THE ACADEMY CORNER

No. 48

Bruce Shawyer

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Here are the problems from the 9$^{\text{th}}$ International Mathematics Competition for University Students, held at Warsaw University in July 2002. Thanks to Moubinool Omarjee for sending them to us.

9$^{\text{th}}$ International Mathematics Competition for University Students

First Day

Problem 1. A standard parabola is the graph of a quadratic polynomial $y = x^2+ax+b$ with leading coefficient 1. Three standard parabolas with vertices V_1, V_2, V_3, intersect pairwise at points A_1, A_2, A_3, respectively. Let $A \mapsto s(A)$ be the reflection of the plane with respect to the x–axis.
Prove that the standard parabolas with vertices $s(A_1)$, $s(A_2)$, $s(A_3)$, intersect pairwise at the points $s(V_1)$, $s(V_2)$, $s(V_3)$, respectively.

Problem 2. Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that, for every $x \in \mathbb{R}$, we have $f(x) > 0$ and $f'(x) = f(f(x))$?

Problem 3. Let n be a positive integer and let

$$a_k \;=\; \frac{1}{\binom{n}{k}}\,, \qquad b_k \;=\; 2^{k-n}\,, \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Show that

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \cdots + \frac{a_n - b_n}{n} \;=\; 0\,.$$

Problem 4. Let $f : [a,b] \to [a,b]$ be a continuous function and let $p \in [a,b]$. Define $p_0 = p$ and $p_{n+1} = f(p_n)$ for $n = 0, 1, 2, \ldots$. Suppose that the set $T_p = \{p_n : n = 0, 1, 2, \ldots\}$ is closed; that is, if $x \notin T_p$, then there is a $\delta > 0$ such that, for all $x' \in T_p$, we have $|x - x'| \geq \delta$.
Show that T_p has finitely many elements.

Problem 5. Prove of disprove the following statements:
(a) There exists a monotone function $f : [0,1] \to [0,1]$ such that, for each $y \in [0,1]$, the equation $f(x) = y$ has uncountably many solutions x.
(b) There exists a continuously differentiable function $f : [0,1] \to [0,1]$ such that, for each $y \in [0,1]$, the equation $f(x) = y$ has uncountably many solutions x.

Problem 6. For an $n \times n$ matrix M with real entries, let $||M|| = \sup\limits_{x \in \mathbb{R}^n \setminus \{0\}} \dfrac{||Mx||_2}{||x||_2}$, where $||\cdot||_2$ denotes the Euclidean norm on $\mathbb{R}^n$. Assume that an $n \times n$ matrix A with real entries satisfies $||A^k - A^{k-1}|| \leq \dfrac{1}{2002n}$ for all positive integers k.
Prove that $||A^k|| \leq 2002$ for all positive integers k.

Second Day

Problem 1.Compute the determinant of the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} (-1)^{|i-j|}, & \text{if } i \neq j, \\ 2, & \text{if } i = j. \end{cases}$$

Problem 2. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was solved correctly by at least one hundred and twenty participants.
Prove that there must be two participants such that every problem was solved by at least one of these two students.

Problem 3. For each $n \geq 1$, let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \qquad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that $a_n \cdot b_n$ is an integer.

Problem 4. In the tetrahedron $OABC$, let $\angle BOC = \alpha$, $\angle COA = \beta$ and $\angle AOB = \gamma$. let σ be the angle between the faces OAB and OAC, and let τ be the angle between the faces OBA and OBC.
Prove that $\gamma > \beta \cdot \cos(\sigma) + \alpha \cdot \cos(\tau)$.

Problem 5. Let A be an $n \times n$ matrix with complex entries and suppose that $n > 1$. Prove that

$$A\overline{A} = I_n \iff \exists\, S \in GL_n(\mathbb{C}) \quad \text{such that} \quad A = S\,\overline{S}^{-1}.$$

(If $A = [a_{ij}]$, then $\overline{A} = [\overline{a_{ij}}]$, where $\overline{a_{ij}}$ is the complex conjugate of a_{ij}. Also, $GL_n(\mathbb{C})$ denotes the set of all $n \times n$ invertible matrices with complex entries, and I_n denotes the identity matrix.)

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a convex function whose gradient vector $\nabla f = \left(\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\right)$ exists at every point of $\mathbb{R}^n$ and satisfies the condition

$$\exists\, L > 0 \;:\; \forall\, x_1, x_2 \in \mathbb{R}^n \quad \|\nabla f(x_1) - \nabla f(x_2)\| \leq L\|x_1 - x_2\|.$$

Prove that

$$\forall\, x_1, x_2 \in \mathbb{R}^n \quad \|\nabla f(x_1) - \nabla f(x_2)\|^2 \leq L\langle \nabla f(x_1) - \nabla f(x_2), x_1 - x_2\rangle.$$

In this formula, $\langle a, b\rangle$ denotes the scalar product of the vector a and b.
Also, $\|(v_1, v_2, \dots, v_n)\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.

THE OLYMPIAD CORNER

No. 224

R.E. Woodrow

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We begin this number with selected problems of the Ukrainian Mathematical Olympiad, April 1999. My thanks go to Ed Barbeau, Canadian team Leader to the IMO in Romania for collecting them.

UKRAINIAN MATHEMATICAL OLYMPIAD
Selected Problems
April 1999

1. (8th grade) Does there exist a 2000–digit integer which is the square of an integer and has at least 1999 5's in its decimal representation?

2. (8th grade). Let us consider the "sunflower" figure (see figure 1). The cells A, B, C in the "sunflower" are marked. The marker is situated in cell A. Each move of the marker may be one of the moves demonstrated in figure 2. In how many different ways can the marker move from A to B if the marker cannot visit C?

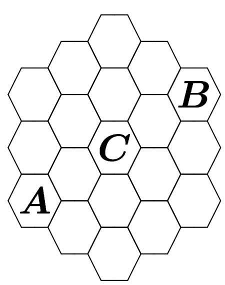


Figure 1

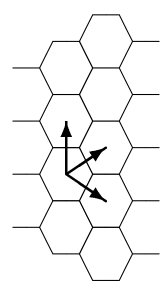


Figure 2

3. (9th grade) Prove that number $9999999 + 1999000$ is composite.

4. (9th grade) The sequence of positive integers a_1, a_2, $\dots$, a_n, $\dots$ is such that $a_{a_n} + a_n = 2n$ for all $n \geq 1$. Prove that $a_n = n$ for all n.

5. (10th grade) Let $P(x)$ be a polynomial with integer coefficients. The sequence of integers $x_1, x_2, \dots, x_n, \dots$ satisfies the conditions $x_1 = x_{2000} = 1999$, $x_{n+1} = P(x_n)$, $n \geq 1$. Find the value

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{1999}}{a_{2000}}\,.$$

6. (10th grade) For real numbers $x_1, x_2, \ldots, x_6 \in [0,1]$ prove the inequality

$$\frac{x_1^3}{x_2^5+x_3^5+x_4^5+x_5^5+x_6^5+5} + \frac{x_2^3}{x_1^5+x_3^5+x_4^5+x_5^5+x_6^5+5} + \cdots + \frac{x_6^3}{x_1^5+x_2^5+x_3^5+x_4^5+x_5^5+5} \le \frac{3}{5}.$$

7. (11th grade) Two players in turn write integers in a row in the following way: the first player writes an arbitrary a_1, the second one — an arbitrary a_2, each following number equals the sum of two previous numbers. The game is finished when the obtained sequence $a_1, a_2, a_3, \ldots,$ first has $i \neq j$ such that $a_i - a_j$ and $a_{i+1} - a_{j+1}$ are multiples of 1999. The player who does the last move is the winner. Which player has a winning strategy?

8. (11th grade) Let AA_1, BB_1, CC_1 be the altitudes of acute triangle ABC, let O be an arbitrary point inside the triangle $A_1B_1C_1$. Let us denote by M and N the bases of perpendiculars drawn from O to lines AA_1 and BC, respectively, by P and Q — ones from O to lines BB_1 and CA, respectively, by R and S — ones from O to lines CC_1 and AB, respectively. Prove that the lines MN, PQ, RS are concurrent.

Next we give selected problems from the 10^{th} and 11^{th}-12^{th} Forms of the Republic of Moldova Mathematical Olympiads, April, 1999. Thanks again go to Ed Barbeau, Canadian Team Leader to the IMO in Romania for collecting them.

THE XLIII MATHEMATICAL OLYMPIAD OF THE REPUBLIC OF MOLDOVA
10^{th} Form – Days 1 and 2
April 1999

1. Let the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2ax - a^2 - \frac{3}{4}$, be considered. Find the values a for which the inequality $|f(x)| \le 1$ is true for every $x \in [0,1]$.

2. Let n be a natural number such that the number $2n^2$ has 28 distinct divisors and the number $3n^2$ has 30 distinct divisors. How many distinct divisors has the number $6n^2$.
[Ed: we suspect that this problem has no solution.]

3. All the natural numbers from 1 to 100 are arranged arbitrarily along a circle. The sum of every three consecutively arranged numbers is calculated. Prove that there exist two such sums, with the difference between them being greater than 2.

4. Let the triangle ABC with height CD be given. It is known that $AB = 1999$, $BC = 1998$ and $AC = 2000$. The circles, inscribed in the triangles ACD and BCD, are tangent to the side CD at the points M and N, respectively. Find the length of MN.

5. Find all the functions $f : \mathbb{R} \to \mathbb{R}$, which satisfy the relation

$$x \cdot f(x) \;=\; \lfloor x \rfloor \cdot f(\{x\}) + \{x\} \cdot f(\lfloor x \rfloor), \quad \forall\, x \in \mathbb{R}\,,$$

where $\lfloor \cdot \rfloor$ and $\{\cdot\}$ denote the integral part and fractional part functions, respectively.

6. Find a polynomial of degree 3 with real coefficients such that each of its roots is equal to the square of one root of the polynomial $P(X) = X^3 + 9X^2 + 9X + 9$.

7. Prove that for all strictly positive numbers a, b, and c the inequality

$$(a+b+x)^{-1} + (b+c+x)^{-1} + (c+a+x)^{-1} \;\leq\; x^{-1}\,,$$

holds, where $x = \sqrt[3]{abc}$.

8. On the sides BC and AB of the equilateral triangle ABC the points D and E, respectively, are taken such that $CD : DB = BE : EA = (\sqrt{5} + 1)/2$. The straight lines AD and CE intersect in the point O. The points M and N are interior points of the segments OD and OC, respectively, such that $MN \| BC$ and $AN = 2OM$. The parallel to the straight line AC, drawn through the point O, intersects the segment MC in the point P. Prove that the half-line AP is the bisectrix of the angle MAN.

11^{th}-12^{th} Form – Days 1 and 2

1. Grandfather distributes n sweets among n grandchildren arranged along a circle: first of all he gives one sweet to some grandchild, then he gives one sweet to the next grandchild, then one sweet skipping one grandchild, then one sweet skipping two grandchildren and so on. The distribution is executed in the same direction. For what values of n does every grandchild get a sweet?

2. Let the number $n \in \mathbb{N}^*$ be given. Denote by M the set of all real numbers x for which there exists a finite sequence (a_p), $p = 1, \ldots, n$, with $a_p \in \{0, 1\}$, $p = 1, \ldots, n$, such that

$$x \;=\; 2^{-1} \cdot a_1 + 2^{-2} \cdot a_2 + \cdots + 2^{-n} \cdot a_n\,.$$

(a) Determine the set M and prove that for every number $x \in M$ there exists a unique finite sequence (a_p), $p = 1, \ldots, n$, with the mentioned property.

(b) Find the function $f : M \to \mathbb{R}$ such that if (a_p) is the sequence defined above by the number x, then

$$f(x) \;=\; 2^{-1} \cdot 2000^{a_1} + 2^{-2} \cdot 2000^{a_2} + \cdots + 2^{-n} \cdot 2000^{a_n}\,, \quad \forall x \in M\,.$$

3. Find the smallest value of $E(x,y) = \sqrt{3+x} + \sqrt{3-xy}$ for the values of x and y, which satisfy the relation $x^2 + y^2 = 9$.

4. Prove that if all faces of a tetrahedron are congruent triangles, then these triangles are acute-angled.

5. Find all the integer values of m, for which the equation

$$\left\lfloor \frac{m^2x - 13}{1999} \right\rfloor = \frac{x-12}{2000}$$

has 1999 distinct real solutions ($\lfloor\cdot\rfloor$ denotes the integral part function).

6. A number, composed from n ($n \geq 2$) digits of 9, is written on the blackboard initially. In each minute one number on the blackboard is decomposed as a product of two natural factors, each factor is increased or decreased independently by 2 such that two strictly positive numbers are obtained; then these numbers are written on the blackboard and the chosen number is erased from the blackboard. For which values of n is it possible to obtain only numbers equal to 9 on the blackboard after a finite time?

7. Prove that the number $a = \dfrac{m^{n+1} + n^{n+1}}{m^m + n^n}$ satisfies the relation $a^m + a^n \geq m^m + n^n$ for non-zero natural numbers m and n.

8. On the sides BC, AC and AB of the equilateral triangle ABC the points M, N and P, respectively, are considered such that $AP : PB = BM : MC = CN : NA = \lambda$. Find all the values λ for which the circle with the diameter AC covers the triangle, bounded by the straight lines AM, BN and CP. (In the case of concurrent straight lines the mentioned triangle degenerates into a point.)

As a final contest this number we give the Team Selection Test written in Cortona, Italy, May 1999.

TEAM SELECTION CONTEST

Cortona, Italy

May 29, 1999 (Time: 4 hours)

1. Prove that for each prime number p the equation

$$2^p + 3^p = a^n$$

has no solutions (a,n), with a and n integers > 1.

2. Points D and E are given on the sides AB and AC of $\triangle ABC$ in such a way that DE is parallel to BC and tangent to the incircle of $\triangle ABC$. Prove that

$$DE \leq \tfrac{1}{8}(AB + BC + CA).$$

3. (a) Determine all the strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + (f(y)) \;=\; f(x) + y, \quad \forall x, y \in \mathbb{R}\,.$$

(b) Prove that for every integer $n > 1$ there do not exist strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) \;=\; f(x) + y^n\,, \quad \forall x, y \in \mathbb{R}\,.$$

4. Let X be a set with n elements, and let A_1, $\ldots$, A_m be subsets of X such that

(i) $|A_i| = 3$ for every $i = 1, \ldots, m$

(ii) $|A_i \cap A_j| \leq 1$ for every $i \neq j$ $(i, j \in \{1, \ldots, m\})$.

Prove that there exists a subset of X with at least $\lfloor\sqrt{2n}\rfloor$ elements, which does not contain A_i for $i = 1, \ldots, m$.

Now we look at readers' solutions to some of the problems of the 1997 Chinese Mathematical Olympiad given [2000 : 200–201].

1. Let x_1, x_2, $\ldots$, x_{1997} be real numbers satisfying the following two conditions:

(a) $-\frac{1}{\sqrt{3}} \leq x_i \leq \sqrt{3}$ $\quad (i = 1, 2, \ldots, 1997)$;

(b) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$.

Find the maximum of $x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$ and give your reason.

Solution by Mohammed Aassila, Strasbourg, France.

Since the function $f(x) := x^{12}$ is convex, the maximum is attained when all the terms, perhaps except one, are equal to $\frac{-1}{\sqrt{3}}$ or $\sqrt{3}$. Suppose that n of the x_i's are equal to $\frac{-1}{\sqrt{3}}$, $1996 - n$ are equal to $\sqrt{3}$ and the last term is equal to

$$-318\sqrt{3} + \frac{n}{\sqrt{3}} - \sqrt{3}(1996 - n)\,.$$

Since 1736 is the only integer which satisfies

$$\frac{-1}{\sqrt{3}} \;\leq\; -318\sqrt{3} + \frac{n}{\sqrt{3}} - \sqrt{3}(1996 - n) \;\leq\; \sqrt{3}\,,$$

we deduce that the maximum is

$$\frac{1736^{12} + 260^{12} \times 3^{12} + 4^6}{3^6}\,.$$

3. Show that there exist infinitely many positive integers n such that one can arrange 1, 2, $\ldots$, $3n$ into the following table

$$\begin{array}{l} a_1,\, a_2,\, \ldots,\, a_n \\ b_1,\, b_2,\, \ldots,\, b_n \\ c_1,\, c_2,\, \ldots,\, c_n \end{array}$$

which satisfies the following two conditions:

(1) $a_1 + b_1 + c_1 \;=\; a_2 + b_2 + c_2 = \cdots = a_n + b_n + c_n$ is a multiple of 6;

(2) $a_1 + \cdots + a_n = b_1 + \cdots + b_n = c_1 + \cdots + c_n$ is also a multiple of 6.

Solution by Mohammed Aassila, Strasbourg, France.

We have $1 + 2 + \cdots + 3n = \dfrac{3n(3n+1)}{2}$. The conditions (1) and (2) impose that $\dfrac{3n(3n+1)}{2}$ is a multiple of $6n$ and 9. Hence, n is a multiple of 3 and is $\equiv 1 \pmod 4$. In fact, all the 9^m $(m \geq 1)$ give an answer to the problem.

4. Quadrilateral $ABCD$ is inscribed in a circle. Line AB meets DC at point P. Line AD meets BC at point Q. Tangent lines QE and QF touch the circle at points E and F, respectively. Prove that points P, E and F are collinear.

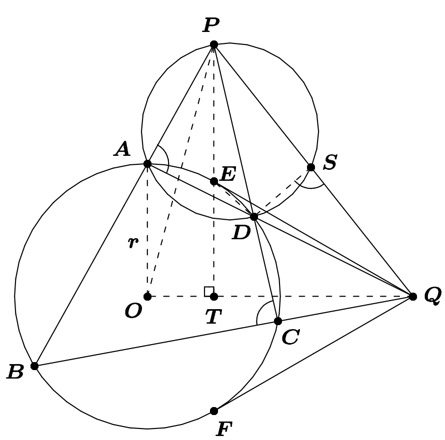
Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.

