THE OLYMPIAD CORNER

No. 221

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First, corrections. AGL should be NGL in line 1 of solution 5 on page 19 of the February 2002 issue. The range given on the second last line of page 75 of the March issue should be $18 \leq x \leq 24$, not $14 \leq x \leq 18$.

We begin this number with the problems of the Swiss Mathematical Contest, May 17, May 20, 1999. Thanks again go to Ed Barbeau for collection the problems when he was Canadian Team Leader to the IMO in Romania.

SWISS MATHEMATICAL CONTEST
May 17, 1999

1. Two circles intersect each other in points M and N. An arbitrary point A of the first circle, which is not M or N, is connected with M and N, and the straight lines AM and AN intersect the second circle again in the points B and C. Prove that the tangent to the first circle at A is parallel to the straight line BC.

2. Is it possible to partition the set $\{1, 2, \ldots, 33\}$ into 11 disjoint subsets, each with three elements, such that in each subset one of the elements is the sum of the other two elements?

3. Determine all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, satisfying

$$\frac{1}{x} f(-x) + f\left(\frac{1}{x}\right) = x \text{ for all } x \in \mathbb{R} \setminus \{0\}.$$

4. Find all solutions $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ of the system

$$\frac{4x^2}{1+4x^2} = y, \qquad \frac{4y^2}{1+4y^2} = z, \qquad \frac{4z^2}{1+4z^2} = x.$$

5. Let $ABCD$ be a rectangle, P a point on the line CD. Let M, N be the mid-points of AD and BC respectively. PM intersects AC in Q. Show that MN is the bisector of the angle QNP.

May 20, 1999

1. Let m and n be two positive integers such that $m^2 + n^2 - m$ is divisible by $2mn$. Prove that m is the square of an integer.

2. A square is partitioned into rectangles whose sides are parallel to the sides of the square. For each rectangle, the ratio of its shorter side to its longer side is determined. Prove that the sum S of these ratios is always at least 1.

3. Determine all integers $n \in \mathbb{N}$ such that there exist positive real numbers $0 < a_1 \le a_2 \le \cdots \le a_n$ satisfying

$$\sum_{i=1}^{n} a_i = 96, \qquad \sum_{i=1}^{n} a_i^2 = 144, \qquad \sum_{i=1}^{n} a_i^3 = 216.$$

4. Prove that for every polynomial $P(x)$ of degree 10 with integer coefficients there is an (in both directions) infinite arithmetic progression which does not contain $P(k)$ for any integer k.

5. Prove that the product of five consecutive positive integers is never a perfect square.

Next we give the problems of the Final Round of the 50$^{\text{th}}$ Polish Mathematical Olympiad, April 14-15, 1999. Thanks go to Ed Barbeau for collecting the contests when he was Canadian Team Leader to the IMO at Bucharest.

50$^{\text{th}}$ POLISH MATHEMATICAL OLYMPIAD
Problems of the Final Round
DAY 1, April 14, 1999 — (Time: 5 hours)

1. Let D be a point on side BC of triangle ABC such that $AD > BC$. Point E on side AC is defined by the equation $\dfrac{AE}{EC} = \dfrac{BC}{AD - BC}$. Show that $AD > BE$.

2. Given are non-negative integers $a_1 < a_2 < a_3 < \cdots < a_{101}$ smaller than 5050. Show that one can choose four distinct integers a_k, a_l, a_m, a_n so that the number $a_k + a_l - a_m - a_n$ is divisible by 5050.

3. Prove that there exist distinct positive integers $n_1, n_2, n_3, \ldots, n_{50}$ such that $n_1 + S(n_1) = n_2 + S(n_2) = n_3 + S(n_3) = \cdots = n_{50} + S(n_{50})$, where $S(n)$ denotes the sum of the digits of n.

DAY 2, April 15, 1999 — (Time: 5 hours)

4. Find all integers $n \geq 2$ for which the system of equations

$$\begin{cases} x_1^2 + x_2^2 + 50 & = & 16x_1 + 12x_2 \\ x_2^2 + x_3^2 + 50 & = & 16x_2 + 12x_3 \\ x_3^2 + x_4^2 + 50 & = & 16x_3 + 12x_4 \\ \vdots & \vdots & \vdots \\ x_{n-1}^2 + x_n^2 + 50 & = & 16x_{n-1} + 12x_n \\ x_n^2 + x_1^2 + 50 & = & 16x_n + 12x_1 \end{cases}$$

has a solution in integers $x_1, x_2, x_3, \ldots, x_n$.

5. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be integers. Prove that

$$\sum_{1 \leq i < j \leq n} (|a_i - a_j| + |b_i - b_j|) \leq \sum_{1 \leq i,j \leq n} |a_i - b_j| .$$

6. In a convex hexagon $ABCDEF$,

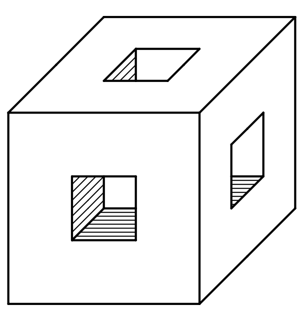
$$\angle A + \angle C + \angle E = 360^\circ , \quad \frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1 .$$

Prove that $\dfrac{AB}{BF} \cdot \dfrac{FD}{DE} \cdot \dfrac{EC}{CA} = 1$.

We now give the problems of the 1996 Chilean Mathematical Olympiad. Thanks go to Raul A. Simon Lamb of Santiago, Chile for forwarding them to us for our use.

CHILEAN MATHEMATICAL OLYMPIAD 1996

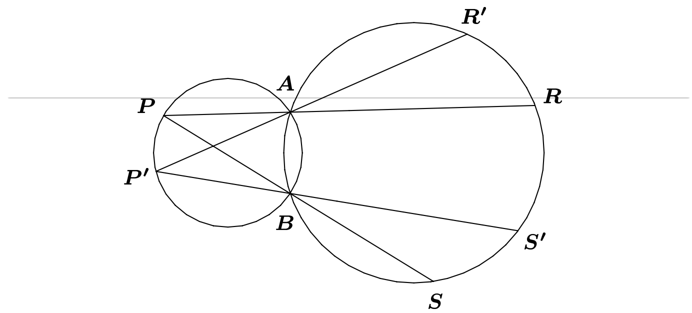
1.



Consider a cube of edge 18 cm. In the centre of three different (and not opposite) faces we bore a square perforation of side 6 cm. That goes across the cube as far as the opposite face. We thus obtain the following figure:

Determine the surface area of the resulting solid.

2. Two circles intersect at A and B. P is a point on arc $\widehat{AB}$ on one of the circles. PA and PB intersect the other circle at R and S (see figure). If P' is any point on the same arc as P and if R' and S' are the points in which $P'A$ and $P'B$ intersect the second circle, prove that $\widehat{RS} = \widehat{R'S'}$.



3. Let a, b, c, d be integers such that $ad \neq bc$.

(a) Prove that it is always possible to write the fraction

$$\frac{1}{(ax+b)(cx+d)}$$

in the form

$$\frac{r}{ax+b} + \frac{s}{cx+d}\,,$$

where r, s are rational numbers.

(b) Calculate the sum

$$\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{1993\cdot 1996}\,.$$

4. Construct $\triangle ABC$, knowing the points on its circumcircle, and points D, E and F where this circumcircle is met by the altitude, the transversal of gravity (median) and the bisectrix (angle bisector), respectively, issuing from C. (Assume $\overline{AC} < \overline{BC}$.)

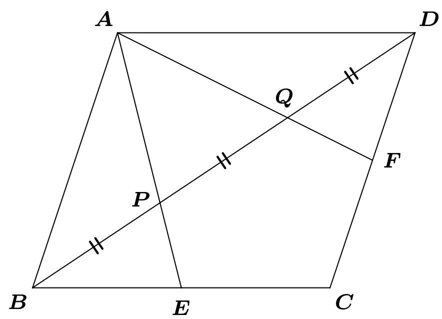
Next we turn to the March 2000 number of the *Corner* and solutions from our readers to problems of the 10^{th} Mexican Mathematical Olympiad National Contest 1996, given [2000 : 71–72].

1. Let $ABCD$ be a quadrilateral and let P and Q be the trisecting points of the diagonal BD (that is, P and Q are the points on the line segment BD for which the lengths BP, PQ and QD are all the same). Let E be the intersection of the straight line through A and P with BC, and let F be the intersection of the straight line through A and Q with DC. Prove the following:

(i) If $ABCD$ is a parallelogram, then E and F are the mid-points of BC and CD, respectively.

(ii) If E and F are the mid-points of BC and CD, respectively, then $ABCD$ is a parallelogram.

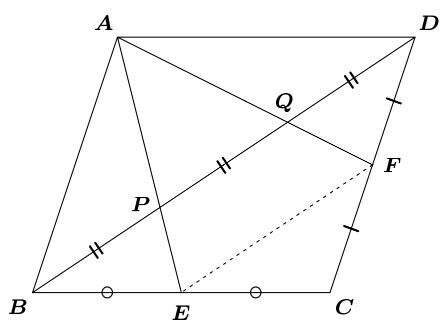
Solutions by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's write-up.



(i) Since $ABCD$ is a parallelogram $AD \| BC$ and $AD = BC$. Thus, we have $BE : AD = BP : PD = 1 : 2$.

Therefore, $BE = \frac{1}{2}AD = \frac{1}{2}BC$, so that $BE = EC$. Similarly, we get $DF : DC = DF : AB = DQ : QB = 1 : 2$. Thus, $DF = \frac{1}{2}DC$; that is $DF = FC$.

Therefore, E and F are mid-points of BC and CD, respectively.



(ii) Since E, F are mid-points of BC, CD, respectively, we get $EF \| BD$ and $EF = \frac{1}{2}BD$.

Since $PQ \| EF$ we have

$$AP : AE = PQ : EF = \frac{1}{3}BD : \frac{1}{2}BD = 2 : 3 = DP : DB .$$

Therefore, $AD \| BE$; that is, $AD \| BC$. Similarly we have $AB \| DC$.

Hence, $ABCD$ is a parallelogram.

3. Prove that it is not possible to cover a 6 cm $\times$ 6 cm square board with eighteen 2 cm $\times$ 1 cm rectangles, in such a way that each one of the interior 6 cm lines that form the squaring goes through the middle of at least one of the rectangles. Prove also that it is possible to cover a 6 cm $\times$ 5 cm square board with fifteen 2 cm $\times$ 1 cm rectangles, in such a way that each one of the interior 6 cm lines that form the squaring and each one of the interior 5 cm lines that form the squaring goes through the middle of at least one of the rectangles.

Comment by Pierre Bornsztein, Pontoise, France.

More generally it is known (see [1]) that an $h \times w$ rectangle has a fault-free tiling with dominoes if and only if hw is even, $h \geq 5$, $w \geq 5$, and $(h, w) \neq (6, 6)$.

Reference

[1] G.E. Martin, "Polynominoes, a Guide to Puzzles and Problems in Tiling", MAA, pp. 17–21.

4. For which integers $n \geq 2$ can the numbers 1 to 16 be written each in one square of a squared 4×4 paper (no repetitions allowed) such that each of the 8 sums of the numbers in rows and columns is a multiple of n, and all of these 8 multiples of n are different from one other?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The only values are $n = 2$ and 4. To see this, note first that the sum of the 16 entries is $\sum_{k=1}^{16} k = 136$. Let R_i and C_i denote the i^{th} row sum and the i^{th} column sum, respectively, $i = 1, 2, 3, 4$. Then, by assumptions, $R_i = a_i n$, $C_i = b_i n$ where the 8 natural numbers a_i's and b_i's are *all* distinct.

Hence, $2 \times 136 = \sum_{i=1}^{4}(R_i + C_i) = n\sum_{i=1}^{4}(a_i + b_i) \geq n\sum_{k=1}^{8} k = 36n$, and thus, $n \leq \lfloor \frac{272}{36} \rfloor = 7$.

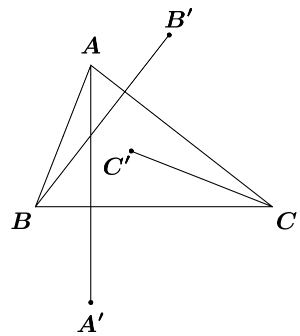
On the other hand, since $n \mid R_i$ for all $i = 1, 2, 3, 4$ we have $n \mid \sum_{i=1}^{4} R_i$ or $n \mid 136$. Since $136 = 2^3 \times 17$, we conclude that the only *possible* values of n are $n = 2$ or 4.

To complete the proof it clearly suffices to display a configuration in which all the 4 row sums and the 4 column sums are *distinct* multiples of 4. The array shown in the figure below is one such configuration since $(R_1, R_2, R_3, R_4) = (16, 20, 48, 52)$ and $(C_1, C_2, C_3, C_4) = (28, 32, 36, 40)$.

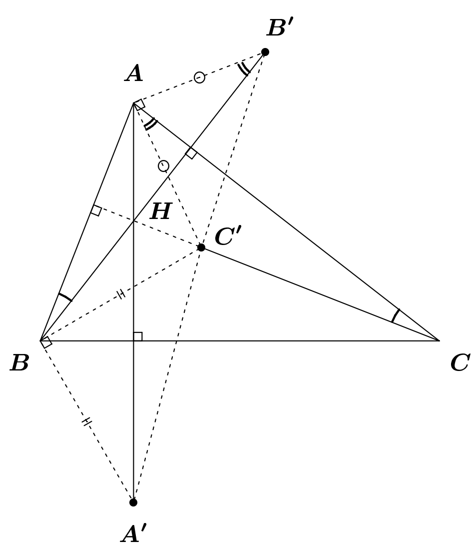
1	3	5	7
2	4	6	8
9	11	13	15
16	14	12	10

6. The picture below shows a triangle $\triangle ABC$ in which the length AB is smaller than that of BC, and the length of BC is smaller than that of

AC. The points A', B' and C' are such that AA' is perpendicular to BC and the length of AA' equals that of BC; BB' is perpendicular to AC and the length of BB' equals that of AC; CC' is perpendicular to AB and the length of CC' equals that of AB. Moreover $\angle AC'B$ is a $90°$ angle. Prove that A', B' and C' are collinear.



Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



AA', BB' and CC' are concurrent at the orthocentre H of $\triangle ABC$. Since $BH \perp AC$ and $CH \perp AB$ we get

$$\angle ABB' = \angle ABH = \angle ACH = \angle ACC' .$$

Since $BA = CC'$ and $BB' = CA$, we have

$$\triangle BAB' \equiv \triangle CC'A .$$

Thus, $AB' = C'A$ and $\angle AB'B = \angle C'AC$. Since $BB' \perp AC$ we have

$$\angle B'AC' \;=\; \angle B'AC + \angle C'AC \;=\; \angle B'AC + \angle AB'B \;=\; 90^\circ\,.$$

Since $AB' = AC'$ and $\angle B'AC' = 90^\circ$ we get $\angle AC'B' = 45^\circ$.

Similarly we have $BA' = BC'$ and $\angle A'BC' = 90^\circ$. Thus, we get $\angle BC'A' = 45^\circ$.

Since $\angle AC'B = 90^\circ$, we have

$$\angle B'C'A' \;=\; \angle AC'B' + \angle AC'B + \angle BC'A' \;=\; 45^\circ + 90^\circ + 45^\circ \;=\; 180^\circ\,.$$

Therefore, A', B' and C' are collinear.

We continue with readers' solutions to problems of the Bi-National Israel-Hungary Competition, 1996, given on [2000 : 73].

1. Find all sequences of integers x_1, x_2, ..., x_{1997} such that

$$\sum_{k=1}^{1997} 2^{k-1}(x_k)^{1997} \;=\; 1996\prod_{k=1}^{1997} x_k\,.$$

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give Bataille's presentation.

We make the following key observation: if $(x_1, x_2, \ldots, x_{1997})$ satisfies the property

$$\sum_{k=1}^{1997} 2^{k-1}(x_k)^{1997} \;=\; 1996\prod_{k=1}^{1997} x_k\,, \tag{1}$$

then x_1 is even.

Substituting $2y_1$ for x_1 in (1) and dividing both sides by 2, we get:

$$(x_2)^{1997} \;+\; 2(x_3)^{1997} \;+\; \cdots \;+\; 2^{1995}(x_{1997})^{1997} \;+\; 2^{1996}(y_1)^{1997} \;=\; 1996\left(\prod_{k=2}^{1997} x_k\right) y_1\,.$$

This means that the sequence $(x_2, \ldots, x_{1997}, y_1)$ also satisfies (1).

Iterating, we obtain successively:

x_2 is even, $x_2 = 2y_2$ and $(x_3, \ldots, x_{1997}, y_1, y_2)$ satisfies (1)
... ...
x_{1997} is even, $x_{1997} = 2y_{1997}$, and $(y_1, y_2, \ldots, y_{1997})$ satisfies (1)

Thus, we have obtained the following result: if $(x_1, x_2, \ldots, x_{1997})$ satisfies (1), then the x_k's are even and $(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_{1997}}{2})$ also satisfies (1). It follows that all the x_k's are 0. Indeed, if $x_j \neq 0$ (say), then we might write $x_j = 2^r \cdot z$ with $r \geq 1$ and z odd; iterating the previous process r times would yield a sequence satisfying (1) and containing the odd term z, which is impossible, as we have seen. Conversely, the null sequence obviously satisfies (1) so that we may conclude that the only sequence satisfying (1) is the null sequence.

2. Let $n > 2$ be an integer, and suppose that n^2 can be represented as the difference of the cubes of two consecutive positive integers. Prove that n is the sum of two squares. Prove that such an n really exists.

Solution by Michel Bataille, Rouen, France.

