1)(h+1)(i+1)} .$$

where $[XYZ]$ represents the area of the figure XYZ.

Remark. As two consequences of this theorem, we have :

- **Ceva's Theorem**. The transversals AD, BE, and CF are concurrent if and only if $ghi = 1$.
- **Menelaus' Theorem**. The three points D, E, and F on the (extended) sides of $\triangle ABC$ are collinear if and only if $ghi = -1$.

Reference.

[1] Z.A. Melzak, *Introduction to Geometry*, John Wiley and Sons, New York, 1983.

Also solved by MICHEL BATAILLE, Rouen, France ; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain ; PIERRE BORNSZTEIN, Pontoise, France ; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC ; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK ; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines ; ZELJKO HANJŠ, University of Zagreb, Zagreb, Croatia ; RICHARD I. HESS, Rancho Palos Verdes, CA, USA ; JOHN G. HEUVER, Grande Prairie, AB ; JOE HOWARD, Portales, NM, USA ; GEOFFREY A. KANDALL, Hamden, CT, USA ; VÁCLAV KONEČNÝ, Big Rapids, MI, USA ; RICK MABRY, LSU, Shreveport, LA, USA ; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA ; TOSHIO SEIMIYA, Kawasaki, Japan ; STAN WAGON, Macalester College, St. Paul, MN, USA ; LI ZHOU, Polk Community College, Winter Haven, FL, USA ; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania ; and the proposer.

About half of the solvers actually gave a proof, sometimes with a reference, and the rest gave multiple references to the literature. Francisco Bellot Rosado is very interested in obtaining a copy of L.A. Graham, *Ingenious Mathematical Problems and Methods*, Dover Publications, NY, 1950, *where the problem is treated in several ways. Can anyone help him? Contact* fbellot@hotmail.com.

2753. [2002 : 329] *Proposed by Mikhail Kotchetov, Memorial University of Newfoundland, St. John's, NL.*

Consider two circles, Γ_1 and Γ_2, centres O_1 and O_2, respectively, of different radii.

The two common tangents, t_1 and t_2, that do not intersect the line segment O_1O_2 meet at Q. A common tangent, t_c that does intersect the line segment O_1O_2 meets the tangents t_1 and t_2 at E_1 and E_2, respectively.

Let P be the mid-point of the line segment O_1O_2.

Prove that P, Q, E_1, and E_2 are concyclic.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Suppose, without loss of generality, that Γ_2 has smaller radius than Γ_1. Consider $\triangle E_1E_1Q$. Its incircle is Γ_2 and its excircle opposite Q is Γ_1.

It is well known that the mid-point of the line segment joining the incentre to an excentre lies on the circumcircle of the triangle. Thus, P, the mid-point of O_1O_2, lies on the circumcircle of $\triangle E_1E_2Q$.

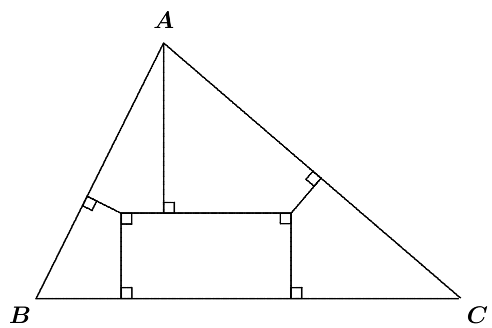
Also solved by MICHEL BATAILLE, Rouen, France; P. BAUTISTA and IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2754. [2002 : 330] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Divide a triangle into five concyclic quadrilaterals.

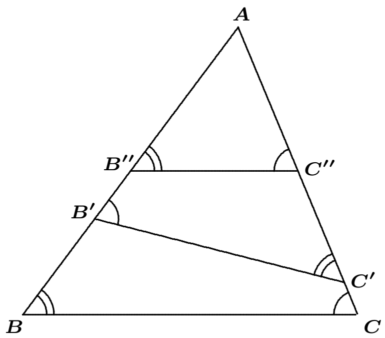
I. *Virtually identical solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let the triangle be ABC; assume that $\angle A$ is the largest angle. Then one class of solutions is given in the diagram :



II. *Combination of solutions by John G. Heuver, Grande Prairie, AB, David Loeffler, student, Trinity College, Cambridge, UK, Peter Y. Woo, Biola University, La Mirada, CA, USA, and Titu Zvonaru, Bucharest, Romania.*

We will again let the triangle be ABC, and assume that $\angle A$ is the largest angle. Let P be any interior point (for example, the incentre of $\triangle ABC$) such that the perpendiculars dropped from P to the sides of $\triangle ABC$ intersect the sides at an interior point of each side. The three quadrilaterals into which these perpendiculars subdivide $\triangle ABC$ are all clearly concyclic.



Now construct any line $B'C'$ where B' lies on AB and C' lies on AC such that $\angle AB'C' = \angle ACB$ and $\angle AC'B' = \angle ABC$. Also construct the line $B''C''$ parallel to BC where B'' lies on AB' and C'' lies on AC'. (See the diagram above.) Clearly, the quadrilaterals $B'C'CB$ and $B''C''C'B'$ are both concyclic. Furthermore, using the argument in the first paragraph we can subdivide $\triangle AB''C''$ into three concyclic quadrilaterals. This gives us a class of solutions to the problem.

Note that the number five in the problem statement can be replaced by any integer greater than or equal to 3.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Woo remarks : "For acute triangles, it seems possible to draw the quadrilaterals so that they have the same area. I have not found out the condition on ABC for which this is possible." *Perhaps a reader can help to answer this question.*

2755. [2002 : 180] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{f_{n+1}^2}{1+f_n f_{n+1}^2 f_{n+2}}\right)$$

where f_n is the n^{th} Fibonacci number (that is, $f_0 = 0$, $f_1 = 1$ and, for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$).

Solution by D. Kipp Johnson, Beaverton, OR, USA.
We have

$$f_{n+1}^2 \;=\; (f_{n+2}-f_n)f_{n+1} \;=\; f_{n+2}f_{n+1}-f_{n+1}f_n\,.$$

Using the trigonometric identity

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) \;=\; \tan^{-1}x-\tan^{-1}y\,,$$

we may write

$$\begin{aligned}\tan^{-1}\left(\frac{f_{n+1}^2}{1+f_nf_{n+1}^2f_{n+2}}\right) &= \tan^{-1}\left(\frac{f_{n+2}f_{n+1}-f_{n+1}f_n}{1+f_{n+2}f_{n+1}\cdot f_{n+1}f_n}\right)\\ &= \tan^{-1}(f_{n+2}f_{n+1})-\tan^{-1}(f_{n+1}f_n)\,.\end{aligned}$$

Let S be the sum we seek. Then

$$\begin{aligned} S &= \lim_{k\to\infty}\sum_{n=1}^{k}\left(\tan^{-1}(f_{n+2}f_{n+1})-\tan^{-1}(f_{n+1}f_n)\right)\\ &= \lim_{k\to\infty}\left(\tan^{-1}(f_{k+2}f_{k+1})-\tan^{-1}(f_2f_1)\right)\,,\end{aligned}$$

since the sum telescopes. Now the product $f_{k+2}f_{k+1}$ increases without bound and $f_2f_1=1$. Hence,

$$S \;=\; \frac{\pi}{2}-\frac{\pi}{4} \;=\; \frac{\pi}{4}\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; OVIDIU FURDUI, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Howard points out that the problem can be solved by an application of the results in L. Bragg, Arctangent Sums, *College Math. J.* **32**(4) (2001), *pp. 255–56.*

2756. [2002 : 330] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Circles $\Gamma_1(O,R)$ and $\Gamma_2(I,r)$ touch line t at D, where $R>r$ and O and I lie on the same side of t. The point A is any point on Γ_1. The tangents to Γ_2 through A intersect t at B and C, respectively. Denote the inradii of $\triangle ABD$ and $\triangle ACD$ by r_1 and r_2, respectively.

