SKOLIAD No. 68

Shawn Godin

Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6 or emailed to

mayhem-editors@cms.math.ca.

We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 September 2003*. A copy of MATHEMATICAL MAYHEM Vol. 2 will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The items in this issue come from the Mandelbrot competition. This competition has four rounds that occur during the school year. Each round has two competitions: an individual test, and a team test written a couple of days later. A school's score is made up of the top four individual scores plus the score of the team (comprised of students selected by the school). Each test is 40 minutes in duration, and no aids of any type are allowed. My thanks go to Sam Vandervelde at Greater Testing Concepts for forwarding the material to me. For more information about the contest you can visit the website

www.mandelbrot.org

The Mandelbrot Competition
Division B Round One Individual Test
November 1997

1. (*) What angle less than 180° is formed by the hands of a clock at 2:30 pm? (Express the answer in degrees.) **(1 point)**

2. (*) If $x = \sqrt{\frac{6}{7}}$, then evaluate $\left(x + \frac{1}{x}\right)^2$. **(1 point)**

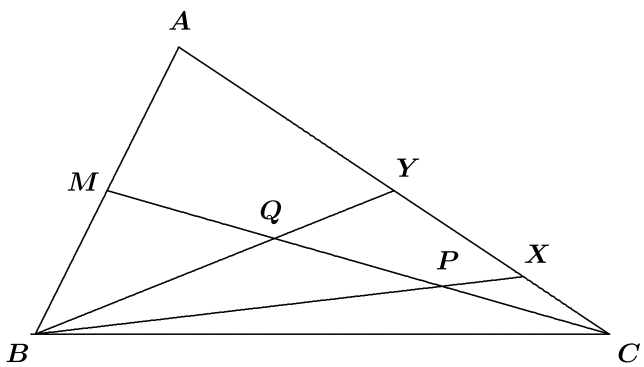
3. (*) How many possible values are there for the sum $a + b + c$ if a, b, and c are positive integers and $abc = 50$? **(2 points)**

4. List the numbers $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[5]{5}$ in order from greatest to least. **(2 points)**

5. Compute 3^A where $A = \dfrac{(\log 1 - \log 4)(\log 9 - \log 2)}{(\log 1 - \log 9)(\log 8 - \log 4)}$. All logarithms are base three. **(2 points)**

6. Joe and Andy are playing a simple game on a circular board with n spaces. First Joe advances five spaces from the starting space, then Andy advances seven, then Joe advances five, then Andy advances seven, and so on. The first player to land back on the original space wins. If n is a random two-digit number, what is the probability that Joe wins? **(3 points)**

7. In the diagram, M is the mid-point of AB and Y is the mid-point of AC. Hence, Q is a trisection point of CM; we call the other trisection point P and extend BP to meet AC at X. Evaluate $(CX + AY)/XY$. **(3 points)**



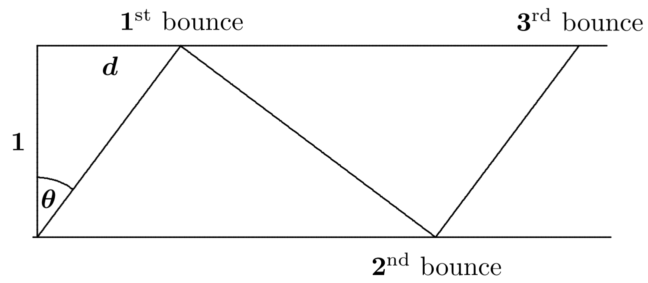
The Mandelbrot Competition
Division B Round One Team Test
November 1997

Facts: The Weighted Power Mean Inequality for two positive variables states that if x_1, x_2, w_1 and w_2 are positive real numbers, and m and n are non-zero integers with $m > n$, then

$$\left(\frac{w_1 x_1^m + w_2 x_2^m}{w_1 + w_2}\right)^{\frac{1}{m}} \geq \left(\frac{w_1 x_1^n + w_2 x_2^n}{w_1 + w_2}\right)^{\frac{1}{n}},$$

which is quite a mouthful. The positive variables are x_1 and x_2. The two sides are equal if and only if $x_1 = x_2$. The numbers w_1 and w_2 "weight" the variables in different proportions. Try $w_1 = w_2 = 1$ to see the standard Power Mean Inequality; then compare with $w_1 = 1$ and $w_2 = 2$, which emphasizes x_2. Finally, the non-zero integers m and n vary the powers. For example, use $m = 1$ and $n = -1$ to obtain the Arithmetic Mean–Harmonic Mean Inequality.

Setup: On the planet Garth a certain laser has the curious property that when reflected off a special mirror it always continues in a direction perpendicular to the original path. Some popular Garthian children's games are based on this phenomenon. The ones described below involve a player situated in the corner of a rectangular mirrored hallway of width one plog (about 7.3 metres), as shown below. The player directs the laser beam an angle of θ away from the left wall, hitting the far wall a distance d from the end of the hall on the first bounce.



Problems: (*Please, no calculus-based solutions.*)

Part i: (4 points) Show that the laser's path length up to the second bounce is $\dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}$.

Part ii: (4 points) The object of one of the simpler games is to have the shortest path length after two bounces. Use the *standard* power mean inequality with $m = 2$, $n = -1$, $x_1 = \cos\theta$, and $x_2 = \sin\theta$ to prove that the shortest possible path length is $2\sqrt{2}$. Invoke the equality condition to show that we need $\tan\theta = 1$ (that is, $d = 1$) to obtain this minimum.

Part iii: (4 points) Show that the total path length after three bounces (a more challenging version) is $2\left(\dfrac{1}{\cos\theta}\right) + \dfrac{1}{\sin\theta}$. To minimize this, we employ a clever strategy. Begin by finding numbers w_1 and λ_1 such that $w_1\lambda_1^2 = 1$ and $w_1/\lambda_1 = 2$.

Part iv: (4 points) Now apply the weighted power mean inequality with $m = 2$ and $n = -1$ as before, using w_1 from **Part iii**, $w_2 = 1$, $x_1 = \lambda_1\cos\theta$, and $x_2 = \sin\theta$, to prove that the minimum path length is $\left(1 + 2^{2/3}\right)^{3/2}$. What value of d should the player aim for?

Part v: (5 points) Use this technique to find the shortest path length after five bounces.

Correction: Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON, points out that our featured solution to problem 8 of the Eighteenth W.J. Blundon Contest [2002 : 529] contains an extraneous root. The value $6 - 2\sqrt{5}$ does not satisfy the original equation. The editor apologizes for letting this go unnoticed.

Next we move on to the solutions to the 2000 Maritime Mathematics Competition presented in the September 2002 issue.

Concours de Mathématiques des Maritimes 2000
2000 Maritime Mathematics Competition

1. At a meeting, one mathematician remarked to another, "There are nine fewer of us here than twice the product of the two digits of our total number." How many mathematicians were at the meeting?

Solution by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

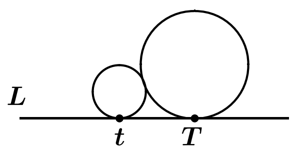
Let AB be the two-digit number. Then we have

$$\begin{aligned} 2 \cdot A \cdot B - 9 &= 10 \cdot A + B \\ 2 \cdot A \cdot B - 10 \cdot A &= B + 9 \\ A &= \frac{B+9}{2B-10} \end{aligned}$$

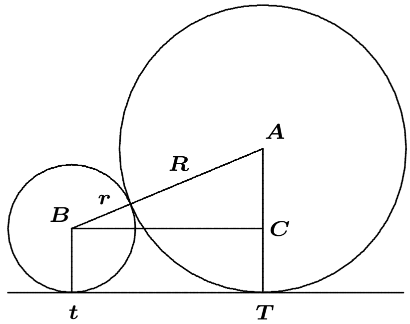
Note that B must be odd, and the only value of B that gives a positive integer for A is $B = 7$, which gives $A = 4$. Thus, there were 47 mathematicians at the meeting.

Also solved by Robert Bilinski, Outremont, QC.

2. Suppose that two circles with radii r and R intersect in a single point and that the straight line L is tangent to both circles at t and T respectively, as in the diagram below. Determine the distance between the points t and T.



Solution by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.



Note that $AB = R + r$ and $AC = R - r$. Thus, by the Pythagorean Theorem

$$\begin{aligned} BC^2 &= AB^2 - AC^2 \\ &= (R+r)^2 - (R-r)^2 \\ &= 4Rr \end{aligned}$$

Thus, $Tt = BC = 2\sqrt{Rr}$.

Also solved by Robert Bilinski, Outremont, QC.

3. Trouver la somme de tous les nombres à quatre chiffres dont les chiffres sont choisis, sans répétition, parmi 1, 2, 3, 4, 5. (Il y en a 120.)

Solution by Robert Bilinski, Outremont, QC.

Parmi les 120 nombres que l'on peut écrire, on en a 24 qui commencent par chaque unité. On peut dire la même chose des autres positions puisque la recherche de nombres est exhaustive et équiprobable. Donc le total de chaque position (chaque colonne dans l'addition) est $24 \times (1+2+3+4+5) = 360$. Ainsi, en additionnant les 120 nombres trouvés, on obtient $360 \times 1111 = 399960$.

Also solved by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

4. A cubic box with edges 1 metre long is placed against a vertical wall. A ladder $\sqrt{15}$ metres long is placed so that it touches the wall as well as the free horizontal edge of the box. Find at what height the ladder touches the wall.

Official Solution

Let x be the height at which the ladder touches the wall. Let y be the distance between the foot of the ladder and the wall. By the Pythagorean Theorem, $x^2 + y^2 = \left(\sqrt{15}\right)^2 = 15$, and, using similar triangles,

$$\frac{x-1}{1} = \frac{1}{y-1} \quad\Longrightarrow\quad (x-1)(y-1) = 1 \quad\Longrightarrow\quad xy = x+y\,.$$

Now

$$15 = x^2 + y^2 = (x^2 + 2xy + y^2) - 2xy = (x+y)^2 - 2(x+y)\,.$$

Thus, letting $z = x + y$, we have

$$\begin{aligned} z^2 - 2z - 15 = 0 \quad &\Longrightarrow \quad (z-5)(z+3) = 0 \\ &\Longrightarrow \quad z = 5 \quad \text{or} \quad z = -3\,. \end{aligned}$$

As x and y are both positive, $z = -3$ is inadmissible. Thus, $5 = z = x + y$, whence $y = 5 - x$. Substituting into $xy = x + y$, we obtain

$$x(5-x) = 5 \quad \Longrightarrow \quad x^2 - 5x + 5 = 0\,.$$

By the Quadratic Formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)} = \frac{5 \pm \sqrt{5}}{2}\,.$$

Therefore, there are two solutions. The ladder touches the wall at a height of either $(5 + \sqrt{5})/2$ metres or $(5 - \sqrt{5})/2$ metres.

5. Une pelouse circulaire de 12 mètres de diamètre est traversée d'une allée de gravier de 3 mètres de large dont un des bords passe par le centre de la pelouse. Trouver l'aire du reste de la pelouse.

Solution by Robert Bilinski, Outremont, QC.

L'aire de la pelouse est donc composée d'un demi-cercle complet, d'aire $\pi r^2/2 = 18\pi$ m^2 , et d'une partie de cercle qui correspond à un secteur angulaire amputé d'un triangle isocèle.

Mettons le cercle de telle sorte que l'allée soit verticale, un de ces côtés est un diamètre, l'autre est une corde verticale. Faisons le rapprochement entre le cercle et un cercle trigonométrique. On remarque que la corde verticale coupe le rayon horizontal en son milieu, donc le cosinus des 2 sommets de la corde verticale sur le cercle est $\frac{1}{2}$. Ainsi l'angle entre les deux sommets est $2\pi/3$. Le secteur angulaire défini par ces deux sommets et le centre du cercle a une aire de 12π m^2.

Mais la partie triangulaire de ce secteur est dans l'allée, le reste étant de la pelouse. Cette partie triangulaire est isocèle de côtés égaux 6 m (le rayon) et d'hauteur principale 3 m (le demi-rayon horizontal). De plus, on sait que le troisième côté égale $6\sqrt{3}$, car la corde verticale correspond au double du sinus de dans un cercle de rayon 6 m. Donc l'aire du triangle est $9\sqrt{3}$ m^2.

Ainsi, la seconde partie de la pelouse a une aire de $(12\pi - 9\sqrt{3})$ m^2. Au total, la pelouse aura une aire de $(30\pi - 9\sqrt{3})$ m^2.

Also solved by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

6. Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following two conditions.

- Each rectangle has the same number of white squares and black squares.
- No two rectangles have the same number of squares.

Find the maximum value of p for which such a decomposition is possible. For this maximum value of p, determine all corresponding decompositions of the chessboard into p rectangles.

Official Solution.

Consider a decomposition of the chessboard into p non-overlapping rectangles subject to the two given conditions. Let a_1, a_2, $\ldots$, a_p be the number of squares in the rectangles in the decomposition.

Because of the second condition, the a_i's are all distinct so we may assume, without loss of generality, that $a_1 < a_2 < \cdots < a_p$. Furthermore, each a_i is even, by the first condition.

We claim that $p \le 7$. To show this, suppose to the contrary that $p \ge 8$. Then the number of squares covered by the rectangles in the decomposition is

$$a_1 + a_2 + \cdots + a_p \;\ge\; 2+4+6+8+10+12+14+16 \;=\; 72\,,$$

which is impossible since the chessboard has only 64 squares.

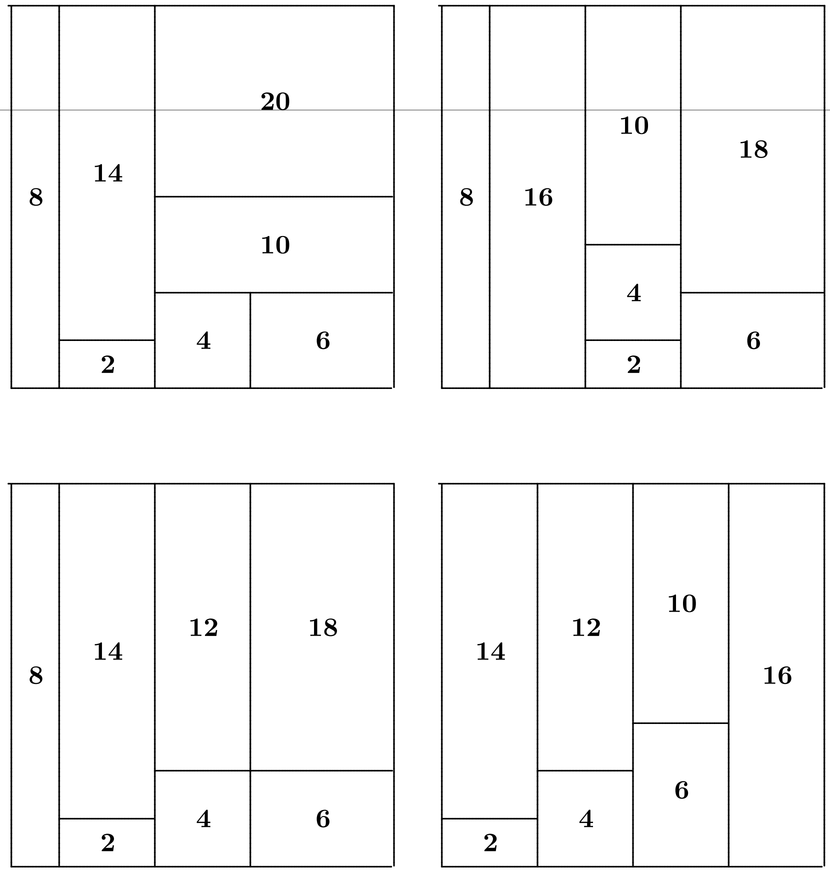
For $p = 7$, we obtain the following five sequences as the only possibilities for $(a_1, a_2, \ldots, a_7)$:

$$\begin{gathered}(2,\,4,\,6,\,8,\,10,\,12,\,22)\,,\\(2,\,4,\,6,\,8,\,10,\,14,\,20)\,,\\(2,\,4,\,6,\,8,\,10,\,16,\,18)\,,\\(2,\,4,\,6,\,8,\,12,\,14,\,18)\,,\\(2,\,4,\,6,\,10,\,12,\,14,\,16)\,.\end{gathered}$$

To establish that the maximum value of p is indeed 7, it remains to show that there is an actual decomposition of the chessboard into 7 rectangles. Now, no rectangle may have 22 squares since it is impossible to find a rectangle contained in the chessboard having dimensions 1×22 or 2×11. There is, therefore, no decomposition of the board corresponding to the first sequence. Each of the other 4 sequences, however, does have a corresponding decomposition, as the diagram on the next page illustrates.

Therefore, the maximum value of p is 7, and there are 4 decompositions of the chessboard into 7 non-overlapping rectangles subject to the given conditions.

Also solved by Robert Bilinski, Outremont, QC; and Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.



That brings us to the end of another number of Skoliad. This issue's winner of a copy of MATHEMATICAL MAYHEM Volume 5 is Alexandre Ortan. Congratulations Alexandre! Please continue to send me your solutions and contests.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada, K1C 2Z7**. The electronic address is

`mayhem-editors@cms.math.ca`

The Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Problems

Envoyez vos propositions et solutions à MATHEMATICAL MAYHEM, 2191 Saturn Crescent, Orleans, ON K4A 3T6, ou par courriel à

`mayhem-editors@cms.math.ca`

N'oubliez pas d'inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d'étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le *1er septembre 2003*. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Hidemitsu Saeki, de l'Université de Montréal, d'avoir traduit les problèmes.

M82. *Proposé par l'équipe de Mayhem.*

En théorie des nombres, on désigne par $\omega(n)$ le nombre de facteurs de n, premiers et *distincts*. Par exemple, $\omega(12) = 2$ puisque $12 = 2 \times 2 \times 3$. Montrer que pour tout nombre entier positif n

$$\ln n \geq \omega(n) \ln 2 .$$

. . . .