Let m be a positive integer such that $n^2 = (m+1)^3 - m^3$. Then we have $(2n)^2 - 3(2m+1)^2 = 1$, so that $(2n, 2m+1)$ is one of the pairs (x, y) satisfying

$$x \text{ is even and } > 4, \quad y \text{ is odd}, \quad x^2 - 3y^2 = 1 .$$

As is well known, the solutions to $x^2 - 3y^2 = 1$ in positive integers are the pairs (x_k, y_k) given by $x_k + y_k\sqrt{3} = (2+\sqrt{3})^k$ or equivalently by the relations

$$\begin{cases} x_{k+1} = 2x_k + 3y_k \\ y_{k+1} = x_k + 2y_k \end{cases}$$

with $x_1 = 2$, $y_1 = 1$. From these recursion formulas, we readily deduce that x_k even, y_k odd occurs precisely when k is odd. Observing that $x_k - y_k\sqrt{3} = (2-\sqrt{3})^k$, we see that

$$2n = x_{2j+1} = \frac{1}{2}\left(\left(2+\sqrt{3}\right)^{2j+1} + \left(2-\sqrt{3}\right)^{2j+1}\right) \quad \text{for some} \quad j > 1 .$$

Denoting $u = 2+\sqrt{3}$, $v = 2-\sqrt{3}$, we have $x_k = \frac{1}{2}(u^k + v^k)$ and $u+v = 4$, $uv = 1$ so that we can easily verify the following relations (valid for all k):

$$x_{k+2} = 4x_{k+1} - x_k ; \quad 2x_k^2 = x_{2k} + 1 ; \quad x_k x_{k+1} = \frac{1}{2}x_{2k+1} + 1 .$$

From this, we get first $n = x_j x_{j+1} - 1$ and then,

$$n = \left(\frac{x_{j+1} - x_j - 1}{2}\right)^2 + \left(\frac{x_{j+1} - x_j + 1}{2}\right)^2 .$$

It remains to prove that $\dfrac{x_{j+1} - x_j - 1}{2}$ is an integer. But

$$x_j + x_{j+1} = 3x_j + 3y_j = 3(y_{j+1} - y_j) .$$

Hence, $x_1 + 2(x_2 + \cdots + x_j) + x_{j+1} = 3y_{j+1} - 3y_1 = 3y_{j+1} - 3$ and

$$\begin{aligned} x_1 + x_2 + \cdots + x_j &= \frac{1}{2}(3y_{j+1} - x_{j+1} - 1) = \frac{1}{2}(x_{j+2} - 3x_{j+1} - 1) \\ &= \frac{1}{2}(4x_{j+1} - x_j - 3x_{j+1} - 1) ; \end{aligned}$$

that is,

$$x_1 + x_2 + \cdots + x_j \;=\; \frac{x_{j+1} - x_j - 1}{2}\,.$$

The result follows.

Examples are obtained by choosing j. For instance, with $j = 1$, $x_3 = 26$ so that $n = 13 = 2^2 + 3^2$ (and $13^2 = 8^3 - 7^3$) or with $j = 2$, $x_5 = 362$ so that $n = 181 = 10^2 + 9^2$ (and $181^2 = 105^3 - 104^3$).

Remark. This problem is close to Problem no. 3 [1999 : 394] and to Problem 2525 [2000 : 116].

3. A given convex polyhedron has no vertex which is incident with exactly 3 edges. Prove that the number of faces of the polyhedron which are triangles, is at least 8.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let E, F, V denote the respective number of edges, faces, and vertices of the polyhedron. As is known, $E + 2 = F + V$. Also, let V_r denote the number of vertices with valence r and F_s the number of faces with s sides. It follows that

$$2E \;=\; 3F_3 + 4F_4 + \cdots + sF_s \;=\; 4V_4 + 5V_5 + \cdots + rV_r\,.$$

Since

$$V \;=\; V_4 + V_5 + \cdots + V_r \quad \text{and} \quad F \;=\; F_3 + F_4 + \cdots + F_s\,,$$

we get from

$$2E + 4 \;=\; 2\left(V_4 + V_5 + \cdots + V_r\right) + 2\left(F_3 + F_4 + \cdots + F_s\right)\,,$$

that

$$F_3 + 2F_4 + \cdots + (s-2)F_s + 4 \;=\; 2\left(V_4 + V_5 + \cdots + V_r\right)$$

and

$$2V_4 + 3V_5 + \cdots + (r-2)V_r + 4 \;=\; 2\left(F_3 + F_4 + \cdots + F_s\right)\,.$$

Adding the last two equations, we get

$$8 + V_5 + 2V_6 + \cdots + (r-4)V_r + F_5 + 2F_6 + \cdots + (s-4)F_s \;=\; F_3\,.$$

Hence, F_3 is at least 8.

4. Let a_1, a_2, $\ldots$, a_n be arbitrary real numbers and b_1, b_2, $\ldots$, b_n real numbers satisfying the condition $1 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$. Prove that there is a positive integer $k \leq n$ for which the inequality $|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leq |a_1 + a_2 + \cdots + a_k|$ holds.

Solution by Michel Bataille, Rouen, France.

Let $A_1 = a_1$, $A_2 = a_1 + a_2$, $\dots$, $A_n = a_1 + a_2 + \cdots + a_n$ and $M = \max(|A_1|, |A_2|, \dots, |A_n|)$. We have to prove that

$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \;\leq\; M\,.$$

Noticing that

$$\begin{aligned} &a_1b_1 + a_2b_2 + \cdots + a_nb_n \\ &\quad= A_1b_1 + (A_2 - A_1)b_2 + \cdots + (A_n - A_{n-1})b_n \\ &\quad= A_1(b_1 - b_2) + A_2(b_2 - b_3) + \cdots + A_{n-1}(b_{n-1} - b_n) + A_nb_n\,, \end{aligned}$$

we obtain:

$$\begin{aligned} &|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \\ &\leq\; |A_1|\,|b_1 - b_2| + |A_2|\,|b_2 - b_3| + \cdots + |A_{n-1}|\,|b_{n-1} - b_n| + |A_n|\,|b_n| \\ &\leq\; M(|b_1 - b_2| + |b_2 - b_3| + \cdots + |b_{n-1} - b_n| + |b_n|) \\ &=\; M(b_1 - b_2 + b_2 - b_3 + \cdots + b_{n-1} - b_n + b_n) \\ &=\; Mb_1 \;\leq\; M\,, \quad \text{as desired.} \end{aligned}$$

Next we move to the April 2000 number of the *Corner* and solutions from our readers to problems of the Finnish High School Mathematics Contest, Final Round, 1997 given [2000 : 132].

1. Determine all numbers a, for which the equation

$$a3^x + 3^{-x} \;=\; 3$$

has a unique solution x.

Solutions by Pierre Bornsztein, Pontoise, France; by Stewart Metchette, Gardena, CA, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We given Bornsztein's solution.

We will prove that the required numbers a are those which satisfy $a \leq 0$ or $a = \frac{9}{4}$. Let a, x be real numbers. Let $f(x) = 3^{1-x} - 3^{-2x}$. Then $a3^x + 3^{-x} = 3 \iff f(x) = a$.

For all real numbers x : $f'(x) = \ln(3) \cdot 3^{-2x}(2 - 3^{1+x})$.

Let $x_0 = \dfrac{\ln\left(\frac{2}{3}\right)}{\ln(3)}$ (that is, $3^{1+x_0} = 2$). We then have

x	$-\infty$		x_0		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	$-\infty$	$\nearrow$	$f(x_0)$	$\searrow$	0

and, since $3^{1+x_0} = 2$, we have $f(x_0) = 3 \cdot \frac{3}{2} - \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

From this, we easily deduce that the equation $f(x) = a$ has a unique solution if and only if $a = f(x_0) = \frac{9}{4}$ or $a \leq 0$. And we are done.

2. Two circles, of radii R and r, $R > r$, are externally tangent. Consider the common tangent of the circles, not passing through their common point. Determine the maximal radius of a circle drawn in the domain bounded by this tangent line and the circles.

Solutions by Michel Bataille, Rouen, France; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's write-up.

We will denote by D the domain bounded by the given circles C, Γ (with respective centres O, Ω) and their common tangent T.

Solution I. We take for granted that the circle γ_m with maximal radius ρ_m contained in D is the one that is tangent to C, Γ and T (which may seem obvious).

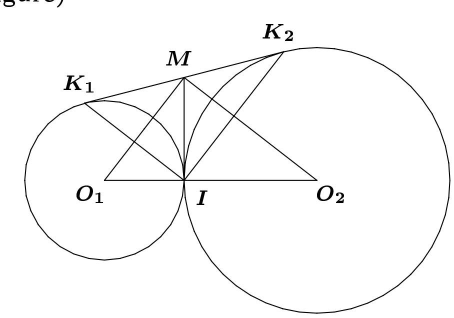
We shall make use of the following result (R) [see proof below]: If C_i, with centre O_i and radius R_i, $(i = 1, 2)$ are two circles tangent externally at I, and T is their common tangent (not through I) touching C_1 at K_1 and C_2 at K_2, then $K_1K_2 = 2\sqrt{R_1R_2}$.

Applying (R) to the three pairs of circles (C, Γ), (γ_m, C), (γ_m, Γ), we readily obtain

$$2\sqrt{rR} \;=\; 2\sqrt{r\rho_m} + 2\sqrt{R\rho_m}\,,$$

so that $\rho_m \;=\; \dfrac{rR}{(\sqrt{r} + \sqrt{R})^2}$.

Proof of (R) (see figure)



We introduce the point M where the tangent at I to the circles intersects T. Then $MI = MK_1 = MK_2$ so that $\triangle K_1IK_2$ is right-angled at I. It follows that $\triangle O_1MO_2$ is right-angled at M. Since MI is the altitude from M in this triangle, we get:

$$IM^2 \;=\; IO_1 \cdot IO_2 \;=\; R_1R_2 \quad \text{and} \quad K_1K_2 \;=\; 2MI \;=\; 2\sqrt{R_1R_2}\,.$$

Solution II. Consider an arbitrary circle γ with radius ρ contained in D. We may suppose that γ is tangent to C and Γ [otherwise we take instead the circle with the same radius ρ and centre at the point of intersection in D of the circles with centres O, Ω and radii $r+\rho$, $R+\rho$].

Now invert the figure in a circle with centre the common point I of C and Γ and radius k, cutting C at A, B and Γ at E, F.

Thus, C, Γ invert into the parallel lines AB, EF respectively, T inverts into a circle τ passing through I and γ into a circle γ'. Note that τ and γ' are both tangent to AB and EF with γ' exterior to τ.

Since $d(I,AB) = \dfrac{k^2}{2r}$ and $d(I,EF) = \dfrac{k^2}{2R}$, the common radius of τ and γ' is given by

$$u \;=\; \frac{1}{2}d(AB,EF) \;=\; \frac{k^2}{2}\left(\frac{1}{2r}+\frac{1}{2R}\right).$$

From a known formula, ρ is related to u by the relation

$$\rho = k^2\frac{u}{d^2-u^2}\,, \tag{1}$$

where d is the distance from I to the centre of γ'. Clearly, the maximal value ρ_m of ρ is obtained when d is minimal; that is, when γ' is tangent (externally) to the circle τ. In this case, some simple calculations give (see figure below)

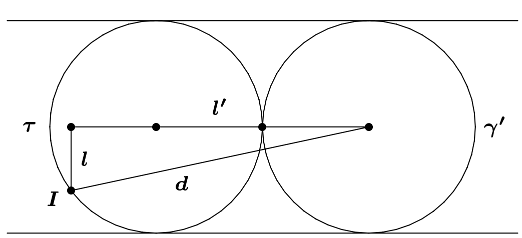
$$l^2 \;=\; \frac{k^4}{4}\left(\frac{1}{2r}-\frac{1}{2R}\right)^2 \quad\text{and}\quad d^2-u^2 \;=\; l^2+(2u+l')^2-u^2$$

where $l'^2 = u^2-l^2 = \dfrac{k^4}{4rR}$. Hence,

$$d^2-u^2 \;=\; \frac{k^4}{4}\left(\frac{1}{r}+\frac{1}{R}\right)\left(\frac{1}{\sqrt{r}}+\frac{1}{\sqrt{R}}\right)^2.$$

With the help of (1), we get

$$\rho_m \;=\; \frac{1}{\left(\dfrac{1}{\sqrt{r}}+\dfrac{1}{\sqrt{R}}\right)^2} \;=\; \frac{rR}{(\sqrt{r}+\sqrt{R})^2}\,.$$



3. Twelve knights sit around a round table. Every knight hates the two knights sitting next to him, but none of the other nine knights. A task group of five knights is to be sent to save a princess in trouble. No two knights who hate each other can be included in the group. In how many ways can the group be selected?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We solve the more general problem when there are n knights and a group of k compatible knights has to be selected for the task. Recall first that by a well-known formula in elementary combinatorics, the number of ways of selecting k objects from a row of n distinct objects so that no two of the selected objects could be adjacent, is given by the binomial coefficient $\binom{n-k+1}{k}$.

Consider one particular knight, say, Sir Lancelot. Then the group to be selected either includes him or excludes him.

If Lancelot is chosen, then his two neighbours cannot be chosen and hence, we must choose $k-1$ more knights from the remaining $n-3$ knights in such a way that no two adjacent knights are selected. By the formula quoted above, this can be done in $\binom{n-3-(k-1)+1}{k-1}$ ways. If Lancelot is not chosen, then we must select all k knights from the $n-1$ remaining knights subject to the same constraint. This can be done in $\binom{n-1-k+1}{k}$ ways.

Therefore, the total number of possible teams is

$$\begin{aligned} f(n,k) &= \binom{n-3-(k-1)+1}{k-1} + \binom{n-1-k+1}{k} \\ &= \binom{n-k-1}{k-1} + \binom{n-k}{k} \\ &= \binom{n-k-1}{k-1} + \frac{n-k}{k}\binom{n-k-1}{k-1} \\ &= \frac{n}{k}\binom{n-k-1}{k-1}. \end{aligned}$$

Note that $f(n,k) \geq 0$ if and only if $k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

For the given problem, $n = 12$, $k = 5$ and thus, the number is $f(12,5) = \frac{12}{5}\binom{6}{4} = 36$.

Remark. This problem can be found on p. 46 of the book *Combinatorics* by N. Ya. Vilenkin (translated by A. Shenitzer and S. Shenitzer).

4. Determine the sum of all 4–digit numbers, all the digits of which are odd.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Stewart Metchette, Gardena, CA, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

We solve the general problem of determining the sum of all n–digit integers, all the digits of which are odd, where $n \in \mathbb{N}$. Let T_n denote the set of all such integers, and let S_n denote the corresponding sum. Clearly, $|T_n| = 5^n$. List all the integers in T_n in a row in increasing order and then in a second row in decreasing order as shown below:

$$1\,1\,1\,\ldots\,1\,,\,1\,1\,1\,\ldots\,3\,,\,\ldots\ldots\,,\,9\,9\,9\,\ldots\,9$$

$$9\,9\,9\,\ldots\,9\,,\,9\,9\,9\,\ldots\,7\,,\,\ldots\ldots\,,\,1\,1\,1\,\ldots\,1$$

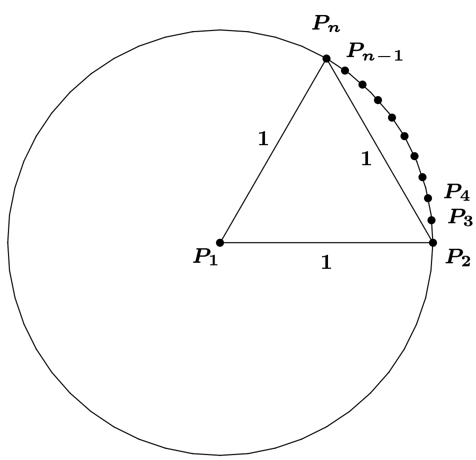
where each integer has exactly n digits.

Then clearly the sum of the two integers in the same column is the constant $1\,1\,1\,\ldots\,1\,0$, where there are n 1's. Hence, $2S_n = 5^n \times 1\,1\,1\,\ldots\,1\,0$ from which it follows that $S_n = 5^n \times 5\,5\,5\,\ldots\,5$ where there are n 5's. In particular, $S_1 = 25$, $S_2 = 5^2 \times 55 = 1375$, $S_3 = 5^3 \times 555 = 69375$, $S_4 = 5^4 \times 5555 = 3471875$.

5. Let $n \geq 3$. Find a configuration of n points in the plane such that the mutual distance of no pair of points exceeds 1 and exactly n pairs of points have a mutual distance equal to 1.

Solutions by Pierre Bornsztein, Pontoise, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

Let $\mathcal{C}$ be the circle with centre P_1 and radius 1. Let P_2, P_n be two distinct points on $\mathcal{C}$ such that $P_1P_2P_n$ is equilateral.



We choose any $n - 3$ points on the arc $\widehat{P_2P_n}$ (see figure above). Then:

$$\begin{aligned} P_1P_i &= 1 \quad \text{for} \quad i = 2, \dots, n \\ P_2P_n &= 1 \end{aligned}$$

and

$$P_iP_j < 1 \quad \text{in all other cases.}$$

Next we turn to readers' solutions to problems of the XI form of the Georgian Mathematical Olympiad, 1997 given [2000 : 133].

2. Two positive numbers are written on a board. At each step you must perform one of the following:

(i) Choose one of the numbers, say a, already written on the board and write down either a^2 or $\frac{1}{a}$ on the board;

(ii) Choose two numbers, say a and b, on the board and write down either $a + b$ or $|a - b|$ on the board.

Obviously, after each step the quantity of numbers on the board increases. How should you proceed in order that the product of the two initial numbers will eventually be written on the board?

Solutions by Pierre Bornsztein, Pontoise, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

Let a and b be the two initial numbers.

Case 1. If $a = b$, the use of (i) gives immediately a^2.