Show that r_1+r_2 is constant as A varies on Γ_1.

Comment.

Several solvers noticed that this problem and its solution have appeared before in Crux as Problem 2320 [1998 : 108 ; 1999 : 126–127].

Solved by MICHEL BATAILLE, Rouen, France ; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain ; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria ; TOSHIO SEIMIYA, Kawasaki, Japan ; PETER Y. WOO, Biola University, La Mirada, CA, USA ; LI ZHOU, Polk Community College, Winter Haven, FL, USA ; and the proposer. Janous has noted that the claim is true only if the point A moves along the arc of the circle Γ_2 lying above the tangent to the circle Γ_1 parallel to the common tangent at D.

2757★. [2002 : 331] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, and C be the angles of a triangle. Show that

$$\sum_{\text{cyclic}} \frac{1}{\tan\left(\frac{A}{2}\right) + 8\tan\left(\frac{\pi - A}{4}\right)^3} \;\le\; \frac{9\sqrt{3}}{11}\,.$$

Solution by Manuel Benito, Oscar Ciaurri and Emilio Fernández, Logroño, Spain (modified by the editor).

Define the function f on $[0,\pi]$ by $f(\pi) = 0$, and for $0 \le x < \pi$,

$$f(x) \;=\; \frac{1}{\tan\left(\frac{x}{2}\right) + 8\tan^3\left(\frac{\pi - x}{4}\right)}$$

Then clearly $f(x) > 0$ for all $x \in [0,\pi]$. We shall prove that the maximum of the function $J(x,y,z) = f(x) + f(y) + f(z)$ over the compact set

$$T \;=\; \{(x,y,z) \in \mathbb{R}^3 : 0 \le x, y, z \le \pi\,;\;\; x + y + z = \pi\}$$

is equal to $\frac{9\sqrt{3}}{11} \approx 1.41713$. Our proof consists of five parts.

I. Variation of $f(x)$ on $[0,\pi]$.

By direct computation, we find that

$$\begin{aligned} f'(x) &= -(f(x))^2\left[\frac{1}{2}\sec^2\left(\frac{x}{2}\right) + 24\tan^2\left(\frac{\pi - x}{4}\right)\sec^2\left(\frac{\pi - x}{4}\right)\left(-\frac{1}{4}\right)\right] \\ &= \frac{1}{2}(f(x))^2\left[12\tan^2\left(\frac{\pi - x}{4}\right)\left(1 + \tan^2\left(\frac{\pi - x}{4}\right)\right) - 1 - \tan^2\left(\frac{x}{2}\right)\right]\,. \quad (1) \end{aligned}$$

Set $r = \tan\left(\frac{x}{2}\right)$, $s = \tan\left(\frac{\pi - x}{4}\right)$ and $t = \tan\left(\frac{x}{4}\right)$. Then for $x \ne \pi$ we have $r = \frac{2t}{1 - t^2}$, and $s = \frac{1 - t}{1 + t}$. Eliminating t, we get $r = \frac{1 - s^2}{2s}$. Since $f(x) > 0$ for $x \in [0,\pi)$, setting $f'(x) = 0$ yields

$$\begin{aligned} 12s^2(1 + s^2) \;&=\; 1 + \left(\frac{1 - s^2}{2s}\right)^2 \;=\; \frac{(1 + s^2)^2}{4s^2} \\ \text{or}\quad (1 + s^2)(48s^4 - s^2 - 1) \;&=\; 0\,. \end{aligned}$$

Solving $48s^4 - s^2 - 1 = 0$, we get $s^2 = \dfrac{1+\sqrt{193}}{96}$. Thus, the only positive root is $s_0 = \sqrt{\dfrac{1+\sqrt{193}}{96}}$. Hence, $\xi_0 = \pi - 4\tan^{-1}(s_0) \approx 1.64076$ is the only critical value of $f(x)$ on $(0,\pi)$. Since $f(0) = 0.125$, $f(\pi) = 0$, and $f(\xi_0) \approx 0.640475$, we conclude that

$$\max\{f(x) : 0 \le x \le \pi\} \;=\; f(\xi_0) = M\,.$$

II. Values of $J(x,y,z)$ on ∂T, the boundary of T.
At the vertices of ∂T, we have

$$J(\pi,0,0) \;=\; J(0,\pi,0) \;=\; J(0,0,\pi) \;=\; 0.25\,.$$

In the interior of ∂T, one of the three coordinates is zero, and therefore, for all $\xi \in (0,\pi)$, we have

$$J(x,y,z) \;=\; f(0) + f(\xi) + f(\pi-\xi) \;\le\; 0.125 + 2M \;<\; 1.406 \;<\; \frac{9\sqrt{3}}{11}\,.$$

III. Variation of $f'(x)$ on $[0,\pi]$.
Differentiating (1) we obtain, after some simplifications,

$$f''(x) \;=\; f(x)\left[f'(x)A(x) + \tfrac{1}{2}f(x)B(x)\right]\,,$$

where

$$A(x) \;=\; 12\tan^4\left(\frac{\pi-x}{4}\right) + 12\tan^2\left(\frac{\pi-x}{4}\right) - \tan^2\left(\frac{x}{2}\right) - 1\,,$$

and

$$\begin{aligned} B(x) \;=\; A'(x) \;=\; -12\tan^5\left(\frac{\pi-x}{4}\right) - 18\tan^3\left(\frac{\pi-x}{4}\right)& \\ - 6\tan\left(\frac{\pi-x}{4}\right) - \tan^3\left(\frac{x}{2}\right) - \tan\left(\frac{x}{2}\right)&\,. \end{aligned}$$

We set $f''(x) = 0$ to search for solutions in $(0,\pi)$. Using the definition of $f(x)$, the fact that $f(x) > 0$, and the right hand side of (1) to substitute for $f'(x)$, we are led to the equation :

$$\begin{aligned} &\left[12\tan^4\left(\frac{\pi-x}{4}\right) + 12\tan^2\left(\frac{\pi-x}{4}\right) - \tan^2\left(\frac{x}{2}\right) - 1\right]^2 \\ &= \left[8\tan^3\left(\frac{\pi-x}{4}\right) + \tan\left(\frac{x}{2}\right)\right]\left[12\tan^5\left(\frac{\pi-x}{4}\right) + 18\tan^3\left(\frac{\pi-x}{4}\right)\right. \\ &\qquad\left. + 6\tan\left(\frac{\pi-x}{4}\right) + \tan^3\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right)\right]\,. \end{aligned} \tag{2}$$

