Letter to the Editor

We have received the following letter from Professor D.J. Smeenk, Zaltbommel, the Netherlands.

I received the September copy of Crux on the 6th of November. As always, it gave me much pleasure and entertainment.

As far as I remember, I have been a subscriber to Crux for about twenty years. In 1981, I retired after having been a school leader for twenty years and then, my good friend, the late Jaap Groenman, advised me to subscribe to Crux. So I did. It has given me much pleasure and entertainment.

To my regret, I noticed that during the last few years, I am the only Dutchman who upholds the credit of the Netherlands!

I should feel very much obliged if, in some way, you can tell me the total number of subscribers to Crux, and how many of them are Dutch. If any, could you not encourage them to send in their solutions and their problems? Crux deserves it, and is worth it.

Your sincerely,

[signed] D.J. Smeenk.

We thank Professor Smeenk for his kind words — it is our aim to be the **best** journal on mathematical problem solving.

In answer to his questions, the total number of subscribers to ***CRUX with MAYHEM*** is 871, and of these, only 3 are sent to addresses in the Netherlands. We look forward to more solvers from the Netherlands!

We are also pleased that some national Mathematical Olympiad committees have taken out bulk subscriptions. There is a discount available for bulk subscriptions sent to a single address — 20% for 25–49 copies and 40% for 50 and more copies.

As was announced in the December 2001 issue, there are some subscriptions awarded to schools in some developing countries, courtesy of a generous donation by a subscriber who wishes to remain anonymous. We again thank this subscriber for this generosity.

Bruce Shawyer
Editor–in–Chief

THE ACADEMY CORNER

No. 46

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In this issue, we present the problems of the 2001 Atlantic Provinces Council on the Sciences, annual mathematics competition for university students, held at St. Francis Xavier University, Antigonish, Nova Scotia. Thanks to Nabil Shalaby for collecting a copy for us. This competition turned out to be quite difficult. If you can solve them, send me your nice solutions!

2001 APICS Math Competition

Time allowed: 3 hours

1. P is a polynomial with integer coefficients. For four distinct integers x, we have $P(x) = 9$. Show that there are no integers x with $P(x) = 16$.

2. Consider the Fibonnacci sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$ with $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 2$.

Show that $\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}$ is equal to $\frac{1}{11}$.

3. Prove that among any 13 distinct real numbers, it is possible to find x and y such that $0 < \frac{x-y}{1+xy} < 2 - \sqrt{3}$.

4. Show that $\lim_{n\to\infty} \sqrt[n]{\frac{(2n)!}{n}} = \frac{4}{e}$.

5. In $\triangle ABC$, R is the mid-point of BC. S is a point on AC such that $CS = 4SA$. T is a point of AB such that the area of $\triangle RST$ is twice the area of $\triangle TBR$. Find $\frac{AT}{TB}$.

6. Determine all functions which are everywhere differentiable and satisfy $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ for all x, $y \in \mathbb{R}$ with $xy \neq 1$.

7. Evaluate the integral $I(k) = \int_0^\infty \frac{\sin(kx)\cos^k(x)}{x}\,dx$, where $k \in \mathbb{N}$.

Hint: recall that $\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$.

8. Find three consecutive integers, the first being a multiple of the square of a prime number, the second being a multiple of the cube of a prime number and the last being a multiple of the fourth power of a prime number.

Next, we present the problems of 8$^{\text{th}}$ International Mathematics Competition, held at the Charles University, Prague, Czech Republic, on 19–25 July 2001. This competition is for university students completing up to their fourth year, and consists of two sessions, each of five hours. Thanks to Moubinool Omarjee for sending them to us.

8$^{\text{th}}$ International Mathematics Competition

Day 1 Problems

Problem 1. Let n be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, \ldots, n^2$ written in order starting top left and moving along each row in turn left to right. We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

Problem 2. Let r, s, t be positive integers which are pairwise relatively prime. If a and b are elements of a commutative multiplicative group with unity element e, and $a^r = b^s = (ab)^t = e$, prove that $a = b = e$.

Does the same conclusion hold if a and b are elements of an arbitrary non-commutative group?

Problem 3. Find $\lim_{t \nearrow 1}(1-t)\sum_{n=1}^{\infty}\frac{t^n}{1+t^n}$, where $t \nearrow 1$ means that t approaches 1 from below.

Problem 4. Let k be a positive integer. Let $p(x)$ be a polynomial of degree n, each of whose coefficients is -1, 1 or 0, and which is divisible by $(x-1)^k$. Let q be a prime such that $\frac{q}{\ln q} < \frac{k}{\ln(n+1)}$. Prove that the complex q^{th} roots of unity are roots of the polynomial $p(x)$.

Problem 5 Let A be an $n \times n$ matrix such that $A \neq \lambda I$ for all $\lambda \in \mathbb{C}$. Prove that A is similar to a matrix having at most one non-zero entry on the main diagonal.

Problem 6.Suppose that the differentiable functions a, b, f, $g : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x) \geq 0\,, \quad f'(x) \geq 0\,, \quad g'(x) > 0 \text{ for all } x \in \mathbb{R}\,,$$

$$\lim_{x\to\infty} a(x) = A > 0\,, \quad \lim_{x\to\infty} b(x) = B > 0\,, \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty\,,$$

and

$$\frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} = b(x)\,.$$

Prove that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 2\frac{B}{A+1}\,.$$

Day 2 Problems

Problem 1 Let r, $s \geq 1$ be integers and $a_0, a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{s-1}$ be real non-negative numbers such that

$$(a_0+a_1x+a_2x^2+\cdots+a_{r-1}x^{r-1}+x^r)(b_0+b_1x+b_2x^2+\cdots+b_{s-1}x^{s-1}+x^s)$$
$$= 1+x+x^2+\cdots+x^{r+s-1}+x^{r+s}.$$

Prove that each a_i and each b_j equals either 0 or 1.

Problem 2 Let $a_0 = \sqrt{2}$, $b_0 = 2$, and $a_{n+1} = \sqrt{2-\sqrt{4-a_n^2}}$, $b_{n+1} = \dfrac{2b_n}{2+\sqrt{4+b_n^2}}$.

(a) Prove that the sequences $\{a_n\}$, $\{b_n\}$ are decreasing and converge to 0.

(b) Prove that the sequence $\{2^n a_n\}$ is increasing, the sequence $\{2^n b_n\}$ is decreasing and that these two sequences converge to the same limit.

(c) Prove that there is a positive constant C such that for all n the following inequality holds: $0 < b_n - a_n < \dfrac{C}{8^n}$.

Problem 3. Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two of these points is strictly greater that $\sqrt{2}$.

Problem 4. Let $A = (a_{k,t})_{k,t=1,\ldots,n}$ be an $n \times n$ complex matrix such that for each $m \in \{1, \ldots, n\}$ and $1 \leq j_1 < \cdots < j_m \leq n$ the determinant of the matrix $(a_{j_k,j_t})_{k,t=1,\ldots,m}$ is zero. Prove that $A^n = 0$ and that there exists a permutation $\sigma \in S_n$ such that the matrix

$$(a_{\sigma(k),\sigma(t)})_{k,t=1,\ldots,n}\,.$$

has all of its non-zero elements above the diagonal.

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Prove that there is no function $f:\mathbb{R}\to\mathbb{R}$ with $f(0)>0$, and such that

$$f(x+y) \;\geq\; f(x) + yf(f(x)) \text{ for all } x, y \in \mathbb{R}\,.$$

Problem 6. For each positive integer n, let $f_n(\theta) = \sin\theta \cdot \sin(2\theta) \cdot \sin(4\theta)\cdots\sin(2^n\theta)$.

For all real θ and all n, and prove that

$$|f_n(\theta)| \;\leq\; \frac{2}{\sqrt{3}}\,|f_n(\pi/3)|\,.$$

THE OLYMPIAD CORNER

No. 219

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

Here we are at the beginning of a new year and a new volume of ***CRUX with MAYHEM***. My thanks go to our readers, problem solvers, commentators, and suppliers of Olympiad materials over the last year (and for some decades!). Among those contributing last year are:

Mohammed Aassila
Miguel Amengual Covas
Ed Barbeau
Michel Bataille
Pierre Bornsztein
René Bornsztein
Christopher J. Bradley
Competitions Committee of the Greek Mathematical Society
George Evagelopoulos
Walther Janous
Athanasios Kalakos
Murray S. Klamkin
Hojoo Lee
Richard Nowakowski
Luyun-Zhong-Qiao
Heinz-Jürgen Seiffert
Toshio Seimiya
Andrei Simion
Raul A. Simon Lamb
Achilleas Sinefakopoulos
Christopher Small
D.J. Smeenk
Edward T.H. Wang

To start the new year we give the problems of the 1999 Vietnamese Mathematical Olympiad. My thanks go to Ed Barbeau, Canadian Team Leader to the IMO in Bucharest for forwarding them to us.

VIETNAMESE MATHEMATICAL OLYMPIAD 1999

Category A

March 12-13, 1999

1. Solve the system of equations

$$\left\{\begin{array}{lcl} (1+4^{2x-y})5^{1-2x+y} & = & 1+2^{2x-y+1} \\ y^3+4x+1+\ln(y^2+2x) & = & 0 \end{array}\right.$$

2. Let A', B', C' be the respective mid-points of the arcs BC, CA, AB, not containing points A, B, C, respectively, of the circumcircle of the triangle ABC. The sides BC, CA and AB intersect the pairs of segments $(C'A', A'B')$, $(A'B', B'C')$ and $(B'C', C'A')$ at the pairs of points (M, N), (P, Q) and (R, S), respectively. Prove that $MN = PQ = RS$ if and only if the triangle ABC is equilateral.

3. Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be two sequences defined recursively as follows

$$\begin{array}{lll} x_0 = 1 & x_1 = 4\,, & x_{n+2} = 3x_{n+1} - x_n\,, \\ y_0 = 1 & y_1 = 2\,, & y_{n+2} = 3y_{n+1} - y_n\,, \end{array}$$

for all $n = 0, 1, 2, \ldots$.

(a) Prove that

$$x_n^2 - 5y_n^2 + 4 \;=\; 0$$

for all non-negative integers n.

(b) Suppose that a, b are two positive integers such that $a^2 - 5b^2 + 4 = 0$. Prove that there exists a non-negative integer k such that $x_k = a$ and $y_k = b$.

4. Let a, b, c be real positive numbers such that $abc + a + c = b$. Determine the greatest possible value of the following expression

$$P \;=\; \frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1}\,.$$

5. In three-dimensional space, let Ox, Oy, Oz, Ot be four non-planar distinct rays such that the angles between any two of them have the same measure.

(a) Determine this common measure.

(b) Let Or be another ray different from the above four rays. Let α, β, γ, δ be the angles formed by Or with Ox, Oy, Oz, Ot, respectively. Put

$$p \;=\; \cos\alpha + \cos\beta + \cos\gamma + \cos\delta\,,$$

$$q \;=\; \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta\,.$$

Prove that p and q are invariant when Or rotates about the point O.

6. Let $\mathbb{T}$ be the set of all non-negative integers not greater than 1999 and $\mathbb{N}$ be the set of all non-negative integers. Find all functions $f : \mathbb{N} \to \mathbb{T}$ satisfying the following conditions

$$\begin{array}{rcll} f(t) & = & t & \text{for all } t \in \mathbb{T} \\ f(m+n) & = & f(f(m)+f(n)) & \text{for all } m,\, n \in \mathbb{N} \end{array}$$

Category B

March 12-13, 1999

1. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence defined by

$$u_1 = 1, \quad u_2 = 2 \quad \text{and} \quad u_{n+2} = 3u_{n+1} - u_n$$

for all $n = 1, 2, \ldots$.

Prove that

$$u_{n+2} + u_n \geq 2 + \frac{u_{n+1}^2}{u_n}$$

for all $n = 1, 2, \ldots$.

2. Let ABC be a triangle inscribed in the circle $\mathcal{O}$. Locate the position of the points P, not lying in the circle $\mathcal{O}$, of the plane (ABC) with the property that the lines PA, PB, PC intersect the circle $\mathcal{O}$ again at points A', B', C' such that $A'B'C'$ is a right-angled isosceles triangle with $\angle A'B'C' = 90°$.

3. Consider real numbers a, b such that all roots of the equation

$$ax^3 - x^2 + bx - 1 = 0$$

are real and positive.

Determine the smallest possible value of the following expression:

$$P = \frac{5a^2 - 3ab + 2}{a^2(b-a)}.$$

4. Let $f(x)$ be a continuous function defined on $[0, 1]$ such that

(i) $f(0) = f(1) = 0$.

(ii) $2f(x) + f(y) = 3f\left(\frac{2x+y}{3}\right) \quad \forall\, x, y \in [0, 1]$.

Prove that $f(x) = 0$ for all $x \in [0, 1]$.

5. The base side and the altitude of a regular hexagonal prism $ABCDEF$, $A'B'C'D'E'F'$ are equal to a and h, respectively. Prove that six planes $(AB'F)$, $(CD'B')$, $(EF'D)$, $(D'EC)$, $(F'AE)$ and $(B'CA)$ are tangent to the same sphere. Determine the centre and the radius of this sphere.

6. Two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are determined recursively by

$$x_1 = 1, \qquad y_1 = 2 \quad \text{and}$$
$$x_{n+1} = 22y_n - 15x_n$$
$$y_{n+1} = 17y_n - 12x_n$$

for all $n = 1, 2, \ldots$.

(a) Prove that

(i) $\{x_n\}$ and $\{y_n\}$ are not equal to zero for all $n = 1, 2, \ldots$.

(ii) The sequences $\{x_n\}$ and $\{y_n\}$ contain infinitely many positive terms and infinitely many negative terms.

(b) Are the $(1999^{1945})^{\text{th}}$ terms of the sequence $\{x_n\}$ and the sequence $\{y_n\}$ divisible by 7 or not?

As a second set for your problem-solving pleasure we give the problems of the 16th Balkan Mathematical Olympiad, 1999, from Ohrid, Macedonia. Thanks again go to Ed Barbeau, Canadian Team leader to the IMO at Bucharest for collecting this set for our use.

16th BALKAN MATHEMATICAL OLYMPIAD

Ohrid, Macedonia, 1999

1. Given an acute-angled triangle ABC, let D be the mid-point of the arc BC of the circumcircle of ABC not containing A. The points which are symmetric to D with respect to the line BC and the centre of the circumcircle are denoted by E and F, respectively. Finally, let K stand for the mid-point of $[EA]$. Prove that:

(a) the circle passing through the mid-points of the edges of the triangle ABC, also passes through K;

(b) the line passing through K and the mid-point of $[BC]$ is perpendicular to AF.

2. Let $p > 2$ be a prime number such that 3 divides $p - 2$. Let

$$S = \{y^2 - x^3 - 1 \mid x, y \text{ are integers}, 0 \le x, y \le p - 1\}.$$

Prove that at most $p - 1$ elements of the set S are divisible by p.

3. Let ABC be an acute-angled triangle; M, N and P are the feet of the perpendiculars from the centroid G of the triangle upon its sides AB, BC and CA respectively. Prove that:

$$\frac{4}{27} < \frac{\text{area}(MNP)}{\text{area}(ABC)} \le \frac{1}{4}.$$

4. Let $0 \le x_0 \le x_1 \le x_2 \le \cdots \le x_n \le \cdots$ be a non-decreasing sequence of non-negative integers such that for every k, $k \ge 0$, the number of terms of the sequence which are less than or equal to k is finite, say y_k. Prove that for all positive integers m, n

$$\sum_{i=0}^{n} x_i + \sum_{j=0}^{m} y_j \ge (n+1)(m+1).$$

Next we have some housekeeping to do — corrections to solutions published last year and an alternate solution. Corrections first!

Murray Klamkin wrote to point out an error in one of his own solutions given last April.

5. *Third Macedonian Mathematical Olympiad* [2001 : 178; 1999 : 198]

Find the biggest number n such that there exist n straight lines in space, $\mathbb{R}^3$, which pass through one point, and the angle between each two lines is the same. (The angle between two intersecting straight lines is defined to be the smaller one of the two angles between these two lines.)

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. Branko Grunbaum pointed out that there are six equi-inclined lines determined by the centre and the vertices of a regular icosahedron.

The result given in the published solution is for concurrent rays, not concurrent lines.

Next we turn to a comment and correction for a solution given in the September number.

3. *St. Petersburg City Mathematical Olympiad, Selective Round, 11*$^{\text{th}}$ *Grade* [2001 : 307; 1999 : 263]

Prove that there are no positive integers a and b such that for all different primes p and q greater than 1000, the number $ap + bq$ is also prime.

Comment and Correction by Greg Martin, Mathematics Department, University of British Columbia.

The solution cites an incorrect version of Dirichlet's Theorem. For a residue class modulo m to contain infinitely many primes, it is not sufficient for the residue class to be non-zero modulo m, as the residue class $2 \pmod 4$ demonstrates. Rather, it is necessary (and sufficient) for the residue class to be *reduced*, that is, for the integers in the residue class to be relatively prime to m. The published solution can be rescued by choosing m to be relatively prime to both a and b, for example, $m = ab + 1$.

Next we give an alternate solution to another problem, from the same set, that is more visual, and thus may be simpler.

1. *St. Petersburg City Mathematical Olympiad, Selective Round, 11*$^{\text{th}}$ *Grade* [200 1: 305-306; 1999 : 263]

It is known about real numbers a_1, $\ldots$, a_{n+1}; b_1, $\ldots$, b_n, that $0 \le b_k \le 1$ ($k = 1, \ldots, n$) and $a_1 \ge a_2 \ge \cdots \ge a_{n+1} = 0$. Prove the inequality:

$$\sum_{i=1}^{n} a_k b_k \;\leq \sum_{k=1}^{[\sum_{j=1}^{n} b_j]+1} a_k \,. \qquad (1)$$

Alternate solution by Greg Martin, Mathematics Department, University of British Columbia.

