Introduction

We are pleased to announce that Jim Totten, Department of Mathematics & Statistics, University College of the Cariboo, Kamloops, BC, has accepted the postition of Editor-in-Chief of ***Crux Mathematicorum with Mathematical Mayhem***, effective 1 January 2003. Most of you will be familiar with Jim, who is a long standing member of the Editorial Board.

Here is a little background information:

> Jim was born in Saskatoon, Saskatchewan, attended school in Regina, and received his B.A. from the University of Sasketchewan (Regina Campus). He attended graduate school at the University of Waterloo, obtaining a Masters of Mathematics (in Computer Science), and a Ph.D. (in finite geometry).
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> He then went to (West) Germany on an NRC Post-Doctorate Fellowship, where he spent two years at the University of Tübingen. On returning to Canada, he joined the faculty at St. Mary's University in Halifax. This was followed by one year at the University of Saskatchewan, and then he moved to Kamloops, British Columbia, joining the faculty at Cariboo College, a Community College established in 1970, which offered a broad spectrum of post secondary education including first and second year university transfer. Cariboo College has since been renamed to the University College of the Cariboo (UCC), as it now offers several bachelor's degrees. He has just completed 23 years there.
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> His mathematical interests include (but are not limited to) geometry and number theory. He has a total of 12 refereed publications, and 2 books on the first 20 years of BC's regional high school mathematics contest.
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> During his tenure at UCC, he has coached the Putnam team almost every year, he has posted and marked a "Problem of the Week" for 10 weeks each semester, organized a regional high school mathematics contest, which he and others have now elevated to a provincial high school contest for British Columbia. He has served on the Executive of the Faculty Association at UCC as Treasurer for 3 years and, later, as chief negotiator for one contract. He served as Chair of the department for a period of 4 years.
>
> His outside interests include many different sports. In the winter he plays ice hockey 2 or 3 times per week, and floor hockey once a week with staff and students at UCC. He golfs extensively in the summer. In addition, he enjoys cross-country skiing and hiking. In the past he has coached both minor hockey and youth soccer, being also involved in the executive of youth soccer for 6 years.
>
> Jim is married to Lynne. They have one son.

Effective immediately, we ask that all proposals and solutions be sent to Jim at the address listed elsewhere. This will assist in a smooth transition.

Introduction

Il nous fait plaisir d'annoncer que Jim Totten, du Département de mathématiques et de statistiques de l'University College of the Cariboo, de Kamloops (C.-B.), a accepté le poste de rédacteur en chef du ***Crux Mathematicorum with Mathematical Mayhem***. Sa nomination entrera en vigueur le 1er janvier 2003. La plupart d'entre vous connaissent probablement Jim, qui fait partie du comité de rédaction depuis longtemps.

Voici quelques renseignements à son sujet :

Jim est né à Saskatoon, en Saskatchewan. Il a fait ses études á Regina et il a reçu un baccalauréat de l'Université de la Sasketchewan (campus de Regina).

Il a fait des études de deuxième et de troisième cycle à l'Université de Waterloo, où il a obtenu une maîtrise en mathématiques (informatique), ainsi qu'un doctorat (géométrie finie).

Il s'est ensuite rendu en Allemagne (de l'Ouest) grâce à une bourse de recherche postdoctorale du CNRC, où il a passé deux ans à l'Université de Tübingen. À son retour au Canada, il s'est joint à l'équipe professorale de l'Université St. Mary's à Halifax. Il est ensuite retourné pendant un an à l'Université de la Saskatchewan, avant de déménager à Kamloops, en Colombie-Britannique, où il a accepté un poste de professeur au Cariboo College, collège communautaire fondé en 1972 offrant un vaste éventail de cours postsecondaires, dont des cours de première et de deuxième année universitaire. Rebaptisé depuis University College of the Cariboo (UCC), le collège offre maintenant plusieurs programmes de baccalauréat. Jim vient de terminer sa 23e année à cet endroit.

Parmi ses nombreux intérêts mathématiques, mentionnons la géométrie et la théorie des nombres. Il a à son actif 12 publications relues par des comités scientifiques, ainsi que deux ouvrages sur les vingt premières années des concours de mathématiques de niveau secondaire en Colombie-Britannique.

Pendant son passage à l'UCC, il a entraîné presque chaque année l'équipe qui a participé au concours Putnam ; il a affiché et corrigé un «problème de la semaine»pendant 10 semaines tous les semestres ; il a organisé un concours régional de mathématiques pour élèves du secondaire, qu'il a, avec des collègues, élevé au rang de concours provincial. Il a aussi été trésorier de son association des professeurs et, plus tard, il a servi de négociateur en chef pour un contrat. Il a en outre été directeur du département pendant quatre ans.

À part les mathématiques, il est un adepte de nombreux sports. En hiver, il joue au hockey sur glace deux ou trois fois par semaine et au hockey en salle une fois par semaine avec des collègues et des étudiants de l'UCC. En été, il passe beaucoup de temps sur les parcours de golf. Entre autres activités, il aime le ski de fond et la randonnée pédestre. Il a déjà été entraîneur au hockey mineur et pour une équipe de soccer de jeunes, en plus de faire partie du comité exécutif d'une association de soccer pendant six ans.

Jim est l'époux de Lynne, et le couple a un fils.

Nous vous demandons de faire parvenir à partir de maintenant vos propositions de problèmes et de solutions à Jim à l'adresse qui figure ailleurs dans cette publication. Vous nous aiderez ainsi à faire la transition tout en douceur.

THE OLYMPIAD CORNER

No. 223

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a problem set in this issue we continue the selection of problems from the St. Petersburg Contests 1965–1984 which we began last issue. These problems were selected (and translated) by two school students and forwarded to me by Andy Liu, University of Alberta, Edmonton, Alberta. I hope you enjoy the choice by Oleg Ivrii and by Robert Barrington Leigh.

ST. PETERSBURG CONTESTS 1965–1984
Problems from Various Contests (continued)

24. An infinite sequence of light bulbs and an infinite sequence of switches are both numbered by the positive integers. Each switch has a finite number of positions. Whether a bulb is on or off depends only on the positions of a finite number of switches. In any setting of the switches, at least one bulb is on. Prove that there exists a finite set of bulbs such that for any setting of the switches, at least one of them is on.

25. The sum of two continuous periodic functions is a nonconstant continuous periodic function. Prove that the periods of these two functions are integral multiples of the period of their sum.

26. Four squares on a 25×25 chessboard are called a quartet if their centres form a rectangle with sides parallel to the sides of the board. What is the maximum number of quartets which do not have any common squares?

27. The intersection of 20 circles consists of more than one point. Prove that the boundary of this intersection is a union of at most 38 circular arcs.

28. An irreducible fraction $\frac{x}{y}$ is called a good approximation of a number c if $\left|c - \frac{x}{y}\right| < \frac{1}{y^{100}}$. Prove that in any interval, there is a number with infinitely many good approximations.

29. From each of k points on a plane, a few rays are drawn. No two rays intersect. Prove that one can choose $k-1$ of the segments connecting these points such that they are disjoint from one another and from any of the rays, except possibly at those k points.

30. There are 25 magnetic tapes on reels and 1 empty reel. One can rewind a tape from a full reel to an empty one, thus, reversing its direction. Can one reverse the direction of every tape while leaving each on its original reel?

31. There are $p-1$ integers none of which is divisible by p, where p is an odd prime. Prove that one can replace some of these numbers with their additive inverses to get $p-1$ numbers whose sum is divisible by p.

32. There are 9 points in a 2×2 square. Prove that the distance between some 2 of these points is not greater than 1.

33. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $f^{(1)}(x) = f(x)$ and let $f^{(n)}(x) = f^{(n-1)}(f(x))$ for any integer $n > 1$. For any polynomial $P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$, define

$$P(f) = a_n f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_1 f(x) + a_0 x .$$

(a) Let $P(t) = t^2 - t + 1$. If $P(f) = 0$, prove that $f^{(7)} = f$.

(b) Let P, Q and R be polynomials such that $Q = PR$. If $P(f) = 0$, prove that $Q(f) = 0$.

(c) Let $P(t) = t^2 - t + 1$. Prove that there exists a function f such that $P(f) = 0$.

(d) Is (c) true if P is an arbitrary polynomial?

(e) Let Q and P be polynomials such that $\{f : P(f) = 0\} \subset \{f : Q(f) = 0\}$. Prove that there exists a polynomial R such that $Q = PR$. In particular, prove this result for $Q(t) = t^n - 1$.

34. A straight line passes through the centre of a regular $2n$-gon. Prove that the sum of distances to this line from the vertices on one side of this line equals the sum of the distances from the remaining vertices.

35. The *luckies* form a finite subset of the positive integers. Let a_k be the largest number of luckies which can be written if we have k copies of each digit. The same lucky may be written several times. Prove that among the numbers $\frac{a_k}{k}$, there exists a maximum.

36. A set X in the coordinate plane intersects any unit square whose vertices have integral coordinates in two parallel segments joining the midpoints of the sides of the square. X is invariant if shifted 25 units horizontally or vertically. Prove that X contains polygonal lines of infinite length.

37. Two points (x_1, y_1) and (x_2, y_2) are said to be *dependable* if $(x_1 - x_2)^2 = (y_1 - y_2)(x_1 y_2 - x_2 y_1)$. Prove that if any two of four points are dependable, then they are collinear.

38. Red, blue and green arcs are used to join pairs of $2n$ points such that there is exactly one arc of each colour at each point. Let a, b and c be the numbers of red-blue, red-green and blue-green cycles. Prove that $n + a \geq b + c$.

39. Each of n lines on a plane is cut by the others into 2 rays and $n - 2$ equal segments. Prove that $n = 3$.

40. A strictly increasing sequence $\{a_n\}$ of positive integers is such that $a_2 = 2$ and $a_{mn} = a_m a_n$ if m and n are relatively prime. Prove that $a_n = n$ for all n.

41. A real number is placed in each square of an infinite chessboard. Numbers in the same row or column at a distance 1982 apart are equal. Each number is the average either of its two horizontal neighbours or of its two vertical neighbours. Prove that either all the numbers in each column are equal or all the numbers in each row are equal.

42. A strait is 10 kilometres wide. The speed of the patrol boat is 7 times the speed of the smugglers' barge. The patrol boat will discover the barge if the distance between them is 1 kilometre or less. Can the patrol boat always discover any barge coming down the strait?

43. H is a given point inside a circle. Prove that a fixed circle passes through the mid-points of the sides of any triangle inscribed in the circle and having H as its orthocentre.

44. A strictly increasing sequence $\{x_n\}$ of positive integers is such that for all $n > 1982$,

$$x_1^3 + x_2^3 + \cdots + x_n^3 = (x_1 + x_2 + \cdots + x_n)^2 .$$

Prove that $x_n = n$ for all n.

45. Let $P(z)$ and $Q(z)$ be complex polynomials, one of which is not constant. Every root of $P(z)$ is also a root of $Q(z)$ and vice versa. Every root of $P(z) - 1$ is also a root of $Q(z) - 1$ and vice versa. Prove that $P = Q$.

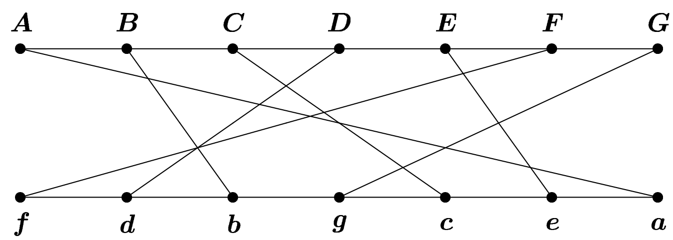
46. A calculator may be used to add two numbers, subtract one number from another, divide a number by any positive integer, and raise any number to the tenth power. Prove that it may be used to find the product of any ten numbers.

47. In a graph, it is possible to go from any vertex to any other passing through at most $n - 1$ vertices in between. The shortest cycle in this graph has length of $2n + 1$. Prove that all the vertices have the same degree.

Next we give a reader's response to a question we posed when giving solutions to problem 5 of the Georg Mohr Konkurrencen I Matematik 1996, [2001 : 239–240; 1999 : 261–262]. The solution we gave provided a negative answer for $n \equiv 2,\ 3 \pmod 4$.

5. In a ballroom 7 gentlemen, A, B, C, D, E, F and G are sitting opposite 7 ladies a, b, c, d, e, f and g in arbitrary order. When the gentlemen walk across the dance floor to ask each of their ladies for a dance, one observes that at least two gentlemen walk distances of equal length. Is that always the case?

The figure shows an example. In this example $Bb = Ee$ and $Dd = Cc$.



Editor's question. If n is congruent to either 0 or 1 modulo 4, is it always possible to arrange the n gentlemen and the n ladies in a way such that the distances are all different?

Solution by Peter de Caux, Eatonton, GA, USA.

We answer the Editor's two questions in the affirmative by constructing functions which specify for each "gentleman" a "lady" to whom he should walk. It will be convenient to call a function f *dispersive* provided:

(a) f is one-to-one and both its range and domain are the same initial segment I of the positive integers and

(b) the set $D = \{|m - f(m)| \mid m \in I\}$ is conumerous with I.

A simple consequence of this definition is that $D = \{m - 1 \mid m \in I\}$ and that for no two m and m' in I is $|m - f(m)| = |m' - f(m')|$.

Suppose now that n is a positive integer.

We build a dispersive function f with domain I consisting of the first $4n+1$ positive integers. We do this in two steps. First we define a sequence S of $4n$ terms whose range is I with $3n+1$ removed as follows:

$$\begin{array}{lrcccccl} \text{for} & 1 & \leq & i & \leq & 2n & : & S(2i-1) = i\,, \\ \text{for} & 1 & \leq & i & \leq & n & : & S(2i) = 4n+2-i\,, \\ \text{for} & n+1 & \leq & i & \leq & 2n & : & S(2i) = 4n+1-i\,. \end{array}$$

Example 1. The consecutive terms of such a sequence if n were 5:

$$1\ \ 21\ \ 2\ \ 20\ \ 3\ \ 19\ \ 4\ \ 18\ \ 5\ \ 17\ \ 6\ \ 15\ \ 7\ \ 14\ \ 8\ \ 13\ \ 9\ \ 12\ \ 10\ \ 11\,.$$

Note that S is one-to-one and that the absolute values of the difference of consecutive terms of S are in descending order the first n positive integers with $2n$ removed.

Step 2 defines f to be the following collection of ordered pairs:

$$\{(S(k),\, S(k+1)) \mid 1 \leq k \leq 4n-1\} \;\cup\; \{(2n+1,1),\, (3n+1,3n+1)\}\,.$$

It is quickly checked that f is dispersive.

For $n = 5$ the values of f would be

$$\begin{array}{lllllllllllll}
f(1) & = & 21\,, & f(21) & = & 2\,, & f(6) & = & 20\,, & f(20) & = & 3\,, \\
f(3) & = & 19\,, & f(19) & = & 4\,, & f(4) & = & 18\,, & f(18) & = & 5\,, \\
f(5) & = & 17\,, & f(17) & = & 6\,, & f(6) & = & 15\,, & f(15) & = & 7\,, \\
f(7) & = & 14\,, & f(14) & = & 8\,, & f(8) & = & 13\,, & f(13) & = & 9\,, \\
f(9) & = & 12\,, & f(12) & = & 10\,, & f(10) & = & 11\,, & f(11) & = & 1\,, \\
f(16) & = & 16\,. & & & & & & & & &
\end{array}$$

If 21 gentlemen are evenly spaced in a straight line along one side of a rectangular dance floor and they are numbered consecutively 1, 2, $\ldots$, 21 and if each of the 21 ladies is evenly spaced on the side of the dance floor opposite the men and each is numbered as the gentleman opposite her is numbered, then, if gentleman m walks directly to lady $f(m)$ and another gentleman m' walks directly to lady $f(m')$, then m and m' walk different distances.

There is another dispersive function $\overline{f}$ with domain the positive integers less than or equal to $4n+1$, I, which can be constructed from a sequence $\overline{S}$ whose range is I with $n+1$ removed. Since the inverse of a dispersive function is again dispersive, we have four distinct dispersive functions with domains I : f, f^{-1}, $\overline{f}$, $\overline{f}^{-1}$.

Question: Are there more than four dispersive functions with domain I?

In an entirely analogous way we construct a dispersive function g with domain I consisting of the positive integers less than or equal to $4n$. We use a sequence of $4n-1$ terms, call it T, which has range I with $3n$ removed. The terms of T, $T(1)$, $T(2)$, $\ldots$, $T(4n-1)$ are

$$1,\ 4n,\ 2,\ 4n-1,\ \ldots,\ n,\ 3n+1,\ n+1,\ 3n-1,\ n+2,\ \ldots,$$

$$2n-2,\ 2n+2,\ 2n-1,\ 2n+1,\ 2n.$$

Using T, the ordered pairs in g are

$$(1,4n),\ (4n,2),\ \ldots,\ (n,3n+1),\ (3n+1,n+1),\ (n+1,3n-1),\ \ldots,$$

$$(2n-1,2n+1),\ (2n+1,2n),\ (2n,1),\ (3n,3n).$$

g is dispersive and together with f settle the editor's two questions.

Again a dispersive function $\overline{g}$ may be built from a sequence $\overline{T}$ whose terms are the first $4n$ positive integers with $n+1$ removed.

Question: Are there more than the four dispersive functions g, g^{-1}, $\overline{g}$, and $\overline{g}^{-1}$ which have domain the positive integers less than or equal to $4n$?

Next we fill in a missing solution when we gave responses to the Second Round of the 13th Iranian Mathematical Olympiad 1996 [1999 : 454–455; 2002 : 11–15].

6. In tetrahedron $ABCD$ let A', B', C', and D' be the circumcentres of faces BCD, ACD, ABD and ABC. We mean by $S(X,YZ)$, the plane perpendicular from point X to the line YZ. Prove that the planes $S(A,C'D')$, $S(B,D'A')$, $S(C,A'B')$, and $S(D,B'C')$ are concurrent.

Solution by Michel Bataille, Rouen, France.