In terms of the variables r, $s \in (0,1)$ introduced earlier, equation (2) becomes

$$(12s^4 + 12s^2 - r^2 - 1)^2 \;=\; (8s^3 + r)(12s^5 + 18s^3 + 6s + r^3 + r)\,. \tag{3}$$

Substituting $r = \dfrac{1-s^2}{2s}$ and $1+r^2 = \dfrac{(1+s^2)^2}{4s^2}$ into (3), we then have

$$\left(12s^4 + 12s^2 - \frac{\left(1+s^2\right)^2}{4s^2}\right)^2 = \left(8s^3 + \frac{1-s^2}{2s}\right)\left(12s^5 + 18s^3 + 6s + \frac{\left(1-s^2\right)\left(1+s^2\right)^2}{8s^3}\right)$$

or $$\left(1+s^2\right)^2\left(48s^4 - s^2 - 1\right)^2 = \left(16s^4 - s^2 + 1\right)\left[8s^3\left(12s^5 + 18s^3 + 6s\right) + \left(1-s^2\right)\left(1+s^2\right)^2\right].$$

Simplifying, we obtain

$$4s^2(s^2+1)(192s^8 + 388s^6 - 60s^4 - 39s^2 + 1) = 0. \qquad (4)$$

[*Ed :* This can be checked and verified easily using MAPLE or some other computer algebra system.]

Equation (4) has exactly two solutions in $(0,1)$, namely : $s_1 \approx 0.595805$ and $s_2 \approx 0.157625$, yielding the following two critical values of $f'(x)$: $\xi_1 = \pi - 4\tan^{-1}(s_1) \approx 0.992276$ and $\xi_2 = \pi - 4\tan^{-1}(s_2) \approx 2.51623$, respectively. Also, for the argument to be used in Part IV below, we need to find the value ξ_3 in $(0,\pi)$ such that $f'(\xi_3) = f'(0) = \frac{23}{128}$. This condition leads to the following equation in s :

$$(s^2+1)(48s^4 - s^2 - 1) = \frac{23}{64}(16s^4 - s^2 + 1)^2$$

or $\quad (1-s^2)(5888s^6 + 2080s^4 - 169s^2 - 87) = 0$.

The second factor in the equation above has a *unique* positive root $s_3 \approx 0.439150$ which yields a unique $\xi_3 = \pi - 4\tan^{-1}(s_3) \approx 1.48641$. Incorporating all the information obtained above, we can summarize the variation of $f'(x)$ on $[0,\pi]$ in the chart below :

x	0	$\cdots$	ξ_1 ≈ 0.99	$\cdots$	ξ_3 ≈ 1.49	$\cdots$	ξ_0 ≈ 1.64	$\cdots$	ξ_2 ≈ 2.51	$\cdots$	π
$f'(x)$	$\frac{23}{128}$	$\nearrow$	≈ 0.449	$\searrow$	$\frac{23}{128}$	$\searrow$	0	$\searrow$	≈ -0.526	$\nearrow$	-0.5

IV. A necessary condition for the relative extrema of $J(x,y,z)$ to occur in the interior of T.

Suppose $J(x,y,z)$ attains a relative extremum at the interior point (x_0,y_0,z_0) of T. Then the method of Lagrange multipliers assures that there exists a $\lambda_0 \in \mathbb{R}$ such that

$$L_x(x_0,y_0,z_0,\lambda_0) = L_y(x_0,y_0,z_0,\lambda_0) = L_z(x_0,y_0,z_0,\lambda_0) = 0,$$

where $L(x,y,z,\lambda) = f(x) + f(y) + f(z) - \lambda(x+y+z-\pi)$. Thus, we have $f'(x_0) = f'(y_0) = f'(z_0) = \lambda_0$ with $0 \le x_0,\ y_0,\ z_0 \le \pi$ such that $x_0 + y_0 + z_0 = \pi$.

We explore the possibilities for these conditions to be satisfied by using the information on the variation of $f'(x)$ to look at the possible location of the value λ_0 in the range of $f'(x)$.

If $f'(\xi_2) \le \lambda_0 < \frac{23}{128} = f'(\xi_3)$, then $f'(x_0) = f'(y_0) = f'(z_0) = \lambda_0$ would imply that $x_0 + y_0 + z_0 > 3\xi_3 > \pi$, which is a contradiction. Furthermore, if $\lambda_0 = f'(\xi_1)$, then $x_0 = y_0 = z_0 = \xi_1$ implies that $3\xi_1 = \pi$, another contradiction. Hence, we must have $\frac{23}{128} \le \lambda_0 < f'(\xi_1)$, in which case $f'(x) = \lambda_0$ has exactly two solutions in $[0, \xi_3]$. Therefore, in order for $f'(x_0) = f'(y_0) = f'(z_0) = \lambda_0$ to hold, at least two of x_0, y_0, and z_0 must be the same, and the problem now reduces to finding the maximum value on $[0, \xi_3]$ of the single variable function $\tilde{J}(x) = 2f(x) + f(\pi - 2x)$.

V. Maximum value of $\tilde{J}(x) = 2f(x) + f(\pi - 2x)$ on $[0, \xi_3]$.

Since $\tilde{J}'(x) = 2f'(x) - 2f'(\pi - 2x)$, we see that $\tilde{J}'(x) = 0$ if and only if $f'(x) = f'(\pi - 2x)$. Using (1), this is equivalent to :

$$\begin{aligned}(f(x))^2 &\left[12\tan^4\left(\frac{\pi - x}{4}\right) + 12\tan^2\left(\frac{\pi - x}{4}\right) - \tan^2\left(\frac{x}{2}\right) - 1\right] \\ &= \left[f(\pi - 2x)\right]^2 \left[12\tan^4\left(\frac{x}{2}\right) + 12\tan^2\left(\frac{x}{2}\right) - \cot^2 x - 1\right]\end{aligned}$$

or

$$\begin{aligned}&\left[\cot x + 8\tan^3\left(\frac{x}{2}\right)\right]^2\left[12\tan^4\left(\frac{\pi - x}{4}\right) + 12\tan^2\left(\frac{\pi - x}{4}\right) - \tan^2\left(\frac{x}{2}\right) - 1\right] \\ &\qquad = \left[\tan\left(\frac{x}{2}\right) + 8\tan^3\left(\frac{\pi - x}{4}\right)\right]^2 \cdot \\ &\qquad\qquad \left[12\tan^4\left(\frac{x}{2}\right) + 12\tan^2\left(\frac{x}{2}\right) - \cot^2 x - 1\right] . \qquad (5)\end{aligned}$$

Again using $r = \tan\left(\frac{x}{2}\right)$ and $s = \tan\left(\frac{\pi - x}{4}\right)$, we have

$$\cot x = \frac{1}{\tan 2\left(\frac{x}{2}\right)} = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{2\tan\left(\frac{x}{2}\right)} = \frac{1 - r^2}{2r} = \frac{-s^4 + 6s^2 - 1}{4s(1 - s^2)} .$$