Let O and r be the centre and radius of the circle. Let S be the second intersection of PQ with the circumcircle of $\triangle PAD$. Then we have

$$\angle DSQ \;=\; \angle PAD \;=\; \angle DCB\,.$$

Thus, C, D, S, Q are concyclic. Hence, $PS \cdot PQ \;=\; PD \cdot PC \;=\; PO^2 - r^2$. Similarly, we have

$$QS{\cdot}QP \;=\; QD{\cdot}QA \;=\; QO^2 - r^2\,.$$



Therefore,

$$\begin{aligned} PQ^2 &= (PS + SQ) \cdot PQ \;=\; PS \cdot PQ + QS \cdot QP \\ &= (PO^2 - r^2) + (QO^2 - r^2) \\ &= PO^2 + QO^2 - 2r^2\,. \end{aligned}$$

Thus,

$$\begin{aligned} 2r^2 &= PO^2 + QO^2 - PQ^2 \\ &= 2PO \cdot QO \cos \angle POQ . \end{aligned}$$

so that we get

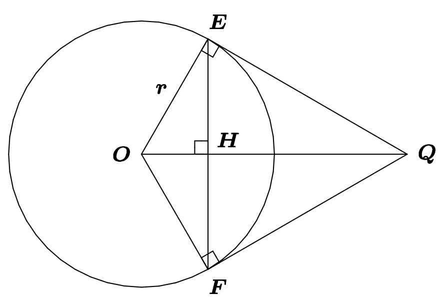
$$r^2 = PO \cdot QO \cos \angle POQ . \tag{1}$$

Let T be the foot of perpendicular from P to OQ, then

$$PO \cos \angle POQ = OT .$$

Thus, we obtain from (1)

$$r^2 = OT \cdot OQ . \tag{2}$$



Let H be the intersection of EF with OQ. Since QE and QF are tangent to the circle at E and F, we have

$$\angle OEQ = \angle OFQ = 90^\circ$$

and

$$EF \perp OQ .$$

Since $\angle OEQ = 90^\circ$ and $EH \perp OQ$, we get

$$OE^2 = OH \cdot OQ ;$$

that is,

$$r^2 = OH \cdot OQ . \tag{3}$$

It follows that from (2) and (3)

$$OT \cdot OQ = OH \cdot OQ .$$

Hence, $OT = OH$. Therefore, T coincides with H. Since $PT \perp OQ$ and $EH \perp OQ$, the line PT coincides with the line EF. This implies that P, E and F are collinear.

6. Let a_1, a_2, $\ldots$ be non-negative numbers which satisfy

$$a_{n+m} \leq a_n + a_m \quad (m, n \in \mathbb{N}) .$$

Prove that

$$a_n \leq m a_1 + \left(\frac{n}{m} - 1\right) a_m$$

for all $n \geq m$.

Solutions by Mohammed Aassila, Strasbourg, France; and by Michel Bataille, Rouen, France. We give Bataille's solution.

From the hypothesis, we immediately get $a_{kn} \leq ka_n$ for all k, $n \in \mathbb{N}$; in particular $a_k \leq ka_1$ for all k.

We will show the desired property by induction on the quotient q in the division of n by m. Assume first that $q = 1$. In other words, $n = m + r$ where $r \in \{0, 1, \ldots, m-1\}$. We have to show that

$$a_{m+r} \leq ma_1 + \frac{r}{m} a_m \, . \tag{1}$$

We have $a_{m+r} \leq a_r + a_m \leq ra_1 + a_m$, but also $a_{m+r} \leq (m+r)a_1$, so that

$$\begin{aligned} ma_{m+r} &= ra_{m+r} + (m-r)a_{m+r} \leq r^2 a_1 + ra_m + (m-r)(m+r)a_1 \\ &= m^2 a_1 + ra_m \, . \end{aligned}$$

Thus, (1) follows immediately.

Suppose now that $a_{qm+r} \leq ma_1 + \left(\frac{qm+r}{m} - 1\right) a_m$ is true for an integer $q \geq 1$ and any $r \in \{0, 1, \ldots, m-1\}$. We prove that

$$a_{(q+1)m+r} \leq ma_1 + \left(\frac{(q+1)m+r}{m} - 1\right) a_m \, . \tag{2}$$

Indeed,

$$\begin{aligned} & a_{(q+1)m+r} \\ &= a_{qm+r+m} \leq a_m + a_{qm+r} \leq a_m + ma_1 + \left(\frac{qm+r}{m} - 1\right) a_m \\ &= ma_1 + \frac{qm+r}{m} a_m = ma_1 + \left(\frac{(q+1)m+r}{m} - 1\right) a_m \, . \end{aligned}$$

Hence, we get (2), and this completes the proof.

Remark. Fixing m, we have, for $n \geq m$, $\frac{a_n}{n} \leq \frac{m}{n} a_1 + \frac{a_m}{m} - \frac{a_m}{n}$. Letting n go to infinity yields: $\limsup \frac{a_n}{n} \leq \frac{a_m}{m}$. Letting now m go to infinity, we obtain $\limsup \frac{a_n}{n} \leq \liminf \frac{a_n}{n}$. Since the sequence $(\frac{a_n}{n})$ is bounded $(0 \leq \frac{a_n}{n} \leq a_1$ for all $n)$, it follows that the sequence $(\frac{a_n}{n})$ is convergent.

Next, we move to solutions for problems from the September 2000 number of the *Corner* and the Swedish Mathematical Competition, Final Round, 1996, given [2000 : 261–262].

1. Through an arbitrary interior point of a triangle lines parallel to the sides of the triangle are drawn dividing the triangle into six regions, three of which are triangles. Let the areas of these three triangles be T_1, T_2, and T_3 and let the area of the original triangle be T. Prove that

$$T \;=\; (\sqrt{T_1}+\sqrt{T_2}+\sqrt{T_3})^2\,.$$

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Dora Ho, student, Calgary, Alberta; by Geoffrey A. Kandall, Hamden, CT, USA; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by April Keller, student, Calgary, Alberta; by Elena Lee, student, Calgary, Alberta; and by Toshio Seimiya, Kawasaki, Japan. We give Bataille's solutions.

Let P be the given interior point of $\mathcal{T} = \triangle ABC$ (with area T) and $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ be the triangles with respective areas T_1, T_2, T_3. We denote by I, J, K the points of intersection of AP with BC, BP with AC, CP with AB, respectively.

We are required to prove that $\sqrt{\frac{T_1}{T}}+\sqrt{\frac{T_2}{T}}+\sqrt{\frac{T_3}{T}} = 1$.
[Ed. T_1 is the area of the small triangle with base on side BC, and so on.]

Solution I. Clearly, triangles $\mathcal{T}_1$ and $\mathcal{T}$ are similar (even homothetic). It follows that $\frac{T_1}{T} = \left(\frac{IP}{IA}\right)^2$. Similar results hold for $\frac{T_2}{T}$ and $\frac{T_3}{T}$, so that

$$\begin{aligned}\sqrt{\frac{T_1}{T}}+\sqrt{\frac{T_2}{T}}+\sqrt{\frac{T_3}{T}} &= \frac{IP}{IA}+\frac{JP}{JB}+\frac{KP}{KC} \\ &= \frac{[PBC]}{[ABC]}+\frac{[PCA]}{[ABC]}+\frac{[PAB]}{[ABC]} \;=\; \frac{[ABC]}{[ABC]} \;=\; 1\end{aligned}$$

(using, for instance, that $\triangle PBC$ and $\triangle ABC$, which share the base BC, have their areas $[PBC]$ and $[ABC]$ in the ratio of their altitudes from P and A, respectively, and this ratio is also $\frac{IP}{IA}$).

Solution II. We use barycentric coordinates with respect to (A,B,C). There exist positive real numbers r, s, t such that $P = rA+sB+tC$ and $r+s+t = 1$. From the hypotheses, it is easy to obtain that the other two vertices of $\mathcal{T}_1$ are given by $(r+s)B+tC$ and $sB+(r+t)C$. It follows that

$$\frac{T_1}{T} \;=\; \left|\det\begin{pmatrix} r & 0 & 0 \\ s & r+s & s \\ t & t & r+t\end{pmatrix}\right| \;=\; |r(r^2+rs+rt)| = r^2\,.$$

Similarly, $\frac{T_2}{T} = s^2$, $\frac{T_3}{T} = t^2$, so that $\sqrt{\frac{T_1}{T}}+\sqrt{\frac{T_2}{T}}+\sqrt{\frac{T_3}{T}} = r+s+t = 1$.

Solution III. The ratio of areas is an affine invariant, so we may as well suppose that $\mathcal{T}$ is equilateral. In that case, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are also equilateral with respective sides a_1, a_2, a_3 that obviously add up to the side a of $\mathcal{T}$. Thus

$$\sqrt{\frac{T_i}{T}} = \sqrt{\frac{a_i^2\sqrt{3}/4}{a^2\sqrt{3}/4}} = \frac{a_i}{a} \qquad (i = 1,\, 2,\, 3)$$

and

$$\sqrt{\frac{T_1}{T}} + \sqrt{\frac{T_2}{T}} + \sqrt{\frac{T_3}{T}} = \frac{a_1 + a_2 + a_3}{a} = 1\,.$$

2. In the country of Postonia one wants to have only two values of stamps. The two values should be integers greater than one, and the difference between the two should be two. It should also be possible to combine, in a precise way, stamps for each letter, the postage of which is greater than or equal to the sum of the two values. What values can be chosen?

Solutions by Peter Du, student, Calgary, Alberta; and by Panos E. Tsaoussoglou, Athens, Greece. We give Du's write-up.

Let the two stamps' values be a and b. Clearly, if a and b are two consecutive even numbers, then it would be impossible to make any postage which is odd. Therefore, a and b must be two consecutive odd integers. By the Problem of Frobenius (in the case for two integers)*, the highest postage which is impossible to make using a stamps and b stamps is $ab - (a+b)$. Note that this requires a and b to be relatively prime, and two consecutive odd integers greater than 1 must be coprime (for otherwise the odd integers would have a common factor of 2, an impossibility).

Without loss of generality, assume that $b = a + 2$. We now prove that $ab - (a+b) > (a+b)$ for all integers $a > 3$. This is clearly the same as proving $ab - 2(a+b) > 0$. Note that $ab - 2(a+b) = ab - 2a - 2b = (a-2)(b-2) - 4$. When a increases as a positive integer, so does the value of $(a-2)(b-2) - 4$, because $(a-2)(b-2) = (a-2)a$. Therefore, $ab - 2(a+b) > 0$ for all integers $a > 3$.

Consequently, there exists a postage greater than the sum of the two stamps which is impossible to make using those stamps, if the smaller stamp has a value greater than 3. When $a = 3$, the largest postage which is impossible to make is $3 \cdot 5 - (3+5) = 7$, and all the postages greater than or equal to $8 = 3 + 5$ would be possible to make using 3 and 5 stamps. This is the only possible case, as 2 and 4 stamps cannot make odd postages.

Note:* Proofs of the Problem of Frobenius for two integers (first proven by Sylvester in 1884) can be found at the following Internet url's:

`www.math.bingham.edu/matthias/papers/frobnote.slides.pdf (page 6)`

`www.math.temple.edu/~matthias/papers/frobnote.pdf (page 5)`

`www.math.swt.edu/~haz/prob_sets/notes/node11.html`

3. For all integers $n \geq 1$ the functions p_n are defined for $x \geq 1$ by

$$p_n(x) \;=\; \tfrac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right) .$$

Show that $p_n(x) \geq 1$ and that $p_{mn}(x) = p_m(p_n(x))$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

Let $m, n \in \mathbb{N}^*$, $x \geq 1$. Let $t = x + \sqrt{x^2-1}$. Then $t \geq 1$ and $\frac{1}{t} = x - \sqrt{x^2-1}$. It follows that

$$p_n(x) \;=\; \tfrac{1}{2}\left(t^n + \tfrac{1}{t^n}\right) \;\geq\; 1$$

since $\alpha + \dfrac{1}{\alpha} \geq 2$ for $\alpha > 0$. Note that equality occurs if and only if $t = 1$. That is, $x = 1$.

Moreover, we have

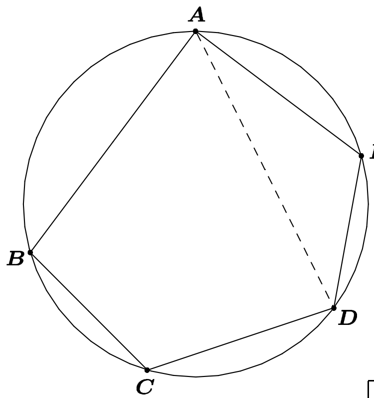
$$\begin{aligned} p_m(p_n(x)) \;&=\; \tfrac{1}{2}\left(\left(\tfrac{1}{2}\left(t^n + \tfrac{1}{t^n}\right) + \sqrt{\tfrac{1}{4}\left(t^{2n} + \tfrac{1}{t^{2n}} + 2\right) - 1}\right)^m\right. \\ &\qquad \left. + \left(\tfrac{1}{2}\left(t^n + \tfrac{1}{t^n}\right) - \sqrt{\tfrac{1}{4}(t^{2n} + \tfrac{1}{t^{2n}} + 2) - 1}\right)^m\right) . \end{aligned}$$

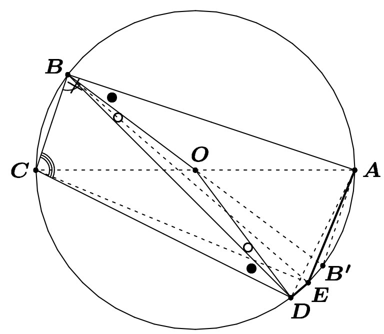
Since $\frac{1}{4}\left(t^{2n} + \frac{1}{t^{2n}} + 2\right) - 1 = \frac{1}{4}\left(t^n - \frac{1}{t^n}\right)^2$ with $t^n - \frac{1}{t^n} \geq 0$ (from $t \geq 1$), we deduce that

$$\begin{aligned} &p_m(p_n(x)) \\ &\quad= \tfrac{1}{2}\left(\left(\tfrac{1}{2}t^n + \tfrac{1}{2t^n} + \tfrac{1}{2}t^n - \tfrac{1}{2t^n}\right)^m + \left(\tfrac{1}{2}t^n + \tfrac{1}{2t^n} - \tfrac{1}{2}t^n + \tfrac{1}{2t^n}\right)^m\right) \\ &\quad= \tfrac{1}{2}\left(t^{nm} + \tfrac{1}{t^{nm}}\right) \\ &\quad= \tfrac{1}{2}((x+\sqrt{x^2-1})^{nm} + (x-\sqrt{x^2-1})^{nm}) \\ &\quad= p_{mn}(x) \quad \text{as desired.} \end{aligned}$$

4. A pentagon $ABCDE$ is inscribed in a circle. The angles at A, B, C, D, E form an increasing sequence. Show that the angle at C is $> \pi/2$. Also, prove that this lower bound is best possible.

Solution by Toshio Seimiya, Kawasaki, Japan.





LEGEND: $\frac{\pi}{2}+\theta$ | $\frac{3\pi}{8}$ | • $\frac{\pi}{8}-\theta$ | ∘ θ

Since $\angle A < \angle B < \angle C$, we have $\angle BAD < \angle BAE < \angle BCD$. Thus, $\angle BAD + \angle BCD < 2\angle BCD$; that is $\pi < 2\angle BCD$. Hence, $\angle BCD > \frac{\pi}{2}$.

We now need to show that $\frac{\pi}{2}$ is the best possible lower bound. Let Γ be a circle with centre O. Let BD be a chord of Γ such that $\angle OBD = \angle ODB = \theta < \frac{\pi}{16}$.

Also, let C be a point on the minor arc BD such that $\angle CBD = \frac{3}{8}\pi$. Since the major angle $BOD = \pi + 2\theta$, we get $\angle BCD = \frac{\pi}{2} + \theta$, and $\angle BDC = \frac{\pi}{8} - \theta$.

Let CA and BB' be diameters of Γ, and let E be a point on the minor arc $B'D$. Since BB' is a diameter, we have $\angle EAB < \angle B'AB = \frac{\pi}{2}$. Since CA is a diameter, $\angle ABC = \frac{\pi}{2}$, so that $\angle EAB < \angle ABC$. Since $\angle BCD = \frac{\pi}{2} + \theta > \frac{\pi}{2}$, we have $\angle BCD > \angle ABC$. Since CA and BB' are diameters, we get

$$\angle ABB' = \angle ABO = \angle BAO = \angle BAC = \angle BDC = \frac{\pi}{8} - \theta\,.$$

Thus, $\angle CDE = \angle CDA + \angle ADE = \frac{\pi}{2} + \angle ABE > \frac{\pi}{2} + \angle ABB' = \frac{\pi}{2} + \left(\frac{\pi}{8} - \theta\right) > \frac{\pi}{2} + \theta$ (since $\theta < \frac{\pi}{16}$).

Thus, $\angle CDE > \angle BCD$.

Since $\angle CDE = \angle CDA + \angle ADE = \frac{\pi}{2} + \angle ABE < \frac{\pi}{2} + \angle ABD = \frac{\pi}{2} + \angle ABB' + \angle B'BD = \frac{\pi}{2} + (\frac{\pi}{8} - \theta) + \theta = \frac{\pi}{2} + \frac{\pi}{8}$, and

$$\angle DEA = \angle DEC + \angle CEA = \angle DBC + \frac{\pi}{2} = \frac{3}{8}\pi + \frac{\pi}{2} > \frac{\pi}{2} + \frac{\pi}{8}\,,$$

we get $\angle CDE < \angle DEA$.

Thus, the inscribed pentagon $ABCDE$ has the property $\angle A < \angle B < \angle C < \angle D < \angle E$. We can choose positive angle θ as small as we like. Therefore, $\frac{\pi}{2}$ is the best possible lower bound of $\angle C$.

5. Let $n \geq 1$. Prove that it is possible to select some of the integers 1, 2, 3, $\ldots$, 2^n so that for all integers $p = 0, 1, \ldots, n-1$ the sum of k^p over all selected integers $k \in \{1, 2, 3, \ldots, 2^n\}$ is the same as the sum of k^p over all nonselected integers $k \in \{1, 2, 3, \ldots, 2^n\}$.

Solution by Pierre Bornsztein, Pontoise, France.

Lemma. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ ($n \in \mathbb{N}^*$) be real numbers and $p \in \mathbb{N}$. For $x \in \mathbb{R}$ and $k \in \mathbb{N}$, denote

$$A_k(x) \;=\; \sum_{i=1}^{n}(a_i + x)^k \quad \text{and} \quad B_k(x) \;=\; \sum_{i=1}^{n}(b_i + x)^k .$$

If $A_k(0) = B_k(0)$ for $k = 0, 1, \ldots, p$, then, for all $x \in \mathbb{R}$,

$$A_k(0) + B_k(x) \;=\; B_k(0) + A_k(x)$$

for $k = 0, 1, \ldots, p+1$.

Proof of the Lemma. Suppose that $A_k(0) = B_k(0)$ for $k = 0, \ldots, p$. Let x be a real number. For $k \in \mathbb{N}$, from the binomial theorem, we have

$$\begin{aligned} A_k(0) + B_k(x) &= A_k(0) + \sum_{i=1}^{n}\sum_{j=0}^{k}\binom{k}{j} b_i^j x^{k-j} \\ &= A_k(0) + \sum_{j=0}^{k}\binom{k}{j} B_j(0) x^{k-j} . \end{aligned}$$

Then, for $k \in \{0, \ldots, p\}$, we have

$$\begin{aligned} A_k(0) + B_k(x) &= B_k(0) + \sum_{j=0}^{k}\binom{k}{j} A_j(0) x^{k-j} \\ &= B_k(0) + A_k(x) . \end{aligned}$$

For $k = p+1$, we also have

$$\begin{aligned} A_{p+1}(0) + B_{p+1}(x) &= A_{p+1}(0) + B_{p+1}(0) + \sum_{j=0}^{p}\binom{p+1}{j} B_j(0) x^{p+1-j} \\ &= B_{p+1}(0) + A_{p+1}(0) + \sum_{j=0}^{p}\binom{p+1}{j} A_j(0) x^{p+1-j} \\ &= B_{p+1}(0) + A_{p+1}(x) . \end{aligned}$$

Thus, the lemma is proved.