Let R_3 and R_4 be the circumradii of $\triangle ABD$ and $\triangle ABC$, respectively. Then $AC' = R_3$ and $AD' = R_4$ so that $A \in \{M : MC'^2 - MD'^2 = R_3^2 - R_4^2\}$. This set of points, which is known to be a plane orthogonal to $C'D'$, is thus, $S(A,C'D')$.

Now let Σ be the circumsphere of the tetrahedron $ABCD$ (centre O, radius R) and Σ' be the circumsphere of the tetrahedron $A'B'C'D'$ (centre O', radius R'). The circumcircle of $\triangle ABC$ is the intersection of Σ with the plane (ABC). Hence, D' is the orthogonal projection of O onto (ABC) and $OD'^2 = AO^2 - AD'^2 = R^2 - R_4^2$. Similarly, $OC'^2 = R^2 - R_3^2$ and therefore, $OC'^2 - OD'^2 = R_4^2 - R_3^2$. Now, let Ω be the reflection of O in O'. Then, using $O'C' = O'D' = R'$,

$$\begin{aligned}
&\Omega C'^2 - \Omega D'^2 \\
&\quad= \Omega O'^2 + O'C'^2 + 2\,\overrightarrow{\Omega O'}\cdot\overrightarrow{O'C'} - \Omega O'^2 - O'D'^2 - 2\,\overrightarrow{\Omega O'}\cdot\overrightarrow{O'D'} \\
&\quad= 2\,\overrightarrow{OO'}\cdot\overrightarrow{O'D'} - 2\,\overrightarrow{OO'}\cdot\overrightarrow{O'C'} \\
&\quad= OD'^2 - OO'^2 - O'D'^2 - (OC'^2 - OO'^2 - O'C'^2) \\
&\quad= R_3^2 - R_4^2\,.
\end{aligned}$$

It follows that $\Omega \in S(A,C'D')$. Similarly, Ω belongs to the planes $S(B,D'A')$, $S(C,A'B')$, and $S(D,B'C')$, and we are done.

We next turn to solutions by readers to problems of the XXXIII Spanish Mathematical Olympiad 1996–97 given in the April 2000 number of the *Corner* [2000 : 196–197].

1. Show that any complex number $z \neq 0$ can be expressed as a sum of two complex numbers such that their difference and their quotient are purely imaginary (that is, with real part zero).

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the write-up by Bornsztein.

Let $z \in \mathbb{C}^*$, $z = a + ib$ with a, $b \in \mathbb{R}$. We are looking for α, $\beta \in \mathbb{C}$ such that

$$\begin{aligned} z &= \alpha + \beta && (1) \\ \alpha - \beta &= \mu i\,, \qquad \text{where } \mu \in \mathbb{R} && (2) \\ \frac{\alpha}{\beta} &= \lambda i\,, \qquad \text{where } \lambda \in \mathbb{R}\,. && (3) \end{aligned}$$

From (1) and (2), we deduce that $\mathfrak{Re}(\alpha) = \mathfrak{Re}(\beta) = \frac{1}{2}\mathfrak{Re}(z) = \frac{a}{2}$. Thus, we suppose that

$$\alpha = \frac{a}{2} + ix\,, \quad \beta = \frac{a}{2} + iy\,, \quad \text{where } x, y \text{ are reals.}$$

Then $x + y = b$.

Since we are looking for some possible values of x and y, we will omit for a time all the cases $a = 0$, $b = 0$, and so on ....

From (3), we have $a + 2ix = -2\lambda y + a\lambda i$. By identifying real parts, and imaginary parts, we may eliminate λ to get:

$$xy = -\frac{a^2}{4}\,.$$

Thus, we obtain

$$x + y = b\,, \quad xy = -\frac{a^2}{4}\,.$$

Since the quadratic $X^2 - bX - \frac{a^2}{4} = 0$ has a discriminant $\Delta = b^2 + a^2 > 0$ (since $z \neq 0$), it gives two real distinct roots:

$$x = \frac{b + \sqrt{b^2 + a^2}}{2} \quad \text{and} \quad y = \frac{b - \sqrt{b^2 + a^2}}{2}\,.$$

Conversely: We have $z = a + ib = \alpha + \beta$ where

$$\alpha = \frac{a}{2} + i\left(\frac{b + \sqrt{b^2 + a^2}}{2}\right) \quad \text{and} \quad \beta = \frac{a}{2} + i\left(\frac{b - \sqrt{b^2 + a^2}}{2}\right)\,.$$

It is clear that $\alpha - \beta$ is purely imaginary. Since $z \neq 0$, we must have

$$b + \sqrt{b^2 + a^2} \neq 0 \quad \text{or} \quad b - \sqrt{b^2 + a^2} \neq 0$$

then $\alpha \neq 0$ or $\beta \neq 0$.

Let us suppose $\beta \neq 0$ (the case $\alpha \neq 0$ is similar). Thus,

$$\frac{\alpha}{\beta} = \frac{a + i(b + \sqrt{b^2 + a^2})}{a + i(b - \sqrt{b^2 + a^2})} = \frac{2a\sqrt{b^2 + a^2}}{a^2 + (b - \sqrt{b^2 + a^2})^2}i$$

is purely imaginary.

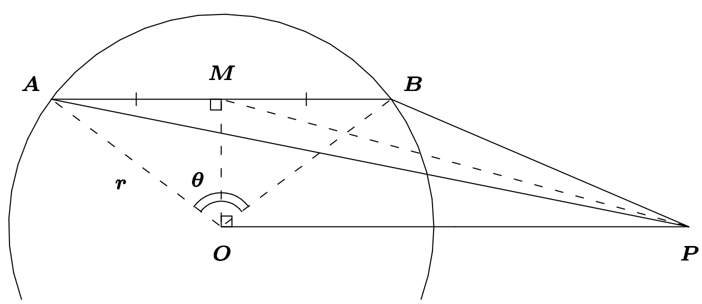
(Note the convention that 0 is both purely real and purely imaginary.)

2. Consider a circle of centre O, radius r, and let P be an external point. We draw a chord AB parallel to OP.

(a) Show that $PA^2 + PB^2$ is constant.

(b) Find the length of the chord AB which maximizes the area of the $\triangle ABP$.

Solutions by Michel Bataille, Rouen, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Toshio Seimiya, Kawasaki, Japan; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Seimiya's write-up.



(a) Let M be the mid-point of AB, then $OM \perp AB$. Since $AB \parallel OP$, we have $OM \perp OP$.

Since M is the mid-point of AB, we get

$$\begin{aligned} PA^2 + PB^2 &= 2(PM^2 + AM^2) \\ &= 2\{(OM^2 + OP^2) + (OA^2 - OM^2)\} \\ &= 2(OP^2 + OA^2) \\ &= 2(OP^2 + r^2) \;=\; \text{constant.} \end{aligned}$$

(b) Since $AB \parallel OP$ we have $[ABP] = [ABO]$, where $[XYZ]$ denotes the area of triangle XYZ.

We put $\angle AOB = \theta$, so

$$[ABO] \;=\; \frac{1}{2}OA \cdot OB \sin\theta \;=\; \frac{1}{2}r^2 \sin\theta \;\leq\; \frac{1}{2}r^2 :$$

equality holds when $\theta = 90^\circ$, and $AB = \sqrt{2}\,r$. Therefore, $[ABP]$ is a maximum when $AB = \sqrt{2}\,r$.

3. Six musicians participate in a music festival. At each concert, some of them play music, and the others listen. What is the minimal number of concerts so that each musician listens to all the others?

Solutions by Christopher J. Bradley, Clifton College, Bristol, UK; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's solution.

Call the musicians A, B, C, D, E, F. In all there have to be at least 30 acts of playing and listening since B, C, D, E, F have all to hear A, etc.

Now at one concert the maximum number of acts of playing and listening is 9, when 3 are playing and 3 listening (splits of $2/4$ give only 8 acts). Therefore, 4 concerts are necessary at a minimum. That it can be done in 4 concerts, see the table below

	Playing	Listening
Concert 1	ABC	DEF
2	AEF	BCD
3	BDF	CAE
4	CDE	ABF

4. The sum of two of the roots of the equation

$$x^3 - 503x^2 + (a+4)x - a = 0$$

is equal to 4. Determine the value of a.

Solutions by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's write-up.

We have $a = 1996$ — what a surprise!

Let α, β, γ denote the roots of $x^3 - 503x^2 + (a+4)x - a = 0$ with $\alpha + \beta = 4$.

Since $\alpha + \beta + \gamma = 503$, we deduce that $\gamma = 499$. Then

$$499^3 - 503 \cdot 499^2 + (a+4)499 - a = 0 .$$

Thus, $a = 1996$ as claimed.

5. If a, b, c are positive real numbers, prove the inequality

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 3(b-c)(a-b) .$$

When is the "$=$" sign valid?

Solutions by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Toshio Seimiya, Kawasaki, Japan; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Seimiya's write-up.

We put $X = a - b$ and $Y = b - c$, so that $X + Y = a - c$. Since

$$\begin{aligned} & 2(a^2 + b^2 + c^2 - ab - bc - ca) - 6(b-c)(a-b) \\ = \quad & (a-b)^2 + (b-c)^2 + (c-a)^2 - 6(b-c)(a-b) \\ = \quad & X^2 + Y^2 + (X+Y)^2 - 6XY \\ = \quad & 2(X^2 + Y^2 - 2XY) \\ = \quad & 2(X-Y)^2 \geq 0 \quad \text{(equality holds when } X = Y\text{)} . \end{aligned}$$

Thus, we have

$$2(a^2+b^2+c^2-ab-bc-ca) \;\geq\; 6(b-c)(a-b)\,.$$

Further, we obtain

$$a^2+b^2+c^2-ab-bc-ca \;\geq\; 3(b-c)(a-b)\,.$$

Equality holds when $X = Y$; that is $a-b = b-c$. This implies that $a+c = 2b$.

6. Find, with reasons, all the natural numbers n such that n^2 has only odd digits.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Stewart Metchette, Gardena, CA, USA. We give Metchette's solution.

If n^2 is to have only odd digits, then it must terminate in an odd integer that is an odd quadratic residue of 10: 1, 5 or 9. Hence, for $n = 1$ or 3, $n^2 = 1$ or 9 and contains only odd digits.

Further, all odd squares > 9 must terminate in one of the 22 quadratic residues of 100, of which 11 are odd:

$$01 \;\; 09 \;\; 21 \;\; 25 \;\; 29 \;\; 41 \;\; 49 \;\; 61 \;\; 69 \;\; 81 \;\; 89\,.$$

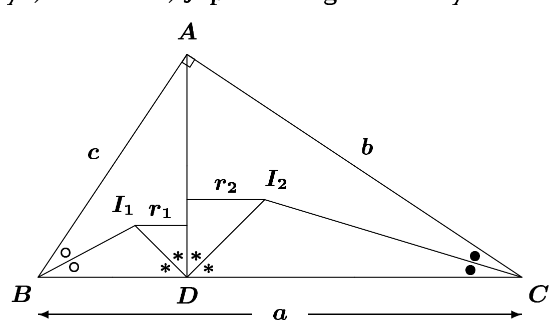
But then the ten's-digit of n^2 is always even and there is no $n^2 > 9$ that contains only odd digits.

Consequently, only for $n = 1$ or 3 does $n^2 = 1$ or 9 have only odd digits.

7. The triangle ABC has $\widehat{A} = 90^\circ$, and AD is the altitude from A. The bisectors of the angles $\widehat{ABD}$ and $\widehat{ADB}$ intersect at I_1; the bisectors of the angles $\widehat{ACD}$ and $\widehat{ADC}$ intersect at I_2.

Find the acute angles of $\triangle ABC$, given that the sum of distances from I_1 and I_2 to AD is $BC/4$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, UK; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's write-up.



We denote the distances from I_1 and I_2 to AD by r_1 and r_2, respectively.

Since I_1 and I_2 are incentres of $\triangle ABD$ and $\triangle ACD$, respectively, then r_1 and r_2 are inradii of $\triangle ABD$ and $\triangle ACD$, respectively.

Since $\angle ADB = \angle ADC = 90°$, we have

$$2r_1 \;=\; AD + BD - AB \quad \text{and} \quad 2r_2 \;=\; AD + DC - AC$$

so that

$$2(r_1 + r_2) \;=\; 2AD + BC - AB - AC\,. \tag{1}$$

We put $BC = a$, $CA = b$ and $AB = c$. Since $\angle BAC = 90°$ and $AD \perp BC$ we get

$$AD \cdot BC \;=\; AB \cdot AC\,.$$

Thus, we have $AD = \frac{bc}{a}$. Since $r_1 + r_2 = \frac{BC}{4}$, we obtain from (1)

$$\frac{a}{2} \;=\; \frac{2bc}{a} + a - b - c\,.$$

Multiplying both sides by $2a$, we get

$$a^2 \;=\; 4bc + 2a^2 - 2a(b+c)\,.$$

Thus, $a^2 - 2a(b+c) + 4bc \;=\; 0$, so that

$$(a - 2b)(a - 2c) \;=\; 0\,.$$

Therefore, either $a - 2b = 0$ or $a - 2c = 0$. If $a - 2b = 0$, then $a = 2b$, since $\angle A = 90°$ so that we get $\angle B = 30°$ and $\angle C = 60°$. If $a - 2c = 0$, similarly, we get $\angle C = 30°$ and $\angle B = 60°$. Therefore, the acute angles of $\triangle ABC$ are $60°$ and $30°$.

8. For each real number x, we denote by $\lfloor x \rfloor$ the biggest integer which is less than or equal to x. We define

$$q(n) \;=\; \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor, \quad n = 1, 2, 3, \ldots.$$

(a) Forming a table with the values of $q(n)$ for $1 \le n \le 25$, make a conjecture about the numbers n for which $q(n) > q(n+1)$.

(b) Determine, with reasons, all the positive integer n such that

$$q(n) \;>\; q(n+1)\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's write-up.

(a) The conjecture must be that $q(n) > q(n+1)$ if and only if $n = m^2 - 1$, where m is an integer ($m > 1$).

(b) If $n = m^2 - 1$, then

$$q(n) = \left\lfloor \frac{m^2-1}{\lfloor\sqrt{m^2-1}\rfloor} \right\rfloor = \left\lfloor \frac{m^2-1}{m-1} \right\rfloor = m+1\,,$$

whereas

$$q(n+1) = \left\lfloor \frac{m^2}{\lfloor\sqrt{m^2}\rfloor} \right\rfloor = \left\lfloor \frac{m^2}{m} \right\rfloor = m < q(n)\,.$$

Apart from these occasional decreases in the value of $q(n)$ when n is a perfect square, it is the case that $q(n+1) \geq q(n)$. To prove this, it is sufficient to show

$$q(m^2+k) \geq q(m^2+k-1) \quad \text{for} \quad 1 \leq k \leq 2m\,.$$

This is in fact trivial, since $m^2 + k > m^2 + k - 1$ and $\left\lfloor\sqrt{m^2+k}\right\rfloor = \left\lfloor\sqrt{m^2+k-1}\right\rfloor = m$ for such values of k.

Next we turn to solutions to some of the problems of the 20th Austrian-Polish Mathematical Competition 1997 given [2000 : 197–199].

1. P is the common point of straight lines l_1 and l_2. Two circles S_1 and S_2 are externally tangent at P and l_1 is their common tangent line. Similarly, two circles T_1 and T_2 are externally tangent at P and l_2 is their common tangent line. The circles S_1 and T_1 have common points P and A, the circles S_1 and T_2 have common points P and B, the circles S_2 and T_2 have common points P and C, and the circles S_2 and T_1 have common points P and D. Prove that the points A, B, C, D lie on a circle if and only if the lines l_1 and l_2 are perpendicular.

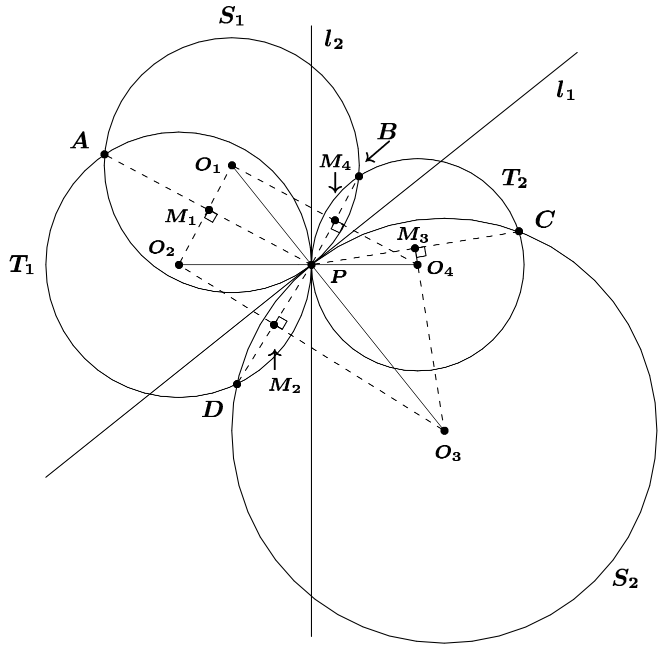
Solution by Toshio Seimiya, Kawasaki, Japan.

See figures on page 301.

Let O_1, O_2, O_3 and O_4 be centres of S_1, T_1, S_2 and T_2, respectively. Since S_1 and S_2 touch l_1 at P, then $O_1P \perp l_1$ and $O_3P \perp l_1$. Thus, O_1, P, and O_3 are collinear and $O_1O_3 \perp l_1$.

Similarly, O_2, P, and O_4 are collinear and $O_2O_4 \perp l_2$. Let PA meet O_1O_2 at M_1. Since S_1, T_1 intersect at P and A, M_1 is the mid-point of PA, and $PM_1 \perp O_1O_2$.

Let M_2, M_3 and M_4 be the intersections of PD, O_2O_3, PC, O_3O_4 and PB, O_4O_1, respectively. Then M_2, M_3 and M_4 are mid-points of O_2O_3,

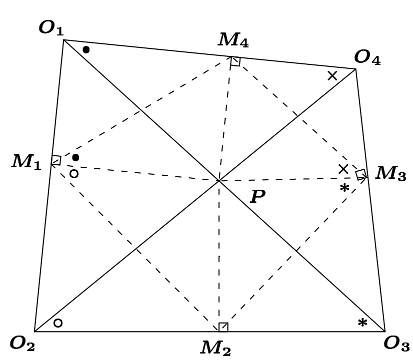


O_3O_4 and O_4O_1, respectively, and $PM_2 \perp O_2O_3$, $PM_3 \perp O_3O_4$, and $PM_4 \perp O_4O_1$.