Consider n rectangles, with heights $a_1, \ldots, a_n$ and widths 1, lined up side by side as in Figure 1. Next, consider n rectangles, with heights a_1, $\ldots, a_n$ and widths $b_1, \ldots, b_n$, lined up side by side as in Figure 2. The total area of this latter set of rectangles is precisely the left-hand side of the putative inequality (1). If we superimpose the two pictures as in Figure 3, where the second set of rectangles has been shaded in, we see that the first set of rectangles completely contains the second. Moreover, not all of the rectangles in the first set are needed — only the first m rectangles, where m is any integer greater than or equal to the width of Figure 2, which is exactly $\sum_{j=1}^{n} b_j$. In particular, if we take $m = [\sum_{j=1}^{n} b_j] + 1$, then the first m rectangles of Figure 1 completely contain the rectangles in Figure 2, which immediately establishes the inequality (1). (In fact, we need only take $m = \lceil \sum_{j=1}^{n} b_j \rceil$ which saves a rectangle if the sum of the b_j's is an integer.)

The only thing that really needs proof in the above argument is that the second set of rectangles will always fit inside the first set of rectangles. It is enough to show that for any y–coordinate between 0 and a_1, the first set of rectangles at height y is at least as wide as the second set of rectangles at the same height. For any such y–coordinate, there is an index $1 \leq i \leq n$ such that $a_i \geq y > a_{i+1}$. It is easy to see, then, that the width of the first set of rectangles at height y is exactly i, while the width of the second set of rectangles at height y is exactly $b_1 + \cdots + b_i \leq 1 + \cdots + 1 = i$. In fact, this argument proves the inequality (1) under an even weaker hypothesis than $b_1, \ldots, b_n \leq 1$: all we really need is that the non-negative numbers b_j satisfy $b_1 + \cdots + b_j \leq j$ for each $1 \leq j \leq n$.

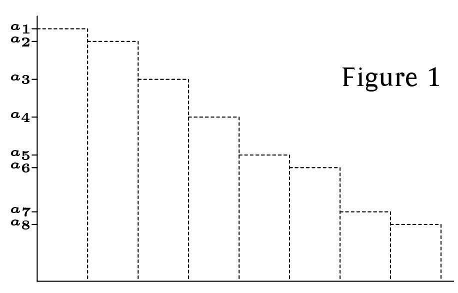


Figure 1

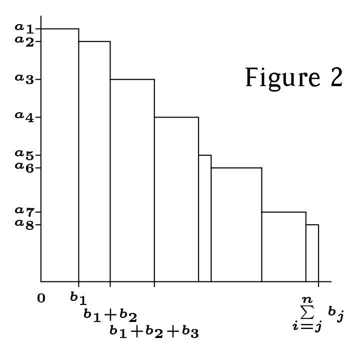


Figure 2

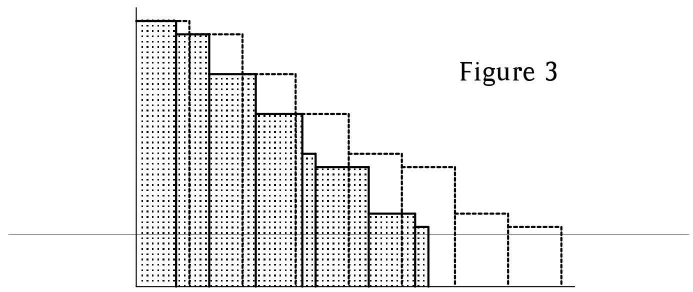


Figure 3

While sorting some solutions, I discovered that Mohammed Aassila had also submitted solutions to problems 4 and 6 of the 1996 Australian Mathematical Olympiad, for which we discussed solutions over a year ago [1999 : 461-462; 1999 : 74-75]. My apologies!

Now we turn to readers' comments and solutions for problems posed in the December 1999 number of the *Corner*. We start with solutions to problems of the 13^{th} Iranian Mathematical Olympiad 1996, Second Round [1999 : 454-455].

1. Prove that for every natural number $n \geq 3$ there exist two sets $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_n\}$ such that

(a) $A \cap B = \emptyset$,

(b) $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$,

(c) $x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2$.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; and by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea. Both Aassila and Lee gave the following solution allowing negative numbers. We give Lee's write-up.

We can choose $(n-1)$ positive numbers $a_1, \ldots, a_{n-1}$ such that $a_1 < a_2 < \cdots < a_{n-1}$. Then we have

$$-\sum_{i=1}^{n-1} a_i < -a_{n-1} < \cdots < -a_2 < -a_1 < 0 < a_1 < a_2 < \cdots < a_{n-1} < \sum_{j=1}^{n-1} a_i.$$

It follows that

$$\{a_1, \ldots, a_{n-1}, -\sum_{i=1}^{n-1} a_i\} \cap \{-a_1, \ldots, -a_{n-1}, \sum_{i=1}^{n-1} a_i\} = \emptyset.$$

Now, let $x_n = -\sum_{i=1}^{n-1} a_i$, $y_n = \sum_{i=1}^{n-1} a_i$ and $x_i = a_i$, $y_i = -a_i$ for $1 \leq i \leq n-1$. Then we get $\sum_{i=1}^{n} x_i = 0 = \sum_{i=1}^{n} y_i$ and

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n-1} a_i^2 + \left(\sum_{i=1}^{n-1} a_i\right)^2 = \sum_{i=1}^{n} y_i^2.$$

Lee's solution also works for $n = 2$. Aassila and Bornsztein also give proofs avoiding negative numbers. We give Bornsztein's solution in positive integers.

We will prove that we may add the condition

(d) "$A \subset \mathbb{N}^*$ and $B \subset \mathbb{N}^*$".

First we note that $A_3 = \{1, 5, 6\}$ and $B_3 = \{2, 3, 7\}$ satisfy the four conditions (a), (b), (c) and (d).

Let n be an integer, with $n \geq 3$.

Suppose that the sets $A_n = \{x_1, \ldots, x_n\}$ and $B_n = \{y_1, \ldots, y_n\}$ satisfy the conditions (a), (b), (c), and (d).

Let $A'_n = \{8x_1, 8x_2, \ldots, 8x_n\}$ and $B'_n = \{8y_1, \ldots, 8y_n\}$. It is clear that A'_n and B'_n satisfy (a), (b), (c) and (d).

Moreover, for each $i \in \{1, \ldots, n\}$, we have

$$8x_i \not\in \{1, \ldots, 7\} \quad \text{and} \quad 8y_i \not\in \{1, \ldots, 7\}\,.$$

It follows that the sets $A_{n+3} = A_3 \cup A'_n$ and $B_{n+3} = B_3 \cup B'_n$ satisfy (a), (b), (c) and (d), with $\text{Card } A_{n+3} = n + 3 = \text{Card } B_{n+3}$.

Thus, from two sets satisfying (a), (b), (c) and (d) for the integer n, we may construct two sets satisfying (a), (b), (c) and (d) for the integer $n + 3$.

We deduce that, using this process, we only have to look at the cases $n = 3, 4, 5$.

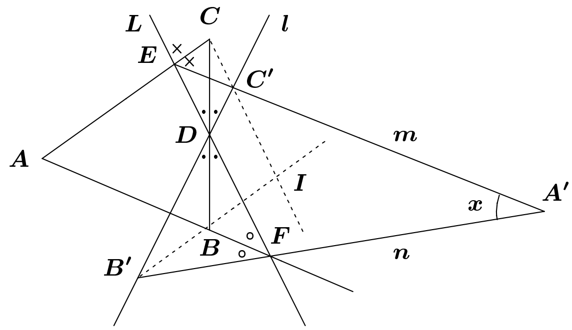
For $n = 3$, we already have found A_3 and B_3.

For $n = 4$, we may choose $A_4 = \{1, 4, 6, 7\}$ and $B_4 = \{2, 3, 5, 8\}$.

For $n = 5$, we may choose $A_5 = \{1, 5, 9, 17, 18\}$ and $B_5 = \{2, 3, 11, 15, 19\}$ and the proof is complete.

2. Let L be a line in the plane of an acute triangle ABC. Let the lines symmetric to L with respect to the sides of ABC intersect each other in the points A', B' and C'. Prove that the incentre of triangle $A'B'C'$ lies on the circumcircle of triangle ABC.

Solutions by Mohammed Aassila, Strasbourg, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



Let D, E, F be the intersections of L with BC, CA, AB respectively.

Let l, m, n be the lines symmetric to L with respect to BC, CA, AB, respectively, and let A', B', C' be the intersections of m, n; n, l; l, m, respectively.

Since BF and BD are bisectors of $\angle B'FD$ and $\angle B'DF$, respectively, we have B is the incentre of $\triangle B'FD$. Hence $B'B$ bisects $\angle FB'D$.

Since DC bisects $\angle EDC'$ and CE bisects the exterior angle of $\angle DEC'$, we have C is the excentre of $\triangle DEC'$ opposite to D. Hence CC' bisects the exterior angle of $\angle DC'E$, so that CC' is the bisector of $\angle B'C'A'$.

Let I be the intersection of BB' and CC'.

Since BB' and CC' are bisectors of $\angle A'B'C'$ and $\angle A'C'B'$ respectively, we have I is the incentre of $\triangle A'B'C'$.

We put $\angle B'A'C' = \angle FA'E = x$.

Since I is the incentre of $\triangle A'B'C'$ we get

$$\begin{aligned} \angle B'IC &= 90^\circ + \frac{x}{2}\,; \quad \text{that is} \\ \angle BIC &= \angle B'IC' \;=\; 90^\circ + \frac{x}{2}\,. \end{aligned} \qquad (1)$$

In triangle $A'EF$, FA and EA are the exterior bisectors of $\angle A'FE$ and $\angle A'EF$, respectively, so that A is the excentre of $\triangle A'EF$ opposite to A'. Thus, we have $\angle FAE = 90^\circ - \frac{x}{2}$; that is

$$\angle BAC \;=\; 90^\circ - \frac{x}{2}\,. \qquad (2)$$

From (1) and (2), it follows that

$$\angle BIC + \angle BAC \;=\; \left(90^\circ + \frac{x}{2}\right) + \left(90^\circ - \frac{x}{2}\right) \;=\; 180^\circ\,.$$

Therefore A, B, I, C are concyclic.

Thus the incentre of $\triangle A'B'C'$ lies on the circumcircle of $\triangle ABC$.

3. $12k$ persons have been invited to a party. Each person shakes hands with $3k+6$ persons. Also, we know that the number of the persons who shake hands with any two persons is constant. Find the number of persons invited.

Solution by Mohammed Aassila, Strasbourg, France.

Let the fixed quantity referred to be n. Now consider a fixed person a. Let B be the set of people who have shaken hands with a, and C the set of those that have not. Then $|B| = 3k+6$ and $|C| = 9k-7$.

For any $b \in B$, people who have shaken hands with a and b must be in B. Hence, b has shaken hands with n people in B, and thus with $3k+5-n$ people in C.

For any $c \in C$, people who have shaken hands with a and c must be in B. Hence, c has shaken hands with n people in B.

The total number of handshakes between B and C is given by $(3k+6)(3k+5-n) = (9k-7)n$, which simplifies to $9k^2 - 12kn + 33k + n + 30 = 0$. It follows that $n = 3m$ for some positive integer m, and $4m = k + 3 + \frac{9k+43}{12k-1}$. Since 3 is not a divisor of 44, we have $\frac{9k+43}{12k-1} \neq 1$ for any choice of k. Furthermore, if $k > 3$ then $2(12k-1) > 9k+43$ and thus, $\frac{9k+43}{12k-1} < 2$. Therefore, we need only consider $k = 1$, 2, 3. Only $k = 3$ yields an integer value for $\frac{9k+43}{12k-1}$, and there are 36 people at the party.

Comment by Pierre Bornsztein, Pontoise, France.

This problem was proposed to, but not used by, the jury of the 36^{th} IMO in Canada (1995). A solution may be found in "*36^{th} International Mathematical Olympiad*" published by the Canadian Mathematical Society, p. 138–139.

4. Let n be a natural number. Prove that n can be written as a sum of some distinct numbers of the form 2^p3^q such that none of them divides any other. For example $19 = 4 + 6 + 9$.

Comments by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Pontoise, France.

This problem is the same as problem 6 of the 8^{th} Korean Mathematical Olympiad, First Round, for which a solution has appeared in ***CRUX with MAYHEM*** [1999 : 462].

5. Prove that for any natural number n

$$\lceil\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}\rceil \;=\; \lceil\sqrt{9n+8}\rceil\,.$$

Comment by Pierre Bornsztein, Pontoise, France.

This problem and its solution may be found in the American Mathematical Monthly, 1988, p. 133–134, Ex. 3010.

Solution by Mohammed Aassila, Strasbourg, France. For $n = 0, 1, 2$ it is easy to see that the relation is true. Now, for $n \geq 3$, we have

$$n(n+1)(n+2) \;>\; \left(n+\frac{8}{9}\right)^3$$

(consider the function $(x-1)x(x+1) - (x-\frac{1}{9})^3$).

By the AM-GM inequality, we have

$$\begin{aligned}\frac{\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}}{3} &> \sqrt[3]{\sqrt{n}\sqrt{n+1}\sqrt{n+2}} \\ &> \sqrt{n+\frac{8}{9}}\,.\end{aligned}$$

Hence,

$$\sqrt{n}+\sqrt{n+1}+\sqrt{n+2} \;>\; \sqrt{9n+8}\,.$$

On the other hand,

$$\frac{\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}}{3} \;<\; \sqrt{\frac{n+n+1+n+2}{3}}\,.$$

Hence,

$$\sqrt{n}+\sqrt{n+1}+\sqrt{n+2} \;<\; \sqrt{9n+9}\,.$$

Consequently,

$$\lceil\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\rceil \;=\; \lceil\sqrt{9n+8}\rceil\,.$$

Next we look at solutions for the Final Round - First Exam, of the 13$^{\text{th}}$ Iranian Mathematical Olympiad given on [1999 : 455].

1. Prove the following inequality

$$(xy+xz+yz)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(x+z)^2}\right) \;\geq\; \frac{9}{4}$$

for positive real numbers x, y, z.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Pontoise, France. We give Aassila's presentation.

Two solutions to this problem have already appeared in ***CRUX with MAYHEM*** [1995 : 205; 1996 : 321]. We present here a new proof.

Reducing to the same denominator, the inequality to be proved is equivalent to

$$A+B+C \;\geq\; 0$$

where

$$A =: \sum_{\text{symmetric}} \left(4x^5y - x^4y^2 - 3x^3y^3\right)$$

$$B =: \sum_{\text{symmetric}} \left(4xy^5 - x^2y^4 - 3x^3y^3\right)$$

$$C =: \sum_{\text{symmetric}} \left(2x^4yz - 2x^3y^2z - 2x^3yz^2 + 2x^2y^2z^2\right)$$

and $\sum_{\text{symmetric}}$ runs over all six permutations of x, y, z.

Thanks to Shur's inequality,

$$x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \;\geq\; 0\,,$$

we have then by multiplying by $2xyz$:

$$C \;=\; \sum_{\text{symmetric}} \left(2x^4yz - 2x^3y^2z - 2x^3yz^2 + 2x^2y^2z^2\right) \;\geq\; 0\,.$$

On the other hand, by the rearrangement inequality, we have

$$\sum_{\text{symmetric}} x^5y \;\geq\; \sum_{\text{symmetric}} x^4y^2\,, \qquad \sum_{\text{symmetric}} x^5y \;\geq\; \sum_{\text{symmetric}} x^3y^3\,,$$

and hence,

$$A \;=\; \sum_{\text{symmetric}} \left(4x^5y - x^4y^2 - 3x^3y^3\right) \;\geq\; 0\,,$$

$$B \;=\; \sum_{\text{symmetric}} \left(4xy^5 - x^2y^4 - 3x^3y^3\right) \;\geq\; 0\,.$$

2. Prove that for every pair m, k of natural numbers, m can be expressed uniquely as

$$m \;=\; \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t}$$

where

$$a_k \;>\; a_{k-1} > \cdots \;>\; a_t \;\geq\; t \;\geq\; 1\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; and by Moubinool Omarjee, Paris, France. We give Omarjee's solution.

Pour l'unicité, si il existe a_k, …, a_t et b_k, …, b_s, en cherchant la première place où il diffère, disons k et $a_k > b_k$, alors

$$m \;\leq\; \binom{b_k}{k} + \cdots + \binom{b_k - k + 1}{1} \;<\; \binom{b_k + 1}{k} \;\leq\; \binom{a_k}{k} \;\leq\; m\,,$$

ce qui est absurde.

Pour l'existence, on cherche le plus grand a_k tel que $\binom{a_k}{k} \le m$, puis on réapplique le même algorithme. On cherche le plus grand a_{k-1} tel que $\binom{a_{k-1}}{k-1} \le m - \binom{a_k}{k}$. La décroissance de a_j suit du fait que $m - \binom{a_k}{k} < \binom{a_k}{k-1}$.

