# THE ACADEMY CORNER

## No. 41

Bruce Shawyer

*All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7*

In this issue, we present some solutions to the First Day problems of the 7$^{\text{th}}$ International Mathematics Competition for University Students, held at University College, London, UK, on 26 July and 31 July 2000 [2000 : 385].

## The International Mathematics Competition for University Students

### First Day Problems, 26 July 2000

1. Is it true that if $f : [0,1] \to [0,1]$ is

   (a) monotone increasing

   (b) monotone decreasing

   then there exists an $x \in [0,1]$ for which $f(x) = x$?

*Solution by Michel Bataille, Rouen, France.*
(a) True. Let $A = \{x \in [0,1] : f(x) \geq x\}$. Then $A$ is non-empty (since $0 \in A$) and bounded above by $1$. It follows that $A$ has a least upper bound $a$ satisfying $a \in [0,1]$. We will show that $f(a) = a$.

For all $x \in A$, we have $f(x) \leq f(a)$ (since $x \leq a$ and $f$ is increasing). Hence, $x \leq f(a)$.

Thus, $f(a)$ is an upper bound of $A$, and, consequently, $a \leq f(a)$. (1)

Observing that $f(a) \in [0,1]$, we have that (1) yields $f(a) \leq f(f(a))$, which shows that $f(a) \in A$. Hence, $f(a) \leq a$. This and (1) give $f(a) = a$, as required.

(b) False. Take $f$ defined by

$$f(x) \;=\; \begin{cases} -\frac{x}{2} + 1 & \text{if} \quad x \in [0, \frac{1}{2}) \quad \text{and} \\ -\frac{x}{2} + \frac{1}{2} & \text{if} \quad x \in [\frac{1}{2}, 1] \end{cases}$$

Then $f : [0,1] \to [0,1]$ is monotone decreasing, and the equation $f(x) = x$ has no solution.

2. Let $p(x) = x^5 + x$ and $q(x) = x^5 + x^2$. Find all pairs $(w, z)$ of complex numbers with $w \neq z$ for which $p(w) = p(z)$ and $q(w) = q(z)$.

*Official solution.*

Let $$P(x,y) \;=\; \frac{p(x)-p(y)}{x-y} \;=\; x^4 + x^3y + x^2y^2 + xy^3 + y^4 + 1,$$
and $$Q(x,y) \;=\; \frac{q(x)-q(y)}{x-y} \;=\; x^4 + x^3y + x^2y^2 + xy^3 + y^4 + x + y.$$

We need those pairs which satisfy $P(w,z) = Q(w,z) = 0$.

From $P - Q = 0$, we have $w + z = 1$. Let $c = wz$. A short calculation shows that $c^2 - 3c + 2 = 0$, which has solutions $c = 1$ and $c = 2$.

From the system $w + z = 1$, $wz = c$, we obtain the following pairs:

$$\left(\frac{1 \pm \sqrt{3}i}{2}, \frac{1 \mp \sqrt{3}i}{2}\right) \qquad \text{and} \qquad \left(\frac{1 \pm \sqrt{7}i}{2}, \frac{1 \mp \sqrt{7}i}{2}\right).$$

*Also solved by MICHEL BATAILLE, Rouen, France.*

3. Suppose that $A$ and $B$ are square matrices of the same size with
$$\mathrm{rank}\,(AB - BA) \;=\; 1\,.$$
Show that $(AB - BA)^2 = 0$.

*Solution by Michel Bataille, Rouen, France.*

Let $C$ be a square matrix with $\mathrm{rank}(C) = 1$. Then, the columns of $C$ are of the form $u_1V$, $u_2V$, $\ldots$, $u_nV$, where $V$ is a non-zero column vector and the $u_k$ are non-zero scalars. Let $U$ be the column vector with the $u_k$ as its entries. Then, $C = VU^T$, where $U^T$ denotes the transpose of $U$.

Observe that $U^TV = \mathrm{tr}(C)$ (where $\mathrm{tr}(C)$ is the trace of the matrix $C$). Thus, we have
$$C^2 \;=\; (VU^T)\,(VU^T) \;=\; V\,(U^TV)\,U^T \;=\; (U^TV)\,VU^T \;=\; \mathrm{tr}(C)C\,.$$
Applying this result to $C = AB - BA$, we obtain that $(AB - BA)^2 = \mathrm{tr}(AB - BA)(AB - BA) = 0$, since $\mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$.