In number theory the function $\omega(n)$ is the number of *distinct* primes dividing n. For example, $\omega(12) = 2$ since $12 = 2 \times 2 \times 3$. Prove that for each positive integer n

$$\ln n \geq \omega(n) \ln 2 .$$

M83. *Proposé par l'équipe de Mayhem.*

On met dans une urne cinq boules numérotées de 1 à 5. On choisit une boule au hasard, on note son numéro et on remet la boule dans l'urne. On répète le même geste encore quatre fois. Si la somme des numéros notés est 15, quelle est la probabilité pour que la boule numéro 3 ait été choisie chaque fois ?

. . . .

Five balls numbered 1 to 5 are put into a box. A ball is drawn at random, its number recorded, and the ball returned to the box. This process is repeated until five numbers have been recorded. If the sum of the recorded numbers is 15, what is the probability that the number 3 was drawn each time?

M84. *Proposé par l'équipe de Mayhem.*

Soit les deux fonctions $f(x) = x^2 - 2ax + 1$ et $g(x) = 2b(a - x)$, où a, b, $x \in \mathbb{R}$. Considérons chaque couple de constantes a et b comme un point (a, b) dans le plan-ab. Soit A l'ensemble des points (a, b) pour lesquels les graphes de $y = f(x)$ et $y = g(x)$ *ne se coupent pas*. Trouver l'aire de A.

. . . .

Consider the two functions $f(x) = x^2 - 2ax + 1$ and $g(x) = 2b(a - x)$, where a, b, $x \in \mathbb{R}$. We will consider each pair of constants a and b as a point (a, b) in the ab-plane. Let A be the set of points (a, b) for which the graphs of $y = f(x)$ and $y = g(x)$ do *not* intersect. Find the area of A.

M85. *Proposé par l'équipe de Mayhem.*

Trouver un triangle dont les côtés sont mesurés par des nombres entiers en progression arithmétique de raison 2 et dont l'aire est 336.

. . . .

Find a triangle whose integer sides are in arithmetic progression with a common difference of 2 and which has an area of 336.

M86. *Proposé par l'équipe de Mayhem.*

Soit deux nombres entiers positifs a et b. Parmi les nombres

$$a,\ 2a,\ 3a,\ \ldots,\ (b-1)a,\ ba$$

combien y en a-t-il qui soient divisibles par b?

. . . .

For any two positive integers a and b, how many of the numbers

$$a,\ 2a,\ 3a,\ \ldots,\ (b-1)a,\ ba$$

are divisible by b?

M87. *Proposé par l'équipe de Mayhem.*

Sept personnes, A, B, C, D, E, F et G, sont sur le bord d'une rivière. Pour la traverser, elles disposent d'une seule barque d'une capacité de deux personnes. Chacune des sept personnes est capable de traverser seule à la rame en 1, 2, 3, 5, 10, 15 et 20 minutes, respectivement. Par contre, avec deux personnes à bord, le temps de traversée est celui de la personne la plus lente. En supposant que personne ne peut traverser sans barque, on demande quel est le temps minimal nécessaire pour que tout le monde puisse passer d'une rive à l'autre.

. . . .

Seven people, A, B, C, D, E, F, G, are on one side of a river. To get across the river they have a rowboat, but it can only fit two people at a time. The times that would be required for the people to row across individually are 1, 2, 3, 5, 10, 15, and 20 minutes, respectively. However, when two people are in the boat, the time it takes them to row across is the same as the time necessary for the slower of the two to row across individually. Assuming that no one can cross without the boat, what is the minimum time for all seven people to get across the river?

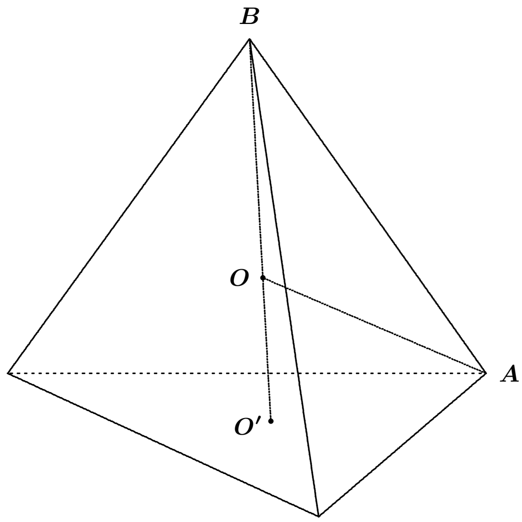
Mayhem Solutions

M31. *Proposed by the Mayhem staff.*

Given four spheres of unit radius, each tangent to the other three, find the radii of the two spheres that are tangent to all four of the unit spheres.

Solution by Antonio Lei, year 12, Colchester Royal Grammar School, Colchester, UK.

The centres of the four spheres make a tetrahedron with side 2. The centre of a sphere that touches the original four must be at the centre of the tetrahedron.



Let A and B be vertices of the tetrahedron, let O be its centre, let O' be the centre of the face opposite B, and let $x = OA = OB$. Then

$$\begin{aligned} O'A &= 2 \times \frac{\sqrt{3}}{3} = \frac{2}{3}\sqrt{3}\,, \\ BO' &= \sqrt{2^2 - \left(\frac{2}{3}\sqrt{3}\right)^2} = \frac{2}{3}\sqrt{6}\,, \\ OO' &= \frac{2}{3}\sqrt{6} - x\,. \end{aligned}$$

Note that $\triangle OO'A$ is right-angled; thus, we have

$$\left(\frac{2}{3}\sqrt{6} - x\right)^2 + \left(\frac{2}{3}\sqrt{3}\right)^2 = x^2\,, \qquad \text{and} \qquad x = \frac{\sqrt{6}}{2}\,.$$

Therefore, the radius of the external sphere is $\frac{\sqrt{6}}{2} + 1$ and the radius of the internal sphere is $\frac{\sqrt{6}}{2} - 1$.

Also solved by Kevin Chung, grade 13, Earl Haig S.S., North York, ON; and Jack Gu, grade 11, Dalian Maple Leaf International School, Dalian, China.

M32. *Proposed by Nicolae Gustia, North York, ON.*

In a triangle with angles A, B and C, if $8\cos A\cos B\cos C = 1$ then prove that $\triangle ABC$ is equilateral.

Solution by Molly Yan, grade 11, Dalian Maple Leaf International School, Dalian, China.

From $\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$ we get

$$\frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]\cos C = \frac{1}{8}\,,$$

$$\cos(A - B) = \frac{1}{4\cos C} - \cos(A + B),$$
$$= \frac{1}{4\cos C} + \cos C.$$

Using the AM–GM inequality we then get

$$\cos(A - B) = \frac{1}{4\cos C} + \cos C \geq 2\sqrt{\left(\frac{1}{4\cos C}\right)(\cos C)} = 1.$$

Therefore, $\cos(A - B) = 1$ which implies that $A = B$. A similar argument with B and C yields $B = C$. Thus, $A = B = C = 60°$ and $\triangle ABC$ is equilateral.

Also solved by Kevin Chung, grade 13, Earl Haig S.S., North York, ON; José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain; Jack Gu, grade 11, Dalian Maple Leaf International School, Dalian, China; Larry Lee, grade 11, Dalian Maple Leaf International School, Dalian, China; Antonio Lei, year 12, Colchester Royal Grammar School, Colchester, UK; Alvin Miao, grade 11, Dalian Maple Leaf International School, Dalian, China; and Corey Zhou, grade 12, Dalian Maple Leaf International School, Dalian, China. [Editor's note: This problem is related to the article which appears on pages 18 and 19 of this issue.]

M33. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS.*

a, b, and c are three consecutive terms of a geometric sequence, where a, b, and c are all integers. If $a + b + c = 7$, determine all possible values of a, b, and c.

Solution by Shaun White, grade 9, Vincent Massey S.S., Windsor, ON.

[*Editor's Note:* The sequences a, b, c found in this solution are, in fact, the only ones, although this is not proved here.]

Let the geometric sequence be represented as: a, $ax = b$, $ax^2 = c$. Then $ax^2 + ax + a = 7$. Factoring and rearranging, we get

$$a(x^2 + x + 1) = 7,$$
$$x^2 + x + 1 = \frac{7}{a},$$
$$x^2 + x + \frac{a-7}{a} = 0,$$

(noting that $a \neq 0$). Because each term in the geometric sequence is an integer, one sequence exists for each rational root of the above equation, with integer a. To make the expression factorable, a may be 1 or 7.

- If $a = 1$, $x^2 + x - 6 = 0 \Longrightarrow x = -3$ or $x = 2$, yielding the sequences $1, -3, 9$ and $1, 2, 4$, respectively.
- If $a = 7$, $x^2 + x = 0 \Longrightarrow x = 0$ or $x = -1$, yielding the sequences $7, 0, 0$ and $7, -7, 7$, respectively.

Noting that the first two sequences have ax^2 terms of 9 and 4, we may find more sequences by setting a to 9 and 4.

- If $a = 9$, $x^2 + x + \frac{2}{9} = 0 \Longrightarrow x = -\frac{2}{3}$ or $x = -\frac{1}{3}$, yielding the sequences 9, -6, 4 and 9, -3, 1, respectively.

- If $a = 4$, $x^2 + x - \frac{3}{4} = 0 \Longrightarrow x = \frac{1}{2}$ or $x = -\frac{3}{2}$, yielding the sequences 4, 2, 1 and 4, -6, 9, respectively.

Therefore, there are eight sequences that satisfy the conditions.

Also solved by Yichuan Wang, grade 11, Lambrick Park S.S., Victoria, BC. Partial solutions received from Robert Bilinski, Outremont, QC; Kevin Chung, grade 13, Earl Haig S.S., North York, ON; Jack Gu, grade 11, Dalian Maple Leaf International School, Dalian, China; Larry Lee, grade 11, Dalian Maple Leaf International School, Dalian, China; Antonio Lei, year 12, Colchester Royal Grammar School, Colchester, UK; Rachel Li, grade 12, Dalian Maple Leaf International School, Dalian, China; Hongge Ren, grade 12, Dalian Maple Leaf International School, Dalian, China; Nancy Teng, grade 11, Dalian Maple Leaf International School, Dalian, China; Alvin Miao, grade 11, Dalian Maple Leaf International School, Dalian, China; Molly Yan, Dalian Maple Leaf International School, Dalian, China; and Corey Zhou, grade 12, Dalian Maple Leaf International School, Dalian, China.

M34. *Proposed by the Mayhem staff.*

The numbers 1 to 2002 are written on a blackboard, so you decide to play a fun game. You flip a coin, then erase two numbers, x and y, from the board. If the coin was heads you write the number $x + y$ on the board; if the coin was tails you write the number $|x - y|$. You continue this process until only one number remains. Prove that the last number is odd.

Solution by Kevin Chung, grade 13, Earl Haig S.S., North York, ON.

Notice that whatever the result of the flipping, the new number to be written on the board has the following property:

- it is even if the numbers x and y have the same parity;

- it is odd if the numbers x and y have opposite parity.

Let e and o denote the total number of even and odd integers, respectively. After each flip,

- e changes to $e' = e - 1$, if the two selected numbers are both even;

- e and o change to $e' = e + 1$ and $o' = o - 2$, if the two numbers are both odd;

- e changes to $e' = e - 1$, if the two numbers have different parity.

We see that the number of odd integers decreases by 2. Therefore, it has the same parity throughout the game. Since there are 1001 odd integers at the beginning, there must be one odd integer at the end.

Also solved by the Austrian IMO team, 2002; and Antonio Lei, year 12, Colchester Royal Grammar School, Colchester, UK.

M35. *Proposed by the Mayhem staff.*

Two sequences are defined by: $x_1 = 4732$, $y_1 = 847$, $x_{n+1} = \frac{x_n+y_n}{2}$ and $y_{n+1} = \frac{2x_ny_n}{x_n+y_n}$. Find

$$\lim_{n\to\infty} x_n \quad \text{and} \quad \lim_{n\to\infty} y_n\,.$$

Solved by Antonio Lei, year 12, Colchester Royal Grammar School, Colchester, UK.

Note that $x_{n+1}y_{n+1} = \dfrac{x_n+y_n}{2} \times \dfrac{2x_ny_n}{x_n+y_n} = x_ny_n$. Thus,

$$x_{n+1}y_{n+1} \;=\; x_ny_n \;=\; \ldots \;=\; x_1y_1 \;=\; 4732\times 847 \;=\; 2002^2\,.$$

Therefore,

$$x_{n+1} \;=\; \frac{x_n+y_n}{2} \;=\; \frac{x_n+\dfrac{2002^2}{x_n}}{2} \;=\; \frac{x_n^2+2002^2}{2x_n}\,.$$

Thus,

$$x_{n+1}+2002 \;=\; \frac{x_n^2+2\times 2002x_n+2002^2}{2x_n} \;=\; \frac{(x_n+2002)^2}{2x_n}\,.$$

Similarly, $x_{n+1}-2002 = \dfrac{(x_n-2002)^2}{2x_n}$. This implies

$$\begin{aligned}
\frac{x_n-2002}{x_n+2002} &= \left(\frac{x_{n-1}-2002}{x_{n-1}+2002}\right)^2 \\
&= \left(\frac{x_{n-2}-2002}{x_{n-2}+2002}\right)^{2\times 2} \\
&\;\;\vdots \\
&= \left(\frac{x_1-2002}{x_1+2002}\right)^{2^{n-1}} \\
&= \left(\frac{4732-2002}{4732+2002}\right)^{2^{n-1}} \;=\; \left(\frac{1365}{3367}\right)^{2^{n-1}}\,.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{x_n-2002}{x_n+2002} \;=\; \lim_{n\to\infty}\left(\frac{1365}{3367}\right)^{2^{n-1}} \;=\; 0\,.$$

Hence,

$$\lim_{n\to\infty} x_n \;=\; 2002\,, \quad \text{and} \quad y_n \;=\; \frac{2002^2}{\lim_{n\to\infty} x_n} \;=\; 2002\,.$$

Also solved by Kevin Chung, grade 13, Earl Haig S.S., North York, ON; José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.

M36. *Proposed by the Mayhem staff.*

In $\triangle ABC$, AM is the median from A. Prove $AM \leq \dfrac{AB + AC}{2}$.

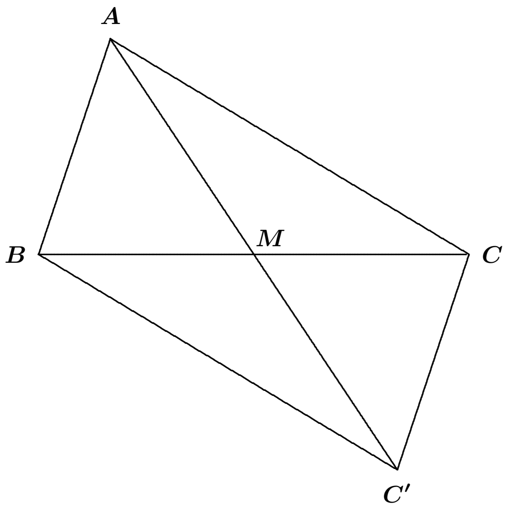
Solution by the Austrian IMO Team 2002.

Since AM is the median from A, we see that M is the mid-point of BC. Therefore, $BM = MC$. Let C' be the point on the line AM extended with $AM = MC'$. Then $\triangle AMC$ is congruent to $\triangle C'MB$ (since $BM = MC$, $AM = MC'$, and $\angle C'MB = \angle AMC$). Therefore, $AC = BC'$. According to the Triangle Inequality, $AB + BC' > AC'$. Since $AM = MC'$,

$$\frac{AC'}{2} = \frac{AM + MC'}{2} = \frac{AM + AM}{2} = AM\,.$$

Hence, we have $(AB + BC')/2 > AM$. Since $BC' = AC$, we get $(AB + AC)/2 > AM$.

Note that $(AB + AC)/2 = AM$ only if $\triangle ABC'$ is not actually a triangle at all (that is, B is a point on AC', which results in $B = M = C$).



Also solved by Marcus Emmanuel Barnes, student, York University, Toronto, ON; Kevin Chung, grade 13, Earl Haig S.S., North York, ON; José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain; Antonio Lei, year 12, Colchester Royal Grammar School, Colchester, UK; and Vedula N. Murty, Dover, PA, USA.

M37. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Find two (different) positive integers less than 100 such that the sum of the digits in both integers is the larger integer and the product of the digits in both integers is the smaller integer.

Solution by Kevin Chung, grade 13*, Earl Haig S.S., North York, ON.*

Let the two positive integers be $10a + b > 10c + d > 0$ where a, b, c, d are non-negative integers ≤ 9. Then we have $10a + b = a + b + c + d$; that is, $9a = c + d < 20$, whence $a = 0$ or 1 or 2.

- If $a = 2$, then $c + d = 18$, forcing $c = d = 9$, which contradicts $10a + b > 10c + d$.
- If $a = 0$, then $c = d = 0$, which contradicts $10c + d > 0$.
- If $a = 1$, then either $c = 1$ and $d = 8$, or $c = 0$ and $d = 0$. The first of these two cases forces $b = 9$, which contradicts $abcd = 10c + d$. The second case gives the two numbers as 9 and 11. These numbers satisfy the conditions, since $11 > 9$, $1 + 1 + 9 = 11$, and $1 \times 1 \times 9 = 9$.

One incorrect solution was received.

Well, our first sets of **MAYHEM TAUNT** problems are starting to see light, and it is time to announce some prizes! From the first two TAUNTs we have Shaun White, a grade 9 student from Vincent Massey S.S. in Windsor, Ontario, as our youngest solver. Also we got consistent solutions from Kevin Chung, a grade 13 student at Earl Haig S.S. in North York, Ontario, and Antonio Lei, a year 12 student at Colchester Royal Grammar School, Colchester, UK. These three students will receive subscriptions to ***CRUX with MAYHEM*** for the year 2003. We also received quite a volume of solutions from students at Dalian Maple Leaf International School in Dalian, China. We will send the school a subscription to ***CRUX with MAYHEM*** for the year 2003, a set of the four volumes of **A.T.O.M.** (A Taste Of Mathematics), as well as the 25^{th} anniversary collection of the Canadian Mathematical Olympiad. Thank you for all your solutions; keep them coming, as there are still prizes to be won!