3. In triangle ABC we have $\angle A = 60°$. Let O, H, I, and I' be the circumcentre, orthocentre, incentre, and the excentre with respect to A of the triangle ABC. Consider points B' and C' on AC and AB such that $AB = AB'$ and $AC = AC'$. Prove that

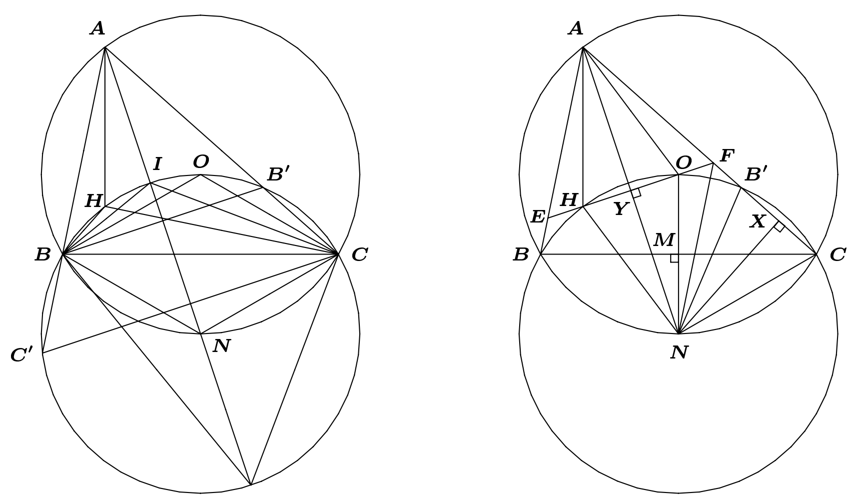
(a) Eight points B, C, H, O, I, I', B', and C' are concyclic.

(b) If OH intersects AB and AC in E and F respectively, then triangle AEF has a perimeter equal to $AC + AB$.

(c) $OH = |AB - AC|$.

Solution by Toshio Seimiya, Kawasaki, Japan.

In the figure we assume that $\triangle ABC$ is acute and $AB < AC$. But the following proof works in other cases with minor changes.



(a) Let N be the second intersection of AI with the circumcircle of $\triangle ABC$. It is well known that $BN = CN = IN$, so that N is the circumcentre of $\triangle IBC$. Since $BH \perp AC$, $CH \perp AB$, we have $\angle BHC = 180° - 60° = 120°$.

Since I is the incentre of $\triangle ABC$, we have $\angle BIC = 90° + \frac{1}{2}\angle A = 90° + 30° = 120°$. Since O is the circumcentre of $\triangle ABC$, we have $\angle BOC = 2\angle BAC = 120°$.

Similarly, $\triangle ACC'$ is equilateral, so that $\angle BC'C = \angle AC'C = 60°$. Also, H, I, O, B' lie on the same side of A with respect to BC, and C', I' lie on the opposite side of A.

Now,

$$\begin{aligned} \angle BHC &= \angle BIC = \angle BOC = \angle BB'C (= 120^\circ)\text{, and} \\ \angle BC'C + \angle BIC &= \angle BI'C + \angle BIC = 60^\circ + 120^\circ = 180^\circ\,. \end{aligned}$$

Hence B, C, H, I, O, B', I' lie on the circle with centre N.

(b) Let M be the intersection of ON with BC. Since $BN = CN$, we have that M is the mid-point of BC and that $OM \perp BC$.

Since $\angle BOM = \frac{1}{2}\angle BOC = \angle BAC = 60^\circ$ and $OB = ON$, we have that $\triangle OBN$ is equilateral. Hence $OM = MN$. As is well known, $AH = 2OM$, so that $AH = ON = OA$. Since H, O lie on the circle with centre N, we have $HN = ON$. Thus, $AH = AO = ON = HN$, so that $AHNO$ is a rhombus. Therefore, HO is the perpendicular bisector of AN.

Since $\angle EAN = \angle FAN$ and $AN \perp EF$, we have $\angle AFE = 90^\circ - \angle NAF = 90^\circ - 30^\circ = 60^\circ$. Since EF is the perpendicular bisector of AN, we get $\angle NFE = \angle AFE = 60^\circ$, so that $\angle NFC = 60^\circ$. Hence, $\angle EFN = \angle NFC$, and $\angle EAN = \angle NAF$, so that N is the excentre of $\triangle AEF$. Let X be the foot of the perpendicular from N to AC. Then X is the point of tangency of the excircle to AC.

Thus, we have $2AX = AE + AF + EF$.

Since $NB' = NC$, X is the mid-point of $B'C$, so $2AX = AB' + AC = AB + AC$. Hence, we have $AE + AF + EF = AB + AC$. Thus, the perimeter of $\triangle AEF$ is equal to $AB + AC$.

(c) Let Y be the intersection of AN with OH. Then $NY \perp OH$. Since $\angle NFY = \angle NFX$, we have $NY = NX$. Since H, O, B', C lie on the circle with centre N, we get from $NY = NX$ that $OH = B'C = AC - AB' = AC - AB$. That is, $OH = |AB - AC|$.

4. Let k be a positive integer. Prove that there are infinitely many perfect squares in the arithmetic progression $\{n \times 2^k - 7\}_{n \ge 1}$.

Solution by Mohammed Aassila, Strasbourg, France.

We first show, by induction on m, that, for every m, there exists a positive number a_m for which $a_m^2 \equiv -7 \pmod{2^m}$.

Note that $a_m = 1$ satisfies the conditions for $m \le 3$. Inductively, let us suppose that $a_m^2 \equiv -7 \pmod{2^m}$. Then $a_m^2 \equiv -7 \left(\mathrm{mod}\ 2^{m+1}\right)$, or $a_m^2 \equiv 2^m - 7 \left(\mathrm{mod}\ 2^{m+1}\right)$.

If $a_m^2 \equiv -7 \left(\mathrm{mod}\ 2^{m+1}\right)$, then we can put $a_{m+1} = a_m$. If $a_m^2 \equiv 2^m - 7 \left(\mathrm{mod}\ 2^{m+1}\right)$, then we put $a_{m+1} = a_m + 2^{m-1}$, and note that

$$\begin{aligned} a_{m+1}^2 &= (a_m + 2^{m-1})^2 = a_m^2 + 2^m a_m + 2^{2m-2} \\ &\equiv a_m^2 + 2^m a_m \left(\mathrm{mod}\ 2^{m+1}\right) \\ &\equiv a_m^2 + 2^m \left(\mathrm{mod}\ 2^{m+1}\right) \equiv -7 \left(\mathrm{mod}\ 2^{m+1}\right) \end{aligned}$$

completing the induction step.

Now since $a_m^2 \geq 2^m - 7$, the sequence (a_m) is unbounded and thus, takes on infinitely many values; that is, there are infinitely many numbers m for which one can find an associated positive number n_m such that $n_m 2^m - 7$ $(= a_m^2)$ is a perfect square.

Now, given our (fixed) number k, we simply consider the infinitely many m for which $m \geq k$, and note that $(n_m \cdot 2^{m-k})2^k - 7$ is a perfect square.

Comment by Pierre Bornsztein, Pontoise, France. This problem was proposed to, and not used by, the jury at the 36th IMO in Canada (1995). A solution may be found in "*36th International Mathematical Olympiad*", published by the Canadian Mathematical Society, p. 332.

5. Let ABC be a non-isosceles triangle. Medians of the triangle ABC intersect the circumcircle in points L, M, N. If L lies on the median of BC and $LM = LN$, prove that $2a^2 = b^2 + c^2$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give Aassila's solution.

Let G be the centroid of $\triangle ABC$. Since $\triangle ACG$ and $\triangle NGL$ are similar, and since $\triangle MLG$ and $\triangle ABG$ are similar, we have

$$\frac{LN}{AC} = \frac{LG}{CG}, \quad \frac{LM}{AB} = \frac{GL}{BG}.$$

Thanks to $LM = LN$, we obtain

$$\frac{AB}{AC} = \frac{BG}{CG}.$$

We have

$$\frac{c^2}{b^2} = \frac{2c^2 + 2a^2 - b^2}{2b^2 + 2a^2 - c^2},$$

which yields

$$(b^2 - c^2)(2a^2 - c^2 - b^2) = 0.$$

Finally, we have $2a^2 = b^2 + c^2$.

That completes this number of the *Corner*. Olympiad season is approaching. Send me your nice solutions as well as Olympiad Contests for use in the *Corner*!

BOOK REVIEWS

JOHN GRANT McLOUGHLIN

ARML-NYSML Contests 1989–1994,
by Lawrence Zimmerman and Gilbert Kessler,
published by MathPro Press, 1995, (Contests in Mathematics, Volume 2),
ISBN 0–9626401–6–6, softcover, 189+ pages.
Reviewed by ***József Pelikán****, Eőtvős, Loránd University, Budapest, Hungary.*

The ARML in the title means American Regions Mathematics League and NYSML means New York State Mathematics League. The first is an annual event for high school students with about 1000 participants coming from all over the United States and Canada and the second one is a similar event, teams coming primarily from New York State.

The format of the two contests is identical and quite different from most other mathematics competitions. 15-member teams are competing in four basic rounds, each round being different in structure.

The TEAM ROUND consists of 10 short answer questions with varying difficulty which the team as a whole has to solve during a given time limit. (They can choose any strategy, for example, working together, or smaller groups working on different problems, etc.)

The POWER QUESTION is a challenging, multi-section problem usually focused about a single mathematical theme. A detailed, well-written solution to it must be produced by the team as a whole within a time limit of one hour.

The INDIVIDUAL ROUND resembles most of the usual mathematical contests. The students, working independently, have to solve eight short answer questions, ten minutes being allowed for each pair of questions.

The most unusual part of the contest is certainly the RELAY ROUND. Here teams split into groups of three. Within each group the first person has to solve a problem, the answer being a number. This number is needed by the second person to be able to solve his or her problem and this solution in turn is needed by the third person. The score then depends on how quickly this third person produces the final answer.

I think, in this contest you can find the most instructive problems among the power questions and the most unusual ones in the relay round. Therefore, I give a sample problem from both types.

Power Question — Lattice Points on a Parabola (ARML 1992)

Throughout this problem, the points $A(a, a^2)$, $B(b, b^2)$, $C(c, c^2)$, and $D(d, d^2)$ represent distinct lattice points on the parabola $y = x^2$.

I. Let the area of the triangle ABC be K. It can be shown that

$$K = \frac{1}{2}\left|(a-b)(b-c)(c-a)\right| .$$

1. Show that K must be an integer.
2. Show that $K = 3$ is the only possible prime value for K.
3. Show that K cannot be the square of a prime.
4. Show that the area of the quadrilateral $ABCD$ cannot be 8.

II. It can be shown that the slope of $\overline{AB}$ is $a + b$.

1. A line passes through the point $(3, 5)$ and through two lattice points on $y = x^2$. Compute the coordinates of these two points. Be sure to find all possible pairs of such points.

2. A line passes through the point $(2, 4)$ and through three other lattice points on the "double parabola" $y^2 = x^4$. Compute the coordinates of these three points. Be sure to find all possible triplets of such points.

III. Consider the quadrilateral $ABCD$. [Remember that the slope of $\overline{AB}$, for example, is $a + b$.]

1. Let the vertices be labelled (alphabetically) in a counterclockwise direction. Show that

$$\tan A = \frac{d-b}{1+(a+b)(a+d)} .$$

2. A quadrilateral is "cyclic" if all four of its vertices lie on the same circle.

   Show that: if quadrilateral $ABCD$ is cyclic, then $a + b + c + d = 0$;

   AND

   if $a + b + c + d = 0$, then quadrilateral $ABCD$ is cyclic.

3. Use the previous result to show that:

   If a circle intersects the graph of $y = x^2$ in four points, and three of them are lattice points, then the fourth must also be a lattice point.

(Note to the reader of this review: The answer is, of course, too long to be given here. Work it out for yourself!)

Relay from NYSML 1990

R1. Compute the area of the smallest square that goes through the points $(0,0)$ and $(4,0)$.

R2. Let $T = \textbf{TNYWR}$, (*Reviewer's remark: this is the acronym used in this competition for 'The Number You Will Receive'.*) and let $K = T - 5$.

The positive integer n is even, and all its divisors (except n itself) divide $n/2$. Compute the largest K–digit number n with this property.

R3. Let $N = \textbf{TNYWR}$, and let K be the sum of the digits of N.

Two secants are drawn to a circle from an outside point, intercepting arcs (between them) of lengths $K\pi$ and 2π. If the angle between the secants is $30°$, compute the radius of the circle.

Solutions

R1. It is clear enough that we get the smallest square if we let the segment $\overline{(0,0),(4,0)}$ be the diagonal of the square. This gives $\mathbf{Area} = \frac{1}{2}(\mathbf{diagonal})^2 = 8$.

R2. (Reviewer's remark: Here comes a tough decision on the part of the competitor which clearly indicates the peculiarities of this competition. With some experimentation you easily come to the conjecture that n must be a power of 2. As $K = T - 5 = 8 - 5 = 3$, your number is then 512. Should you pass it quickly on to the third person or spend some time trying to find a rigorous proof that your conjecture is indeed true? There was some slight indication of the possibility of such a dilemma already in R1. There the phrase 'clear enough' would not be a satisfactory explanation in some rigorous mathematical competitions, but in R1 the situation was intuitively so clear that it would have been a serious mistake — competition-wise — to try to find a rigorous proof.)

The conjecture that n must be a power of 2 is true, and a rigorous proof is actually quite easy: if n had an odd prime divisor p then n/p —although a proper divisor— would not divide $n/2$.

R3. As $N = 512$, $K = 8$. Since the difference in the degree measures of the arcs must be $2 \cdot 30° = 60°$, which corresponds to an arc length $2\pi r/6$, we have $8\pi - 2\pi = 2\pi r/6$ which gives $r = 18$.

The book covers the problems (with solutions) of ARML 1989–1994, NYSML 1989–1992 and tiebreakers of NYSML and ARML in the period 1983–1994. (These were used when ties occurred among top individual scores.) It also contains a list of ARML and NYSML winners (both teams and individuals) for the given periods and a Glossary of some of the less common mathematical terms used in the book.

I found the book well-written, the problems in general interesting, and I can recommend it to anyone having an interest in contests in mathematics.

If (a, b, c) is Heron, can $(s - a, s - b, s - c)$ also be Heron?

K.R.S. Sastry

Heron's name should be familiar to those who use the formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,, \quad \text{where} \quad s = (a+b+c)/2\,,$$

to calculate the area of a triangle in terms of the lengths a, b, c of its sides. According to mathematical historians Heron lived in Alexandria in the first century, and according to [3] he introduced the definitions of point, straight line, etc. into Euclid's *Elements*. His name is further associated with the observation that the triangle with side lengths 13, 14, and 15 has area 84. We use the quadruple $(a, b, c; \Delta)$ to denote the sides and the area of the triangle, and when a, b, c, and Δ are all integers we call the triangle a *Heron* triangle.

Here we look at the triple $a' = s - a$, $b' = s - b$, $c' = s - c$, and ask when these numbers can be the sides of a triangle, which we shall call the *derived triangle*. Moreover, we will be interested when the triangle derived from a Heron triangle is itself a Heron triangle. Certainly not always: in the case of Heron's own triangle $(13, 14, 15; 84)$ we have $(a', b', c'; \Delta') = (8, 7, 6; \frac{21}{4}\sqrt{15})$. Here (a', b', c') is not Heron because Δ' is not an integer. Of course, (a', b', c') may even fail to form a triangle! (Consider $(a, b, c) = (4, 13, 15)$.) In the next section we derive a condition on (a, b, c) so that (a', b', c') too forms a triangle.

There are isosceles Heron triangles that provide an affirmative answer to our question. But the determination of solutions of other types is an open problem. Our aim is to determine the isosceles Heron triangles (a, b, c) for which $(a', b', c') = (s - a, s - b, s - c)$ is also Heron. Also, we show that a Heron triangle whose sides form an arithmetic progression cannot be a solution to our problem.

Necessary Conditions

We first establish a known fact about *primitive* Heron triangles; that is, about Heron triangles that have the gcd of the sides equal to 1.

Theorem 1. In a primitive Heron triangle exactly one side is even.

Proof. Since a, b, c are integers, $s = (a + b + c)/2$ is either an integer or a half-integer. If s is an integer, then $a + b + c$ must be even. Since

$\gcd(a, b, c) = 1$, we must have that one of a, b, or c is even and the other two are odd.

If s is a half-integer, then $s - a$, $s - b$, $s - c$ are all half-integers. In this case Δ cannot be an integer. Hence the proof is complete.

The answer to the question, when do both (a, b, c) and (a', b', c') form triangles is provided by

Theorem 2. If the inequalities $s/2 < a,\ b,\ c < s$ hold, then (a', b', c') forms a triangle.

Proof. When the triple (a', b', c') forms a triangle then, necessarily, $a' + b' > c'$. This requires that $(s - a) + (s - b) > s - c$; that is, that $s/2 < c$. Also, $s - c = (a + b - c)/2 > 0$; that is, $s > c$. Repetition of these two facts using a and b in place of c and noting that the necessary conditions are also sufficient completes the proof.

The next theorem shows that for the present purposes we need s to be an even integer.

Theorem 3. If both (a, b, c) and (a', b', c') are Heron, then s is an even integer.

Proof. Since $s' = (a' + b' + c')/2 = s/2$ is an integer, s must be even.

On two occasions we require the solutions (x, y, z) of the Diophantine equation $x^2 = y^2 + kz^2$. Here k is a given natural number. We refer the reader to [2], p. 420 and p. 426 for a discussion of this. We merely state the solution which is easy to derive anyway:

$$\left.\begin{array}{lll} x = \lambda(u^2 + kv^2)\,, & & y = \lambda|u^2 - kv^2|\,, \\ z = \lambda(2uv)\,, & & \lambda = 1,\ 2,\ 3,\ \ldots\ . \end{array}\right\} \tag{1}$$

Earlier we saw that Heron's own (a, b, c) did not yield (a', b', c') Heron. The next theorem shows more generally that a Hoppe's triangle (that is, a triangle whose sides form an arithmetic progression, see [2], p. 197) does not make (a', b', c') Heron.