Case 2. If $a \neq b$ we proceed as follows:

— with (i), we write a^2, b^2;

— with (ii), we write $a + b$; then with (i), we write $(a + b)^2$;

— with (ii), we write $|(a + b)^2 - a^2| = b^2 + 2ab$;

— with (ii), we write $|(b^2 + 2ab) - b^2| = 2ab$;

— with (i), we write $\frac{1}{2ab}$, two times;

— with (ii), we write $\frac{1}{2ab} + \frac{1}{2ab} = \frac{1}{ab}$;

— with (i), we write $\frac{1}{\frac{1}{ab}} = ab$; and we are done.

4. We say that there is an algebraic operation defined on the closed interval $[0, 1]$ if there is a rule that corresponds to every pair (a, b) of numbers from this interval a new number c from the same interval. We denote it by $c = a \otimes b$. Find all positive k with the property that there exists an algebraic operation defined on $[0,1]$ such that for any x, y, z from $[0,1]$ the following equalities hold:

(i) $x \otimes 1 = 1 \otimes x = x$,

(ii) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$,

(iii) $(zx) \otimes (zy) = z^k(x \otimes y)$.

For all such k define the corresponding algebraic operation.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give Bataille's write-up.

We first show that 1 and 2 are the only possible values for the positive number k.

Firstly, for any a, b (with $b \neq 0$) such that $0 \leq a \leq b \leq 1$, we have

$$a \otimes b = ab^{k-1} \tag{1}$$

Indeed, $\frac{a}{b} \in [0,1]$ and $a \otimes b = \left(b\frac{a}{b}\right) \otimes (b1) = b^k\left(\frac{a}{b} \otimes 1\right) = ab^{k-1}$. Now, choose a, b such that $0 < a < b < 1$ with $a < b^{2k-1}$ and $a < b^{k^2-k+1}$. Then,

$$(a \otimes b) \otimes (b \otimes b) = (ab^{k-1}) \otimes b^k = ab^{k-1}(b^k)^{k-1} = ab^{(k-1)(k+1)}. \tag{2}$$

[since $ab^{k-1} < b^k$].

On the other hand $(a \otimes b) \otimes (b \otimes b) = a \otimes (b \otimes b \otimes b)$ with either,

if $k \leq 1$, $b \otimes b \otimes b = b \otimes (b \otimes b) = b \otimes b^k = bb^{k(k-1)} = b^{k^2-k+1}$ (since $b \leq b^k$)

or,

if $k \geq 1$, $b \otimes b \otimes b = (b \otimes b) \otimes b = b^k \otimes b = b^k b^{k-1} = b^{2k-1}$ (since $b^k \leq b$).

This gives $(a \otimes b) \otimes (b \otimes b) = a(b^{k^2-k+1})^{k-1}$ (if $k \leq 1$), or $a\left(b^{2k-1}\right)^{k-1}$ (if $k \geq 1$).

Comparing with (2) immediately yields $k = 1$ or $k = 2$.

Using (1), we note that necessarily $a \otimes b = \min(a,b)$ if $k = 1$ and $a \otimes b = ab$ if $k = 2$.

Conversely, it is readily checked that the algebraic operation defined by $a \times b = \min(a,b)$ [respectively, $a \otimes b = ab$] is actually defined on $[0,1]$ and satisfies (i), (ii) and (iii) with $k = 1$ [respectively, with $k = 2$].

In conclusion, the desired algebraic operation exists if and only if

$k = 1$ and then $a \otimes b = \min(a,b)$ for all $(a,b) \in [0,1] \times [0,1]$

or $k = 2$ and then $a \otimes b = ab$ for all $(a,b) \in [0,1] \times [0,1]$.

That completes the *Corner* for this issue. Send me your nice comments and generalizations as well as Olympiad contests!

BOOK REVIEWS

JOHN GRANT McLOUGHLIN

Symmetry by Hans Walser,
translated from the German by Peter Hilton and Jean Pedersen,
published by The Mathematical Association of America (Spectrum Series), 2001,
ISBN # 0-88385-532-1, softcover, 120 pages, $23.50 (US).
Reviewed by **Peter Hilton**, *SUNY, Binghamton, NY, USA.*

One of the biggest, and most difficult, jobs facing mathematicians and teachers of mathematics is to restore the study of geometry to its proper role in the curriculum. For the uncomfortable fact we have to face is that, in the United States at least, the teaching of geometry is largely neglected at the pre-college level. Not only do our students arrive at university often unable to carry out a geometric proof or even to recognize a valid (or invalid) one when they see it, but they are also largely without the benefit of very reliable and useful geometric intuition.

A reason why this sorry state has come about is that our mathematical education is highly compartmentalized, so that it is administratively very convenient to neglect a subject which seems to stand alone. Indeed, we should recognize that we have, for a long time, failed to recognize the fundamental unity of mathematics and thus to understand that we should present geometry primarily as a source of ideas and questions and not of methods and answers. Thus we should not, at the secondary and early undergraduate levels, regard geometry and algebra as two distinct subjects, but rather see geometry as providing the concepts and the questions, and algebra as providing the methods and the answers. Thus, geometry and algebra are interactive and complementary.

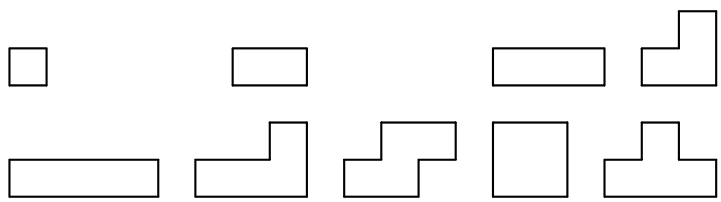
That is the point of view adopted by Hans Walser, the author of this very stimulating monograph. He is very largely concerned with symmetry as a geometric concept, although his last two chapters deal with certain aspects of its applicability outside geometry. But the methods he employs to study the symmetry concept are certainly not confined to those of synthetic geometry. Thus, in the reviewer's judgment, the author is correctly assigning to geometry its proper role in the integrated curriculum. The book is full of applications of the symmetry concept within mathematics and in the real world.

The reviewer must now declare an interest — it was I who, with the assistance of the geometer, Jean Pedersen, translated Walser's text from German into English. Moreover it was I who recommended that the Mathematical Association of America should authorize a translation and place it in their excellent Spectrum Series. In fact, my enthusiastic endorsement of the translation project is published on the back cover of the book. Thus I cannot claim to have approached the writing of this review in an objective spirit. Nevertheless, I do claim that English-speaking, non-German-speaking students must surely benefit greatly from the availability of this very stimulating text with its fascinating and unusual examples of symmetry. The mathematics will not be difficult for the bright high school student: but the wealth of applications — to mirrors, centres of gravity, parquet floors, error-correcting, minimal supply channels, palindromes and rhyming schemes — will surely provide a real, and welcome, challenge.

Polyomino Number Theory (I)

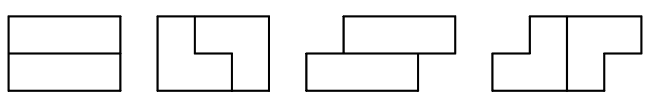
Andris Cibulis, Andy Liu and Bob Wainwright

Polyominoes are connected plane figures formed of joining unit squares edge to edge. We have a monomino, a domino, two trominoes named $\boldsymbol{I}$ and $\boldsymbol{V}$, and five tetrominoes named $\boldsymbol{I}$, $\boldsymbol{L}$, $\boldsymbol{N}$, $\boldsymbol{O}$ and $\boldsymbol{T}$, respectively.

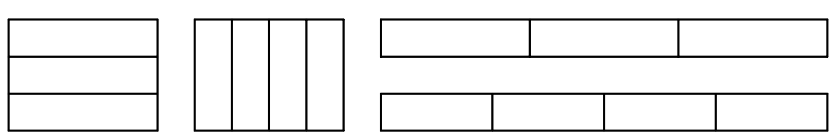


A polyomino $\boldsymbol{A}$ is said to **divide** another polyomino $\boldsymbol{B}$ if a copy of $\boldsymbol{B}$ may be assembled from copies of $\boldsymbol{A}$. We also say that $\boldsymbol{A}$ is a **divisor** of $\boldsymbol{B}$, $\boldsymbol{B}$ is **divisible** by $\boldsymbol{A}$, and $\boldsymbol{B}$ is a **multiple** of $\boldsymbol{A}$. The monomino divides every polyomino.

A polyomino is said to be a **common divisor** of two other polyominoes if it is a divisor of both. It is said to be a **greatest common divisor** if no other common divisor has greater area. Note that we say a greatest common divisor rather than the greatest common divisor since it is not necessarily unique. For instance, the two hexominoes below have both the $\boldsymbol{I}$–tromino and the $\boldsymbol{V}$–tromino as their greatest common divisors.



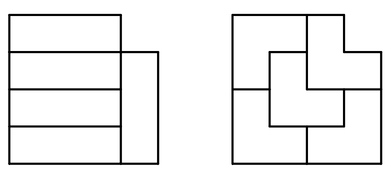
When two polyominoes have at least two greatest common divisors, each greatest common divisor is clearly not divisible by any of the others. However, even if a unique greatest common divisor exists, it is still not necessarily divisible by the other common divisors. For instance, the two dodecominoes below have the $\boldsymbol{I}$–tetromino as their unique greatest common divisor, but it is not divisible by the $\boldsymbol{I}$–tromino which is also a common divisor.



Any two polyominoes have a greatest common divisor, since we can always fall back on the monomino. When the greatest common divisor is the monomino, we say that these two polyominoes are **relatively prime** to each other. The monomino is relatively prime to every other polyomino. A **prime** polyomino is one which is divisible only by itself and the monomino, and it is also relatively prime to every other polyomino. Note that the monomino is not considered to be a prime polyomino.

If the area of a polyomino is a prime number, then it must be a prime itself. The converse is not true. The smallest counter-example is the $\boldsymbol{T}$–tetromino. It has area 4, but is a prime polyomino.

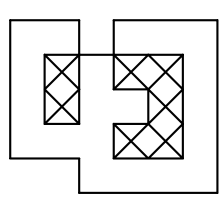
A polyomino is said to be a **common multiple** of two other polyominoes if it is a multiple of both. If two polyominoes have common multiples, they are said to be **compatible**. A **least common multiple** of two compatible polyominoes is a common multiple with minimum area. As shown earlier, the $\boldsymbol{I}$–tromino and the $\boldsymbol{V}$–tromino have at least two least common multiples. Clearly, neither multiple divides the other. These two trominoes even have a common multiple whose area is not divisible by $\mathbf{6}$, the area of their least common multiple.



However, the area of every common multiple of the $\boldsymbol{I}$–tromino and the $\boldsymbol{I}$–tetromino must be a multiple of $\mathbf{12}$, the area of their least common multiple.

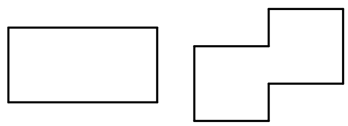
Given two small polyominoes, it is a trivial matter to determine all common divisors of them. It is a different situation with common multiples. To determine whether they are even compatible is often an interesting question. Finding the area of a least common multiple of two compatible polyominoes can also be challenging.

The monomino is trivially compatible with every polyomino. This property is not shared even by the domino, which is incompatible with the icosomino below. Thus compatibility is not a transitive relation.

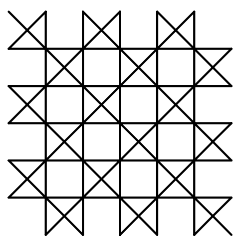


Suppose we wish to find a least common multiple of the O–tetromino with either the T–tetromino or the N–tetromino. Clearly, the area of any multiple of a tetromino is a multiple of 4. Since the tetrominoes in question are distinct, the smallest possible area of a common multiple is 8.

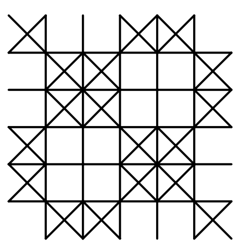
Note that two copies of the O–tetromino can abut in essentially two ways as shown below. Neither figure can be assembled from copies of either the T–tetromino or the N–tetromino. Hence a common multiple has area at least 12.



If we paint the squares of the infinite grid black and white in the usual checkerboard fashion as shown below, then three copies of the O–tetromino always cover an even number of white squares, while three copies of the T–tetromino always cover an odd number of white squares. Hence they have no common multiples with area 12.

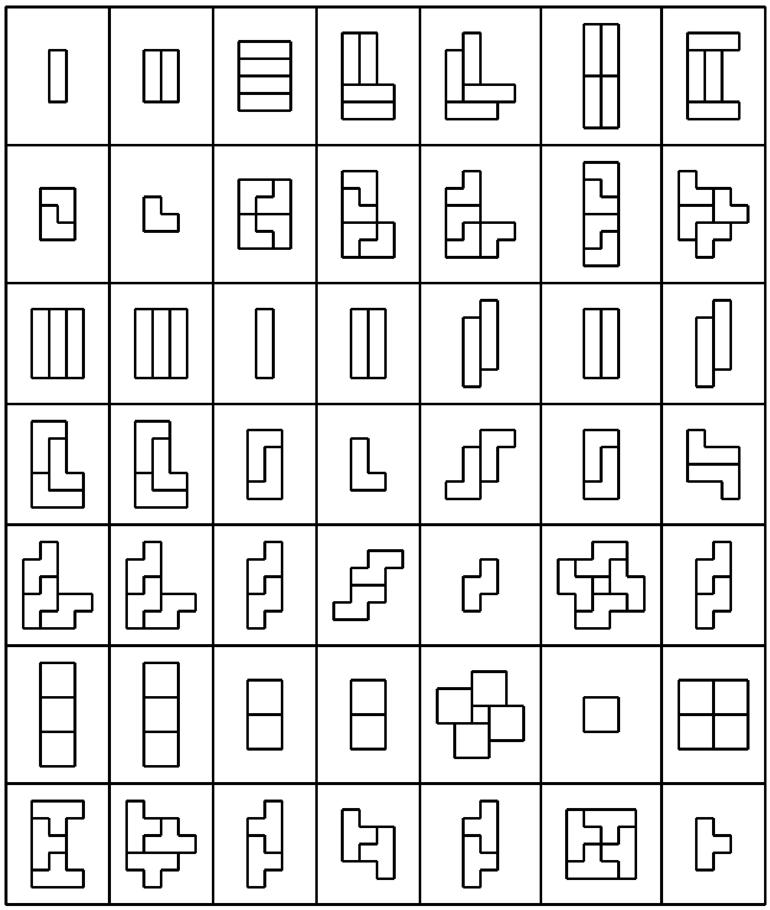


If we paint the squares of the infinite grid black and white in the checkered pattern shown below, then three copies of the O–tetromino always cover an even number of white squares, while three copies of the N–tetromino always cover an odd number of white squares. Hence they have no common multiples with area 12 either.



It follows that in both cases, the minimum area of a common multiple is 16. It turns out that such least common multiples exist.

The chart below gives a least common multiple of each pair of trominoes and tetrominoes. The polyominoes are featured along the main diagonal. The figure in the i^{th} row and the j^{th} column shows how a least common multiple of the i^{th} and j^{th} polyominoes can be constructed from the i^{th} polyomino.



Note that the minimum possible area is attained in all but two cases, between the O–tetromino on the one hand, and the T–tetromino and the N–tetromino on the other. We have dealt with these cases earlier.

The structure considered in this paper is an example of a *Normed Division Domain* considered by Solomon W. Golomb in his paper with that title, published on pages 680 to 686 in Volume 88 of the American Mathematical Monthly in 1981.

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Note that the fifth character in "line1ine" is the digit 1 and not the letter 'l'

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7 (NEW!)**. The electronic address is

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The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The other staff member is Jimmy Chui (University of Toronto).

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1**, or emailed to

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Please include in all correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 October 2002*. Solutions received after this time will be considered only if there is time before publication of the solutions.

Starting this issue, problems will be printed in English and French.

To be eligible for this month's MAYHEM TAUNT, solutions must be postmarked *before* 1 August 2002.

M39. Proposed by the Mayhem staff.

Given x is a positive real number and

$$x = 2002 + \cfrac{1}{2002 + \cfrac{1}{2002 + \cfrac{1}{2002 + \cfrac{1}{2002 + \frac{1}{x}}}}},$$

find x.

. . . .

Trouver x si x est un nombre réel positif et

$$x = 2002 + \cfrac{1}{2002 + \cfrac{1}{2002 + \cfrac{1}{2002 + \cfrac{1}{2002 + \frac{1}{x}}}}}.$$

M40. Proposed by Louis-François Préville-Ratelle, student, Cégep Régional de* Lanaudière à L'Assomption, Joliette, Québec.

Suppose a and b are two divisors of the integer n, with $a < b$. Prove:

$$\left\lfloor \frac{n}{a+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{b} \right\rfloor = \left\lfloor \frac{n}{\frac{n}{b}+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{\frac{n}{a}} \right\rfloor$$

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

For example, if $n = 24$, $a = 3$, and $b = 6$, this says:

$$\left\lfloor \frac{24}{4} \right\rfloor + \left\lfloor \frac{24}{5} \right\rfloor + \left\lfloor \frac{24}{6} \right\rfloor = \left\lfloor \frac{24}{5} \right\rfloor + \left\lfloor \frac{24}{6} \right\rfloor + \left\lfloor \frac{24}{7} \right\rfloor + \left\lfloor \frac{24}{8} \right\rfloor,$$

which evaluates to the identity $6 + 4 + 4 = 4 + 4 + 3 + 3$.

. . . .

Soit a et b deux diviseurs de l'entier n tels que $a < b$. Montrer que

$$\left\lfloor \frac{n}{a+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{b} \right\rfloor = \left\lfloor \frac{n}{\frac{n}{b}+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{\frac{n}{a}} \right\rfloor$$

Ici, $\lfloor x \rfloor$ désigne le plus grand entier plus petit ou égal à x.