Four Proofs of the Inequality $-1 < \cos A \cos B \cos C \leq \frac{1}{8}$

Vedula N. Murty

Problem.

Let A, B, and C denote the angles of a triangle ABC. The following inequality is known:

$$-1 \ < \ \cos A \cos B \cos C \ \leq \ \frac{1}{8} \,. \tag{1}$$

We present one solution to the left-hand side of the inequality, and four solutions to the right-hand side, one of which is believed to be new.

Proof 1.

Let a, b, c denote the side lengths of BC, CA, and AB, respectively. We have the known inequality

$$\left(b^2+c^2-a^2\right)\left(c^2+a^2-b^2\right)\left(a^2+b^2-c^2\right) \ \leq \ a^2b^2c^2 \,. \tag{2}$$

Using the known relations

$$\begin{aligned} b^2+c^2-a^2 &= 2bc\cos A\,, \\ c^2+a^2-b^2 &= 2ca\cos B\,, \qquad \text{and} \\ a^2+b^2-c^2 &= 2ab\cos C \end{aligned}$$

in (2), we establish the right-hand side of (1). □

Proof 2.

Let O denote the circumcentre and let H denote the orthocentre of the triangle ABC. Then we have the following formula for the square of the distance between O and H.

$$OH^2 \ = \ R^2\left(1-8\cos A\cos B\cos C\right) \ , \tag{3}$$

where R is the circumradius of the triangle. The right-hand side of (1) follows immediately since $OH^2 \geq 0$. □

Most classical textbooks on Trigonometry, such as [1] and [2], have a proof of (3).

Proof 3.

If the triangle is right-angled, then the product of the cosines of the three angles is zero. If the triangle is obtuse, then this product is negative. Hence, in these two cases the right-hand side of (1) is established. If the triangle is acute, all the cosines are positive and the AM–GM inequality gives us

$$(\cos A \cos B \cos C)^{\frac{1}{3}} \;\leq\; \tfrac{1}{3}(\cos A + \cos B + \cos C)\;. \tag{4}$$

Jensen's Inequality gives us

$$\tfrac{1}{3}(\cos A + \cos B + \cos C) \;\leq\; \cos\left(\tfrac{1}{3}(A+B+C)\right) \;=\; \tfrac{1}{2}\,. \tag{5}$$

Then (4) and (5) establish the right-hand side of (1). ☐

Proof 4. (We believe this proof is new.)

Let the semi-perimeter, inradius and circumradius of triangle ABC be denoted by s, r, R, respectively. Let $y = s/R$ and $x = r/R$. This notation is familiar to readers of ***CRUX with MAYHEM***. Then, the following equality is known and can be easily proved:

$$y^2 - (x+2)^2 \;=\; 4\cos A \cos B \cos C\,. \tag{6}$$

Also

$$y^2 \;\leq\; -x^2 + 8x + 3\,. \tag{7}$$

We now prove that the left-hand side of (6) is less than or equal to $\frac{1}{2}$, which immediately establishes the right-hand side of (1). For this, we observe that

$$y^2 - (x+2)^2 \;\leq\; \tfrac{1}{2} \qquad \Longleftrightarrow \qquad 2y^2 \;\leq\; 2x^2 + 8x + 9\,.$$

In view of (7) it will suffice to prove $2x^2 + 8x + 9 \geq 2(-x^2 + 8x + 3)$. This last inequality is equivalent to $(2x-1)(2x-3) \geq 0$, which is clearly true. This completes our proof. ☐

Finally, the left-hand side of (1) is seen by observing that

$$\sin^2 A + \sin^2 B + \sin^2 C = 2\,(1 + \cos A \cos B \cos C) > 0\,.$$

References:

[1] S.L. Loney, *Plane Trigonometry*, Cambridge At The University Press, 1900.

[2] E.W. Hobson, *A Treatise On Plane Trigonometry*, Cambridge At The University Press, 1891.

Vedula N. Murty
3595 Cypress Court
Dover, PA, USA 17315

Pólya's Paragon

Paul Ottaway

This month we're going to have a little fun and play some games. The area of mathematics called *combinatorial game theory* is of particular interest to me, so I thought I would share some of it with my readers.

Before we start actually playing some games, I had better make sure everyone understands what I mean by a combinatorial game. Every game that I talk about adheres to the following properties:

1. There are exactly 2 players, who move alternately.
2. The game ends in a finite number of moves, regardless of their sequence.
3. There is perfect information (everything known to one player is also known to the other).
4. The game does not involve any chance devices (such as cards or dice).
5. The last player to move determines the winner.

In particular, under *normal play* the last player to make a legal move wins, and under *misère play* the last player to make a legal move loses. Note that games with these properties can never end in a draw. Unless otherwise stated, all games I introduce will use normal play.

At first glance, many people probably would not be able to name a single combinatorial game. There are, however, many common games that are almost combinatorial games—ones that break only one or two of the rules listed above. For example, chess breaks only the second rule and the oriental game of Go breaks only the fifth rule.

I will now introduce one of the most commonly known combinatorial games—Nim. The game involves piles of beans. On each player's turn (s)he selects one of the remaining piles, then removes any number of beans from it. Since we are using normal play, the player who takes the last bean wins.

This is where we need to put on our thinking caps! Since we have perfect information and there is no element of chance, we should be able to tell from the outset of the game who will win. Let us examine some simple positions.

Any game with n piles can be easily expressed as an unordered n-tuple. For example, a game with just one pile of 5 beans can be denoted by (5). Who wins this game? Well, this one is easy. Since there is only one pile, whoever moves first can simply select it and remove all five beans, thus winning the game. Clearly, any game that starts with only one pile will end very quickly when the first player removes all the beans!

Now the question becomes: What happens if there are two or more piles? Consider the game $(4, 3)$. If either player empties a pile, the other player will win by removing the entire other pile. If you play first, how can you guarantee that your opponent must empty a pile first? Answer: remove one bean from the pile of 4. Before reading on, see if you can figure out why this is a good move (and the *only* move which lets you win!).

After taking one bean from the pile of 4, both piles will be of equal size. From that point on, no matter how your opponent moves, you can always make the same move using the other pile. Since this guarantees a move for you regardless of what your opponent does, he can never make the last move. Therefore, you must eventually make the last move and win the game.

This type of logical reasoning is the basis for much of combinatorial game theory. If you want a challenge, you can try to work out who would win the game given by $(7, 5, 3, 1)$ or other such positions that involve more than 2 piles. There exists an easy method to analyze such games with any number of piles. Without proof I will now explain it.

First, write each pile as a number in binary (base 2). For example, 5 becomes 101, since $5 = 1(2^2) + 0(2^1) + 1(2^0)$. Next, calculate the **XOR** sum of these numbers. This means that you look at each column of binary digits; if there is an odd number of 1s you get a 1 in the sum, and if there is an even number of 1s you get a 0 in the sum. Now look at your sum. If it is all 0s there is nothing you can do and your opponent will win the game (as long as he knows how to play as well as you do!). Otherwise, you have to reduce one of the piles so that the new **XOR** sum is all 0s.

Here is an example with the game $(3, 6, 8)$:

$$\begin{array}{ll} 3 & 0011 \\ 6 & 0110 \\ 8 & \underline{1000} \\ & 1101 \end{array}$$

Since you want an even number of 1s in each column, you must remove beans from the pile of 8. This is because there is no way to add a one to the left-most column. You need to create a new 1 in both the 4s and 1s column. Therefore, you should change the 8 to 5, giving the following position:

$$\begin{array}{ll} 3 & 0011 \\ 6 & 0110 \\ 5 & \underline{0101} \\ & 0000 \end{array}$$

Now it is your opponent's turn. It is easy to see that any move he makes cannot possibly leave this sum as all 0s. In fact, it is impossible for him to win because eventually all the piles will be empty, which would give an **XOR** sum of all 0s. Since your opponent can never move to an all 0s position, you must eventually make the last move.

Problems:

Here are the rules for a couple of combinatorial games. Try playing with friends to learn more about good strategies and winning positions. Since these are combinatorial games, the outcome can be determined before any play begins! Of course, these games are far more complicated and may not have an easy solution.

Game 1 - Domineering: This game is played on an 8×8 board. Each player is given an unlimited supply of dominoes that each cover a 1×2 area of the board. The players alternately place dominoes on the board so that they do not overlap or hang off the board. The first player is only allowed to place dominoes so that they are oriented North-South and the second player is only allowed to place dominoes so that they are oriented East-West. When a player is unable to place a domino according to these rules, he loses the game.

Game 2 - Clobber: This game is played on any size board (though 4×7 is most common). White and black pieces are placed so as to fill the board in a checkerboard pattern. The first player plays white. On each player's turn, he moves one of his pieces to an adjacent space (not diagonally, though) which must contain an opponent's piece. The opponent's piece is replaced by the piece that is moving. The first player unable to make a move loses.

THE OLYMPIAD CORNER

No. 228

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, AB, Canada. T2N 1N4.

We begin this number with problems of the 2000 Belarusian Mathematical Olympiad. My thanks go to Andy Liu, Canadian Team Leader to the IMO in Korea, for collecting them.

2000 BELARUSIAN MATHEMATICAL OLYMPIAD

1. Pete and Bill play the following game. At the beginning, Pete chooses a number a, then Bill chooses a number b, and then Pete chooses a number c. Can Pete choose his numbers in such a way that the three equations $x^3+ax^2+bx+c=0$, $x^3+bx^2+cx+a=0$ and $x^3+cx^2+ax+b=0$ have a common

(a) real root?

(b) negative root?

2. How many pairs (n,q) satisfy $\{q^2\}=\left\{\frac{n!}{2000}\right\}$, where n is a positive integer and q is a non-integer rational number such that $0<q<2000$? [*Editor's comment:* $\{r\}$ means the "fractional part" of r.]

3. Given a fixed integer $N\geq 5$, and any sequence e_1, e_2, $\ldots$, e_N, where $e_i\in\{1,-1\}$ for $i=1$, 2, $\ldots$, N, a move is made by choosing any five consecutive terms and changing their signs. Two such sequences are said to be similar if one of them can be obtained from the other in a finite number of moves. Find the maximal number of sequences no two of which are similar to each other.

4. Let $ABCD$ be a quadrilateral with AB parallel to DC. A line ℓ intersects AD, AC, BD, and BC, forming three segments of equal length between consecutive points of intersection. Does it follow that ℓ is parallel to AB?

5. Nine points are given on a plane, no three on a line. Every pair of points is connected by a segment. Is it possible to colour these segments by some colours so that for each colour used, there are exactly three segments of this colour, and these three form a triangle with vertices among the given points?

6. A vertex of a tetrahedron is called *perfect* if one can construct a triangle using edges from this vertex as its sides. What are the possible numbers of perfect vertices a tetrahedron can have?

7. (a) Find all positive integers n such that $(a^a)^n = b^b$ has at least one solution in integers a and b, both exceeding 1.

(b) Find all positive integers a and b such that $(a^a)^5 = b^b$.

8. To any triangle ABC with $AB = c$, $BC = a$, $CA = b$, $\angle A = \alpha$, $\angle B = \beta$, and $\angle C = \gamma$, we assign the sextuple $(a, b, c, \alpha, \beta, \gamma)$, where the angles are measured in radians. Find the minimal value of n for which there is a non-isosceles triangle ABC such that there are exactly n distinct numbers in $(a, b, c, \alpha, \beta, \gamma)$.

As a second problem set we give the 2000 Taiwanese Mathematical Olympiad. Again, thanks to Andy Liu for collecting them for our use while Team Leader for the Canadian Team to the IMO in Korea.

2000 TAIWANESE MATHEMATICAL OLYMPIAD

1. Find all pairs (x, y) of positive integers such that $y^{x^2} = x^{y+2}$.

2. In an acute triangle ABC, $AC > BC$ and M is the mid-point of AB. Let AP be the altitude from A. Let BQ be the altitude from B meeting AP at H. Let the lines AB and PQ meet at R. Prove that the lines RH and CM are perpendicular to each other.

3. Let $S = \{1, 2, 3, \ldots, 100\}$, and let $\mathcal{P}$ denote the family of all subsets T of S with $|T| = 49$. For each set T in $\mathcal{P}$, we label it with a number chosen at random from $\{1, 2, \ldots, 100\}$. Prove that there exists a subset M of S with $|M| = 50$ such that for each $x \in M$, $M - \{x\}$ is not labelled with x.

4. Let $\phi(k)$ denote the number of positive integers $n \leq k$ such that $\gcd(n, k) = 1$. Suppose that $\phi(5^m - 1) = 5^n - 1$ for some positive integers m and n. Prove that $\gcd(m, n) > 1$.

5. Let $A = \{1, 2, 3, \ldots, n\}$, where n is a positive integer. A subset of A is said to be *connected* if it consists of one element or some consecutive integers. Determine the greatest integer k for which A contains k distinct subsets such that the intersection of any two of them is connected.

6. Let f be a function from the set of positive integers to the set of non-negative integers such that $f(1) = 0$ and

$$f(n) = \max\{f(j) + f(n-j) + j\}$$

for all $n \geq 2$. Determine $f(2000)$.

Now we turn to the problems of the Composition de Mathématiques, Classe Terminale S given [2000 : 453–454]. We received reader solutions by Mohammed Aassila, Strasbourg, France. However, we will only present here the web address, provided by Pierre Bornsztein, Pontoise, France, where the official solutions may be found. Go to

`www.ac-poitiers.fr/voir.asp?r=88`

Then click successively on "Pour s'y retrouver", "Plan du site", "Concours général", "Concours 97: indications et corrigé", and finally on "Corrigé des 5 exercices".

Next, we give solutions to the Ukrainian Mathematical Olympiad, Selected Problems 1997 [2001 : 5–6].

1. (9th Grade) Cells of some rectangular board are coloured as chessboard cells. In each cell an integer is written. It is known that the sum of the numbers in each row is even and the sum of numbers in each column is even. Prove that the sum of all numbers in the black cells is even.

Solved by Bruce Crofoot, University College of the Cariboo, Kamloops, BC; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We present the solution by Crofoot.

First observe that in any particular row or column, the sum of the numbers in the black cells has the same parity (even or odd) as the sum of the numbers in the white cells, because the sum over all cells in the column is even.

Now consider the sum over all the black cells on the board. We imagine this sum to be calculated by columns. Since we are interested only in the parity of the sum, we are free to sum over all the white cells instead of the black cells in any given column. Thus, we sum over the black cells in the odd-numbered columns and over the white cells in the even-numbered columns. Thinking now in terms of rows, our sum is effectively over *all* cells (both black and white) in every second row. This sum is clearly even, since the sum over all the cells in any row is even.

2. (10th Grade) Solve the system in real numbers

$$\left\{\begin{array}{rcl} x_1+x_2+\cdots+x_{1997} & = & 1997 \\ x_1^4+x_2^4+\cdots+x_{1997}^4 & = & x_1^3+x_2^3+\cdots+x_{1997}^3\,. \end{array}\right.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; and Panos E. Tsaoussoglou, Athens, Greece. We first give the solution by Bornsztein.

More generally, let n be a positive integer. We will prove that the unique solution in real numbers of the system

$$\left\{\begin{array}{rcl} x_1+x_2+\cdots+x_n & = & n \\ x_1^4+x_2^4+\cdots+x_n^4 & = & x_1^3+x_2^3+\cdots+x_n^3 \end{array}\right.$$

is $(1,1,\ldots,1)$. It is easy to see that $(1,1,\ldots,1)$ is a solution of the system. Conversely, if x_1, x_2, $\ldots$, x_n are real numbers satisfying the system, then

$$\begin{aligned} 0 &= \sum_{i=1}^n x_i^4 - \sum_{i=1}^n x_i^3 - \sum_{i=1}^n x_i + n = \sum_{i=1}^n (x_i^4 - x_i^3 - x_i + 1) \\ &= \sum_{i=1}^n (x_i-1)^2(x_i^2+x_i+1) = \sum_{i=1}^n (x_i-1)^2\left(\left(x_i+\frac{1}{2}\right)^2+\frac{3}{4}\right). \end{aligned}$$

Since, for each i, we have $(x_i-1)^2\left((x_i+\frac{1}{2})^2+\frac{3}{4}\right) \geq 0$, the equality occurs only if $x_i = 1$ for $i = 1, 2, \ldots, n$.

Next we give an alternate approach by Maragoudakis.

By Chebyshev's inequality, if $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_n$, then

$$n(a_1b_1+a_2b_2+\cdots+a_nb_n) \geq (a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n).$$

Equality occurs if and only if at least one sequence is constant [2001 : 514].

Without loss of generality, we assume that $x_1 \leq x_2 \leq \cdots \leq x_{1997}$. Then $x_1^3 \leq x_2^3 \leq \cdots \leq x_{1997}^3$. Therefore,

$$1997(x_1^4+x_2^4+\cdots+x_{1997}^4) \geq (x_1+x_2+\cdots+x_{1997})(x_1^3+x_2^3+\cdots+x_{1997}^3).$$

Since $x_1+x_2+\cdots+x_{1997} = 1997$ and

$$x_1^4+x_2^4+\cdots+x_{1997}^4 = x_1^3+x_2^3+\cdots+x_{1997}^3,$$

the above inequality becomes equality. Thus,

$$x_1 = x_2 = \cdots = x_{1997} \quad \text{or} \quad x_1^3 = x_2^3 = \cdots = x_{1997}^3.$$

In either case, $x_1 = x_2 = \cdots = x_{1997} = 1$.

3. (10th Grade) Let $d(n)$ denote the greatest odd divisor of the natural number n. We define the function $f : \mathbb{N} \to \mathbb{N}$ as follows: $f(2n-1) = 2^n$, $f(2n) = n + \dfrac{2n}{d(n)}$ for all $n \in \mathbb{N}$.
Find all k such that $f(f(\ldots f(1)\ldots)) = 1997$, where f is iterated k times.