Let $E_1 = \{1\}$, $F_1 = \{2\}$ and for $n \in \mathbb{N}^*$, if $E_n = \{a_1, \ldots, a_{2^{n-1}}\}$ and $F_n = \{b_1, b_2, \ldots, b_{2^{n-1}}\}$, then $E_{n+1} = E_n \cup \{b_1 + 2^n, b_2 + 2^n, \ldots, b_{2^{n-1}} + 2^n\}$ and $F_{n+1} = F_n \cup \{a_1 + 2^n, a_2 + 2^n, \ldots, a_{2^{n-1}} + 2^n\}$.

By induction, it is easy to verify that, for all $n \in \mathbb{N}^*$, $E_n \cap F_n = \emptyset$ and $E_n \cup F_n = \{1, 2, 3, \ldots, 2^n\}$. Moreover,

$$\sum_{a_i \in E_n} a_i^k = \sum_{b_i \in F_n} b_i^k \quad \text{for} \quad k = 0 .$$

Then, by induction and using the lemma (for $n = 2, 2^2, 2^3, \ldots$), we easily deduce that, for all $n \in \mathbb{N}^*$,

$$\sum_{a_i \in E_n} a_i^k = \sum_{b_i \in F_n} b_i^k \quad \text{for} \quad k = 0, 1, \ldots, n-1 ,$$

and the proof is complete.

6. Tiles of dimension 6×1 are used to construct a rectangle. Prove that one of the sides has a length divisible by 6.

Comment by Pierre Bornsztein, Pontoise, France. More generally, we may prove that: "Let $a, b \in \mathbb{N}^*$. An $a \times b$ rectangle can be tiled using $n \times 1$ tiles if and only if n divides one of the sides of the rectangle."

A proof of this claim can be found in [1]. It is a consequence of a theorem of De Bruijn which states that: "whenever a rectangle is tiled by rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side".

For fourteen (!) proofs of the De Bruijn theorem, see [2]. The links between these two results may also be found in [1].

References:

[1] S.W. Golomb, "Polyominoes, puzzles, patterns, problems and packing", Princeton University Press (1994), p. 116, 118, and 121.

[2] S. Wagon, "Fourteen proofs of a result about tiling a rectangle", A.M.M. (1987), p. 601–617.

Next we turn to solutions by our readers to problems of the 48th Polish Mathematical Olympiad, Final Round, 1997 given [2000 : 262–263].

1. The positive integers $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ satisfy the conditions:

$$x_6 = 144 \quad \text{and} \quad x_{n+3} = x_{n+2}(x_{n+1} + x_n) \quad \text{for} \quad n = 1, 2, 3, 4 .$$

Compute x_7.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Robert Bilinski, Outremont, Quebec; by Pierre Bornsztein, Pontoise, France; by Brett Link, student, Calgary, Alberta; by Geoffrey A. Kandall, Hamden, CT, USA; by Moubinool Omarjee, Paris, France; and by Panos E. Tsaoussoglou, Athens, Greece. We give Aassila's comment.

This problem was proposed at the Dutch Mathematical Olympiad. A solution appeared in ***Crux Mathematicorum*** [1986 : 76].

We give the (slightly edited) *solution of Omarjee which differs from the solution already given.*

If we substitute the equations for $n = 1$ and 2 into the equation for $n = 3$, we get

$$x_3(x_1+x_2)(x_2+x_3)(x_3+x_4) \;=\; 144\,.$$

Now we can get lower bounds

$$\begin{aligned} x_4 &= x_3(x_1+x_2) \;\geq\; 2x_3\,, \\ x_5 &= x_4(x_2+x_3) \;\geq\; 2x_3(1+x_3) \;>\; 2x_3^2\,, \\ 144 &\geq 2x_3(3x_3^2) \;=\; 6x_3^3\,. \end{aligned}$$

Thus, x_3 is 1 or 2.

Suppose that $x_3 = 1$. We get

$$(x_1+x_2)(1+x_1+x_2)(x_2+1) \;=\; 144$$

using the fact that x_1+x_2 and $1+x_1+x_2$ are consecutive factors in

$$144 \;=\; (1\times 2)\times 72 \;=\; (2\times 3)\times 24 \;=\; (3\times 4)\times 12 \;=\; (8\times 9)\times 2\,.$$

An easy calculation shows us that the only possible value for x_1+x_2 is 8. This gives $x_1 = 7$, $x_2 = 1$, $x_3 = 1$, $x_2 = 8$, $x_5 = 16$, $x_6 = 144$ and $x_7 = 3456$.

Suppose that $x_3 = 2$. We get

$$(x_1+x_2)(1+x_1+x_2)(x_2+2) \;=\; 36\,.$$

The same ideas as above give us $x_1 = 1$, $x_2 = 2$, $x_3 = 2$, $x_4 = 6$, $x_5 = 18$, $x_6 = 144$, and $x_7 = 3456$.

2. Solve the following system of equations in real numbers x, y, z:

$$\begin{aligned} 3(x^2+y^2+z^2) &= 1 && (1) \\ x^2y^2+y^2z^2+z^2x^2 &= xyz(x+y+z)^3\,. && (2) \end{aligned}$$

Solutions by Mohammed Aassila, Strasbourg, France; by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Moubinool Omarjee, Paris, France; and by Panos E. Tsaoussoglou, Athens, Greece. We give Amengual Covas' solution.

Since all the terms in the sum of the left hand side of (2) are non-negative, we have $xyz(x+y+z)^3 \geq 0$ and hence $xyz(x+y+z) \geq 0$.

If $xyz(x+y+z)=0$, equation (2) implies that $xy=yz=zx=0$. From this and (1), it follows that exactly two of the numbers x, y, z are equal to zero.

If $x=y=0$, substitution in (1) yields $z=\pm\frac{1}{\sqrt{3}}$.

Similarly, if $y=z=0$ then $x=\pm\frac{1}{\sqrt{3}}$. If $z=x=0$, then $y=\pm\frac{1}{\sqrt{3}}$.

If $xyz(x+y+z)>0$, we consider the inequality

$$(u+v+w)^2 \;\le\; 3(u^2+v^2+w^2) \tag{3}$$

which is a special case of Cauchy's Inequality

$$(\alpha u+\beta v+\gamma w)^2 \;\le\; (\alpha^2+\beta^2+\gamma^2)(u^2+v^2+w^2)$$

with $\alpha=\beta=\gamma=1$.

By (3) applied with $u=xy$, $v=yz$, $w=zx$, we have

$$(xy+yz+zx)^2 \;\le\; 3(x^2y^2+y^2z^2+z^2x^2)$$

which simplifies to

$$\begin{aligned} xyz(x+y+z) &\le x^2y^2+y^2z^2+z^2x^2 \\ &= xyz(x+y+z)^3 \quad \text{(by (2).)} \end{aligned}$$

Dividing by the positive factor $xyz(x+y+z)$, we get

$$\begin{aligned} 1 &\le (x+y+z)^2 \\ &\le 3(x^2+y^2+z^2) \quad \text{(by (3))} \quad = 1 \quad \text{(by (1).)} \end{aligned}$$

Therefore,

$$(x+y+z)^2 \;=\; 3(x^2+y^2+z^2)\,,$$

which is equivalent to

$$(x-y)^2+(y-z)^2+(z-x)^2 \;=\; 0\,,$$

and this holds if and only if $x=y=z$.

Solving for x, y and z, we immediately obtain

$$x \;=\; y \;=\; z \;=\; \frac{1}{3} \quad \text{or} \quad x \;=\; y \;=\; z \;=\; -\frac{1}{3}\,.$$

Thus, the general solution is constituted by the following triples (x,y,z):

$$\left(\frac{1}{\sqrt{3}},0,0\right),\left(-\frac{1}{\sqrt{3}},0,0\right),\left(0,\frac{1}{\sqrt{3}},0\right),\left(0,-\frac{1}{\sqrt{3}},0\right),$$

$$\left(0,0,\frac{1}{\sqrt{3}}\right),\left(0,0,-\frac{1}{\sqrt{3}}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3}\right).$$

3. In a triangular pyramid $ABCD$, the medians of the lateral faces ABD, ACD, BCD drawn from vertex D form equal angles with the corresponding edges AB, AC, BC. Prove that the area of each lateral face is less than the sum of the areas of the two other lateral faces.

Solutions by Michel Bataille, Rouen, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bataille's solution.

Let A', B', C' be the respective mid-points of BC, CA, AB, and θ be the common angle formed by DA' and BC, DB' and CA, DC' and AB. We denote by $[KLM]$ the area of $\triangle KLM$.

Since $[DA'B] = [DA'C]$, we have

$$[DBC] \;=\; 2[DA'B] \;=\; 2\times\frac{1}{2}\times\frac{BC}{2}\times DA'\times\sin\theta \;=\; B'C'\cdot DA'\cdot\sin\theta\,.$$

Similar relations hold for $[DCA]$ and $[DAB]$, and so we are required to prove that, for instance,

$$B'C'\cdot DA' \;<\; A'C'\cdot DB' + A'B'\cdot DC'\,. \tag{1}$$

Denoting by X, Y, Z, the vectors $\overrightarrow{DA'}$, $\overrightarrow{DB'}$, $\overrightarrow{DC'}$, respectively, it amounts to proving that

$$\|X\|\cdot\|Z-Y\| \;<\; \|Y\|\cdot\|Z-X\| + \|Z\|\cdot\|X-Y\|$$

or, since X, Y, Z are non-zero vectors, that

$$\frac{\|Z-Y\|}{\|Z\|\;\|Y\|} \;<\; \frac{\|Z-X\|}{\|Z\|\;\|X\|} + \frac{\|X-Y\|}{\|X\|\;\|Y\|}\,. \tag{2}$$

From $\left\|\frac{1}{\|X\|^2}X\right\| = \frac{1}{\|X\|}$ and $\|X-Y\|^2 = \|X\|^2 + \|Y\|^2 - 2X\cdot Y$ (where $X\cdot Y$ is the scalar product of X and Y), we deduce that

$$\frac{\|X-Y\|^2}{\|X\|^2\|Y\|^2} \;=\; \frac{1}{\|Y\|^2} + \frac{1}{\|X\|^2} - 2\frac{X\cdot Y}{\|X\|^2\|Y\|^2} \;=\; \left\|\frac{1}{\|X\|^2}X - \frac{1}{\|Y\|^2}Y\right\|^2\,.$$

Thus, (2) may be rewritten as

$$\|Z'-Y'\| \;<\; \|Z'-X'\| + \|X'-Y'\|\,, \tag{3}$$

where $X' = \frac{1}{\|X\|^2}X$, $Y' = \frac{1}{\|Y\|^2}Y$, $Z' = \frac{1}{\|Z\|^2}Z$.

Since (3) results from the triangular inequality (a strict inequality since X', Y', Z' are not coplanar), (2) and (1) follow and the proof is complete.

Remark. (1) would be Ptolemy's inequality, were D, A', B', C' coplanar. Clearly, the proof above also provides a proof of this inequality.

4. The sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 0 \quad a_n = a_{\lfloor n/2 \rfloor} + (-1)^{n(n+1)/2} \quad \text{for} \quad n > 1\,.$$

For every integer $k \geq 0$, find the number of all n such that

$$2^k \leq n < 2^{k+1}\,, \quad a_n = 0$$

($\lfloor n/2 \rfloor$ denotes the greatest integer not exceeding $n/2$).

Comments by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Pontoise, France. We give Bornsztein's remarks.

For every integer k the number of n such that $2^k \leq n < 2^{k+1}$ and $a_n = 0$ is $\binom{k}{k/2}$ if k is even, and 0 if k is odd. It easily follows from the solutions of problem No. 8 proposed to the IMO but not used in 1996 (see [1998 : 466–470] and [1999 : 136–137]). In particular, $a_n = b_n - c_n$, where b_n (respectively c_n) denotes the total number of pairs 00, 11 (respectively, 01, 10) in the binary expansion of n. Noting that $2^k \leq n < 2^{k+1}$ if and only if the binary expansion of n has exactly $k+1$ digits, and that the first digit is always 1, we have $a_n = 0$ if and only if $b_n = c_n$. That is we have exactly $\frac{k}{2}$ changes of digits (10 or 01) between two consecutive digits.

5. Given is a convex pentagon $ABCDE$ with

$$DC = DE \quad \text{and} \quad \angle DCB = \angle DEA = 90^\circ\,.$$

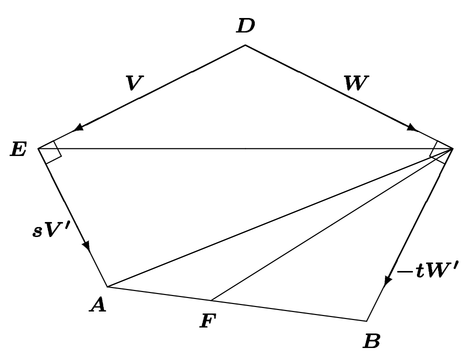
Let F be the point on AB such that $AF : BF = AE : BC$. Show that

$$\angle FCE = \angle ADE \quad \text{and} \quad \angle FEC = \angle BDC\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; and by Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's write-up.

Let $X \mapsto X'$ be the linear transformation that rotates each non-zero vector counterclockwise by 90° while preserving its length. This transformation preserves dot products: $X' \cdot Y' = X \cdot Y$.

Let $V = \overrightarrow{DE}$, $W = \overrightarrow{DC}$ ($|V| = |W|$). There exist $x, t > 0$ such that $\overrightarrow{EA} = sV'$, $\overrightarrow{CB} = -tW'$.



Note that $\tan\angle ADE = \frac{EA}{DE} = \frac{s|V'|}{|V|} = s$, giving us that $\cos\angle ADE = \frac{1}{\sqrt{s^2+1}}$.

We have $\overrightarrow{CA} = \overrightarrow{CE} + \overrightarrow{EA} = V - W + sV'$. Since

$$\frac{AF}{BF} = \frac{AE}{BC} = \frac{s|V'|}{t|W'|} = \frac{s}{t},$$

it follows that

$$\overrightarrow{CF} = \frac{1}{s+t}(s\overrightarrow{CB} + t\overrightarrow{CA}) = \frac{t}{s+t}((V-W) + s(V-W)').$$

Therefore, $\overrightarrow{CF}\cdot\overrightarrow{CE} = \frac{t}{s+t}|V-W|^2$. Also,

$$|\overrightarrow{CF}|^2 = \overrightarrow{CF}\cdot\overrightarrow{CF} = \frac{t^2}{(s+t)^2}(s^2+1)|V-W|^2.$$

Thus,

$$\cos\angle FCE = \frac{\overrightarrow{CF}\cdot\overrightarrow{CE}}{|\overrightarrow{CF}|\cdot|\overrightarrow{CE}|} = \frac{1}{\sqrt{s^2+1}} = \cos\angle ADE.$$

Hence, $\angle FCE = \angle ADE$.

The second part of the conclusion is analogous to the first.

6. Consider n points ($n \geq 2$) on the circumference of a circle of radius 1. Let q be the number of segments having those points as endpoints and having length greater than $\sqrt{2}$. Prove that $3q \leq n^2$.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Pontoise, France. We give Aassila's remark.

The estimate $3q \leq n^2$ follows from Turán's theorem (compare Ex. 30, p. 68 of "Combinatorial Problems and Exercises" by L. Lovász). Let us prove that there does not exist a complete graph on 4 vertices [with edges of length greater that $\sqrt{2}$]. Assume, by contradiction, that $ABCD$ is a complete graph with edges of length greater than $\sqrt{2}$. Hence,

$$\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA} > \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} > 2\pi.$$

Contradiction.

For completeness we give Bornsztein's solution.

Mark the n points on the circle Γ with center O and radius 1. For every $M, P \in \Gamma : MP > \sqrt{2}$ if and only if $\widehat{MOP} > \frac{\pi}{2}$. Since the number of marked points is finite, we may divide Γ into four arcs by two perpendicular diameters $[AC]$ and $[BD]$ such that none of the points A, B, C, D is marked.

Denote by a, b, c, d the number of marked points in the arcs AB, BC, CD, DA, respectively. Thus,

$$a+b+c+d = n\,. \tag{1}$$

With no loss of generality, we may suppose that $ab = \max(ab, bc, cd, da)$. There are $\binom{n}{2} - q$ segments having the marked points as endpoints and having length not greater than $\sqrt{2}$.

First note that if M, P are two marked points in the same arc then $\widehat{MOP} < \frac{\pi}{2}$ and then $MP < \sqrt{2}$. Thus, we may count $\binom{a}{2} + \binom{b}{2} + \binom{c}{2} + \binom{d}{2}$ such segments.

Secondly, there are $abcd$ choices for four marked points M, N, P, Q such that $M \in \widehat{AB}$, $N \in \widehat{BC}$, $P \in \widehat{CD}$, $Q \in \widehat{DA}$. Note that for such a choice, O is interior to the convex quadrilateral $MNPQ$. Thus, $\widehat{MON} + \widehat{NOP} + \widehat{POQ} + \widehat{QOM} = 2\pi$. It follows that at least one of these four angles is not greater than $\frac{\pi}{2}$. Thus, for every choice of four such points we may find a segment whose length is not greater than $\sqrt{2}$. Such a segment must have its endpoints belonging to two consecutive arcs. For example, $NP \le \sqrt{2}$. This segment may be counted at most ad times in the $abcd$ segments above. And $ad \le ab$. Thus, each of the segments is counted at most ab times. It follows that, in this case, we may find at least $\frac{abcd}{ab} = cd$ segments having length not greater than $\sqrt{2}$. It follows that

$$\binom{n}{2} - q \;\ge\; cd + \binom{a}{2} + \binom{b}{2} + \binom{c}{2} + \binom{d}{2}\,.$$

Using (1), we deduce that

$$-q \;\ge\; cd - \frac{1}{2}n^2 + \frac{1}{2}(a^2+b^2+c^2+d^2)\,.$$

Thus,

$$\begin{aligned} n^2 - 3q &\ge -\frac{n^2}{2} + 3cd + \frac{3}{2}(a^2+b^2+c^2+d^2) \\ &= \left(\frac{1}{2}(a+b) - (c+d)\right)^2 + \frac{3}{4}(a-b)^2 \quad \text{(from (1)} \\ &\ge 0\,, \end{aligned}$$

and we are done.

That again completes yet another *Olympiad Corner* for this issue. Send me your nice solutions as well as Olympiad Contests for use in the *Corner*!