Substituting into (5) and simplifying, we obtain the following equation in s, where $s \in (0, 1)$,

$$\begin{aligned}&(4 - 17s^2 + 30s^4 - 17s^6 + 4s^8)^2(48s^6 + 47s^4 - 2s^2 - 1) \\ &= s^2(1 - s^2 + 16s^4)^2 \cdot \\ &\quad \left[12(1 - s^2)^6 + 48s^2(1 - s^2)^4 - s^2(-s^4 + 6s^2 - 1)^2 - 16s^4(1 - s^2)^2\right] ,\end{aligned}$$

or, on further simplifying and factoring,

$$\begin{aligned}4(s^2 + 1)(1 - 3s^2)(192s^{18} - 212s^{16} - 1617s^{14} + 4406s^{12} \\ - 5404s^{10} + 3258s^8 - 1136s^6 + 154s^4 + 15s^2 - 4) = 0 .\end{aligned}$$

On the interval $(0, 1)$, the last equation has exactly three solutions, namely, $\sigma_0 = \frac{1}{\sqrt{3}}$, $\sigma_1 \approx 0.521949$ and $\sigma_2 \approx 0.477039$ yielding the three critical values of $\tilde{J}(x)$ in $(0, \xi_3)$:

$$\begin{aligned} x_0 &= \pi - 4\tan^{-1}(\sigma_0) = \frac{\pi}{3}\,, \\ x_1 &= \pi - 4\tan^{-1}(\sigma_1) \approx 1.21738\,, \\ \text{and} \qquad x_2 &= \pi - 4\tan^{-1}(\sigma_2) \approx 1.36115\,. \end{aligned}$$

By direct computations, we find that

$$\tilde{J}(x_0) \;=\; \frac{9\sqrt{3}}{11}\,, \quad \tilde{J}(x_1) \approx 1.41514\,, \quad \text{and} \quad \tilde{J}(x_2) \approx 1.41615\,.$$

Since $\tilde{J}(0) = 2f(0)+f(\pi) = 0.25$ and $\tilde{J}(\xi_3) = 2f(\xi_3)+f(\pi-\xi_3) \approx 1.4116$, we finally conclude that the maximum value of $\tilde{J}(x)$ on $[0, \xi_3]$, and consequently of the function $J(x,y,z)$ over T, is $\frac{9\sqrt{3}}{11}$, which is attained if and only if $x = y = z = \frac{\pi}{3}$.

There was one incorrect solution.

2758. [2002 : 331] *José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

If $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \neq 2$, determine all real numbers x, y, z such that

$$\begin{aligned} 0 \;=\; & (1+2a^2)x^2 + (1+2b^2)y^2 + (1+2c^2)z^2 \\ & + 2xy(ab-a-b) + 2yz(bc-b-c) + 2zx(ca-c-a)\,. \end{aligned}$$

Solution by Michel Bataille, Rouen, France.

The given equation can be written as

$$(ax+by-z)^2 + (ax-y+cz)^2 + (-x+by+cz)^2 \;=\; 0\,.$$

It follows that its solutions are those of the homogeneous linear system

$$\begin{aligned} ax+by-z &= 0\,, \\ ax-y+cz &= 0\,, \\ -x+by+cz &= 0\,. \end{aligned}$$

The determinant of this system is equal to $1 - ab - bc - ca - 2abc$. The condition

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \;\neq\; 2$$

is equivalent to $1 - ab - bc - ca - 2abc \neq 0$. Therefore, $x = y = z = 0$ is the only solution of the homogeneous system and the only solution of the given equation.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; JOE HOWARD, Portales, NM, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposers.

2759. [2002 : 331] *Proposed by Michel Bataille, Rouen, France.*

On the line segment AB, let C, D be such that $\frac{AC}{CB} = \frac{BD}{DA} = \frac{1}{3}$. Distinct points M_1, M_2, M_3 lie on a circle passing through B and C and are such that $\angle M_1BC = 2\angle M_1CB$, $\angle M_2BC = 2\angle M_2CB$, and $\angle M_3AD = 2\angle M_3DA$. Show that $\triangle M_1M_2M_3$ is equilateral.

Solution by Toshio Seimiya, Kawasaki, Japan.

Lemma 1. Let $\triangle ABC$ be a triangle with $\angle B = 2\angle C$; let D be the foot of the perpendicular from A to BC; and let M be the mid-point of BC. Then $AB = 2DM$.

Proof. We set $\alpha = \angle C$. Then $\angle B = 2\alpha$. Let N be the mid-point of AC. Then NM is parallel to AB, $NM = \frac{1}{2}AB$, and $ND = NA = NC$. Thus, $\angle NDC = \angle NCD = \alpha$. Since $NM \| AB$, $\angle NMC = \angle ABC = 2\alpha$. Therefore, $\angle MND = \angle NMC - \angle NDC = 2\alpha - \alpha = \alpha = \angle MDN$, so that $DM = NM = \frac{1}{2}AB$; that is, $AB = 2DM$.

Lemma 2. Let $\triangle ABC$ be a triangle with $AB < AC$; let D be the foot of the perpendicular from A to BC; and let M be the mid-point of BC. If $AC = 2DM$, then $\angle ABC = 90^\circ + \frac{1}{2}\angle ACB$.

Proof. Let N be the mid-point of AC. Then NM is parallel to AB, and $ND = NA = NC$. Since $DM = \frac{1}{2}AC = DN$, we deduce that $\angle NMC = 90^\circ + \frac{1}{2}\angle NDM$. Since $ND = NC$, we obtain $\angle NDM = \angle NCD = \angle ACB$, so that $\angle NMC = 90^\circ + \frac{1}{2}\angle ACB$. Since $NM \| AB$, we have $\angle NMC = \angle ABC$. Thus, $\angle ABC = 90^\circ + \frac{1}{2}\angle ACB$.

Now we turn to the original problem. We assume that M_1 and M_2 lie on the major and minor arcs BC, respectively. Then M_3 lies on the major arc BC. We set $\alpha = \angle M_1CB$ and $\beta = \angle M_2CB$. Then $\angle M_1BC = 2\alpha$ and $\angle M_2BC = 2\beta$. Since $\angle M_1CM_2 + \angle M_1BM_2 = 180^\circ$, we have $(\alpha+\beta)+(2\alpha+2\beta) = 180^\circ$. Thus $\alpha+\beta = 60^\circ$. We set $AB = 4a$; then $AC = a$, $CD = 2a$, and $DB = a$. Let T be the foot of the perpendicular from M_3 to AC. We set $x = TC$. Let M be the mid-point of AD, then $CM = \frac{1}{2}(CD - AC) = \frac{1}{2}(2a - a) = \frac{a}{2}$. Since $\angle M_3AD = 2\angle M_3DA$, we have (by virtue of lemma 1) $M_3A = 2TM = 2\left(x + \frac{1}{2}a\right) = 2x + a$. Since $M_3T \perp AB$, we get $M_3B^2 - M_3A^2 = TB^2 - TA^2$. Thus, we have

$$\begin{aligned} M_3B^2 &= M_3A^2 + TB^2 - TA^2 \\ &= (2x+a)^2 + (x+3a)^2 - (a-x)^2 \\ &= 4x^2 + 12ax + 9a^2 \;=\; (2x+3a)^2 \,. \end{aligned}$$

Hence, $M_3B = 2x + 3a$.