Since M_1, M_2, M_3, and M_4 are mid-points of PA, PD, PC and PB, respectively, we have

$$\begin{aligned} A,\ B,\ C,\ D \text{ are concyclic} &\iff M_1,\ M_2,\ M_3,\ M_4 \text{ are concyclic.} \\ &\iff \angle M_2M_1M_4 + \angle M_2M_3M_4 = 180^\circ . \end{aligned}$$

Since $\angle PM_1O_1 = \angle PM_4O_1 = 90^\circ$, it follows that O_1, M_1, P, M_4 are concyclic, so that $\angle PM_1M_4 = \angle PO_1M_4 = \angle PO_1O_4$. Similarly, we have $\angle PM_1M_2 = \angle PO_2O_3$, $\angle PM_3M_2 = \angle PO_3O_2$ and $\angle PM_3M_4 = \angle PO_4O_1$.



Thus,

$$\begin{aligned}
&\angle M_2M_1M_4 + \angle M_2M_3M_4 \\
&\quad= \angle PM_1M_4 + \angle PM_1M_2 + \angle PM_3M_2 + \angle PM_3M_1 \\
&\quad= \angle PO_1O_4 + \angle PO_2O_3 + \angle PO_3O_2 + \angle PO_4O_1 \\
&\quad= (\angle PO_1O_4 + \angle PO_4O_1) + (\angle PO_2O_3 + \angle PO_3O_2) \\
&\quad= \angle O_1PO_2 + \angle O_1PO_2 \;=\; 2\angle O_1PO_2\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
\angle M_2M_1M_4 + \angle M_2M_3M_4 \;=\; 180^\circ \quad &\Longleftrightarrow \quad 2\angle O_1PO_2 \;=\; 180^\circ \\
&\Longleftrightarrow \quad \angle O_1PO_2 \;=\; 90^\circ\,.
\end{aligned}$$

Since $O_1P \perp l_1$ and $O_2P \perp l_2$, we have that l_1 and l_2 coincide with O_2P and O_1P, respectively.

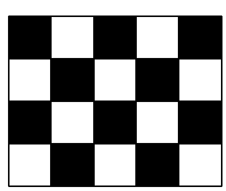
Thus, $\angle O_1PO_2 = 90^\circ$ if and only if $l_1 \perp l_2$. Therefore, we have that A, B, C, D are concyclic if and only if $l_1 \perp l_2$.

2. Let m, n, p, q be positive integers. We have a rectangular chess-board of dimensions $m \times n$ divided into $m \cdot n$ equal squares. The squares are referred to by their coordinates (x, y), where $1 \le x \le m$, $1 \le y \le n$. There is a piece on each square. A piece can be moved from the square (x, y), to the square (x', y') if and only if $|x - x'| = p$ and $|y - y'| = q$. We want to move each piece simultaneously so as to get again one piece on each square. Find the number of ways in which such a multiple move can be made.

Solution by Mohammed Aassila, Strasbourg, France.

If m is odd, then there are more black squares in the top row than in the second one (see the figure below). Hence, the move is impossible (for $p = q = 1$). Similarly, if n is odd, the move is impossible. If m and n are even, then the move is possible in only one way.

If p and q are general positive integers. We divide the $m \times n$ squares into $p \times q$ classes based on their x–coordinate $\pmod p$ and y–coordinate $\pmod q$. Hence, the problem is reduced to the case when $p = q = 1$. Thus, if $2p \mid m$ and $2q \mid n$, there is only one way in which such a move can be made, and it is impossible otherwise.



3. On the blackboard there are written the numbers 48, 24, 16, $\dots$, $\frac{48}{97}$; that is, rational numbers $\frac{48}{k}$ with $k = 1$, 2, 3, $\dots$, 97. In each step two arbitrarily chosen numbers a and b are cancelled and the number $2ab - a - b + 1$ is written on the blackboard. After 96 steps there is only one number on the blackboard. Determine the set of numbers which are possible outcomes of the procedure.

Solution by Pierre Bornsztein, Pontoise, France.

The only possible outcome is $\frac{1}{2}$.

Let E_i be the set of all numbers written on the blackboard after the i^{th} step. First note that $\frac{1}{2} = \frac{48}{96} \in E_0$.

Let $i \geq 1$ be a fixed integer. Suppose that $\frac{1}{2} \in E_{i-1}$. At the i^{th} step, we choose a and b.

- if neither a nor b is equal to $\frac{1}{2}$, then $\frac{1}{2} \in E_i$.
- if $a = \frac{1}{2}$, for example, then $2ab - a - b + 1 = \frac{1}{2}$.

Thus, $\frac{1}{2} \in E_i$. Then, in all cases $\frac{1}{2} \in E_i$.

We easily deduce by induction that $\frac{1}{2} \in E_i$ for $i = 0, 1, \ldots, 96$. Since E_{96} contains only one number, we must have $E_{96} = \left\{\frac{1}{2}\right\}$, as claimed.

4. In a convex quadrilateral $ABCD$ the sides AB and CD are parallel, the diagonals AC and BD intersect at point E and points F and G are the orthocentres of the triangles EBC and EAD, respectively. Prove that the midpoint of the segment GF lies on the line k perpendicular to AB such that $E \in k$.

Solutions by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.

We denote

$$\angle DAB = \alpha, \quad \angle ABC = \beta, \quad BC = b, \quad AD = d.$$

$$AB \parallel CD \implies d \sin\alpha = b \sin\beta \tag{1}$$

Write $\angle AED = \angle BEC = \varepsilon$ and let G', E', F' be the projections onto AB of G, E, and F, respectively. Then $\angle EGG' = \alpha$, $\angle EFF' = \beta$.

Denote by R_1 the circumradius of $\triangle AED$, and by R_2 that of $\triangle BEC$.

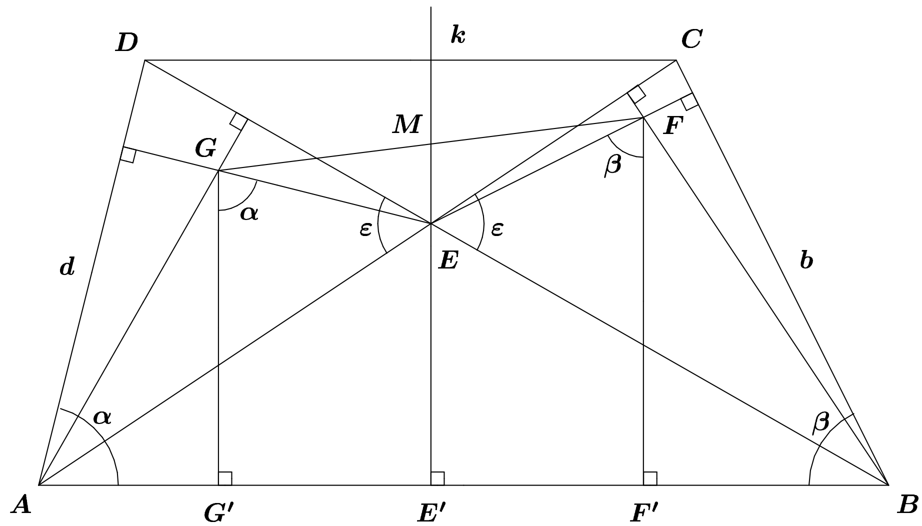
$$d = 2R_1 \sin\varepsilon \implies R_1 = \frac{d}{2\sin\varepsilon}; \; EG = 2R_1\cos\varepsilon = d\cot\varepsilon$$
$$\implies E'G' = EG\sin\alpha = d\cot\varepsilon\sin\alpha. \tag{2}$$

In the same way:

$$E'F' = b\cot\varepsilon\sin\beta \tag{3}$$

$$\text{(1), (2) and (3)} \implies E'G' = E'F'. \tag{4}$$

(4) $\implies$ M, the mid-point of GF, lies on EE' $(= k)$. (See figure on page 304.)



5. Let p_1, p_2, p_3, and p_4 be four distinct prime numbers. Prove that there does not exist a cubic polynomial $Q(x) = ax^3 + bx^2 + cx + d$ with integer coefficients such that

$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3\,.$$

Solution by Pierre Bornsztein, Pontoise, France.
Let p_1, p_2, p_3, p_4 be four distinct prime numbers.

Suppose, for a contradiction, that there exists $Q(x) = ax^3+bx^2+cx+d$ with $a, b, c, d \in \mathbb{Z}$ and $a \neq 0$, such that $|Q(p_i)| = 3$ for $i = 1, 2, 3, 4$. With no loss of generality, we may suppose that at least two of the numbers $Q(p_i)$ are equal to 3 (otherwise we use $-Q$). Define $R(x) = Q(x) - 3$. Then $R \in \mathbb{Z}[x]$ and R has degree 3.

If $Q(p_i) = 3$ for each i: then the polynomial R has four distinct roots, which is impossible, since the degree of R is 3.

If exactly three of the $Q(p_i)$ are equal to 3: with no loss of generality, we may suppose that $Q(p_1) = Q(p_2) = Q(p_3) = 3$ and $Q(p_4) = -3$. Then p_1, p_2, p_3 are the roots of $R(n)$, and we have

$$R(x) = a(x - p_1)(x - p_2)(x - p_3)\,.$$

Thus, $|R(p_4)| = |a|\,|p_4 - p_1|\,|p_4 - p_2|\,|p_4 - p_3| = 6 = 2 \times 3$, with $|a|$, $|p_4 - p_1|$, $|p_4 - p_2|$, $|p_4 - p_3| \in \mathbb{N}^*$. It follows that at least two of these four numbers are equal to 1.

But if $|p_i - p_j| = 1$, then the integers p_i and p_j are consecutive. And, if they are primes, they have to be 2 and 3. Since p_1, p_2, p_3, p_4 are distinct such a situation can occur at most one time. It follows that:

$|a| = 1$ and (by symmetry) $|p_4 - p_1| = 1, \ |p_4 - p_2| = 2, \ |p_4 - p_3| = 3$.

But the difference between two primes is odd if and only if one of these primes is 2. Then $p_4 = 2$. It follows that $p_2 \in \{0; 4\}$, which is impossible.

If exactly two of the $Q(p_i)$ are equal to 3, we may suppose that

$$Q(p_1) = Q(p_2) = 3 \quad \text{and} \quad Q(p_3) = Q(p_4) = -3\,.$$

Let α be the third root of $R(x)$. We then have $p_1 + p_2 + \alpha = -\frac{b}{a}$. Thus, $a\alpha$ is an integer.

Moreover:

$$\begin{aligned} |R(p_3)| &= |p_3 - p_1|\,|p_3 - p_2|\,|ap_3 - a\alpha| = 6 \\ |R(p_4)| &= |p_4 - p_1|\,|p_4 - p_2|\,|ap_4 - a\alpha| = 6 \end{aligned}$$

with $|p_3 - p_1|$, $|p_3 - p_2|$, $|ap_3 - a\alpha|$, $|p_4 - p_1|$, $|p_4 - p_2|$, $|ap_4 - a\alpha| \in \mathbb{N}^*$.

As above, we deduce that:

(a) at least one of the integers $|p_4 - p_1|$ and $|p_4 - p_2|$ is equal to 1 or 3, and therefore, is odd.

(b) At least one of the integers $|p_3 - p_1|$ and $|p_3 - p_2|$ is equal to 1 or 3, and therefore, is odd.

Then:

• if $p_4 = 2$, we have $p_2 \neq 2$ and $p_1 \neq 2$. From (b), we must have $p_3 = 2 = p_4$. A contradiction.

• If $p_4 \neq 2$, then we may suppose that $p_1 = 2$ (from (a)). From (a) we deduce that $p_4 \in \{3, 5\}$, and from (b) that $p_3 \in \{3, 5\}$. We may suppose that $p_3 = 3$ and $p_4 = 5$. Then $p_2 \geq 7$ and $|p_2 - p_3| \geq 4$. Since $|p_2 - p_3|$ divides 6, we must have $p_2 - p_3 = 6$. Thus, $p_2 = 9$, which is not a prime, a contradiction.

And the proof is complete.

That completes the *Corner* for this issue. Send me your nice solutions and Olympiad materials.

BOOK REVIEWS

JOHN GRANT McLOUGHLIN

Mathematical Reflections: In a Room with Many Mirrors, by Peter Hilton, Derek Holton and Jean Pedersen, published by Springer-Verlag, 1997, ISBN 0-387-94770-1, hardcover, 351 pages, 138 illus., US$44.95.

Mathematical Vistas: From a Room with Many Windows, by Peter Hilton, Derek Holton and Jean Pedersen, published by Springer-Verlag, 2002, ISBN 0-387-95064-8, hardcover, 350 pages, 162 illus., US$59.00.
Reviewed by ***G. Eric Moorhouse****, University of Wyoming, Laramie, WY, USA.*

Contributions to the literature of recreational mathematics have proliferated in recent years, although most works in this genre offer less depth than breadth.

These two books—we will call them *Reflections* and *Vistas*—help to fill that void by presenting a limited selection of mathematical topics of general interest (some of which I have listed below) in sufficient detail to satisfy the serious undergraduate student's penchant for completeness and concreteness, as well as the instructor's regard for correctness. Although technically demanding upon our eyes and fingers as we verify the mechanical steps throughout, at least nothing has been swept under the rug in these *Rooms*.

But readers will find in *Reflections* and *Vistas* more than another couple of collections of popular tidbits. They in fact offer us an informal glimpse into the minds of three professional mathematicians. Indeed, their first stated purpose is to realise what the authors call the *basic principle of mathematical instruction,* asserting that "mathematics must be taught so that students comprehend how and why mathematics is done by those who do it successfully." Their second (and equally important) stated purpose, is to attract its readers, by impressing them with the sheer delight of mathematics, to the engagement of mathematical activity. (That is *activity,* not just *study,* a distinction maintained by the authors throughout.) These goals are admirably achieved by a lively selection of mathematical topics delivered in a casual writing style, in which technical arguments are richly interspersed with comments on the guiding principles of mathematical investigation (such as looking for symmetry when possible, favouring conceptual proofs over less illuminating proofs, etc.)

These concrete principles—the authors' advice on how mathematics should be "made", "done", written and taught (thereby transcending the mere "reading" and "learning" of mathematics)—are then listed and explained in the last chapter of *Reflections*. My favourite of these, "Be Optimistic!", encourages us to approach all mathematical endeavours, everything from homework to independent research, with the faith that it all makes sense. Observance of this principle will, in particular, encourage us to state mathematical conjectures in the greatest generality possible.

Reflections and *Vistas* are aimed at "secondary students of mathematics, undergraduate students of mathematics, [and] adults wishing to update and upgrade their mathematical competence." They hit very near the intended level, I think. The reader may approach these books with little more than some appreciation for beauty in mathematics, and moderate patience and modest skills with algebraic manipulation. Groups, graphs, and equivalence relations are introduced as the need arises. Although no comparable preparation is supplied within for the use of mathematical induction, appearances of induction are limited to a couple instances clearly labeled as supplements for the more experienced reader (for example, the "difficult argument" of *Reflections,* pp. 200–202).

The mathematical content of *Reflections* and *Vistas* includes topics not only with strong popular appeal, but also well within the areas of mathematical expertise of the authors (often in fact from areas in which the authors have published research papers individually or jointly). These topics include

- Fibonacci and Lucas numbers^{RV};
- Paper-Folding Constructions of Polygons and Polyhedra^{RV};
- Graph Theory^{V}: Planarity, Chromatic Numbers, Ramsey Theory^{V};
- Infinite Cardinals^{R};
- Fractals and Dimension^{R};
- Binomial and Multinomial Coefficients^{RV}, Catalan numbers^{V};
- Paradoxes^{V};
- Symmetry^{RV}, Groups and Polya Counting^{V};

and much more. (The superscripts ^{R}, ^{V} and ^{RV} indicate topics found in *Reflections, Vistas* or both. Note the significant overlap between these two books, sometimes bordering on redundancy.) Not much harm would result from reading the chapters in a different order than that in which they appear, or from omitting certain chapters entirely. Yet the presentation is nicely unified, in both style and content—surprisingly so, considering the triple authorship. For example, their study of folded paper models provides concrete examples of discrete symmetry groups, but also (more surprisingly) is shown to relate to their earlier work on Fibonacci numbers (since in both settings one finds an integer sequence for which a_m divides a_n whenever m divides n—and this observation inspires an investigation of divisibility properties in more general integer sequences). "All good ideas in mathematics show up in a variety of mathematical and real-world contexts" (*Reflections,* p. 328).

Finally, a word of praise is due for the many illustrations appearing throughout in black and white, or in greyscale; these are carefully rendered and appropriate, never gratuitous. These books will be enjoyed by all, and in particular by the more visual and kinesthetic learners, through the exercises in hand-generating spirals, folding paper and sewing surfaces. Enjoy!

Some generalizations of an inequality from IMO 2001

Oleg Mushkarov and Nikolai Nikolov

The purpose of this paper is to consider some natural generalizations of Problem 2 from IMO 2001 which states:

Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}} \;\geq\; 1\,,$$

where a, b and c are arbitrary positive numbers.

Many different proofs of this inequality were given during the Olympiad and it was also shown by the first author that

$$\frac{a}{\sqrt{a^2+\lambda bc}}+\frac{b}{\sqrt{b^2+\lambda ac}}+\frac{c}{\sqrt{c^2+\lambda ab}} \;\geq\; \frac{3}{\sqrt{1+\lambda}}$$

for arbitrary a, b, $c > 0$ and $\lambda \geq 8$. It is easy to see that the latter inequality is not true for $0 < \lambda < 8$. Moreover, it can be shown that in this case

$$\frac{a}{\sqrt{a^2+\lambda bc}}+\frac{b}{\sqrt{b^2+\lambda ac}}+\frac{c}{\sqrt{c^2+\lambda ab}} \;>\; 1\,,$$

and the lower bound is sharp.

We now prove a general inequality that encompasses all of these results.