Theorem 4. Let (a, b, c) be a Heron triangle in which the sides are in arithmetic progression. Then (a', b', c') is not Heron.

Proof. For definiteness, we let $a = 2a_1$ be the even side and b and c be odd. If d is the common difference of the progession, then $b = 2a_1 - d$ and $c = 2a_1 + d$. This shows that d is odd. However, $s = 3a_1$ must be even by Theorem 3. Let $a_1 = 2a_2$. This gives $a = 4a_2$, $b = 4a_2 - d$, $c = 4a_2 + d$ and $\Delta = 2a_2\sqrt{3(4a_2^2 - d^2)}$. Since Δ is an integer, we must have

$$4a_2^2 - d^2 = 3p^2 \quad \text{or} \quad (2a_2)^2 = d^2 + 3p^2\,.$$

The solution of the above equation for $k = 3$ from (1) is

$$2a_2 = \lambda(u^2 + 3v^2), \quad d = \lambda|u^2 - 3v^2|, \quad p = \lambda(2uv).$$

The first of these shows that $\lambda(u^2 + 3v^2)$ is even. Hence, at least one of λ, $u^2 + 3v^2$ must be even. But $u^2 + 3v^2 = (u^2 - 3v^2) + (6v^2)$ shows that $u^2 + 3v^2$ and $u^2 - 3v^2$ are together both even or both odd. In any case, this contradicts the fact that d must be odd. Hence the claim of the theorem is true.

We prove our main result in the next section. However, we begin the next section with a discussion on isosceles Heron triangles.

Isosceles Heron Triangles

Our interest lies in a solution to our problem by determining primitive Heron triangles. Carlson [1] and Singmaster [8] show that isosceles Heron triangles result when we juxtapose two identical copies of a primitive Pythagorean triangle (a right triangle with integer sides). It is known ([2] pp. 165, 169) that the sides of primitive Pythagorean triangles are completely described by

$$m^2 - n^2, \quad 2mn, \quad m^2 + n^2, \tag{2}$$

where m, n are natural numbers such that $m > n$, $\gcd(m, n) = 1$ and one of m, n is even and the other is odd. We can generate an isosceles Heron triangle in two ways. They are illustrated below.

The first juxtaposition given by Figure 1 has $(a, b, c) = (2(m^2 - n^2), m^2 + n^2, m^2 + n^2)$, $s = 2m^2$ and $(a', b', c') = (2n^2, m^2 - n^2, m^2 - n^2)$.

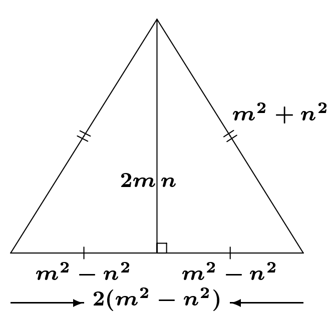


Figure 1

We note that s and a' are even and b' and c' are odd in agreement with Theorems 1 and 3. Hence we may hope for a solution to our problem.

The second juxtaposition given by Figure 2 has $(a, b, c) = (4mn, m^2 + n^2, m^2 + n^2)$, $s = (m + n)^2$, and $(a', b', c') = ((m - n)^2, 2mn, 2mn)$.

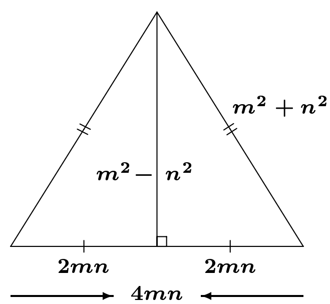


Figure 2

We note that s and a' are odd and b' and c' are even, contradicting Theorems 1 and 3. Hence this case does not lead to primitive solutions at all.

We are now in a position to establish our main result.

Theorem 5. Both the triangle $(a, b, c) = (2(m^2 - n^2), m^2 + n^2, m^2 + n^2)$ and its derived triangle $(a', b', c') = (2n^2, m^2 - n^2, m^2 - n^2)$ are Heron if and only if $m = u^2 + 2v^2$ and $n = 2uv$, where u is odd and $\gcd(u, v) = 1$.

Proof. Heron's formula yields $\Delta' = mn^2\sqrt{m^2 - 2n^2}$. Therefore, Δ' will be an integer if and only if $m^2 - 2n^2 = l^2$ is a perfect square. That is

$$m^2 = l^2 + 2n^2 ,$$

an instance of equation (1) when $k = 2$. Hence the solution

$$m = \lambda(u^2 + 2v^2) , \quad l = \lambda|u^2 - 2v^2| , \quad n = \lambda(2uv) .$$

Our interest is in the primitive solutions (a, b, c). Hence the presence of λ is unnecessary. Furthermore, if u is even, then l, m, n have $\gcd 2$. Therefore, we require that u be odd. Thus

$$m = u^2 + 2v^2 , \quad n = 2uv . \tag{3}$$

We leave the verification that (a, b, c) and (a', b', c') as determined by (3) are both Heron to the reader. This completes the proof.

We illustrate Theorem 5 with a couple of examples. If we put $u = v = 1$ in (3), we get $m = 3$, $n = 2$, $(a, b, c) = (10, 13, 13)$ and $(a', b', c') = (8, 5, 5)$. Also $u = 3$, $v = 1$ gives $m = 11$, $n = 6$, $(a, b, c) = (170, 157, 157)$ and $(a', b', c') = (72, 85, 85)$. One can also observe that if u is even, say $u = 2$, $v = 1$, then $m = 6$, $n = 4$, $(a, b, c) = (40, 52, 52)$ and $(a', b', c') = (32, 20, 20)$. This is just a multiple of our first illustration.

Conclusion. The present discussion determined a partial solution to our general problem: For which Heron triangles (a, b, c) is (a', b', c') also Heron. We invite the reader to provide other solutions to our general problem. This may take the form of determining another class of Heron triangles

(a, b, c) that also have (a', b', c') Heron. A proof that the Pythagorean triangles $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$ can or cannot have (a', b', c') Heron would be another step. By chance if one meets with a Heron triangle (p, q, r) for which $pqr(p + q + r)$ is a perfect square, then that would be an example to our problem: Take $(a', b', c') = (p, q, r)$. Then one has $(a, b, c) = (q + r, r + p, p + q)$. (Can you see why?) In the references the reader can find many interesting problems on Heron triangles.

Acknowledgement: The author thanks the referee and the editor for their suggestions to improve the presentation.

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MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7 (NEW!)**. The electronic address is
mayhem-editors@cms.math.ca

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The other staff member is Jimmy Chui (University of Toronto).

MAYHEM TAUNT

As promised, at various times in 2001, 2002 is going to be a year of prizes here at ***MAYHEM***. Since we have decided that the focus of ***MAYHEM*** will be on pre-university mathematics, the prizes will be awarded to students enrolled in elementary or secondary schools or equivalent. Solutions from other people are always welcomed, and the featured solution may not necessarily come from this age group.

To be eligible, each solution to a ***MAYHEM*** **Problem** must be **handwritten** on a sheet of paper (one question per sheet). Attached to each solution of each problem must be a completed student information sheet signed by the student and a representative of the student's school (teacher or administrator). A copy of the information sheet is included in this issue, and on the Canadian Mathematical Society's web site at `journals.cms.math.ca/CRUX/MAYHEM-Taunt`.

Students may work alone, or in groups. If the problem is solved by a group please submit **one** solution with the information of all the group members.

Prizes will be awarded based on the following criteria:

- first solution to a problem;
- most correct solutions from a single person;
- youngest solver;
- most elegant solution.

Other prizes may be awarded at the discretion of the ***MAYHEM*** Editors. In all cases **the decision of the *MAYHEM* Editors is *final*!**

Our purpose is also to support schools, and as a result we will have some prizes for schools. To aid us in this process, we would ask that the school information for the student is filled out with care.

Prizes will range from past copies of ***MAYHEM*** to subscriptions to ***CRUX with MAYHEM*** to book prizes from the Canadian Mathematical Society. The prizes are made possible from a grant from the Endowment Fund of the Canadian Mathematical Society, and we thank the board of the Endowment Grants Committee for providing us with the money to make this possible.

Problems and information will be available on the Canadian Mathematical Society's web site at `journals.cms.math.ca/CRUX/MAYHEM-Taunt`.

LE DÉFI MAYHEM

Comme nous l'avons promis à quelques reprises en 2001, le ***MAYHEM*** remettra plusieurs prix en 2002. Puisque nous avons décidé que le ***MAYHEM*** mettrait l'accent cette année sur les mathématiques préuniversitaires, des prix seront remis aux élèves inscrits à des écoles primaires ou secondaires (ou l'équivalent). Les solutions d'autres personnes sont toujours les bienvenues, toutefois, car la solution présentée ne sera pas nécessairement celle d'une personne du groupe d'âge ciblé.

Pour être acceptable, une solution à un **problème du *MAYHEM*** doit être écrite **à la main** sur papier (une question par feuille). À chaque problème présenté devra être annexée une fiche de renseignements de l'élève, dûment remplie et signée par l'élève et un représentant de son école (membre du personnel enseignant ou de la direction). La fiche de renseignements est reproduite dans le présent numéro et paraît sur le site Web de la Société au `journals.smc.math.ca/CRUX/MAYHEM-defi`.

Les élèves peuvent travailler seuls ou en groupe. Si un problème est résolu en groupe, prière de remetttre **une** solution accompagnée d'une fiche de renseignements pour chaque membre du groupe.

Les prix seront attribués dans les catégories suivantes :

- première solution à un problème;
- plus grand nombre de bonnes solutions présentées par une personne;
- élève le plus jeune ayant résolu un problème;
- solution la plus élégante.

D'autres prix pourront être remis à la discrétion de la rédaction du Mayhem. Dans tous les cas, **la décision des rédacteurs du *MAYHEM* est *finale*!**

Comme notre objectif est aussi d'encourager les écoles, nous remettrons aussi des prix aux établissements. Pour nous faciliter la tâche, nous vous demandons de remplir avec soin l'information sur l'école sur la fiche de renseignements de l'élève.

Au nombre des prix, il y aura des anciens numéros du ***MAYHEM***, des abonnements à ***Crux with Mayhem*** et des ouvrages de la Société mathématique du Canada. Ces prix ont été achetés grâce à une bourse du fonds de dotation de la SMC. Nous remercions le Comité d'attribution des bourses du fonds de dotation de nous avoir remis la somme nécessaire à notre concours.

Les problèmes et tout autre renseignement seront publiés sur le site Web de la Société mathématique du Canada au `journals.smc.math.ca/CRUX/MAYHEM-defi`.

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1**, or emailed to

mayhem-editors@cms.math.ca

Please include in all correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 August 2002*. Solutions received after this time will be considered if there is time before publication of the solutions.

Starting this issue, problems will be printed in English and French.

To be eligible for this month's MAYHEM TAUNT, solutions must be postmarked *before* 1 June 2002.

M29. *Proposed by the Mayhem staff.*

Define the "silly product" of two numbers as the sum of the product of all the corresponding digits. So $235 \times_s 718 = 2 \times 7 + 3 \times 1 + 5 \times 8 = 57$. Find two numbers A and B so that $A \times_s B = 2002$ and $A + B$ is a minimum.

On définit le "produit singulier" de deux nombres comme la somme des produits de leurs chiffres respectifs. Par exemple : $235 \times_s 718 = 2 \times 7 + 3 \times 1 + 5 \times 8 = 57$. Trouver deux nombres A et B tels que $A \times_s B = 2002$ et $A + B$ soit minimale.

M30. *Proposed by Haralampy Steryion, Chalkis, Greece.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the property

$$f(x+y) = f(x)e^{f(y)-1} \qquad \text{for every } x, y \in \mathbb{R}.$$

Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ satisfaisant la condition

$$f(x+y) = f(x)e^{f(y)-1} \qquad \text{pour tout } x, y \in \mathbb{R}.$$

M31. *Proposed by the Mayhem staff.*

Given four spheres of unit radius, each tangent to the other three, find the radii of the two spheres that are tangent to all four of the unit spheres.

On considère quatre sphères de rayon unité, chacune tangente aux trois autres. Trouver le rayon des deux sphères qui sont simultanément tangentes aux quatre sphères données.

M32. *Proposed by Nicolae Gustia, North York, Ontario.*

In a triangle with angles A, B and C, if $8\cos A\cos B\cos C = 1$ then prove that $\triangle ABC$ is equilateral.

Montrer que si dans un triangle, les angles A, B et C satisfont la condition $8\cos A\cos B\cos C = 1$, alors le triangle est équilatéral.

M33. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

a, b and c are three consecutive terms of a geometric sequence, where a, b and c are all integers. If $a + b + c = 7$, determine all possible values of a, b and c.

Les entiers a, b et c sont trois termes consécutifs d'une suite géométrique. Si $a + b + c = 7$, trouver toutes les valeurs possibles de a, b et c.

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem.

How many ordered triples of integers (a, b, c) satisfy $|a + b| + c = 19$ and $ab + |c| = 97$?

(1997 AHSME, Problem 28)

Solution. We know that $a + b = \pm(19 - c)$ and $ab = 97 - |c|$. We can view a and b as the solutions to the quadratic $t^2 \mp (19-c)t + (97-|c|) = 0$.

Now, for this equation to have integer solutions, a necessary condition is that the discriminant is a perfect square.

Hence, $D = (19 - c)^2 - 4(97 - |c|) = c^2 - 38c + 4|c| - 27$ must be a perfect square.

If $c \geq 0$, then $D = c^2 - 34c - 27 = (c - 17)^2 - 316$. Let this be m^2, and after rearranging, we have $(c + m - 17)(c - m - 17) = 316 = 2^2 \cdot 79$. Let $c + m - 17$ be x and $c - m - 17$ be y, so that $xy = 316$. Now, we note that $x + y = 2c - 34$, which is independent of the introduced variable m. From this, we can tell that x and y must have the same parity (they add to an even number). From this, we know that $\{x, y\} = \{\pm 2, \pm 158\}$. (Order is not relevant when we are solving for c.) Then, $x+y = 2c-34 = \pm 160$. Solving for c, the only value satisfying $c \geq 0$ is $c = 97$. However, from the original equations, this value is impossible, since that would mean that $|a + b|$ is a negative number.

The other case is if $c < 0$.

Then the discriminant is $D = c^2-42c-27 = (c-21)^2-468$. Let this be m^2, and after rearranging, we get $(c+m-21)(c-m-21) = 468 = 2^2 \cdot 3^2 \cdot 13$. From similar reasoning as before, $x = c + m - 21$ and $y = c - m - 21$ multiply to 468, add to $2c - 42$, and are of the same parity. The only values that x and y can take are $\{x, y\} = \{\pm 2, \pm 234\}$, $\{\pm 6, \pm 78\}$, $\{\pm 18, \pm 26\}$. These values correspond to $x + y = 2c - 42 = \pm 236, \pm 84, \pm 44$. The only values of c that satisfy $c < 0$ are $c = -97, -21, -1$.

When $c = -1$, we have the two equations $|a + b| = 20$ and $ab = 96$. This leads to the four solutions $\{a, b\} = \{\pm 8, \pm 12\}$. (These solutions can be found from, for example, $t^2 \pm 20t + 96 = 0$.) When $c = -21$, we have the four solutions $\{a, b\} = \{\pm 2, \pm 38\}$. When $c = -97$, we have four more solutions $\{a, b\} = \{\pm 116, 0\}$. Hence, the original set of equations have 12 solutions in total.

Note. An alternate solution can be obtained by first eliminating c from the original set of equations. This can be done by making the two cases $c \geq 0$ and $c < 0$. The objective is then to work with equations in terms of a and b.

High School Solutions

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H283. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a_1, a_2, $\ldots$, a_n be positive real numbers in arithmetic progression. Prove that

$$\sum_{k=1}^{n} \frac{1}{a_k a_{n-k+1}} > \frac{4n}{(a_1+a_n)^2}.$$

I. *Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

The strict inequality should be replaced by "$\geq$" since equality is possible. Let d denote the common difference of the arithmetic progression. Then for all $k = 1, 2, \ldots, n$, we have $a_k = a_1 + (k-1)d$ and $a_{n-k+1} = a_1 + (n-k)d$. Hence, $a_k + a_{n-k+1} = 2a_1 + (n-1)d = a_1 + a_n$ which implies

$$\sum_{k=1}^{n} \frac{1}{(a_k + a_{n-k+1})^2} = \frac{n}{(a_1+a_n)^2} \tag{1}$$

Since $4a_k a_{n-k+1} \leq (a_k + a_{n-k+1})^2$ we have

$$\sum_{k=1}^{n} \frac{1}{a_k a_{n-k+1}} \geq \sum_{k=1}^{n} \frac{4}{(a_k + a_{n-k+1})^2} \tag{2}$$

From (1) and (2), the result follows.

II. *Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina.*

Suppose that $n > 1$. Thus, there exists a non-zero real number r such that $a_k = a_1 + (k-1)r$ for $k = 1, \ldots, n$. By the AM–GM Inequality, since the a_k's are different numbers, we have,

$$\sqrt{a_k a_{n-k+1}} < \frac{a_k + a_{n-k+1}}{2} = \frac{2a_1 + (n-1)r}{2} = \frac{a_1 + a_n}{2},$$

whence it follows that

$$\sum_{k=1}^{n} \frac{1}{a_k a_{n-k+1}} > \sum_{k=1}^{n} \frac{4}{(a_1+a_n)^2} = \frac{4n}{(a_1+a_n)^2}.$$

[Note that, as pointed out by Wang, the a_k's do not have to be distinct; thus, the "$>$" must be replaced by "$\geq$". *Ed.*]

Also solved by Mihály Bencze, Brasov, Romania and the proposer.