Let N be the mid-point of CB; then $TN = TC + CN = x + \frac{3a}{2}$, or $2TN = 2x + 3a = M_3B$. Set $\gamma = \angle M_3BC$. By lemma 2, we get $\angle M_3CB = 90^\circ + \frac{1}{2}\gamma$. Since $\angle M_3CM_2 + \angle M_3BM_2 = 180^\circ$, we have

$$\left(90^\circ + \tfrac{1}{2}\gamma + \beta\right) + (\gamma + 2\beta) \;=\; 180^\circ\,;$$

that is, $180^\circ + 3(\gamma + 2\beta) = 360^\circ$, from which we get $\gamma + 2\beta = 60^\circ$. Therefore, $\angle M_3M_1M_2 = \angle M_3BM_2 = \gamma + 2\beta = 60^\circ$, which implies that $\angle M_1M_3M_2 = \angle M_1CM_2 = \alpha + \beta = 60^\circ$. Hence, $\triangle M_1M_2M_3$ is equilateral.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2760. [2002 : 332] (corrected in [2002 : 396]) *Proposed by Michel Bataille, Rouen, France.*

Suppose that A, B, C are the angles of a triangle. Prove that

$$\begin{aligned} 8(\cos A + \cos B + \cos C) \;&\le\; 9 + \cos(A-B) + \cos(B-C) + \cos(C-A) \\ &\le\; \csc^2(A/2) + \csc^2(B/2) + \csc^2(C/2)\,. \end{aligned}$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Since $f(x) = \csc^2 x$ is convex on $(0,\pi)$ [*Ed :* Direct computations show that $f''(x) = 2\csc^4 x + 4\csc^2 x\cot^2 x > 0$], we have

$$\begin{aligned} \csc^2\left(\frac{A}{2}\right) + \csc^2\left(\frac{B}{2}\right) + \csc^2\left(\frac{C}{2}\right) \;&\ge\; 3\csc^2\left(\frac{A+B+C}{6}\right) \;=\; 12 \\ &\ge\; 9 + \cos(A-B) + \cos(B-C) + \cos(C-A)\,. \end{aligned}$$

For the left inequality, note first that

$$\begin{aligned} &\cos A + \cos B + \cos C \\ &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\left(\frac{C}{2}\right) \\ &= 1 + 2\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]\cos\left(\frac{A+B}{2}\right) \\ &= 1 + 2\left[2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right]\sin\left(\frac{C}{2}\right) \\ &= 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \qquad (1) \end{aligned}$$

Also,

$$
\begin{aligned}
&\cos(A-B)+\cos(B-C)+\cos(C-A)\\
&= 2\cos\left(\frac{A-C}{2}\right)\cos\left(\frac{A+C-2B}{2}\right)+2\cos^2\left(\frac{C-A}{2}\right)-1\\
&= 2\cos\left(\frac{C-A}{2}\right)\left[\cos\left(\frac{A+C-2B}{2}\right)+\cos\left(\frac{C-A}{2}\right)\right]-1\\
&= 4\cos\left(\frac{C-A}{2}\right)\cos\left(\frac{C-B}{2}\right)\cos\left(\frac{A-B}{2}\right)-1 \qquad (2)
\end{aligned}
$$

From (1) and (2), we see that the left inequality is equivalent to

$$
8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \;\le\; \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right),
$$

which has been shown at least three times previously in Crux : 585 [1981 : 303], 2472 [2000 : 440–441], and 2717 [2003 : 119–120].

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; SCOTT H. BROWN, Auburn University, Montgomery, AL, USA; JOE HOWARD, Portales, NM, USA (2 solutions); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was one incorrect solution.

Most solvers used various known results from the classic book "Geometric Inequalities" by O. Bottema et al. Of course, identity (1) in the solution featured above can also be found in this book (§2.16, p. 22). In particular, Klamkin, Lau, and Loeffler all cited Gerretsen's Inequality, $s^2 \ge 16rR - 5r^2$*, and the famous Euler Inequality,* $2r \le R$*, where* r *and* R *denote the inradius and circumradius of triangle* ABC*, respectively, and* s *denotes its semiperimeter.*

Bencze gave various refinements which were results that he obtained in 1995–96; for example, he showed that the left inequality can be refined to :

$$
8F(a,b,c)\cdot\sum_{\text{cyclic}}\cos A \;\le\; 1+8F(a,b,c)+\sum_{\text{cyclic}}\cos(A-B)\,, \qquad (*)
$$

where

$$
F(a,b,c) \;=\; F \;=\; \left(1+\frac{(\sqrt{a}-\sqrt{b})^2}{8\sqrt{ab}}\right)\left(1+\frac{(\sqrt{b}-\sqrt{c})^2}{8\sqrt{bc}}\right)\left(1+\frac{(\sqrt{c}-\sqrt{a})^2}{8\sqrt{ca}}\right).
$$

Clearly, $F(a,b,c) \;\ge\; 1$*. Since it is well known that* $\sum_{\text{cyclic}}\cos A \;>\; 1$*, we conclude that* $8(F-1)\cdot\sum_{\text{cyclic}}\cos A \;\ge\; 8(F-1)$ *which implies by* $(*)$ *that*

$$
8\sum_{\text{cyclic}}\cos A \;\le\; 8-8F+8F\cdot\sum_{\text{cyclic}}\cos A \;\le\; 9+\sum_{\text{cyclic}}\cos(A-B)\,.
$$

2761★. [2002 : 332] *Proposed by Edgar G. Goodaire, Memorial University, St. John's, NF.*

Give a proof by vectors that the medians of a triangle have a common point of intersection : a proof, however, ***which does not presuppose the answer***.

The vector proofs of this result with which I am familiar answer the question posed this way :

Prove that the medians of $\triangle ABC$ intersect at $\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)$, where O is the origin.

The proof, of course, then amounts simply to showing that this point is on each median.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let A be the origin, and let $\overrightarrow{b}$ and $\overrightarrow{c}$ be representatives of the vectors AB and AC. The midpoints L, M, and N of sides BC, CA, and AB have position vectors $\frac{1}{2}(\overrightarrow{b}+\overrightarrow{c})$, $\frac{1}{2}\overrightarrow{c}$ and $\frac{1}{2}\overrightarrow{b}$, respectively. The vector equations of AL, BM, and CN are as follows :

$$\begin{aligned} AL: &\quad \overrightarrow{r} = \tfrac{1}{2}s\left(\overrightarrow{b}+\overrightarrow{c}\right) \\ BM: &\quad \overrightarrow{r} = \overrightarrow{b}+t\left(\tfrac{1}{2}\overrightarrow{c}-\overrightarrow{b}\right) \\ CN: &\quad \overrightarrow{r} = \overrightarrow{c}+u\left(\tfrac{1}{2}\overrightarrow{b}-\overrightarrow{c}\right). \end{aligned}$$