Solution by Pierre Bornsztein, Pontoise, France.

Let (x_k) be the sequence defined by $x_1 = 1$ and $x_{k+1} = f(x_k)$ for all $k \geq 1$. We want to find the integers k such that $x_{k+1} = 1997$.

The first terms of the sequence (x_k) are 1, 2, 3, 4, 6, 5, 8, 12, 10, 7, 16, 24, ... We present them in successive rows, R_1, R_2, R_3, ..., where R_j contains exactly j terms:

$$\begin{array}{l} R_1 : 1 \\ R_2 : 2,\ 3 \\ R_3 : 4,\ 6,\ 5 \\ R_4 : 8,\ 12,\ 10,\ 7 \\ R_5 : 16,\ 24,\ 20,\ 14,\ 9 \\ \quad\vdots \end{array}$$

We will prove that, for all positive integers i, j, with $j \leq i$, the j^{th} number in R_i is $(2j-1)2^{i-j}$.

This is clearly true for $i = 1$. Let $i \geq 1$ be fixed. Suppose that the result is true for R_i. Then the last term of R_i (the one at the right) is $2i-1$. It follows that the first term of R_{i+1} is $f(2i-1) = 2^i = (2 \times 1 - 1)2^{i+1-1}$. Thus, the desired formula is true for this first term.

Suppose that the result is true for the j^{th} number in row R_{i+1}, where $1 \leq j < i+1$. Then the following term in R_{i+1} is:

$$f\big((2j-1)2^{i+1-j}\big) \;=\; (2j-1)2^{i-j} + 2^{i-j+1} \;=\; \big(2(j+1)-1\big)2^{i+1-(j+1)}\,.$$

Therefore, the formula is true for the value $j+1$. By induction, it is true for all $j \in \{1, 2, \ldots, i+1\}$.

Thus, the formula is true for all of R_{i+1}. By induction, it is true for all the rows.

Let $a = m2^n$, where m, n are non-negative integers and m is odd. Then $m2^n = (2j-1)2^{i-j}$ if and only if

$$\left\{\begin{array}{rcl} m &=& 2j-1 \\ n &=& i-j \end{array}\right. ; \qquad \text{that is,} \qquad \left\{\begin{array}{l} k = \dfrac{m+1}{2} \\ i = n + \dfrac{m+1}{2} \end{array}\right. .$$

It follows that a appears exactly once in the sequence, in position $\dfrac{m+1}{2}$ in $R_{n+\frac{m+1}{2}}$.

If $a = 1997$, then $m = 1997$ and $n = 0$. Thus, $k = i = 999$. Therefore, $x_k = 1997$ if and only if x_k is the last term of R_{999}, in which case

$$k \;=\; 1+2+3+\cdots+999 \;=\; 999 \cdot 500 \;=\; 499500\,.$$

5. (11th Grade) It is known that the equation $ax^3 + bx^2 + cx + d = 0$ with respect to x has three distinct real roots. How many roots does the equation $4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2$ have?

Comment by Murray S. Klamkin, University of Alberta, Edmonton, AB.

As stated, the problem is trivial. Since the degree of the given equation is 4, it must have 4 roots. Perhaps the original version asked for the number of real roots?

Solution by Bruce Crofoot, University College of the Cariboo, Kamloops, BC.

Let $p(x) = ax^3 + bx^2 + cx + d$, and let

$$f(x) = 4(ax^3 + bx^2 + cx + d)(3ax + b) - (3ax^2 + 2bx + c)^2 .$$

We are told that the equation $p(x) = 0$ has three distinct real roots, and we are asked about the roots of the equation $f(x) = 0$. Clearly, the problem is interested in real roots only. Note that $f(x) = 2p(x)p''(x) - [p'(x)]^2$. Hence, $f'(x) = 2p(x)p'''(x) = 12ap(x)$ and $f''(x) = 12ap'(x)$.

Let the roots of $p(x)$ be r_1, r_2, and r_3, where $r_1 < r_2 < r_3$. Since these roots are distinct, $p'(r_i) \neq 0$ (for $i = 1$, 2, 3). Since there are three roots, the degree of $p(x)$ cannot be less than three. Therefore, $a \neq 0$. We can assume $a > 0$. (Otherwise we replace $p(x)$ by $-p(x)$, with no effect on $f(x)$.) Then $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = -\infty$. Hence $p'(r_1) > 0$, $p'(r_2) < 0$ and $p'(r_3) > 0$.

For each i, $f'(r_i) = 12ap(r_i) = 0$, and there are no other points where $f'(x) = 0$. Since $f''(x)$ has the same sign as $p'(x)$, we have $f''(r_1) > 0$, $f''(r_2) < 0$ and $f''(r_3) > 0$. Thus, f has local minima at r_1 and r_3, a local maximum at r_2, and no other local maxima or minima. For each i, $f(r_i) = -[p'(r_i)]^2 < 0$. Therefore, $f(x) < 0$ on an interval containing r_1, r_2 and r_3. Furthermore, $\lim_{x\to\pm\infty} f(x) = \infty$ (since the highest-degree term in $f(x)$ is $3a^2x^4$). All of this implies that the equation $f(x) = 0$ has exactly two real roots, one of which is less than r_1 (where $f(x)$ changes from positive to negative) and the other greater than r_3 (where $f(x)$ changes from negative to positive).

6. (11th Grade) Let $\mathbb{Q}^+$ denote the set of all positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$:

(a) $f(x + 1) = f(x) + 1$,

(b) $f(x^2) = (f(x))^2$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the write-up of Bataille.

Let f be any such function. By property (a) and an immediate induction, we get

$$f(x + n) = f(x) + n \quad \text{for all } x \in \mathbb{Q}^+ \text{ and } n \in \mathbb{N} .$$

On the other hand,

$$\begin{aligned} \big(f(x+n)\big)^2 &= f\big((x+n)^2\big) = f(x^2 + 2nx + n^2) \\ &= f(x^2 + 2nx) + n^2 . \end{aligned}$$

Comparing, we obtain

$$f(x^2 + 2nx) \;=\; f(x^2) + 2nf(x) \qquad (1)$$

for all $x \in \mathbb{Q}^+$ and $n \in \mathbb{N}$.

Now, let $r = \frac{p}{q}$ be any element of $\mathbb{Q}^+$, where $p \in \mathbb{N}$ and $q \in \mathbb{N}$. From (1), with $x = r$ and $n = q$, we get $f(r^2 + 2p) = f(r^2) + 2qf(r)$. Then

$$f(r^2) + 2p \;=\; f(r^2) + 2qf(r)\,,$$

which yields $f(r) = \frac{2p}{2q} = r$. Therefore, f is the identity on $\mathbb{Q}^+$.

Conversely, the identity of $\mathbb{Q}^+$ clearly satisfies conditions (a) and (b), whence it is the unique solution.

7. (11th Grade) Find the smallest n such that among any n integers there are 18 integers whose sum is divisible by 18.

Solved by Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bornsztein's account.

The smallest n is 35.

Erdős has proved, more generally, that for any given integer $k > 1$, among any $2k - 1$ integers there are k integers whose sum is divisible by k (see [1]). Consider any $2k - 2$ integers such that $k - 1$ of them are equal to 0 modulo k, and the other $k - 1$ are equal to 1 modulo k. It is easy to see that, among these $2k - 2$ integers, we cannot find k integers whose sum is divisible by k. Thus, the value $2k - 1$ is indeed minimal.

Reference:

[1] R. Graham, *The Mathematical Intelligencer*, 1979, p. 250.

Next on the list are solutions to problems of the Tenth Irish Mathematical Olympiad 1997 given [2001 : 6–8].

1. Find (with proof) all pairs of integers (x, y) satisfying the equation

$$1 + 1996x + 1998y \;=\; xy\,.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Panos E. Tsaoussoglou, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution and comment by Amengual Covas.

We find all pairs of integers (x, y) satisfying the more general equation

$$1 + (p-1)x + (p+1)y \;=\; xy\,,$$

where $p > 1$ is a prime number. This equation is equivalent to

$$px + py \;=\; xy + x - y - 1\,,$$

which can be rewritten as

$$p\big((x-1) + (y+1)\big) \;=\; (x-1)(y+1)\,. \tag{1}$$

We observe that $(x, y) = (1, -1)$ is a solution and that no other candidates for (x, y) with $x = 1$ or $y = -1$ can satisfy (1).

Now suppose that $x \neq 1$ and $y \neq -1$, and denote by d the greatest common divisor of $x - 1$ and $y + 1$. We have

$$x - 1 \;=\; du, \quad y + 1 \;=\; dv\,, \tag{2}$$

where u and v are relatively prime integers. Substituting these expressions for $x - 1$ and $y + 1$ into (1) and dividing both sides by d gives

$$p(u + v) \;=\; duv\,. \tag{3}$$

Hence, uv divides the product $p(u + v)$ and is relatively prime to $u + v$. By the Fundamental Theorem of Arithmetic, uv is a divisor of p. Thus,

$$uv \;\in\; \{1, -1, p, -p\}\,.$$

Since $p > 0$ and $d > 0$, it follows from (3) that $u + v$ and uv agree in sign. This leads to the following possibilities:

- $u = v = 1$. Then, by (3), $d = 2p$. Substituting these values into (2), we find that
$$x \;=\; 2p + 1\,, \quad y \;=\; 2p - 1\,.$$
- $u = 1$, $v = p$. This yields $d = p + 1$ and
$$x \;=\; p + 2\,, \quad y \;=\; p(p+1) - 1\,.$$
- $u = 1$, $v = -p$. This yields $d = p - 1$ and
$$x \;=\; p\,, \quad y \;=\; -p(p-1) - 1\,.$$
- $u = p$, $v = 1$. This yields $d = p + 1$ and
$$x \;=\; p(p+1) + 1\,, \quad y \;=\; p\,.$$
- $u = -p$, $v = 1$. This yields $d = p - 1$ and
$$x \;=\; 1 - p(p-1)\,, \quad y \;=\; p - 2\,.$$

We conclude that the set of solutions for (x,y) is

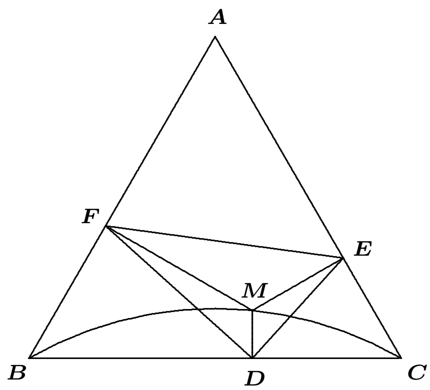
$$\{(1,-1)\,,\quad (2p+1,2p-1)\,,\quad (p+2,p(p+1)-1)\,,\quad (p,-p(p-1)-1)\,,$$
$$(p(p+1)+1,p)\,,\quad (1-p(p-1),p-2)\}\,.$$

The given problem is the special case when $p=1997$.

Comment. For two different methods of finding the integer solutions of the equation $p(x+y)=xy$, see problem 4 of the 31^{st} Spanish Mathematical Olympiad given in the *Corner* in the May 2000 number [2000 : 206] and Problem E:10322 in the Romanian Gazeta Matematica Nr 5/1992, pp. 186–187.

2. Let ABC be an equilateral triangle. For a point M inside ABC, let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right angle.

Solution by Michel Bataille, Rouen, France.



For all interior points M whose projections onto BC, CA, AB are D, E, F, respectively, points B, D, M, F are concyclic (they lie on the circle with diameter BM). Similarly, M, D, C, E are concyclic. It follows that $\angle FBM=\angle FDM$ and $\angle ECM=\angle EDM$. Therefore,

$$\angle FDE \;=\; \angle FBM+\angle ECM\,.$$

Thus,

$$\begin{aligned}\angle FDE \;=\; 90^\circ \quad &\Longleftrightarrow\quad \angle FBM+\angle ECM \;=\; 90^\circ\\ &\Longleftrightarrow\quad \angle MBD+\angle MCD \;=\; 30^\circ\\ &\qquad\qquad (\text{since } \angle B \;=\; \angle C \;=\; 60^\circ)\\ &\Longleftrightarrow\quad \angle BMC \;=\; 150^\circ\,.\end{aligned}$$

We may now conclude that the locus of M is the arc of the circle interior to $\triangle ABC$ subtending 150° on the line segment BC (as shown in the figure).

3. Find all polynomials $p(x)$ satisfying the equation

$$(x-16)p(2x) \;=\; 16(x-1)p(x)$$

for all x.

Solved by Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use the solution by Andrieux.

Posons

$$p(x) \;=\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \,,$$

avec $a_n \neq 0$. Dans $(x-16)p(2x) = 16(x-1)p(x)$, l'égalité des coefficients des termes de plus haut degré donne $2^n a_n \;=\; 16 a_n$. On en déduit donc que $n = 4$. D'où

$$p(x) \;=\; a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \,.$$

On a

$$\begin{aligned}(x-16)p(2x) \;=\;& 16a_4x^5 + (8a_3 - 256a_4)x^4 + (4a_2 - 128a_3)x^3 \\ & +(2a_1 - 64a_2)x^2 + (a_0 - 32a_1)x - 16a_0\end{aligned}$$

et

$$\begin{aligned}16(x-1)p(x) \;=\;& 16a_4x^5 + (16a_3 - 16a_2)x^4 + (16a_2 - 16a_3)x^3 \\ & +(16a_1 - 16a_2)x^2 + (16a_0 - 16a_1)x - 16a_0 \,.\end{aligned}$$

On obtient par identification des coefficients

$$a_3 \;=\; 30a_4 \,,\quad a_2 \;=\; 280a_4 \,,\quad a_1 \;=\; -960a_4 \,,\quad a_0 \;=\; 1024a_4 \,,$$

d'où

$$\begin{aligned}p(x) \;&=\; a_4(x^4 - 30x^3 + 280x^2 - 960x + 1024) \\ &=\; a_4(x-2)(x-4)(x-8)(x-16) \,.\end{aligned}$$

4. Let a, b, c be non-negative real numbers such that $a+b+c \geq abc$. Prove that $a^2+b^2+c^2 \geq abc$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Heinz-Jürgen Seiffert, Berlin, Germany; Panos E. Tsaoussoglou, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Seiffert's solution.

More generally, let $n \geq 2$ be an integer and $1 \leq p \leq n$. We claim that if $a_1, a_2, \ldots, a_n$ are non-negative real numbers such that $\sum a_i \geq \prod a_i$, then

$$\sum a_i^p \;\geq\; n^{(p-1)/(n-1)} \prod a_i \,.$$

(Here and below, all sums and products are extended over $i = 1, 2, \ldots, n$.)

Proof. We consider two cases:

Case 1. $\prod a_i \leq n^{n/(n-1)}$.

From the AM-GM Inequality, we have

$$\begin{aligned} \sum a_i^p &\geq n\left(\prod a_i^p\right)^{1/n} = n\left(\prod a_i\right)^{(p-n)/n} \prod a_i \\ &\geq n^{(p-1)/(n-1)} \prod a_i \,, \end{aligned}$$

because $p \leq n$.

Case 2. $\prod a_i > n^{n/(n-1)}$.

Using the Power-Mean Inequality and the condition $\sum a_i \geq \prod a_i$, we obtain

$$\left(\frac{1}{n}\sum a_i^p\right)^{1/p} \;\geq\; \frac{1}{n}\sum a_i \;\geq\; \frac{1}{n}\prod a_i \,,$$

which implies

$$\begin{aligned} \sum a_i^p &\geq n^{1-p}\left(\prod a_i\right)^p = n^{1-p}\left(\prod a_i\right)^{p-1} \prod a_i \\ &\geq n^{(p-1)/(n-1)} \prod a_i \,. \end{aligned}$$

This completes the proof of the claim.

Taking $n = 3$ and $p = 2$, and renaming a_1, a_2, a_3 by a, b, c, we see that under the conditions given in the proposal, there holds the better inequality $a^2 + b^2 + c^2 \geq \sqrt{3}\, abc$. This inequality is stronger than the one proposed.

We also give Klamkin's solution and remarks.

We need only consider two cases.

(1) If a, b, c are all ≥ 1, then clearly $a^2 + b^2 + c^2 \geq a + b + c \geq abc$.

(2) If at least one of a, b, $c \leq 1$ (say $c \leq 1$), then $a^2 + b^2 \geq ab \geq abc$.

Comment. The same problem without the condition that a, b, c be non-negative is given as a problem without solution in D. Fomin, S. Genkin, I. Itenberg, *Mathematical Circles*, Amer. Math. Soc., 1996, p. 185.

We now conclude the proof by allowing negative numbers. If just one of a, b, c is negative or if all three are negative, then $abc \leq 0$, in which case the result is immediate. Thus, we may assume that only two of them, say b and c, are negative. Then, letting $x = -b$ and $y = -c$, we have to show that when $a \geq x + y + axy$, we can conclude that $a^2 + x^2 + y^2 \geq axy$.

The assumed inequality implies that $xy \leq 1$. If $a \geq 1$, then $a^2 \geq axy$, which implies the desired result. If $a \leq 1$, then $x \leq 1$ and $y \leq 1$. In this case $a > x + y \geq xy$, since the latter inequality can be rewritten as $1 \geq (1-x)(1-y)$. Thus, we still have $a^2 \geq axy$, with the same conclusion as before.

5. Let S be the set of all odd integers greater than one. For each $x \in S$, denote by $\delta(x)$ the unique integer satisfying the inequality

$$2^{\delta(x)} < x < 2^{\delta(x)+1}.$$

For a, $b \in S$, define

$$a * b = 2^{\delta(a)-1}(b-3) + a.$$

[For example, to calculate $5 * 7$ note that $2^2 < 5 < 2^3$, so that $\delta(5) = 2$, and hence, $5 * 7 = 2^{2-1}(7-3) + 5 = 13$. Also $2^2 < 7 < 2^3$, so that $\delta(7) = 2$ and $7 * 5 = 2^{2-1}(5-3) + 7 = 11$].