H284. Prove that for any positive integer n,

$$1 \;\geq\; \frac{n^n}{(n!)^2} \;\geq\; \frac{(4n)^n}{(n+1)^{2n}}\,.$$

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

Note that, if $k \leq n-1$, we get $nk \geq (k+1)k$ and consequently $nk - k^2 - k \geq 0$. From that follows

$$nk - k^2 - k + n \;=\; (n-k)(k+1) \;\geq\; n\,,$$

and finally

$$\prod_{k=0}^{n-1}(n-k)(k+1) \;=\; (n!)^2 \;\geq\; n^n\,.$$

Thus, the left inequality holds.

It follows from some short calculations that the right inequality

$$\frac{n^n}{(n!)^2} \;\geq\; \frac{(4n)^n}{(n+1)^{2n}}$$

is equivalent to

$$\left(\frac{n+1}{2}\right)^{2n} \;\geq\; (n!)^2\,.$$

It follows from the AM–GM Inequality, that, for $0 \leq k \leq n-1$,

$$\left(\frac{n+1}{2}\right)^2 \;=\; \left(\frac{(n-k)+(k+1)}{2}\right)^2 \;\geq\; (n-k)(k+1)\,,$$

so that

$$\prod_{k=0}^{n-1}(n-k)(k+1) \;=\; (n!)^2 \;\leq\; \left(\frac{n+1}{2}\right)^{2n}\,.$$

Also solved by Mihály Bencze, Brasov, Romania; Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina, Henry J. Pan, student East York C.I., Toronto and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

H285. Four people, A, B, C, D, are on one side of a river. To get across the river they have a rowboat, but it can fit only two people at a time. A, B, C, D, could each row across the river in the boat individually in 1, 2, 5, and 10 minutes respectively. However, when two people are on the boat, the time it takes them to row across the river is the same as the time necessary to row across for the slower of the two people. Assuming that no one can cross without the boat, and everyone is to get across, what is the minimum time for all four people to get across the river?

Solution by the editors.
The minimum time is 17 minutes.

The following trip takes 17 minutes. A and B cross together (2 **min**). A returns alone (1 **min**). C and D cross together (10 **min**). B returns alone (2 **min**). A and B cross together (2 **min**).

Now we show it cannot take less than 17 minutes. To get everyone across, a total of 5 trips must be made. Notice:

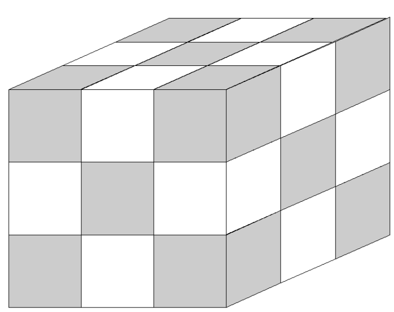
1. C and D cross together; otherwise 15 minutes is used for two of the trips and the total time is at least 18 minutes.

2. C and D travel only once; otherwise at least 15 minutes is used for at most three trips, and the total time is at least 17 minutes.

3. Now with 10 minutes taken up for C and D's one crossing, we have four trips that must be done. If the total time is less than 17 minutes, then the four trips must be done in 6 minutes or less. For this to be done, at least two trips have to take 1 minute; that is, A travelling back **alone**. (If A travels with anyone else it takes more than 1 minute). But if A comes back twice, he must cross over three times (to end up on the other side). Thus, A must go on all five trips, so that C and D cannot cross together. Thus, it is impossible to make the trip in less than 17 minutes.

H286. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunnelling through all of the 27 $1 \times 1 \times 1$ sub-cubes. If he starts at one of the corner sub-cubes and always moves onto an uneaten adjacent sub-cube can he finish at the center of the cube? (Assume that he can tunnel through walls but not edges or corners.)

Solution by the editors.
Colour the 3×3 cube in this manner: colour each corner sub-cube black. Every other sub-cube is coloured black or white so that each sub-cube is a different colour than all the other sub-cubes that it shares a face with. Thus, we end up with the alternating cube pictured below.



Now, notice that the corner sub-cube is black, and the centre sub-cube is white. But as the mouse goes through from sub-cube to sub-cube, the destination sub-cube is a **different** colour from his original cube. Since there are 13 white cubes and 14 black cubes, the mouse's path must go $BWBW\ldots BWB$. The last cube must be black. Thus, he **cannot** end up in the centre sub-cube last.

H287. Suppose we want to construct a solid polyhedron using just n pentagons and some unknown number of hexagons (none of which need be regular), so that exactly three faces meet at every vertex on the polyhedron. For what values of n is this feasible?

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

Let the number of the hexagonal faces of the polyhedron be m, and let be v, f, and e the numbers of the vertices, faces and edges of the polyhedron, respectively. Then we have, according to Euler's formula,

$$v + f - e \;=\; 2\,. \tag{1}$$

Since all the faces of the polyhedron must be either pentagons or hexagons,

$$f \;=\; n + m\,.$$

Each of the n pentagonal faces has 5 edges; each of the m hexagonal faces has 6 edges. Taking into account that every edge belongs to 2 faces, this yields

$$e \;=\; \frac{5n+6m}{2}\,.$$

Since exactly three faces meet at every vertex, three edges meet at every vertex. Each of the edges connects two vertices of the polyhedron, so that

$$v \;=\; \frac{2e}{3} \;=\; \frac{10n+12m}{6}\,.$$

Plugging that into (1), we finally get

$$\frac{10n+12m}{6} + n + m - \frac{5n+6m}{2} \;=\; 2\,,$$

which simplifies to

$$n \;=\; 12\,.$$

Two examples of polyhedra that meet the requirements are the pentagon-dodecahedron (where $m = 0$) and the truncated icosahedron (with $m = 20$).

H288. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

If x, y, z are positive real numbers, show that

$$\frac{1}{\sinh(x+z)}\left(\frac{\cosh y\cosh z}{\cosh(x+y+z)}-\cosh x\right) = \frac{1}{\sinh(y+z)}\left(\frac{\cosh x\cosh z}{\cosh(x+y+z)}-\cosh y\right)$$

Solution by the proposer.

After reducing to a common denominator, the left hand side of the above identity can be written as

$$\frac{1}{\sinh(x+z)}\left[\frac{\cosh y\ \cosh z-\cosh(x+y+z)\ \cosh x}{\cosh(x+y+z)}\right]. \qquad (1)$$

Taking into account the identities

$$\sinh a\sinh b = \frac{1}{2}\left[\cosh(a+b)-\cosh(a-b)\right]$$

$$\cosh a\cosh b = \frac{1}{2}\left[\cosh(a+b)+\cosh(a-b)\right],$$

expression (1) is equal to

$$\frac{1}{2\sinh(x+z)}\left[\frac{\cosh(y-z)+\cosh(2x+y+z)}{\cosh(x+y+z)}\right] = \frac{\sinh(x+y)}{\cosh(x+y+z)}. \qquad (2)$$

Since (2) is symmetric in x and y, from it we obtain the right side of our identity and we are done.

SKOLIAD No. 59

Shawn Godin

Effective this issue, the Skoliad Corner is now incorporated into ***Mathematical Mayhem***. This is to consolidate High School level material into one section. We envisage little change in the actual material!

Solutions may be sent to Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5, or emailed to

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Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 June 2002*. A copy of **MATHEMATICAL MAYHEM Vol. 1** will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

This issue's items come to us from South Africa. My thanks go out to John Webb of the University of Cape Town for forwarding the material to me. For more information on the math competitions visit the University of Cape Town Mathematics Department's web site

`http://www.mth.uct.ac.za`

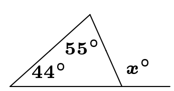
The first contest is the 2001 contest for grades 9 and 10 students. Students are given 75 minutes, no calculators are allowed. Participants are awarded 0 points for an incorrect answer, 1 point for each question not answered and for correct answers the points are: 5 points for questions 1 - 10, 6 points for questions 11 - 20 and 7 points for questions 21 - 30.

The UCT Mathematics Competition
Grades 9 and 10 : 2001

1. $\frac{8}{5}$ is equal to

(1) 0.625 (2) 1.667 (3) 1.8 (4) 1.6 (5) 0.6

2. In the diagram, the value of x is



(1) 99 (2) 98 (3) 101 (4) 109 (5) 111

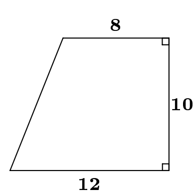
3. $70 \div 0.02$ is equal to

(1) 3.5 (2) 35 (3) 350 (4) 0.35 (5) 3500

4. $(3y + x) - (y - 2x) + (x - y)$ is equal to

(1) $2y + 4x$ (2) $y + 4x$ (3) $2y$ (4) y (5) $5y$

5. The area of the figure, in square centimetres, is



(1) 80 (2) 88 (3) 100 (4) 120 (5) 160

6. What is the last digit of 2^{2001}?

(1) 1 (2) 2 (3) 4 (4) 6 (5) 8

7. A large reservoir can be emptied by four sets of cylindrical pipes, at the same level.
Set $\boldsymbol{P}$ has one pipe, of diameter $\mathbf{40}$ cm.
Set $\boldsymbol{Q}$ has two pipes, of diameter $\mathbf{20}$ cm.
Set $\boldsymbol{R}$ has three pipes, of diameter $\mathbf{16}$ cm.
Set $\boldsymbol{S}$ has five pipes, of diameter $\mathbf{10}$ cm.
Which set will empty the reservoir in the shortest time?

(1) $\boldsymbol{P}$ (2) $\boldsymbol{Q}$ (3) $\boldsymbol{R}$ (4) $\boldsymbol{S}$ (5) They will all take the same time.

8. In the multiplication shown, the value of $\boldsymbol{A}+\boldsymbol{B}$ is

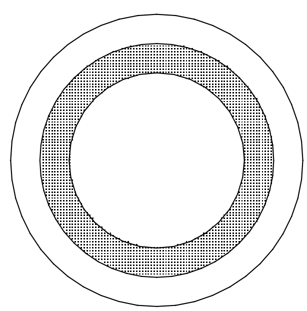
$$\begin{array}{ccc} & \mathbf{2} & \boldsymbol{A} \\ \times & \boldsymbol{B} & \mathbf{3} \\ \hline & \mathbf{6} & \mathbf{9} \\ \mathbf{9} & \mathbf{2} & \\ \hline \mathbf{9} & \mathbf{8} & \mathbf{9} \end{array}$$

(1) $\mathbf{7}$ (2) $\mathbf{11}$ (3) $\mathbf{13}$ (4) $\mathbf{14}$ (5) $\mathbf{16}$

9. What is the sum of all the prime numbers which are greater than $\mathbf{20}$ and less than $\mathbf{40}$?

(1) $\mathbf{115}$ (2) $\mathbf{120}$ (3) $\mathbf{131}$ (4) $\mathbf{133}$ (5) $\mathbf{140}$

10. The three circles in the figure have the same centre; their radii are $\mathbf{3}$ cm, $\mathbf{4}$ cm and $\mathbf{5}$ cm. What percentage of the large circle is shaded?



(1) $\mathbf{20}\%$ (2) $\mathbf{25}\%$ (3) $\mathbf{28}\%$ (4) $\mathbf{30}\%$ (5) $\mathbf{33\frac{1}{3}}\%$

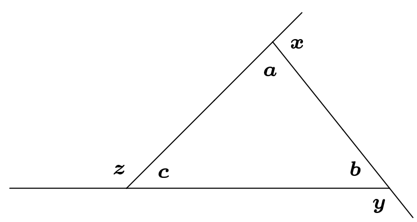
11. On a $\mathbf{26}$ question test, $\mathbf{8}$ points were given for each correct answer and $\mathbf{5}$ points were deducted for each wrong answer. Tom answered all the questions and scored zero. How many questions did he get correct?

(1) $\mathbf{8}$ (2) $\mathbf{9}$ (3) $\mathbf{10}$ (4) $\mathbf{12}$ (5) $\mathbf{13}$

12. Ahmed, Bongani, Cyril, Delia and Evan have their birthdays on successive days, but not necessarily in that order. Ahmed's birthday is as many days before Cyril's as Bongani's is after Evan's. Delia is two days

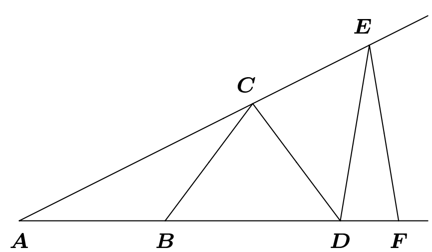
older than Evan. Cyril's birthday is on a Wednesday. On what day of the week is Evan's birthday?
(1) Tuesday (2) Monday (3) Thursday (4) Friday (5) Sunday

13. If the exterior angles x, y, z of the triangle are in the ratio $4 : 5 : 6$, then the interior angles a, b, c are in the ratio



(1) $7 : 5 : 3$ (2) $3 : 2 : 1$ (3) $4 : 2 : 1$ (4) $8 : 5 : 2$ (5) $6 : 5 : 4$

14. In the figure $AB = BC = CD = DE = EF$ and $AE = AF$. What is the size of $\angle EAF$?



(1) $10°$ (2) $15°$ (3) $20°$ (4) $30°$ (5) Not enough information.

15. The highest common factor of two numbers is 4. The lowest common multiple of these two numbers is 24. What are the possibilities for the sum of the two numbers?
(1) 20 only (2) 28 only (3) 20 or 28 (4) 36 only (5) 20 or 36

16. One and a half litres of water are poured into jugs A and B. If jug A contains 50% more water than jug B, how much water is in jug A?
(1) 1000 ml (2) 900 ml (3) 750 ml (4) 600 ml (5) 500 ml

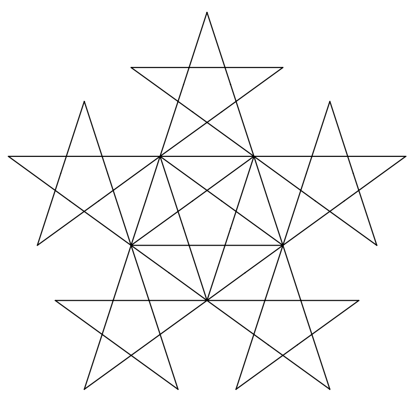
17. A die rests on a table. Ali, sitting on one side of the table, sees two sides of the die and its top, and can see $\mathbf{7}$ dots altogether. Benni, sitting on the opposite side of the table, can see the top and the other two faces and sees $\mathbf{11}$ dots altogether. How many dots are on the bottom face of the die resting on the table?

(1) $\mathbf{1}$ (2) $\mathbf{2}$ (3) $\mathbf{3}$ (4) $\mathbf{4}$ (5) $\mathbf{5}$

18. A point $\boldsymbol{P}$ is inside a regular octagon $\boldsymbol{ABCDEFGH}$, such that triangle $\boldsymbol{ABP}$ is equilateral. What is the size of $\angle \boldsymbol{APC}$?

(1) $\mathbf{120^\circ}$ (2) $\mathbf{135^\circ}$ (3) $\mathbf{90^\circ}$ (4) $\mathbf{112\frac{1}{2}^\circ}$ (5) $\mathbf{97\frac{1}{2}^\circ}$

19. The figure shown is to be drawn without lifting the pencil from the paper and without going over any line twice. What is the smallest number of straight line strokes needed to draw the figure?



(1) $\mathbf{20}$ (2) $\mathbf{22}$ (3) $\mathbf{25}$ (4) $\mathbf{30}$ (5) $\mathbf{35}$

20. Sizwe and Thabo sit down to eat sausages. Sizwe has four sausages and Thabo has three. Vuyo joins them. He has no sausages, but has $\mathbf{R7}$ [**means 7 Rand, the South African currency. Ed**.]. He offers to give the money to Sizwe and Thabo if they will share the sausages with him. They agree, and the three boys cut up the sausages and share them equally. When Vuyo gives the $\mathbf{R7}$ to Sizwe and Thabo, how should they divide the money?

(1)	(2)	(3)	(4)	(5)
Sizwe $\mathbf{R6}$, Thabo $\mathbf{R1}$	Sizwe $\mathbf{R5.50}$, Thabo $\mathbf{R1.50}$	Sizwe $\mathbf{R5}$, Thabo $\mathbf{R2}$	Sizwe $\mathbf{R4.50}$, Thabo $\mathbf{R2.50}$	Sizwe $\mathbf{R4}$, Thabo $\mathbf{R3}$

21. A fraction has a four-digit numerator and a five-digit denominator and simplifies to exactly $\frac{1}{2}$. The nine digits are all different. Which of the following could be the numerator of the fraction?

(1) $\mathbf{5314}$ (2) $\mathbf{6729}$ (3) $\mathbf{7341}$ (4) $\mathbf{7629}$ (5) $\mathbf{8359}$

22. If A, B, C, D and E are five points in the same plane, with $AB = 21$, $BC = 17$, $CD = 14$, $DE = 67$ and $EA = 15$, then AD is equal to

(1) 52 (2) 63 (3) 73 (4) 43 (5) Not enough information.

23. Four-fifths of the children in a school are boys. Three-quarters of the boys are expelled for misbehaviour, but none of the girls. What fraction of the children remaining are girls?