These three lines pass through a common point if and only if there exist values of s, t, and u such that

$$\tfrac{1}{2}s(\overrightarrow{b}+\overrightarrow{c}) = (1-t)\overrightarrow{b}+\tfrac{1}{2}t\overrightarrow{c} = (1-u)\overrightarrow{c}+\tfrac{1}{2}u\overrightarrow{b}.$$

Since the vectors $\overrightarrow{b}$ and $\overrightarrow{c}$ are linearly independent, the above system of vector equations is equivalent to

$$\begin{aligned} \tfrac{1}{2}s &= (1-t) = \tfrac{1}{2}u, \\ \tfrac{1}{2}s &= \tfrac{1}{2}t = (1-u). \end{aligned}$$

The only solution is $s=u=t=\frac{2}{3}$.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; M. Benito, O. Ciaurri and E. Fernández, Logroño, Spain; ELIAS BUISSANT DES AMORIE, CJ Castricum, the Netherlands; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, NS; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, *AB; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. Most of the submitted solutions are similar to the above.*

Bellot has located two solutions published earlier: E. Donath, Die merkwürdigen Punkte und Linien des ebenen Dreiecks, *VEB, Berlin, 1969 (in German), and E.M. Patterson,* Vector Algebra, *Oliver & Boyd, London, 1968.*

2762. [2002 : 332] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Quadrilateral $ABCD$ is inscribed in circle Γ. The tangents at A, B, C, D to Γ are t_A, t_B, t_C, t_D, respectively. Given that BD, t_A, and t_C are concurrent, prove that AC, t_B, and t_D are concurrent.

Initial comment. Since t_B and t_D can be parallel, the problem's conclusion should be that these two tangents are concurrent with AC or are parallel to it.

I. *Nearly identical solutions by Michel Bataille, Rouen, France; David Loeffler, student, Trinity College, Cambridge, UK; and Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let BD, t_A, and t_C be concurrent at E. As the point of intersection of t_A and t_C, E is the pole of AC with respect to Γ. Since the pole of AC lies on BD, the pole of BD, call it F, lies on AC (because a polarity is an incidence-preserving involution). But F is the point where t_B and t_D meet. Thus, AC, t_B, and t_D are concurrent at F. Note that for this projective argument, Γ can be any conic, not just a circle.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Suppose that BD, t_A, and t_C intersect at E. [Without any loss of generality, we may assume that B and E are on the same side of AC.] Let O be the centre of Γ, and suppose that OE intersects AC at M. Then $OE \perp AC$. If BD is a diameter of Γ, then AC, t_B, and t_D are all perpendicular to OE, in which case $AC \| t_B \| t_D$. Otherwise, AC intersects t_B at some point F. Then O, M, B, F are on a circle Ω with OF as a diameter. Hence, $\angle OFB = \angle EMB$. On the other hand, $EB \cdot ED = EC^2 = EO \cdot EM$; thus, $\triangle EBM \sim \triangle EOD$. Consequently, $\angle EDO = \angle EMB = \angle OFB$, which implies that D is on Ω as well. Therefore $OD \perp FD$, which means that FD coincides with t_D.

III. *Solution by Christopher J. Bradley, Clifton College, Bristol, UK.*

Take the circle Γ to have equation $x^2 + y^2 = 1$, and let

$$T = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

be a point on the circumference. The equations of the tangent at A [$t = a$], the tangent at C [$t = c$], and the line BD [from $t = b$ to $t = d$], respectively, are

$$\begin{array}{rrcl} t_A : & (1-a^2)x + 2ay & = & 1+a^2\,; \\ t_C : & (1-c^2)x + 2cy & = & 1+c^2\,; \\ BD : & (1-bd)x + (b+d)y & = & 1+bd\,. \end{array}$$

If these are concurrent, then

$$\det \begin{bmatrix} 1-a^2 & 2a & -(1+a^2) \\ 1-c^2 & 2c & -(1+c^2) \\ 1-bd & b+d & -(1+bd) \end{bmatrix} \;=\; 0\,.$$

That is,

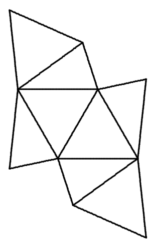
$$-2(a-c)(ab+ad+cb+cd-2ac-2bd) \;=\; 0\,.$$

Since $a \neq c$, we deduce that $(a+c)(b+d) = 2(ac+bd)$. This expression is symmetrical — it does not change when the pair of letters a and c is interchanged with the pair b and d. Consequently, AC, t_B, and t_D are concurrent (or parallel).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria ; TOSHIO SEIMIYA, Kawasaki, Japan ; PETER Y. WOO, Biola University, La Mirada, CA, USA (a second solution) ; TITU ZVONARU, Bucharest, Romania ; and the proposer.

2763. [2002 : 397] *Proposé par Izidor Hafner, Faculty of Electrical Engineering, Ljubljana, Slovénie.*

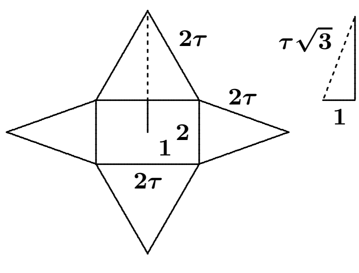
Le développement montré à la droite consiste en 4 triangles équilatéraux dont les côtés mesurent 2τ (deux fois le nombre d'or, $\tau = \frac{\sqrt{5}+1}{2}$), et 4 triangles isocèles dont le petit côté mesure 2. Noter qu'en pliant le développement, on peut obtenir deux polyèdres convexes. Ont-ils le même volume ?



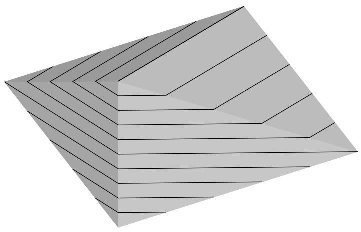
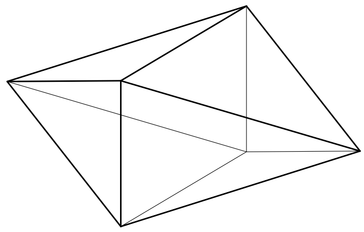
Solution de Robert Bilinski, Outremont, QC.

La première configuration est un octaèdre formé de 2 pyramides collées à la base. Le patron de chaque pyramide est formé d'un rectangle ayant pour côtés 2 et 2τ, d'une paire de triangles équilatéraux et d'une paire de triangles isocèles.

La hauteur des triangles équilatéraux mesure $\tau\sqrt{3}$. En utilisant la symétrie des pyramides, on peut calculer leur hauteur en formant un triangle rectangle composé d'une hauteur de triangle équilatéral et une demi-largeur allant de la base de la hauteur du côté au centre du rectangle.



La hauteur de la pyramide est donc $\sqrt{3\tau^2 - 1}$ par pythagore.