BOOK REVIEWS

JOHN GRANT McLOUGHLIN

1000 Play Thinks: Puzzles, Paradoxes, Illusions and Games
by Ivan Moscovich, illustrated by Tim Robinson with a foreword by Ian Stewart, published by Thomas Allen and Son Limited, 2001,
ISBN 0-7611-1826-8, softcover, 420 + ix pages, US$29.95
Reviewed by ***Amar Sodhi****, Sir Wilfred Grenfell College, Corner Brook, NF.*

During the Edo period (1603-1867) in Japan, many a shrine or temple would have, hanging from its roof, a wooden tablet engraved with a beautifully drawn coloured drawing. The drawing would describe a problem in geometry and people from all walks of life would take delight in attempting to solve the problem. The wooden tablet was called a *Sangaku* and it was the *Sangaku* which inspired Ivan Moscovich to create the book *1000 Play Thinks*.

A typical *Play Think* consists of a problem accompanied by a colourful drawing which has the potential to intrigue anyone with an interest in recreational mathematics; so in this sense a *Play Think* is similar to a *Sangaku*. However, unlike a *Sangaku*, a *Play Think* is not restricted to theorems in geometry. A number puzzle, a logic problem, or even an optical illusion, can be considered *Play Thinks*. Most of the activities in this vast collection of exercises for the mind are puzzles which can be solved using clever reasoning and perhaps the use of pen and paper. However, there are also puzzles which require the reader to show dexterity in manipulating figures as well as a selection of games having some underlying mathematical theme. Challenging problems in the book include three *Sangakus* (*Play Thinks 2, 259* and *339*) and *Appollonius' Problem* (*Play Think 242*).

Play Thinks is divided into fourteen chapters. The first and last chapters contain a miscellaneous selection of *Play Thinks*, but the other chapters are more thematic. For example, a chapter on graphs and networks exposes the reader to Hamiltonian and Eulerian graphs, crossing numbers and Ramsey's Theorem. There are various chapters devoted to themes in geometry. Patterns and symmetry are all explored, and there is a wealth of number puzzles for those who enjoy challenges of an arithmetic nature. A diligent reader of the book will discover theorems in geometry and topology such as Pappus' Theorem (*Play Think 135*) and the Jordan Curve Theorem (*Play Think 134*). Scattered through each chapter are vignettes which provide a gentle digression from problem solving. Through the vignettes, the reader is given a glimpse of certain mathematical concepts, discoveries and insights pertinent to problem solving.

Anybody who enjoys puzzles should enjoy this book. Of course, an experienced solver of mathematical puzzles will find many old friends in this work, but old friends in a new setting are always welcome. The book itself is eye-catching and will feel at home on any coffee table as it waits for people of all ages to peruse its large colourful pages. People with a more professional interest in mathematics will find this book a useful resource; a teacher who wishes to conduct a mathematics workshop for primary to high-school students should gain many ideas from the pages of this book.

The Zen of Magic Squares, Circles, and Stars: An Exhibition of Surprising Structures across Dimensions
by Clifford A. Pickover, published by Princeton University Press, 2002,
ISBN 0-691-07041-5, hardcover, 405 + xx pages, US$29.95.
Reviewed by **Monte J. Zerger**, Adams State College, Alamosa, Colorado.

If you have read any of Pickover's numerous books (for example, *Keys to Infinity, The Loom of God, Wonders of Numbers*), you know he is a remarkably creative thinker, from whom you quickly grow to expect the unexpected. His writings have often subtly revealed that he has a mystical side, but this book even more clearly demonstrates it. If you don't pick up that clue from the title, you surely will when you read that the book is dedicated "not to a person but to a meditative aid, the Durga Mantra, to which numbers can be applied in magic ways."

When most of us hear the term *magic square* we think of an $n \times n$ square array whose n^2 cells are filled with the numbers from 1 to n^2 in such a way that the sum of the numbers in each row, column, and the two main diagonals are the same. However, as this book quickly reveals, that is only the tip of the iceberg. Pickover goes on to define and discuss many forms of magic squares, as well as magic cubes, circles, spheres, stars, hexagons, etc., and even four-dimensional magic tesseracts.

Readers are not only taught how to construct, classify, and "play with" magic squares, but also treated to the rich and colorful history which surrounds them. As Pickover skillfully points out, throughout history many were convinced that magic squares held the secret of the universe. Although a classic work on the subject (*Magic Squares and Cubes* by Andrews) exists, Pickover's work expands, augments, enlivens, and probably most importantly, updates that book. Pickover has a way of presenting the material which will engage a wide variety of readers. One does not have to be a mathematician, or even have dabbled in magic squares before, to understand and appreciate the contents.

I sometimes wonder if Pickover writes too rapidly, often a problem with highly creative people who would rather go on to their next challenge than to refine and carefully hone their present one. The book often seems disjointed, reading a bit like a collection of notes, too hastily thrown together. I believe the contents could have been more carefully arranged, and then presented in a smoother, more continuous, flowing manner. This is surely the most comprehensive book on this subject in decades, and will probably become the standard encyclopedia for the subject. As the esteemed mathematician Sherman Stein put it, "Every generation seems to demand its own updated book dedicated to magic squares." Pickover's work meets the needs of the present generation.

An Industrial Application of Spherical Inversion

Edward Crane

A local engineer, Noel Stephens, built an instrument to locate the position of a solitary light source of unknown brightness and position in three-dimensional space. The instrument consisted of an arrangement in the plane of n fixed brightness sensors; the light source was always to lie above this plane. In this problem there are four real unknowns: the brightness of the source and three position co-ordinates, so that at least four sensors are necessary. Noel knew that it would be pointless to place the sensors all in a line, for then rotating the light source about that line would not alter the sensor readings. Thus, for his first experiment, he placed the sensors at the vertices of a square. To Noel's surprise, his computer was unable to determine the position of the source. He tried a rectangle, but again the computer was unable to fix the source. The sensors were working, and there was no mistake in the code — the data were simply insufficient to determine the position of the source uniquely! After some computer experimentation, Noel conjectured the following theorem, which we prove below.

Theorem 1 . Let $Q_0, \dots, Q_{n-1}$ lie on a circle C in $\mathbb{R}^3$, and let P be any other point of $\mathbb{R}^3$. For each $i \neq j$ let T_{ij} be the centre of the Apollonian sphere S_{ij} consisting of the points X such that $\dfrac{Q_iP}{Q_jP} = \dfrac{Q_iX}{Q_jX}$. Then

1. the centres T_{ij} lie on a line, and
2. there is a circle through P that is contained in all the spheres S_{ij}.

Although the brightness of the light source at P is unknown, if we know the *ratio* of intensities measured by two different sensors at Q_i and Q_j then we know a sphere of Apollonius on which P must lie. The theorem says that if you place all of your sensors on a circle then you have no hope of locating a light source at P, regardless of how many sensors you have, because two different sources on the common circle of those spheres and of appropriate brightnesses will give exactly the same data from the sensors. We assume that the source emits light isotropically (the same intensity in all directions), and that the sensors respond isotropically.

The proof uses spherical inversion. Inversion in a plane means reflection in that plane. In this article we write *sphere* to mean "sphere or plane" and *circle* to mean "circle or line". A line or plane may be characterised as a circle or sphere that passes through ∞, the point at infinity. For example, some of the Apollonian spheres S_{ij} could be planes; the centre in such a case is $T_{ij} = \infty$.

It is natural to use spherical inversion because of the following well-known lemma:

Lemma 1 .

1. The sphere or plane S is a sphere of Apollonius of two points A and B if and only if inversion in S swaps A with B.

2. If A and B are inverse points with respect to a sphere S and I is any spherical inversion, then $I(A)$ and $I(B)$ are inverse points with respect to $I(S)$.

It follows that any spherical inversion sends Apollonian spheres to Apollonian spheres.

Proof.

1. A simple exercise using similar triangles. See [1], p. 89 (section 6.6) and p. 92 (section 6.82).

2. $I(A)$, $I(B)$ are inverse with respect to $I(S)$ if and only if every sphere T passing through both $I(A)$ and $I(B)$ is orthogonal to $I(S)$. Since inversion is its own inverse and preserves angles, the image spheres $I(T)$ all pass through A and B and are orthogonal to S.

∎

Now we can prove the theorem.

Proof. Since we know that all the spheres S_{ij} contain P, it is clear that the two parts of the conclusion are equivalent: the spheres contain the circle through P in the plane perpendicular to the line of their centres.

Invert the figure about some point X on C: we will denote images under this inversion by dashes. The points Q'_i all lie on the line C'. By the lemma, the image sphere S'_{ij} is the Apollonian sphere for Q'_i and Q'_j that passes through P', whose centre lies on the line $Q'_iQ'_j$, which is C'. Since the centres of the S'_{ij} are collinear, the S'_{ij} all contain a certain circle E', passing through P'. Therefore, the S_{ij} all contain the image of E' under inversion about X, which is a circle through P. ∎

Remark: the centre of S_{ij} never inverts to the centre of S'_{ij}, which is why for an inversive proof it is best to aim for the second part of the conclusion. We may sum the proof up as follows:

> Because the original problem has a description that is invariant under the Möbius group (generated by all inversions), the set of useless positionings of the sensors must also be invariant under the Möbius group.

Here is an alternative proof, using a different inversion, in which we aim directly for the first conclusion of the theorem.

Proof. Call the plane of the n–gon H, and let R be the reflection of P in H. Then all the spheres S_i pass through P and also through R. Invert the figure about P; again we will use dashes to denote components of the inverse figure. H' is a sphere through $P = \infty'$. By the lemma, P and R are inverses with respect to H, so that $P' = \infty$ and R' are inverses with respect to H', so that R' is the centre of H'. The spheres of Apollonius invert to spheres S'_{ij}, which all pass through both R' and $P' = \infty$, so that they are planes through R'. The points Q_i and Q_j are inverse with respect to S_{ij}, so that Q'_i and Q'_j are inverse with respect to S'_{ij}: they are reflections of each other in the plane S'_{ij}.

Now, the points Q_i all lie on a circle C in the plane H, which inverts to a circle C' on the sphere H'. Since C' contains a point and its reflection in the plane S'_{ij}, (namely Q'_i and Q'_j), the plane of C' must be perpendicular to each S_{ij}. Now we know that the planes S'_{ij} all pass through the centre R' of the sphere H', and that they have a common perpendicular plane (the plane of C'). Therefore, they all contain a certain diameter of H', say L'.

The points T_{ij} are the centres of the S_{ij}, which are the points inverse to infinity with respect to S_{ij}. Thus, $P = \infty'$ and T'_{ij} are inverse with respect to S'_{ij}, so that they are each other's reflections in S'_{ij}. The sphere H' passes through P, and the planes S'_{ij} all contain a diameter L' of H', so that the reflections T_{ij} of P in those planes all lie in a circle D' passing through $P = \infty'$ whose plane is perpendicular to L'. Thus, the circle D' inverts to a line D, containing all the T_{ij}. ∎

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1. H.S.M. Coxeter, *Introduction to Geometry*, (Wiley Classics Library), John Wiley and Sons, New York, 1969.

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MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7 (NEW!)**. The electronic address is

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Mayhem Problems

Envoyez vos propositions et solutions à MATHEMATICAL MAYHEM, Faculté de mathématiques, Université de Waterloo, 200 University Avenue West, Waterloo, ON, N2L 3G1, ou par courriel à

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N'oubliez pas d'inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d'étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le *1er april 2003*. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Pour être admissibles au DÉFI MAYHEM de ce mois-ci, les solutions doivent avoir été postées *avant le* 1er février 2003, cachet de la poste faisant foi.

M57. Proposé par J. Walter Lynch, Athens, GA, USA.

Quatre points sont egalement espacés autour d'un cercle ayant un radius r. Le cercle est donc divisé par 4 arcs égaux. Renversez les arcs en laissant le point du bout en place. Trouvez l'aire de la figure ainsi obtenue.

. . . .

Four points are equally spaced around a circle with radius r. This divides the circle into 4 equal arcs. Flip over each arc, leaving the endpoints in place. Find the area enclosed by the figure thus obtained.

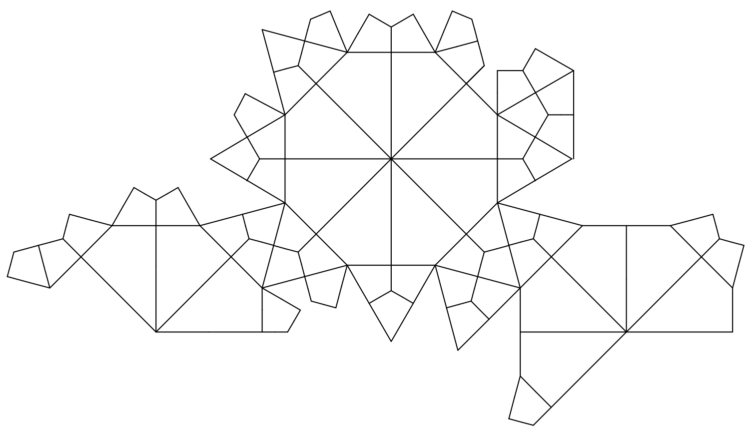
M58. Proposé par l'équipe de Mayhem.

Trouvez tous les entiers positifs x et y qui satisfont l'équation $x^y = y^x$.

. . . .

Find all positive integers x and y which satisfy the equation $x^y = y^x$.

M59. Proposé par Izidor Hafner, Tržaška 25, Ljubljana, Slovenia. Le diagramme ci-dessous représente le développement d'un polyèdre sur un plan. Les faces du solide sont divisées en polygones plus petits. Le problème consiste à colorer les polygones (ou á les numéroter) de telle sorte que chaque face du solide original soit d'une couleur différente.



The diagram above represents the net of a polyhedron in which the faces of the solid are divided into smaller polygons. The task is to colour the polygons (or number them), so that each face of the original solid is a different colour.

M60. Proposé par Mihàly Bencze, Brasov, Romania

Déterminez tous les entiers positifs dont $\left\lfloor \sum_{k=1}^{n} \sqrt{k} \right\rfloor = n$, et que $\lfloor x \rfloor$ est le plus grand entier plus petit ou égal à x.

. . . .

Determine all positive integers for which $\left\lfloor \sum_{k=1}^{n} \sqrt{k} \right\rfloor = n$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

M61. Proposé par l'équipe de Mayhem.

On vous donne 54 poids qui pèsent 1^2, 2^2, 3^2, $\ldots$, 54^2. Regroupez ceux-ci en trois groupes de poids égales.

. . . .

You are given 54 weights which weigh 1^2, 2^2, 3^2, $\ldots$, 54^2. Group these into three sets of equal weight.

M62. Proposé par Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.

Disons que $ABCD$ est un trapezoid dont les cotés AB et CD sont parallèles et que les diagonals AC et BC se croisent au point P. Supposons que $AB = 50$, $CD = 160$, et l'aire du triangle PAD est 2000. Trouvez l'aire du trapezoid.

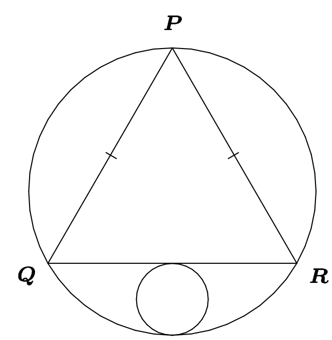
. . . .

Let $ABCD$ be a trapezoid where sides AB and CD are parallel and the diagonals AC and BC intersect at point P. Suppose $AB = 50$, $CD = 160$, and the area of triangle PAD is 2000. Determine the area of the trapezoid.

Mayhem Problem Solutions

M7. *From the Australian Mathematics Trust problems set.*

A circle of radius 6 has an isosceles triangle PQR inscribed in it, where $PQ = PR$. A second circle touches the first circle and the mid-point of the base QR of the triangle as shown. The side PQ has length $4\sqrt{5}$. Find the radius of the smaller circle.



Solutions by Robert Bilinski, Outremont, Quebec, and Paul Jefferys, Berkhamsted Collegiate School, UK. We give the solution of Jeffreys.

Let r be the radius of the smaller circle. Let X be the mid-point of QR. Let O be the centre of the larger circle. Then,

$$\begin{aligned} PQ^2 - QX^2 &= (12 - 2r)^2 \\ 80 - QX^2 &= 144 - 48r + 4r^2 \\ -QX^2 &= 64 - 48r + 4r^2 \,. \end{aligned} \qquad (1)$$

Similarly,

$$\begin{aligned} OQ^2 - QX^2 &= (6-2r)^2 \\ 36 - QX^2 &= 36 - 24r + 4r^2 \,. \end{aligned}$$

Substituting (1):

$$\begin{aligned} 36 + 64 - 48r + 4r^2 &= 36 - 24r + 4r^2 \\ 64 &= 24r \,. \end{aligned}$$

Therefore, the radius of the smaller circle is $\frac{8}{3}$.

M8. *Proposed by the Mayhem staff.*

Find all right-angled triangles with integer sides if one of the sides is 2001 units long.

Solution by Paul Jefferys, Berkhamsted Collegiate School, UK.

We consider the formula for Pythagorean triplets: $(x^2-y^2)k$, $(2xy)k$, and $(x^2+y^2)k$, where x, y, and k are positive integers and $x > y$.

Now the side length given by $(2xy)k$ is even, so that we need only consider the cases in which $(x^2-y^2)k = 2001$ and $(x^2+y^2)k = 2001$.

Case 1: $(x^2-y^2)k = 2001$. Then k is either 1, 3, 23, 29, 69, 87, 667, or 2001, since k divides 2001.

If $k = 2001$, then $x^2 - y^2 = 1$, so that $(x-y)(x+y) = 1$ forcing $y = 0$, a contradiction since y must be a positive integer.

If $k = 667$, then $x^2-y^2 = 3$ and we must have $x+y = 3$ and $x-y = 1$, giving the triplet $(2001, 2668, 3335)$.

If $k = 87$, then $x^2 - y^2 = 23$ and we must have $x + y = 23$ and $x - y = 1$, giving the triplet $(2001, 22968, 23055)$.

If $k = 69$, then $x^2 - y^2 = 29$ and we must have $x + y = 29$ and $x - y = 1$, giving the triplet $(2001, 28980, 29049)$.

If $k = 29$, then $x^2 - y^2 = 69 = 3 \times 23$. Then either $x + y = 69$ and $x - y = 1$, giving $x = 35$ and $y = 34$, or $x + y = 23$ and $x - y = 3$, giving $x = 13$ and $y = 10$. This gives the pair of triplets $(2001, 69020, 69049)$ and $(2001, 7540, 7801)$.

Similarly, $k = 23$ gives the pair of triplets $(2001, 87032, 87055)$ and $(2001, 9568, 9775)$; and $k = 3$ gives the pair of triplets $(2001, 667332, 667335)$ and $(2001, 468, 2055)$.

If $k = 1$, then $x^2 - y^2 = 2001 = 3 \times 23 \times 29$. Then there are four ways to factor 2001 as the product of two factors: $2001 = 2001 \times 1 = 667 \times 3 = 87 \times 23 = 69 \times 29$. These four factorizations yield the four triplets $(2001, 2002000, 2002001)$, $(2001, 222440, 222449)$, $(2001, 3520, 4049)$, and $(2001, 1960, 2801)$.