Prove that if a, b, $c \in S$, then

(a) $a * b \in S$ and

(b) $(a * b) * c = a * (b * c)$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

(a) Since $b-3$ is even, $a*b$ is clearly odd. Also, since $2^{\delta(a)-1}(b-3) \geq 0$, we have $a * b \geq a > 1$. Hence, $a * b \in S$.

(b) Note first that

$$\begin{aligned} (a*b)*c &= 2^{\delta(a*b)-1}(c-3) + (a*b) \\ &= 2^{\delta(a*b)-1}(c-3) + 2^{\delta(a)-1}(b-3) + a \end{aligned} \qquad (1)$$

and

$$\begin{aligned} a*(b*c) &= a*(2^{\delta(b)-1}(c-3) + b) \\ &= 2^{\delta(a)+\delta(b)-2}(c-3) + 2^{\delta(a)-1}(b-3) + a. \end{aligned} \qquad (2)$$

By (1) and (2) it clearly suffices to show that

$$\delta(a*b) = \delta(a) + \delta(b) - 1. \qquad (3)$$

By definition, $2^{\delta(a)} < a < 2^{\delta(a)+1}$ and $2^{\delta(b)} < b < 2^{\delta(b)+1}$.

Using the inequalities $a < 2^{\delta(a)+1}$ and $b < 2^{\delta(b)+1}$, we have

$$\begin{aligned} 2^{\delta(a)-1}(b-3) + a &< 2^{\delta(a)-1}(b-3) + 2^{\delta(a)+1} \\ &= 2^{\delta(a)-1}(b-3+4) \\ &= 2^{\delta(a)-1}(b+1) \leq 2^{\delta(a)-1} \cdot 2^{\delta(b)+1}; \end{aligned}$$

that is,

$$a * b \ < \ 2^{\delta(a)+\delta(b)} \,. \tag{4}$$

Similarly, using the inequalities $2^{\delta(a)} < a$ and $2^{\delta(b)} < b$, we have

$$\begin{aligned} 2^{\delta(a)-1}(b-3)+a &> 2^{\delta(a)-1}(b-3)+2^{\delta(a)} \\ &= 2^{\delta(a)-1}(b-3+2) \\ &= 2^{\delta(a)-1}(b-1) \ \geq \ 2^{\delta(a)-1}\cdot 2^{\delta(b)}\,; \end{aligned}$$

that is,

$$a * b \ > \ 2^{\delta(a)+\delta(b)-1} \,. \tag{5}$$

From (4) and (5) we have $2^{\delta(a)+\delta(b)-1} < a * b < 2^{\delta(a)+\delta(b)}$. Therefore,

$$\delta(a * b) \ = \ \delta(a)+\delta(b)-1 \,,$$

which is (3). This completes the proof.

6. Given a positive integer n, denote by $\sigma(n)$ the sum of all the positive integers which divide n. [For example, $\sigma(3) = 1+3 = 4$, $\sigma(6) = 1+2+3+6 = 12$, $\sigma(12) = 1+2+3+4+6+12 = 28$.]

We say that n is abundant if $\sigma(n) > 2n$. (Thus, for example, 12 is abundant). Let a, b be positive integers and suppose that a is abundant. Prove that ab is abundant.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Maragoudakis' solution.

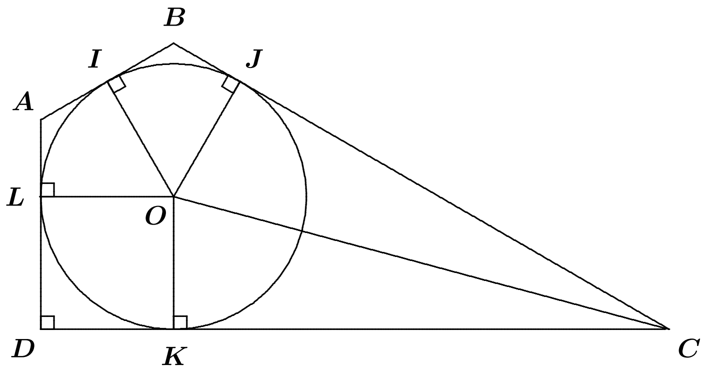
Let d_1, d_2, $\ldots$, d_k be all the positive integers which divide a. Then $d_1+d_2+\cdots+d_k > 2a$. Also, d_1b, d_2b, $\ldots$, d_kb are all different positive integers that divide ab. Thus,

$$\sigma(ab) \ \geq \ d_1b+d_2b+\cdots+d_kb \ > \ 2ab \,,$$

whence ab is abundant.

7. $ABCD$ is a quadrilateral which is circumscribed about a circle Γ (that is, each side of the quadrilateral is tangent to Γ). If $\angle A = \angle B = 120^\circ$, $\angle D = 90^\circ$ and BC has length 1, find, with proof, the length of AD.

Solved by Michel Bataille, Rouen, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Pavlos Maragoudakis, Pireas, Greece. We give the solution by Bataille.



Let O be the centre of Γ, and let I, J, K, and L be the points at which AB, BC, CD, and DA touch Γ, respectively. Triangle IBJ is isosceles with $\angle B = 120^\circ$. Therefore, $\angle BIJ = \angle BJI = 30^\circ$; whence, $\angle OIJ = 60^\circ$. It follows that $\triangle IOJ$ is equilateral and, consequently, $IJ = OI = OJ = R$, the radius of Γ. Since $\frac{\sqrt{3}}{2} = \cos 30^\circ = \frac{IJ/2}{BJ}$, we get $BJ = \frac{R}{\sqrt{3}}$.

Now, observing that $OKDL$ is a square and that $\triangle IOJ$ and $\triangle IOL$ are equilateral triangles, we obtain $\angle KOJ = 150^\circ$. Then $\angle OCJ = 15^\circ$; whence, $2 - \sqrt{3} = \tan 15^\circ = \frac{OJ}{CJ}$. This implies that $CJ = R(2+\sqrt{3})$. The relation $1 = BC = BJ + CJ$ now yields $R = \frac{\sqrt{3}}{4+2\sqrt{3}}$. Using $DL = R$ and $AL = BJ = \frac{R}{\sqrt{3}}$, we can compute $AD = AL + DL = \frac{R}{\sqrt{3}} + R$, which readily gives $AD = \frac{\sqrt{3}-1}{2}$.

8. Let A be a subset of $\{0, 1, 2, 3, \ldots, 1997\}$ containing more than 1000 elements. Prove that either A contains a power of 2 (that is, a number of the form 2^k with k a non-negative integer) or there exist two distinct elements $a, b \in A$ such that $a + b$ is a power of 2.

Solved by Pierre Bornsztein, Pontoise, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

Let $S = \{0, 1, 2, \ldots, 1997\}$. We prove the following stronger result: if A is a subset of S containing more than 997 elements, then A contains a power of 2, or A contains two distinct elements a, b such that $a + b$ is a power of 2. Furthermore, the bound 997 is the best possible.

Suppose $A \subseteq S$ does *not* have the described property. We will show that $|A| \leq 997$.

Note that $2^{10} = 1024 < 1997 < 2048 = 2^{11}$. We arrange the numbers in $S - \{0, 1, 2, 8, 32, 1024\}$ into an array as follows:

3	13	14	50	51	1997
4	12	15	49	52	1996
5	11	16	48	53	1995
6	10	17	47	54	1994
7	9	18	46	55	1993
		·	·	·	·
		·	·	·	·
		31	33	·	·
				·	·
				·	·
				1023	1025

Let C_i denote the i^{th} column of the array ($i = 1, 2, \ldots, 6$). Note that in $C_1 \cup C_2$, the two numbers in the same row add up to 16; in $C_3 \cup C_4$, the two numbers in the same row add up to 64; and in $C_5 \cup C_6$, the two numbers in the same row add up to 2048. Since 16, 64, and 2048 are powers of 2, and since $|C_1| = |C_2| = 5$, $|C_3| = |C_4| = 18$, and $|C_5| = |C_6| = 973$, we have $|A \cap (C_1 \cup C_2)| \le 5$, $|A \cap (C_3 \cup C_4)| \le 18$, and $|A \cap (C_5 \cup C_6)| \le 973$. Allowing $0 \in A$, we find that $|A| \le 1 + 5 + 18 + 973 = 997$, as claimed.

Furthermore, if we take $A = \{0\} \cup C_2 \cup C_4 \cup C_6$, then clearly $|A| = 997$, and it is easy to check that A does not contain a power of 2 or two elements which add up to a power of 2. Thus, the upper bound of 997 is the best possible.

Remark: A very interesting problem "deserving" a nice solution, which hopefully is from "The Book" (by Erdős' definition).

9. Let S be the set of all natural numbers n satisfying the following conditions:

(a) n has 1000 digits,

(b) all the digits of n are odd, and

(c) the absolute value of the difference between adjacent digits of n is 2.

Determine the number of distinct elements of S.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Pontoise, France. We give Bornsztein's solution.

Let k be a positive integer. Denote by S_k the set of all k-digit natural numbers n satisfying (b) and (c). Let U_k be the set of elements of S_k ending in 1 or 9, V_k the set of elements of S_k ending in 3 or 7, and W_k the set of elements of S_k ending in 5. Let $s_k = |S_k|$, $u_k = |U_k|$, $v_k = |V_k|$, and $w_k = |W_k|$.

Note that each element of S_{k+1} is obtained from an element of S_k by adding an odd number at the right of its decimal expansion, following (c). Any $n \in U_k$ may be used to construct exactly one number of S_{k+1} because the only possible digit that may be added to n is 3 if the rightmost digit of n is 1, and 7 if the rightmost digit of n is 9. Then we obtain u_k elements belonging to V_{k+1}. Similarly, any $n \in V_k$ may be used to construct exactly one number of U_{k+1} and one number of W_{k+1}; and any $n \in W_k$ may be used to construct exactly two numbers of V_{k+1}.

Since n cannot belong to more than one of the cases above, the construction leads to different numbers in S_{k+1}. Thus,

$$\left\{ \begin{array}{rcl} u_{k+1} & = & v_k \\ v_{k+1} & = & u_k + 2w_k \\ w_{k+1} & = & v_k \end{array} \right. ,$$

with $u_1 = v_1 = 2$ and $w_1 = 1$. For $k \geq 2$, we have $u_k = w_k = v_{k-1}$, which leads to $v_{k+1} = 3v_{k-1}$. Therefore, $v_{2k+1} = 3^k v_1 = 2 \times 3^k$ and $v_{2k+2} = 3^k v_2 = 4 \times 3^k$ for all $k \geq 0$. Then $u_{2k+1} = w_{2k+1} = 4 \times 3^{k-1}$ and $u_{2k} = w_{2k} = 2 \times 3^{k-1}$ for all $k \geq 1$.

We deduce that, for all $k \geq 1$,

$$\begin{array}{rcl} s_{2k} & = & 2 \times 3^{k-1} + 4 \times 3^{k-1} + 2 \times 3^{k-1} \;=\; 8 \times 3^{k-1}\,, \\ s_{2k+1} & = & 4 \times 3^{k-1} + 2 \times 3^{k} + 4 \times 3^{k-1} \;=\; 14 \times 3^{k-1}\,. \end{array}$$

In particular, $s_{1000} = 8 \times 3^{499}$.

10. Let p be a prime number and n a natural number, and let $T = \{1, 2, 3, \ldots, n\}$. Then n is called *p-partitionable* if there exist non-empty subsets $T_1, T_2, \ldots, T_p$ of T such that

(i) $T = T_1 \cup T_2 \cup \cdots \cup T_p$,

(ii) $T_1, T_2, \ldots, T_p$ are disjoint (that is, $T_i \cap T_j$ is the empty set for all i, j with $i \neq j$), and

(iii) the sum of the elements in T_i is the same for $i = 1, 2, \ldots, p$.

[For example, 5 is 3-partitionable since, if we take $T_1 = \{1, 4\}$, $T_2 = \{2, 3\}$, $T_3 = \{5\}$, then (i), (ii) and (iii) are satisfied. Also 6 is 3-partitionable since, if we take $T_1 = \{1, 6\}$, $T_2 = \{2, 5\}$, $T_3 = \{3, 4\}$, then (i), (ii) and (iii) are satisfied.]

(a) Suppose that n is p-partitionable. Prove that p divides n or $n+1$.

(b) Suppose that n is divisible by $2p$. Prove that n is p-partitionable.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's solution.

(a) Denote by $s(A)$ the sum of the elements of the finite subset A of $\mathbb{N}$. Then

$$\frac{n(n+1)}{2} \;=\; s(T) \;=\; \sum_{i=1}^{p} s(T_i) \;=\; pm\,,$$

where $m = s(T_i)$, $i = 1, 2, \ldots, p$. Thus, $2pm = n(n+1)$. The prime number p, which divides the product $n(n+1)$, must divide n or $n+1$.

(b) Let k be the integer such that $n = 2pk$, and set

$$\begin{aligned}
T_1 &= \{1, 2, \ldots, k\} \cup \{(2p-1)k+1, (2p-1)k+2, \ldots, 2pk\}, \\
T_2 &= \{k+1, k+2, \ldots, 2k\} \cup \\
&\qquad \{(2p-2)k+1, (2p-2)k+2, \ldots, (2p-1)k\}, \\
&\vdots \\
T_{j+1} &= \{jk+1, \ldots, (j+1)k\} \cup \{(2p-j)k+1, \ldots, \big(2p-(j-1)\big)k\}, \\
&\vdots \\
T_p &= \{(p-1)k+1, \ldots, pk, pk+1, pk+2, \ldots, (p+1)k\}.
\end{aligned}$$

Now we have $s(T_i) = k(2pk+1)$ for $i = 1, 2, \ldots, p$, and clearly (i) and (ii) are also satisfied. It follows that n is p-partitionable.

That completes this number of the *Corner*. This is Olympiad Season—send me Olympiad Contests as well as your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

Proofs Without Words II: Exercises in Visual Thinking
by Roger B. Nelsen, published by The Mathematical Association of America, 2000, ISBN 0-883-85721-9, softcover, 142 pages, US$26.95.
Reviewed by **Bernardo Recamán Santos**, *Ministry of Education, Bogotá, Columbia.*

"A being whose intellect was infinitely powerful would take no interest in logic and mathematics", wrote the philosopher Alfred J. Ayer. Fortunately, we human beings have powerful but limited intellects, and can thus still enjoy and appreciate this fine collection of mathematical results, all of which are proved here *almost* entirely without words. Presumably, a superior being, upon glancing at any of these often beautiful and awesome diagrams, would immediately utter, like a certain London detective, "Elementary!"

This, however, is not how the author expects ordinary readers to react to the contents of this book. Some of the results included here, and their "proofs", will take some time to decode; others will force the reader to get hold of pencil and paper and start experimenting for herself until finally she will be able to run down the street, preferably clothed, shouting Eureka! A few will surprise the reader, a handful might even annoy her, but all will "stimulate mathematical thought", just as the editor promises they will do.

This second collection of *Proofs Without Words* is divided into five chapters arranged according to topics ranging from Geometry and Algebra to Infinite Series and Linear Algebra. They have been collected from a wide variety of sources, including the MAA's journals and the World Wide Web. Quite aptly, the first chapter begins with six more proofs of the Pythagorean Theorem, including one by Leonardo da Vinci. (Six others appeared in the first book in the series, also published by the MAA.) A favourite theorem of this reviewer, and some of his students, the *Pizza Theorem*, appears in this chapter together with an intricate visual proof by Stan Wagon and Larry Carter. This remarkable theorem, first discovered by L.J. Upton in 1968, states that if a pizza is divided into eight slices making cuts at 45° angles from an arbitrary point in the pizza, then the sums of the areas of alternate slices are equal.

Not all of the proofs here are straight out of *The Book*, Paul Erdős' legendary book in God's hand, which is supposed to contain the best proof of every single theorem. However, most of them are real gems, and will serve the additional purpose of reminding the reader of long forgotten results, or may even introduce her to new and elegant results. All in all, this is a book worth keeping at close range.

Hypermath : 120 Exercices de haut vol
par Pierre Bornsztein (preface de Johan Yebbou), Editeur Librairie Vuibert, 2001, ISBN 2-7117-5301-8, Paris, broché X + 230 pages
Rapporteur : **André Ronveaux**, *Facultés Universitaires N-D de la Paix, Namur, Belgium*

L'auteur examine dans ce petit ouvrage 60 problèmes d'arithmétique et 60 problèmes de géométrie. Les outils utilisés dans les solutions proposées restent élémentaires : trigonométrie, géométrie du triangle par exemple, mais tous ces problèmes sont difficiles. Si les énoncés remplissent en effet 26 pages, les solutions s'étalent sur 200 pages ! Ces problèmes sont en grande partie tirés de compétitions mathématiques et donc de niveau fin du secondaire ou d'un premier cycle universitaire. Les solutions de l'auteur sont détaillées, souvent astucieuses, et parfois precédées de lemmes non triviaux. Des références complètent assez souvent les démonstrations.

Parfois l'auteur résout un problème plus général ou signale une extension ou un prolongement. Dans l'esprit des Olympiades, le lecteur doit souvent examiner et discuter différentes situations, en éliminer plusieurs au moyen d'arguments parfois pointus, énumerer les solutions possibles, et calculer numériquement chaque situation retenue. Un exemple typique extrait des problèmes arithmétique, et proposé en 1994 par la Lituanie est le suivant : Résoudre l'équation : $1! + 2! + 3! + \ldots + n! = m^k$ en entiers naturels non nuls, avec $n, k > 1$.

Examinant les cas $k = 2, k = 3$ et $3 < k$, on montre d'abord qu'il n'y a pas de solution pour n plus grand ou égal à 8. On calcule alors les sommes de factorielle pour $n = 2, 3, 4, 5, 6, 7$, on vérifie qu'aucun de ces nombres n'est une puissance k-ième pour $3 = k$, et $3 < k$. La seule solution est donc $(n, m, k) = (3, 3, 2)$ Les problèmes géométriques nécessitent également la même gymnastique. Un problème, proposé aux Olympiades mathématique (Colorado 1988) est particuliérement ardu. Soient 5 points dans ou sur les bords d'un triangle d'aire 1. Prouver que 3 d'entre eux sont les sommets d'un triangle dont l'aire ne dépasse pas 1/4. La solution, découpée en 3 lemmes, prend 5 pages, comprenant également des remarques, des généralisations et des références, et introduit les constantes de Heilbronn, décrites sur le site suivant :

`http://www.stetson.edu/~efriedma/heilbronn/`

L'auteur, familier de la revue Crux Mathematicorum, annonce une deuxième partie, qui, si elle est calquée sur cette première partie ne devrait pas être triste !