Proposition 1. For any positive integers n and m, and any positive numbers x_1, x_2, $\ldots$, x_n with $x_1x_2\ldots x_n = \lambda^n$ $(\lambda > 0)$, we have the following sharp inequality:

$$\sum_{i=1}^{n}\frac{1}{(1+x_i)^{\frac{1}{m}}} \;\geq\; \min\left(1, \frac{n}{(1+\lambda)^{\frac{1}{m}}}\right)\,. \tag{1}$$

Proof. Set

$$d \;=\; \min\left(1, \frac{n}{(1+\lambda)^{\frac{1}{m}}}\right).$$

Multiplying both sides of (1) by $\prod_{i=1}^{n}(1+x_i)^{\frac{1}{m}}$ and then taking the m^{th} power we see that (1) is equivalent to the inequality

$$\sum_{i=1}^{n}\prod_{k=1,k\neq i}^{n}(1+x_k) + T \;\geq\; d^m\prod_{i=1}^{n}(1+x_i)\,, \tag{2}$$

where

$$T \;=\; \left[\sum_{i=1}^{n}\prod_{k=1,k\neq i}^{n}(1+x_k)^{\frac{1}{m}}\right]^m - \sum_{i=1}^{n}\prod_{k=1,k\neq i}^{n}(1+x_k)\,.$$

Denote by $\sigma_1, \sigma_2, \ldots, \sigma_n$, the elementary symmetric functions of the x_i and set $\sigma_0 = 1$. Then it is easy to check that

$$\prod_{i=1}^{n}(1+x_i) \;=\; \sum_{i=0}^{n}\sigma_i \quad \text{and} \quad \sum_{i=1}^{n}\prod_{k=1,k\neq i}^{n}(1+x_k) \;=\; \sum_{i=0}^{n-1}(n-i)\sigma_i\,.$$

Hence, (2) can be rewritten as

$$\sum_{i=0}^{n-1}(n-i-d^m)\sigma_i + T \;\geq\; d^m\sigma_n$$

By the AM-GM inequality we have

$$\sigma_i \geq \binom{n}{i}(\sigma_n)^{\frac{i}{n}} \;=\; \binom{n}{i}\lambda^i\,, \quad 0 \leq i \leq n\,, \tag{3}$$

and, therefore,

$$\prod_{i=1}^{n}(1+x_i) \;=\; \sum_{i=0}^{n}\sigma_i \geq \sum_{i=0}^{n}\binom{n}{i}\lambda^i = (1+\lambda)^n\,. \tag{4}$$

To estimate the term T we use the following inequality

$$\left(\sum_{i=1}^{n}a_i\right)^m \;\geq\; \sum_{i=1}^{n}a_i^m + (n^m-n)\left(\prod_{i=1}^{n}a_i\right)^{\frac{m}{n}} \quad \text{for} \quad a_i > 0\,, \tag{5}$$

which follows easily by induction on m. Setting

$$a_i \;=\; \prod_{k=1,k\neq i}^{n}(1+x_k)^{\frac{1}{m}}$$

in (5) gives

$$T \geq (n^m - n) \prod_{i=1}^{n} (1 + x_i)^{\frac{n-1}{n}},$$

and, therefore, (4) implies that

$$T \geq (n^m - n)(1 + \lambda)^{n-1}. \tag{6}$$

In view of (3), (6) and the fact that $d \leq 1$, to prove (2), it is sufficient to show that

$$d^m \lambda^n - (n^m - n)(1 + \lambda)^{n-1} - \sum_{i=0}^{n-1} (n - i - d^m) \binom{n}{i} \lambda^i \leq 0. \tag{7}$$

But the left hand side of (7) is equal to $(1 + \lambda)^{n-1}(d^m(1 + \lambda) - n^m)$ (this can be seen, for example, by comparing the coefficients of the powers of λ in both expressions) and the inequality (7) follows since

$$d \leq \frac{n}{(1 + \lambda)^{\frac{1}{m}}}.$$

Note that, if $\lambda \geq n^m - 1$, then $d = n(1 + \lambda)^{-\frac{1}{m}}$ and (1) tells us that

$$\sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\frac{1}{m}}} \geq \frac{n}{(1 + \lambda)^{\frac{1}{m}}}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n = \lambda$. On the other hand, if $\lambda < n^m - 1$, then $d = 1$ and (1) takes the form

$$\sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\frac{1}{m}}} > 1.$$

To see that the latter inequality is sharp, set $x_1 = x_2 = \cdots = x_{n-1} = \frac{1}{t}$ and $x_n = t^{n-1}\lambda^n$, where $t \to 0$.

Now, we shall show that the inequality (1) still holds if we replace the power $\frac{1}{m}$ by any real number $\alpha \in (0, 1]$. In this case, however, it is not possible to proceed as in the proof of Proposition 1, since inequality (5) is not true for any real number $m > 1$ and any positive integer n (take, for example, $m = \frac{3}{2}$, $n = 2$, $x_1 = 1$, $x_2 = \frac{1}{16}$). Instead, we shall use the powerful Lagrange multiplier criterion.

Proposition 2. For any $\alpha \in (0, 1]$ and any positive numbers $x_1, x_2, \ldots, x_n$ with $x_1 x_2 \ldots x_n = \lambda^n$ ($\lambda > 0$), we have the following sharp inequality:

$$\sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\alpha}} \geq \min\left(1, \frac{n}{(1 + \lambda)^{\alpha}}\right). \tag{8}$$

Proof. Denote by d the infimum of the function

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \frac{1}{(1+x_i)^\alpha}$$

on the set

$$A = \{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 x_2 \ldots x_n = \lambda^n , x_1, x_2, \ldots , x_n > 0 \} .$$

Suppose first that this infimum is not attained at a point of A. Then, $d = \lim_{k\to\infty} f(x_1^{(k)}, \ldots, x_n^{(k)})$, where, for example, $\lim_{k\to\infty} x_n^{(k)} = 0$ or $+\infty$. Then, for example, $\lim_{k\to\infty} x_1^{(k)} = +\infty$ or 0 and, in both cases, we see that $d \geq 1$. Note that if $\lim_{k\to\infty} x_s^{(k)} = +\infty$ for $s = 1, 2, \ldots, n-1$ and $\lim_{k\to\infty} x_n^{(k)} = 0$, then $\lim_{k\to\infty} f(x_1^{(k)}, \ldots, x_n^{(k)}) = 1$, which shows that $d = 1$. Now, let d be attained at a point of A. Consider the function

$$F(x_1, x_2, \ldots, x_n, \mu) = f(x_1, x_2, \ldots, x_n) + \mu(x_1 x_2 \ldots x_n - \lambda^n) .$$

Then the Lagrange multiplier criterion says that d is attained at a point $(x_1, x_2, \ldots, x_n) \in A$ such that

$$\frac{\partial F}{\partial x_i} = -\frac{\alpha}{(1+x_i)^{\alpha+1}} + \frac{\mu x_1 \ldots x_n}{x_i} = 0 ;$$

that is, when

$$\frac{x_i}{(1+x_i)^{\alpha+1}} = \frac{x_j}{(1+x_j)^{\alpha+1}} , \; 1 \leq i, j \leq n . \tag{9}$$

Consider the function

$$g(x) = \frac{x}{(1+x)^{\alpha+1}} .$$

Then,

$$g'(x) = \frac{1-\alpha x}{(1+x)^{\alpha+2}} ,$$

and, therefore, $g(x)$ takes each of its values at most twice. Hence, (9) shows that $x_1 = \cdots = x_k = x$ and $x_{k+1} = \cdots = x_n = y$ for some $1 \leq k \leq n$. If $k = n$, then $x_1 = x_2 = \cdots = x_n = \lambda$ and

$$f(x_1, x_2, \ldots, x_n) = \frac{n}{(1+\lambda)^\alpha} .$$

If $k < n$, then

$$f(x_1, x_2, \ldots, x_n) = \frac{k}{(1+x)^\alpha} + \frac{n-k}{(1+y)^\alpha} \geq \frac{1}{(1+x)^\alpha} + \frac{1}{(1+y)^\alpha} .$$

To prove Proposition 2 it is sufficient to show that

$$\frac{1}{(1+x)^\alpha} + \frac{1}{(1+y)^\alpha} > 1 \tag{10}$$

provided that

$$\frac{x}{(1+x)^{\alpha+1}} = \frac{y}{(1+y)^{\alpha+1}}, \; x \neq y. \tag{11}$$

Set $\beta = 1/\alpha \geq 1$, $z = (1+x)^\alpha$ and $t = (1+y)^\alpha$. Then (10) and (11) can be written, respectively, as $z + t > zt$ and

$$(zt)^\beta = \frac{z^{\beta+1} - t^{\beta+1}}{z - t}.$$

Thus, we have to prove that

$$(z+t)^\beta > \frac{z^{\beta+1} - t^{\beta+1}}{z - t}. \tag{12}$$

Assume that $z < t$ and set $u = z/t < 1$. Applying Bernoulli's inequality twice, we obtain

$$(1+u)^\beta \geq 1 + \beta u > \frac{1 - u^{\beta+1}}{1 - u},$$

which is just the inequality (12).

Remark. Using similar arguments to the ones used in the proof of Proposition 2, one can show that the inequality (8) holds also in the case $\alpha > 1$ and $n \geq \alpha + 1$. Note that if $\alpha > 1$ but $n < \alpha + 1$, then this inequality is not true in general (take, for example, $\alpha = n = 2$, $x_1 = 8$, $x_2 = \frac{1}{50}$).

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MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7 (NEW!)**. The electronic address is

`mayhem-editors@cms.math.ca`

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The other staff member is Paul Ottaway (Dalhousie University).

Editorial

We have a change in the staff here at MAYHEM. Jimmy Chui, author of The Problem of The Month, has left us. Jimmy will be entering fourth year at the University of Toronto and wishes to have a bit more time to concentrate on his undergraduate thesis. We will all miss Jimmy here at MAYHEM, and appreciate the fine work he has done over the years. Best of luck Jimmy!

Joining us, and taking over the column Polya's Paragon, is Paul Ottaway. Paul is starting a Masters program at Dalhousie University in Halifax this fall. He has just completed concurrent B.Math. and B.Ed. degrees at the University of Waterloo and Queen's University, respectively. He comes to us with a lot of experience from participating in mathematics competitions since grade 7, up to working with the CMC on their annual summer problem seminars and working on contest problem setting committees. We welcome Paul to MAYHEM, and know that he is going to bring our readers some interesting material while he is here with us.

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1**, or emailed to

`mayhem-editors@cms.math.ca`

Please include in all correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 March 2003*. Solutions received after this time will be considered only if there is time before publication of the solutions.

To be eligible for this month's MAYHEM TAUNT, solutions must be postmarked *before* 1 January 2003.

NOTE: We will also accept entries to the MAYHEM TAUNT that have been handwritten, scanned, and emailed as long as they are legible.

M51. Proposed by the Mayhem Staff.

You have a deck with cards numbered 1 through 25. You perform the following operations on the deck:

- you place the top card on the bottom of the deck.
- you place the new top card on the bottom of the deck.
- you flip the new top card face up on the table.

You continue this process until all cards are face up on the table. Find the order of the cards in the deck if, when the process is performed, the cards get laid out on the table in the order 1, 2, 3, $\ldots$, 25.

. . . .

Sur une pile de cartes numérotées de 1 à 25, on effectue les opérations suivantes :

– on place la carte du dessus en-dessous de la pile.

– on place la nouvelle carte du dessus en-dessous de la pile.

– on retourne la nouvelle carte du dessus et on la pose sur la table.

On continue ainsi de suite jusqu'à ce que toutes les cartes soient retournées sur la table. Trouver l'ordre des cartes dans la pile si, une fois le processus terminé, les cartes qu'on a retournées sont dans l'ordre 1, 2, 3, $\ldots$, 25.

M52. Proposed by J. Walter Lynch, Athens, GA, USA.

You have two coins. One is a normal half dollar and the other is a fake half dollar with a head on both sides. You randomly toss one of the coins into a drawer and the other coin into another drawer. A man comes into the room and opens one of the drawers. He looks in and sees a head. Question: What is the probability that he is seeing the coin with two heads?

. . . .

On a deux pièces de monnaie. L'une est une pièce d'un dollar normale et l'autre une fausse pièce d'un dollar, avec deux faces. On jette au hasard chacune des pièces dans deux tiroirs différents. Quelqu'un entre dans la chambre et ouvre un des tiroirs et aperçoît une pièce, côté face. Quelle est la probabilité que cette pièce soit celle à deux faces ?

M53. Proposed by the Mayhem Staff.

A circular path is surrounded by 17 stepping stones numbered 0, 1, 2, $\dots$, 16. Sally starts on stone 0 and moves 1 step to stone 1, then 4 steps to stone 5, then 9 steps to step 14 and continues in the following pattern until at last she moves 2002^2 steps and stops (to rest). What stone is Sally standing on while she rests?

. . . .

Un sentier circulaire est entouré de 17 marches numérotées 0, 1, 2, $\dots$, 16. Sophie commence sur la marche 0 et fait 1 pas jusqu'à la marche 1, puis 4 pas jusqu'à la marche 5, puis 9 pas jusqu'à la marche 14 et ainsi de suite, jusqu'à ce qu'elle se déplace de 2002^2 pas et s'arrête (pour se reposer). Quel est le numéro de la marche sur laquelle Sophie se repose?

M54. Proposed by Gary Tupper, Pedagoguery Software Inc., Terrace, BC.

An ellipse with major axis AB and foci F and F' is inscribed in a circle with diameter AB and centre C. P is a point on the ellipse and D is a point on the circle so that radius CD bisects FP. Show that line DP is tangent to the ellipse.

Pedagoguery Software has offered a copy of their software GrafEq *to the first correct solution received by the MAYHEM problems editor.*

. . . .

Une ellipse de grand axe AB et de foyers F et F' est inscrite dans un cercle de diamètre AB et de centre C. Soit P un point sur l'ellipse et D un point sur le cercle de sorte que le rayon CD coupe FP en son milieu. Montrer que la droite DP est tangente à l'ellipse.

Pedagoguery Software a offert une copie de leur logiciel GrafEq *à l'auteur de la première solution correcte reçue par l'éditeur des problèmes du MAYHEM.*

M55. Proposed by the Mayhem Staff.

Find the sum of the first 2002 terms in the following sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \dots .$$

. . . .

Trouver la somme des 2002 premiers termes de la suite suivante

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \dots .$$

M56. Proposed by Vedula N. Murty, Dover, PA, USA.
Prove the identity

$$\left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 = 4 \cos A \cos B \cos C\,,$$

where the sums are cyclic and $A + B + C = \pi$.

Démontrer l'identité

$$\left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 = 4 \cos A \cos B \cos C\,,$$

où les sommes sont cycliques et $A + B + C = \pi$.

Mayhem Problem Solutions

The solutions in this issue are to Australian Mathematics Trust Questions.

M1. [2001 : 322] Four singers take part in a musical round of 4 equal lines, each finishing after singing the round through three times. The second singer begins when the first singer begins the second line, the third singer begins when the first singer begins the third line, the fourth singer begins when the first singer begins the fourth line. Find the fraction of the total singing time that all four are singing at the same time.

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.
Let x be the time it takes to sing each line. Then the total time spent singing is:

Time when the last singer is not singing	$+$	Time when the last singer is singing	$=$	$3x + 12x$
			$=$	$15x$.

The time for which all four singers are singing is:

Time when the first singer stops singing	$-$	Time when the last singer begins singing	$=$	$12x - 3x$
			$=$	$9x$.

Therefore, the fraction of the total singing time that all four are singing at the same time is $\frac{9x}{15x} = \frac{3}{5}$.

M2. [2001 : 322] When 5 new classrooms were built for Wingecarribee School the average class size was reduced by 6. When another 5 classrooms were built, the average class size reduced by another 4. If the number of students remained the same throughout the changes, how many students were there at the school?

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

Let y be the original number of classrooms, and let x be the number of students.

The addition of the first five new classrooms gives $\frac{x}{y} = \frac{x}{y+5} + 6$. Multiplying both sides first by $y(y+5)$ gives

$$\begin{aligned} x(y+5) &= xy + 6y(y+5) \\ 5x &= 6y(y+5)\,. \end{aligned} \tag{1}$$

The second addition of five new classrooms gives $\frac{x}{y+5} = \frac{x}{y+10} + 4$, which simplifies to

$$5x = 4(y+5)(y+10)\,. \tag{2}$$

Equating the righthand sides of (1) and (2), we get:

$$\begin{aligned} 6y(y+5) &= 4(y+5)(y+10) \\ 6y &= 4y + 40 \\ 2y &= 40 \\ y &= 20\,. \end{aligned}$$

Substituting $y = 20$ into (1) or (2) gives $x = 600$, and so there were 600 students at the school.

One incorrect solution was received.

M3. [2001 : 323] How many years in the 21^{st} century will have the property that, dividing their year number by each of 2, 3, 5, and 7 always leaves a remainder of 1?

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

Suppose that n has the desired property. Then $n-1$ is divisible by 2, 3, 5, and 7, and hence, by $2\times3\times5\times7 = 210$. Hence, x is one greater than some multiple of 210. But 2101 satisfies this condition, and thus, no number from 2001 to 2100 inclusive will also satisfy the condition. Therefore, no years in the 21^{st} century will have the stated property.

M4. [2001 : 323] We write down all the numbers 2, 3, $\ldots$, 100, together with all their products taken two at a time, their products taken three at a time, and so on up to and including the product of all 99 of them. Find the sum of the reciprocals of all the numbers written down.

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

When adding the reciprocals of all the numbers, we use the common denominator $2 \times 3 \times \cdots \times 100$. The numerator of the sum of the reciprocals is the straightforward sum of all the numbers written down plus 1 (excluding the entire product). Add and subtract 1 to this sum (keeping the sum unchanged), making sure to write the first 1 as $\frac{2\times3\times\cdots\times100}{2\times3\times\cdots\times100}$. Then, the numerator factors and we obtain the following expression for the sum:

$$\frac{(2+1)(3+1)\dots(100+1)}{2\times3\times\cdots\times100} - 1 = \frac{2+1}{2}\times\frac{3+1}{3}\times\cdots\times\frac{100+1}{100} - 1$$
$$= \frac{3}{2}\times\frac{4}{3}\times\cdots\times\frac{101}{100} - 1 = \frac{101}{2} - 1 = \frac{99}{2}.$$

M5. [2001 : 323] The ratio of the speeds of two trains is equal to the ratio of the time they take to pass each other going in the same direction to the time they take to pass each other going in the opposite directions. Find the ratio of the speeds of the two trains.