Case 2: $(x^2 + y^2)k = 2001$. Again, k is either 1, 3, 23, 29, 69, 87, 667, or 2001. But $x^2 + y^2$ must leave a remainder of 1 upon division by 4, and 2001 also leaves a remainder of 1 upon division by 4, so that k itself must leave a remainder of 1 upon division by 4. Thus, the possibilities for k are 2001, 69, 29, and 1.

If $k = 2001$, then $x^2 + y^2 = 1$, but this is impossible when x and y are positive integers.

If $k = 69$, then $x^2 + y^2 = 29$, and checking gives the only solution $x = 5$ and $y = 2$, giving the triplet $(1449, 1380, 2001)$.

If $k = 29$, then $x^2 + y^2 = 69$, and checking shows there are no integer solutions.

If $k = 1$, then $x^2 + y^2 = 2001$, and checking shows there are no integer solutions.

M9. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Find integers a, b, and c (not all equal) with $a + b + c = 2001$, such that a, b, and c form an arithmetic sequence (in that order) and $a + b$, $b + c$, and $c + a$ form a geometric sequence (in that order).

Solutions by Mihály Bencze, Brasov, Romania, and Paul Jefferys, Berkhamsted Collegiate School, UK. One incorrect solution was received. We give the solution of Bencze.

Since a, b, and c form an arithmetic series, let $a = b - r$ and $c = b + r$ for some integer r.

Then

$$\begin{aligned} a + b + c &= 2001 \\ b - r + b + b + r &= 2001 \\ 3b &= 2001 \\ b &= 667\,. \end{aligned}$$

Then, since $a + b$, $b + c$, and $c + a$ form a geometric sequence we have

$$\begin{aligned} (a + b)(c + a) &= (b + c)^2 \\ (2b - r)(2b) &= (2b + r)^2 \\ 4b^2 - 2br &= 4b^2 + 4br + r^2 \\ 6br + r^2 &= 0 \\ r(6b + r) &= 0\,. \end{aligned}$$

Either $r = 0$ (inadmissible since a, b, and c are not all equal), or $r = -6b = -4002$.

Then we get $a = 4669$, $b = 667$, and $c = -3335$.

M10. *Proposed by Nicolae Gustia, North York, Ontario.*
(a) Factor fully $2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$.

(b) Find the geometric interpretation of the above expression if a, b, and c are sides of a non-degenerate triangle.

I. *Solution to* (a) *by Mihály Bencze, Brasov, Romania.*

$$\begin{aligned}
&2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \\
&\quad= 4a^2b^2 - 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \\
&\quad= 4a^2b^2 - (2a^2b^2 - 2b^2c^2 - 2c^2a^2 + a^4 + b^4 + c^4) \\
&\quad= 4a^2b^2 - (a^2 + b^2 - c^2)^2 \\
&\quad= (2ab - (a^2 + b^2 - c^2))(2ab + (a^2 + b^2 - c^2)) \\
&\quad= (c^2 - (a^2 - 2ab + b^2))((a^2 + 2ab + b^2) - c^2) \\
&\quad= (c - (a - b))(c + (a - b))((a + b) - c)((a + b) + c) \\
&\quad= (a + b - c)(b + c - a)(c + a - b)(a + b + c) .
\end{aligned}$$

II. *Solution to* (b) *by proposer.*
If a, b, c are the sides of a scalene triangle, then if we let $a+b+c = 2s$, we get

$$\begin{aligned}
(b + c - a) &= 2(s - a) \\
(c + a - b) &= 2(s - b) \\
(a + b - c) &= 2(s - c) ,
\end{aligned}$$

so that $(a+b-c)(b+c-a)(c+a-b)(a+b+c) = 16s(s-a)(s-b)(s-c)$. But from Heron's formula, we know $A = \sqrt{s(s-a)(s-b)(s-c)}$. Thus, $2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = 16A^2$ where a, b, c are the sides of a scalene triangle with area A.

[Ed. This is still valid without the restriction "scalene".]

M11. *Proposed by the Mayhem staff.*
Two sequences a_1, a_2, $\ldots$, a_{2001} and b_1, b_2, $\ldots$, b_{2001} are formed by the following rules:

- $a_1 = 5$ and $a_2 = 3$,
- $b_1 = 9$ and $b_2 = 7$,
- $\dfrac{a_n}{b_n} = \dfrac{a_{n-1} + a_{n-2}}{b_{n-1} + b_{n-2}}$ for $n > 2$ such that each $\dfrac{a_n}{b_n}$ is in lowest terms.

What is the smallest fraction of the form $\dfrac{a_n}{b_n}$?

Solutions by Mihály Bencze, Brasov, Romania, and Geneviève Lalonde, Massey, ON. One incorrect solution was received. We give the solution of Lalonde.

First note that if $\frac{a}{b} < \frac{c}{d}$, where both fractions are in lowest terms (and a, b, c, d are positive integers), then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof: Since $\frac{a}{b} < \frac{c}{d}$ then $ad < cb$, giving

$$\begin{aligned} \frac{a+c}{b+d} &= \frac{b(a+c)}{b(b+d)} = \frac{ab+cb}{b(b+d)} \\ &> \frac{ab+ad}{b(b+d)} = \frac{a(b+d)}{b(b+d)} \\ &= \frac{a}{b}\,. \end{aligned}$$

We can similarly prove the other inequality. Thus, if we let $c_n = \frac{a_n}{b_n}$, we see that every time we create a new term, it is in between the two previous terms. Therefore, we get

$$c_2 < c_4 < c_6 < c_8 < \cdots < c_{2n} < \cdots < c_{2n-1} < \cdots < c_7 < c_5 < c_3 < c_1\,,$$

so that the smallest fraction is $c_2 = \frac{3}{7}$.

M12. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Determine all ordered pairs (x, y) with $\gcd(x, y) = 1$, and $x < y$ such that $2000\left(\frac{x}{y} + \frac{y}{x}\right)$ is an odd integer.

Solution by Paul Jefferys, Berkhamsted Collegiate School, UK.

$$2000\left(\frac{x}{y} + \frac{y}{x}\right) = 2000\left(\frac{x^2+y^2}{xy}\right)\,.$$

Since $\gcd(x, y) = 1$, xy shares no common factors with $x^2 + y^2$. Thus, if $2000\left(\frac{x^2+y^2}{xy}\right)$ is to be an integer, xy must divide 2000. Further, 16 must divide xy, since the expression is to be odd. Then, since $\gcd(x, y) = 1$, we have either 16 divides x or 16 divides y.

Case 1: 16 divides x.

If $x > 16$, then since x divides 2000, x is at least $16 \times 5 = 80$, in which case x would be greater than y since xy must divide 2000. Thus, $x = 16$. Then $y > x$ and y divides 2000. But y is also odd, so that the possibilities for y are 25 and 125.

Case 2: 16 divides y.

If 5 divides x, then y divides 2000 but 5 does not divide y. Thus, $y = 16$ and $x = 5$ since $x < y$.

If 5 divides y, then $x = 1$ since x divides 2000 but neither 2 nor 5 divide x. Then the possibilities for y are 80, 400, and 2000.

If 5 divides neither x nor y, then $x = 1$ and $y = 16$.

Therefore, in the end, we have the pairs $(16, 25)$, $(16, 125)$, $(5, 16)$, $(1, 16)$, $(1, 80)$, $(1, 400)$, and $(1, 2000)$.

Advanced Solutions

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A264. *Proposed by Mohammed Aassila, Strasbourg, France.*
Prove that for any integer a and natural number m,

$$a^m \equiv a^{m-\phi(m)} \pmod{m} .$$

(This is a generalization of Euler's theorem.)

Solution by Kenneth Williams, Carleton University, Ottawa.
For $\mathbb{N} = \{1, 2, 3, \ldots, \}$ and a, $m \in \mathbb{N}$, we are required to solve

$$a^m \equiv a^{m-\phi(m)} \pmod{m} .$$

If $a = 1$ or $m = 1$, the result is trivial. We may suppose $a > 1$ and $m > 1$.

Let

$$m_1 = \prod_{\substack{p \mid m \\ p \mid a}} p^{\nu_p(m)} , \qquad m_2 = \prod_{\substack{p \mid m \\ p \nmid a}} p^{\nu_p(m)} ,$$

where $p^{\nu_p(m)} \mid m$, and p is prime.

Clearly, $m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N}$, $m_1 m_2 = m$, $(m_1, m_2) = 1$, and $(m_2, a) = 1$.

Since $(m_2, a) = 1$, we have $a^{\phi(m_2)} \equiv 1 \pmod{m_2}$.

Since $(m_1, m_2) = 1$, we have $\phi(m) = \phi(m_1 m_2) = \phi(m_1)\phi(m_2)$, so that

$$a^{\phi(m_1 m_2)} = a^{\phi(m_1)\phi(m_2)} = \left(a^{\phi(m_2)}\right)^{\phi(m_1)} \equiv 1 \pmod{m_2} .$$

Hence, $m_2 \mid a^{\phi(m)} - 1$.

Next, let p be a prime dividing m_1, so that

$$\nu_p(m) \geq 1,\ \nu_p(a) \geq 1,\ p^{\nu_p(m)} \mid m_1 .$$

Then,

$$\begin{aligned}
\nu_p(m) &\le 2^{\nu_p(m)-1} \;\le\; p^{\nu_p(m)-1} \\
&\le (p-1)p^{\nu_p(m)-1} \\
&\le (p-1)p^{\nu_p(m)-1}\phi\left(\frac{m}{p^{\nu_p(m)}}\right) \\
&= p^{\nu_p(m)}\phi\left(\frac{m}{p^{\nu_p(m)}}\right) - p^{\nu_p(m)-1}\phi\left(\frac{m}{p^{\nu_p(m)}}\right) \\
&\le p^{\nu_p(m)}\left(\frac{m}{p^{\nu_p(m)}}\right) - \phi\left(p^{\nu_p(m)}\left(\frac{m}{p^{\nu_p(m)}}\right)\right) \\
&= m - \phi(m) \;\le\; \nu_p(a)\left(m - \phi(m)\right) ,
\end{aligned}$$

so that

$$m_1 = \prod_{\substack{p \mid m \\ p \mid a}} p^{\nu_p(m)} \left| \prod_{\substack{p \mid m \\ p \mid a}} p^{(m-\phi(m))\nu_p(a)} \right| \prod_{p \mid a} p^{(m-\phi(m))\nu_p(a)} ,$$

giving $m_1 \mid a^{m-\phi(m)}$.

Hence,

$$m_1 m_2 \mid a^{m-\phi(m)}\left(a^{\phi(m)} - 1\right) ,$$

so that

$$m \mid a^m - a^{m-\phi(m)} .$$

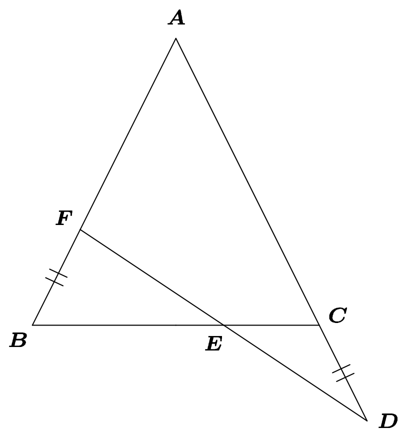
Also solved by MICHEL BATAILLE, Rouen, France.

Polya's Paragon

Paul Ottaway

I would like to begin by examining a problem that I left at the end of last month's article. Here is the problem again to refresh your memory:

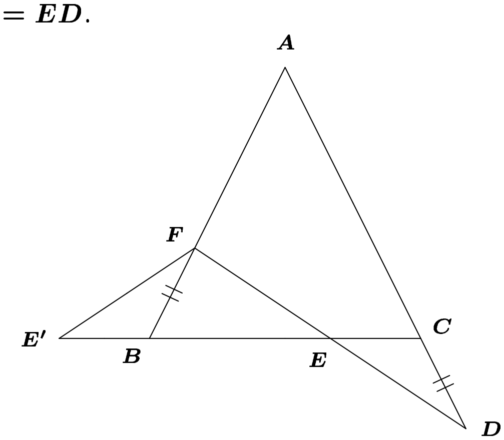
In the given diagram, triangle ABC is isosceles with $AB = AC$. We extend AC to D and construct F on AB such that $CD = BF$. E is the intersection of DF and BC. We would like to show that $EF = ED$.



The reason I am revisiting this problem is because of the variety of solutions. One in particular uses a very clever technique. Here are a couple of the solutions that I have discovered:

Solution #1:

Extend CB to E' such that triangle ECD is congruent to triangle $E'BF$. Now, $\angle FE'E = \angle DEC = \angle FEE'$. Therefore, triangle FEE' is isosceles with $FE = FE' = ED$.



Solution #2:

Using the sine law multiple times, we discover the following:

$$\frac{\sin(\angle EBF)}{EF} = \frac{\sin(\angle BEF)}{BF} = \frac{\sin(\angle CED)}{CD} = \frac{\sin(\angle ECD)}{ED}$$

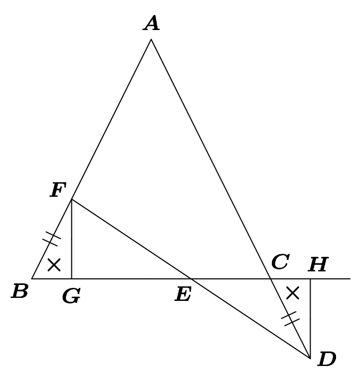
We also know that $\angle EBF = \angle ECA$, $\angle ECD = 180° - \angle ECA$ and that $\sin(x) = \sin(180° - x)$ so that $\sin(\angle EBF) = \sin(\angle ECD)$. Plugging that into our previous relation we see that

$$1 = \frac{\sin(\angle EBF)}{\sin(\angle ECD)} = \frac{EF}{ED}$$

Therefore, $EF = ED$ as desired.

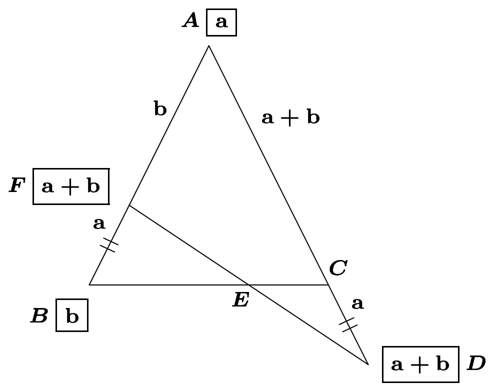
Solution #3:

Drop perpendiculars from D and F to the line BC extended as shown. Note that since $\angle FBG = \angle HCD$ and $FB = CD$ then it must be the case that the triangles FBG and DCH are congruent. Triangle EGF has all its angles the same as in triangle EHD and $FG = HD$ from our previous work, so that we know these two triangles are also congruent. It therefore follows that $FE = ED$.



There are many other solutions possible, each with its own uniqueness. The next (and last) solution I will present needs a little bit of background first. A high school student I met over the summer showed this solution to me. He used a truly beautiful method most commonly known as "mass points".

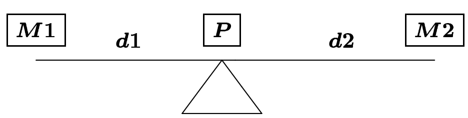
The idea is simple. We assign a "mass" to a vertex and balance, like a teeter-totter, the other vertices by assigning them "masses" as well. To begin, we need to know a little about physics and how to balance a teeter-totter. The following diagram shows two masses M_1 and M_2 balanced perfectly about a pivot point (fulcrum) P. In order for the system to be balanced $M_1 \times d_1 = M_2 \times d_2$. This should hopefully make some intuitive sense. If one object is heavier than the other it would have to be placed closer to the fulcrum in order for the balance to be maintained. The other piece of information that can be gained is the mass at P. If we were holding this contraption at the point P, it would naturally seem to have the mass $M_1 + M_2$. For simplicity, I will ensure that masses appear in boxes so that they are not confused with distances or labels for points.



I will not prove these results, but they are true and can be used in a variety of geometry problems, making them much easier to solve.

Here is an example of how mass points can be used to solve a rather difficult problem:

In the given diagram, we know that $AP : PB = 3 : 4$ and $AQ : QC = 3 : 2$. Prove that C is the mid-point of BD.



Solution:

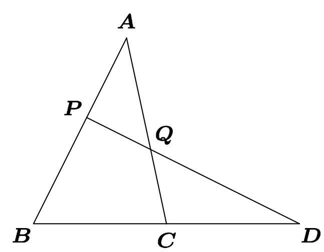
We begin by assigning a mass of 4 at the point A. We can say that $AP = 3x$ and $PB = 4x$. From this, in order to have BPA balanced at P, we need a mass of 3 at B. We can also conclude that P has a mass of 7 (the sum of the masses at A and B). If we repeat this process with the line AQC we let $AQ = 3y$ and $QC = 2y$. Now, we see that to have balance we need a mass of 6 at C and therefore, a mass of 10 at Q. Since we know that any point between two others must have the same mass as the sum of the other two we can examine the line PQD to find that the mass at D must be 3. Finally, looking along BCD, we see that all of the masses have already been determined and that $B = 3$, $C = 6$ and $D = 3$. Since the mass at B and D are equal and C is their sum, we must conclude that $BC = CD$ and thus, C is at the mid-point.

This solution seems much simpler to me than any alternative that comes to mind. Can you see another way to solve the problem? I assure you there is!

Let us now return to our original problem to see how it can be solved using mass points.

Solution #4:

Since we do not know specific distances or ratios, we begin by assigning the variable a to lengths CD and BF. We also let $AF = b$ and therefore, $AC = a + b$. Now, we assign a mass of a at point A. We continue by finding the mass at B is b and the mass at F is $a + b$ using the line AFB. Now, using the line ACD we find that the mass at D is $a + b$. At this point, we can stop and examine the line EFD. Since both endpoints have the same mass, they must both be the same distance from the fulcrum at E. Therefore, $FE = ED$ as required.



If you feel so inclined, continue to fill in the masses of the remaining points just to make sure the system as a whole is sound.

Overall, this is a technique that is rarely seen these days and has the power to deliver surprisingly simple solutions. It is an excellent tool that can help approach geometry problems that can sometimes seem too tough to tackle. What I have shown here is only a very brief introduction to the topic and those that find this interesting should look for more information from further sources.

Here are a couple more problems for you to try on your own:

1. The medians of a triangle concur at a point called the centroid or centre of mass. Show that this point divides each of the medians into $2 : 1$ ratios.

2. A tetrahedron is a three dimensional object that has 4 triangular faces. On each face, construct the centroid. From each centroid, construct the line connecting it to the opposite vertex of the tetrahedron. These 4 lines intersect at a point. Find the ratio in which these 4 lines are divided by this point.

3. (**AHSME 1964 #35**) - The sides of a triangle are of lengths 13, 14, and 15. The altitudes of the triangle meet at point H. If AD is the altitude to the side of length 14, what is the ratio $HD : HA$?

SKOLIAD No. 64

Shawn Godin

Solutions may be sent to Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5, or emailed to

mayhem-editors@cms.math.ca.