(1) $\frac{1}{2}$ (2) $\frac{1}{20}$ (3) $\frac{1}{5}$ (4) $\frac{1}{10}$ (5) $\frac{1}{4}$

24. In an acute-angled triangle each angle is a whole number of degrees and the smallest angle is one-sixth of the largest angle. What is the sum of the two smaller angles?

(1) $96°$ (2) $90°$ (3) $102°$ (4) $84°$ (5) $108°$

25. If $a = 2^{250}$, $b = 3^{200}$ and $c = 5^{150}$, which of the following is true?

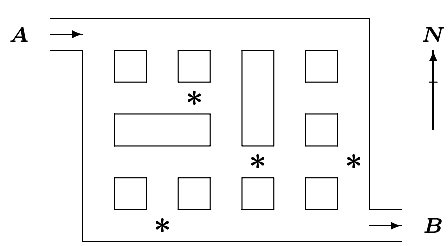
(1) $a > b > c$ (2) $a > c > b$ (3) $c > a > b$ (4) $b > c > a$ (5) $c > b > a$

26. The natural numbers are arranged in the pattern below. In which row does 2001 lie?

Row 1			3					11							19									
Row 2		2		6		10				14				18				22						
Row 3	1		5		9			13				17			21									
Row 4		4		8		12				16				20				24						
Row 5			7					15							23									

(1) Row 1 (2) Row 2 (3) Row 3 (4) Row 4 (5) Row 5

27. Moving East or South all the time, how many routes are there from A to B through at most one star?



(1) 14 (2) 12 (3) 11 (4) 10 (5) 9

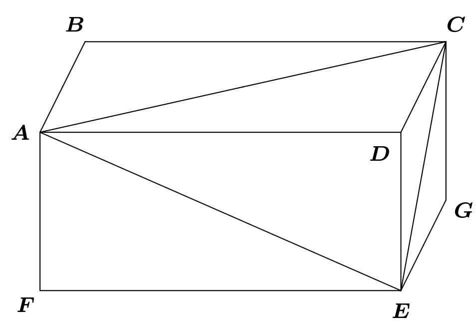
28. In triangle ABC, $AB = 25$, $BC = 23$ and $AC = 24$. A perpendicular BD is dropped onto AC, with D on AC. Then $AD - DC$ is equal to

(1) 4 (2) $3\sqrt{2}$ (3) $1 + 2\sqrt{3}$ (4) $\sqrt{17}$ (5) $1 + \sqrt{2} + \sqrt{3}$

29. A number of unit cubes are put together to make a larger cube and then some of the faces of the larger cube are painted. After the paint dries the larger cube is taken apart. It is found that 45 small cubes have no paint on any face. How many faces of the large cube were painted?

(1) 1 (2) 2 (3) 3 (4) 4 (5) 5

30. A rectangular prism has a tetrahedron $ACED$ cut out of it. The ratio of the volume of the tetrahedron to the volume of the prism is



(1) $\frac{1}{3}$ (2) $\frac{1}{4}$ (3) $\frac{1}{12}$ (4) $\frac{1}{8}$ (5) $\frac{1}{6}$

Next is the Interprovincial Mathematics Olympiad. This is written by teams of ten. 30 minutes are allowed, and no calculators. Each correct answer receives 100 points.

INTERPROVINCIAL MATHEMATICS OLYMPIAD: 2001
TEAM PAPER: SENIORS

S1. Find the largest integer which cannot be expressed in the form $7a + 11b + 13c$, where a, b and c are integers, with $a \geq 0$, $b \geq 0$ and $c \geq 0$.

S2. The 5–digit number $32..1..$ is divisible by 156. What is the number?

S3. Eight boxes, each in a unit cube, are packed in a $2 \times 2 \times 2$ crate, open at the top. The boxes are taken out one by one. In how many ways can this be done? (Remember that a box in the bottom layer can only be removed after the box above it has been removed.)

S4. How many integers between 1 and 1000 cannot be expressed as the difference between the squares of two integers?

S5. Find the smallest positive integer which has a factor ending in 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

S6. A circle is inscribed in quadrilateral $ABCD$. The sides BC and DA have the same lengths, and the sides AB and CD are parallel, with lengths 9 and 16, respectively. What is the radius of the circle?

S7. For how many integers n between 1 and 2002 is the improper fraction

$$\frac{n^2+4}{n+5}$$

NOT in lowest terms?

S8. Solve the inequality $\log_{\sqrt{3}}(2-x)+4\log_9(6-x)>2$.

S9. $ABCD$ is a 2×2 square and E and F are the mid-points of AB and BC, respectively. If AF intersects ED and BD at G and H, respectively, what is the area of quadrilateral $BEGH$?

S10. Points A, B and C lie on a circle. The line AP is perpendicular to BC, with P on BC. If $AP=6$, $BP=4$ and $CP=17$, find the radius of the circle.

Next we give the solutions to the contest presented in [2001 : 315]. First is the 2000 National Bank Junior Mathematics Competition.

1. In this problem, we will be placing various arrangements of 10c and 20c coins on the nine squares of a 3×3 grid. Exactly one coin will be placed in each of the nine squares. The grid has four 2×2 subsquares each containing a corner, the centre, and the two squares adjacent to these.

(a) Find an arrangement where the totals of the four 2×2 subsquares are 40c, 60c, 60c and 70c in any order.

Solution

10	10	20
10	10	20
20	20	20

.

(b) Find an arrangement where the totals of the four 2×2 subsquares are 50c, 60c, 70c and 80c in any order.

Solution

10	10	10
10	20	20
20	20	20

.

(c) What is the maximum amount of money which can be placed on the grid so that each of the 2×2 subsquares contains exactly 50c?

Solution. The maximum amount is $1.30

20	10	20
10	10	10
20	10	20

.

(d) What is the minimum amount of money which can be placed on the grid so that the average amount of money in each of the 2×2 subsquares is exactly 60c?

Solution. The minimum amount is $1.20

10	20	10
10	20	10
10	20	10

.

2. (*Note: In this question an "equal division" is one where the total* ***weight*** *of the two parts is the same.*)

(a) Belinda and Charles are burglars. Among the loot from their latest caper is a set of 12 gold weights of 1g, 2g, 3g, and so on, through to 12g. Can they divide the weights equally between them? If so, explain how they can do it; if not, why not?

Solution. **Yes, it can be done**. There are many possible solutions, for example Belinda gets 1g, 3g, 5g, 6g, 7g, 8g, 9g and Charles gets the rest.

(b) When Belinda and Charles take the remainder of the loot to Freddy the Fence, he demands the 12g weight as his payment. Can Belinda and Charles divide the remaining 11 weights equally between them? If so, explain how they can do it; if not, why not?

Solution. **Yes, it can be done**. There are many possible solutions, for example Belinda gets 1g, 3g, 5g, 7g, 8g, 9g and Charles gets the rest.

(c) Belinda and Charles also have a set of 150 silver weights of 1g, 2g, 3g, and so on, through to 150g. Can they divide these weights equally between them? If so, explain how they can do it; if not, why not?

Solution. **No, it cannot be done**. There are 75 even weights and 75 odd weights so that the total weight is odd. Thus, they cannot split it up evenly .

3. Humankind was recently contacted by three alien races: the Kweens, the Ozdaks, and the Merkuns. Little is known about these races except:

- Kweens always speak the truth.
- Ozdaks always lie.
- In any group of aliens a Merkun will never speak first. When it does speak, it tells the truth if the previous statement was a lie, and lies if the previous statement was truthful.

Although the aliens can readily tell one another apart, of course to humans all aliens look the same.

A high-level delegation of three aliens has been sent to Earth to negotiate our fate. Among them is at least one Kween. On arrival they make the following statements (in order):

Statement A (First Alien): The second alien is a Merkun.

Statement B (Second Alien): The third alien is not a Merkun.

Statement C (Third Alien): The first alien is a Merkun.

Which alien or aliens can you be certain are Kween?

Solution.

The first alien cannot be a Merkun, so that statement C is a lie. If the third alien is a Merkun (who lies), then statement B must be true, but this is impossible. Thus, the third alien must be an Ozdak. If the first alien is a Kween, then the second is a Merkun who lies, but this is impossible, so that the first alien is an Ozdak. Now the second alien cannot be a Merkun. If the second alien is an Ozdak then the third is a Merkun, which is impossible. Thus, the second alien must be a Kween.

Only the second alien is a Kween.

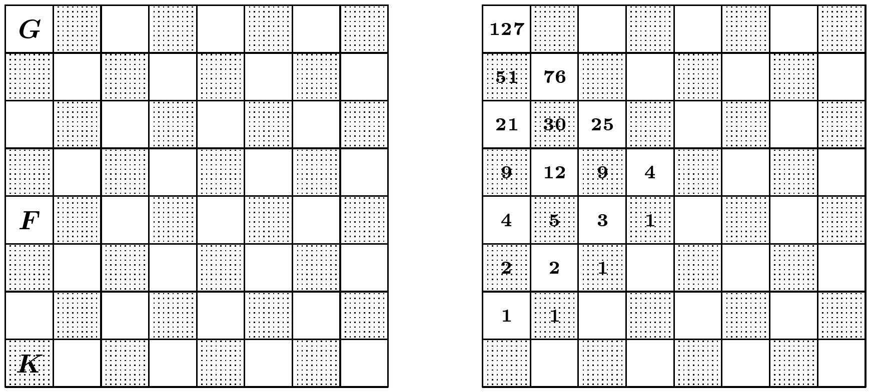
4. A chessboard is an 8×8 grid of squares. One of the chess pieces, the king, moves one square at a time in any direction, including diagonally.

(a) A king stands on the lower left corner of a chessboard (marked **K**). It has to reach the square marked **F** in exactly **3** moves. Show that the king can do this in exactly *four* different ways.

Solution. By counting paths as in the diagram to the solution to part (b).

(b) Assume that the king is placed back on the bottom left corner. In how many ways can it reach the upper left corner (marked **G**) in exactly *seven* moves?

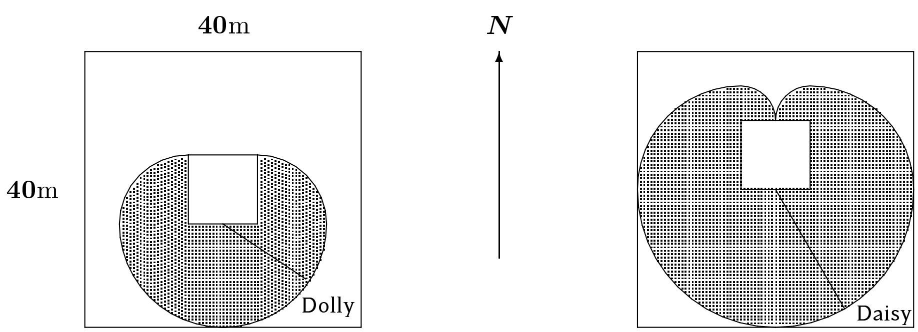
Solution. There are 127 possible paths (see diagram).



5. (*Note: For this question answers containing expressions such as* $\frac{4\pi}{13}$ *are acceptable.)*

(a) The Jones family lives in a perfectly square house, 10m by 10m, which is placed exactly in the middle of a 40m by 40m section, entirely covered (except for the house) in grass. They keep the family pet, Dolly the sheep, tethered to the middle of one side of the house on a 15m rope. What is the area of the part of the lawn (in m^2) in which Dolly is able to graze? (See shaded area.)

Solution. The area is made up of a semicircle of radius 15m and two quarter circles of radius 10m. Thus, $\mathbf{Area} = \frac{1}{2}\pi 15^2 + 2 \times \frac{1}{4}\pi 10^2 = \frac{325\pi}{2}$.



The Jones

The Smiths

(b) The Jones' neighbours, the Smiths, have an identical section to the Jones but their house is located 5m to the North of the centre. Their pet sheep, Daisy, is tethered to the middle of the southern side of the house on a 20m rope. What is the area of the part of the lawn (in m^2) in which Daisy is able to graze?

Solution. The area that Daisy can graze is given above. It is made up of one semicircle and four quarter circles. Thus, $\text{Area} = \frac{1}{2}\pi 20^2 + 2 \times \frac{1}{4}\pi 15^2 + 2 \times \frac{1}{4}\pi 5^2 = 325\pi$.

Finally, we give the solutions to the 2001 BC Colleges Senior High School Mathematics Contest, part A [2001 : 318].

1. The difference of squares $x^2 - y^2$ factors into $(x-y)(x+y)$. Since x and y are positive integers, we know that $x > y$. Thus, $2001 = (x-y)(x+y)$ for four different sets of integers x and y. This requires that we determine how 2001 factors. With a little effort we see that $2001 = 3 \times 23 \times 29$. Thus, the factorizations of 2001 are 1×2001, 3×667, 23×87, and 29×69. For $2001 = a \times b$ with $a < b$, we have $x - y = a$ and $x + y = b$, which means that $x = \frac{1}{2}(a+b)$. Thus, for our four factorizations of 2001 we have $x = \frac{1}{2}(1 + 2001) = 1001$, $\frac{1}{2}(3 + 667) = 335$, $\frac{1}{2}(23 + 87) = 55$, and $\frac{1}{2}(29 + 69) = 49$, respectively. Therefore, the sum of the these four values is $1001 + 335 + 55 + 49 = 1440$. The answer is **d**

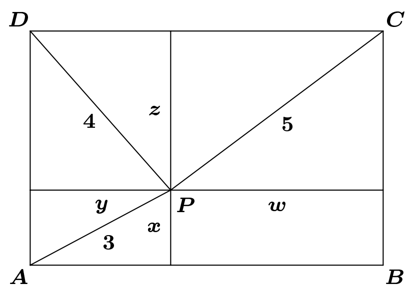
2. Let x and y be the number of pears and peaches respectively that Antonino purchases. Then (in cents) he spends $18x + 33y = 2001$, which simplifies to $6x + 11y = 667$ or $6x = 667 - 11y$. Clearly the maximum number of fruits he could buy occurs when he maximizes the number of peaches (since they are cheaper), which means he should buy as few pears as possible. Thus, let us try successive small values of y, starting at $y = 0$ to determine when $667 - 11y$ is first a multiple of 6. For $y = 0$, 1, 2, 3, 4, and 5 we get $667 - 11y = 667$, 656, 645, 634, 623, and 612. It is easy to check that 612 is the first of these which is a multiple of 6. Thus, $y = 5$ and $x = 612/6 = 102$ Therefore, $x + y = 107$. The answer is **b**

3. Set $x = \sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$. Then

$$\begin{aligned} x^2 &= 3 + 2\sqrt{2} - 2\sqrt{\left(3 + 2\sqrt{2}\right)\left(3 - 2\sqrt{2}\right)} + 3 - 2\sqrt{2} \\ &= 6 - 2\sqrt{3^2 - (2\sqrt{2})^2} \;=\; 6 - 2\sqrt{9 - 8} \;=\; 6 - 2 \;=\; 4\,. \end{aligned}$$

Thus, $x = \pm 2$. However, it is clear from the definition of x that it is positive, since $3 + 2\sqrt{2} > 3 - 2\sqrt{2}$. Therefore, $x = 2$. The answer is **b**

4. Draw lines through P parallel to the sides of the rectangle $ABCD$, cutting off lengths x, y, z, w, as shown in the diagram.

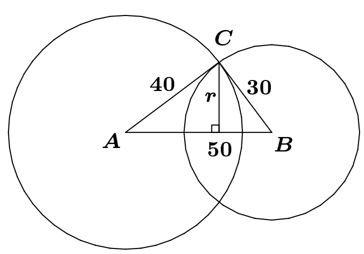


Then from the Theorem of Pythagoras we have

$$\begin{aligned} x^2 + y^2 &= 9\,, & (1)\\ y^2 + z^2 &= 16\,, & (2)\\ z^2 + w^2 &= 25\,. & (3) \end{aligned}$$

If we now subtract (2) from the sum of (1) and (3) we get: $x^2 + w^2 = 18$. But we also have (from the Theorem of Pythagoras): $x^2 + w^2 = \overline{PB}^2$, whence, PB has length $\sqrt{18} = 3\sqrt{2}$. The answer is **b**

5. Consider a cross-section through the centres of the two spheres as shown in the diagram below. Let A be the centre of the sphere of radius 40mm, and let B be the centre of the sphere of radius 30mm. Let C be one of the points in this cross-section which lie where the two spheres join. Since the distance AB is 50mm, we see by the Theorem of Pythagoras that $\triangle ABC$ is right angled with the right angle at point C. The altitude of this triangle is clearly the radius of the circle of intersection of the two bubbles. Let us denote this altitude by r.



Then the area (in mm^2) of $\triangle ABC$ can be computed in 2 different ways:

$$A = \frac{1}{2} \cdot 30 \cdot 40 = \frac{1}{2} \cdot 50 \cdot r\,,$$

from which we see that $r = 24$mm. Thus, the diameter of the circle of intersection of the spheres is 48mm. The answer is **b**

6. If we let those people in line possessing only a toonie be denoted by T, and those possessing a loonie be denoted by L, then our problem can be translated to: what is the probability of a random list of four Ls and four Ts having the property that in moving from the beginning of the list to the end of the list we will have always encountered at least as many Ls as Ts. To begin we will first determine the total number of possible random orderings of four Ls and four Ts. Clearly there are eight positions in the list, four of which must be set aside for L, with the remainder having T. This means there are in total $\binom{8}{4} = 70$ such random orderings of four Ls and four Ts. Now let us try to determine the number of such orderings satisfying the additional condition that in moving from the beginning of the list to the end of the list we always encounter at least as many Ls as Ts. Let us examine the first four positions in the list. We note that there must be at least two Ls in these first four positions. We also note that however many Ls there are among the first four positions, there are the same number of Ts among the last four positions of the list.