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

Let x be the speed of the faster train, and let y be the speed of the slower train. Now, the ratio of the time taken for the two trains to pass each other is equal to the ratio of their relative speeds as they pass each other, since in both cases the faster train covers the same distance relative to the slower train. Thus,

$$\frac{x}{y} = \frac{x+y}{x-y}\,, \qquad x(x-y) = y(x+y)\,,$$
$$x^2 - 2xy = y^2\,, \qquad x^2 - 2xy + y^2 = 2y^2\,,$$
$$(x-y) = \sqrt{2}y\,, \qquad x = (1+\sqrt{2})y\,;$$

whereupon, $\frac{y}{x} = \sqrt{2} - 1$.

M6. A city railway network has for sale one-way tickets for travel from one station to another station. Each ticket specifies the origin and destination. Several new stations were added to the network, and an additional 76 different ticket types had to be printed. How many new stations were added to the network?

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

If x is the initial number of stations, then we start with $x(x-1)$ tickets. Then, if y new stations are built, there are $(x+y)(x+y-1)$ tickets in all. Thus, there are $2xy + y^2 - y$ new tickets, and we get $y(2x+y-1) = 76$, so that y must divide 76. Because $x > 0$, we have $y < 2x + y - 1$; thus, y is the smaller factor in the equation $y(2x+y-1) = 76$. Hence, y is 1, 2, or 4. But y cannot be 1 since several new stations were added. $y = 2$ forces x to be a fraction, which is not possible. $y = 4$ gives $x = 8$, so that 4 new stations were added to the network.

Pólya's Paragon

Paul Ottaway

"Beauty" and "Mathematics" are two words that you do not often hear in the same sentence — unless, of course, you are talking to a mathematician. Many people will call mathematics useful, difficult, mechanical or even boring — but beautiful? I hope to show through a couple of examples where I think this beauty lies.

One of the most famous mathematical theorems in the world is one attributed to Pythagoras dealing with the relationship between the sides of a right-angled triangle. No doubt the phrase "$a^2+b^2=c^2$" has a permanent place in the mind of most of my readers. This theorem has been examined by a countless number of people ever since it was first conceived. Moreover, dozens of proofs to his theorem have been found. You might ask yourself, why after all this time do people still examine this theorem and try to find other unique proofs. To a mathematician, this would be equivalent to asking a painter why he would bother painting a sunset, since so many have been painted in the past. Each proof has its own beauty (or lack thereof) due to its elegance, succinctness, imagination or any number of other traits.

To further illustrate my point, here is the problem that was the inspiration for this article:

Prove that, for all $n \in \mathbb{N}$, the sum of the elements in the n^{th} row of Pascal's triangle is 2^n.

Pascal's Triangle

Row 1			1		1		
Row 2		1		2		1	
Row 3	1		3		3		1
Etc.							

Each row begins and ends with a 1. Every other element is formed by taking the sum of the two numbers above to the right and above to the left of its position.

Solution # 1:

We know that the numbers in the n^{th} row of Pascal's triangle are the elements

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}.$$

Therefore, the problem is equivalent to showing that their sum is 2^n.

From the binomial theorem, we know that

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n .$$

If we now let $x = 1$, we get the desired result.

Solution # 2:

Imagine we have n objects and we want to create a subset of a particular size.

There are $\binom{n}{0}$ ways of making a subset of 0 elements.

There are $\binom{n}{1}$ ways of making a subset of 1 element.

$\vdots$

There are $\binom{n}{n}$ ways of making a subset of n elements.

By summing these terms, we will have counted all possible ways of making an arbitrary subset of the n objects. Another way of counting the total number of subsets would be as follows: Each object is either in the subset or it is not, so that we have 2 choices for each of the n elements. Therefore, there are 2^n subsets possible. We have now counted these subsets two different ways to get the following relationship:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n ,$$

and we arrive at the desired result.

Solution # 3:

When constructing Pascal's Triangle, each number is added twice to the following row to help form the numbers down and to the right and down and to the left (you can think of the 1's as being $(0+1)$ from the previous row). Therefore, the sum of the elements of any row must be twice the sum of the numbers in the previous row. Here is an example showing how row 4 is constructed from row 3:

Row 3		1		3		3		1		
Summations	$(0+1)$		$(1+3)$		$(3+3)$		$(3+1)$		$(1+0)$	
Row 4	1		4		6		4		1	

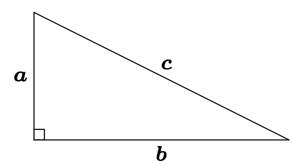
It is easy to see that the sum of the numbers in the first row is $2 = 2^1$, so that, by induction, it must be the case that the n^{th} row has a sum of 2^n.

These three proofs use almost entirely unrelated methods to come up with the same result. I am sure that there are at least two or three other really amazing proofs that I have never seen. This is one of the aspects of mathematics that I find the most aesthetically pleasing. It is also the reason why people will pursue a problem even when a solution is known. It would be a very boring world indeed if artists only ever painted things that had never been painted before.

Problems for you to try:

The following problems have many different solutions, so that, even if you find a good proof, try to find another! If you get stuck on a problem put it aside for a while and try again later with a different approach. Some solutions are much easier to come by than others.

1. (Pythagorean Theorem) In the right angle triangle shown, prove that $a^2 + b^2 = c^2$.



   (Try to find at least three different proofs. Hundreds have been found so far. It is not cheating to look them up as long as you understand the solution!)

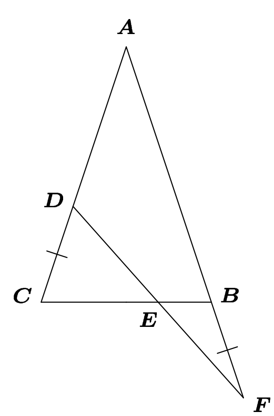
2. For all positive values of n we define $h(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

   Prove that $n + h(1) + h(2) + h(3) + \cdots + h(n-1) = nh(n)$.

   (I have discovered a nice direct proof as well as one that uses induction — try to find them both and more!)

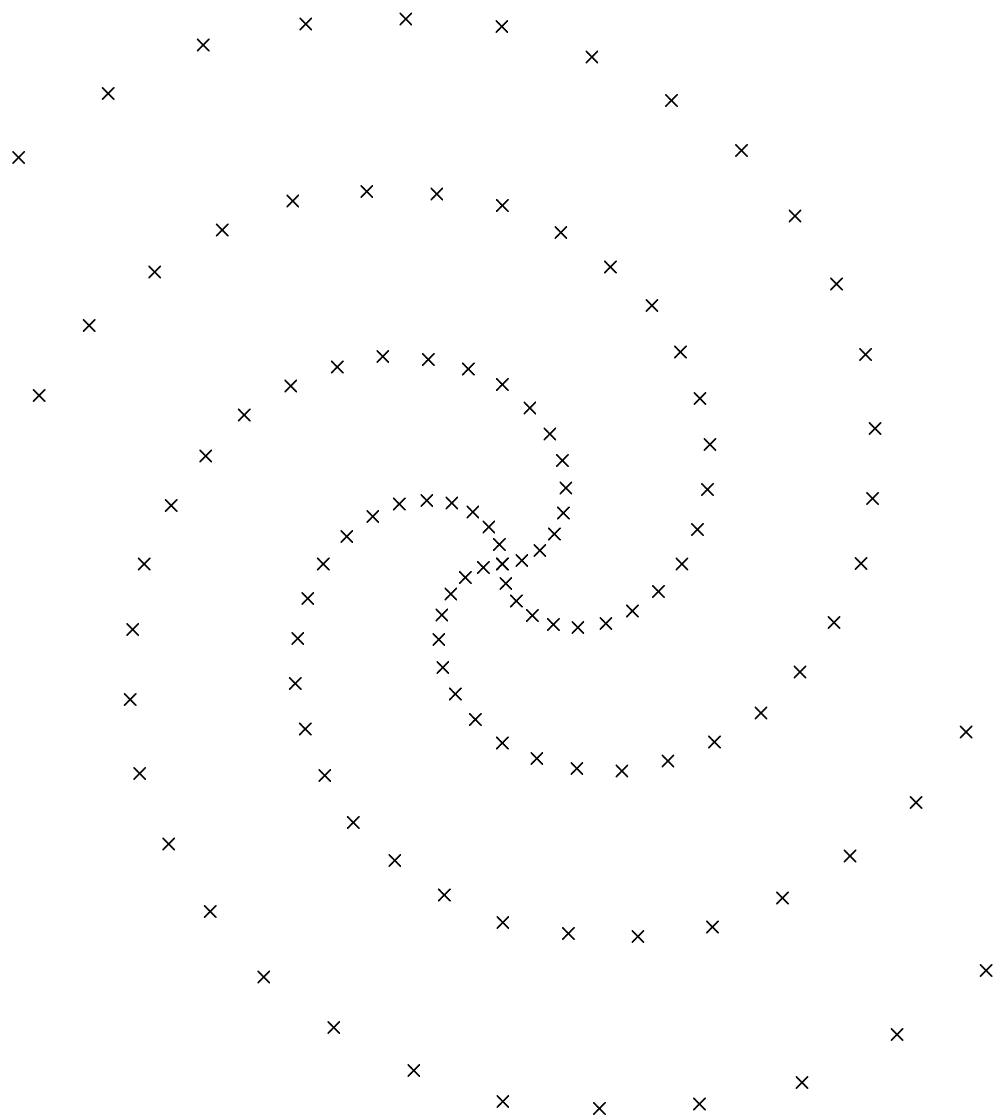
3. In the given diagram, $AC = AB$ and $CD = BF$.

   Show that $DE = EF$.



So that you can try the problem without the diagram, triangle ABC is isosceles with $AC = AB$. Extend AB to a point F. Draw a line from F to a point D on AC such that $CD = BF$. Label the intersection of this line with BC as the point E.

(I have found at least five or six proofs to this one using trigonometry, constructions, rotations or even co-ordinate geometry. Needless to say, some proofs are more elegant than others!)



SKOLIAD No. 63

Shawn Godin

Solutions may be sent to Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5, or emailed to
mayhem-editors@cms.math.ca.

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 January 2002*. A copy of **MATHEMATICAL MAYHEM Vol. 5** will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

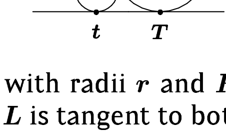
Our item this issue is the 2000 Concours de Mathématiques des Maritimes / Maritime Mathematics Competition. My thanks go out to David Horrocks at the University of Prince Edward Island for forwarding the material to me.

Concours de Mathématiques des Maritimes 2000
2000 Maritime Mathematics Competition

1. Lors d'une réunion de mathématiciens, un des participants remarque que le nombre total de personnes présentes à la réunion est neuf de moins que deux fois le produit des deux chiffres formant ce nombre. Combien de personnes ont assisté à la réunion ?

   At a meeting, one mathematician remarked to another, "There are nine fewer of us here than twice the product of the two digits of our total number." How many mathematicians were at the meeting?

2. Si deux cercles de rayons r et R se coupent en un seul point, et la droite L est tangente aux deux cercles en t et T, respectivement, tel qu'indiqué dans la figure ci-dessous, quelle est la distance entre les points t et T ?



   Suppose that two circles with radii r and R intersect in a single point and that the straight line L is tangent to both circles at t and T, respectively, as in the diagram below. Determine the distance between the points t and T.

3. Trouver la somme de tous les nombres à quatre chiffres dont les chiffres sont choisis, sans répetition, parmi 1, 2, 3, 4, 5. (Il y en a 120.)

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There are 120 four digit numbers that contain only the digits 1, 2, 3, 4, 5, each at most once. Find the sum of all such numbers.

4. Une boîte cubique d'un mètre d'arête est placée contre un mur vertical. Une échelle longue de $\sqrt{15}$ mètres est appuyée contre le mur de telle sorte qu'elle s'appuie également contre l'arête libre du cube. À quelle hauteur l'échelle touche-t-elle au mur ?

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A cubic box with edges 1 metre long is placed against a vertical wall. A ladder $\sqrt{15}$ metres long is placed so that it touches the wall as well as the free horizontal edge of the box. Find at what height the ladder touches the wall.

5. Une pelouse circulaire de 12 mètres de diamètre est traversée d'une allée de gravier de 3 mètres de large dont un des bords passe par le centre de la pelouse. Trouver l'aire du reste de la pelouse.

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A circular grass plot 12 metres in diameter is cut by a straight gravel path 3 metres wide, one edge of which passes through the centre of the plot. Determine the number of square metres in the remaining grass area.

6. Considérons les décompositions d'un échiquier 8×8 en p rectangles, sans chevauchement, et telles que les conditions suivantes soient satisfaites.

   – Chaque rectangle comporte le même nombre de cases blanches et de cases noires.

   – Il n'y a pas deux rectangles qui ont le même nombre de cases.

   Trouver la valeur maximale de p pour laquelle une telle décomposition soit possible. Pour cette valeur maximale de p, déterminer toutes les décompositions correspondantes.

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Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following two conditions.

- Each rectangle has the same number of white squares and black squares.
- No two rectangles have the same number of squares.

Find the maximum value of p for which such a decomposition is possible. For this maximum value of p, determine all corresponding decompositions of the chessboard into p rectangles.

Lastly, here are the official solutions to the 2001 Maritime Mathematics Contest from the December 2001 issue [2001 : 521].

2001 Maritime Mathematics Contest

1. Alice and Bob were comparing their stacks of pennies. Alice said "If you gave me a certain number of pennies from your stack, then I'd have six times as many as you, but if I gave you that number, you'd have one-third as many as me." What is the smallest number of pennies that Alice could have had?

Solution: Let a and b be, respectively, the number of pennies that Alice and Bob had, and let x be the certain number of pennies. From the given information, we obtain the following two equations.

$$\begin{aligned} a + x &= 6(b - x)\,, \\ a - x &= 3(b + x)\,. \end{aligned}$$

From the first equation, $a = 6b - 7x$, and, from the second equation, $a = 3b + 4x$. Therefore, $6b - 7x = 3b + 4x$, so that $b = \frac{11}{3}x$.

Since a, b, and x are required to be positive integers, the smallest possible value for x is 3. Then $b = 11$ and $a = 6(11) - 7(3) = 45$. Therefore, 45 is the smallest number of pennies that Alice could have had.

2. The infinite sequence

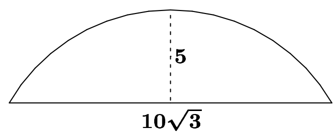
$$1\;2\;3\;4\;5\;6\;7\;8\;9\;1\;0\;1\;1\;1\;2\;1\;3\;1\;4\;1\;5\;1\;6\;1\;7\;1\;8\;1\;9\;2\;0\;2\;1\;2\;2\;2\;3\ldots$$

is obtained by writing the positive integers in order. What is the 2001^{st} digit in this sequence?

Solution: The digits 1, 2, $\ldots$, 9 occupy 9 positions, and the digits in the numbers 10, 11, $\ldots$, 99 occupy $2 \times 90 = 180$ positions. Further, the digits in the numbers 100, 101, $\ldots$, 199 occupy $100 \times 3 = 300$ positions. Similarly, 300 positions are required for the numbers 200 to 299, *et cetera*.

Therefore, the digits in the numbers up to and including 699 occupy the first $9 + 180 + (6 \times 300) = 1989$ positions. A further 12 positions are required to write 700, 701, 702, 703, so that the 2001^{st} digit is the "3" in 703.

3. The maximum height of a railway tunnel is 5 metres and the width of the tunnel is $10\sqrt{3}$ metres. The outline of the tunnel is in the form of a segment of a circle as shown below. Determine the area of a cross-section of the tunnel.



Solution: Consider a circle with centre O and let AB be a chord (which is not a diameter) of the circle. Let P be the point on the circumference of the circle such that OP is the perpendicular bisector of AB. Finally, let X be the point of intersection of AB and OP. Suppose that $|XP| = 5$ and $|AX| = |BX| = 5\sqrt{3}$. The chord AB divides the circle into two sections; the problem is to determine the area of the smaller section. Let r be the radius of the circle so that $|OB| = r$ and $|OX| = r - 5$. Applying the Pythagorean Theorem to $\triangle OXB$, we obtain $(r-5)^2 + \left(5\sqrt{3}\right)^2 = r^2$, from which we get $r = 10$.

Therefore, the lengths of the sides of $\triangle OXB$ are 5, $5\sqrt{3}$, and 10; that is, in the ratio $1 : \sqrt{3} : 2$. Therefore, $\angle XOB = 60°$, so that the area of the sector OAB of the circle is

$$\frac{1}{3} \times \text{area of the entire circle} = \frac{1}{3}\pi(10)^2 = \frac{100\pi}{3}.$$

The area of $\triangle AOB$ is

$$\frac{1}{2} \times |AB| \times |OX| = \frac{1}{2}\left(10\sqrt{3}\right)(5) = 25\sqrt{3},$$

so that the required area is

$$\frac{100\pi}{3} - 25\sqrt{3} \text{ square units.}$$

4. Which of the following numbers is greater?

$$A = \frac{2.0000004}{(1.0000004)^2 + 2.0000004} \quad \text{or} \quad B = \frac{2.0000002}{(1.0000002)^2 + 2.0000002}$$

Solution: Consider the numbers

$$a = \frac{2+2x}{(1+2x)^2 + (2+2x)} = \frac{2+2x}{3+6x+4x^2} \quad \text{and}$$

$$b = \frac{2+x}{(1+x)^2 + (2+x)} = \frac{2+x}{3+3x+x^2},$$

where $x > 0$. Now $a < b$ is equivalent to

$$(2+2x)(3+3x+x^2) \;<\; (2+x)(3+6x+4x^2)\,.$$

Expanding both sides, we obtain

$$6+12x+8x^2+2x^3 \;<\; 6+15x+14x^2+4x^3\,;$$

that is, $2x^3+6x^2+3x > 0$. This inequality is true for any $x > 0$ so, for all such x, $a < b$. Setting $x = 0.0000002$, we have $a = A$ and $b = B$, so that $A < B$; that is, B is greater.