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 February 2003*. A copy of **MATHEMATICAL MAYHEM Vol. 6** will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

Our item this issue is the 1991 Canadian Mathematical Society Prize Exam. The exam was written by students in grades 11 and 12. No calculators, tables or drawing instruments are allowed. All questions are of equal value. My thanks go out to Michael Nutt at Acadia University for forwarding the material to me.

Canadian Mathematical Society Prize Exam

Nova Scotia: Friday, April 26, 1991

9:00 - 11:30 am

1. Show that $1-\frac{2}{3}+\frac{3}{9}-\frac{4}{27}+\frac{5}{81}-\dots\frac{100}{3^{99}}=\frac{9}{16}\left[1-\frac{403}{3^{101}}\right]$.
2. Solve for all real x: $\sqrt{x^2-x+2}+\sqrt{x^2-x-2}=1$.
3. Two circles of equal radii pass through each other's centres. What are
   (a) the perimeter and
   (b) the area of the whole region enclosed by the circles?
4. When the polynomial $x^4+ax^3-7x^2+bx-49$ is divided by $(x-3)$ the remainder is 53, and by $(x+2)$ the remainder is -87. Find a and b.
5. From the letters of the word "antenna", we want to make all possible four letter "words" (they may be nonsensible, for example, "aann"). How many can we make?
6. If a and b are positive integers larger than 2, prove that (2^a+1) cannot be divisible by (2^b-1).

Next we return to the contests from South Africa given in the February issue. First, we give the answers to the University of Cape Town Mathematics Competition for grades 9 and 10 [2002 : 37].

1.	(4)	**2**.	(1)	**3**.	(5)	**4**.	(2)	**5**.	(3)
6.	(2)	**7**.	(1)	**8**.	(1)	**9**.	(2)	**10**.	(3)
11.	(3)	**12**.	(1)	**13**.	(1)	**14**.	(3)	**15**.	(3)
16.	(2)	**17**.	(5)	**18**.	(4)	**19**.	(2)	**20**.	(3)
21.	(2)	**22**.	(1)	**23**.	(1)	**24**.	(1)	**25**.	(5)
26.	(3)	**27**.	(2)	**28**.	(1)	**29**.	(4)	**30**.	(5)

Next we present the solutions to the Interprovincial Mathematics Olympiad from 2001 [2002 : 42].

S1. Find the largest integer which cannot be expressed in the form $7a + 11b + 13c$, where a, b and c are integers, with $a \geq 0$, $b \geq 0$ and $c \geq 0$.

Solution.

First, note that it is impossible to express 30 in the form $7a+11b+13c$, since if 30 could be expressed in such a way, then one or more of 23, 19, and 17 could also be expressed in the form $7a + 11b + 13c$. A straightforward check reveals that it is impossible to express 23, 19, and 17 in the desired manner.

Next, observe the following expressions:

$$7 + 11 + 13 = 31;\ 7 \times 3 + 11 = 32;\ 7 + 13 \times 2 = 33;\ 7 \times 3 + 13 = 34;$$
$$7 \times 5 = 35;\ 7 \times 2 + 11 \times 2 = 36;\ 11 + 13 \times 2 = 37.$$

We can express seven consecutive integers in the desired manner. Since we can add multiples of seven to these consecutive integers as we please, we can express every integer greater than 30 in the form $7a + 11b + 13c$.

S2. The 5–digit number $32\ldots1\ldots$ is divisible by 156. What is the number?

Solution.

$156 \times 211 = 32916$. The number is 32916.

S3. Eight boxes, each a unit cube, are packed in a $2 \times 2 \times 2$ crate, open at the top. The boxes are taken out one by one. In how many ways can this be done? (Remember that a box in the bottom layer can only be removed after the box above it has been removed.)

Solution.

Suppose we number the cubes on the top layer 1, 2, 3, 4 and the ones underneath 1, 2, 3, 4, so that the cubes on the bottom layer have the same number as the cube above them. Thus, as we take the cubes out the top 1 must come out before the bottom 1. Thus, if we look at the number of distinct permutations of 1, 1, 2, 2, 3, 3, 4, 4, it will describe each possible way to unpack the cubes (the first occurrence of a number is the cube from the top row and the second is from the bottom). Thus, the total number of ways to unpack the crate is $\frac{8!}{2!2!2!2!} = 2520$.

S4. How many integers between 1 and 1000 cannot be expressed as the difference between the squares of two integers?

Solution.

Suppose for n between 1 and 1000 that $n = x^2 - y^2$. Then $n = (x - y)(x + y)$. Thus, if we can find an integer solution to the system

$$x - y = a \qquad (1)$$
$$x + y = b \qquad (2)$$

for any pair (a, b) of factors for n, then we can write n as the difference between the squares of two integers. Now, (1) + (2) gives $2x = a + b$, whereupon the system has integer solutions so long as we can find a pair of factors for n with an even sum. We can certainly do this if n is odd by taking 1 and n as the factors. We can also do this whenever n has two even factors; that is, whenever n is divisible by 4. Thus, we are unable to write n as the difference between the squares of two integers only when n is an even integer not divisible by 4. Between 1 and 1000, these numbers are

$$2,\ 6,\ 10,\ \ldots,\ 998.$$

There are 250 such numbers.

S5. Find the smallest positive integer which has a factor ending in 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Solution.

Since 1 divides any positive integer, that is taken care of. To end with 0, 2, 4, 6, or 8 the number must be even so that 2 must be a factor. Similarly to end with 5, we must have 5 as a factor. Since 2 and 5 are factors of the number, then so is 10, which ends with 0. To get a factor ending with 3, 7 or 9, we need another odd prime factor other than 5. If we choose 3, we also take care of 6.

Thus, so far we have $2 \times 3 \times 5 = 30$ which has factors ending with 0, 1, 2, 3, 5, 6. To get 7, introduce a 7, but we also get 4 since $7 \times 2 = 14$ is also a factor. All that is left is 8 and 9. If we introduce another factor of 3 we get 9 and 8 (since $3^2 = 9$ and $2 \times 3^2 = 18$). Thus, $2 \times 3^2 \times 5 \times 7 = 630$ is the smallest number with factors ending in 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

S6. A circle is inscribed in quadrilateral $ABCD$. The sides BC and DA have the same lengths, and the sides AB and CD are parallel, with lengths 9 and 16, respectively. What is the radius of the circle?

Solution.

Let the points of contact between the circle and quadrilateral $ABCD$ be E, F, G, and H on AB, BC, CD, and DA respectively. Then E is the mid-point of AB and G is the mid-point of CD. Then, since EB and BF are both tangent to the circle, and since $EB = 4.5$, we get $BF = 4.5$. Similarly, $FC = 8$. Thus, $BC = 12.5$ and hence $DA = 12.5$.

Drop a perpendicular from B to CD at I. Then clearly $EG = BI$. But

$$BI^2 \;=\; BC^2 - CI^2 \;=\; 12.5^2 - 3.5^2 \;=\; 144\,.$$

Hence, $BI = EG = 12$. Since EG is a diameter of the circle, the radius of the circle is 6.

S7. For how many integers n between 1 and 2002 is the improper fraction $\dfrac{n^2+4}{n+5}$ **NOT** in lowest terms?

Solution.

Suppose, for some n between 1 and 2002, that $\dfrac{n^2+4}{n+5}$ is not in lowest terms. Then there is some integer d greater than 1 such that d divides both n^2+4 and $n+5$. Now, d divides $n+5$ implies d divides $(n+5)^2 = n^2+10n+25$. Hence, d divides $n^2+10n+25-(n^2+4) = 10n+21$.

Continuing along: d divides $n+5$ implies d divides $10n+50$, and then d divides $10n+50-(10n+21) = 29$. Since $d > 1$ we get $d = 29$.

Thus, $\dfrac{n^2+4}{n+5}$ is not in lowest terms only if 29 divides $n+5$.

Now, suppose 29 divides $n+5$. Then $n+5 = 29k$ for some positive integer k, and we get $n = 29k-5$, so that $n^2 = 841k^2 - 145k + 25$, and then $n^2+4 = 841k^2-145k+29 = 29(29k^2-5k+1)$, and we see that n^2+4 is automatically divisible by 29.

Hence, $\dfrac{n^2+4}{n+5}$ is not in lowest terms if and only if n is divisible by 29. There are 69 multiples of 29 between 1 and 2002.

S8. Solve the inequality $\log_{\sqrt{3}}(2-x) + 4\log_9(6-x) > 2$.

Solution.

Let $y = 4\log_9(6-x)$. That is, $9^y = (6-x)^4$, or $\left(\sqrt{3}\right)^y = 6-x$. From the definition of logarithms, then, we get that $y = \log_{\sqrt{3}}(6-x)$.

Hence, $\log_{\sqrt{3}}(2-x) + 4\log_9(6-x) > 2$ implies
$\log_{\sqrt{3}}(2-x) + \log_{\sqrt{3}}(6-x) > 2$
or $\log_{\sqrt{3}}(2-x)(6-x) > 2$ or $(2-x)(6-x) > \left(\sqrt{3}\right)^2$
or $12 - 8x + x^2 > 3$ or $x^2 - 8x + 9 > 0$.

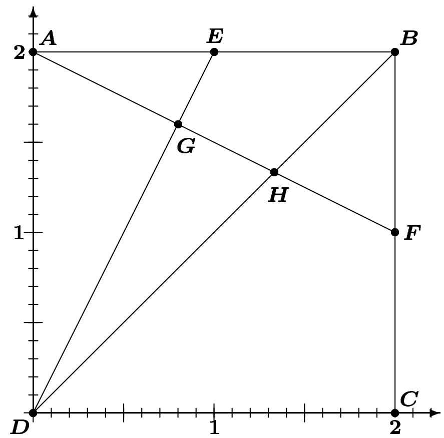
The quadratic equation $x^2-8x+9=0$ has roots $4+\sqrt{7}$ and $4-\sqrt{7}$, so that the inequality is satisfied whenever $x < 4-\sqrt{7}$ or $x > 4+\sqrt{7}$.

S9. $ABCD$ is a 2×2 square and E and F are the mid-points of AB and BC, respectively. If AF intersects ED and BD at G and H, respectively, what is the area of quadrilateral $BEGH$?

Solution.

If we draw the square on a grid as in the diagram below we see that the slopes of AF and DE are $-\frac{1}{2}$ and 2 respectively. Thus, when they meet at G, the distance from G to AB and CD is in the ratio of $1 : 4$. Therefore, the height of $\triangle AGE$ is $\frac{2}{5}$ and its area is $\frac{1}{5}$.

Similarly, the distance from H to DA and BC is in the ratio of $2 : 1$. Thus, the height of $\triangle BHF$ is $\frac{2}{3}$ and its area is $\frac{1}{3}$. Thus, since the area of $\triangle ABF$ is 1, the area of quadrilateral $BEGH$ is $1 - \dfrac{1}{5} - \dfrac{1}{3} = \dfrac{7}{15}$.



S10. Points A, B and C lie on a circle. The line AP is perpendicular to BC, with P on BC. If $AP = 6$, $BP = 4$ and $CP = 17$, find the radius of the circle.

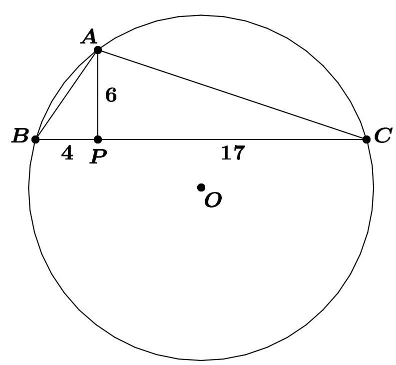
Solution.
The extended Law of Sines tells us

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where R is the radius of the circumscribing circle.

Thus, from our information, we get

$$\begin{aligned} R &= \frac{a}{2\sin A} \\ &= \frac{21}{2(\sin(\angle BAP + \angle PAC))} \\ &= \frac{21}{2\left(\dfrac{4}{2\sqrt{13}}\dfrac{6}{5\sqrt{13}} + \dfrac{6}{2\sqrt{13}}\dfrac{17}{5\sqrt{13}}\right)} \\ &= \frac{65}{6}. \end{aligned}$$



PROBLEMS

Faire parvenir les propositions de problèmes et les solutions à Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's (Terre-Neuve), Canada, A1C 5S7. Les propositions de problèmes doivent être accompagnées d'une solution ainsi que de références et d'autres indications qui pourraient être utiles à la rédaction. Si vous envoyez une proposition sans solution, vous devez justifier une solution probable en fournissant suffisamment d'information. Un numéro suivi d'une astérisque ($\star$) *indique que le problème a été proposé sans solution.*

Nous sollicitons en particulier des problèmes originaux. Cependant, d'autres problèmes intéressants pourraient être acceptables s'ils ne sont pas trop connus et si leur provenance est précisée. Normalement, si l'auteur d'un problème est connu, il faut demander sa permission avant de proposer un de ses problèmes.

Pour faciliter l'étude de vos propositions, veuillez taper ou écrire à la main (lisiblement) chaque problème sur une feuille distincte de format $8\frac{1}{2}$"$\times$11" *ou* A4, *la signer et la faire parvenir au rédacteur en chef. Les propositions devront lui parvenir au plus tard le* **1er avril 2003**. *Vous pouvez aussi les faire parvenir par courriel à* **crux-editors@cms.math.ca**. (*Nous apprécierions de recevoir les problèmes et solutions envoyés par courriel au format* LaTeX). *Les fichiers graphiques doivent être de format « epic » ou « eps » (encapsulated postscript). Les solutions reçues après la date ci-dessus seront prises en compte s'il reste du temps avant la publication. Veuillez prendre note que nous n'acceptons pas les propositions par télécopieur.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

2760. [2002 : 332] Correction *Proposed by Michel Bataille, Rouen, France.*

Suppose that A, B, C are the angles of a triangle. Prove that

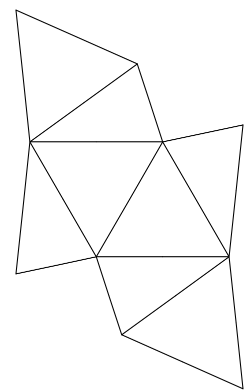
$$\begin{aligned} 8(\cos A+\cos B+\cos C) &\leq 9+\cos(A-B)+\cos(B-C)+\cos(C-A) \\ &\leq \csc^2(A/2)+\csc^2(B/2)+\csc^2(C/2)\,. \end{aligned}$$

. . . .

Soit A, B, C les angles d'un triangle. Montrer que

$$\begin{aligned} 8(\cos A+\cos B+\cos C) &\leq 9+\cos(A-B)+\cos(B-C)+\cos(C-A) \\ &\leq \csc^2(A/2)+\csc^2(B/2)+\csc^2(C/2)\,. \end{aligned}$$

2763. *Proposé par Izidor Hafner, Faculty of Electrical Engineering, Ljubljana, Slovénie.*



Le développement montré à la droite consiste en 4 triangles équilatéraux dont les côtés mesurent 2τ (deux fois le nombre d'or, $\tau = \frac{\sqrt{5}+1}{2}$), et 4 triangles isocèles dont le petit côté mesure 2. Noter qu'en pliant le développement, on peut obtenir deux polyèdres convexes. Ont-ils le même volume ?

. . . .

The net shown to the right consists of four equilateral triangles whose sides have length 2τ (twice the golden ratio, $\tau = \frac{\sqrt{5}+1}{2}$), and four isosceles triangles whose short side has length 2. Observe that there are two convex polyhedra that can be obtained by folding up the net. Do they have equal volumes ?

2764. *Proposé par Christopher J. Bradley, Clifton College, Bristol, UK.*

Trouver un triangle scalène à côtés entiers dans lequel toutes les bissectrices intérieures ont des longueurs entières

. . . .

Find a integer-sided scalene triangle in which the lengths of the internal bisectors all have integer values.

2765. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Trouver les longueurs des côtés de la famille des triangles de Héron ABC pour laquelle le centre du cercle des 9–points est situé sur le côté BC. (Les côtés et l'aire d'un triangle de Héron sont des entiers.) [Voir problème 2525(Avril) [2001 : 177 ; 2001 : 270]

. . . .

Derive a set of side length expressions for the family of Heron triangles ABC in which the nine-point centre V lies on side BC. (A Heron triangle has integer sides and integer area.) [See problem 2525(April) [2000 : 177; 2001 : 270].]

2766. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Dans un triangle de Héron, on suppose que les côtés a, b, c satisfont l'équation $b = a(a - c)$. Montrer que le triangle est isocèle. (Les côtés et l'aire d'un triangle de Héron sont des entiers.)

. . . .

In a Heron triangle, the sides a, b, c satisfy the equation $b = a(a - c)$. Prove that the triangle is isosceles. (A Heron triangle has integer sides and integer area.)

2767. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

On donne les points $O(0,0)$, $A(1,0)$, $B(0,1)$. Soit $D(a,0)$, $E(1-a,a)$, $F(0,1-a)$ des points variables sur les côtés du triangle OAB $(0 < a < 1)$. Soit P le point d'intersection des trois cercles. Déterminer le lieu de P.

. . . .

The points $O(0,0)$, $A(1,0)$, $B(0,1)$ are given. Let $D(a,0)$, $E(1-a,a)$, $F(0,1-a)$ be variable points on the sides of $\triangle OAB$ $(0 < a < 1)$. Let P denote the point of concurrence of the circles ODF, DEA and BFE. Determine the locus of P.

2768. *Proposé par Mohammed Aassila, Strasbourg, France.*

Soit x_1, x_2, $\ldots$, x_n n nombres réels positifs. Montrer que

$$\frac{x_1}{\sqrt{x_1x_2+x_2^2}}+\frac{x_2}{\sqrt{x_2x_3+x_3^2}}+\cdots+\frac{x_n}{\sqrt{x_nx_1+x_1^2}} \;\geq\; \frac{n}{\sqrt{2}}\,.$$

. . . .

Let x_1, x_2, $\ldots$, x_n be n positive real numbers. Prove that

$$\frac{x_1}{\sqrt{x_1x_2+x_2^2}}+\frac{x_2}{\sqrt{x_2x_3+x_3^2}}+\cdots+\frac{x_n}{\sqrt{x_nx_1+x_1^2}} \;\geq\; \frac{n}{\sqrt{2}}\,.$$

2769. *Proposé par Aram Tangboondouangjit, étudiant, University of Maryland, College Park, Maryland USA.*

Dans un triangle ABC, on suppose que $\cos B - \cos C = \cos A - \cos B \geq 0$. Montrer que

$$\left(b^2+c^2\right)\cos A - \left(a^2+b^2\right)\cos C \;\geq\; \left(c^2-b^2\right)\sec B\,.$$

. . . .

In $\triangle ABC$, suppose that $\cos B - \cos C = \cos A - \cos B \geq 0$. Prove that

$$\left(b^2+c^2\right)\cos A - \left(a^2+b^2\right)\cos C \;\geq\; \left(c^2-b^2\right)\sec B\,.$$

2770. *Proposé par Aram Tangboondouangjit, étudiant, University of Maryland, College Park, Maryland USA.*

Dans un triangle ABC, on suppose que $a \leq b \leq c$ et $\angle ABC \neq \frac{\pi}{2}$. Montrer que

$$2+\sec B \;\leq\; \left(1+\frac{b}{a}\right)\left(1+\frac{b}{c}\right)\,.$$

. . . .