Case (i): there are four Ls among the first four positions. In this case there is only one possibility, namely $LLLLTTTT$.

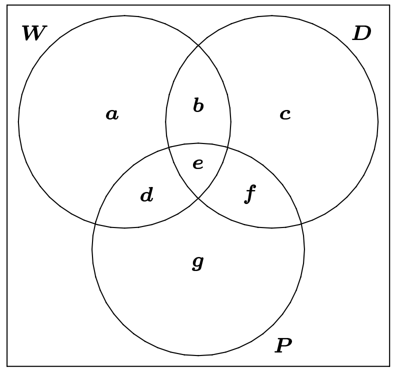
Case (ii): there are three Ls among the first four positions. Since the first position must be L, there are three places where one can put the T that belongs to the first four positions. Thus, there are three possible arrangements for the first four positions. But by symmetry, there are also three possible arrangements for the last four positions, and the first four positions and the last four positions can be arranged independently, for a total of $3 \times 3 = 9$ possibilities for case (ii).

Case (iii): there are two Ls among the first four positions. Again the first position must be L. It is easy to see that there are only two possible arrangements among the first four positions, namely $LLTT$ and $LTLT$. By symmetry, we have the same number of possibilities for the last four positions, for a total of $2 \times 2 = 4$ arrangements for case (iii).

Thus, we have a total of $1 + 9 + 4 = 14$ acceptable arrangements, and the probability we seek is $14/70 = 1/5$. The answer is **d**

7. Consider the diagram below, where the three circles represent the applicants with design skills (D), writing skills (W), and programming skills (P). We have used the letters a through g to represent the various subsets of these people having different combinations (or lack) of skills. We are interested in the value of e.

Since 80% of the 45 applicants have at least one of the desired skills, there are 36 such applicants. From the remaining information in the problem statement we conclude that



$$\begin{aligned}
a+b+c+d+e+f+g &= 36\\
b+c\quad +e+f &= 20\\
a+b\quad +d+e &= 25\\
d+e+f+g &= 21\\
b\quad +e &= 12\\
d+e &= 14\\
e+f &= 11
\end{aligned}$$

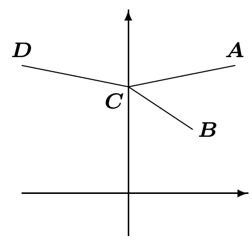
Adding the second, third, and fourth equations above and subtracting the first we get $b+d+2e+f=30$, while adding the last three equations together yields $b+d+3e+f=37$. Comparing these we see that $e=7$. This is all we need to answer the question. However, the interested reader may be curious to find all the remaining values as well; therefore, we continue. With this value of e we can use the last three equations displayed above to determine $b=5$, $d=7$, and $f=4$. With these values we can use the second, third, and fourth equations displayed above to determine $c=4$, $a=6$, and $g=3$. One can simply check that the first equation is satisfied for these values. The answer is **b**

8. Let us label the critical equation:

$$a \circledS (b \circledS c) \;=\; (a \circledS b) \,Ⓛ\, (a \circledS c)\,, \tag{1}$$

where we are assuming that a, b, c are three distinct numbers. Clearly, the left hand side of this expression is always the smallest of the three values a, b, c. If the smallest of the three values is b, then the left hand side of (1) is b, while the right hand side simplifies to $b\,Ⓛ\,(a \circledS c)$, which is definitely a or c; thus, b cannot be the smallest of the three values. Similarly, c cannot be the smallest of the three values. This means that a is the smallest values. This eliminates all but choice (a) and choice (e) from the set of possible answers. Now, if we examine (1) with a the smallest value, then both sides resolve to a, and we have (1) holding true. This describes choice (a). We note that choice (e) imposes a further restriction, namely $b < c$, which is unnecessary. We are asked to determine which *must* hold. Thus, our solution is simply that a must be the smallest of the three values. The answer is **a**

9. Consider also the point D with coordinates $(-7, 4)$ (see diagram below).



Clearly $\overline{AC} = \overline{DC}$. Thus, we must find k which minimizes the sum $\overline{DC} + \overline{BC}$. This sum is obviously minimized when B, C, and D are collinear (that is, when they lie on one line). This occurs when the slope of BC is equal to the slope of BD. The slope of BD is $-\frac{3}{10}$ and the slope of BC is $(1-k)/3$. Setting these equal yields $1 - k = -\frac{9}{10}$, or $k = 1.9$. The answer is **c**

10. Observe that the unshaded portion of the quarter circle is also $\frac{1}{2}$ of its area. Let us then compute the area of the unshaded regions. We will solve the more general problem using a radius of r units. Clearly, the area of triangle CBX is $\frac{1}{2}xr$. Now drop a perpendicular from A to the line CD meeting it at E. Since $\angle ACD = 30°$, we see that $\overline{AE} = \frac{1}{2}r$, and by the Theorem of Pythagoras we then get $\overline{CE} = \frac{\sqrt{3}}{2}r$. Thus, the area of triangle AXE is $\frac{1}{2}\left(\frac{\sqrt{3}}{2}r - x\right)\cdot\frac{1}{2}r$. The remaining unshaded region is the curved piece AED. This is obviously the difference between the circular sector ACD and triangle ACE. The sector ACD has area $\frac{1}{12}\pi r^2$, since it is one twelfth part of a circle of radius r. The triangle ACE has area $\frac{1}{2}\cdot\frac{\sqrt{3}}{2}r\cdot\frac{1}{2}r$. Putting all the pieces together we see that the area of the unshaded part is:

$$A = \frac{xr}{2} + \left(\frac{\sqrt{3}r^2}{8} - \frac{xr}{4}\right) + \left(\frac{\pi r^2}{12} - \frac{\sqrt{3}r^2}{8}\right) = \frac{xr}{4} + \frac{\pi r^2}{12}.$$

But we are given that A is one half the area of the quarter circle; that is, one half of $\frac{1}{4}\pi r^2$. Thus, we have

$$\frac{xr}{4} + \frac{\pi r^2}{12} = \frac{\pi r^2}{8}, \quad \frac{xr}{4} = \frac{\pi r^2}{24}, \quad x = \frac{\pi r}{6}.$$

Since $r = 1$ we have $x = \pi/6$. The answer is **c**

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 September 2002**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in LaTeX*). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Starting with this issue, we will be giving each problem twice, once in each of the official languages of Canada, English and French. In issues 1, 3, 5 and 7, English will precede French, and in issues 2, 4, 6 and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

2701★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Do there exists infinitely many triplets $(n, n+1, n+2)$ of adjacent natural numbers such that all of them are sums of two positive perfect squares?

(Examples are $(232, 233, 234)$, $(520, 521, 522)$ and $(808, 809, 810)$.)

Compare the 2000 Putnam problem A2 [2001 : 3]

. . . .

Existe-t-il une infinité de triplets $(n, n+1, n+2)$ de nombres naturels consécutifs qui soient tous la somme de deux carrés parfaits non nuls ?

(Exemples : $(232, 233, 234)$, $(520, 521, 522)$ et $(808, 809, 810)$.)

Voir le Putnam 2000, problème A2 [2001 : 3]

2702. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let λ be an arbitrary real number. Show that

$$\left(\frac{s}{r}\right)^{2\lambda} s^2 \ \geq\ 3^{3\lambda+1}\left(s^2 - 8Rr - 2r^2\right) \ ,$$

where R, r and s are the circumradius, the inradius and the semi-perimeter of a triangle, respectively.

Determine the cases of equality.

. . . .

Soit λ un nombre réel arbitraire. Montrer que

$$\left(\frac{s}{r}\right)^{2\lambda} s^2 \ \geq\ 3^{3\lambda+1}\left(s^2 - 8Rr - 2r^2\right) \ ,$$

où R, r et s sont respectivement le rayon du cercle circonscrit, le rayon du cercle inscrit et le demi-périmètre d'un triangle.

Trouver les cas d'égalité.

2703. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that a, b, c, d, u, v $\in$ $\mathbb{R}$ and $a + c \neq 0$. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(ax+b) + f(cx+d) = ux + v$.

. . . .

Soit a, b, c, d, u, $v \in \mathbb{R}$ et $a + c \neq 0$. Trouver toutes les fonctions continues $f : \mathbb{R} \to \mathbb{R}$ pour lesquelles $f(ax+b) + f(cx+d) = ux + v$.

2704. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$R - 2r \ \geq\ \frac{1}{12}\left(\sum_{\text{cyclic}} \sqrt{2(b^2+c^2) - a^2} - \frac{s^2 + r^2 + bRr}{R}\right) \ \geq\ 0\,,$$

where a, b and c are the sides of a triangle, and R, r and s are the circumradius, the inradius and the semi-perimeter of a triangle, respectively.

. . . .

Montrer que

$$R - 2r \ \geq\ \frac{1}{12}\left(\sum_{\text{cyclique}} \sqrt{2(b^2+c^2) - a^2} - \frac{s^2 + r^2 + bRr}{R}\right) \ \geq\ 0\,,$$

où a, b et c sont les côtés d'un triangle, et R, r et s sont respectivement le rayon du cercle circonscrit, le rayon du cercle inscrit et le demi-périmètre d'un triangle.

2705. *Proposed by Angel Dorito, Geld, Ontario.*

The interior of a rectangular container is 1 metre wide and 2 metres long, and is filled with water to a depth of $\frac{1}{2}$ metre. A cube of gold is placed flat in the tub, and the water rises to exactly the top of the cube without overflowing.

Find the length of the side of the cube.

L'intérieur d'un bassin rectangulaire mesure $1m$ de largeur et $2m$ de longueur ; il est rempli d'eau jusqu'à une hauteur d'un demi-mètre. Un cube en or est posé au fond du bassin et le niveau d'eau monte jusqu'à coïncider exactement avec la hauteur du cube.

Trouver la longueur de l'arête du cube.

2706. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that Γ_1 and Γ_2 are two circles having at least one point S in common. Take an arbitrary line ℓ through S. This line intersects Γ_k again at P_k (if ℓ is tangent to Γ_k, then $P_k = S$).

Let λ be a (fixed) real number, and let $R_\lambda = \lambda P_1 + (1-\lambda)P_2$.

Determine the locus of R_λ as ℓ varies over all possible lines through S.

. . . .

Soient Γ_1 et Γ_2 deux cercles ayant au moins un point S en commun. Une droite ℓ passant passant S coupe Γ_k en un point P_k (si ℓ est tangente à Γ_k, alors $P_k = S$).

Soit λ un nombre réel fixe, et soit $R_\lambda = \lambda P_1 + (1-\lambda)P_2$.

Trouver le lieu des points R_λ lorsque ℓ parcourt l'ensemble des droites par S.

2707. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let ABC be a triangle and P a point in its plane. The feet of the perpendiculars from P to the lines BC, CA and AB are D, E and F respectively.

Prove that

$$\frac{AB^2 + BC^2 + CA^2}{4} \leq AF^2 + BD^2 + CE^2\,,$$

and determine the cases of equality.

. . . .

Soit P un point dans le plan d'un triangle ABC. Soit D, E et F respectivement, les pieds des perpendiculaires menées de P sur les droites BC, CA et AB.

Montrer que

$$\frac{AB^2 + BC^2 + CA^2}{4} \leq AF^2 + BD^2 + CE^2\,,$$

et déterminer les cas d'égalité.

2708. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that

1. O is the intersection of diagonals AC and BD of quadrilateral $ABCD$,

2. $OA < OC$ and $OD < OB$,
3. M and N are the mid-points of AC and BD, respectively,
4. MN meets AB and CD at E and F, respectively, and
5. P is the intersection of BF and CE.

Prove that OP bisects the line segment EF.

....

On suppose que

1. O est l'intersection des diagonales AC et BD d'un quadrilatère $ABCD$,
2. $OA < OC$ et $OD < OB$,
3. M et N sont respectivement les points milieu de AC et BD,
4. MN coupe AB et CD en E et F, respectivement, et
5. P est l'intersection de BF avec CE.

Montrer que OP coupe le segment EF en son milieu.

2709. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that

1. P is an interior point of $\triangle ABC$,
2. AP, BP and CP meet BC, CA and AB at D, E and F, respectively,
3. A' is a point on AD produced beyond D such that $DA' : AD = \kappa : 1$, where κ is a fixed positive number,
4. B' is a point on BE produced beyond E such that $EB' : BE = \kappa : 1$, and
5. C' is a point on CF produced beyond F such that $FC' : CF = \kappa : 1$.

Prove that $[A'B'C'] \leq \frac{(3\kappa+1)^2}{4}[ABC]$, where $[PQR]$ denotes the area of $\triangle PQR$.

....

On suppose que

1. P est un point intérieur du triangle ABC,
2. AP, BP et CP coupent BC, CA et AB en D, E et F, respectivement,
3. A' est un point sur AD situé au-delà de D de sorte que $DA' : AD = \kappa : 1$, où κ est un nombre positif fixe,
4. B' est un point sur BE situé au-delà de E de sorte que $EB' : BE = \kappa : 1$, et
5. C' est un point sur CF situé au-delà de F de sorte que $FC' : CF = \kappa : 1$.

Montrer que $[A'B'C'] \leq \frac{(3\kappa+1)^2}{4}[ABC]$, où $[PQR]$ désigne l'aire du $\triangle\ PQR$.

2710. *Proposed by Jaroslav Švrček, Palacký University, Olomouc, Czech Republic.*

Determine the point P on the semicircle Γ, constructed externally over the side AB of the square $ABCD$, such that $AP^2 + CP^2$ is maximal.

. . . .

Sur le demi-cercle Γ construit sur le côté AB, à l'extérieur du carré $ABCD$, trouver le point P tel que $AP^2 + CP^2$ soit maximal.

2711★. *Proposed by Catherine Shevlin, Wallsend, England.*

Two circles, centres O_1 and O_2, of radii R_1 and R_2 ($R_1 > R_2$), respectively, are externally tangent at P. A common tangent to the two circles, not through P, meets O_1O_2 produced at Q, the circle with centre O_1 at A_1 and the circle with centre O_2 at A_2.

Prove or disprove that there exist simultaneously integer triangles QO_1A_1 and QO_2A_2.

. . . .

Deux cercles, de centre O_1 et O_2, de rayon respectif R_1 et R_2 ($R_1 > R_2$), sont extérieurement tangents en P. Une tangente commune aux deux cercles et ne passant par P coupe la droite O_1O_2 en Q et rencontre le cercle de centre O_1 en A_1 et celui de centre O_2 en A_2.

Montrer si oui on non il existe simultanément deux triangles QO_1A_1 et QO_2A_2 dont les côtés sont des entiers.

2712. *Proposed by Antreas P. Hatzipolakis, Athens, Greece; and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given $\triangle ABC$, let Y and Z be the feet of the altitudes from B and C. Suppose that the bisectors of $\angle BYC$ and $\angle BZC$ meet at X. Prove that $\triangle BXC$ is isosceles.

. . . .

On donne un triangle ABC et soit Y et Z les pieds des perpendiculaires abaissées des sommets B et C. Soit X le point d'intersection des bissectrices de $\angle BYC$ et $\angle BZC$. Montrer que le triangle BXC est isocèle.

Professor Jordi Dou

We always like to recognise milestones. We have just discovered that we missed Professor Jordi Dou's ninetieth birthday last year. It will be nice to have some problems dedicated to Jordi this year. Please send proposals post haste to the Editor-in-Chief.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

My, the gremlins have been at work! We apologise to Michel Bataille, Rouen, France, for omitting his name as a solver of problem 2571; to David Loeffler, student, Trinity College, Cambridge, UK, for omitting his name as a solver of problem 2563; and to Walther Janous, Ursulinengymnasium, Innsbruck, Austria, for omitting his name as a solver of problems 2495, 2559, 2569 and 2572.

2572. [2000 : 374, 2001 : 473] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, c be positive real numbers. Prove that

$$a^b b^c c^a \;\leq\; \left(\frac{a+b+c}{3}\right)^{a+b+c} .$$

[Compare problem **2394** [1999 : 524], note by V.N. Murty on the generalization.]

III. *Remarks by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Howard's question (in the editorial remarks given after the solutions [2001 : 473]) unfortunately has a negative answer. Indeed, let $n = 5$ and note that we must have

$$(a+b+c+d+e)^2 - 5(a\cdot b + b\cdot c + c\cdot d + d\cdot e + e\cdot a) \;\geq\; 0 .$$

But the following self-explanatory substitutions of $a, \ldots, e$, yield the claimed contradiction; that is

(i) $(3+4+2+1+2)^2 - 5(3\cdot 4 + 4\cdot 2 + 2\cdot 1 + 1\cdot 2 + 2\cdot 3) \;=\; -6 \;<\; 0$;

(ii) $(10+4+2+1+2)^2 - 5(10\cdot 4 + 4\cdot 2 + 2\cdot 1 + 1\cdot 2 + 2\cdot 10) \;=\; 1 \;>\; 0$.

Next, let us proceed more systematically. We shall prove the following interesting result.

Theorem. Suppose that $n \geq 4$. Then the best constant λ_n such that the inequality

$$(x_1 + \cdots + x_n)^2 \;\geq\; \lambda_n \cdot (x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1) \qquad (1)$$

holds true whenever $x_1, x_2, \ldots, x_n \geq 0$ occurs when $\lambda_n \leq 4$.

Proof. Putting $x_1 = x_2 = 1$ and $x_3 = \cdots = x_n = t$ in (1), we obtain

$$\left(2 + (n-2)t\right)^2 \;\geq\; \lambda_n \left(1 + 2t + (n-3)t^2\right) .$$

Thus, letting $t \to 0$, it follows that $\lambda_n \leq 4$.