Par exemple, si $n = 24$, $a = 3$, et $b = 6$, ceci signifie :

$$\left\lfloor \frac{24}{4} \right\rfloor + \left\lfloor \frac{24}{5} \right\rfloor + \left\lfloor \frac{24}{6} \right\rfloor = \left\lfloor \frac{24}{5} \right\rfloor + \left\lfloor \frac{24}{6} \right\rfloor + \left\lfloor \frac{24}{7} \right\rfloor + \left\lfloor \frac{24}{8} \right\rfloor,$$

qui se réduit à l'identité $6 + 4 + 4 = 4 + 4 + 3 + 3$.

M41. Proposed by J. Walter Lynch, Athens, GA, USA

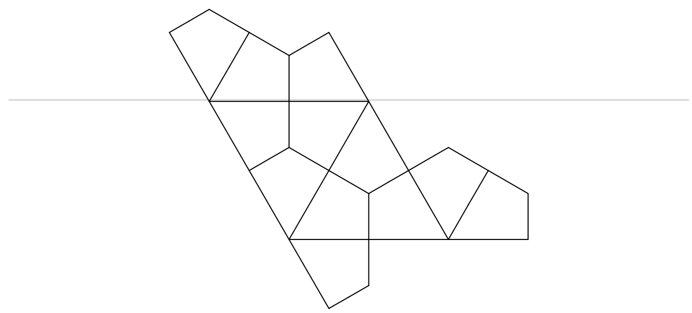
Find the number of orders of wins and losses that can occur in a World Series. For example if the series ends after five games there are eight possible orders: ANNNN NANNN NNANN NNNAN NAAAA ANAAA AANAA AAANA where A is for an American League win and N is for a National League win. Note that the series ends as soon as one team wins four games.

. . . .

Trouver le nombre d'ordres des victoires et des défaites possibles dans une Série Mondiale. Par exemple, si la série s'achève après cinq parties, il y a huit ordres possibles :ANNNN NANNN NNANN NNNAN NAAAA ANAAA AANAA AAANA où A désigne une victoire de la Ligue Americaine et N celle de la Ligue Nationale. Noter que la série se termine dès qu'une équipe gagne quatre parties.

M42. Proposed by Izidor Hafner, Tržaška 25, Ljubljana, Slovenia.

The diagram below represents the net of a polyhedron. The faces of the solid are divided into smaller polygons. The task is to colour the polygons (or number them), so that each face of the original solid is a different colour.



. . . .

Le diagramme ci-dessus représente le développement d'un polyèdre sur un plan. Les faces du solide sont divisées en polygones plus petits. Le problème consiste à colorer les polygones (ou á les numéroter) de telle sorte que chaque face du solide original soit d'une couleur différente.

M43. Proposed by the Mayhem staff.

Prove that

$$\frac{29-5\sqrt{29}}{58}\left(\frac{7+\sqrt{29}}{2}\right)^{2002}+\frac{29+5\sqrt{29}}{58}\left(\frac{7-\sqrt{29}}{2}\right)^{2002}$$

is an integer.

. . . .

Montrer que

$$\frac{29-5\sqrt{29}}{58}\left(\frac{7+\sqrt{29}}{2}\right)^{2002}+\frac{29+5\sqrt{29}}{58}\left(\frac{7-\sqrt{29}}{2}\right)^{2002}$$

est un entier.

M44. Proposed by K.R.S. Sastry, Bangalore, India.

$ABCD$ is a Heron parallelogram (in which the sides, the diagonals and the area are natural numbers). The diagonals AC and BD have measures 85 and 41 respectively. Determine the measures of the sides AB and BC.

. . . .

Soit $ABCD$ un parallelogramme de Heron (dont les côtés, les diagonales et l'aire sont des nombres naturels). Les diagonales AC et BD mesurent respectivement 85 et 41. Trouver les longueurs des côtés AB et BC.

Challenge Board Solutions

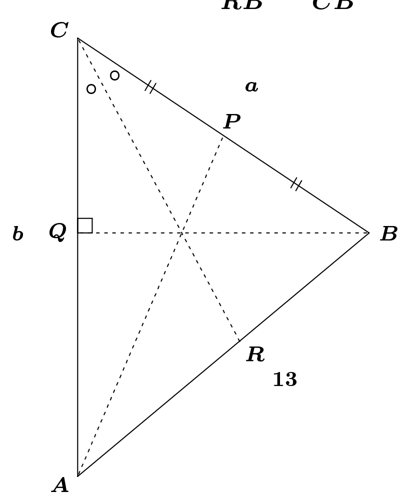
Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

In this issue we present the remainder of the solutions to the Konhauser Problemfest presented in the April 2001 issue [2001 : 204].

9. Gail was giving a class on triangles, and she was planning to demonstrate on the blackboard that the three medians, the three angle bisectors, and the three altitudes of a triangle each meet at a point (the centroid, incentre, and orthocentre of the triangle, respectively). Unfortunately, she got a little careless in her example, and drew a certain triangle ABC with the median from vertex A, the altitude from vertex B, and the angle bisector from vertex C. Amazingly, just as she discovered her mistake, she saw that the three segments met at a point anyway! Luckily it was the end of the period, so no one had a chance to comment on her mistake. In recalling her good fortune later that day, she could only remember that the side across from vertex C was 13 inches in length, that the other two sides also measured an integral number of inches, and that none of the lengths were the same. What were the other two lengths?

Solution: By Ceva's Theorem the segments AP, BQ, CR meet at a point if and only if $\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=1$. Since AP is a median, $BP=PC$ and we get $\frac{AR}{RB}\cdot\frac{CQ}{QA}=1$

Now let the unknown lengths BC, CA be a, b, respectively. Since CR is the angle bisector from C, we have $\frac{AR}{RB}=\frac{AC}{CB}=\frac{b}{a}$ and thus $\frac{b}{a}\cdot\frac{CQ}{QA}=1$.



On the other hand, $CQ+QA=b$ and combining the last two equations soon leads to $CQ = \dfrac{ab}{a+b}$, $QA = \dfrac{b^2}{a+b}$. Now, we use that BQ is an altitude, so that by Pythagoras' Theorem, $BC^2 - CQ^2 = AB^2 - QA^2$, which leads to

$$(a-13)(a+13)(a+b) \;=\; b^2(a-b)\,.$$

We need to find a solution to this equation in positive integers a, b so that there exists a triangle with sides 13, a, b, and 13, a, b are all distinct. Note that any prime factor, p, of $a+b$ must be a factor of $b^2(a-b)$ and hence such a prime must divide either b or $a-b$. This suggests that a and b might have common factors. On the other hand, if p is a common factor of a and b, then p^3 divides $b^2(a-b) = (a-13)(a+13)(a+b)$, and unless $p = 13$ (which is easily seen to be impossible) p cannot divide $a-13$ or $a+13$, so that p^3 divides $a+b$. This suggests that the common factors of a and b cannot be too large!

Specifically, if we try $p = 3$, we need 27 to divide $a+b$. Let us try $a+b = 27$, $a = 3x$, $b = 27-3x$. Substituting this into our equation yields

$$\begin{aligned}(3x-13)(3x+13)27 &= (27-3x)^2(6x-27)\\ (3x-13)(3x+13) &= (9-x)^2(2x-9)\,,\end{aligned}$$

which is correct for $x = 4$. Thus, $a = 12$, $b = 15$ is a solution. (Note that this solution is unique.)

10. An infinite sequence of digits "1" and "2" is determined uniquely by the following properties:

(i) The sequence is built up by stringing together pieces of the form "12" and "112".

(ii) If we replace each "12" piece with a "1" and each "112" piece with a "2", then we get the original sequence back.

(a) Write down the first dozen digits in the sequence. At which place will the 100^{th} "1" occur? What is the 1000^{th} digit?

Solution: Clearly, the sequence starts with a "1". Now if we think of the replacement process described in (ii) *backwards*, we see that the initial "1" must have started as "$1\ 2$" before the replacement. And, in turn, the "1" of "$1\ 2$" started as "$1\ 2$" and the "2" started as "$1\ 1\ 2$". Therefore, before the replacement, "$1\ 2$" was "$1\ 2\ 1\ 1\ 2$". Continuing in this way, we can reconstruct as much of the sequence as we want. Let us call the finite sequence we get after k steps of this backward replacement S_k; let x_k, y_k, t_k be the number of 1's, 2's and the total number of digits respectively (so that $x_k + y_k = t_k$) in S_k.

Since at each step, 1 gets replaced by $1\ 2$ and 2 gets replaced by $1\ 1\ 2$, we have $x_{k+1} = x_k + 2y_k$, $y_{k+1} = x_k + y_k = t_k$. Now we can start answering the questions.

Since $t_4 = 12$, the first dozen digits form $S_4 = 121121212112$. As for the 100^{th} "1", if we calculate x_k, y_k, t_k a bit further, we get $x_7 = 99$, $y_7 = 70$, $t_k = 169$. Thus, after 169 digits, S_7 is complete and we have 99 "1". The next digit must be a "1" (it is the start of a new "$1\ 2$" or "$1\ 1\ 2$"), so that it is the 100^{th} "1" and the 170^{th} digit.

To help find the 1000^{th} digit, we can use the recurrence:

$$t_{k+1} \;=\; x_{k+1} + y_{k+1} \;=\; 2t_k + t_{k-1}\,.$$

We can use this to get t_k quickly. A little calculation gives $t_9 = 985$, and the 1000^{th} digit will be 15 digits beyond the end of sequence S_9.

A second observation is that we can think of $S_3 = 12112$ as consisting of not two parts "$1\ 2$" and "$1\ 1\ 2$", but as three parts "$1\ 2$", "1" and "$1\ 2$". As we construct S_4, S_5, etc, from S_3, these three parts do not influence each other. Since "$1\ 2$" is actually S_2 and "1" is actually S_1, we have $S_{n+1} = S_n S_{n-1} S_n$, so that $S_{10} = S_9 S_8 S_9$.

Therefore, the 15^{th} digit beyond the end of S_9, which is the digit that we want, is the same as the 15^{th} digit of S_8. Since S_5 already has $t_5 = 29$ digits and S_8 is just an extension of S_5, our digit is also the 15^{th} digit of $S_5 = S_4 S_3 S_4$; since S_4 has 12 digits, we are looking for the 3^{rd} digit in S_3 which is a "1".

(b) Let A_n be the number of "1"s among the first n digits of the sequence. Given that the ratio A_n/n approaches a limit, find that limit.

Solution: Given that $\lim\limits_{x\to\infty} \dfrac{A_n}{n}$ exists, we can find this limit by looking only at the integers $n = t_k$, which has the advantage that $A_n = A_{t_k} = x_k$. Thus, we are looking for $\lim\limits_{k\to\infty} \dfrac{x_k}{t_k}$. Since $x_k = t_k - y_k = t_k - t_{k-1}$, this limit equals:

$$\lim_{k\to\infty} \frac{t_k - t_{k-1}}{t_k} = \lim_{k\to\infty} \left(1 - \frac{t_{k-1}}{t_k}\right) = 1 - \lim_{k\to\infty} \frac{t_{k-1}}{t_k}\,. \tag{1}$$

Since $t_{k+1} = 2t_k + t_{k-1}$, we have $t_{k+1} > 2t_k$ but also $t_{k+1} = 2t_k + t_{k-1} < 2t_k + t_k = 3t_k$, so that $2 < \dfrac{t_{k+1}}{t_k} < 3$, or $\dfrac{1}{3} < \dfrac{t_k}{t_{k+1}} < \dfrac{1}{2}$.

Thus, there is no danger that the ratios $\dfrac{t_{k+1}}{t_k}$ will approach 0, and by the given properties, and from (1), $L = \lim\limits_{k\to\infty} \dfrac{t_{k-1}}{t_k}$ and $\dfrac{1}{L} = \lim\limits_{k\to\infty} \dfrac{t_k}{t_{k-1}} = \lim\limits_{k\to\infty} \dfrac{t_{k+1}}{t_k}$ both exist.

Now since $t_{k+1} = 2t_k + t_{k-1}$, dividing by t_k gives

$$\frac{t_{k+1}}{t_k} \;=\; 2 + \frac{t_{k-1}}{t_k}\,.$$

Taking the limit as $k \to \infty$ yields: $\frac{1}{L} = 2 + L$.

Therefore, $L^2 + 2L - 1 = 0$, $L = -1 \pm \sqrt{2}$, but since $\frac{1}{3} < L < \frac{1}{2}$, we have $L = -1 + \sqrt{2}$. Thus, our final answer is $\lim_{n \to \infty} \frac{A_n}{n} = 1 - L = 2 - \sqrt{2}$.

(c) (Tiebreaker) Show that the limit from part (b) actually exists.

Solution: Consider two sequences (α_k) and (β_k) for $k \geq 4$ such that:

(1) (α_k) is (monotonically) increasing and (β_k) is decreasing;

(2) For all n with $t_{k-1} + 1 \leq n \leq t_k$, we have $\alpha_k \leq \frac{A_n}{n} \leq \beta_k$.

Furthermore, it will turn out that

(3) For $f(x) = \frac{2-x}{3-x}$, $\alpha_{k+1} = f(\beta_k)$ and $\beta_{k+1} = f(\alpha_k)\left(1 + \frac{1}{t_k + 1}\right)$.

Once we succeed in doing all this, we can argue as follows. The bounded monotonic sequences (α_k) and (β_k) have limits, say α and β respectively. By (3), the continuity of f, and the fact that $1 + \frac{1}{t_k + 1} \to 1$ as $k \to \infty$, we get $\alpha = f(\beta)$ and $\beta = f(\alpha)$, so that $\alpha = f(f(\alpha))$. This gives $\alpha = 2 \pm \sqrt{2}$, but since $\alpha \leq 1$, we must have $\alpha = 2 - \sqrt{2}$ and $\beta = f(\alpha) = 2 - \sqrt{2}$.

But (2) now shows that *all* values of $\frac{A_n}{n}$ approach $2 - \sqrt{2}$ as $n \to \infty$, so that we will be done once we prove that (1), (2) and (3) can be arranged.

To get started, we let α_4, β_4 be the minimum, maximum values of $\frac{A_n}{n}$ for $t_3 + 1 \leq n \leq t_4$, respectively. Recall that $t_3 = 5$, $t_4 = 12$, and hence, we get $\alpha_4 = \frac{5}{9}$ and $\beta_4 = \frac{2}{3}$.

We now construct the sequences (α_k) and (β_k) inductively. Suppose we have α_k and β_k for some k, and we want to define α_{k+1} and β_{k+1}; in particular, we want to find bounds for A_n where $t_k + 1 \leq n \leq t_{k+1}$. Note that the lowest value of $\frac{A_n}{n}$ on that interval occurs for some n so that the n^{th} digit is a 2. Such an n corresponds to the end of a "piece" $1\ 2$ or $1\ 1\ 2$; in fact, it corresponds to the end of the m^{th} piece for some m with $t_{k-1} + 1 \leq m \leq t_k$. In the sequence through the m^{th} digit we have A_m "1" and $m - A_m$ "2", which convert to A_m pieces 12 and $m - A_m$ pieces 112 for a total of $2m - A_m$ "1" and m "2" in the sequence through the n^{th} digit. Thus,

$$\frac{A_n}{n} = \frac{2m - A_m}{3m - A_m} = \frac{2 - \frac{A_m}{m}}{3 - \frac{A_m}{m}} = f\left(\frac{A_m}{m}\right).$$

Since the function f is decreasing and $\frac{A_m}{m} \leq \beta_k$, we have $\frac{A_n}{n} \geq f(\beta_k)$. Thus, as stated in (2) and (3), we can take $\alpha_{k+1} = f(\beta_k)$ as a lower bound for all $\frac{A_n}{n}$ with $t_k + 1 \le n \le t_{k+1}$. The proof that we can take $\beta_{k+1} = f(\alpha_k)\left(1 + \frac{1}{t_k + 1}\right)$ as an upper bound for those same $\frac{A_n}{n}$ is similar, using the fact that the highest value for $\frac{A_n}{n}$ on the interval occurs *just before* a digit 2 and that $1 + \frac{1}{n} \le 1 + \frac{1}{t_k + 1}$ on the interval.

From $\alpha_4 = \frac{5}{9}$, $\beta_4 = \frac{2}{3}$ we get $\alpha_5 = \frac{4}{7}$, $\beta_5 = \frac{7}{11}$. Note that $\alpha_4 < \alpha_5$ and $\beta_4 > \beta_5$. We can now show by induction that both sequences are monotonic. Since f is decreasing, $\beta_k > \beta_{k+1}$ implies $f(\beta_k) < f(\beta_{k+1})$; that is, $\alpha_{k+1} < \alpha_{k+2}$. Similarly, $\alpha_k < \alpha_{k+1}$ implies $f(\alpha_k) > f(\alpha_{k+1})$ and since $1 + \frac{1}{t_k + 1} > 1 + \frac{1}{t_{k+1} + 1}$, it follows that $\beta_{k+1} > \beta_{k+2}$. This concludes the verification of properties (1), (2) and (3), and thus, the proof.

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. (a) The equation $y = x^2 + 2ax + a$ represents a parabola for all real values of a. Prove that each of these parabolas passes through a common point and determine the coordinates of this point.

(b) The vertices of the parabolas in part (a) lie on a curve. Prove that this curve is itself a parabola whose vertex is the common point found in part (a).

(1999 Euclid, Problem 8)

Solution.

(a) The point $(-1/2, 1/4)$ satisfies the given equation, and thus, all the parabolas pass through that point.

Reasoning: We want to find an (x, y) that satisfies the equation of the parabola, that does not rely on the variable a. Noting that the given equation can be rewritten as $y = x^2 + a(2x + 1)$, the only way of eliminating a is to let $x = -1/2$. Upon doing this, we find that $(x, y) = (-1/2, 1/4)$ satisfies the equations of the family of parabolas.

(b) The equation can be rewritten as $y = (x + a)^2 + a - a^2$. Hence the vertex is at $(-a, a - a^2)$. Now, since $x = -a$, we have $a = -x$, and $y = a - a^2 = -x - x^2$. Therefore, the vertex lies on the parabola $y = -x - x^2$, the vertex of which is $(-1/2, 1/4)$.