In $\triangle ABC$, suppose that $a \leq b \leq c$ and $\angle ABC \neq \frac{\pi}{2}$. Prove that

$$2+\sec B \;\leq\; \left(1+\frac{b}{a}\right)\left(1+\frac{b}{c}\right)\,.$$

2771★. *Proposé par Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Trouver toutes les paires d'entiers positifs a et b telles que

$$(a+b)^b = a^b + b^a.$$

. . . .

Find all pairs of positive integers a and b such that

$$(a+b)^b = a^b + b^a.$$

2772. *Proposé par Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$f(x)\, f(yf(x)-1) = x^2 f(y) - f(x) \text{ pour tous les réels } x \text{ et } y.$$

. . . .

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)\, f(yf(x)-1) = x^2 f(y) - f(x) \text{ for all real } x \text{ and } y.$$

2773. *Proposé par Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China et Wu Kang, South China Normal University, Guang Zhou City, Guang Dong Province, Chine.*

(a) Trouver toutes les suites de nombres naturels x_1, x_2, $\ldots$, x_n, $\ldots$, telles que ij divise $x_i + x_j$ pour toute paire d'entiers positifs i et j.

. . . .

(a) Find all sequences of integers x_1, x_2, $\ldots$, x_n, $\ldots$, such that ij divides $x_i + x_j$ for any two distinct positive integers i and j.

2774. *Proposé par Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Soit x un nombre réel tel que $0 < x \leq \frac{2}{9}\pi$. Montrer que

$$(\sin x)^{\sin x} \; < \; \cos x\,.$$

(Ceci est une généralisation du problème 10261 dans le American Mathematical Monthly [1992 : 872, 1994 : 690]).

. . . .

Let x be a real number such that $0 < x \leq \frac{2}{9}\pi$. Prove that

$$(\sin x)^{\sin x} \; < \; \cos x\,.$$

(This is a generalization of Problem 10261 in the American Mathematical Monthly [1992 : 872, 1994 : 690]).

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2568. [2000 : 373, 2001 : 416] *Proposed by K.R.S. Sastry, Bangalore, India.*

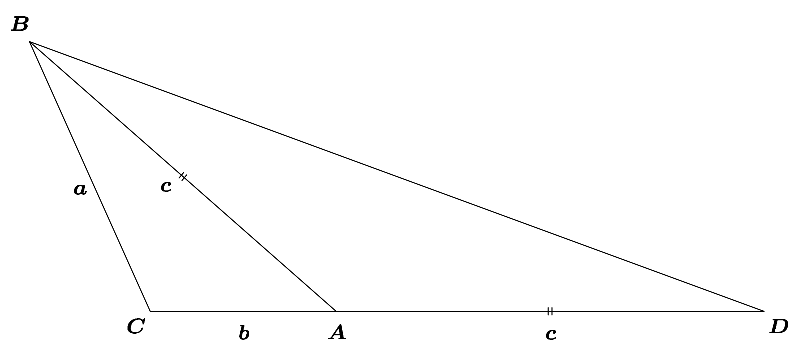
The sides a, b and c of a non-degenerate triangle ABC satisfy the relations $b^2 = ca + a^2$ and $c^2 = ab + b^2$. Find the measures of the angles of triangle ABC.

[Editor's note: A solution to this problem by Václav Konečný appeared in CRUX [2001 : 416]. The solution relied heavily on trigonometry. The following solution was submitted later and is featured here since it is quite elegant and does not use trigonometry.]

Solution by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

In [1] the following problem and solution appears (loosely translated from the Spanish by the editors):

349. Show that if among the sides a, b, and c of a triangle the relation $a^2 = b^2 + bc$ holds, then the angles A and B opposite the sides a and b, respectively, satisfy the equation $\angle A = 2\angle B$.



Proof: Let $\triangle ABC$ be given (as in the diagram above). Let D be on the side AC produced in the direction of A such that $AD = c$. From the equation $a^2 = b^2 + bc$, it follows that

$$\frac{a}{b} = \frac{b+c}{a}.$$

This implies that $\triangle CAB$ and $\triangle CBD$ are similar and $\angle A = \angle CBD$. Furthermore, $\angle B = \angle BDA = \angle DBA$.

Consequently, $\angle A = \angle B + \angle DBA = 2\angle B$.

A solution to #2568 now proceeds as follows:

$$\begin{aligned} b^2 &= ca + a^2 &\Longrightarrow\quad \angle B &= 2\angle A \\ c^2 &= ab + b^2 &\Longrightarrow\quad \angle C &= 2\angle B = 4\angle A\,. \end{aligned}$$

Since $\pi - \angle A = \angle B + \angle C = 6\angle A$, we conclude that

$$\angle A = \frac{\pi}{7}\,, \qquad \angle B = \frac{2\pi}{7}\,, \qquad \angle C = \frac{4\pi}{7}\,.$$

[1] V. Lidski y otros: Problemas de Matemáticas Elementales, Mir, Moscú, 1978 (p.56, 236)

2645. [2001 : 269, 2002 : 278] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{2\left(a^3+b^3+c^3\right)}{abc} + \frac{9(a+b+c)^2}{a^2+b^2+c^2} \;\geq\; 33\,.$$

V. *Editor's comment.*

Šefket Arslanagić was the first to note the (obvious) error in solution II [2002 : 279]. The inequality $\frac{1}{3}(a+b+c)^2 \geq a^2+b^2+c^2$ is incorrect, which makes the entire solution II incorrect. The correct inequality is $\frac{1}{3}(a+b+c)^2 \leq a^2+b^2+c^2$ (AM–QM). The editors apologize for the error and promise to be constantly on high alert.

2656★. [2001 : 336] *Proposed by Vedula N. Murty, Dover, PA, USA.*

For positive real numbers a, b and c, show that

$$\frac{(1-b)(1-bc)}{b(1+a)} + \frac{(1-c)(1-ca)}{c(1+b)} + \frac{(1-a)(1-ab)}{a(1+c)} \;\geq\; 0\,.$$

Editor's comment.

It turns out that this inequality is false as stated. Several solvers sent us their counterexample. Perhaps readers might try restricting the reals a, b and c to the interval $[0,1]$.

2657. [2001 : 336] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Prove that

$$\sum_{n=0}^{2k-1} \tan\left(\frac{(4n-1)\pi + (-1)^n 4\theta}{8k}\right) \;=\; \frac{2k}{1+(-1)^{k+1}\sqrt{2}\sin\theta}\,.$$

Solution by Stanley Rabinowitz, Westford, MA, USA.

We start with the formulas for $\sin(nA)$ and $\cos(nA)$ which can be found in any book on advanced trigonometry, such as [1], page 52:

$$\begin{aligned}\cos(nA) &= (\cos^n A)\left[\binom{n}{0} - \binom{n}{2}t^2 + \binom{n}{4}t^4 - \cdots\right],\\ \sin(nA) &= (\cos^n A)\left[\binom{n}{1}t - \binom{n}{3}t^3 + \binom{n}{5}t^5 - \cdots\right],\end{aligned}$$

where $t = \tan A$. [Ed: These are immediate consequences of the binomial theorem and De Moivre's formula.] Dividing, we get

$$\begin{aligned}\cot(nA) &= \frac{\binom{n}{0} - \binom{n}{2}t^2 + \binom{n}{4}t^4 - \cdots}{\binom{n}{1}t - \binom{n}{3}t^3 + \binom{n}{5}t^5 - \cdots}\\ &= \frac{x^n - \binom{n}{2}x^{n-2} + \binom{n}{4}x^{n-4} - \cdots}{nx^{n-1} - \binom{n}{3}x^{n-3} + \binom{n}{5}x^{n-5} - \cdots},\end{aligned}$$

where $x = \frac{1}{t} = \cot A$. [Ed: Assuming $\sin A \neq 0$ and $\cos A \neq 0$.] Note that the equation

$$\begin{aligned}&x^n - \binom{n}{2}x^{n-2} + \binom{n}{4}x^{n-4} - \cdots\\ &\quad = (\cot(nA))\left[nx^{n-1} - \binom{n}{3}x^{n-3} + \binom{n}{5}x^{n-5} - \cdots\right],\end{aligned} \tag{1}$$

is an n^{th} degree polynomial equation in x. Since

$$\cot\left(n\left(A + \frac{j\pi}{n}\right)\right) = \cot(nA + j\pi) = \cot(nA)$$

for $j = 0, 1, 2, \ldots, n-1$, and since $\cot(A + \frac{j\pi}{n})$ are all distinct, they are exactly the n roots of the equation in (1). Hence, we have

$$\sum_{j=0}^{n-1} \cot\left(A + \frac{j\pi}{n}\right) = n\cot(nA). \tag{2}$$

Let S denote the summation on the left side of the given identity. Setting $n = 2r$ for the even-indexed terms, $n = 2r+1$ for the odd-indexed terms, and separating these terms, we have

$$\begin{aligned}S &= \sum_{r=0}^{k-1} \tan\left(\frac{(8r-1)\pi + 4\theta}{8k}\right) + \sum_{r=0}^{k-1} \tan\left(\frac{(8r+3)\pi - 4\theta}{8k}\right)\\ &= \sum_{r=0}^{k-1} \tan\left(\frac{r\pi}{k} + \frac{4\theta - \pi}{8k}\right) + \sum_{r=0}^{k-1} \tan\left(\frac{r\pi}{k} + \frac{3\pi - 4\theta}{8k}\right).\end{aligned}$$

Since $\tan x = -\cot(x+\frac{\pi}{2})$, we have by (2) that

$$\begin{aligned}
S &= -\sum_{r=0}^{k-1}\cot\left(\frac{r\pi}{k}+\frac{4\theta-\pi}{8k}+\frac{\pi}{2}\right)-\sum_{r=0}^{k-1}\cot\left(\frac{r\pi}{k}+\frac{3\pi-4\theta}{8k}+\frac{\pi}{2}\right)\\
&= -k\cot\left(k\left(\frac{4\theta-\pi}{8k}+\frac{\pi}{2}\right)\right)-k\cot\left(k\left(\frac{3\pi-4\theta}{8k}+\frac{\pi}{2}\right)\right)\\
&= -k\left(\cot\left(\frac{4\theta-\pi}{8}+\frac{k\pi}{2}\right)+\cot\left(\frac{3\pi-4\theta}{8}+\frac{k\pi}{2}\right)\right).
\end{aligned}$$

Using the half-angle formula $\cot(\frac{x}{2}) = \frac{1+\cos x}{\sin x}$, we then have

$$\begin{aligned}
S &= -k\left(\frac{1+\cos\left(\theta-\frac{\pi}{4}+k\pi\right)}{\sin\left(\theta-\frac{\pi}{4}+k\pi\right)}+\frac{1+\cos\left(\frac{3\pi}{4}-\theta+k\pi\right)}{\sin\left(\frac{3\pi}{4}-\theta+k\pi\right)}\right)\\
&= -k\left(\frac{1+(-1)^k\cos\left(\theta-\frac{\pi}{4}\right)}{(-1)^k\sin\left(\theta-\frac{\pi}{4}\right)}+\frac{1+(-1)^k\cos\left(\frac{3\pi}{4}-\theta\right)}{(-1)^k\sin\left(\frac{3\pi}{4}-\theta\right)}\right)\\
&= -k\left(\frac{(-1)^k+\cos\left(\theta-\frac{\pi}{4}\right)}{\sin\left(\theta-\frac{\pi}{4}\right)}+\frac{(-1)^k+\sin\left(\theta-\frac{\pi}{4}\right)}{\cos\left(\theta-\frac{\pi}{4}\right)}\right)\\
&= -k\left(\frac{(-1)^k[\sin\left(\theta-\frac{\pi}{4}\right)+\cos\left(\theta-\frac{\pi}{4}\right)]+1}{\sin\left(\theta-\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right)}\right). \qquad (3)
\end{aligned}$$

Since

$$\sin\left(\theta-\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right) = \frac{1}{2}\sin\left(2\theta-\frac{\pi}{2}\right) = -\frac{1}{2}\cos(2\theta)$$

and

$$\begin{aligned}
&\sin\left(\theta-\frac{\pi}{4}\right)+\cos\left(\theta-\frac{\pi}{4}\right)\\
&\quad= \sqrt{2}\left(\sin\left(\theta-\frac{\pi}{4}\right)\cos\frac{\pi}{4}+\cos\left(\theta-\frac{\pi}{4}\right)\sin\frac{\pi}{4}\right)\\
&\quad= \sqrt{2}\sin\theta,
\end{aligned}$$

we get from (3) that

$$S = 2k\left(\frac{(-1)^k\sqrt{2}\sin\theta+1}{\cos 2\theta}\right). \qquad (4)$$

But

$$\begin{aligned}
\cos 2\theta &= 1-2\sin^2\theta = 1-(\sqrt{2}\sin\theta)^2\\
&= \left(1-(-1)^k\sqrt{2}\sin\theta\right)\left(1+(-1)^k\sqrt{2}\sin\theta\right),
\end{aligned}$$

so that we finally obtain from (4) that

$$S = \frac{2k}{1-(-1)^k\sqrt{2}\sin\theta} = \frac{2k}{1+(-1)^{k+1}\sqrt{2}\sin\theta}.$$

Reference

[1] E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover Publications, Inc. New York 1957.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Both Bataille and Woo used complex numbers in their solutions which are shorter than, but not as elementary as the one published above. Liu was the only one who pointed out explicitly that the proposed identity makes sense only if $\theta \neq \frac{(8m-1)\pi}{4}$*, or* $\frac{(8m-3)\pi}{4}$ *when* k *is odd, and* $\theta \neq \frac{(8m+1)\pi}{4}$*, or* $\frac{(8m+3)\pi}{4}$ *when* k *is even, where* $m \in \mathbb{Z}$.

2659. [2001 : 337] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, the side BC is fixed. A is a variable point. Assume that $AC > AB$. Let M be the mid-point of BC, let O be the circumcentre of $\triangle ABC$, let R be the circumradius, let G be the centroid and H the orthocentre. Assume that the Euler line, OH, is perpendicular to AM.

1. Determine the locus of A.
2. Determine the range of $\angle BGC$.

Solution by Toshio Seimiya, Kawasaki, Japan.

1. Since G is the intersection of AM and OH, we have $\angle AGO = 90^\circ$. Since $OA = OC$ and $OM \perp MC$, we also have

$$AG^2 - GM^2 = AO^2 - OM^2 = OC^2 - OM^2 = MC^2.$$

Since $AG = 2GM$ it follows that $3GM^2 = MC^2$. Thus, $\sqrt{3}\,GM = MC$. Therefore, $AM = 3GM = \sqrt{3}\,MC$. Let $\triangle PBC$ and $\triangle QBC$ be equilateral (with P and Q on opposite sides of BC). Then $MP = MQ = \sqrt{3}\,MC$, and $MP \perp BC$, $MQ \perp BC$, whence P, M, Q are collinear. Thus, we have $MA = MP = MQ$, which means that A is a point on the circle with diameter PQ. Since $AB < AC$, A is a point on the same side as B with respect to the line PQ. Therefore, the locus of A is a semi-circle with diameter PQ, which lies on the same side of PQ as B, excluding the intersection with BC (say R, as in figure 1 below).

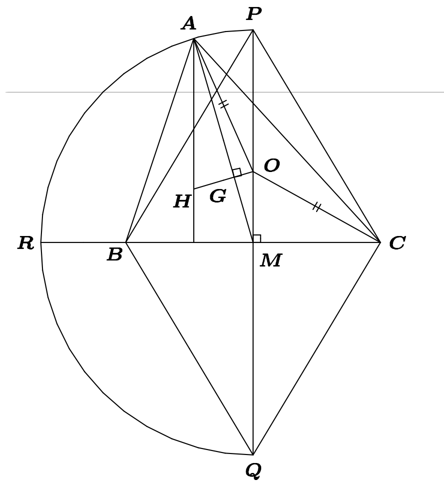


Figure 1.

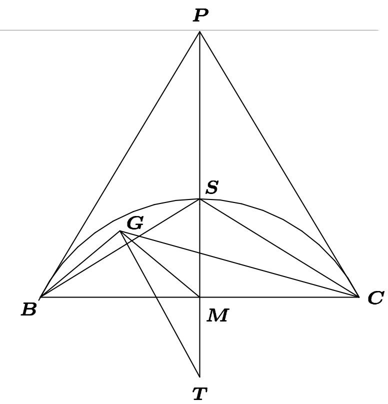


Figure 2.

2. We assume that A, P lie on the same side of BC. Let S be the centroid of $\triangle PBC$. Then $\angle BSC = 120^\circ$ and $MS = \frac{1}{\sqrt{3}}MC = MG$. Let T be the circumcentre of $\triangle SBC$. (See figure 2 above.) Then S, M, T are collinear and

$$GT \;<\; MG + MT \;=\; MS + MT \;=\; TS\,.$$

Thus, G is an interior point of segment BSC, whence

$$\angle BGC \;>\; \angle BSC = 120^\circ\,.$$

Therefore, $120^\circ < \angle BGC < 180^\circ$.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Almost all solutions were variations of the above. Some solvers chose not to eliminate the degenerate triangle which occurs when $A = R$; others ignored the condition $AC > AB$.

2661. [2001 : 337] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let H be the orthocentre of acute-angled $\triangle ABC$ in which $\tan\left(\frac{A}{2}\right) = \frac{1}{2}$. Show that the sum of the radii of the incircles of $\triangle AHB$ and $\triangle AHC$ is equal to the inradius of $\triangle ABC$.

Is the converse true?

Solution by Michel Bataille, Rouen, France.