5. Alice and Bob play the following game with a pile of 2001 beans. A move consists of removing one, two, or three beans from the pile. The players move alternately, beginning with Alice. The person who takes the last bean in the pile is the winner. Which player has a winning strategy for this game and what is that strategy?

Solution: Alice has the following winning strategy. On her first move, she takes one bean. On subsequent moves, Alice removes $4-x$ beans, where x is the number of beans removed by Bob on the preceding turn.

We now prove that the above strategy guarantees a win for Alice. After Alice's first move the pile contains 2000 beans. Moreover, after every pair of moves, a move by Bob followed by a move by Alice, the pile decreases by exactly 4 beans. Therefore, after every move by Alice the number of beans in the pile is a multiple of 4. Eventually, after a move by Alice, there will be 4 beans left in the pile. After Bob removes one, two, or three beans, Alice takes the remainder and wins the game.

6. Show that, regardless of what integers are substituted for x and y, the expression

$$x^5-x^4y-13x^3y^2+13x^2y^3+36xy^4-36y^5$$

is never equal to 77.

Solution: The given expression may be factored as follows.

$$\begin{aligned} N &= x^5-x^4y-13x^3y^2+13x^2y^3+36xy^4-36y^5 \\ &= x^4(x-y)-13x^2y^2(x-y)+36y^4(x-y) \\ &= (x-y)(x^4-13x^2y^2+36y^4) \\ &= (x-y)(x^2-4y^2)(x^2-9y^2) \\ &= (x-y)(x+2y)(x-2y)(x+3y)(x-3y)\,. \end{aligned}$$

If $y = 0$ then $N = x^5$ which is not equal to 77 for any integer x. On the other hand, if $y \neq 0$ then the five factors of N are all distinct. However, any expression of 77 as a product of distinct integers contains at most four factors, specifically as $(1)(-1)(7)(-11)$ or $(1)(-1)(-7)(11)$. Therefore, for any choice of x and y, N is never equal to 77.

PROBLEMS

Problem proposals and solutions should be sent to Jim Totten, Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, BC, Canada. V2C 4Z9. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 March 2003**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5 and 7, English will precede French, and in issues 2, 4, 6 and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

5 problems dedicated to Professor Jordi Dou, belatedly for his 90$^{\text{th}}$ birthday.

5 problèmes dédiés tardivement au Professeur Jordi Dou, pour son 90-ième anniversaire.

2751. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

On each side of $\triangle ABC$, draw squares outwards to create six new points, D, E, F, G, H and I. Characterise those triangles such that the points D, E, F, G, H and I are concyclic.

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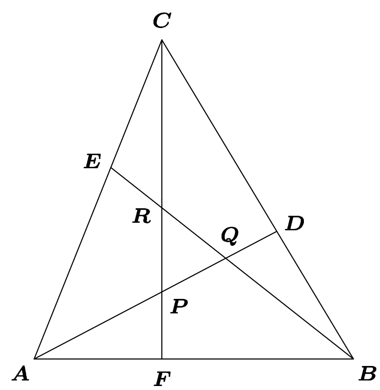
Sur chaque côté d'un triangle ABC, on dessine vers l'extérieur des carrés créant ainsi six nouveaux points D, E, F, G, H et I. Caractériser les triangles tels que les points D, E, F, G, H et I soient sur un même cercle.

2752. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, Newfoundland and Labrador.*

A generalization of Putnam 2001, question A4.

Suppose that $\dfrac{AF}{FB} = i$, $\dfrac{BD}{DC} = g$ and $\dfrac{CE}{EA} = h$. Determine the area of $\triangle PQR$ as a proportion of the area of $\triangle ABC$.



. . . .

Une généralisation du Putnam 2001, question A4.

Supposons que $\dfrac{AF}{FB} = i$, $\dfrac{BD}{DC} = g$ et $\dfrac{CE}{EA} = h$. Determiner l'aire de $\triangle PQR$, une proportion de l'aire de $\triangle ABC$.

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2753. *Proposed by Mikhail Kotchetov, Memorial University of Newfoundland, St. John's, Newfoundland and Labrador.*

Consider two circles, Γ_1 and Γ_2, centres O_1 and O_2, respectively, of different radii.

The two common tangents, t_1 and t_2, that do not intersect the line segment O_1O_2 meet at Q. A common tangent, t_c that does intersect the line segment O_1O_2 meets the tangents t_1 and t_2 at E_1 and E_2, respectively.

Let P be the mid-point of the line segment O_1O_2.

Prove that P, Q, E_1 and E_2 are concyclic.

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Considérons deux cercles Γ_1 et Γ_2, de centres respectifs O_1 and O_2 et de rayons différents.

Soit Q l'intersection des deux tangentes communes t_1 et t_2 qui ne coupent pas le segment O_1O_2. Désignons par t_c une tangente commune coupant O_1O_2, et par E_1 et E_2 les points d'intersection de t_c avec t_1 et t_2, respectivement.

Soit P le point milieu du segment O_1O_2.

Montrer que P, Q, E_1 et E_2 sont sur un même cercle.

2754. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Divide a triangle into five concyclic quadrilaterals.

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Diviser un triangle en cinq quadrilatères, chacun inscriptible dans un cercle.

2755. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{f_{n+1}^2}{1+f_n f_{n+1}^2 f_{n+2}}\right)$$

where f_n is the n^{th} Fibonacci number (that is, $f_0 = 0$, $f_1 = 1$ and, for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$).

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Evaluer

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{f_{n+1}^2}{1+f_n f_{n+1}^2 f_{n+2}}\right)$$

où f_n est le n-ième nombre de Fibonacci (c'est-à-dire, $f_0 = 0$, $f_1 = 1$ et, pour $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$).

2756. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Circles $\Gamma_1(O, R)$ and $\Gamma_2(I, r)$ touch line t at D, where $R > r$ and O and I lie on the same side of t. The point A is any point on Γ_1. The tangents to Γ_2 through A intersect t at B and C, respectively. Denote the inradii of $\triangle ABD$ and $\triangle ACD$ by r_1 and r_2, respectively.

Show that $r_1 + r_2$ is constant as A varies on Γ_1.

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Les cercles $\Gamma_1(O, R)$ et $\Gamma_2(I, r)$ touchent la droite t en D, $R > r$, O et I sont du même côté de t.

Le point A est quelconque sur Γ_1. Les tangentes à Γ_2 passant par A coupent t en B et C, respectivement. Notons r_1 et r_2 les rayons respectifs des cercles inscrits aux triangles $\triangle ABD$ et $\triangle ACD$.

Montrer que $r_1 + r_2$ reste constant lorsque A parcourt Γ_1.

2757★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B and C be the angles of a triangle. Show that

$$\sum_{\text{cyclic}} \frac{1}{\tan\left(\frac{A}{2}\right) + 8\tan\left(\frac{\pi - A}{4}\right)^3} \le \frac{9\sqrt{3}}{11} .$$

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Soit A, B et C les angles d'un triangle. Montrer que

$$\sum_{\text{cyclique}} \frac{1}{\tan\left(\frac{A}{2}\right) + 8\tan\left(\frac{\pi - A}{4}\right)^3} \le \frac{9\sqrt{3}}{11} .$$

2758. *José Luis Díaz and Juan José Egozcue, Universitat Politècnica de Catalunya, Terrassa, Barcelona. Spain.*

If $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \neq 2$, determine all real numbers x, y, z such that

$$\begin{aligned} 0 &= (1+2a^2)x^2 + (1+2b^2)y^2 + (1+2c^2)z^2 \\ &\quad + 2xy(ab-a-b) + 2yz(bc-b-c) + 2zx(ca-c-a) . \end{aligned}$$

. . . .

Si $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \neq 2$, déterminer tous les nombres réels x, y, z tels que

$$\begin{aligned} 0 &= (1+2a^2)x^2 + (1+2b^2)y^2 + (1+2c^2)z^2 \\ &\quad + 2xy(ab-a-b) + 2yz(bc-b-c) + 2zx(ca-c-a) . \end{aligned}$$

2759. *Proposed by Michel Bataille, Rouen, France.*

On the line segment AB, let C, D be such that $\frac{AC}{CB} = \frac{BD}{DA} = \frac{1}{3}$. Distinct points M_1, M_2, M_3 lie on a circle passing through B and C and are such that $\angle M_1BC = 2\angle M_1CB$, $\angle M_2BC = 2\angle M_2CB$, and $\angle M_3AD = 2\angle M_3DA$. Show that $\triangle M_1M_2M_3$ is equilateral.

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Sur le segment de droite AB, soit C, D tels que $\frac{AC}{CB} = \frac{BD}{DA} = \frac{1}{3}$. Des points distincts M_1, M_2, M_3 situés sur un cercle passant par B and C sont tels que $\angle M_1BC = 2\angle M_1CB$, $\angle M_2BC = 2\angle M_2CB$, et $\angle M_3AD = 2\angle M_3DA$. Montrer que le triangle $M_1M_2M_3$ est équilatéral.

2760. *Proposed by Michel Bataille, Rouen, France.*

Suppose that A, B, C are the angles of a triangle. Prove that

$$\begin{aligned} 8(\cos A+\cos B+\cos C) &\leq 9+\cos(A+B)+\cos(B+C)+\cos(C+A) \\ &\leq \csc^2(A/2)+\csc^2(B/2)+\csc^2(C/2)\,. \end{aligned}$$

. . . .

Soit A, B, C les angles d'un triangle. Montrer que

$$\begin{aligned} 8(\cos A+\cos B+\cos C) &\leq 9+\cos(A+B)+\cos(B+C)+\cos(C+A) \\ &\leq \csc^2(A/2)+\csc^2(B/2)+\csc^2(C/2)\,. \end{aligned}$$

2761★. *Proposed by Edgar G. Goodaire, Memorial University, St. John's, Newfoundland and Labrador.*

Give a proof by vectors that the medians of a triangle have a common point of intersection: a proof, however, ***which does not presuppose the answer***.

The vector proofs of this result with which I am familiar answer the question posed this way:

> Prove that the medians of $\triangle ABC$ intersect at $\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)$, where O is the origin.

The proof, of course, then amounts simply to showing that this point is on each median.

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En utilisant les vecteurs, montrer que les médianes d'un triangle se coupent en un point, ***sans toutefois présupposer***.

Les démonstrations vectorielles avec lesquelles je suis familier répondent à cette question posée de la manière suivant :

> Montrer que les médianes du triangle ABC se coupent en $\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)$, où O est l'origine.

La démonstration, bien entendu, se résume alors à montrer que ce point est sur chacune des médianes.

2762. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Quadrilateral $ABCD$ is inscribed in circle Γ. The tangents at A, B, C, D to Γ are t_A, t_B, t_C, t_D, respectively. Given that BD, t_A and t_C are concurrent, prove that AC, t_B and t_D are concurrent.

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Soit $ABCD$ un quadrilatère inscrit dans un cercle Γ. Soit t_A, t_B, t_C, t_D les tangentes respectives à Γ. Sachant que BD, t_A et t_C sont concourantes, montrer que AC, t_B et t_D le sont aussi.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We inadvertantly omitted Murray S. Klamkin, University of Alberta, Edmonton, Alberta, from the list of solvers of 2649, and Michel Bataille, Rouen, France, from the list for 2626. We also erred in the name of Marcelo R. de Souza in the list of solvers for 2645. Sorry, Murray, Michel and Marcelo.

2572. [2000 : 374, 2001 : 473, 2002 : 57] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, c be positive real numbers. Prove that

$$a^b b^c c^a \;\leq\; \left(\frac{a+b+c}{3}\right)^{a+b+c} .$$

[Compare problem **2394** [1999 : 524], note by V.N. Murty on the generalization.]

IV. Murray S. Klamkin has pointed out that in Walter Janous's remarks, he proves the inequality:

$$\left(x_1+x_2+\cdots+x_n\right)^2 \;\geq\; 4\left(x_1x_2+x_2x_3+\cdots+x_nx_1\right) .$$

This inequality from the 1984 Moscow Olympiad appears with solution as #15 in [1985 : 288–289].

2651★. [2001 : 335] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* Dedicated to Professor M.V. Subbarao on the occasion of his 80th birthday. (Professor Klamkin offers a prize of $100 for the first correct solution received by the Editor-in-Chief.)

Let P be a non-exterior point of a regular n–dimensional simplex $A_0A_1A_2\ldots A_n$ of edge length e. If

$$F \;=\; \sum_{k=0}^{n} PA_k + \min_{0\leq k\leq n} PA_k\,, \qquad F' \;=\; \sum_{k=0}^{n} PA_k + \max_{0\leq k\leq n} PA_k\,,$$

determine the maximum and minimum values of F and F'.

This problem was suggested by problem 2594 for a general triangle, and the proposer was trying to obtain a stronger inequality by finding the maximum of F.

Solution by Fedor Petrov, Saint Petersburg, Russian Federation; revised by Rudolf Fritsch, Munich, Germany

The crucial notion for this solution is the notion of a *convex function* $f : \mathcal{P} \to \mathbb{R}$; that is, a function f whose domain is a compact convex subset of some Euclidean space $\mathbb{R}^n$ and which satisfies

$$f\left(\frac{M+N}{2}\right) \;<\; \frac{f(M)+f(N)}{2}$$

for all pairs of points M, $N \in \mathcal{P}$, $M \neq N$. The usual definition of a convex function is slightly different, but in the following just the given property is needed. In our case the domain will be either the simplex $\mathcal{T}$ with the vertices A_0, A_1, $\ldots$, A_n or a compact, convex subset of $\mathcal{T}$. For a convex function f (with a convex domain) there is at most one place where f takes a minimal value. Indeed, if there were two different places M, N with this property, then the mid-point of the line segment $[MN]$ would yield a smaller value in contradiction to the assumption. On the other hand, a maximal place of a convex function must be a vertex of the domain $\mathcal{P}$. Recall that a point P is a *vertex* or *extreme point* of a compact convex subset $\mathcal{P}$ of some Euclidean space $\mathbb{R}^n$, if it is not an interior point of a line segment contained in $\mathcal{P}$. Indeed, assume $f : \mathcal{P} \to \mathbb{R}$ to be a convex function and consider a point $P \in \mathcal{P}$ which is not a vertex. Then, P is an interior point of a line segment contained in $\mathcal{P}$ and we can find a smaller line segment $[MN]$ having P as mid-point. By convexity we get

$$f(P) \;<\; \frac{f(M)+f(N)}{2} \;\leq\; \frac{2\cdot\max\{f(M),f(N)\}}{2} \;=\; \max\{f(M),f(N)\}\,;$$

thus, the point P cannot be a maximal place. Note that the vertices of a simplex are the extreme points of the simplex in the sense just described.

Now we turn to the given problem. To begin with, observe that the triangle inequality

$$(M+N)A \;\leq\; MA+NA$$

yields by means of division by 2:

$$KA \;\leq\; \frac{MA+NA}{2}$$

for all triples of points A, M, $N \in \mathbb{R}^n$, $M \neq N$, where K denotes the mid-point of the line segment $[MN]$. Equality holds if and only if the point A belongs to line MN but is not an interior point of the line segment $[MN]$.

From this we conclude that the function $F' : \mathcal{T} \to \mathbb{R}$ under consideration is convex. In fact, given a pair of points M, $N \in \mathbb{R}^n$, $M \neq N$, note first

$$KA_j \;\leq\; \frac{MA_j+NA_j}{2} \;=\; \frac{MA_j}{2}+\frac{NA_j}{2} \;\leq\; \max_{0\leq i\leq n}\frac{MA_i}{2}+\max_{0\leq i\leq n}\frac{NA_i}{2}$$

for all $j \in \{0, 1, \dots, n\}$, implying that

$$\max_{0\le i\le n} KA_i \;\le\; \max_{0\le i\le n} \frac{MA_i}{2} \;+\; \max_{0\le i\le n} \frac{NA_i}{2} i\,.$$

Secondly, there is at least one vertex A_i which is not on the line MN and thus,

$$\begin{aligned} F'(K) &= \sum_{i=0}^{n} KA_i + \max_{0\le i\le n} KA_i \\ &< \sum_{i=0}^{n} \frac{MA_i + NA_i}{2} + \max_{0\le i\le n} \frac{MA_i}{2} + \max_{0\le i\le n} \frac{NA_i}{2} i \\ &= \frac{F'(M) + F'(N)}{2}\,. \end{aligned}$$

The function $F' : \mathcal{T} \to \mathbb{R}$ has a compact domain and is continuous; thus, it has a maximal and a minimal value. Let M_1' denote a point where F' takes the maximal value and M_2' the unique point where F' takes the minimal value.

By the convexity of F' the point M_1' must be a vertex of the simplex $\mathcal{T}$; that is, there is an index $j \in \{0, 1, \dots, n\}$ such that

$$F'(M_1') \;=\; F'(A_j) = (n+1)\cdot e\,.$$

Therefore,

$$(n+1)\cdot e$$

is the maximal value of the function F'.

By definition, the function F' is invariant under a permutation of the vertices $A_0, A_1, \dots, A_n$. If M_2' were different from the circumcentre of $\mathcal{T}$, then a point M_2'' obtained by a permutation of the barycentric coordinates of M_2 would be also a minimal place, in contradiction to the uniqueness of the minimal place of a convex function. Thus, the point M_2' is the circumcentre C of T having equal distances from all vertices of the simplex $\mathcal{T}$. Since the circumradius of a regular n–simplex with edge length e is computed to be

$$\rho_n \;=\; e\cdot\sqrt{\frac{n}{2(n+1)}}$$

the minimal value of F' is

$$(n+2)\cdot CA_0 \;=\; (n+2)\cdot e\cdot\sqrt{\frac{n}{2(n+1)}}\,.$$

Now we discuss the function F. For $j \in \{0, 1, \dots, n\}$ define

$$\mathcal{T}_j \;=\; \{P \in \mathcal{T} \mid PA_j \;=\; \min_{0\le i\le n} PA_i\}\,.$$

Since by definition the function F is invariant under a permutation of the vertices $A_0, A_1, \ldots, A_n$, it is sufficient to compute the extreme values of the restriction $F_0 = F|\mathcal{T}_0$. The proof of the convexity of the function F' given above shows that the function F_0 is convex. But F_0 is also continuous, so that it has at least one maximal place and a unique minimal place M_1.