A Trigonometric Equation

Nicolea Gusita

Consider the equation:

$$a\sin x + b\cos x \;=\; c \qquad \text{with} \qquad a,\, b \neq 0\,. \tag{1}$$

Solution 1: Let us divide both sides by a in (1) and then denote

$$\frac{b}{a} \;=\; \tan\phi\,. \tag{2}$$

This yields $\sin x + \tan\phi\cos x = \frac{c}{a}$ or

$$\frac{\sin(x+\phi)}{\cos\phi} \;=\; \frac{c}{a}\,. \tag{3}$$

Since $1 + \tan^2\phi = \frac{1}{\cos^2\phi}$, we have $\cos\phi = \frac{\pm 1}{\sqrt{1+\tan^2\phi}}$, which becomes $\cos\phi = \frac{\pm a}{\sqrt{a^2+b^2}}$, taking into consideration (2).

Therefore, equation (3) becomes:

$$\sin(x+\phi) \;=\; \frac{c\cos\phi}{a} \;=\; \frac{\pm c}{\sqrt{a^2+b^2}}\,, \tag{4}$$

with the solution $x + \phi = \pm\arcsin\frac{c}{\sqrt{a^2+b^2}} + 2k\pi$ or $x = -\arctan\frac{b}{a} \pm \arcsin\frac{c}{\sqrt{a^2+b^2}} + 2k\pi$, where $k \in \mathbb{Z}$. From (4) there is the restriction $-1 \leq \frac{c}{\sqrt{a^2+b^2}} \leq 1$ so that $-\sqrt{a^2+b^2} \leq c \leq \sqrt{a^2+b^2}$ or simply

$$a^2 + b^2 \;\geq\; c^2\,. \tag{5}$$

We notice that equations (4) and (1) are equivalent since:

$$\begin{aligned}
\text{LS} \;&=\; a\sin x + b\cos x \;=\; a\left(\sin x + \frac{b}{a}\cos x\right) \\
&=\; a(\sin x + \tan\phi\cos x) \;=\; \frac{a\sin(x+\phi)}{\cos\phi} \\
&=\; \pm\sqrt{a^2+b^2}\,\sin(x+\phi)\,.
\end{aligned}$$

What happens if we divide both sides in equation (1) by b? We will leave this as an exercise for the reader.

Solution 2: In this method let us put

$$\tan\frac{x}{2} \;=\; t \qquad \text{or} \qquad \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} \;=\; t\,. \tag{6}$$

Therefore, $x \neq (2k+1)\pi$.

Case 1: Equation (1) does not have solutions of the form $x = (2k+1)\pi$. Therefore, $a\sin(2k+1)\pi + b\cos(2k+1)\pi \neq c$. Since we have $\sin(2k+1)\pi = \sin(\pi+2k\pi) = \sin\pi = 0$ and $\cos(2k+1)\pi = \cos(\pi+2k\pi) = \cos\pi = -1$, the last relation will be converted to

$$b + c \neq 0\,. \tag{7}$$

But, $\sin x = \frac{2t}{1+t^2}$, and $\cos x = \frac{1-t^2}{1+t^2}$. Therefore, we have $a\frac{2t}{1+t^2} + b\frac{1-t^2}{1+t^2} = c$ or $(b+c)t^2 - 2at - b + c = 0$; that is, a quadratic equation since $b + c \neq 0$ from (7). This quadratic equation will have real roots if the discriminant $\Delta \geq 0$. This gives us:

$$\Delta = (-a)^2 + (b+c)(b-c) = a^2 + b^2 - c^2 \geq 0$$

or $a^2 + b^2 \geq c^2$, exactly the same restriction we found in solution 1.

The solutions then would be: $\tan\frac{x_1}{2} = t_1$, or $x_1 = 2\arctan t_1 + 2k_1\pi$ and $\tan\frac{x_2}{2} = t_2$ or $x_2 = 2\arctan t_2 + 2k_2\pi$ with $k_1, k_2 \in \mathbb{Z}$.

Case 2: Equation (1) does have a solution of the form $x = (2k+1)\pi$. Thus, $a\sin(2k+1)\pi + b\cos(2k+1)\pi = c$, or $b + c = 0$, so that $c = -b$, which yields $a\sin x + b(1+\cos x) = 0$.

But, $\sin x = \sin 2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $1 + \cos x = 2\cos^2\frac{x}{2}$, so that we have:

$$\begin{aligned} 2a\sin\frac{x}{2}\cos\frac{x}{2} + 2b\cos^2\frac{x}{2} &= 0\,, \\ 2\cos\frac{x}{2}\left(a\sin\frac{x}{2} + b\cos\frac{x}{2}\right) &= 0\,. \end{aligned}$$

Therefore, the roots of equation (1) in this case are obtained thus: $\cos\frac{x_1}{2} = 0$, so that $\frac{x_1}{2} = (2k+1)\frac{\pi}{2}$; that is, $x_1 = (2k+1)\pi, k \in \mathbb{Z}$.

We also have $\tan\frac{x_2}{2} = -\frac{b}{a}$, so that $x_2 = 2\arctan\left(-\frac{b}{a}\right) + 2k\pi$.

Solution 3: We can determine $\sin x = X$ and $\cos x = Y$ from the algebraic system:

$$aX + bY = c\,, \qquad X^2 + Y^2 = 1\,.$$

I will leave this solution for the readers to have fun. Good luck!

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SKOLIAD No. 61

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mayhem-editors@cms.math.ca

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 August 2002*. A copy of ***MATHEMATICAL MAYHEM* Vol. 3** will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

Our item this issue is the 2001 W.J. Blundon Mathematics Contest. My thanks go out to Don Rideout of Memorial University for forwarding the material to me.

THE EIGHTEENTH W.J. BLUNDON MATHEMATICS CONTEST

Sponsored by The Canadian Mathematical Society, in cooperation with The Department of Mathematics and Statistics, Memorial University of Newfoundland

February 21, 2001

1. (a) At a meeting of 100 people, every person shakes hands with every other person exactly once. How many handshakes are there in total?

(b) How many four-digit numbers are divisible by 5?

2. Show that $n^2 + 2$ is divisible by 4 for no integer n.

3. Prove that the difference of squares of two odd integers is always divisible by 8.

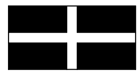
4. The inscribed circle of a right triangle ABC is tangent to the hypotenuse AB at D. If $AD = x$ and $DB = y$, find the area of the triangle in terms of x and y.

5. Find all integers x and y such that

$$2^x + 3^y = 3^{y+2} - 2^{x+1}.$$

6. Find the number of points (x, y), with x and y integers, that satisfy the inequality $|x| + |y| < 100$.

7. A flag consists of a white cross on a red field.



The white stripes are of the same width, both vertical and horizontal. The flag measures 48 cm $\times$ 24 cm. If the area of the white cross equals the area of the red field, what is the width of the cross?

8. Solve $\frac{x+1}{2+\sqrt{x}} - \frac{1}{2-\sqrt{x}} = 3$.

9. Let $P(x)$ and $Q(x)$ be polynomials with "reversed" coefficients

$$\begin{aligned} P(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 , \\ Q(x) &= a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-2} x^2 + a_{n-1} x + a_n , \end{aligned}$$

where $a_n \neq 0$, $a_0 \neq 0$. Show that the roots of $Q(x)$ are the reciprocals of the roots of $P(x)$.

10. If 1997^{1998} is multiplied out, what is the units digit of the final product?

Next we turn to solutions to the contests presented in the November 2001 issue. Following are the official solutions to the 2001 British Columbia Colleges mathematics competitions. [2001 : 440–445]

BRITISH COLUMBIA COLLEGES

Junior High School Mathematics Contest, 2001

Final Round – Part A

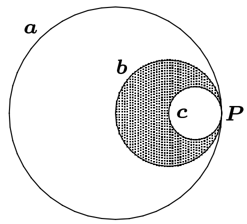
Friday May 4, 2001

1. The integer 9 is a perfect square that is both two greater than a prime number, 7, and two less than a prime number, 11. Another such perfect square is:

(a) 25 (b) 49 (c) 81 (d) 121 (e) 169

Soln. If $n = 3k \pm 1$, then $n^2 + 2 = 9k^2 \pm 6k + 3 = 3(3k^2 \pm 2k + 1)$, which is **not** prime. Thus we must have n a multiple of 3 in order to have $n^2 + 2$ prime. This eliminates all but 81. If we check 81, we see that 79 and 83 are both prime. $\boxed{\text{c}}$

2. Three circles, a, b, and c, are tangent to each other at point P, as shown.

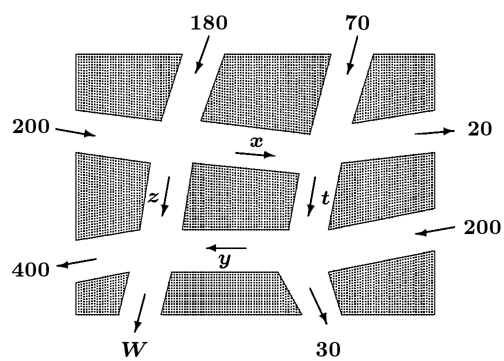


The centre of b is on c and the centre of a is on b. The ratio of the area of the shaded region to the total area of the unshaded regions enclosed by the circles is:

(a) $3:13$ (b) $1:3$ (c) $1:4$ (d) $2:9$ (e) $1:25$

Soln. Let the radius of c be r. Then the radii of b and a are $2r$ and $4r$, respectively. Therefore, the areas of a, b, and c are $16\pi r^2$, $4\pi r^2$, and πr^2, respectively. Then the area of the shaded region is $4\pi r^2 - \pi r^2 = 3\pi r^2$, and the area of the unshaded region is $16\pi r^2 - 3\pi r^2 = 13\pi r^2$. The ratio of shaded to unshaded areas is then $3:13$. $\boxed{\text{a}}$

3. Here is a diagram of part of the downtown in a medium sized town in the interior of British Columbia. The arrows indicate one-way streets. The numbers or letters by the arrows represent the number of cars that travel along that portion of the street during a typical week day.



Assuming that no car stops or parks and that no cars were there at the beginning of the day, the value of the variable W is:

(a) 30 (b) 200 (c) 250 (d) 350 (e) 600

Soln. Clearly the number of cars entering the diagram must equal the number of cars exiting the diagram; that is,

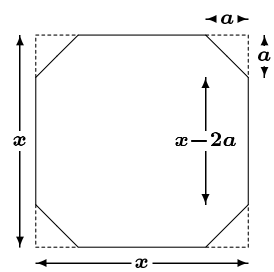
$$\begin{aligned} 200+180+70+200 &= 20+30+W+400 \\ 650 &= W+450 \\ W &= 200\,. \qquad \boxed{\text{b}} \end{aligned}$$

4. The corners of a square of side x are cut off so that a regular octagon remains. The length of each side of the resulting octagon is:

(a) $\frac{\sqrt{2}}{2}x$ (b) $2x\left(2+\sqrt{2}\right)$ (c) $\frac{x}{\sqrt{2}-1}$

(d) $x\left(\sqrt{2}-1\right)$ (e) $x\left(\sqrt{2}+1\right)$

Soln. Let the length of the removed corner piece be a (see diagram below). Then a side of the resulting octagon is equal to $x-2a$.



Using the Theorem of Pythagoras on the right-angled triangle in any corner gives us:

$$\begin{aligned} (x-2a)^2 &= a^2+a^2 = 2a^2 \\ x-2a &= a\sqrt{2} \\ x &= a(2+\sqrt{2}) \\ a &= \frac{x}{2+\sqrt{2}}\,. \end{aligned}$$

We are interested in the length of the side of the octagon:

$$\begin{aligned} x-2a &= x-\frac{2x}{2+\sqrt{2}} = \frac{2x+x\sqrt{2}-2x}{2+\sqrt{2}} = \frac{x\sqrt{2}}{2+\sqrt{2}} \\ &= \frac{x\sqrt{2}(2-\sqrt{2})}{4-2} = \frac{x(2\sqrt{2}-2)}{2} = x(\sqrt{2}-1)\,. \end{aligned}$$

Alternate approach: Let b be the side length of the regular octagon. Since the removed corners are $45°$-$45°$-$90°$ triangles, the legs have length $b/\sqrt{2}$. Thus

$$\begin{aligned} x &= \frac{2b}{\sqrt{2}}+b = \left(\frac{2+\sqrt{2}}{\sqrt{2}}\right)b \\ \text{or}\qquad b &= \left(\frac{\sqrt{2}}{2+\sqrt{2}}\right)x\,. \end{aligned}$$

Rationalizing the denominator we get:

$$b = \frac{\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}}x = \left(\frac{2\sqrt{2}-2}{4-2}\right)x = \left(\sqrt{2}-1\right)x\,. \qquad \boxed{\text{d}}$$

5. The value of $(0.0\overline{1})^{-1}+1$ is: (The line over the digit 1 means that it is repeated indefinitely.)

(a) $\frac{1}{91}$ (b) $\frac{90}{91}$ (c) $\frac{91}{90}$ (d) 10 (e) 91

Soln. Let $x = 0.0\overline{1}$. Then we note that $10x = 0.\overline{1}$, which can also be written as $10x = 0.1\overline{1}$. Comparing this form of $10x$ with x we see that the decimal fraction expansions agree except for the first digit following the decimal point. Thus we may subtract to obtain $9x = 0.1$, which means that $x = 1/90$. Then

$$(0.0\overline{1})^{-1}+1 = \left(\frac{1}{90}\right)^{-1}+1 = 90+1 = 91\,.$$

Alternate method: We first note that $0.01 < x < 0.02$ or $\frac{1}{100} < x < \frac{1}{50}$, which means that $100 > \frac{1}{x} > 50$, and there is only one possible answer in this range. $\boxed{\text{e}}$

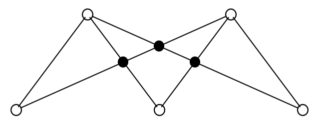
6. The people living on Sesame Street all decide to buy new house numbers from the same store, and they purchase the digits for their house numbers in the order of their addresses: $1, 2, 3, \ldots$. If the store has 100 of each digit, then the first address which cannot be displayed occurs at house number:

(a) 100 (b) 101 (c) 162 (d) 163 (e) 199

Soln. In order to cover the addresses from 1 to 99, we need 20 of each nonzero digit and 9 zeros. From 100 to 199, we will have the greatest call on the digit 1 since every such address will have at least one digit 1 in it. Therefore, let us examine only the digit 1 first. From 100 to 109 we use 11 ones; from 110 to 119 we use 21 ones; for each subsequent group of ten (up to the address 199) we use a further 11 ones. Thus we want $20+11+21+k(11) \leq 100$, implying that $k \leq 4$. That is, up to address 159 we have $20+11+21+4(11) = 96$ ones. Addresses 160, 161, and 162 use up a further 4 ones and we have exhausted the 100 ones we started with. Thus the first address which cannot be displayed is 163. $\boxed{\text{d}}$

7. Given p dots on the top row and q dots on the bottom row, draw line segments connecting each top dot to each bottom dot. (In the diagram below, the dots referred to are the small open circles.) The dots must be arranged such that no three line segments intersect at a common point

(except at the ends). The line segments connecting the dots intersect at several points. (In the diagram below, the points of intersection of the line segments are the small filled circles.) For example, when $p = 2$ and $q = 3$ there are three intersection points, as shown below.



When $p = 3$ and $q = 4$ the number of intersections is:

(a) 7 (b) 12 (c) 18 (d) 21 (e) 27

Soln. If we first consider $p = 2$ and $q = 4$, we easily see that there are $1 + 2 + 3 = 6$ points of intersection. If we now consider $p = 3$ and $q = 4$ we see that by considering any pair of the $p = 3$ dots, together with the $q = 4$ dots opposite, we get 6 points of intersection. Now there are three such distinct pairs which gives us a total of 18 points of intersection. $\boxed{c}$

8. At one time, the population of Petticoat Junction was a perfect square. Later, with an increase of 100, the population was 1 greater than a perfect square. Now, with an additional increase of 100, the population is again a perfect square. The original population was a multiple of:

(a) 3 (b) 7 (c) 9 (d) 11 (e) 17

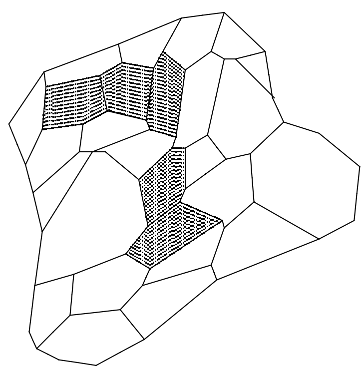
Soln. Let the first-mentioned population be n. Then $n = a^2$ for some integer a. We then also have $n + 100 = b^2 + 1$ and $n + 200 = c^2$ for some integers b and c. That is, $a^2 + 99 = b^2$ and $a^2 + 200 = c^2$, or $b^2 - a^2 = 99$ and $c^2 - a^2 = 200$. Subtracting these, we get $c^2 - b^2 = 101$. Thus $(c - b)(c + b) = 101$. Since 101 is prime we see that $c - b = 1$ and $c + b = 101$, whence $c = 51$ and $b = 50$. Thus $n = a^2 = b^2 - 99 = 2401 = 49^2 = 7^4$. $\boxed{b}$

9. The cashier at a local movie house took in a total of \$100 from 100 people. If the rates were \$3 per adult, \$2 per teenager and 25 cents per child, then the smallest number of adults possible was:

(a) 0 (b) 2 (c) 5 (d) 13 (e) 20

Soln. Let a be the number of adults, t be the number of teenagers, and c be the number of children attending the movie. Then $a+t+c = 100$ is the number of persons attending the movie, and $3a+2t+c/4 = 100$ is the number of dollars taken in by the movie house. Multiplying the second equation by 4 to clear the fractions, and subtracting the first equation we get: $11a + 7t = 300$, or $t = (300 - 11a)/7$. Since we are seeking integer solutions and we want the smallest possible value for a, we may simply examine successive values of a starting with $a = 0$ until we find an integer solution for t. The first (that is, the smallest) value of a is $a = 5$, which gives $t = 35$ (and $c = 60$). $\boxed{\text{c}}$

10. The island of Aresia has 27 states, each of which belongs to one of two factions, the white faction and the grey faction, who are sworn enemies. The United Nations wishes to bring peace to Aresia by converting one state at a time to the opposite faction; that is, converting one state from white to grey or from grey to white, so that eventually all states belong to the same faction. In doing this they must guarantee that no single state is completely surrounded by states of the **opposite faction**. Note that a coastal state can never be completely surrounded, and that it may be necessary to convert a state from one faction to the other at one stage and then convert it back to its original faction later. A map of the state of Aresia is shown.