Let r, r', and r'' denote the inradii of $\triangle ABC$, $\triangle AHB$, and $\triangle AHC$, respectively. Recall the following formulas (in which R is the circumradius and s the semiperimeter of $\triangle ABC$):

$$\begin{aligned} r &= (s-a)\tan\left(\frac{A}{2}\right)\text{, and} && (1)\\ &= 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\,. && (2)\end{aligned}$$

Moreover, $\triangle AHB$ has circumradius R (equal to the circumradius of the original triangle) and angles $90^\circ - B$, $180^\circ - C$, $90^\circ - A$ (in the order A, H, B). Thus (by the Sine Law), $AH = 2R\cos A$, $BH = 2R\cos B$, and $\angle AHB = 180^\circ - C$. It follows (using (1) and $c = 2R\sin C$) that

$$r' \;=\; \left(\frac{AH+BH+c}{2} - c\right)\tan(90^\circ - C/2) \;=\; \frac{R(\cos A + \cos B - \sin C)}{\tan\left(\frac{C}{2}\right)}\,.$$

Similarly,

$$r'' \;=\; \frac{R(\cos A + \cos C - \sin B)}{\tan\left(\frac{B}{2}\right)}\,.$$

With the help of the appropriate trigonometric formulas we calculate

$$\begin{aligned} r' + r'' &= R\left(\frac{2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)}{\tan\left(\frac{C}{2}\right)}\right.\\ &\qquad\left. + \frac{2\sin\left(\frac{B}{2}\right)\cos\left(\frac{A-C}{2}\right) - 2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)}{\tan\left(\frac{B}{2}\right)}\right)\\ &= R\left(2\cos\left(\frac{C}{2}\right)\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{C}{2}\right)\right]\right.\\ &\qquad\left. +2\cos\left(\frac{B}{2}\right)\left[\cos\left(\frac{A-C}{2}\right) - \cos\left(\frac{B}{2}\right)\right]\right)\\ &= R\left[4\cos\left(\frac{C}{2}\right)\sin\left(45^\circ - \frac{A}{2}\right)\sin\left(45^\circ - \frac{B}{2}\right)\right.\\ &\qquad\left. +4\cos\left(\frac{B}{2}\right)\sin\left(45^\circ - \frac{A}{2}\right)\sin\left(45^\circ - \frac{C}{2}\right)\right]\\ &= 4R\frac{\sqrt{2}}{2}\left[\cos\left(\frac{A}{2}\right) - \sin\left(\frac{A}{2}\right)\right]\frac{\sqrt{2}}{2}\left[2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) - \sin\left(\frac{B+C}{2}\right)\right]\\ &= 2R\left[\cos\left(\frac{A}{2}\right) - \sin\left(\frac{A}{2}\right)\right]\left[\cos\left(\frac{B-C}{2}\right) + \sin\left(\frac{A}{2}\right) - \cos\left(\frac{A}{2}\right)\right]\,.\end{aligned}$$

Now let $t = \tan\left(\frac{A}{2}\right)$. Using (2) we easily obtain

$$\frac{r'+r''}{r} \;=\; \frac{(1-t)\left[\cos\left(\frac{B-C}{2}\right) + (t-1)\cos\left(\frac{A}{2}\right)\right]}{t\left[\cos\left(\frac{B-C}{2}\right) - t\cos\left(\frac{A}{2}\right)\right]}\,.$$

Therefore,

$$\frac{r'+r''}{r} \;=\; 1 \quad \text{if and only if} \quad (2t-1)\left[\cos\left(\frac{B-C}{2}\right)-\cos\left(\frac{A}{2}\right)\right] \;=\; 0\,.$$

Note that $\cos\left(\frac{B-C}{2}\right)=\cos\left(\frac{A}{2}\right)$ holds for $\triangle ABC$ if and only if B or C is a right angle. Consequently, we conclude that

$$r'+r'' \;=\; r \textit{ if and only if } B \textit{ or } C \textit{ is a right angle or } \tan\left(\frac{A}{2}\right) \;=\; 1/2\,.$$

Since our problem asks what happens when all angles of $\triangle ABC$ are acute, the answer is ***yes***, the converse is true: for an acute triangle, $r'+r''=r$ if and only if $\tan\left(\frac{A}{2}\right)=1/2$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2662. [2001 : 337] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that $\triangle ABC$ is acute-angled, has inradius r and has area Δ. Prove that

$$\left(\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}\right)^2 \;\le\; \frac{\Delta}{r^2}\,.$$

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Note that

$$\cot A \;=\; \frac{\cos A}{\sin A} \;=\; \frac{b^2+c^2-a^2}{2bc\sin A} \;=\; \frac{R}{abc}(b^2+c^2-a^2)\,.$$

We must show that

$$\frac{R}{abc}\left(\sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2}+\sqrt{a^2+b^2-c^2}\right)^2 \;\le\; \frac{\Delta}{r^2}\,,$$

or,

$$\left(\sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2}+\sqrt{a^2+b^2-c^2}\right)^2 \;\le\; \frac{\Delta abc}{r^2R} \;=\; (2s)^2\,,$$

or,

$$\sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2}+\sqrt{a^2+b^2-c^2} \;\le\; a+b+c\,.$$

The last inequality is identical to the first of the April issue's Five Klamkin Quickies [2001 : 166]; the proof is in [2001 : 299]. Equality holds for an equilateral triangle.

Also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; RICHARD EDEN, Ateneo de Manila University, Philippines; VINAYAK GANESHAN, student, University of Waterloo, Waterloo, Ontario; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HENRY LIU, student, University of Memphis, Memphis, TN, USA; VEDULA N. MURTY, Dover, PA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Klamkin and Bencze proved the stronger inequality

$$3(\cot A + \cot B + \cot C) \;\leq\; \frac{\Delta}{r^2}\,.$$

2663. [2001 : 337] *Proposed by Antreas P. Hatzipolakis, Athens, Greece; and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Suppose that the incircle of $\triangle ABC$ is tangent to the circle with BC as diameter. Show that the excircle on BC has radius equal to BC.

I. *Solution by Michel Bataille, Rouen, France.*

Let D and D' be the points of tangency of BC with the incircle and the excircle on BV, respectively.

Then $BD = \frac{r}{\tan(B/2)}$ and $BD' = \frac{r_a}{\tan((\pi - B)/2)} = r_a \tan(B/2)$. (Here, r and r_a denote the inradius and the radius of the excircle on BC, respectively.) Making use of signed distances, it follows that

$$DB \cdot DC \;=\; DB \cdot BD' \;=\; -r \cdot r_a\,. \tag{1}$$

Now, under the inversion with centre D and (negative) power $-r \cdot r_a$, the circle with diameter BC is invariant (because of (1)) and the incircle is transformed into a line L perpendicular to DI; hence, parallel to BC. The distance between L and BC is $d(D, L) = \frac{|-r \cdot r_a|}{2r} = \frac{r_a}{2}$, and the hypothesis implies that L is tangent to the circle with diameter BC. Thus, $\frac{r_a}{2} = \frac{a}{2}$; that is, $r_a = a$.

II. *Solution by Vinayak Ganeshan, student, University of Waterloo, Waterloo, Ontario*. [Ed.: The solver uses standard triangle notation.]

Since the two circles touch [internally], the difference of their radii must be equal to the distance between their centres:

$$\frac{a}{2} - r \;=\; \sqrt{\left(s - c - \frac{a}{2}\right)^2 + r^2}\,.$$

Squaring yields

$$\frac{a^2}{4} - ar \;=\; \left(\frac{b-c}{2}\right)^2, \qquad \text{so that} \qquad ar \;=\; (s-c)(s-b)\,.$$

But, $rs = r_a(s-a) = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$, so that $ar = \dfrac{\Delta^2}{s(s-a)} = rr_a$, giving $r_a = a = BC$.

Also solved by MICHEL BATAILLE, Rouen, France (2nd solution); FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; ANDREI SIMION, student, Cornell University, Ithaca, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

2664. [2001 : 403] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Let a, b and c be positive real numbers such that $a+b+c = abc$. Prove that $a^5(bc-1) + b^5(ca-1) + c^5(ab-1) \;\geq\; 54\sqrt{3}$.

I. *Solution by Kee-Wai Lau, Hong Kong, China.*

By the AM–GM inequality, we have $abc = a+b+c \geq 3(abc)^{\frac{1}{3}}$ so that $abc \geq 3\sqrt{3}$.

Hence,

$$\begin{aligned}
&a^5(bc-1) + b^5(ca-1) + c^5(ab-1)\\
&\quad= \;abc(a^4+b^4+c^4) - a^5 - b^5 - c^5\\
&\quad= \;(a+b+c)(a^4+b^4+c^4) - a^5 - b^5 - c^5\\
&\quad= \;a(b^4+c^4) + b(c^4+a^4) + c(a^4+b^4)\\
&\quad\geq \;a(2b^2c^2) + b(2c^2a^2) + c(2a^2b^2)\\
&\quad\geq \;6\left((ab^2c^2)(bc^2a^2)(ca^2b^2)\right)^{\frac{1}{3}}\\
&\quad= \;6(abc)^{\frac{5}{3}} \geq 6(3\sqrt{3})^{\frac{5}{3}} \;=\; 54\sqrt{3}\,.
\end{aligned}$$

II. *Generalization by Murray S. Klamkin, University of Alberta, Edmonton, Alberta (expanded slightly by the editor).*

We establish the following more general result:

Let $n \in \mathbb{N}$, $n > 1$ and let $a_k > 0$, $k = 1, 2, \ldots, n$. Suppose that $S = P$ where $S = \sum_{k=1}^{n} a_k$ and $P = \prod_{k=1}^{n} a_k$. Then, for all real numbers m and r, we have

$$\sum_{k=1}^{n} a_k^m \left(\frac{P}{a_k} - 1\right)^r \;\geq\; n(n-1)^r \cdot n^{\frac{m}{n-1}}$$

with equality if and only if all the a_k's are equal.

To prove this, we apply the AM–GM inequality twice to get

$$
\begin{aligned}
&\sum_{k=1}^{n} a_k^m \left(\frac{P}{a_k} - 1\right)^r \\
&\quad = \sum_{k=1}^{n} a_k^{m-r}(P-a_k)^r \;\geq\; n\left(P^{m-r}\cdot\prod_{k=1}^{n}(P-a_k)^r\right)^{\frac{1}{n}} \\
&\quad = nP^{\frac{m-r}{n}}\cdot\prod_{k=1}^{n}(S-a_k)^{\frac{r}{n}} \\
&\quad \geq nP^{\frac{m-r}{n}}\left(\prod_{k=1}^{n}\left((n-1)\left(\frac{P}{a_k}\right)\right)^{\frac{1}{n-1}}\right)^{\frac{r}{n}} \\
&\quad = nP^{\frac{m-r}{n}}(n-1)^r\cdot P^{\frac{r}{n}} \;=\; n(n-1)^r P^{\frac{m}{n}} \qquad (1)
\end{aligned}
$$

Now, by AM–GM inequality again, we have $P = S \geq nP^{\frac{1}{n}}$ or $P^{n-1} \geq n^n$. Hence,

$$P \;\geq\; n^{\frac{n}{n-1}}\,. \qquad (2)$$

From (1) and (2), our claim follows.

Note that the given inequality is the special case when $n = 3$, $m = 5$ and $r = 1$.

III. *Generalization by Walther Janous, Ursulinengymnasium, Innsbruck, Austria* (*modified slightly by the editor*).

We prove the following more general result which contains the given inequality as a special case.

Theorem: Let a, b, c be non-negative reals and let $r \in \mathbb{R}$ with $r \geq 2$.

Then

$$a^r(b+c) + b^r(c+a) + c^r(a+b) \;\geq\; \frac{2}{3^{\frac{(r-1)}{2}}}(ab+bc+ca)^{\frac{(r+1)}{2}}\,.$$

Proof: If $ab+bc+ca = 0$ then clearly equality holds.

Thus, we may assume that $ab+bc+ca > 0$.

Since $r \geq 2$ the function $f(x) = x^{r-1}$ is convex on $[0,\infty)$.

Hence, by Jensen's Inequality, we have

$$\begin{aligned}
&\frac{a^r(b+c)+b^r(c+a)+c^r(a+b)}{2(ab+bc+ca)} \\
&= \frac{(ab+ca)a^{r-1}+(bc+ab)b^{r-1}+(ca+bc)c^{r-1}}{2(ab+bc+ca)} \\
&\geq \left(\frac{(ab+ca)a+(bc+ab)b+(ca+bc)c}{2(ab+bc+ca)}\right)^{r-1} \\
&= \left(\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{2(ab+bc+ca)}\right)^{r-1} \qquad (3)
\end{aligned}$$

Next, we establish the following inequality

$$a^2(b+c)+b^2(c+a)+c^2(a+b) \;\geq\; \frac{2}{\sqrt{3}}(ab+bc+ca)^{\frac{3}{2}} \qquad (4)$$

Note that (4) is equivalent to

$$(a+b+c)(ab+bc+ca) \;\geq\; \frac{2}{\sqrt{3}}(ab+bc+ca)^{\frac{3}{2}}+3abc \qquad (5)$$

By the AM–GM inequality, we have

$$\frac{1}{3}(a+b+c)(ab+bc+ca) \;\geq\; 3abc \qquad (6)$$

Furthermore, from $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$ we have

$$(a+b+c)^2 \;\geq\; 3(ab+bc+ca)$$

or

$$a+b+c \;\geq\; \sqrt{3}(ab+bc+ca)^{\frac{1}{2}}\,.$$

Hence,

$$\frac{2}{3}(a+b+c)(ab+bc+ca) \;\geq\; \frac{2}{\sqrt{3}}(ab+bc+ca)^{\frac{3}{2}} \qquad (7)$$

Adding up (6) and (7) yields (5) and hence, (4). Substituting (4) into (3) we then have

$$\begin{aligned}
&a^r(b+c)+b^r(c+a)+c^r(a+b) \\
&\geq 2(ab+bc+ca)\left(\frac{2}{\sqrt{3}}\,\frac{(ab+bc+ca)^{\frac{3}{2}}}{2(ab+bc+ca)}\right)^{r-1} \\
&= \frac{2}{3^{\frac{(r-1)}{2}}}(ab+bc+ca)^{\frac{r+1}{2}}\,,
\end{aligned}$$

completing the proof.

Finally, we show that the given inequality is a special case of our theorem. Let $r = 4$ and assume that $a + b + c = abc$. Note first that $(a+b+c)(ab+bc+ca) \geq 9abc$ from (6) and hence, $ab + bc + ca \geq 9$.

Using the theorem above, we then have

$$\begin{aligned} & a^5(bc-1) + b^5(ca-1) + c^5(ab-1) \\ = \;& a^4(b+c) + b^4(c+a) + c^4(a+b) \\ \geq \;& \frac{2}{3^{\frac{3}{2}}}(9)^{\frac{5}{2}} \;=\; 2 \cdot 3^{\frac{7}{2}} \;=\; 54\sqrt{3}\,. \end{aligned}$$

Remark: Clearly equality holds in our theorem if $r = 1$. This leads to the natural question: What happens for $r \in (1, 2)$?

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ATENEO PROBLEM-SOLVING GROUP, Ateneo de Manila University, The Philippines; DIONNE T. BAILEY, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Courdimanche, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADAM BROWN, student, Balliol College, Oxford, UK; MARCELO R. DE SOUZA, Rio De Janero, Brazil; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, Portales, NM, USA; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; D. KIPP JOHNSON, Beaverton, OR, USA; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; VEDULA N. MURTY, Visakhapatnam, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ANDREI SIMION, student, Cornell University, Ithaca, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Most of the solutions are either variations of or very similar to, solution I above. Bencze, Bornsztein, and Seiffert also gave various generalizations, but they are all special cases of the more general result obtained by Klamkin in solution II above.

2665. [2001 : 403] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In $\triangle ABC$, we have $\angle ACB = 90^\circ$ and sides $AB = c$, $BC = a$ and $CA = b$. In $\triangle DEF$, we have $\angle EFD = 90^\circ$, $EF = (a+c)\sin\left(\frac{B}{2}\right)$ and $FD = (b+c)\sin\left(\frac{A}{2}\right)$. Show that $DE \geq c$.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK; and Heinz-Jürgen Seiffert, Berlin, Germany.

The claim of the problem is incorrect. We prove that

$$\frac{c}{\sqrt{2}} \;<\; DE \;\leq\; c\sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}\,,$$

with equality on the right-hand side only when $A = 45^\circ$. Without loss of generality, take $c = 1$. Then $a = \sin A$ and $b = \cos A$. In addition, $\sin^2\frac{A}{2} = \frac{1}{2}(1-\cos A)$ and $\sin^2\frac{B}{2} = \frac{1}{2}(1-\cos B) = \frac{1}{2}(1-\sin A)$. It follows that

$$\begin{aligned}
DE^2 &= EF^2 + FD^2 \\
&= \frac{1}{2}(1+\sin A)^2(1-\sin A) + \frac{1}{2}(1+\cos A)^2(1-\cos A) \\
&= \frac{1}{2}\cos^2 A(1+\sin A) + \frac{1}{2}\sin^2 A(1+\cos A) \\
&= \frac{1}{2} + \frac{1}{2}\sin A\cos A(\sin A + \cos A) \\
&= \frac{1}{2} + \frac{\sqrt{2}}{4}\sin 2A\cos(45^\circ - A)\,.
\end{aligned}$$

Clearly,

$$\frac{1}{\sqrt{2}} < DE \le \sqrt{\frac{1}{2}+\frac{\sqrt{2}}{4}}\,,$$

with equality only if $A = 45^\circ$, as claimed.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ATENEO PROBLEM-SOLVING GROUP, Ateneo de Manila University, The Philippines; ELSIE CAMPBELL, Angelo State University, San Angelo, TX, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; VEDULA N. MURTY, Dover, PA, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

The range of DE has also been found by Hess and Janous. All of the other solvers have noted that the stated inequality is incorrect and either proved that $DE < c$ or found a counterexample to the statement $DE \ge c$.

2667. [2001 : 404] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

You are given a circle Γ and two points A and B outside of Γ such that the line through A and B does not intersect Γ. Let X be any point on Γ.

Determine at which point X on Γ the sum $AX + XB$ attains its minimum value.

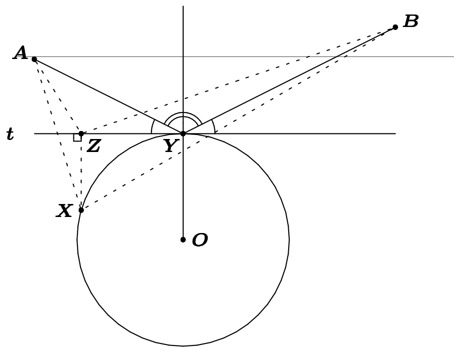
Solution by Toshio Seimiya, Kawasaki, Japan.

Let O be the centre of the circle Γ and let Y be the point on Γ such that:

(1) the line OY is the bisector of $\angle AYB$;

(2) if t is the tangent line to Γ at Y, then A and B lie on the opposite side of Γ with respect to t.

We claim that $AY + YB < AX + XB$ for every point X of Γ, $X \neq Y$.



First, we note that the angles between t and AY and between t and BY are equal. Let X be any point on Γ other than Y. Let Z be the foot of the perpendicular from X to the tangent line t. Then $\angle AZX > 90^\circ$ and $\angle BZX > 90^\circ$, so that $AZ + BZ < AX + BX$. It is well-known that $AY + BY \leq AZ + BZ$ for every point Z on the line t.

Hence, $AY + BY < AX + BX$, which proves our claim. Therefore, the point Y is the desired one.

The problem of constructing the point Y is a classical problem, also known as the Alhasen's Problem. We refer the reader to [1] for further discussion.

Reference

[1] George Martin, Geometric Constructions, Springer-Verlag 1998, pp. 137–139.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; VICTOR PAMBUCCIAN, ASU West, Phoenix, Arizona, USA. There were also two incorrect (incomplete) solutions submitted.

Congratulations

We note that the MAA has announced its 2002 awards for distinguished teaching. Included in the list are two *Cruxers*; Andy Liu and Li Zhou. Our hearty congratulations to both.