In order to show that $\lambda_n = 4$, we proceed by induction ; that is, we will show

$$(x_1 + \cdots + x_n)^2 \;\ge\; 4(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1)\,. \qquad (2)$$

If all the x_i's are equal to zero, the inequality is clear. If not, we may put (due to homogenuity) $x_1 + \cdots + x_n = 1$, and (2) then reads

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \;\le\; \frac{1}{4}\,. \qquad (3)$$

The case of $n = 4$ was settled already by Howard.

Therefore, suppose that (3) is valid up to n. Now, suppose that $x_1 + \cdots + x_n + x_{n+1} = 1$. Because the LHS of (3) is cyclically homogenous, we may let $x_{n+1} = \max\{x_1, \ldots, x_{n+1}\}$.

By the induction hypothesis, we have (on "gluing together" x_1 and x_2 !)

$$(x_1 + x_2)x_3 + x_2x_3 + \cdots + x_nx_{n+1} + x_{n+1}(x_1 + x_2) \;\le\; \frac{1}{4}\,;$$

that is,

$$x_2x_3 + x_2x_3 + \cdots + x_nx_{n+1} + x_{n+1}x_1 + x_1x_3 + x_{n+1}x_2 \;\le\; \frac{1}{4}\,.$$

But $x_1x_3 + x_{n+1}x_2 \ge x_{n+1}x_2 \ge x_1x_2$, and the proof is complete.

2601. [2001 : 48] *Proposed by Michel Bataille, Rouen, France.*

Sequences $\{u_n\}$ and $\{v_n\}$ are defined by $u_0 = 4$, $u_1 = 2$, and for all integers $n \ge 0$, $u_{n+2} = 8t^2u_{n+1} + \left(t - \frac{1}{2}\right)u_n$, $v_n = u_{n+1} - u_n$. For which t is $\{v_n\}$ a non-constant geometric sequence ?

Amalgamated solutions of Christopher J. Bradley, Clifton College, Bristol, UK and David Loeffler, student, Cotham School, Bristol, UK.

Now $v_0 = u_1 - u_0 = 2 - 4 = -2$. Thus, for $\{v_n\}$ to be a non-constant geometric sequence we must have $v_n = -2r^n$ with $r \ne 1$ for $n \ge 0$. Then

$$\begin{aligned} u_{n+1} - u_n &= -2r^n \\ u_n - u_{n-1} &= -2r^{n-1} \\ &\cdots \\ u_1 - u_0 &= -2 \end{aligned}$$

Adding these we get $u_{n+1} = u_0 - 2\left(\dfrac{r^{n+1}-1}{r-1}\right) = \dfrac{4r - 2r^{n+1} - 2}{r-1}$. Substituting this expression into the recurrence and multiplying by $r - 1$, we obtain :

$$4r - 2r^{n+2} - 2 \;=\; 8t^2(4r - 2r^{n+1} - 2) + \left(t - \tfrac{1}{2}\right)(4r - 2r^n - 2)$$

for all $n \geq 0$. This can be rearranged as

$$\begin{aligned} r^n(-2r^2+16t^2r+2t-1) &= 3+32t^2r-16t^2+4rt-2t-6r \\ \text{or} \quad r^n(-2r^2+16t^2r+2t-1) &= (2r-1)(2t+1)(8t-3) \end{aligned}$$

for all $n \geq 0$. We now see that both sides of this equation must be equal to zero, since if the bracketed term on the left were non-zero the left hand side would vary with n while the right would not, a contradiction. Thus we have either $r = \frac{1}{2}$, $t = -\frac{1}{2}$, or $t = \frac{3}{8}$.

If $r = \frac{1}{2}$, the bracketed term on the left is $-\frac{1}{2}+8t^2+2t-1$; this must be zero, giving the solutions $t = \dfrac{\pm\sqrt{13}-1}{8}$.

If $t = -\frac{1}{2}$, the bracketed term becomes $-2r^2+4r-2 = -2(r-1)^2$. To make this zero, we would have to take $r = 1$, contradicting the requirement that $\{v_n\}$ be non-constant. Thus this value of t may be rejected.

If $t = \frac{3}{8}$ the bracketed term is $-2r^2+\frac{9}{4}r-\frac{1}{4} = -\frac{1}{4}(8r-1)(r-1)$. Thus this is also a valid solution with $r = \frac{1}{8}$.

Thus, the possible values of t are $\frac{3}{8}$ and $(\pm\sqrt{13}-1)/8$.

Also solved by AUSTRIAN IMO TEAM 2001; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; VINAYAK GANESHAN, student, University of Waterloo, Waterloo, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong, China; HENRY LIU, student, University of Memphis, Memphis, TN, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. There were four incorrect solutions.

2602★. [2001 : 48] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For integers a, b and c, let $Q(a,b,c)$ be the set of all numbers an^2+bn+c, where $n \in \mathbb{N} = \{0, 1, \ldots, \}$.

(a) Show that $Q(6,3,-2)$ is square-free.

(b) Determine other infinite sets $Q(a,b,c)$ with the same property.

Solution by Manuel Benito and Emilio Fernández, I.B. Praxedes Mateo Sagasta, Logroño, Spain.

(a) In order to show that the equation $6n^2+3n-2 = u^2$ has no solution on the non-negative integers n and u, let us multiply by 24 and rewrite it then as $(12n+3)^2-57 = 24u^2$. By putting $x = 12n+3$ and $y = 2u$, the equation for x and y is

$$x^2-6y^2 = 57\,. \tag{1}$$

The fundamental (that is, minimal over positive integers) solution of Pell's equation $x^2-6y^2 = 1$ is $x_1 = 5$, $y_1 = 2$.

Let us apply Theorem 108 on page 205 of Nagell, T., *Introduction to Number Theory*, Chelsea, 1981 :

If $u + v\sqrt{D}$ is the fundamental solution of any class of the equation $u^2 - Dv^2 = N$ and if $x_1 + y_1\sqrt{D}$ is the fundamental solution of equation $x^2 - Dy^2 = 1$, we have the inequalities

$$0 \;\le\; v \;\le\; \frac{y_1}{\sqrt{2(x_1+1)}}\sqrt{N} \qquad (2)$$

and

$$0 \;<\; |u| \;\le\; \sqrt{\frac{1}{2}(x_1+1)N}\,.$$

From (2) we get, for our equation (1), the inequality

$$0 \;\le\; y \;\le\; \frac{2}{\sqrt{2\cdot 6}}\sqrt{6} \;=\; \sqrt{2}\,.$$

But (1) has no solution when $y = 0$ or 1. Thus, it has no solution at all.

(b) Let $\sqrt{D} = [a_0; \overline{a_1, a_2, \ldots, a_n}]$ be the continued fraction development for $\sqrt{D}$; it is known (see, for example, Baker, A., *A Concise Introduction in the Theory of Numbers*, Cambridge, 1984, p. 122) that if n is even, the equation $x^2 - Dy^2 = -1$ has no solution in integers, so that $Q(D, 0, -1)$ shall be square–free in such cases.

Examples.

1. $Q(15, 0, -1)$ is square–free, because $\sqrt{15} = [3; \overline{1, 6}]$.
2. For α, $\beta \in \mathbb{N}$, let $D = \alpha^2\beta^2 + 2\alpha$; since $\sqrt{D} = [\alpha\beta; \overline{\beta, 2\alpha\beta}]$, we have that $Q(\alpha^2\beta^2 + 2\alpha, 0, -1)$ is square–free.
3. For α, $\beta \in \mathbb{N}$, let $D = \alpha^2\beta^2 + \alpha$; since $\sqrt{D} = [\alpha\beta; \overline{2\beta, 2\alpha\beta}]$, we have that $Q(\alpha^2\beta^2 + \alpha, 0, -1)$ is also square–free.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK part (a) only; *RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JOEL SCHLOSBERG, student, New York University, NY, USA.*

2603. [2001 : 48] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that A, B and C are the angles of a triangle. Prove that

$$\sin A + \sin B + \sin C \;\le\; \sqrt{\frac{15}{4} + \cos(A-B) + \cos(B-C) + \cos(C-A)}\,.$$

Solution by Henry Liu, University of Memphis, Tennessee, USA.

Since $\sin A + \sin B + \sin C > 0$ and

$$\frac{15}{4} + \cos(A-B) + \cos(B-C) + \cos(C-A) \;>\; 0\,,$$

it suffices to show that

$$(\sin A + \sin B + \sin C)^2 \;\le\; \frac{15}{4} + \cos(A-B) + \cos(B-C) + \cos(C-A)\,.$$

We have

$$
\begin{aligned}
\left(\sum_{\text{cyclic}} \sin A\right)^2 &\le \frac{15}{4} + \sum_{\text{cyclic}} \cos(A-B) \\
\iff \sum_{\text{cyclic}} \sin^2 A + 2\sum_{\text{cyclic}} \sin A \sin B & \\
&\le \frac{15}{4} + \sum_{\text{cyclic}} (\cos A\cos B + \sin A \sin B) \\
\iff 3 - \sum_{\text{cyclic}} \cos^2 A &\le \frac{15}{4} + \sum_{\text{cyclic}} (\cos A\cos B - \sin A \sin B) \\
\iff \sum_{\text{cyclic}} \cos(A+B) + \sum_{\text{cyclic}} \cos^2 A + \frac{3}{4} &\ge 0 \\
\iff \sum_{\text{cyclic}} \cos(\pi - A) + \sum_{\text{cyclic}} \cos^2 A + \frac{3}{4} &\ge 0 \\
\iff -\sum_{\text{cyclic}} \cos A + \sum_{\text{cyclic}} \cos^2 A + \frac{3}{4} &\ge 0 \\
\iff \sum_{\text{cyclic}} \left(\cos A - \frac{1}{2}\right)^2 &\ge 0\,.
\end{aligned}
$$

Clearly, the last inequality is true, so that the initial inequality is also true. Equality holds when $\cos A = \cos B = \cos C = \frac{1}{2}$; that is, when $A = B = C = 60^\circ$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO-TEAM 2001; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SCOTT BROWN, Auburn University at Montgomery, Montgomery, Alabama, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; VINAYAK GANESHAN, student, University of Waterloo, Waterloo, Ontario, Canada; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; KEE-WAI LAU, Hong Kong; DAVID LOEFFLER, student, Cotham School, Bristol, UK; RÉVAI MATH CLUB, Győr, Hungary; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, student, New York University, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; STEPHEN WEBER, Georg-Cantor-Gymnasium Halle, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incomplete solution submitted.

2604. [2001 : 49] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Determine the upper and lower bounds of $\dfrac{a}{a+b}+\dfrac{b}{b+c}-\dfrac{a}{a+c}$ for all positive real numbers a, b and c.

(b)* Determine the upper and lower bounds (as functions of n) of

$$\sum_{j=1}^{n-1}\frac{x_j}{x_j+x_{j+1}} \;-\; \frac{x_1}{x_1+x_n}$$

for all positive real numbers x_1, x_2, $\ldots$, x_n.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

(a) Let $f(a,b,c)=\dfrac{a}{a+b}+\dfrac{b}{b+c}-\dfrac{a}{a+c}$. Then

$$f(a,b,c) \;=\; \frac{a^2b+b^2c+c^2a+abc}{a^2b+b^2a+b^2c+c^2b+c^2a+a^2c+2abc}\,.$$

Hence, $0<f(a,b,c)<1$. Now, $f(a,b,c)$ can be made arbitrarily close to 1 by letting $c=\epsilon b$ and $b=\epsilon a$, when ϵ is sufficiently small.

Further, $f(a,b,c)$ can be made arbitrarily close to 0 by letting $a=\epsilon b$ and $b=\epsilon c$, when ϵ is sufficiently small.

Therefore, the greatest lower bound and least upper bound values of $f(a,b,c)$ are 0 and 1, respectively.

(b) Let

$$\begin{aligned} g &= \frac{x_1}{x_1+x_2}+\frac{x_2}{x_2+x_3}+\cdots+\frac{x_{n-1}}{x_{n-1}+x_n}-\frac{x_1}{x_1+x_n} \\ &= f(x_1,x_2,x_3)+f(x_1,x_3,x_4)+f(x_1,x_4,x_5)+\cdots+f(x_1,x_{n-1},x_n)\,, \end{aligned}$$

which has values between 0 and $n-2$.

By letting $x_2=\epsilon x_1$, $x_2=\epsilon x_2$, $\ldots$, we can make g arbitrarily close to $n-2$, when ϵ is sufficiently small.

Similarly, by letting $x_1=\epsilon x_2$, $x_2=\epsilon x_3$, $\ldots$, we can make g arbitrarily close to 0, when ϵ is sufficiently small.

Therefore, the greatest lower bound and least upper bound values of g are 0 and $n-2$, respectively.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; OLEG IVRII, Cummer Valley Middle School, North York, Ontario; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; DAVID LOEFFLER, student, Cotham School, Bristol, UK; RÉVAI MATH CLUB, Győr, Hungary; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany (part (a) only); CHRIS WILDHAGEN, Rotterdam, the Netherlands (2 solutions to part (a) only); LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect and one incomplete solution.

The solution by the RÉVAI MATH CLUB was a very good solution, but was in such detail that it would have required four pages in print! This editor also awarded an $A+$ grade to the solutions of Christopher J. Bradley and of Manuel Benito and Emilio Fernández Moral.

2606. [2001 : 49] *Proposed by K.R.S. Sastry, Bangalore, India.*

A Gergonne cevian connects the vertex of a triangle to the point at which the incircle is tangent to the opposite side.

Determine the unique triangle ABC (up to similarity) in which the Gergonne cevian BE bisects the median AM, and the Gergonne cevian CF bisects the median NB.

I. *Solution independently submitted by Nikolaos Dergiades, Thessaloniki, Greece and by D.J. Smeenk, Zaltbommel, the Netherlands.*

If s is the semiperimeter of triangle ABC then $AE = s - a$. If G is the mid-point of EC then MG is parallel to BE [since M is the mid-point of BC]; moreover, since BE bisects AM, E is the mid-point of AG. Thus,

$$AE \;=\; \frac{1}{3}AC\,, \quad \text{or} \quad s - a \;=\; \frac{1}{3}b\,, \quad \text{or}$$

$$3(b + c - a) \;=\; 2b\,. \qquad (1)$$

Similarly,

$$3(c + a - b) \;=\; 2c\,. \qquad (2)$$

Solving the system (1) and (2) we find that

$$\frac{a}{5} \;=\; \frac{b}{6} \;=\; \frac{c}{3}\,.$$

II. *Solution by Vinayak Ganeshan, student, University of Waterloo.*

Applying Menelaus's theorem to $\triangle AMC$ with BE as transversal $\left[\text{that is, } \frac{AP}{PM} \cdot \frac{MB}{BC} \cdot \frac{CE}{EA} = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{s-c}{s-a} = 1 \quad (P \text{ being where } AM \text{ intersects } BE)\right]$, we get

$$(s - a) \cdot 2 \cdot 1 = (s - c) \cdot 1 \cdot 1\,,$$

which implies that $b = 3a - 3c$. Similarly [using $\triangle BNA$ with transversal CF], $c = 3b - 3a$. Together, these equations give $a : b : c = 5 : 6 : 3$, which solves the problem.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GEOFFREY A. KANDALL, Hamden, CT, USA; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and by the proposer.

2610. [2001 : 50] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Let $\{f_n\}$ be the Fibonacci sequence given by $f_0 = 0$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Prove that, for $n \geq 1$,

$$f_{2n} \mid (f_{3n} + (-1)^n f_n) \; .$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA..

Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Then $a + b = 1$ and $ab = -1$.

Since $a^n + b^n = (a+b)(a^{n-1} + b^{n-1}) - ab(a^{n-2} + b^{n-2})$, we see easily by induction that $a^n + b^n$ is an integer for $n \geq 1$. By Binet's formula,

$$\begin{aligned} f_{3n} + (-1)^n f_n &= \frac{1}{\sqrt{5}}(a^{3n} - b^{3n}) + \frac{1}{\sqrt{5}}(-1)^n(a^n - b^n) \\ &= \frac{1}{\sqrt{5}}\left(a^{3n} - b^{3n} + (ab)^n(a^n - b^n)\right) \\ &= \frac{1}{\sqrt{5}}(a^n + b^n)(a^{2n} - b^{2n}) \;=\; (a^n + b^n)f_{2n}\,, \end{aligned}$$

from which the conclusion follows.

Also solved by the AUSTRIAN IMO-TEAM, 2001; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SCOTT H. BROWN, Auburn University at Montgomery, Montgomery, AL, USA; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; VINAYAK GANESHAN, student, University of Waterloo, Waterloo; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK;DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; REVAÌ MATH CLUB, Györ, Hungary; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, student, New York University, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, KS, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA (*second solution*) *; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

Many solvers proved the statement by showing that $F_{3n} + (-1)^n F_n = L_n F_{2n}$ where L_n is the n^{th} term of the Lucas sequence $\{L_n\}$ defined by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$, and $L_1 = 1$, $L_2 = 3$. Since it is also well known that $L_n = a^n + b^n$, the solution given above yields the same result. However, some solvers only showed that $F_{3n} + (-1)^n F_n = (a^n + b^n)F_{2n}$ and then claimed that the conclusion follows. This logic is clearly flawed since it is not obvious that $a^n + b^n$ is an integer, though it is clear that it must be a rational number. Nonetheless, we are willing to give the benefit of doubt to these solvers since as it turns out, $a^n + b^n$ is *an integer.*