The five shaded states belong to the grey faction, and all of the unshaded states belong to the white faction. The minimum number of conversions necessary to completely pacify Aresia is:

(a) 5 (b) 7 (c) 9 (d) 10 (e) 15

Soln. We can first make a shaded "chain" to the coast by shading one coastal region at the top left and one of the two interior unshaded regions linking the two shaded regions. This requires 2 conversions. This shaded chain of 7 states can now be unshaded one at a time working from the interior to the coast, requiring another 7 conversions for a total of 9 conversions. To see that there can never be fewer than 9 conversions, we note first that we must convert the shaded states to unshaded in order to minimize the number of conversions, and secondly that it is necessary to convert at least one unshaded coastal state and one unshaded interior state to shaded in order to avoid a shaded state being ultimately surrounded by unshaded states. This means that we would have a minimum of 7 shaded states to be converted to unshaded (in addition to the minimum of 2 unshaded that need to be converted to shaded). Thus we require at least 9 conversions. Thus, 9 is the minimum number of conversions needed to pacify Aresia. **C**

BRITISH COLUMBIA COLLEGES

Junior High School Mathematics Contest, 2001

Final Round – Part B

Friday May 4, 2001

1. Find the smallest 3–digit integer which leaves a non-zero remainder when divided by any of 2, 3, 4, 5, or 6 but not when divided by 7.

Soln. Let n be the 3–digit integer in question. Clearly n is a multiple of 7. Since it has three digits, we may start with the smallest 3–digit multiple of 7 and examine successive multiples of 7 until the conditions are satisfied. The first 3–digit multiple of 7 is 105, which is also a multiple of 5; the next is 112, which is a multiple of 2; the next is $119 = 7 \times 17$, and this leaves a non-zero remainder when divided by any of 2, 3, 4, 5, or 6. Thus $n = 119$.

Alternate Solution: There was at least one student who, in reading the problem, recognized (unlike the problem posers!) that nowhere is there a mention that the smallest 3–digit integer had to be positive. Since any negative number is smaller than any positive one, the student then found the smallest negative 3–digit integer satisfying the conditions. Since 994 is a multiple of 7, so is -994. Thus this represents the starting point. Since -994 is a multiple of 2, it is eliminated; the next candidate is -987, which is a multiple of 3 and is also eliminated; then comes -980, which is a multiple of 2 again; the next one is -973, and it satisfies all the conditions.

Strictly speaking, -973 is the only correct answer! However, since most solvers, as well as the problem posers, read "positive" into the problem, we also allowed 119 as a correct answer.

2. Assume that the land within two kilometres of the South Pole is flat. There are points in this region where you can travel one kilometre south, travel one kilometre east along *one* circuit of a latitude, and finally travel one kilometre north, and thus arrive at the point where you started. How far is such a point from the South Pole?

Soln. In the vicinity of the South Pole all east-west travel is on a circle centred about the South Pole. Since we wish to have the circumference of such a circle equal to 1 km, we must have the radius equal to $1/2\pi$ km. The original point from which the trip starts must be located a further 1 km away from the south pole. Thus we must start $1 + (1/2\pi)$ km from the South Pole.

3. Café de la Pêche offers three fruit bowls:

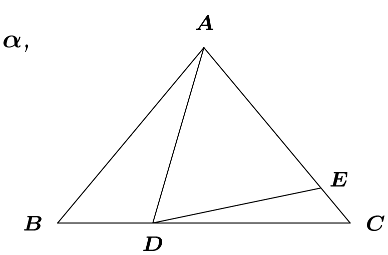
   - Bowl A has two apples and one banana;
   - Bowl B has four apples, two bananas, and three pears;
   - Bowl C has two apples, one banana, and three pears.

   Your doctor tells you to eat exactly 16 apples, 8 bananas and 6 pears each day. How many of each type of bowl should you buy so there is no fruit left over? Find all possible answers. (The numbers of bowls must be non-negative integers.)

Soln. Since the number of apples is twice the number of bananas in each bowl as well as in the doctor's dictum, we can ignore the apple constraint, and simply solve the problem for bananas and pears. Since we have in each bowl either 0 or 3 pears, we see that the condition on the pears can be met in exactly one of three ways: two of bowl B and none of bowl C; one of each of bowls B and C; or none of bowl B and two of bowl C. In each case we can then add the number of A bowls to fill out the requirements. Thus, there are three solutions: $(A, B, C) = (4, 2, 0)$, $(5, 1, 1)$, and $(6, 0, 2)$.

4. In the triangle shown, $\angle BAD = \alpha$, $\overline{AB} = \overline{AC}$ and $\overline{AD} = \overline{AE}$.

   Find $\angle CDE$ in terms of α.



Soln. Let $\angle B = \angle C = x$. Let $\angle CDE = y$. Since $\angle AED$ is an exterior angle to $\triangle EDC$, we have $\angle AED = x + y$. Since $\triangle ADE$ is isosceles, we also have $\angle ADE = x + y$, whence $\angle ADC = x + 2y$. But $\angle ADC$ is an exterior angle of $\triangle ABD$, which means that $\angle ADC = x + \alpha$. Thus, we have $x + 2y = x + \alpha$, or $y = \frac{1}{2}\alpha$. Therefore, $\angle CDE = \frac{1}{2}\alpha$.

5. In the multiplication below each of the letters stands for a distinct digit. Find all values of $JEEP$.

$$\begin{array}{r} JEEP \\ \times\ JEEP \\ \hline BEEBEEP \end{array}$$

Soln. Since $P^2 = P + 10k$ for some integer k, $0 \le k \le 8$, we see that P is one of 0, 1, 5, or 6. Now by considering the last two digits of each factor and the product we have $(10E + P)^2 = 100n + 10E + P$ for some integer $n < 100$.

This means that $20PE + P^2 - 10E - P = 10E(2P - 1) + P(P - 1)$ is a multiple of 100. Let us consider $P = 6$. Then $110E + 30$ is a multiple of 100, implying that $E = 7$. This means that 776^2 must end in the digits 776, but 776^2 actually ends in the digits 176. Thus, $P \neq 6$.

Next try $P = 5$. Then $90E + 20$ is a multiple of 100, implying that $E = 2$. This means that 225^2 must end in the digits 225, but 225^2 actually ends in the digits 625. Thus $P \neq 5$. Therefore, $P = 0$ or 1. In either case we have $P(P - 1) = 0$, which means that $10E(2P - 1)$ is a multiple of 100. Since $2P - 1 = \pm 1$, we conclude that $E = 0$, implying that $P = 1$, since it must be different from E. Thus, we have $P = 1$ and $E = 0$. Then we have $B = J^2$ and $B = 2J$, since $(J001)^2 = (J)^2 00(2J)001 = B00B001$. Since $J^2 = 2J$ and $J \neq 0$, we conclude that $J = 2$, whence $JEEP = 2001$.

BRITISH COLUMBIA COLLEGES

Senior High School Mathematics Contest, 2001

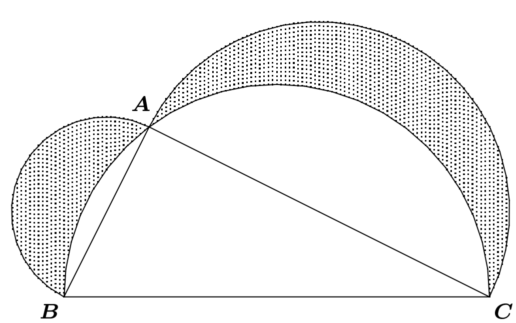
Final Round – Part B

Friday May 4, 2001

1. See question #4 above.

Soln. See question #4 on the Junior Final (Part B).

2. A semicircle BAC is mounted on the side BC of the triangle ABC. Semicircles are also mounted outwardly on the sides BA and AC, as shown in the diagram. The shaded crescents represent the area inside the smaller semicircles and outside the semicircle BAC. Show that the total shaded area equals the area of the triangle ABC.



Soln. To obtain the area of the shaded region we will compute the sum of the areas of triangle ABC, the area of the semicircle on AB, and the area of the semicircle on AC, and then we will subtract from this sum the area of the semicircle on BC. Let a, b, and c be the lengths of the sides BC, AC, and AB, respectively. Since $\triangle ABC$ is inscribed in a semicircle on BC, we see that $\angle BAC = 90°$. Thus by the Theorem of Pythagoras we have $a^2 = b^2 + c^2$. If we now denote the area of $\triangle ABC$ by $[ABC]$, then our desired area is:

$$\begin{aligned} \mathcal{A} &= [ABC] + \tfrac{1}{2}\pi\left(\tfrac{c}{2}\right)^2 + \tfrac{1}{2}\pi\left(\tfrac{b}{2}\right)^2 - \tfrac{1}{2}\pi\left(\tfrac{a}{2}\right)^2 \\ &= [ABC] + \tfrac{1}{8}\pi(b^2 + c^2 - a^2) \;=\; [ABC]\,. \end{aligned}$$

3. Five schools competed in the finals of the British Columbia High School Track Meet. They were Cranbrook, Duchess Park, Nanaimo, Okanagan Mission, and Selkirk. The five events in the finals were: the high jump, shot put, 100–metre dash, pole vault and 4–by–100 relay. In each event the school placing first received five points; the one placing second, four points; the one placing third, three points; and so on. Thus, the one placing last received one point. At the end of the competition, the points of each school were totalled, and the totals determined the final ranking.

   (a) Cranbrook won with a total of 24 points.

   (b) Sally Sedgwick of Selkirk won the high jump hands down (and feet up), while Sven Sorenson, also of Selkirk, came in third in the pole vault.

   (c) Nanaimo had the same number of points in at least four of the five events.

   Each school had exactly one entry in each event. Assuming there were no ties and the schools ended up being ranked in the same order as the alphabetical order of their names, in what position did Doug Dolan of Duchess Park rank in the high jump?

Soln. Since each school had exactly one entry in each event, we conclude by (a) that Cranbrook had four first place finishes and one second place finish. By (b) it becomes clear that the one second place finish they had was in the high jump. Thus Doug Dolan of Duchess Park could finish no higher than third place in the high jump. Of the total of 75 available points, 24 went to Cranbrook, which leaves 51 points to be shared by the other four schools.

Since they all received different totals, Duchess Park, who came in second must have obtained at least 15 points (since $14+13+12+11 = 50$, which is too small). A similar argument shows that last place Selkirk must have obtained at most 11 points (since $12 + 13 + 14 + 15 = 54$, which is too large). Since Selkirk obtained 5 points for the high jump and 3 points for the pole vault by (b), and at least 1 point for each of the other three events, they must have a total of at least 11. This, together with our previous remark shows that Selkirk had exactly 11 points. This leaves only 40 points to be shared by Duchess Park, Nanaimo, and Okanagan Mission, and each of them must have at least 12 points.

The only possibility is that Duchess Park had 15 points, Nanaimo had 13 points and Okanagan Mission had 12 points. Since Nanaimo received the same number of points in four of the five events and had a total of 13 points, they must have finished third four times and last once (since four second place finishes would give them too many points, while four fourth place finishes would require them to finish first in the other event to get 13 points, but all the first place finishes went to Cranbrook and Selkirk).

Thus Nanaimo had to finish last in the pole vault, as Selkirk finished third. At this point we have determined that all 1–point, 3–point, and 5–point finishes (except for last place in the high jump) have gone to one of Cranbrook, Nanaimo, or Selkirk. Since the only remaining odd point will generate an odd total, it must go to Duchess Park, which has a total of 15 points.

Thus Doug Dolan of Duchess Park must have finished last in the high jump.

4. A box contains tickets of two different colours: blue and green. There are 3 blue tickets. If two tickets are to be drawn together at random from the box, the probability that there is one ticket of each colour is exactly $\frac{1}{2}$. How many green tickets are in the box? Give all possible solutions.

Soln. Let g be the number of green tickets in the box. Then the total number of tickets in the box is $g+3$. The number of ways of drawing two tickets from the box (together) is $\binom{g+3}{2} = \frac{(g+3)(g+2)}{2}$. The number of ways of drawing one ticket of each colour is by drawing one of 3 blue tickets and one of g green tickets, which is $3 \cdot g$. Thus the probability of drawing one of each colour when drawing two tickets together is

$$\frac{3g}{(g+3)(g+2)/2} = \frac{6g}{(g+3)(g+2)}\,.$$

We are told that this probability is $\frac{1}{2}$. Therefore, we have

$$\begin{aligned} \frac{1}{2} &= \frac{6g}{g^2+5g+6} \\ g^2+5g+6 &= 12g \\ g^2-7g+6 &= 0 \\ (g-6)(g-1) &= 0\,, \end{aligned}$$

which means that $g = 1$ or $g = 6$. Both of these solutions can be verified.

5. In (a), (b), and (c) below the symbols m, h, t, and u can represent any integer from 0 to 9 inclusive.

   (a) If $h - t + u$ is divisible by 11, prove that $100h + 10t + u$ is divisible by 11.

   (b) If $h + u = m + t$, prove that $1000m + 100h + 10t + u$ is divisible by 11.

   (c) Is it possible for $1000m + 100h + 10t + u$ to be divisible by 11 if $h + u \neq m + t$? Explain.

Soln. (a) Note that

$$100h + 10t + u \;=\; 99h + 11t + (h - t + u) \;=\; 11(9h + t) + (h - t + u)\,.$$

Clearly, if $h - t + u$ is divisible by 11, the entire right hand side is also divisible by 11, which means that $100h + 10t + u$ is divisible by 11.
(b) In this case we observe

$$\begin{aligned} &1000m + 100h + 10t + u \\ &\quad= 1001m + 99h + 11t + (-m + h - t + u) \\ &\quad= 11(91m + 9h + t) + (h + u - m - t) \qquad (1) \end{aligned}$$

Again, if $h + u = m + t$ we see that the right hand side is simply $11(91m + 9h + t)$, which is clearly divisible by 11, implying that the right hand is also divisible by 11.
(c) If we examine (1) in part (b) above, we see that to have the right hand side divisible by 11 all we need is that $h + u - m - t$ is divisible by 11. This can happen without $h + u = m + t$ as the following example shows: $h = 9$, $u = 8$, $m = 4$, and $t = 2$; $h + u = 17$ and $m + t = 6$, which means that $h + u \neq m + t$, but $h + u - m - t = 11$ and the expression in (1) above is then divisible by 11.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"$\times 11$" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 November 2002**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5 and 7, English will precede French, and in issues 2, 4, 6 and 8, French will precede English. In the solutions section, the problem will be given in the language of the primary featured solution.

2704. *Proposed by Mihály Bencze, Brasov, Romania.* Correction.

Prove that

$$R-2r \;\geq\; \frac{1}{12}\left(\sum_{\text{cyclic}}\sqrt{2(b^2+c^2)-a^2}-\frac{s^2+r^2+4Rr}{R}\right) \;\geq\; 0\,,$$

where a, b and c are the sides of a triangle, and R, r and s are the circumradius, the inradius and the semi-perimeter of the triangle, respectively.

. . . .

Montrer que

$$R-2r \;\geq\; \frac{1}{12}\left(\sum_{\text{cyclique}}\sqrt{2(b^2+c^2)-a^2}-\frac{s^2+r^2+4Rr}{R}\right) \;\geq\; 0\,,$$

où a, b et c sont les côtés d'un triangle, et R, r et s sont respectivement le rayon du cercle circonscrit, le rayon du cercle inscrit et le demi-périmètre du triangle.

2724★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle and h_a, h_b, h_c, respectively, the corresponding altitudes. Prove that the maximum range of validity of the inequality

$$\left(\frac{h_a^t + h_b^t + h_c^t}{3}\right)^{1/t} \le \frac{\sqrt{3}}{2}\left(\frac{a^t + b^t + c^t}{3}\right)^{1/t},$$

where $t \neq 0$ is $\dfrac{-\ln 4}{\ln 4 - \ln 3} < t < \dfrac{\ln 4}{\ln 4 - \ln 3}$.

....

Soit a, b, c les côtés d'un triangle et h_a, h_b, respectivement, les hauteurs correspondantes. Montrer que le domaine de validité maximal de l'inégalité

$$\left(\frac{h_a^t + h_b^t + h_c^t}{3}\right)^{1/t} \le \frac{\sqrt{3}}{2}\left(\frac{a^t + b^t + c^t}{3}\right)^{1/t},$$

où $t \neq 0$ est $\dfrac{-\ln 4}{\ln 4 - \ln 3} < t < \dfrac{\ln 4}{\ln 4 - \ln 3}$.

2725. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $k \geq 1$, let $S_k(n) = \sum\limits_{j=1}^{n}(2j-1)^k$ be the sum of the k^{th} powers of the first n odd numbers.

1. Show that the sequence $\{S_3(n), n \geq 1\}$ contains *infinitely many squares*.
2. ★ Prove that this sequence contains only *finitely many squares* of other exponents k.

....

Pour $k \geq 1$, soit $S_k(n) = \sum\limits_{j=1}^{n}(2j-1)^k$ la somme des k-ièmes puissances des n premiers nombres impairs.