An Inversion Formula for Putnam Data

Keith A. Brandt and Donald L. Vestal, Jr.

Below are the score and rank data from the 62^{nd} William Lowell Putnam Mathematical Competition, held in December, 2001. The table is reproduced as it is presented in the mailings from the Competition organizers.

Score	Rank	Score	Rank	Score	Rank	Score	Rank
101	1	58	59.5	38	190.5	21	414.5
100	2	57	62.5	37	195	20	494
86	3	55	64	36.9	197	19	555
80	4.5	54	66	36.8	198	18	570.5
79	6	53	69	36.7	199	17	575
77	7.5	52	72.5	36.6	200	15	577
73	9	51	78	36	202	14	586.5
72	11	50	93.5	35	204	13	606
71	14	49	109	34	205	12	655
70	16.5	48	115	33	209	11	759
69	19	47	118.5	32	225.5	10	970
68	23.5	46	120.5	31	252	9	1130.5
67	27.5	45	122	30	287.5	8	1154
66	29	44	123	29	314.5	5	1162.5
63	30	43	125	28	324.5	4	1166
62	32	42	130.5	26	330	3	1181.5
61	37	41	143.5	24	332.5	2	1252
60	47.5	40	164	23	340	1	1469.5
59	56	39	181.5	22	365	0	2292

Table 1. Putnam Scores and Ranks

The table does not include the frequencies of the scores. This omission is interesting because the frequencies are used to determine the rankings. Suppose a student, Putnam Pat, scored 20. According to the table, Pat's is the 59^{th} highest score, giving a rank of 494. Can anything more be said? Our goal in this note is to recover the frequencies from the rankings.

We first choose some notation and explain how the rankings are determined. For $i = 1, 2, \ldots, 76$ (76 being the number of different scores), let s_i denote the i^{th} score (where s_1 is the highest), let n_i be the number of students with score s_i, and let c_i be the number of students with score s_i or higher. Thus, n_i is the i^{th} frequency and c_i is the i^{th} cumulative frequency. Finally, let r_i be the rank of the score s_i.

Since the rankings for the top three scores are 1, 2, and 3, it is clear that these scores were obtained by one student each. The rank 4.5 assigned to the fourth score of 80 indicates that that two students had this score, since 4.5 is the average of the places 4 and 5. Here is the formula for the rankings:

$$r_i = \sum_{k=1}^{i-1} n_k + \frac{n_i + 1}{2} = c_{i-1} + \frac{n_i + 1}{2} \qquad (1 \le i \le 76)\,. \tag{1}$$

For example, $r_6 = 7.5$ means that exactly two students scored 77, since $7.5 = 6 + \frac{2+1}{2}$.

We will derive a simple formula for n_i in terms of the r_i. First note that $r_i - r_{i-1} = \frac{1}{2}(n_i + n_{i-1})$. Therefore, $n_i + n_{i-1} = 2(r_i - r_{i-1})$. Since $n_i - n_{i-2} = (n_i + n_{i-1}) - (n_{i-1} + n_{i-2})$, we have

$$n_i - n_{i-2} = 2(r_i - 2r_{i-1} + r_{i-2})\ .$$

Similarly, since $n_i + n_{i-3} = (n_i - n_{i-2}) + (n_{i-2} + n_{i-3})$, we have

$$n_i + n_{i-3} = 2(r_i - 2r_{i-1} + 2r_{i-2} - r_{i-3})\ .$$

Continuing in this fashion, we obtain

$$n_i + (-1)^i n_1 = 2\left(r_i - 2r_{i-1} + 2r_{i-2} - \cdots + (-1)^i 2r_2 + (-1)^{i-1} r_1\right)\ .$$

Finally, since $n_1 = 2r_1 - 1$, we have

$$\begin{aligned} n_i &= 2\left(r_i - 2r_{i-1} + 2r_{i-2} - \cdots + (-1)^i 2r_2 + (-1)^{i-1} 2r_1\right) + (-1)^i \\ &= 2r_i + 4\sum_{k=1}^{i-1} (-1)^k r_{i-k} + (-1)^i \qquad (1 \le i \le 76)\,. \end{aligned} \tag{2}$$

With this formula (and a little help from a computer), we obtain the data in Table 2 on the next page.

Now we can easily say more about the results of the competition. For example, here is the information relevant to Putnam Pat: there were 99 students who had a score of 20, since $n_{59} = 99$; there were 444 students who had a score above 20, since $c_{58} = 444$; there were $2954 - 543 = 2411$ students who scored below 20, since $c_{59} = 543$ out of 2954 students participating in the competition.

Furthermore, the scores resulting in prize money are more apparent. The Putnam fellows (the top five) had scores 101, 100, 86, and 80. The \$1000 winners (the next ten) had scores 79, 77, 73, 72, and 71. The \$250 winners (the next eleven) had scores 70, 69, and 68.

Score	Rank	Freq.	Cum. Freq.	Score	Rank	Freq.	Cum. Freq.
101	1	1	1	38	190.5	6	193
100	2	1	2	37	195	3	196
86	3	1	3	36.9	197	1	197
80	4.5	2	5	36.8	198	1	198
79	6	1	6	36.7	199	1	199
77	7.5	2	8	36.6	200	1	200
73	9	1	9	36	202	3	203
72	11	3	12	35	204	1	204
71	14	3	15	34	205	1	205
70	16.5	2	17	33	209	7	212
69	19	3	20	32	225.5	26	238
68	23.5	6	26	31	252	27	265
67	27.5	2	28	30	287.5	44	309
66	29	1	29	29	314.5	10	319
63	30	1	30	28	324.5	10	329
62	32	3	33	26	330	1	330
61	37	7	40	24	332.5	4	334
60	47.5	14	54	23	340	11	345
59	56	3	57	22	365	39	384
58	59.5	4	61	21	414.5	60	444
57	62.5	2	63	20	494	99	543
55	64	1	64	19	555	23	566
54	66	3	67	18	570.5	8	574
53	69	3	70	17	575	1	575
52	72.5	4	74	15	577	3	578
51	78	7	81	14	586.5	16	594
50	93.5	24	105	13	606	23	617
49	109	7	112	12	655	75	692
48	115	5	117	11	759	133	825
47	118.5	2	119	10	970	289	1114
46	120.5	2	121	9	1130.5	32	1146
45	122	1	122	8	1154	15	1161
44	123	1	123	5	1162.5	2	1163
43	125	3	126	4	1166	5	1168
42	130.5	8	134	3	1181.5	26	1194
41	143.5	18	152	2	1252	115	1309
40	164	23	175	1	1469.5	320	1629
39	181.5	12	187	0	2292	1325	2954

Table 2. Frequencies and Cumulative Frequencies of the Putnam Scores

The inversion formula (2) can be derived using linear algebra. We start with a matrix form of equation (1): $\overrightarrow{r} = A\overrightarrow{n} + \overrightarrow{1/2}$, where

$$\overrightarrow{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{76} \end{bmatrix}, \quad \overrightarrow{n} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_{76} \end{bmatrix}, \quad \overrightarrow{1/2} = \begin{bmatrix} 1/2 \\ 1/2 \\ \vdots \\ 1/2 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 1/2 & 0 & 0 & \cdots & 0 \\ 1 & 1/2 & 0 & \cdots & 0 \\ 1 & 1 & 1/2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1/2 \end{bmatrix}.$$

Solving the linear system for $\overrightarrow{n}$ yields $\overrightarrow{n} = A^{-1}\left(\overrightarrow{r} - \overrightarrow{1/2}\right)$. Because of the simple form of the matrix A, we can easily calculate

$$A^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ -4 & 2 & 0 & 0 & \cdots & 0 \\ 4 & -4 & 2 & 0 & \cdots & 0 \\ -4 & 4 & -4 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4 & 4 & -4 & 4 & \cdots & 2 \end{bmatrix}.$$

The equation for $\overrightarrow{n}$ becomes

$$\overrightarrow{n} = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ -4 & 2 & 0 & 0 & \cdots & 0 \\ 4 & -4 & 2 & 0 & \cdots & 0 \\ -4 & 4 & -4 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4 & 4 & -4 & 4 & \cdots & 2 \end{bmatrix} \begin{bmatrix} r_1 - 1/2 \\ r_2 - 1/2 \\ \vdots \\ r_{76} - 1/2 \end{bmatrix},$$

which is a matrix form for the inversion formula (2).

Keith Brandt
Department of Mathematics,
Computer Science, and Physics
Rockhurst University
Kansas City, MO 64110
Keith.Brandt@Rockhurst.edu

Donald L. Vestal, Jr.
Department of Computer Science,
Mathematics, and Physics
Missouri Western State College
Saint Joseph, MO 64507
vestal@mwsc.edu

PROBLEMS

Faire parvenir les propositions de problèmes et les solutions à Jim Totten, Département de mathématiques et de statistique, University College of the Cariboo, Kamloops, BC V2C 5N3. Les propositions de problèmes doivent être accompagnées d'une solution ainsi que de références et d'autres indications qui pourraient être utiles à la rédaction. Si vous envoyez une proposition sans solution, vous devez justifier une ~~solution probable en fournissant suffisamment d'information.~~ Un numéro suivi d'une astérisque ($\star$) *indique que le problème a été proposé sans solution.*

Nous sollicitons en particulier des problèmes originaux. Cependant, d'autres problèmes intéressants pourraient être acceptables s'ils ne sont pas trop connus et si leur provenance est précisée. Normalement, si l'auteur d'un problème est connu, il faut demander sa permission avant de proposer un de ses problèmes.

Pour faciliter l'étude de vos propositions, veuillez taper ou écrire à la main (lisiblement) chaque problème sur une feuille distincte de format $8\frac{1}{2}''\times 11''$ *ou* A4, *la signer et la faire parvenir au rédacteur en chef. Les propositions devront lui parvenir au plus tard le* **1er octobre 2003**. *Vous pouvez aussi les faire parvenir par courriel à* **crux-editors@cms.math.ca**. (*Nous apprécierions de recevoir les problèmes et solutions envoyés par courriel au format* LaTeX). *Les fichiers graphiques doivent être de format « epic » ou « eps » (encapsulated postscript). Les solutions reçues après la date ci-dessus seront prises en compte s'il reste du temps avant la publication. Veuillez prendre note que nous n'acceptons pas les propositions par télécopieur.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Hidemitsu Saeki, de l'Université de Montréal, d'avoir traduit les problèmes.

2814. *Proposé par Juan José Egozcue et José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Terrassa, Espagne.*

On suppose que a, b, et c sont des nombres réels positifs tels que $a+b+c=abc$. Trouver la valeur minimale de

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}}\,.$$

. . . .

Let a, b, and c be positive real numbers such that $a+b+c=abc$. Find the minimum value of

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}}\,.$$

2815. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit $\Gamma(O, R)$ le cercle circonscrit au triangle ABC tel que $\angle ACB \neq 60^\circ$. Supposons que AB est fixe et que le point C se déplace sur Γ (en restant du même côté de AB).

On construit deux triangles équilatéraux BCD et ACE tels que A et D (respectivement B et E) sont de part et d'autre de BC (respectivement de AC).

(a) Montrer que CD et CE coupent respectivement Γ en des points fixés F et G. Caractérisez ces points.

(b) Complétez le parallélogramme $DCEH$. Montrer que H est un point fixe. Caractérisez H.

(c) Si K est le point d'intersection de CF et DE, déterminer le lieu de K lorsque C varie.

....

Suppose that $\Gamma(O, R)$ is the circumcircle of $\triangle ABC$, where $\angle ACB \neq 60^\circ$. Suppose that side AB is fixed and that C varies on Γ (always on the same side of AB).

Construct equilateral triangles BCD and ACF such that A and D are on opposite sides of BC, and B and E are on opposite sides of AC.

(a) Show that CD and CE intersect Γ at fixed points F and G, respectively. Characterize these points.

(b) Complete the parallelogram $DCEH$. Show that H is a fixed point. Characterize H.

(c) If K is the point of intersection of CF and DE, determine the locus of K as C varies.

2816. *Proposé par Boris Harizanov, étudiant, Stara Zagora, Bulgarie.*

Deux triangles isocèles d'angles aigus $A_1B_1C_1$ (avec $A_1C_1 = B_1C_1$) et $A_2B_2C_2$ (avec $A_2C_2 = B_2C_2$) sont tels que $A_1C_1 = A_2C_2$. Pour ces triangles $A_kB_kC_k$, $k = 1, 2$, on considère les cercles inscrits, de centre I_k, de rayon r_k, et les cercles circonscrits, de centre O_k, radius R_k.

Si $I_1O_1 = I_2O_2$, peut-on affirmer que ces deux triangles sont congruents ?

....

In acute-angled isosceles triangles $A_1B_1C_1$ (with $A_1C_1 = B_1C_1$) and $A_2B_2C_2$ (with $A_2C_2 = B_2C_2$), we have $A_1C_1 = A_2C_2$. For $k = 1, 2$, we have a circle with centre I_k and radius r_k inscribed in $\triangle A_kB_kC_k$, and a circle with centre O_k and radius R_k circumscribed around $\triangle A_kB_kC_k$.

If $I_1O_1 = I_2O_2$, is it true that $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ must be congruent?

2817. *Proposé par Vedula N. Murty, Dover, PA, USA.*
Soit A, B, et C les angles du triangle ABC. On définit

$$\begin{aligned} L &= 4\cos^2\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right); \\ M &= \left(\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)\right) \\ & \qquad \prod_{\text{cyclique}}\left(\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)-\cos\left(\frac{A}{2}\right)\right). \end{aligned}$$

Montrer que $L = M$.

. . . .

Suppose that A, B, and C are the angles of $\triangle ABC$. Define

$$\begin{aligned} L &= 4\cos^2\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right); \\ M &= \left(\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)\right) \\ & \qquad \prod_{\text{cyclic}}\left(\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)-\cos\left(\frac{A}{2}\right)\right). \end{aligned}$$

Show that $L = M$.

2818. *Proposé par Mihály Bencze, Brasov, Roumanie.*
Soit n, $k \geq 2$ des entiers tels que $(n + k^n, k) = 1$.

Montrer qu'au moins un des deux nombres $n + k^n$ et $n\,k^{(k^n - 1)} + 1$ n'est pas premier.

. . . .

Suppose that n, $k \geq 2$ are integers such that $(n + k^n, k) = 1$.

Prove that at least one of $n + k^n$ and $n\,k^{(k^n - 1)} + 1$ is not prime.

2819. *Proposé par Mihály Bencze, Brasov, Roumanie.*
Montrer que si la fonction $f : \mathbb{R} \to \mathbb{R}$ satisfait, pour tout x et y réels, $f\left(\frac{2x+y}{3}\right) \geq f\left(\sqrt[3]{x^2y}\right)$, alors f est décroissante sur $(-\infty, 0]$ et croissante sur $[0, \infty)$.

. . . .

Let $f : \mathbb{R} \to \mathbb{R}$ satisfy, for all real x and y, $f\left(\frac{2x+y}{3}\right) \geq f\left(\sqrt[3]{x^2y}\right)$.

Prove that f is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

2820. *Proposé par Christopher J. Bradley, Clifton College, Bristol, Grande-Bretagne.*

On suppose que Q est un point quelconque dans le plan du triangle ABC. On suppose de plus que AQ, BQ, et CQ, coupent respectivement BC, CA, et AB en D, E, et F ; que L, M, et N sont respectivement les points milieu de BC, CA, et AB ; et finalement que U, V, et W sont respectivement les points milieu de AQ, BQ, et CQ.

On sait qu'une conique Σ passe par les points D, E, F, L, M, N, U, V, et W. Il est clair que si Σ est agrandie par un facteur 2 avec Q comme centre d'agrandissement, alors la conique résultante Σ_Q passe par A, B, et C.

Supposons alors que P soit un point quelconque sur Σ_Q, et que les droites passant par P et parallèles à AQ, BQ, et CQ coupent les côtés BC, CA, et AB en R, S, et T, respectivement.

Montrer que R, S et T sont colinéaires.

. . . .

Suppose that Q is any point in the plane of $\triangle ABC$. Suppose that AQ, BQ, CQ meet BC, CA, AB at D, E, F, respectively; that L, M, N are the mid-points of BC, CA, AB, respectively; and that U, V, W are the mid-points of AQ, BQ, CQ, respectively.

It is known that a conic Σ passes through D, E, F, L, M, N, U, V, and W. Clearly, if Σ is enlarged by a factor of 2, with Q as the centre of enlargement, then the resulting conic Σ_Q passes through A, B, and C.

Suppose that P is any point on Σ_Q, and that lines through P parallel to AQ, BQ, CQ meet the sides BC, CA, AB at R, S, T, respectively.

Prove that R, S, and T are collinear.

2821. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

Dans un triangle ABC, soit w_a, w_b, et w_c les longueurs des bissectrices des angles intérieures et r le rayon du cercle inscrit. Montrer que

$$\frac{1}{{w_a}^2}+\frac{1}{{w_b}^2}+\frac{1}{{w_c}^2} \leq \frac{1}{3r^2},$$

où l'égalité a lieu si et seulement si le triangle ABC est équilatéral.

. . . .

In triangle $\triangle ABC$, let w_a, w_b, w_c be the lengths of the interior angle bisectors, and r the inradius. Prove that

$$\frac{1}{{w_a}^2}+\frac{1}{{w_b}^2}+\frac{1}{{w_c}^2} \leq \frac{1}{3r^2},$$

with equality if and only if $\triangle ABC$ is equilateral.