We claim that M_1 must belong to the line segment $[A_0C]$ where C, as above, denotes the circumcentre of the simplex $\mathcal{T}$. An arbitrary point $M \in \mathcal{T}_0$ has a unique representation of the form

$$M = \sum_{i=0}^{n} r_i A_i$$

with $r_i \in [0,1]$ for all $i \in \{0, 1, \ldots, n\}$, such that $\sum_{i=0}^{n} r_i = 1$, and $r_0 = \max\{r_i | i \in \{0, 1, \ldots, n\}\}$; it belongs to the line segment $[A_0C]$ if and only if $r_1 = r_2 = \cdots = r_n$. The scalars r_i are the so-called barycentric coordinates of the point M. Now consider a point M not belonging to the line segment $[A_0C]$. Then there are indices j, k with $1 \leq j < k \leq n$ and $r_j \neq r_k$. Form the point

$$N = \sum_{i=0}^{n} \tilde{r}_i A_i$$

by taking

$$\tilde{r}_i = \begin{cases} r_i, & j \neq i \neq k, \\ r_k, & i = j, \\ r_j, & i = k. \end{cases}$$

The points M, N and K, where K denotes, as before, the mid-point of the line segment $[MN]$, all belong to $\mathcal{T}_0$. We have

$$F_0(M) = F(M) = F(N) = F_0(N)$$

and by convexity

$$F_0(K) = F(K) < F(M) = F_0(M).$$

Thus, M cannot be a minimal place of the function F_0.

It follows that the minimal place M_1 of the function F_0 has a representation of the form

$$M_1 = (1-r)A_0 + rC$$

with $r \in [0,1]$. Denote by h_n the altitude of the regular simplex $\mathcal{T}$ (that is,

$$h_n = \frac{n+1}{n} \cdot \rho_n = e \cdot \sqrt{\frac{n+1}{2n}} = \sqrt{\frac{n+1}{n-1}} \cdot \rho_{n-1},)$$

and by s the distance of M_1 from the circumcentre of the face of $\mathcal{T}$ opposite to the vertex A_0 (that is,

$$s = h_n - M_1A_0 = h_n - r \cdot \rho_n) .$$

We compute

$$\begin{aligned} M_1A_1 &= \sqrt{s^2 + \rho_{n-1}^2} = M_1A_2 = M_1A_3 = \cdots = M_1A_n , \\ F_0(M_1) &= F(M_1) = 2 \cdot M_1A_0 + n \cdot M_1A_1 \\ &= 2 \cdot (h_n - s) + n \cdot \sqrt{s^2 + \rho_{n-1}^2} . \end{aligned}$$

Thus, s is a place in the interval $[h_n - \rho_n, h_n]$ where the function

$$g : [h_n - \rho_n, h_n] \to \mathbb{R} , t \mapsto 2 \cdot (h_n - t) + n \cdot \sqrt{t^2 + \rho_{n-1}^2}$$

takes its absolute minimal value. To see that this condition determines s uniquely and to compute the corresponding value of s, we form the derivative

$$g'(t) = -2 + \frac{n \cdot t}{\sqrt{t^2 + \rho_{n-1}^2}} .$$

Now we have two cases.

1. If $n = 2$, then g' is negative on the interval under consideration. Thus, the function g is monotonic decreasing and takes its minimal value at the upper end of the interval; that is, for $t = h_n$. Then, $M_1 = A_0$ and

   $$F_0(M_1) = 2e$$

   is the minimal value of F.

2. If $n > 2$, we find the unique minimal place

   $$s = \frac{2\rho_{n-1}}{\sqrt{n^2 - 4}}$$

   and thus, the minimal value of F is

   $$\begin{aligned} F_0(M_1) &= \rho_{n-1} \left(2\sqrt{\frac{n+1}{n-1}} + \sqrt{n^2 - 4} \right) \\ &= \frac{e}{\sqrt{2n}} \cdot (2\sqrt{n+1} + \sqrt{(n-1)(n-2)(n+2)}) . \end{aligned}$$

Finally, we are looking for the maximal value of F. Let M_2 be a maximal place for the function F_0. It is a vertex of the convex polytope $\mathcal{T}_0$. The vertices of this polytope are the vertex A_0 of the simplex $\mathcal{T}$ and the circumcentres of the faces of the simplex $\mathcal{T}$ containing the vertex A_0 (to be proved in the appendix). If C_k is a circumcentre of a k–dimensional face of this sort, then it has $k+1$ barycentric coordinates with value $\frac{1}{k+1}$ and the remaining $n-k$ barycentric coordinates vanish. Without loss of generality it suffices to consider

$$C_k \;=\; \frac{1}{k+1}\sum_{i=0}^{k} A_i$$

including $C_0 = A_0$ and $\rho_0 = 0$ and to compute

$$\begin{aligned}
C_kA_0 \;&=\; \rho_k \;=\; C_kA_1 \;=\; \cdots \;=\; C_kA_k\,,\\
C_kA_{k+1} \;&=\; e\cdot\sqrt{\frac{k+2}{2(k+1)}} \;=\; C_kA_{k+2} \;=\; \cdots \;=\; C_kA_n\,,\\
F_0(C_k) \;&=\; F(C_k) \;=\; (k+2)\cdot C_kA_0 + (n-k)\cdot C_kA_{k+1}\\
&=\; \frac{e}{\sqrt{2(k+1)}}\cdot\left((k+2)\sqrt{k} + (n-k)\sqrt{k+2}\right).
\end{aligned}$$

Thus, it remains to check, given n fixed, for which $k \in \{0,\, 1,\, \ldots,\, n\}$ the value $F_0(C_k)$ is maximal. There are several cases to distinguish.

1. In the case $n \geq 5$, we claim

   $$F_0(C_k) \;\leq\; F_0(C_0) \;=\; n\cdot e$$

   for all $k \in \{0,\, 1,\, \ldots,\, n\}$. This implies that $n\cdot e$ is the maximal value of the function F in the case $n \geq 5$. To see this, transform the desired inequality to

   $$(k+2)\sqrt{k} + (n-k)\sqrt{k+2} \;\leq\; n\cdot\sqrt{2(k+1)}\,.$$

   For $k = 1$ this inequality becomes

   $$3 + (n-1)\sqrt{3} \;\leq\; n\cdot 2\,,$$

   which is equivalent to $n \geq 1$. Thus, we can restrict our attention to the cases with $2 \leq k \leq n$ and transform the inequality into

   $$(k+2)\sqrt{k} - k\sqrt{k+2} \;\leq\; n\cdot\left(\sqrt{2(k+1)} - \sqrt{k+2}\right).$$

   Multiplication by $\sqrt{2(k+1)} + \sqrt{k+2}$ and division by k yield

   $$\left(\frac{k+2}{\sqrt{k}} - \sqrt{k+2}\right)\cdot\left(\sqrt{2(k+1)} + \sqrt{k+2}\right) \;\leq\; n\,.$$

   Next consider the following transformations and estimations:

   $$\begin{aligned}
   \frac{k+2}{\sqrt{k}} - \sqrt{k+2} \;&=\; \sqrt{1+\frac{2}{k}}\cdot\left(\sqrt{k+2} - \sqrt{k}\right),\\
   \sqrt{2(k+1)} \;&<\; \sqrt{2}\sqrt{k+2}\,,\\
   \sqrt{k+2} \;&\leq\; \sqrt{2}\sqrt{k} \qquad \text{(in view of } k \geq 2\text{)}\,,\\
   \sqrt{2(k+1)} + \sqrt{k+2} \;&<\; \sqrt{2}\left(\sqrt{k+2} + \sqrt{k}\right).
   \end{aligned}$$

These yield

$$\left(\frac{k+2}{\sqrt{k}} - \sqrt{k+2}\right)\cdot\left(\sqrt{2(k+1)} + \sqrt{k+2}\right) \;<\; 2\cdot\sqrt{2+\tfrac{4}{k}} \;\leq\; 4 \;<\; n\,,$$

as desired.

2. For the remaining cases we compute $\frac{F_0(C_k)}{e}$ in the following table where $n \in \{1, 2, 3, 4\}$ is the row index and $k \in \{0, 1, 2, 3, 4\}$ is the column index.

	0	1	2	3	4
1	1	$\frac{3}{2}$			
2	2	$\frac{3+\sqrt{3}}{2} \approx 2.37$	$\frac{4\sqrt{3}}{3} \approx 2.31$		
3	3	$\frac{3}{2} + \sqrt{3} \approx 3.23$	$\frac{4+\sqrt{2}}{\sqrt{3}} \approx 3.13$	$\frac{5\sqrt{3}}{2\sqrt{2}} \approx 3.06$	
4	4	$\frac{3+3\sqrt{3}}{2} \approx 4.10$	$\frac{4+2\sqrt{2}}{\sqrt{3}} \approx 3.94$	$\frac{5\sqrt{3}+\sqrt{5}}{2\sqrt{2}} \approx 3.85$	$\frac{6\sqrt{2}}{\sqrt{5}} \approx 3.79$.

This shows that, for regular simplices up to dimension 4 (that is, for line segments, equilateral triangles, regular tetrahedra and regular 4–simplices), the function F takes its maximal values at the mid-points of one-dimensional edges. These maximal values are

$$\frac{3}{2}e\,,\quad \left(\frac{3+\sqrt{3}}{2}\right)e\,,\quad \left(\frac{3}{2}+\sqrt{3}\right)e\,,\quad \left(\frac{3+3\sqrt{3}}{2}\right)e\,.$$

Appendix

To make the presentation self-contained we add a proof for the fact that the vertices of the compact convex set $\mathcal{T}_0$ are just the circumcentres of the faces of the simplex $\mathcal{T}$ containing the vertex A_0. Before doing this, note that this fact depends on the regularity of the simplex $\mathcal{T}$. For general simplices one must take the centroids instead of the circumcentres. The circumcentres are characterized by the fact that their barycentric coordinates have at most two values, 0 and $\frac{1}{k+1}$ if dealing with a face of dimension k; in particular the 0^{th} barycentric coordinate has the value $\frac{1}{k+1}$.

As noted above, the points

$$M \;=\; \sum_{i=0}^{n} r_i A_i$$

of $\mathcal{T}_0$ are characterized among all points of the simplex $\mathcal{T}$ by the condition

$$r_0 \;=\; \max\{r_i | i \in \{0, 1, \ldots, n\}\}\,.$$

If such a point is not a circumcentre, then there is an index $j \in \{1, 2, \dots, n\}$ with $0 < r_j < r_0$. We choose μ such that r_μ is the largest barycentric coordinate of M which is smaller than r_0. Further, let $\mathcal{L}$ denote the set of indices l with $r_l = r_0$, $k+1$ the number of elements of $\mathcal{L}$ and define

$$C_k = \frac{1}{k+1} \sum_{i \in \mathcal{L}} A_i .$$

Note that the presence of r_μ implies $r_0 < \frac{1}{k+1}$. Consider now the ray emanating from C_k and passing through M. Its points are represented in the form

$$\sum_{i \in \mathcal{L}} \left(\frac{1-t}{k+1} + t r_i \right) A_i + t \sum_{i \notin \mathcal{L}} r_i A_i .$$

Specializing to

$$t = \frac{1}{1-(k+1)(r_0 - r_\mu)} > 1 ,$$

we obtain the point

$$N = t r_\mu \sum_{i \in \mathcal{L}} A_i + t \sum_{i \notin \mathcal{L}} r_i A_i$$

on this ray which still belongs to $\mathcal{T}_0$ and therefore, M is an interior point of the line segment $[C_k N]$. Thus, this point M is not a vertex of $\mathcal{T}_0$.

On the other hand we show that the points C_k of the form

$$C_k = \frac{1}{k+1} \sum_{i \in \mathcal{L}} A_i ,$$

(where $\mathcal{L}$ denotes a set of $k+1$ indices) are vertices. To this end, assume that $M, N \in \mathcal{T}_0$ are different with C_k belonging to the line segment $[MN]$. We shall show that either $C_k = M$ or $C_k = N$, which proves that C_k is a vertex. There is a $t \in [0,1]$ such that $C_k = (1-t)M + tN$. Let us fix the barycentric coordinates of M and N:

$$M = \sum_{i=0}^{n} r_i A_i ,$$
$$N = \sum_{i=0}^{n} s_i A_i .$$

which implies

$$(1-t) \cdot r_i + t s_i = \begin{cases} \frac{1}{k+1}, & i \in \mathcal{L} , \\ 0 , & \text{otherwise.} \end{cases}$$

We have two cases.

1. If $k < n$ then there is an index $j \in \{1, 2, \dots, n\} \backslash \mathcal{L}$ and thus, 0 belongs to the interval $[r_j, s_j]$ with non-negative end-points. Thus, either r_j or s_j must vanish. If $r_j = 0 \neq s_j$ it follows $t = 0$ and thus, $C_k = M$; if $r_j \neq 0 = s_j$ we get $t = 1$ and $C_k = N$.

   It remains for us to consider the case in which $r_j = 0 = s_j$ for all $j \in \{1, 2, \dots, n\} \backslash \mathcal{L}$. From M, $N \in \mathcal{T}_0$ it follows that r_0, $s_0 \geq \frac{1}{k+1}$, but we have either $r_0 \leq \frac{1}{k+1} \leq s_0$ or $s_0 \leq \frac{1}{k+1} \leq r_0$. Thus, we get three possibilities:

   (a) $r_0 = \frac{1}{k+1} < s_0$: In this case we have $t = 0$ and $C_k = M$.

   (b) $s_0 = \frac{1}{k+1} < r_0$: In this case we have $t = 1$ and $C_k = N$.

   (c) $s_0 = \frac{1}{k+1} = r_0$: From $0 \leq r_i$, $s_i \leq \frac{1}{k+1}$ for all $i \in \mathcal{L}$ and $\sum\limits_{i \in \mathcal{L}} r_i = \sum\limits_{i \in \mathcal{L}} s_i = 1$, we obtain $r_i = s_i = \frac{1}{k+1}$ for all $i \in \mathcal{L}$ and thus, the contradiction $M = N$. This case does not occur.

2. The case $k = n$ follows in the same way as the last part of the previous case.

This finishes the proof.

Also solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

2652★. [2001 : 336] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let d, e and f be the sides of the triangle determined by the three points at which the internal angle-bisectors of given $\triangle ABC$ meet the opposite sides. Prove that

$$d^2 + e^2 + f^2 \;\leq\; \frac{s^2}{3},$$

where s is the semiperimeter of $\triangle ABC$.

Show also that equality occurs if and only if the triangle is equilateral.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Let the internal bisectors of $\angle BAC$, $\angle CBA$ and $\angle ACB$ meet BC, CA and AB at D, E and F, respectively, and let s be the semiperimeter of $\triangle ABC$. It is possible to obtain a proof of Janous' inequality by adapting Bataille's proof of the weaker inequality $d + e + f \leq s$ [2001 : 53–54]. In the proof, Bataille obtains the inequality

$$DF^2 \;\leq\; \frac{ab^2c}{(a+b)(b+c)}.$$

Adding the corresponding inequalities for FE and ED and applying the AM–GM inequality, we obtain

$$\begin{aligned} d^2+e^2+f^2 &\le \frac{abc[b(a+c)+a(b+c)+c(a+b)]}{(a+b)(b+c)(c+a)} \\ &= \frac{2abc(ab+bc+ca)}{(a+b)(b+c)(c+a)} \\ &\le \frac{2abc(ab+bc+ca)}{(2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca})} \\ &= \tfrac{1}{4}(ab+bc+ca) \;\le\; \tfrac{1}{12}(a+b+c)^2\,. \end{aligned}$$

(The last inequality follows from $2\left((a+b+c)^2-3(ab+bc+ca)\right) = (a-b)^2+(b-c)^2+(c-a)^2 \ge 0$.) Therefore, $d^2+e^2+f^2 \le \frac{s^2}{3}$. Equality occurs if and only if $\triangle ABC$ is equilateral.

Also solved by GEORGE BALOGLOU, SUNY Oswego, Oswego, NY, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; TODOR MITEV, University of Rousse, Rousse, Bulgaria; C.R. PRANESACHAR, Department of Mathematics, Indian Institute of Science, Bangalore, India; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

2653. [2001 : 336] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For whole numbers $n \ge 0$ and $N \ge 1$, evaluate the (combinatorial) sum

$$S_N(n) := \sum_{k\ge n} \binom{N}{2k}\binom{k}{n}\,.$$

I. *Comment by Michel Bataille, Rouen, France, (expanded slightly by the editor)*.

This problem is not new. It was proposed by Frank Gerrish as problem 78F in the July, 1994 issue of the Mathematical Gazette [Math. Gaz. 78 (482) 199]. Two solutions appeared in the March, 1995 issue [ibid. 79 (484) 129–132]. The first one was a fairly complicated combinatorial argument by J.K.R. Barnett and the second one was a more straightforward proof given by Nick Lord which used Newton's Generalized Binomial Theorem. Later on, a second and shorter combinatorial proof given by Chris Norman appeared in the November, 1995 issue [ibid. 79 (486) 587–588].

[Ed: Below we present a solution which is different from the ones that have appeared before and is also interesting and self-contained.]

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA, (modified slightly by the editor).*

Clearly, $S_N(n) = 0$, if $N < 2n$. We shall show that

$$S_N(n) \;=\; T_N(n) \quad \text{if} \quad N \geq 2n\,, \tag{1}$$

where $T_N(n) = \dfrac{2^{N-2n-1}N}{N-n}\dbinom{N-n}{n}$.

When $n = 0$, we have $S_N(0) = \sum\limits_{k\geq 0}\binom{N}{2k} = 2^{N-1} = T_N(0)$.

If $N = 1$, then $n = 0$. Hence (1) holds for $N = 1$. Note also that $S_{2n}(n) = 1 = T_{2n}(n)$ for all $n \geq 1$ and thus, in particular, (1) holds for $N = 2$.