1. Montrer que la suite $\{S_3(n), n \geq 1\}$ contient une *infinité de carrés*.
2. ★ Montrer que cette suite ne contient qu'un *nombre fini de carrés* d'autres exposants k.

2726. *Proposed by Armend Shabini, University of Prishtina, Prishtina, Kosovo, Serbia.*

Given the finite sequence of real numbers, $\{a_k\}$, $1 \leq k \leq 2n$, where the terms satisfy

$$a_{2k} - a_{2k-1} = d\,,\ 1 \leq k \leq n\,, \qquad \text{and} \qquad \frac{a_{2k+1}}{a_{2k}} = q\,,\ 1 \leq k \leq n-1\,,$$

prove that, when $q \neq 1$,

(a) $\sum_{k=1}^{2n} a_k \;=\; \dfrac{2qa_{2n} - 2a_1 - nd(1+q)}{q-1}$, and

(b) $a_{2n} \;=\; a_1\, q^{\frac{n-2}{n}} + d\left(\dfrac{1-q^{\frac{n}{2}}}{1-q}\right)$.

. . . .

On donne la suite finie de nombres réels $\{a_k\}$, $1 \le k \le 2n$, satisfaisant

$$a_{2k} - a_{2k-1} \;=\; d\,,\;\; 1 \le k \le n\,, \qquad \text{et} \qquad \frac{a_{2k+1}}{a_{2k}} \;=\; q\,,\;\; 1 \le k \le n-1.$$

Montrer que si $q \neq 1$,

(a) $\sum_{k=1}^{2n} a_k \;=\; \dfrac{2qa_{2n} - 2a_1 - nd(1+q)}{q-1}$, et

(b) $a_{2n} \;=\; a_1\, q^{\frac{n-2}{n}} + d\left(\dfrac{1-q^{\frac{n}{2}}}{1-q}\right)$.

2727. *Proposed by Armend Shabini, University of Prishtina, Prishtina, Kosovo, Serbia.*

Given the finite sequence of real numbers, $\{a_k\}$, $1 \le k \le n$, where the terms satisfy

$$a_k - a_{k-1} \;=\; a_{k-1} - a_{k-2} + d\,, \quad k > 2\,,\;\; d \in \mathbb{R}\,,$$

find a closed form expression for $\sum_{k=1}^{n} a_k$.

Use this to find the value of $\sum_{k=0}^{n-1} \binom{2k+2}{2k}$.

. . . .

On donne la suite finie de nombres réels $\{a_k\}$, $1 \le k \le n$, satisfaisant

$$a_k - a_{k-1} \;=\; a_{k-1} - a_{k-2} + d\,, \quad k > 2\,,\;\; d \in \mathbb{R}.$$

Trouver une expression explicite pour $\sum_{k=1}^{n} a_k$.

Utiliser le résultat pour trouver la valeur de $\sum_{k=0}^{n-1} \binom{2k+2}{2k}$.

2728. *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

The distance between two well-known points in $\triangle ABC$ is

$$\frac{bc}{a+b+c}\sqrt{2(\cos A+1)}.$$

What are the points?

. . . .

La distance de deux points bien connus dans un triangle ABC est

$$\frac{bc}{a+b+c}\sqrt{2(\cos A+1)}.$$

Quels sont ces points?

2729. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Let $Z(n)$ denote the number of trailing zeros of $n!$, where $n \in \mathbb{N}$.

(a) Prove that $\dfrac{Z(n)}{n} < \dfrac{1}{4}$.

(b)⋆ Prove or disprove that $\lim\limits_{n\to\infty} \dfrac{Z(n)}{n} = \dfrac{1}{4}$.

. . . .

Soit $Z(n)$ le nombre de zéros apparaissant en queue de $n!$, où $n \in \mathbb{N}$.

(a) Montrer que $\dfrac{Z(n)}{n} < \dfrac{1}{4}$.

(b)⋆ Montrer si oui on non, $\lim\limits_{n\to\infty} \dfrac{Z(n)}{n} = \dfrac{1}{4}$.

2730. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $\mathrm{AM}(x_1, x_2, \ldots, x_n)$ and $\mathrm{GM}(x_1, x_2, \ldots, x_n)$ denote the arithmetic mean and the geometric mean of the real numbers x_1, x_2, ..., x_n, respectively.

Given positive real numbers a_1, a_2, ..., a_n, b_1, b_2, ..., b_n, prove that

(a) $\mathrm{GM}(a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$
$\geq \mathrm{GM}(a_1, a_2, \ldots, a_n) + \mathrm{GM}(b_1, b_2, \ldots, b_n)$.

For each real number $t \geq 0$, define $f(t) = \mathrm{GM}(t+b_1, t+b_2, \ldots, t+b_n) - t$.

(b) Prove that $f(t)$ is a monotonic increasing function of t, and that

$$\lim_{t\to\infty} f(t) = \mathrm{AM}(b_1, b_2, \ldots, b_n).$$

Soit $\mathrm{AM}(x_1, x_2, \ldots, x_n)$ et $\mathrm{GM}(x_1, x_2, \ldots, x_n)$ la moyenne arithmétique, respectivement la moyenne géométrique des nombres réels x_1, x_2, $\ldots$, x_n. Etant donné des nombres réels positifs a_1, a_2, $\ldots$, a_n, b_1, b_2, $\ldots$, b_n, montrer que

(a) $\mathrm{GM}(a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$
$\geq \mathrm{GM}(a_1, a_2, \ldots, a_n) + \mathrm{GM}(b_1, b_2, \ldots, b_n)$.

Pour tout nombre réel $t \geq 0$, soit $f(t) = \mathrm{GM}(t+b_1, t+b_2, \ldots, t+b_n) - t$.

(b) Montrer que $f(t)$ est une fonction monotone croissante de t, et que

$$\lim_{t\to\infty} f(t) \;=\; \mathrm{AM}(b_1, b_2, \ldots, b_n)\,.$$

2731. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let C be a conic with foci F_1, F_2, and directrices D_1, D_2, respectively.

Given any point M on the conic, draw the line passing through M, perpendicular to the directrices, intersecting D_1, D_2, at M_1, M_2, respectively. Let R be the point of intersection of the lines M_1F_1 and M_2F_2. Prove that

(a) $\dfrac{\overline{F_1R}}{\overline{M_1R}}$ is independent of the choice of M;

(b) the normal to the conic at M passes through R.

. . . .

Soit C une conique de foyers F_1, F_2 et de directrices D_1, D_2, respectivement. Par un point donné quelconque M sur la conique, on dessine la perpendiculaire aux directrices, et soit M_1, M_2 les points d'intersection respectifs. Si R est le point d'intersection des droites M_1F_1 et M_2F_2, montrer que

(a) $\dfrac{\overline{F_1R}}{\overline{M_1R}}$ est indépendant du choix de M;

(b) la normale à la conique en M passe par R.

2732. *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle with sides a, b, c, medians m_a, m_b, m_c, altitudes h_a, h_b, h_c, and area Δ. Prove that

$$a^2+b^2+c^2 \;\geq\; 4\sqrt{3}\Delta \max\left\{\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right\}\,.$$

. . . .

Soit ABC un triangle de côtés a, b, c, de médianes m_a, m_b, m_c, de hauteurs h_a, h_b, h_c, et d'aire Δ. Montrer que

$$a^2+b^2+c^2 \;\geq\; 4\sqrt{3}\Delta \max\left\{\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right\}\,.$$

2733★. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

It is a known result that if O is the circumcentre of $\triangle A_1A_2A_3$, and if O_1, O_2, O_3, are the circumcentres of $\triangle OA_2A_3$, $\triangle OA_3A_1$, $\triangle OA_1A_2$, respectively, then the lines A_1O_1, A_2O_2 and A_3O_3 are concurrent.

Does the corresponding result hold for simplexes? That is, if O is the circumcentre of a simplex $A_0A_1\dots A_n$ and O_k is the circumcentre of the simplex determined by O and the face opposite A_k, are the lines O_kA_k, $k = 0, 1, \dots, n$, concurrent?

. . . .

On connaît le résultat affirmant que si O est le centre du cercle circonscrit du triangle $A_1A_2A_3$, et si respectivement O_1, O_2, O_3 sont les centres des cercles circonscrits aux triangles OA_2A_3, OA_3A_1 et OA_1A_2, alors les droites A_1O_1, A_2O_2 et A_3O_3 sont concourantes.

Le résultat ci-dessus reste-t-il valable pour des simplexes? En d'autres termes, si O est le centre de la sphère circonscrite au simplexe $A_0A_1\dots A_n$ et si O_k est le centre de la sphère circonscrite au simplexe déterminé par O et la face opposée A_k, est-ce que les droites O_kA_k, $k = 0, 1, \dots, n$ sont concourantes?

2734. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Prove that

$$(bc)^{2n+3} + (ca)^{2n+3} + (ab)^{2n+3} \;\geq\; (abc)^{n+2}\left(a^n + b^n + c^n\right) ,$$

where a, b, c, are non-negative reals, and n is a non-negative integer.

. . . .

Montrer que

$$(bc)^{2n+3} + (ca)^{2n+3} + (ab)^{2n+3} \;\geq\; (abc)^{n+2}\left(a^n + b^n + c^n\right) ,$$

où a, b, c sont des réels non négatifs et n un entier non négatif.

2735★. *Proposed by Richard I. Hess, Rancho Palos Verdes, CA, USA.*

Given three Pythagorean triangles with the same hypotenuse, is it possible that the area of one triangle is equal to the sum of the areas of the other two triangles?

. . . .

Etant donné trois triangles pythagoriciens ayant la même hypoténuse, est-il possible que l'aire de l'un des triangles soit égale à la somme des aires des deux autres?

2736. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $ABCD$ be a convex quadrilateral. From points A and B, draw lines parallel to sides BC and AD, respectively, giving points G and F on CD, respectively.

Let P and Q be the points of the intersection of the diagonals of the trapezoids $ABFD$ and $ABCG$, respectively.

Prove that $PQ \parallel CD$.

. . . .

Soit $ABCD$ un quadrilatère convexe. Des sommets A et B, on trace des parallèles aux côtés BC et AD, déterminant respectivement des points G et F sur CD.

Soit P et Q les points d'intersection des diagonales des trapèzes $ABFD$ et $ABCG$, respectivement.

Montrer que PQ et CD sont parallèles.

2737. *Proposed by Lyubomir Lyubenov, teacher, and Ivan Slavov, student, Foreign Language High School "Romain Rolland", Stara Zagora, Bulgaria.*

Find all solutions of the equation

$$x^n - 2nx^{n-1} + 2n(n-1)x^{n-2} + ax^{n-3} + bx^{n-4} + \cdots + c \;=\; 0\,,$$

given that there are n real roots.

. . . .

Trouver toutes les solutions de l'équation

$$x^n - 2nx^{n-1} + 2n(n-1)x^{n-2} + ax^{n-3} + bx^{n-4} + \cdots + c \;=\; 0\,,$$

sachant qu'il y a n racines réelles.

2738. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let x, y and z be positive real numbers satisfying $x^2 + y^2 + z^2 = 1$. Prove that

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \;\geq\; \frac{3\sqrt{3}}{2}\,.$$

. . . .

Soit x, y et z des nombres réels positifs satisfaisant $x^2 + y^2 + z^2 = 1$. Montrer que

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \;\geq\; \frac{3\sqrt{3}}{2}\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2608*. [2001 : 49] *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x, y, $z \geq 0$ and $x^2+y^2+z^2=1$. Prove or disprove that

(a) $1 \;\leq\; \dfrac{x}{1-yz}+\dfrac{y}{1-zx}+\dfrac{z}{1-xy} \;\leq\; \dfrac{3\sqrt{3}}{2}$;

(b) $1 \;\leq\; \dfrac{x}{1+yz}+\dfrac{y}{1+zx}+\dfrac{z}{1+xy} \;\leq\; \sqrt{2}$.

I. *Solution to* (a) *by Michel Bataille, Rouen, France.*

Let $S=\dfrac{x}{1-yz}+\dfrac{y}{1-zx}+\dfrac{z}{1-xy}$. We show that $S \leq \dfrac{3\sqrt{3}}{2}$.

If one of x, y or z is 0, say, $x=0$, then $S=y+z<2<\dfrac{3\sqrt{3}}{2}$.

Now, suppose that $xyz \neq 0$, so that x, y, $z \in (0,1)$.

Note that $\dfrac{x}{1-yz}=x+\dfrac{xyz}{1-yz}$. Hence,

$$S \;=\; x+y+z+xyz\left(\frac{1}{1-yz}+\frac{1}{1-zx}+\frac{1}{1-xy}\right)\,. \tag{1}$$

Since

$$\begin{aligned} 1-yz &\geq 1-\frac{1}{2}\left(y^2+z^2\right) \;=\; \frac{1}{2}\left(1+x^2\right) \\ &= \frac{1}{2}\left(2x^2+y^2+z^2\right) \;\geq\; 2\sqrt[4]{x^2x^2y^2z^2} \;=\; 2x\sqrt{yz}\,, \end{aligned}$$

we have, using the AM–GM Inequality several times, that

$$\begin{aligned} & xyz\left(\frac{1}{1-yz}+\frac{1}{1-zx}+\frac{1}{1-xy}\right) \\ \leq\;& \frac{xyz}{2}\left(\frac{1}{x\sqrt{yz}}+\frac{1}{y\sqrt{zx}}+\frac{1}{z\sqrt{xy}}\right) \\ =\;& \frac{1}{2}\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right) \;\leq\; \frac{1}{2}\left(\frac{y+z}{2}+\frac{z+x}{2}+\frac{x+y}{2}\right) \\ =\;& \frac{1}{2}(x+y+z)\,. \end{aligned} \tag{2}$$

From (1) and (2), we have, using the Cauchy-Schwarz Inequality, that

$$S \;\leq\; \frac{3}{2}(x+y+z) \;\leq\; \frac{3}{2}\left(1^2+1^2+1^2\right)^{\frac{1}{2}}\left(x^2+y^2+z^2\right)^{\frac{1}{2}} \;=\; \frac{3\sqrt{3}}{2}\,.$$

[Ed : Bataille also gave a proof of the left inequality of part (b), which, obviously, implies the left inequality of part (a).]

II. *Composite of essentially the same solution to* (a) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, and the proposers.*

Let $f(x,y,z) = \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy}$. The left inequality is trivial since the given assumptions imply that $0 \le x,\ y,\ z \le 1$, and thus, $f(x,y,z) \ge x+y+z \ge x^2+y^2+z^2 = 1$. Clearly, equality holds if and only if $(x,y,z) = (1,0,0)$, $(0,1,0)$ or $(0,0,1)$.

For the right inequality, first note that $1 - yz \ge 1 - \frac{1}{2}\left(y^2+z^2\right) = \frac{1}{2}\left(1+x^2\right)$ and, hence, $\frac{x}{1-yz} \le \frac{2x}{1+x^2}$. Therefore,

$$f(x,y,z) \ \le\ \frac{2x}{1+x^2} + \frac{2y}{1+y^2} + \frac{2z}{1+z^2}\,. \qquad (3)$$

Next, we claim that

$$\frac{2x}{1+x^2} \le \frac{3\sqrt{3}}{8}\left(1+x^2\right)\ . \qquad (4)$$

To show that (4) holds, note that

$$\begin{aligned}\left(1+x^2\right)^2 - \frac{16\sqrt{3}}{9}x &= \frac{1}{9}\left(9x^4 + 18x^2 - 16\sqrt{3}x + 9\right)\\ &= \frac{1}{9}\left(3x^2 - 2\sqrt{3}x + 1\right)\left(3x^2 + 2\sqrt{3}x + 9\right)\\ &= \frac{1}{9}\left(\sqrt{3}x - 1\right)^2\left(3x^2 + 2\sqrt{3}x + 9\right) \ \ge\ 0\,.\end{aligned}$$

Hence, (4) holds, and we have equality if and only if $x = \frac{\sqrt{3}}{3}$.

Summing (4) with the two corresponding inequalities in y and z then yields

$$\frac{2x}{1+x^2} + \frac{2y}{1+y^2} + \frac{2z}{1+z^2} \ \le\ \frac{3\sqrt{3}}{8}\left(3+x^2+y^2+z^2\right) \ =\ \frac{3\sqrt{3}}{2}\,. \qquad (5)$$

From (3) and (5), we conclude that $f(x,y,z) \le \frac{3\sqrt{3}}{2}$, with equality if and only if $x = y = z = \frac{\sqrt{3}}{3}$.

III. *Solution to the left inequality in* (b) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Note first that

$$x + xyz \ \le\ x + \frac{1}{2}x\left(y^2+z^2\right) \ =\ x + \frac{1}{2}x\left(1-x^2\right) \ =\ \frac{1}{2}\left(3x - x^3\right) \ \le\ 1\,,$$

since $x^3 - 3x + 2 = (x-1)^2(x+2) \ge 0$.

Hence, with all summations being cyclic, we have

$$\sum \frac{x}{1+yz} \;=\; \sum \frac{x^2}{x+xyz} \;\geq\; \sum x^2 \;=\; 1\,.$$

IV. *Solution to the right inequality in* (b) *by the proposers, modified slightly by the editor*.

Due to complete symmetry in x, y and z, we may assume, without loss of generality, that $x \le y \le z$. Then,

$$\frac{x}{1+yz} + \frac{y}{1+zx} + \frac{z}{1+xy} \;\le\; \frac{x+y+z}{1+xy}\,.$$

Hence, it suffices to prove that $\frac{x+y+z}{1+xy} \le \sqrt{2}$, or

$$x+y+z-\sqrt{2}xy \;\le\; \sqrt{2}\,. \tag{6}$$

We now use the method of Lagrange Multipliers to determine the extreme values of the function $g(x,y,z) = x+y+z-\sqrt{2}xy$ in the region

$$B \;=\; \{(x,y,z) \mid x,\, y,\, z \ge 0 \;\text{ and }\; x^2+y^2+z^2 = 1\}\,.$$

We let $G(x,y,z) = x+y+z-\sqrt{2}xy - \lambda\left(x^2+y^2+z^2-1\right)$, and set $\frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = \frac{\partial G}{\partial z} = 0$. Then, we have

$$1-\sqrt{2}y-2\lambda x \;=\; 0\,, \tag{7}$$
$$1-\sqrt{2}x-2\lambda y \;=\; 0\,, \tag{8}$$
$$1-2\lambda z \;=\; 0\,, \tag{9}$$
$$x^2+y^2+z^2 \;=\; 1\,. \tag{10}$$

From (7) and (8), we get $\left(\sqrt{2}-2\lambda\right)(x-y) = 0$, and hence, either $\lambda = \frac{\sqrt{2}}{2}$ or $x = y$.