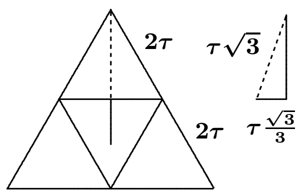
 

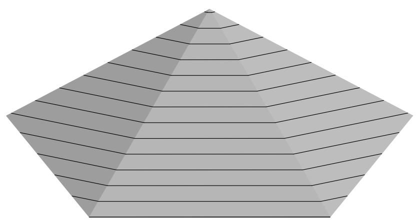
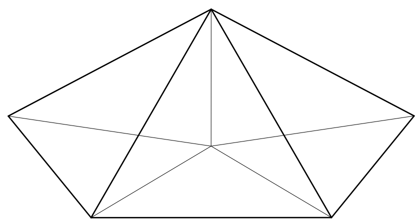
On a donc

$$\text{Volume Pyramide } = \frac{\text{Base}\cdot\text{hauteur}}{3} = \frac{4\tau\sqrt{3\tau^2-1}}{3}$$
$$\text{et Volume Octaèdre } = \frac{8\tau\sqrt{3\tau^2-1}}{3} \approx 11{,}2962 \text{ unité}^2\,.$$

La deuxième configuration est formée de trois tétraèdres. Il y en a un qui est régulier d'arête 2τ que nous appèlerons Volume A et 2 tétraèdres formés chacun de 2 triangles équilatéraux et 2 triangles isocèles que nous appèlerons volume B.

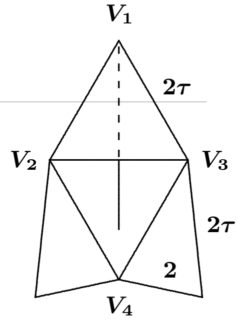
En utilisant la symétrie du tétraèdre A, on peut calculer sa hauteur en formant un triangle rectangle formé avec une hauteur de triangle équilatéral et un tiers de hauteur de triangle équilatéral allant de la base de la hauteur du côté au centre de gravité de la base.



La hauteur de la pyramide est donc $\frac{2\tau\sqrt{6}}{3}$. La base étant un triangle équilatéral a une aire de $\tau^2\sqrt{3}$. Ainsi, le volume $A = \frac{2\tau^3\sqrt{2}}{3}$.

Pour trouver le volume du tétraèdre B, on utilisera le déterminant de Cayley-Menger (voir [1]). Numérotons les sommets du tétraèdre V_1, V_2, V_3 et V_4. Alors les valeurs d_{ij} dans le déterminant représentent les longueurs des côtés du tétraèdre.



$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix} .$$

Cela devient $288V^2 = 384\tau^4 - 128\tau^2$ qui se simplifie en

$$\text{Volume } B = \frac{2}{3}\tau\sqrt{3\tau^2 - 1} .$$

Ainsi la configuration 2 a un volume total de

$$\frac{2\tau^3\sqrt{2}}{3} + \frac{4}{3}\tau\sqrt{3\tau^2 - 1} \approx 9,64189 \text{ unité}^2 .$$

On voit immédiatement que la première configuration est plus volumineuse.

Référence :

[1] http://mathworld.wolfram.com/Tetrahedron.html

Solutioné aussi par MICHEL BATAILLE, Rouen, France ; D. KIPP JOHNSON, Beaverton, OR, USA ; PETER Y. WOO, Biola University, La Mirada, CA, USA ; et le proposeur.

2764. [2002 : 397] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Find an integer-sided scalene triangle in which the lengths of the internal bisectors all have integer values.

Solution by D. Kipp Johnson, Beaverton, OR, USA.

We can find such a triangle by considering Heronian triangles (which have integer sides and integer area). It is known that if p, q, r are positive integers with $r^2 > pq$, then the following formulas produce a Heronian triangle with sides of length a, b, and c:

$$a = p(q^2 + r^2) , \quad b = q(p^2 + r^2) , \quad c = (p + q)(r^2 - pq) .$$

It is also known that the three angle bisectors of the triangle with sides of length a, b, c are

$$w_a = \sqrt{bc\left(1 - \frac{a^2}{(b+c)^2}\right)} , \quad w_b = \sqrt{ac\left(1 - \frac{b^2}{(a+c)^2}\right)} ,$$

$$w_c \;=\; \sqrt{ab\left(1-\frac{c^2}{(a+b)^2}\right)}\,.$$

Substituting the above values for the sides of a Heronian triangle and simplifying gives

$$\begin{aligned} w_a &= \frac{2qr(r^2-pq)(p+q)}{pr^2+2qr^2-pq^2}\sqrt{p^2+r^2}\,, \\ w_b &= \frac{2pr(r^2-pq)(p+q)}{2pr^2-p^2q+qr^2}\sqrt{q^2+r^2}\,, \\ w_c &= \frac{2pqr}{r^2+pq}\sqrt{(p^2+r^2)(q^2+r^2)}\,. \end{aligned}$$

(The denominators are positive if $r^2 > pq$.) The three angle bisectors will have rational lengths if p^2+r^2 and q^2+r^2 are perfect squares. If the sides or angle bisectors are then rational but not integral, we can scale up the entire triangle by an appropriate factor to make them all integers. It is not hard to find some acceptable values for p, q, r: $p = 7$, $q = 32$, $r = 24$ (with $r^2 - pq = 352 > 0$ as required), making $p^2 + r^2 = 25^2$ and $q^2 + r^2 = 40^2$. These values yield $a = 11200$, $b = 20000$, $c = 13728$.

We substitute these values of a, b, c into the formulas above for the angle bisectors and get rational numbers with denominators of 527, 779, and 1. Scaling by a factor of $529 \cdot 779$ yields a solution to the problem: $a = 4\,597\,969\,600$, $b = 8\,210\,660\,000$, $c = 4\,635\,797\,024$, which yield angle bisectors of $w_a = 256\,658\,688$, $w_b = 75\,963\,888$, $w_c = 5\,517\,563\,520$.

Also solved by the proposer, whose example had sides of length $315\,409\,500$, $388\,584\,504$, *and* $426\,433\,644$. *For this triangle the internal bisectors are* $312\,405\,600$, $278\,555\,200$, *and* $375\,350\,976$. *As a bonus, the inradius of the triangle is also an integer:* $r = 104\,085\,135$.

2765. [2002 : 397] *Proposed by K.R.S. Sastry, Bangalore, India.*

Derive a set of side length expressions for the family of Heron triangles ABC in which the nine-point centre V lies on side BC. (A Heron triangle has integer sides and integer area.) [See problem 2525(April) [2000 : 177; 2001 : 270].]

Solution by Michel Bataille, Rouen, France.

From problem 2525 we know that V is on the line BC if and only if $\cos(B-C) = 0$; that is, $B = C + \frac{\pi}{2}$ or $C = B + \frac{\pi}{2}$. We will determine the sides of the Heron triangles satisfying $B = C + \frac{\pi}{2}$; those satisfying $C = B + \frac{\pi}{2}$ are obtained by interchanging b and c.