We claim that for $N \geq 3$ and $n \geq 1$, $S_N(n)$ satisfies the recurrence relation:

$$S_N(n) \;=\; 2S_{N-1}(n) + S_{N-2}(n-1)\,. \tag{2}$$

Indeed, since

$$\begin{aligned}
2\binom{N-1}{2k} &= \binom{N}{2k} - \binom{N-1}{2k-1} + \binom{N-1}{2k} \\
&= \binom{N}{2k} - \left(\binom{N-2}{2k-2} + \binom{N-2}{2k-1}\right) + \left(\binom{N-2}{2k} + \binom{N-2}{2k-1}\right) \\
&= \binom{N}{2k} - \binom{N-2}{2k-2} + \binom{N-2}{2k},
\end{aligned}$$

[Ed: Note that $\binom{N-1}{2k-1} = \binom{N-2}{2k-2} = \binom{N-2}{2k-1} = 0$ if $k = 0$, by convention.]

we have

$$\begin{aligned}
&2S_{N-1}(n) + S_{N-2}(n-1) \\
&\quad= \sum_{k\geq 0}\binom{N}{2k}\binom{k}{n} - \sum_{k\geq 1}\binom{N-2}{2k-2}\binom{k}{n} + \sum_{k\geq 0}\binom{N-2}{2k}\left(\binom{k}{n} + \binom{k}{n-1}\right) \\
&\quad= S_N(n) - \sum_{j\geq 0}\binom{N-2}{2j}\binom{j+1}{n} + \sum_{k\geq 0}\binom{N-2}{2k}\binom{k+1}{n} = S_N(n)\,,
\end{aligned}$$

which establishes (2). Since $S_N(n)$ and $T_N(n)$ have the same initial values, it suffices to show that $T_N(n)$ also satisfies the same recurrence relation. Indeed,

$$\begin{aligned}
&2T_{N-1}(n) + T_{N-2}(n-1) \\
&= \frac{2^{N-2n-1}}{N-n-1}\left[\binom{N-n-1}{n}(N-1) + \binom{N-n-1}{n-1}(N-2)\right]
\end{aligned}$$

$$= \frac{2^{N-2n-1}}{N-n-1}\left[\left(\binom{N-n}{n} - \binom{N-n-1}{n-1}\right)(N-1) + \binom{N-n-1}{n-1}(N-2)\right]$$
$$= \frac{2^{N-2n-1}}{N-n-1}\left[\binom{N-n}{n}(N-1) - \binom{N-n-1}{n-1}\right]$$
$$= \frac{2^{N-2n-1}}{N-n-1}\binom{N-n}{n}\left[N-1-\frac{n}{N-n}\right]$$
$$= \frac{2^{N-2n-1}}{N-n-1}\binom{N-n}{n}\frac{N(N-n-1)}{N-n}$$
$$= \frac{2^{N-2n-1}N}{N-n}\binom{N-n}{n} = T_N(n)\,,$$

and our proof is complete.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was also one incomplete solution.

Though it is obvious that $S_N(n) = 0$ for $N < 2n$, only Guersenzvaig, Seiffert and the proposer pointed this out explicitly in their solutions. Checking the lists of the solvers of the original problem in the Gazette and the current problem reveals that there is exactly one person in the intersection. Since this solver did not mention anything about the original problem, the editor can only assume that after eight years, he has completely forgotten about it.

2654. [2001 : 336] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that $\triangle ABC$ has medians AD, BE and CF. Suppose that L, M and N are points on the sides BC, CA and AB, respectively.

Prove that the line through L parallel to AD, the line through M parallel to BE and the line through N parallel to CF are concurrent if and only if

$$\frac{BL^2}{BC^2} + \frac{CM^2}{CA^2} + \frac{AN^2}{AB^2} = \frac{LC^2}{BC^2} + \frac{MA^2}{CA^2} + \frac{NB^2}{AB^2}\,.$$

I. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

We apply a linear transformation to $\triangle ABC$, transforming it into an equilateral triangle $A'B'C'$. Clearly, the original lines concur if and only if their images do so. Specifically, the lines through L', M' and N' are parallel to the images of the medians of ABC, which are the medians of $A'B'C'$ (since the transformation preserves ratios of distances). These are now perpendicular to the sides, so the lines concur if and only if

$$B'L'^2 + C'M'^2 + A'N'^2 = L'C'^2 + M'A'^2 + N'B'^2$$

by Carnot's theorem. Since $A'B' = B'C' = C'A'$ we may write this as

$$\frac{B'L'^2}{B'C'^2} + \frac{C'M'^2}{C'A'^2} + \frac{A'N'^2}{A'B'^2} = \frac{L'C'^2}{B'C'^2} + \frac{M'A'^2}{C'A'^2} + \frac{N'B'^2}{A'B'^2}$$

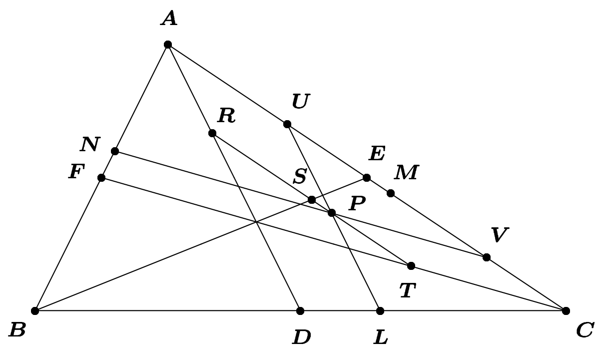
However, we have $\frac{B'L'^2}{B'C'^2} = \frac{BL^2}{BC^2}$, etc., since the transformation preserves ratios of distances. Hence, concurrency occurs if and only if

$$\frac{BL^2}{BC^2} + \frac{CM^2}{CA^2} + \frac{AN^2}{AB^2} = \frac{LC^2}{BC^2} + \frac{MA^2}{CA^2} + \frac{NB^2}{AB^2}$$

as required

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

For concreteness, we give the solution for the configuration in the accompanying figure. The proof for other positions of P is similar. [*Editor's comment:* in fact, the proof is valid for all P if one uses directed distances.]



First, we note that the given equation is equivalent to

$$\frac{BL - LC}{BC} + \frac{CM - MA}{CA} + \frac{AN - NB}{AB} = 0\,.$$

Since D is the mid-point of BC, $\frac{BL - LC}{BC} = \frac{DL}{DC} = \frac{AU}{AC}$, and likewise $\frac{CM - MA}{CA} = \frac{-2EM}{AC}$ and $\frac{AN - NB}{AB} = \frac{-VC}{AC}$. Suppose that PR is parallel to AC. If the three lines concur at P, then

$$AU - 2EM - VC = RP - 2SP - PT = RS - ST = 0$$

as desired. Conversely, if the given equation holds then $EM = SP$, which implies that MP is parallel to BE.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2655. [2001 : 336] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let a, b and c be the sides of $\triangle ABC$ and let s be its semiperimeter. Given that

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} \;=\; s\,,$$

show that $\triangle ABC$ is equilateral.

I. *Solution by Richard Eden, Ateneo de Manila University, Philippines.*

By the AM–HM inequality,

$$\frac{x+y}{2} \;\geq\; \frac{2}{\frac{1}{x}+\frac{1}{y}}$$

if x, $y > 0$, with equality if and only if $x = y$. Thus,

$$\begin{aligned} a+b+c \;&=\; \frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2} \;\geq\; \frac{2}{\frac{1}{a}+\frac{1}{b}}+\frac{2}{\frac{1}{b}+\frac{1}{c}}+\frac{2}{\frac{1}{c}+\frac{1}{a}} \\ &=\; \frac{2ab}{a+b}+\frac{2bc}{b+c}+\frac{2ca}{c+a} \\ \text{or} \qquad s \;&=\; \frac{a+b+c}{2} \;\geq\; \frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a} \end{aligned}$$

with equality if and only if $a = b = c$, when the triangle is equilateral.

II. *Solution by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let us denote $b+c = 2p$, $c+a = 2q$, and $a+b = 2r$. Then

$$\begin{aligned} a+b+c \;&=\; p+q+r \\ a \;&=\; -p+q+r \\ b \;&=\; p-q+r \\ c \;&=\; p+q-r\,. \end{aligned}$$

The given hypothesis now successively implies:

$$\begin{aligned} \frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} \;&=\; \frac{1}{2}(a+b+c) \\ \frac{p^2-(q-r)^2}{2p}+\frac{q^2-(r-p)^2}{2q}+\frac{r^2-(p-q)^2}{2r} \;&=\; \frac{1}{2}(p+q+r) \\ \frac{(q-r)^2}{p}+\frac{(r-p)^2}{q}+\frac{(p-q)^2}{r} \;&=\; 0\,. \end{aligned}$$

This forces $p = q = r$, which means that $a = b = c$.

III. *Solution by Panos E. Tsaoussoglou, Athens, Greece.*

$$\begin{aligned}
\frac{a+b+c}{2} = \frac{2(a+b+c)}{2\times 2} &= \frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} \\
\frac{a+b}{4}-\frac{ab}{a+b}+\frac{b+c}{4}-\frac{bc}{b+c}+\frac{c+a}{4}-\frac{ca}{c+a} &= 0 \\
\frac{(a+b)^2-4ab}{4(a+b)}+\frac{(b+c)^2-4bc}{4(b+c)}+\frac{(c+a)^2-4ca}{4(c+a)} &= 0 \\
\frac{(a-b)^2}{4(a+b)}+\frac{(b-c)^2}{4(b+c)}+\frac{(c-a)^2}{4(c+a)} &= 0\,.
\end{aligned}$$

Therefore, $a=b=c$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 proofs); MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA(2 proofs); CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; JOSÉ LUIS DÍAZ-BARRERO and JUAN JOSÉ EGOZCUE, Universitat Politècnica de Catalunya, Barcelona, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; VINAYAK GANESHAN, University of Waterloo, Waterloo, Ontario; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; ZELJKO HANJS, University of Zagreb, Zagreb, Croatia; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFREYS, student, Berkhamsted Collegiate School, UK; D. KIPP JOHNSON, Beaverton, OR, USA (3 proofs); KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA (2 proofs); DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROBERT McGREGOR, Auburn, Alabama, USA; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; STANLEY RABINOWITZ, Westford, MA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ANDREI SIMION, student, Cornell University, Ithaca, NY, USA; TREY SMITH, Angelo State University, San Angelo, TX, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

Bencze established a nice generalization: Let a_k, $k=1,2,\ldots,n$ be the sides of the convex polygon $A_1A_2\cdots A_n$ and let s be the semiperimeter. Given that

$$\frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}}+\frac{1}{\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}}+\cdots+\frac{1}{\frac{1}{a_n}+\frac{1}{a_1}+\cdots+\frac{1}{a_{n-2}}} = \frac{2s}{n-1}\,,$$

show that $A_1A_2\cdots A_n$ is equilateral. We will let the interested reader prove it.

2658. [2001 : 337] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $\triangle ABC$ have $\angle BCA = 90^\circ$. Squares $ACDE$ and $CBGF$ are drawn externally to the triangle. Suppose that AG and BE intersect at M. Show that M lies on the altitude CN.

I. *Solution by John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta.*

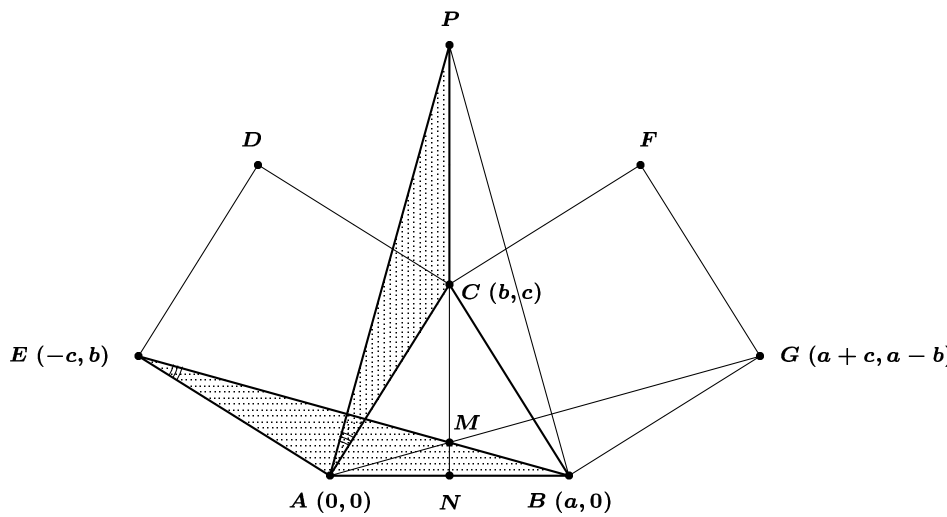
We do not need the condition that $\angle BCA = 90° - M$ lies on the altitude CN for any angle at C. Let $A = (0,0)$, $B = (a,0)$, $C = (b,c)$ be the coordinates of the triangle's vertices. Then $G = (a+c, a-b)$ and $E = (-c, b)$. The equations of AG and BE are

$$y = \frac{x(a-b)}{a+c} \quad \text{and} \quad y = \frac{b(x-a)}{-c-a}.$$

The coordinates of M are

$$\left(b, \frac{b(a-b)}{a+c}\right).$$

Thus, M lies on the altitude CN.



II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Even more generally, the squares on the sides CA and CB can be replaced by similar rectangles — *we assume that $\angle BCA$ is arbitrary and that $ACDE$ and $CBGF$ are rectangles such that $AC : AE = BC : BG$.*

Let P be a point on the altitude CN produced beyond C such that

$$CP : AB = AC : AE = BC : BG. \qquad (1)$$

Since $\angle PCA$ is an exterior angle of $\triangle CNA$ we have

$$\begin{aligned} \angle PCA &= \angle CAN + \angle CNA = \angle CAN + 90° \\ &= \angle CAN + \angle CAE = \angle NAE = \angle BAE. \qquad (2) \end{aligned}$$

From (1) and (2), we have $\triangle PCA \sim \triangle BAE$, so that $\angle PAC = \angle BEA$. Thus,

$$\angle BEA + \angle PAE = \angle PAC + \angle PAE = \angle CAE = 90°.$$

We therefore deduce that $PA \perp BE$. Similary, $PB \perp AG$. Therefore the three altitudes BE, AG, and PN are concurrent at the orthocentre of $\triangle PAB$. This implies that M lies on CN.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); ∗MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD B. EDEN, Philippines; VINAYAK GANESHAN, Waterloo, ON; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; D. KIPP JOHNSON, Beaverton, OR, USA; GERRY LEVERSHA, St. Paul's School, London, England; ∗HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ∗WOLFGANG LUDWICKI, Stendal Germany, and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; ROBERT McGREGOR, Auburn AL; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria (multiple solutions); ∗STANLEY RABINOWITZ, Westford, MA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. An asterisk indicates that the solver used an arbitrary angle at C.

Noting the many possible elementary approaches to the problem, two readers said that they intended to use the $90°$ version as an exercise in their classes. Evidently the problem has been popular for a long time — it has appeared in journals over the past 200 years, both with and without the condition requiring the $90°$ angle at C. Bataille came across it in [3]*, where it is proposed as a problem set by Dr. Porson (1759–1808). Rabinowitz found it in* [1]*. Further 19*th *century references are provided in* [2]*; the proof that appears there dates from 1879 and resembles our solution II, but without Seimiya's generalization to rectangles.*

The configuration of squares on the sides of a triangle has other nice properties that have been featured before in **CRUX with MAYHEM**. *See "One problem — Six Solutions" by Georg Gunther* [1998 : 81–87]*, and also* #1493 [1991 : 52–53] *and* #1496 [1991 : 56–57]. *Among other things, the altitude of* $\triangle ABC$ *turns out to be the median of* $\triangle CFD$.

References

[1] M.N. Aref and William Wernick, *Problems and Solutions in Euclidean Geometry*. Dover Publications 1968. Page 21, problem 1.27.

[2] F.G.-M., *Exercices de Géométrie*. Page 225, section 446.

[3] *Math. Gazette* **76** n^o 477 (Nov.1992) p. 455, and **77** n^o 479 (July 1993) pp. 295–296.

2660. [2001 : 337] . *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let z_1, z_2, ..., z_n be distinct non-zero complex numbers. Prove that

$$\sum_{j=1}^{n} z_j^{n-1}\left(1+\prod_{\substack{k=1\\k\neq j}}^{n} z_k\right)\prod_{\substack{k=1\\k\neq j}}^{n}\frac{1}{z_k-z_j}$$

is a real number, and determine its value.

Composite of solutions by David Loeffler, student, Trinity College, Cambridge, UK and Heinz-Jürgen Seiffert, Berlin, Germany (modified slightly by the editor).

Let L_n denote the given sum. We show that $L_n = (-1)^{n-1}$ for all $n \geq 2$. [Ed: for $n = 1$ both products in L_n are "empty" and thus the value of L_n is subject to interpretation.] The claim follows immediately from Lambrou's general result given in his solution to problem # 2487 [Ed: Also by the same proposer of the present problem.] [1999 : 431; 2000 : 512]. Using his notation, let

$$S_n(m) = \sum_{j=1}^{n} \frac{z_j^m}{\prod\limits_{\substack{k=1\\k\neq j}}^{n} (z_j - z_k)} .$$

Lambrou proved that

$$S_n(m) = \begin{cases} 0 & \text{if } 0 \leq m \leq n-2, \\ 1 & \text{if } m = n-1, \\ \sum\limits_{j=1}^{n} z_j & \text{if } m = n. \end{cases}$$

Since

$$\begin{aligned} L_n &= \sum_{j=1}^{n} \frac{z_j^{n-1}}{\prod\limits_{\substack{k=1\\k\neq j}}^{n} (z_k - z_j)} + \sum_{j=1}^{n} \frac{z_j^{n-2}\left(\prod\limits_{k=1}^{n} z_k\right)}{\prod\limits_{\substack{k=1\\k\neq j}}^{n} (z_k - z_j)} \\ &= (-1)^{n-1} S_n(n-1) + \left(\prod_{k=1}^{n} z_k\right) S_n(n-2) , \end{aligned}$$

it follows immediately that $L_n = (-1)^{n-1}$ as claimed.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were two incorrect solutions.

Both Deiermann and the proposer used complex integration and the theory of residue to obtain the result.