2822. *Proposé par Peter Y. Woo, Biola University, La Mirada, CA, USA.*

On suppose que Π est un parallélogramme de côtés $2a$ et $2b$, d'angle intérieur aigu α, et que F et F' sont les foyers de l'ellipse Λ qui, elle, est tangente aux quatre côtés de Π en leur point milieu.

(a) Trouver le grand et le petit demi-axe de Λ en fonction de a, b, et α.

(b) Trouver une construction avec la règle et le compas pour F et F'.

. . . .

Suppose that Π is a parallelogram with sides of lengths $2a$ and $2b$ and with acute interior angle α, and that F and F' are the foci of the ellipse Λ that is tangent to the four sides of Π at their mid-points.

(a) Find the major and minor semi-axes of Π in terms of a, b, and α.

(b) Find a straight-edge and compass construction for F and F'.

2823. *Proposé par Christopher J. Bradley, Clifton College, Bristol, Grande-Bretagne.*

Supposons que L, M, et N sont, dans l'ordre, des points sur les côtés BC, CA et AB d'un triangle ABC, et qu'ils sont distincts de A, B et C. Supposons aussi que :

$$\frac{BL}{LC} = \frac{1-\lambda}{\lambda}, \quad \frac{BM}{MA} = \frac{1-\mu}{\mu}, \quad \text{et} \quad \frac{AN}{NB} = \frac{1-\nu}{\nu},$$

et que les cercles AMN, BNL, and CLM se coupent au point de Miquel P.

Trouver $[BCP] : [CAP] : [ABP]$ en fonction de λ, μ, ν et des côtés du triangle ABC.

. . . .

Suppose that L, M, N are points on BC, CA, AB, respectively, and are distinct from A, B and C. Suppose further that

$$\frac{BL}{LC} = \frac{1-\lambda}{\lambda}, \quad \frac{BM}{MA} = \frac{1-\mu}{\mu}, \quad \text{and} \quad \frac{AN}{NB} = \frac{1-\nu}{\nu},$$

and that the circles AMN, BNL, and CLM meet at the Miquel point P.

Find $[BCP] : [CAP] : [ABP]$ in terms of λ, μ, ν and the side lengths of $\triangle ABC$.

2824. *Proposé par Eckard Specht, Université Otto-von-Guericke, Magdeburg, Allemagne.*

Deux segments perpendiculaires AB et CD se coupent en S. Soit K, L, M, N les symétriques respectifs de S par rapport aux droites AC, BC, BD, AD. Supposons que le cercle circonscrit au triangle SKL coupe la droite AL en E, et que le cercle circonscrit au triangle SMN coupe la droite AM en F.

Montrer que le quadrilatère $KEFN$ est cyclique.

. . . .

Two perpendicular line segments AB and CD intersect at S. Denote by K, L, M, N the reflections of S in the lines AC, BC, BD, AD, respectively. Suppose that the circumcircle of $\triangle SKL$ meets the line AL again in E, and that the circumcircle of $\triangle SMN$ meets the line AM again in F.

Prove that quadrilateral $KEFN$ is cyclic.

2825★. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

Soit $\mathcal{R}_n$ un n–gone ($n \geq 3$) (polygone à n côtés) régulier et $\mathcal{P}_n$ l'ensemble de tous les points P dans $\mathcal{R}_n$ tel que les pieds de toutes les n perpendiculaires abaissées de P sur les côtés de $\mathcal{R}_n$ soient à l'intérieur des côtés respectifs. Ces pieds, les extrémités des côtés respectifs et le point P forment $2n$ triangles (rectangles). On désigne par S_1 et S_2 les sommes des aires des n triangles qui n'ont alternativement que le point P en commun.

Montrer que $S_1 = S_2$ pour tous les points P dans $\mathcal{P}_n$.

. . . .

Let $\mathcal{R}_n$ be a regular n–gon ($n \geq 3$), and let $\mathcal{P}_n$ be the set of all points P in $\mathcal{R}_n$ such that all n perpendiculars from P to the sides of $\mathcal{R}_n$ have feet lying in the interior of the respective sides. These feet, the endpoints of the respective sides, and the point P form $2n$ (right-angled) triangles. Let S_1 and S_2 be the sums of the areas of n triangles each, using alternate triangles.

Prove that $S_1 = S_2$ for all points P in $\mathcal{P}_n$.

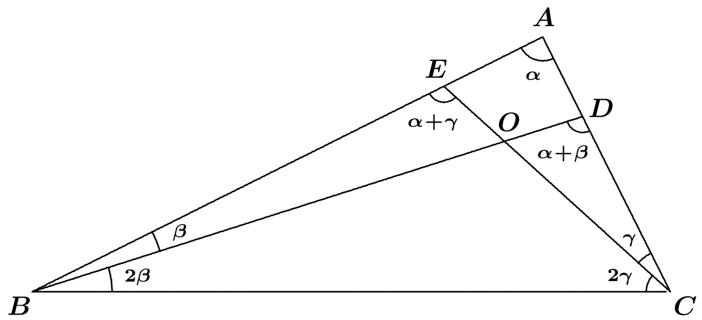
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2714. [2002 : 111] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that D and E are points on sides AC and AB, respectively, of $\triangle ABC$, such that $\angle DBC = 2\angle ABD$ and $\angle ECB = 2\angle ACE$. Suppose that BD and CE meet at O, and that $OD = OE$. Characterize $\triangle ABC$.

Solution by Michel Bataille, Rouen, France.



Let $\alpha = \angle A$, $\beta = \angle ABD$ $(= \angle B/3)$, and $\gamma = \angle ACE$ $(= \angle C/3)$. Then

$$\angle BEC = 180^\circ - \angle AEC = \alpha + \gamma .$$

Similarly, $\angle BDC = \alpha + \beta$. From the Law of Sines in triangles BOE and COD, we obtain

$$\frac{OE}{\sin\beta} = \frac{BE}{\sin(\alpha+\beta+\gamma)} \quad \text{and} \quad \frac{OD}{\sin\gamma} = \frac{CD}{\sin(\alpha+\beta+\gamma)} .$$

Hence,

$$\frac{OE}{OD} = \frac{\sin\beta}{\sin\gamma}\cdot\frac{BE}{CD} . \tag{1}$$

Similarly, from triangles BDC and BEC,

$$\frac{CD}{\sin(2\beta)} = \frac{BC}{\sin(\alpha+\beta)} \quad \text{and} \quad \frac{BE}{\sin(2\gamma)} = \frac{BC}{\sin(\alpha+\gamma)} .$$

Hence,

$$\frac{BE}{CD} = \frac{\sin(2\gamma)}{\sin(2\beta)}\cdot\frac{\sin(\alpha+\beta)}{\sin(\alpha+\gamma)} .$$

Substituting in (1), we obtain

$$\frac{OE}{OD} = \frac{\cos\gamma}{\cos\beta}\cdot\frac{\sin(\alpha+\beta)}{\sin(\alpha+\gamma)} = \frac{\sin\alpha+\tan\beta\cos\alpha}{\sin\alpha+\tan\gamma\cos\alpha} .$$

It follows that

$$\begin{aligned} OE \;=\; OD \quad &\Longleftrightarrow \quad \tan\beta\cos\alpha \;=\; \tan\gamma\cos\alpha \\ &\Longleftrightarrow \quad \alpha \;=\; 90^\circ \quad \text{or} \quad \beta \;=\; \gamma\,. \end{aligned}$$

Therefore, $\triangle ABC$ is either right-angled at A or isosceles with $AB = AC$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer. There were also 4 incomplete solutions submitted.

2715. [2002 : 111] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that the convex quadrilateral $ABCD$ has an incircle with centre O. Let E and F be the incentres of $\triangle ABC$ and $\triangle ADC$, respectively. Prove that A, O and the circumcentre of $\triangle AEF$ are collinear.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Since the circle with centre O is inscribed in $ABCD$, then AO is the bisector of $\angle BAD$ and $AB + CD = BC + DA$. Thus,

$$DA - CD \;=\; AB - BC\,. \tag{1}$$

If H is the point of contact of the incircle of triangle ADC with AC, then $2AH = AC + DA - CD$. Then, using (1), $2AH = AC + AB - BC$, which implies that H is the point of contact of the incircle of triangle ABC with AC. Consequently, AH is an altitude of triangle AEF. If K is the circumcentre of triangle AEF, then AK is the isogonal conjugate of AH. Hence, $\angle EAK = \angle HAF = \frac{1}{2}\angle HAD$. Also, $\angle BAE = \frac{1}{2}\angle BAH$. This implies

$$\angle BAK \;=\; \angle BAE + \angle EAK \;=\; \frac{1}{2}\angle BAH + \frac{1}{2}\angle HAD \;=\; \frac{1}{2}\angle BAD\,.$$

Thus, AK is the bisector of $\angle BAD$, and hence the points A, K, O are collinear.

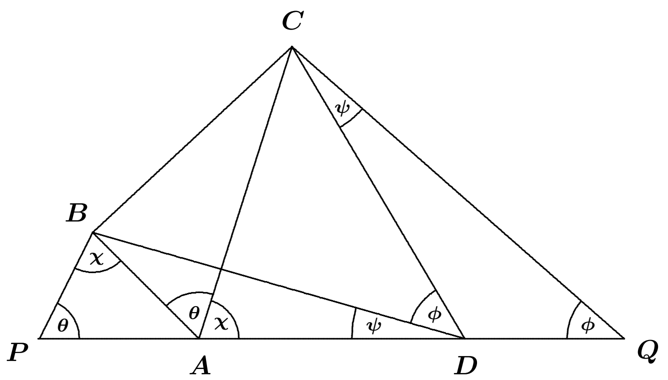
Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; BOGDAN IONIŢĂ and TITU ZVONARU, Bucharest, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; D.J. SMEENK, Zaltbommel, the Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2716. [2002 : 111] *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that

1. convex quadrilateral $ABCD$ is given,
2. P is a point of AD produced beyond A such that $\angle APB = \angle BAC$,
3. Q is a point on AD produced beyond D such that $\angle DQC = \angle BDC$, and
4. $AP = DQ$.

Characterize quadrilateral $ABCD$.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK; and by Peter Y. Woo, Biola University, La Mirada, CA, USA. The solutions by D.J. Smeenk, Zaltbommel, the Netherlands and by Nikolaos Dergiades, Thessaloniki, Greece were very similar.



Angles marked χ in the diagram are equal, since each is equal to $180^\circ - \theta - \angle PAB$. Angles marked ψ in the diagram are equal, since each is equal to $180^\circ - \phi - \angle QDC$.

From the Sine Rule for triangles APB and DQC, we have

$$\frac{AP}{\sin\chi} = \frac{AB}{\sin\theta} \qquad \text{and} \qquad \frac{DQ}{\sin\psi} = \frac{CD}{\sin\phi}\,.$$

From the Sine Rule for triangles BAD and CAD, we have

$$\frac{AB}{\sin\psi} = \frac{AD}{\sin(\psi+\chi+\theta)} \qquad \text{and} \qquad \frac{CD}{\sin\chi} = \frac{AD}{\sin(\psi+\chi+\phi)}\,.$$

But $AP = DQ$, and hence,

$$\frac{AB}{CD} = \frac{\sin\psi\sin\theta}{\sin\chi\sin\phi} = \frac{\sin\psi\sin(\psi+\chi+\phi)}{\sin\chi\sin(\psi+\chi+\theta)}\,.$$

Hence, $\sin\theta\sin(\psi+\chi+\theta) = \sin\phi\sin(\psi+\chi+\phi)$. Therefore,

$$\begin{aligned}\cos(\psi+\chi)-\cos(2\theta+\psi+\chi) &= \cos(\psi+\chi)-\cos(2\phi+\psi+\chi)\,,\\ \cos(2\theta+\psi+\chi) &= \cos(2\phi+\psi+\chi)\,.\end{aligned}$$

Thus, $\theta=\phi$ or $\psi+\chi+\theta+\phi=180^\circ$.

Hence, $ABCD$ is a cyclic quadrilateral or $AB \parallel CD$.

Also solved by DAVID LOEFFLER, student, Trinity College, Cambridge, UK; GEORGE TSINTSIFAS, Thessaloniki, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Loeffler stated that he thought he had seen a partial converse somewhere—if $ABCD$ is a trapezoid, then $AP = DQ$—but could not recall the reference. He congratulated Professor Seimiya for discovering "this entertaining result".

2717. *Proposed by Mihály Bencze, Brasov, Romania.*

For any triangle ABC, prove that

$$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \;\le\; \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right)\,.$$

[*Editor's comments:* Seiffert pointed out that the given inequality was part of **CRUX** problem #585 [1980 : 284], for which a solution by Klamkin was published in [1981 : 303]. Interestingly, Klamkin was also a solver of the current problem, and his solution this time is different from the one in 1981. His 1981 solution used the Law of Sines and some elementary inequalities.]

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Since $\cos\frac{A}{2}$, $\cos\frac{B}{2}$, $\cos\frac{C}{2} > 0$, the given inequality is equivalent to

$$\sin A\sin B\sin C \;\le\; \prod_{\text{cyclic}}\cos\left(\frac{A-B}{2}\right)\cos\frac{C}{2}$$

or

$$\sin A\sin B\sin C \;\le\; \prod_{\text{cyclic}}\frac{1}{2}\left[\cos\left(\frac{A-B+C}{2}\right)+\cos\left(\frac{A-B-C}{2}\right)\right]\,.$$

But $\cos\left(\frac{A-B+C}{2}\right) = \cos\left(\frac{\pi-2B}{2}\right) = \sin B$, and

$$\cos\left(\frac{A-B-C}{2}\right) \;=\; \cos\left(\frac{B+C-A}{2}\right) \;=\; \cos\left(\frac{\pi-2A}{2}\right) \;=\; \sin A\,.$$

Thus, we have

$$\begin{aligned}\prod_{\text{cyclic}}\frac{1}{2}\left[\cos\left(\frac{A-B+C}{2}\right)+\cos\left(\frac{A-B-C}{2}\right)\right] &= \prod_{\text{cyclic}}\frac{(\sin B+\sin A)}{2}\\ &\ge \prod_{\text{cyclic}}\sqrt{\sin B\sin A}\\ &= \sin A\sin B\sin C\,,\end{aligned}$$

which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SCOTT H. BROWN, Auburn University at Montgomery, Montgomery, AL, USA; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Terrassa, Spain and JUAN JOSÉ EGOZCUE, Applied Mathematics III, UPC, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; TOSHIO SEIMIYA, Kawasaki, Japan; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Murty remarked that the given inequality implies the well known Euler inequality which states that if r and R denote the inradius and circumradius of $\triangle ABC$, then $2r \leq R$. The implication is a consequence of the known formula $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{2r}{R}$ and the fact that $\prod_{\text{cyclic}} \cos\left(\frac{A-B}{2}\right) \leq 1$.

2719. [2002 : 112] *Proposed by Antal E. Fekete, Memorial University, St. John's, NF.*

Let $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Show that

$$\sum_{j=0}^{\infty} \frac{(k+j)^n}{j!} \;=\; (-1)^n \sum_{j=0}^{\infty} \frac{(-k-2+j)^n}{j!}$$

and

$$\sum_{j=0}^{\infty} (-1)^j \frac{(k+j)^n}{j!} \;=\; (-1)^n \sum_{j=0}^{\infty} (-1)^j \frac{(-k+2+j)^n}{j!}$$

for $n = 1$ and $n = 2$.

Are these equalities true or false for other positive integral values of n?

[*Editor's note:* There was a typographical error in the original statement of the problem. The "-2" in the second equation should be a "$+2$". The problem, as received from the proposer, was correct; the editor accepts responsibility for the error. The corrected problem is displayed above.]

Solution by Southwest Missouri State University Problem Solving Group, Southwest Missouri State University, Springfield MO, USA.

We will show, more generally, that for any real number c

$$\sum_{j=0}^{\infty} c^j \frac{(k+j)^n}{j!} \;=\; (-1)^n \sum_{j=0}^{\infty} c^j \frac{(-k-2c+j)^n}{j!}$$

for both $n = 1$ and $n = 2$, but for no other positive integers n (unless $c = 0$).

Let $F(n,k) = \sum_{j=0}^{\infty} c^j \frac{(k+j)^n}{j!}$. It is straightforward to verify that $F(1,k) = e^c(k+c)$ and $F(2,k) = e^c(k^2+2ck+c^2+c)$ (compare to Problem 2720). Hence,

$$F(1,k) = -F(1,-k-2c) \qquad \text{and} \qquad F(2,k) = F(2,-k-2c)\,,$$

and the result follows.

To show that the result does not hold for $n > 2$, consider $F(n,k)$ and $(-1)^n F(n,-k-2c)$ as polynomials in k. The coefficient of k^{n-3} in $F(n,k)$ is

$$\binom{n}{3}\sum_{j=0}^{\infty}\frac{c^j j^3}{j!} = \binom{n}{3}e^c(c^3+3c^2+c)\,,$$

but the coefficient of k^{n-3} in $(-1)^n F(n,-k-2c)$ is

$$\binom{n}{3}\sum_{j=0}^{\infty}\frac{c^j(2c-j)^3}{j!} = \binom{n}{3}e^c(c^3+3c^2-c)\,,$$

and these are not equal for $c \neq 0$.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; PIERRE BORNSZTEIN, Pontoise, France; KEITH EKBLAW, Walla Walla, WA, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and the proposer.

Janous points out that a recent paper [Lewis, Barry, Partitioning a set, *Mathematical Gazette,* **86**, *No. 505 (March 2002) 51–58] studies series of the form above, expressing them using the Bell and Stirling numbers. In fact, one submitted solution used the Bell and Stirling number form of the series.*

One solution noted that the result holds for any $k \in \mathbb{C}$. The editor notes that the proof of non-equality holds for any real $n \neq 0, 1, 2$ by the general Binomial Theorem.

2720. [2002 : 113] *Proposed by Antal E. Fekete, Memorial University, St. John's, Newfoundland.*

Let k be an integer and n be a non-negative integer.

(a) Show that $\sum_{j=0}^{\infty}\frac{(k+j)^n}{j!}$ is an integral multiple of e, and find the sum.