If $\lambda = \frac{\sqrt{2}}{2}$, then $z = \frac{\sqrt{2}}{2}$ from (9). Hence, from (10), we get

$$x^2+y^2 \;=\; \frac{1}{2}\,. \tag{11}$$

On the other hand, from (7), we have $1-\sqrt{2}(x+y) = 0$, and hence,

$$x+y \;=\; \frac{\sqrt{2}}{2}\,. \tag{12}$$

From (11) and (12), we easily have $x = 0$ and $y = \frac{\sqrt{2}}{2}$.

Clearly, equality holds in (6) when $x = 0$ and $y = z = \frac{\sqrt{2}}{2}$.

If $x = y$, then (6) becomes

$$2x + z - \sqrt{2}x^2 \;\leq\; \sqrt{2}\,. \tag{13}$$

Since $(2x+z)^2 \leq 2\left(4x^2+z^2\right)$, we have

$$2x + z - \sqrt{2}x^2 \;\leq\; \sqrt{2}\sqrt{4x^2+z^2} - \sqrt{2}x^2\,,$$

and hence, (13) is true if

$$\sqrt{4x^2+z^2} \;\leq\; x^2+1\,,$$

or

$$2x^2+x^2+y^2+z^2 \;\leq\; \left(x^2+1\right)^2\,,$$

or

$$2x^2+1 \;\leq\; x^4+2x^2+1\,,$$

which is clearly true. This shows that (13) is true, and hence, (6) holds.

Finally, we check the points on the boundary of B. Without loss of generality, we may assume that $x = 0$. Then,

$$x+y+z-\sqrt{2}xy \;=\; y+z \;\leq\; \sqrt{2\left(y^2+z^2\right)} \;=\; \sqrt{2}\,.$$

Hence, (6) holds.

Therefore, the given inequality is valid, and it is easy to see that equality holds for $(x,y,z) = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, or $\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$, or $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$.

Also solved (both parts) *by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain. Part* (a) *only was also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece ; DAVID LOEFFLER, student, Cotham School, Bristol, UK ; JOEL SCHLOSBERG, student, New York University, NY, USA ; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.*

There were also two imcomplete solutions, both of which employed the method of Lagrange Multipliers, but claimed, without proof, that the solutions to $\frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = \frac{\partial G}{\partial z} = 0$ *must be* $x = y = z$ *by symmetry.*

2619. [2001 : 137] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, dedicated to Murray S. Klamkin, on his 80th birthday.*

For natural numbers n, define functions f and g by $f(n) = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rfloor$ and $g(n) = \left\lceil \frac{n}{\lceil\sqrt{n}\rceil} \right\rceil$. Determine all possible values of $f(n) - g(n)$, and characterize all those n for which $f(n) = g(n)$. [See [2000 : 197], Q. 8.]

Solution by David Loeffler, student, Cotham School, Bristol, UK.

If n is a perfect square, say $n = m^2$, then $f(m^2) = g(m^2) = m$. If this is not the case, set $n = m^2 + k$, where $0 < k < 2m + 1$. We then have

$$\begin{aligned} f(m^2+k) &= \left\lfloor \frac{m^2+k}{\lfloor\sqrt{m^2+k}\rfloor} \right\rfloor = \left\lfloor \frac{m^2+k}{m} \right\rfloor = m + \left\lfloor \frac{k}{m} \right\rfloor \\ g(m^2+k) &= \left\lceil \frac{m^2+k}{\lceil\sqrt{m^2+k}\rceil} \right\rceil = \left\lceil \frac{m^2+k}{m+1} \right\rceil = m - 1 + \left\lceil \frac{k+1}{m+1} \right\rceil \end{aligned}$$

If $k < m$, then $\left\lfloor \frac{k}{m} \right\rfloor = 0$ and $\left\lceil \frac{k+1}{m+1} \right\rceil = 1$ and we get $f(m^2+k) = g(m^2+k) = m$.

If $k = m$, then both fractions are 1, which implies $f(m^2+k) = m+1$ and $g(m^2+k) = m$.

If $m < k < 2m$, then we have $1 < \frac{k}{m} < 2$, so that $\left\lfloor \frac{k}{m} \right\rfloor = 1$ and $f(m^2+k) = m+1$; this also implies $m+1 < k+1 < 2m+1 < 2(m+1)$, implying $1 < \frac{k+1}{m+1} < 2$. Thus $\left\lceil \frac{k+1}{m+1} \right\rceil = 2$ and $g(m^2+k) = m+1$.

If $k = 2m$, then we have $\left\lfloor \frac{k}{m} \right\rfloor = 2$ and $\left\lceil \frac{k+1}{m+1} \right\rceil = 2$, yielding $f(m^2+k) = m+2$ and $g(m^2+k) = m+1$.

Thus, $f(n) - g(n)$ is always either 0 or 1. Also, $f(n) = g(n)$ for all integers n which are not of the form $m^2 + m$ or $m^2 + 2m$ for some integer m, (that is, those n for which neither $n+1$ nor $4n+1$ are perfect squares).

Comment : We may extend the problem by defining

$$h(n) = \left\lceil \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rceil \qquad \text{and} \qquad j(n) = \left\lfloor \frac{n}{\lceil\sqrt{n}\rceil} \right\rfloor .$$

This case may be analysed in exactly the same way as above; we find that $h(n) - j(n) \in \{0, 1, 2\}$, with $h(n) = j(n)$ if and only if n is a perfect square. Also $h(n) - j(n) = 1$ if and only if $n = m^2 + m$ for some m.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; OLEG IVRII, Cummer Valley Middle School, Toronto, Ont; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFREYS, student, Berkhamsted Collegiate School, UK; KEE-WAI LAU, Hong Kong; CRAIG CHAPMAN, RICHARD CRAMMER, students, and CARL LIBIS, Richard Stockton Collegiate of HJ, Pomona, NJ; HENRY LIU, student, University of Memphis, TN; WILLIAM MOSER, McGill University, Montreal, Que; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TREY SMITH, Angelo State University, San Angelo, TX; SOUTHWEST MISSOURI STATE PROBLEM SOLVING GROUP; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, KS, USA; LI ZHOU, Polk Community College, Winter Haven, FL; and the proposer.

Zhou points out that the above solution also shows that $f(n) > f(n+1)$ if and only if $n = k^2 - 1$, which answers problem 8 from the Olympiad Corner from 2000, p. 197. Seiffert also considered the extended problem defined in Loeffler's comment.

2621. [2001 : 137] *Proposed by J. Chris Fisher, University of Regina, Regina, Saskatchewan, and Bruce Shawyer, Memorial University of Newfoundland, St. John's, Newfoundland, dedicated to Murray S. Klamkin, on his 80th birthday.*

You are given :

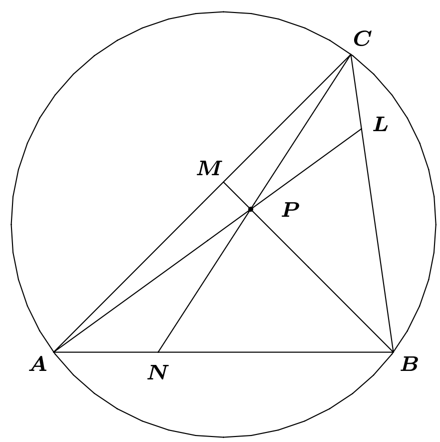
(a) fixed real numbers λ and μ in the open interval $(0, 1)$;

(b) circle ABC with fixed chord AB, variable point C, and points L and M on BC and CA, respectively, such that $BL : LC = \lambda : (1 - \lambda)$ and $CM : MA = \mu : (1 - \mu)$;

(c) P is the intersection of AL and BM.

Find the locus of P as C varies around the circle ABC. (If $\lambda = \mu = \frac{1}{2}$, it is known that the locus of P is a circle.)

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.



Let the lines CP and AB intersect at N. Then by Ceva's Theorem,

$$\frac{AN}{NB} = \frac{AM}{MC} \cdot \frac{CL}{LB} = \frac{(1-\mu)(1-\lambda)}{\mu\lambda},$$

which is a constant. Hence, N is a fixed point ; moreover,

$$\frac{AN}{AB} = \frac{(1-\mu)(1-\lambda)}{(1-\mu)(1-\lambda)+\mu\lambda}.$$

By Menelaus's Theorem applied to $\triangle NBC$ with line AL,

$$\frac{NP}{PC} = \frac{NA}{AB} \cdot \frac{BL}{LC} = \frac{(1-\mu)\lambda}{(1-\mu)(1-\lambda)+\mu\lambda}.$$

It follows that

$$\frac{NP}{NC} = \frac{(1-\mu)\lambda}{(1-\mu)(1-\lambda)+\mu\lambda+(1-\mu)\lambda} = \frac{(1-\mu)\lambda}{1-\mu+\mu\lambda}$$

is a constant. Hence, as C travels along circle ABC, P will travel along a circle whose circumference is divided by line AB into the same ratio as is the original circle; specifically, the locus of P is obtained from the locus of C by a dilatation centred at N with ratio of magnitude

$$\frac{NP}{NC} = \frac{(1-\mu)\lambda}{1-\mu+\mu\lambda}.$$

QED.

Editor's comments. Technically speaking, P is not defined for positions of C at A or B, so the locus of P is a circle minus its two points on AB. Two solvers noted that more generally, were C constrained to move about an arbitrary curve, the above argument establishes that P would trace out a homothetic image of that curve shrunk by the factor

$$\frac{NP}{NC} = \frac{(1-\mu)\lambda}{1-\mu+\mu\lambda}.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; and by the proposers.

2622. [2001 : 139] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Find the exact value of $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}}{(2n+1)\binom{2n}{n}}$.

A combination of almost identical solutions by Jan Ciach, Ostrowiec Świętokrzyski, Poland and Kee-Wai Lau, Hong Kong, China.

From the identity $(\arcsin x)^2 = \sum\limits_{n=0}^{\infty} \frac{2^{2n}(n!)^2 x^{2n+2}}{(2n+1)!(n+1)}$, which is valid for $|x| \le 1$, we have, by differentiation,

$$4x\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{2^{2n+2}(n!)^2 x^{2n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2x)^{2n+2}}{(2n+1)\binom{2n}{n}}.$$

By taking $x = \frac{\sqrt{2}}{2}$, we conclude that $\sum\limits_{n=0}^{\infty} \frac{2^{n+1}}{(2n+1)\binom{2n}{n}} = \pi$.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; CARL LIBIS, Richard Stockton College of NJ, Pomona, NJ, USA; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands (two solutions); PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Most solutions submitted were similar to the above one. Janous started with a more general power series representation and mentioned other results that can be obtained from it. Manes gave an expanded version of a solution by G. Klambauer in Problems and Propositions in Analysis*, Marcel Dekker, Inc., New York, 1979, problem 151, pp. 317–318. Stan Wagon wrote to point out that there are algorithms for doing problems such as these, and his philosophy, still a minority in the problem-solving community, is to use computers to do such problems.*

2624. [2001 : 138] *Proposed by H.A. Shah Ali, Tehran, Iran.*

Let n black objects and n white objects be placed on the circumference of a circle, and define any set of m consecutive objects from this cyclic sequence to be an m–chain.

(a) Prove that, for each natural number $k \leq n$, there exists at least one $2k$–chain consisting of k black objects and k white objects.

(b) Prove that, for each natural number $k \leq \sqrt{2n+5} - 2$, there exist at least two such disjoint $2k$–chains.

Solution by Elsie Campbell and Trey Smith, Angelo State University, San Angelo, TX, USA.

For ease of notation, a $2k$–chain will be called *good* if it consists of k black and k white objects. We may label the objects $M(1)$, $M(2)$, $\ldots$, $M(2n)$ where $M(1)$ is arbitrarily chosen and the objects are taken in a clockwise order. Then we observe that for $k \leq n$ there are $2n$ different $2k$–chains which will be labelled

$$\begin{aligned} C(1) &= \big(M(1), M(2), \ldots, M(2k)\big) \\ C(2) &= \big(M(2), M(3), \ldots, M(2k+1)\big) \\ \cdots & \quad \cdots \\ C(2n) &= \big(M(2n), M(1), \ldots, M(2k-1)\big) . \end{aligned}$$

For each chain $C(i)$ let $W(i)$ and $B(i)$ be the number of white objects and the number of black objects, respectively, in the chain.

Notice that each object will be in exactly $2k$ different $2k$–chains. Since there are n white objects we have that

$$W(1) + W(2) + \cdots + W(2n) \;=\; n \cdot 2k \;=\; 2kn .$$

We may now prove :

Theorem (a) : Fix natural numbers n and k with $k \leq n$. Then there exists at least one good $2k$–chain.

Proof: Suppose that no good $2k$–chain exists. We must derive a contradiction. Arbitrarily fix a $2k$–chain $C(1)$. We may assume, without loss of generality, that $W(1) > B(1)$. Since there are no good $2k$–chains, we conclude that for all natural numbers i such that $1 \leq i \leq 2n$ we have $W(i) > B(i)$. If this were not the case, then we could let j be the least natural number such that

$W(j) < B(j)$. Then $W(j-1) > B(j-1)$, but this is impossible since $C(j)$ could have at most 1 white object less than, and 1 more black object, than $C(j-1)$, and

$$W(j) \ < \ B(j) \qquad \Longrightarrow \qquad W(j) + 2 \ \leq \ B(j)\,,$$

since $W(i)$ and $B(i)$ have the same parity for all i, $1 \leq i \leq 2n$. Thus, we have $W(i) \geq k+1$ for every i, $1 \leq i \leq 2n$. This implies that

$$W(1) + W(2) + \cdots + W(2n) \ \geq \ 2n(k+1) \ > \ 2kn,$$

which is a contradiction. Therefore, there is a good $2k$–chain.

Theorem (b) : Fix natural numbers k and n with $k \leq \sqrt{2n+5} - 2$. Then there exist at least two good disjoint $2k$–chains.

Proof : If $2n - 4k + 1 < 0$, then

$$\begin{aligned} 2n \ < \ 4k - 1 \quad &\Longrightarrow \quad 2n + 5 \ < \ 4k + 4 \\ &\Longrightarrow \quad 2n + 5 \ < \ k^2 + 4k + 4 \\ &\Longrightarrow \quad \sqrt{2n+5} \ < \ k + 2 \\ &\Longrightarrow \quad \sqrt{2n+5} - 2 \ < \ k\,, \end{aligned}$$

which is a contradiction. Thus we may assume (since k and n are integers) $2n - 4k + 1 > 0$.

By the previous theorem we may fix a good $2k$–chain and label it $C(1)$. Observe that there are $2n - 4k + 1$ many $2k$–chains which are disjoint from $C(1)$. They are, in particular,

$$C(2k+1),\ C(2k+2),\ \ldots,\ C(2n-2k),\ C(2n-2k+1).$$

Now suppose that none of these are good. We must derive a contradiction. By the same argument as that used in the previous theorem, it cannot be the case that some of the chains have more black objects than white objects while others have more white objects than black objects. We may assume, without loss of generality, that all of the chains have at least $k+1$ many whites. Thus,

$$W(2k+1) + W(2k+2) + \cdots + W(2n-2k+1) \ \geq \ (k+1)(2n-4k+1)\,.$$

In addition, each of the k white objects in $C(1)$ will be in $2k$ many $2k$–chains. Hence the sum of the white objects from $C(1)$ that are in the chains

$$C(2n-2k+2),\ \ldots,\ C(2n),\ C(1),\ \ldots,\ C(2k)$$

is $k \cdot 2k = 2k^2$.

We must, finally, sum up all of the white objects that fall outside of $C(1)$ from the chains

$$C(2n-2k+2),\ \ldots,\ C(2n),\ C(1),\ \ldots,\ C(2k).$$

Since $W(2k+1) \geq k+1$, then

$$\begin{array}{rl} C(2k) & \text{contains at least } k \text{ white objects outside of } C(1), \\ C(2k-1) & \text{contains at least } k-1 \text{ white objects outside of } C(1), \\ \cdots & \cdots \\ C(2k-(k-1)) & \text{contains at least 1 white object outside of } C(1), \end{array}$$

for a total of at least $\frac{1}{2}k(k+1)$. Similarly, the sum of the white objects outside of $C(1)$ from

$$C(2n-2k+2),\ C(2n-2k+3),\ \ldots,\ C(2n)$$

is at least $\frac{1}{2}k(k+1)$.

Using the fact that $W(1)+W(2)+\cdots+W(2n)=2kn$, we get

$$\begin{aligned} (k+1)(2n-4k+1)+2k^2+\tfrac{1}{2}k(k+1)+\tfrac{1}{2}k(k+1) &\leq 2kn \\ 2kn+2n-k^2-2k+1 &\leq 2kn \\ 2n &\leq k^2+2k-1 \\ 2n &< k^2+4k-1 \\ 2n+5 &< k^2+4k+4 \\ \sqrt{2n+5}-2 &< k\,, \end{aligned}$$

which is a contradiction. Thus, there are at least two good disjoint $2k$–chains.

Also solved by HENRY LIU, student, University of Memphis, Memphis, TN, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Part (a) *only was solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JOEL SCHLOSBERG, student, New York University, NY, USA.*