Let K be the area of some suitable $\triangle ABC$. Then the Law of Sines yields $\dfrac{b}{\cos C} = \dfrac{c}{\sin C} = \dfrac{abc}{2K}$, so that $\cos C = \dfrac{2K}{ac}$ and $\sin C = \dfrac{2K}{ab}$. Hence,

$4K^2(b^2+c^2) = a^2b^2c^2$, and we see that b^2+c^2 is a perfect square, say $b^2+c^2=\lambda^2$. Furthermore,

$$\frac{b^2+c^2-a^2}{2bc} = \cos A = \cos\left(\frac{\pi}{2}-2C\right)$$
$$= \sin 2C = 2\sin C\cos C = \frac{8K^2}{a^2bc};$$

hence, $a^2(b^2+c^2-a^2) = 16K^2$. Thus, $b^2+c^2-a^2=\mu^2$ for some positive integer μ. Note that $2\lambda K = abc$ and $4K = \mu a$, so that $\lambda\mu = 2bc$. These results easily lead to

$$a^2 = \lambda^2-\mu^2\,,\quad (b-c)^2 = \lambda(\lambda-\mu)\,,\quad (b+c)^2 = \lambda(\lambda+\mu)\,.$$

Now, from $\lambda^2 = a^2+\mu^2$, we have either

(1) $a=2dmn$, $\mu = d(m^2-n^2)$, $\lambda = d(m^2+n^2)$, or

(2) $a = d(m^2-n^2)$, $\mu = 2dmn$, $\lambda = d(m^2+n^2)$,

for some positive integers d, m, n such that m, n are coprime, of opposite parity, and $m>n$.

In case (1), $(b-c)^2 = 2d^2n^2(m^2+n^2)$, which calls for $2(m^2+n^2)=k^2$ for some positive integer k. But this is impossible since m^2+n^2 is odd. Thus, we must be in case (2), which leads to $(b-c)^2 = d^2(m-n)^2(m^2+n^2)$, and $(b+c)^2 = d^2(m+n)^2(m^2+n^2)$.

Hence, $m^2+n^2=k^2$ for some positive integer k. This gives $b=dkm$ and $c=dkn$ and we may conclude that the sides a, b, c are given by

$$a = d(m^2-n^2)\,,\quad b = dkm\,,\quad c = dkn$$

where (m,n,k) is a primitive Pythagorean triple (with $m>n$) and d a positive integer.

Conversely, suppose that a, b, and c satisfy these relations. From $\frac{a}{b-c} = \frac{m+n}{k}$, and $\frac{b+c}{a} = \frac{k}{m-n}$, and $(m-n)^2 < k^2 < (m+n)^2$, we easily deduce that $b-c<a<b+c$, and ABC is actually a triangle. Heron's formula gives $4K^2 = d^4m^2n^2(m^2-n^2)^2$; whence, K is an integer (because m or n is even). Hence, $\triangle ABC$ is a Heron triangle. Moreover, the relation $\frac{a^2+c^2-b^2}{2ac} = -\frac{2K}{ab}$ is easily checked and means that $\cos B = -\sin C$; that is, $B = C+\frac{\pi}{2}$ and V is on BC.

Note. V is on the line *segment* BC if the additional condition $\cos(A-B)\cdot\cos(C-A)>0$ holds (because V, supposed on the line BC, has $(0, b\cos(C-A), a\cos(A-B))$ for *areal* coordinates relative to (A,B,C)). The condition may also be written as $\cos(C-B)+\cos(3A-\pi)>0$ or

$\cos(3A) < 0$. This yields $C < \frac{\pi}{6}$ or $\cos C > \sqrt{3/2}$. From the results above, it is easy to see that this will be the case if $m^2 > 3n^2$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

For explicit examples one can set m equal to the larger of $u^2 - v^2$ or $2uv$ (where u, v are relatively prime positive integers, $v < u < (\sqrt{2}+1)v$), and n equal to the smaller. Again d is an arbitrary positive integer. Then

$$a = d|u^4 - 6u^2v^2 + v^4|\,, \quad b = d(u^4 - v^4)\,, \quad c = 2duv(u^2 + v^2)\,.$$

The area is $uv(u^2 - v^2)|u^4 - 6u^2v^2 + v^4|$.

Sastry, in addition to providing a solution to his problem and an alternative treatment of problem 2525, showed that the following properties are equivalent:

(1) The nine-point centre is on BC.

(2) $|B - C| = \frac{\pi}{2}$.

(3) $\tan B \tan A = -1$.

(4) $OA \| BC$.

(5) AH is tangent to the circumcircle of $\triangle ABC$.

(6) AH is tangent to the nine-point circle of $\triangle ABC$.

(7) BC bisects AH.

(8) $AN = \frac{1}{2}OH$.

(9) AC, BC trisect $\angle OCH$.

2767. [2002 : 398] *Proposed by K.R.S. Sastry, Bangalore, India.*

The points $O(0,0)$, $A(1,0)$, $B(0,1)$ are given. Let $D(a,0)$, $E(1-a,a)$, $F(0,1-a)$ be variable points on the sides of $\triangle OAB$ ($0 < a < 1$). Let P denote the point of concurrence of the circles ODF, DEA, and BFE. Determine the locus of P.

Combination of solutions by Robert Bilinski, Outremont, QC and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Because P is opposite the right angle at O on the circle $ODPF$ (which implies that FD is the diameter), we see that $\left(\frac{a}{2}, \frac{1-a}{2}\right)$ must be the centre; hence, its equation is

$$x^2 - ax + y^2 - (1-a)y \;=\; 0\,. \tag{1}$$

For circle AED, its centre K is the corner of an isosceles right triangle KNM with right angle at the mid-point $M\left(\dfrac{1+a}{2}, 0\right)$ of DA, while N (where the perpendicular bisector KN of AE meets OA) is $(1-a, 0)$. Hence, K is $\left(\dfrac{1+a}{2}, \dfrac{3a-1}{2}\right)$, and the circle is

$$x^2 - (1+a)x + y^2 - (3a-1)y \;=\; -a\,. \tag{2}$$

For the locus of P we eliminate a by subtracting (2) from (1) to get $a = \dfrac{x - 2y}{1 - 4y}$, then substituting back into (1). Routine algebra leads to

$$\left(x - \frac{3}{8}\right)^2 + \left(y - \frac{3}{8}\right)^2 = \frac{1}{32}; \qquad (3)$$

hence, P lies on the circle with centre $\left(\frac{3}{8}, \frac{3}{8}\right)$ and radius $1/(4\sqrt{2})$.

We next investigate what portion of the circle (3) is traced by P as a runs from 0 to 1. Note that the point $U\left(\frac{2}{5}, \frac{1}{5}\right)$ on the median through B satisfies equation (2) for all a, so that U is on all the circles AED. By symmetry, one can argue that the point $\left(\frac{1}{5}, \frac{2}{5}\right)$ on the median through A lies on BFE for all values of a. It follows that the locus can be described as the set of points (other than E) where a circle in the pencil of circles through A and U intersects the corresponding circle of the pencil through B and V. It is easier, however, to obtain the coordinates of P as the point other than $D(a, 0)$ satisfying (1) and (2) simultaneously:

$$P = \left(\frac{4a^2 - 5a + 2}{16a^2 - 16a + 5}, \frac{4a^2 - 3a + 1}{16a^2 - 16a + 5}\right).$$

One simply checks that $P = U$ when $a = 0$, P is the mid-point of AB when $a = \frac{1}{2}$, and $P = V$ when $a = 1$. Thus, P sweeps out the arc of the circle (3) above the line UV from U to V as a goes from 0 to 1.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; D. KIPP JOHNSON, Beaverton, OR, USA; Mª JESÚS VILLAR RUBIO, Santander, Spain; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.