(b) Show that $\sum_{j=0}^{\infty}(-1)^j\frac{(k+j)^n}{j!}$ is an integral multiple of $\frac{1}{e}$, and find the sum.

Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina.

For any non-negative integers i and j,

$$j^i \;=\; \sum_{t=0}^{i} \left\{ {i \atop t} \right\} j(j-1)\dots(j-t+1)\,,$$

where $\left\{ {i \atop t} \right\}$ is a Stirling number of the second kind. These Stirling numbers are positive integers. They count the number of ways to partition a set of i elements into t nonempty subsets. (See [1].)

By taking into account that the series involved are absolutely convergent, we get

$$\begin{aligned}
\sum_{j=0}^{\infty} (\pm 1)^j \frac{(k+j)^n}{j!} &= \sum_{i=0}^{n} \binom{n}{i} k^{n-i} \sum_{j=0}^{\infty} (\pm 1)^j \frac{j^i}{j!} \\
&= \sum_{i=0}^{n} \binom{n}{i} k^{n-i} \sum_{t=0}^{i} \left\{ {i \atop t} \right\} \left(\sum_{j=t}^{\infty} (\pm 1)^j \frac{1}{(j-t)!} \right) \\
&= \sum_{i=0}^{n} \binom{n}{i} k^{n-i} \sum_{t=0}^{i} (\pm 1)^t \left\{ {i \atop t} \right\} \left(\sum_{j=0}^{\infty} \frac{(\pm 1)^j}{j!} \right) \\
&= \left(\sum_{i=0}^{n} \binom{n}{i} \sum_{t=0}^{i} (\pm 1)^t \left\{ {i \atop t} \right\} k^{n-i} \right) e^{\pm 1}\,.
\end{aligned}$$

Note that the expression multiplying $e^{\pm 1}$ is an integer.

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; KEITH EKBLAW, Walla Walla, WA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA (part (a) only); *SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP; and the proposer. There was one incorrect solution.*

Most solvers presented a solution similar to our featured solution. Many added that the integers $\sum_{t=0}^{i} \left\{ {i \atop t} \right\}$ *are called* Bell numbers; *see the web page*

`http://mathworld.wolfram.com/BellNumber.html.`

Bataille and the proposer referred to [2], *where the integers that multiply* e *are called* extended Bell numbers, *while the multipliers of* e^{-1} *are called* extended Rényi numbers. *The SWMSU Problem Solving Group showed (with almost no extra effort) that more generally,*

$$\sum_{j=0}^{\infty} c^j \frac{(k+j)^n}{j!} \;=\; b_n(c,k)e^c,$$

where the $b_n(c,k)$ *are polynomials in* c *and* k *with integer coefficients.*

Both Klamkin and Loeffler made use of the connection between this problem and CRUX 2719; in fact, Klamkin solved 2719 using the formulas he obtained for 2720.

References

[1] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.

[2] Antal E. Fekete, Apropos Bell and Stirling Numbers, *Crux with Mayhem*, 25:5 (September 1999), 274-281.

2721. [2002 : 113] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Consider the cubic equation $x^3 - 19x + 30 = 0$. It is easily verified that the roots of this equation are -5, 2, and 3. If one tries to solve the above equation using trigonometry, the roots come out as

$$-2\rho^{\frac{1}{3}}\cos\left(\frac{\theta}{3}\right), \quad -2\rho^{\frac{1}{3}}\cos\left(\frac{2\pi+\theta}{3}\right), \quad \text{and} \quad -2\rho^{\frac{1}{3}}\cos\left(\frac{4\pi+\theta}{3}\right),$$

where $\rho^{\frac{1}{3}} = \sqrt{\frac{19}{3}}$ and $\cos\theta = 15\left(\frac{3}{19}\right)^{\frac{3}{2}}$.

Show, without the use of a calculator, that

$$-2\rho^{\frac{1}{3}}\cos\left(\frac{\theta}{3}\right) = -5\,, \; -2\rho^{\frac{1}{3}}\cos\left(\frac{2\pi+\theta}{3}\right) = 2\,, \text{ and } -2\rho^{\frac{1}{3}}\cos\left(\frac{4\pi+\theta}{3}\right) = 3\,.$$

[*Editor's Note:* We are told that the roots of the given equation are -5, 2 and 3, and that they are also $-2\rho^{1/3}\cos(\theta/3)$, $-2\rho^{1/3}\cos[(\theta+2\pi)/3]$ and $-2\rho^{1/3}\cos[(\theta+4\pi)/3]$. The roots given in terms of θ must match up somehow with the roots given as integers, but what is the exact correspondence? Our featured solution sorts out the possibilities.]

Solution by Michel Bataille, Rouen, France.

For any given θ satisfying

$$\cos\theta \;=\; 15\left(\frac{3}{19}\right)^{3/2}, \tag{1}$$

let a, b, and c be defined as follows:

$$\begin{aligned}
a &= -2\rho^{1/3}\cos\left(\frac{\theta}{3}\right), \\
b &= -2\rho^{1/3}\cos\left(\frac{\theta+2\pi}{3}\right) = \rho^{1/3}\left[\cos\left(\frac{\theta}{3}\right) + \sqrt{3}\sin\left(\frac{\theta}{3}\right)\right], \\
c &= -2\rho^{1/3}\cos\left(\frac{\theta+4\pi}{3}\right) = \rho^{1/3}\left[\cos\left(\frac{\theta}{3}\right) - \sqrt{3}\sin\left(\frac{\theta}{3}\right)\right].
\end{aligned}$$

First consider $\theta = \theta_0$, the solution of (1) in the interval $(0, \pi/2)$. Then $\cos(\theta/3) > 0$, and therefore $a = -5$. Furthermore,

$$\begin{aligned}
b - c \;=\; 2\sqrt{3}\rho^{1/3}\sin(\theta/3) &= 2\sqrt{3}\sqrt{\rho^{2/3}[1-\cos^2(\theta/3)]} \\
&= 2\sqrt{3}\sqrt{(19/3)-(5/2)^2} \;=\; 1\,,
\end{aligned}$$

and thus $b = 3$ and $c = 2$. All the results in the table below now follow immediately. Note that the formulas stated in the problem are valid for $\theta = -\theta_0$. [The values of a, b, and c are obviously invariant under a transformation $\theta \to \theta + 6\pi$.]

θ	a	b	c
θ_0	-5	3	2
$\theta_0 + 2\pi$	3	2	-5
$\theta_0 + 4\pi$	2	-5	3
$-\theta_0$	-5	2	3
$-\theta_0 + 2\pi$	2	3	-5
$-\theta_0 + 4\pi$	3	-5	2

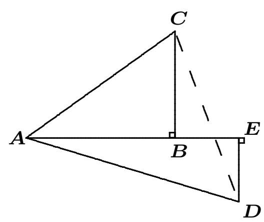
Also solved by BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; and CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK. Six solutions were incorrect or incomplete.

Most solvers interpreted the problem as asking for more than just a matching of roots. Indeed, the intent of the problem seems to be to find a direct and simple demonstration of the proposed formulas by some alternative approach that does not simply solve the given equation. Among the solutions submitted, the most common approach was to apply the trigonometric identity $\cos(3x) = 4\cos^3 x - 3\cos x$ *with* $x = \theta/3$ *to show that the equation* $4u^3 - 3u = \cos\theta$ *has the roots* $\cos(\theta/3)$, $\cos[(\theta+2\pi)/3]$, *and* $\cos[(\theta+4\pi)/3]$. *This implies that the equation in the problem has the roots* $-2\rho^{1/3}\cos(\theta/3)$, $-2\rho^{1/3}\cos[(\theta+2\pi)/3]$, *and* $-2\rho^{1/3}\cos[(\theta+4\pi)/3]$. *(The two equations are related by the change of variable* $x = -2\rho^{1/3}u$, *and their roots are related accordingly.)*

This problem raises a large issue. Is there a way to recognize when the trigonometric form for a root simplifies to a rational number? For example, if we try to find the cube roots of the complex number $18 + 26i$ *by the standard method, using the polar form of the number, we obtain ugly trigonometric expressions. One of the roots is* $3 + i$, *but this is not obvious. Perhaps some reader can suggest a reference that sheds light on such matters.*

2722. [2002 : 114] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Consider two Pythagorean triangles as indicated in the figure. The lengths of AC, CB, AD and DE are odd integers and the lengths of the overlapping sides, AE and AB, are even integers.



Does there exist such a configuration of Pythagorean triangles such that the length of CD is an integer?

Solution by Titu Zvonaru, Bucharest, Romania.

The answer is yes. Let CD meet AE at S. Then, from similar triangles, we have $SB/SE = BC/DE$, which implies

$$\frac{SB}{BE} = \frac{SB}{SB+SE} = \frac{BC}{BC+DE}.$$

Hence,

$$SB = \frac{BC \cdot BE}{BC + DE} = \frac{BC \cdot (AE - AB)}{BC + DE}. \qquad (1)$$

Similarly, from $SE/SB = DE/BC$ we get

$$\frac{SE}{BE} = \frac{SE}{SB + SE} = \frac{DE}{BC + DE}.$$

Hence,

$$SE = \frac{DE \cdot BE}{BC + DE} = \frac{DE \cdot (AE - AB)}{BC + DE}. \qquad (2)$$

From (1) and (2) we then have

$$\begin{aligned} CS &= \sqrt{(BC)^2 + \frac{(BC)^2 \cdot (AE - AB)^2}{(BC + DE)^2}} \\ &= \frac{BC}{BC + DE}\sqrt{(BC + DE)^2 + (AE - AB)^2} \end{aligned}$$

and

$$\begin{aligned} DS &= \sqrt{(DE)^2 + \frac{(DE)^2(AE - AB)^2}{(BC + DE)^2}} \\ &= \frac{DE}{BC + DE}\sqrt{(BC + DE)^2 + (AE - AB)^2}. \end{aligned}$$

Note that $8^2 + 15^2 = 17^2$ and $40^2 + 9^2 = 41^2$. If we take $AB = 8$, $BC = 15$, $AC = 17$, $AE = 40$, $DE = 9$, and $AD = 41$, then $BC + DE = 24$ and $AE - AB = 32$. Hence, $(BC + DE)^2 + (AE - AB)^2 = 24^2 + 32^2 = 40^2$. It follows that $CS = \frac{15}{24}(40) = 25$, $DS = \frac{9}{24}(40) = 15$, and $CD = 40$.

Also solved by CHARLES ASHBACHER, Hiawatha, IA, USA (2 solutions); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; K.R.S. SASTRY, Bangalore, India (2 solutions); HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA (2 solutions); PANOS E. TSAOUSSOGLOU, Athens, Greece; TITU ZVONARU, Bucharest, Romania (a second solution).

The configuration in the featured solution was also obtained by Ashbacher and SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP. There were nine other configurations obtained. Among them, the one with the smallest value of CD (equal to 20) was given by Smeenk. These configurations can be represented by using the 7-tuple $(AC, BC, AD, DE, AB, AE, CD)$ *as follows:* $(15, 9, 25, 7, 12, 24, 20)$ *(Smeenk);* $(25, 15, 39, 15, 20, 36, 34)$ *(Southwest Missouri State University Problem Solving Group);* $(13, 5, 45, 27, 12, 36, 40)$ *(Seiffert);* $(5, 3, 51, 45, 4, 24, 52)$ *(Ashbacher);* $(25, 15, 65, 33, 20, 56, 60)$ *(Sastry);* $(65, 63, 65, 33, 16, 56, 104)$ *(Diminnie and Zvonaru);* $(325, 125, 595, 91, 300, 588, 360)$ *(Sastry);* $(375, 351, 765, 117, 132, 756, 780)$ *(Bradley); and* $(1305, 153, 5353, 5015, 1296, 1872, 5200)$ *(Tsaoussoglou). Both Diminnie and Sastry showed that there are infinitely many solutions.*

2723. [2002 : 114] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $1 \leq k \leq N$, let $n_1, n_2, \ldots, n_k$ be non-negative integers such that $n_1 + n_2 + \cdots + n_k = N$. Determine the minimum value of the sum $\sum\limits_{j=1}^{k} \binom{n_j}{m}$ when (a) $m = 2$; (b)* $m \geq 3$.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

(a) If k divides N, the answer is obvious: Noting that the function $f(x) = \binom{x}{2} = \frac{1}{2}x(x-1)$ is convex, we have

$$\sum_{j=1}^{k} f(n_j) \;\geq\; kf\left(\frac{1}{k}\sum_{j=1}^{k} n_j\right) \;=\; kf\left(\frac{N}{k}\right) \;=\; \frac{1}{2}N\left(\frac{N}{k}-1\right).$$

If k does not divides N, set $N = ak + b$, where $0 \leq b < k$. Now, let

$$r_j \;=\; \begin{cases} a+1 & \text{for } j \leq b, \\ a & \text{for } j > b. \end{cases}$$

We claim that

$$\sum_{j=1}^{k} f(r_j) \;\leq\; \sum_{j=1}^{k} f(n_j)$$

for any n_j such that $\sum\limits_{j=1}^{n} n_j = N$.

Proof: This may be proved using Karamata's Majorization Inequality, which states that given two sequences, $\{c_i\}_{i=1}^n$, $\{d_i\}_{i=1}^n$, arranged (without loss of generality) in non-increasing order, then a necessary and sufficient condition that $\sum\limits_{i=1}^{n} f(c_i) \geq \sum\limits_{i=1}^{n} f(d_i)$ for all convex functions f is that

$$\sum_{i=1}^{n} c_i \;=\; \sum_{i=1}^{n} d_i\,,$$

and for any integer t, such that $1 \leq t < n$,

$$\sum_{i=1}^{t} c_i \;\geq\; \sum_{i=1}^{t} d_i\,.$$

We need only show that $\{r_j\}_{j=1}^n$ is majorized by an arbitrary sequence of integers $\{n_j\}_{j=1}^n$ having sum N.

Suppose instead that there is some t ($1 \leq t < n$) such that

$$\sum_{j=1}^{t} n_j \;<\; \sum_{j=1}^{t} r_j\,,$$

and consider the least such t. Then

$$n_1 + \cdots + n_t \;<\; r_1 + \cdots + r_t \,,$$

but

$$n_1 + \cdots + n_{t-1} \;\geq\; r_1 + \cdots + t_{t-1}$$

(considering both sides of the last inequality to be 0 for $t = 1$). Thus, we have $n_t < r_t \leq a + 1$. Hence, $n_i \leq a \leq r_i$ for $i \geq t$, which implies

$$N \;=\; \sum_{j=1}^{k} n_j \;\leq\; \sum_{j=1}^{t} n_j + \sum_{j=t+1}^{k} r_j \;<\; \sum_{j=1}^{k} r_j \;=\; N \,,$$

a contradiction.

It follows that the n's majorize the r's, and hence that

$$\sum_{j=1}^{n} \binom{n_j}{2} \;\geq\; b\binom{a+1}{2} + (k-b)\binom{a}{2} \,.$$

(b) We may extend this analysis to $m \geq 3$ using a simple trick. Let

$$f_m(x) \;=\; \begin{cases} \frac{1}{m!}x(x-1)\cdots(x-m+1) & \text{for } x > m-1, \\ 0 & \text{for } 0 \leq x \leq m-1\,. \end{cases}$$

Then for all integers n, we have $f_m(n) = \binom{n}{m}$. We claim that f_m is convex on $[0, \infty)$.

Proof: Clearly, the polynomial $g(x) = x(x-1)\cdots(x-m+1)$ is of degree m and has m simple zeroes at the integers 0, 1, ..., $m-1$. Hence, its derivative must have $m-1$ zeroes, by Rolle's Theorem, one in each of the intervals $(r, r+1)$ for $0 \leq r \leq m-2$.

Each of these zeroes must also be simple, because otherwise $g'(x)$ would have too many zeroes for its degree. We may repeat the process to see that $g''(x)$ must have $m-2$ simple zeroes, each in an interval between consecutive zeroes of $g'(x)$.

Hence, in the interval $[m-1, \infty)$ there are no zeroes of $g''(x)$. Since $g(x)$ has positive leading coefficient, so does $g''(x)$. Thus, $g''(x)$ tends to $+\infty$ for large x; but as it is continuous, it must be positive on this interval. Therefore, $g(x)$, and thus $f_m(x)$, is convex on the interval.

However, since $g'(x) \neq 0$ on this interval and $g'(x)$ is continuous and tends to ∞ for large x, we see that $g'(m-1)$ is positive. Since $f_m(x)$ is zero on the interval $[0, m-1]$, it follows that $f_m(x)$ is convex everywhere.

By the Majorization Inequality, we have

$$\sum_{j=1}^{n} \binom{n_j}{m} \;\geq\; b\binom{a+1}{m} + (k-b)\binom{a}{m}$$

for all such k-tuples n_j.

Comment: It is obvious that the k-tuple $[N, 0, \ldots, 0]$ majorizes any other k-tuple of non-negative integers with sum N, so the maximum value is clearly $\binom{N}{m}$.

Also solved by PIERRE BORNSZTEIN, Pontoise, France (part (a) only); NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; VEDULA N. MURTY, Dover, PA, USA (part (a) only); and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

Guersenzvaig notes that in the cases $m = 0$ and $m = 1$ the minimum sums are k and N, respectively. Moreover, he states that it can be proved, via translations, that for any positive integer k, and any non-negative integers m, N, c, and $d \leq N/k$, the minimum value, M, of the sum

$$\sum_{j=1}^{k} \binom{c+n_j}{m},$$

where the n_j's are integers satisfying $n_j \geq d$ and $n_1 + \cdots + n_k = N$, is given by

$$M = \sum_{j=1}^{k} \binom{c+n_j^*}{m} = r\binom{c+d+q}{m-1} + k\binom{c+d+q}{m},$$

where q and r denote the quotient and remainder, respectively, on division of $N - kd$ by k, and

$$(n_1^*, \ldots, n_k^*) = (\underbrace{q+1, \ldots, q+1}_{r}, \underbrace{q, \ldots, q}_{k-r}).$$