4. (a) Show that, if $\{x_k\}$ is a decreasing sequence of positive numbers, then
$$\left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} \;\leq\; \sum_{k=1}^{n} \frac{x_k}{\sqrt{k}}\,.$$

(b) Show that there is a constant $C$ such that, if $\{x_k\}$ is a decreasing sequence of positive numbers, then
$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left(\sum_{k=m}^{\infty} x_k^2\right)^{\frac{1}{2}} \;\leq\; C \sum_{k=1}^{\infty} x_k\,.$$

*Official solution.*

(a)

$$\left(\sum_{i=1}^{n}\frac{x_i}{\sqrt{i}}\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_ix_j}{\sqrt{i}\sqrt{j}} \geq \sum_{i=1}^{n}\frac{x_i}{\sqrt{i}}\sum_{j=1}^{i}\frac{x_i}{\sqrt{j}} \geq \sum_{i=1}^{n}\frac{x_i}{\sqrt{i}}i\frac{x_i}{\sqrt{i}} = \sum_{i=1}^{n}x_i^2 .$$

(b) Using a generalization of (a), we obtain

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\left(\sum_{k=m}^{\infty}x_k^2\right)^{\frac{1}{2}} &\leq \sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\sum_{i=m}^{\infty}\frac{x_i}{\sqrt{i-m+1}}\\ &= \sum_{i=1}^{\infty}x_i\sum_{m=1}^{i}\frac{1}{\sqrt{m}\sqrt{i-m+1}} .\end{aligned}$$

By checking that $\sup\limits_{i}\sum\limits_{m=1}^{i}\dfrac{1}{\sqrt{m}\sqrt{i-m+1}} \leq \displaystyle\int_0^{i+1}\frac{dx}{\sqrt{x}\sqrt{i+1-x}} = \pi$, the result is complete.

5. Let $R$ be a ring of characteristic zero (not necessarily commutative). Let $e$, $f$ and $g$ be idempotent elements of $R$ satisfying $e+f+g=0$. Show that $e=f=g=0$.

   ($R$ is of characteristic zero means that, if $a \in R$ and $n$ is a positive integer, then $na \neq 0$ unless $a=0$. An idempotent $x$ is an element satisfying $x=x^2$.)

*Solution by Michel Bataille, Rouen, France.*

From $eg+fg+g^2 = 0g = 0 = g0 = ge+gf+g^2$ and $g^2=g$, we obtain

$$eg+fg = -g = ge+gf. \tag{1}$$

Similarly,

$$ef+gf = -f = fe+fg \tag{1'}$$

and

$$fe+ge = -e = ef+eg. \tag{1''}$$

Developing $0=(e+f+g)^2$, and using (1), we readily obtain

$$ef+fe = 2g. \tag{2}$$

It follows from (2) that $ef+efe=2eg$ and $efe+fe=2ge$, so that, by differencing and using (1”), we have $ef-fe = 2(eg-ge) = -2(ef-fe)$. Hence, $3(ef-fe)=0$ and so, $ef-fe=0$.

Thus, we have $ef = fe$, and, with (2), we get $2ef = 2g$, or $2(ef-g)=0$. It follows that $ef=fe=g$, and, similarly, $fg=gf=e$ and $eg=ge=f$.

Now, $efg = e(fg) = e^2 = e$ and $efg = (ef)g = g^2 = g$, so that $g=e$. Similarly, $f=g$, and we conclude that $e=f=g$, and further, $e=f=g=0$, since $0=e+f+g=3e=3f=3g$.

6. Let $f : \mathbb{R} \to (0, \infty)$ be an increasing differentiable function for which $\lim_{x\to\infty} f(x) = \infty$ and $f'$ is bounded.

   Let $F(x) = \int_0^x f(t)\,dt$. Define the sequence $\{a_n\}$ inductively by

$$a_0 = 1\,, \quad a_{n+1} = a_n + \frac{1}{f(a_n)}\,, \tag{1}$$

and the sequence $\{b_n\}$ simply by $b_n = F^{-1}(n)$.

   Prove that $\lim_{n\to\infty}(a_n - b_n) = 0$. [Ed. The original said $\infty$, but the solution proves 0.]

*Official solution.* From the conditions, it is clear that $F$ is increasing and that $\lim_{n\to\infty} b_n = \infty$.

By Lagrange's Theorem and the recursion in (1), for all non-negative integers $k$, there exists a real number $\xi \in (a_k, a_{k+1})$ such that

$$F(a_{k+1}) - F(a_k) \;=\; f(\xi)(a_{k+1} - a_k) \;=\; \frac{f(\xi)}{f(a_k)}\,. \tag{2}$$

By the monotonicity, $f(a_k) \le f(\xi) \le f(a_{k+1})$. Thus,

$$1 \;\le\; F(a_{k+1}) - F(a_k) \;\le\; \frac{f(a_{k+1})}{f(a_k)} \;=\; 1 + \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}\,. \tag{3}$$

Summing (3) over $k$ and substituting $F(b_n) = n$, we have

$$\begin{aligned} F(b_n) \;&<\; n + F(a_0) \;\le\; F(a_n) \\ &\le\; F(b_n) + F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}\,. \end{aligned} \tag{4}$$

From the first two inequalities, we already have $a_n > b_n$ and $\lim_{n\to\infty} a_n = \infty$.

Let $\epsilon > 0$. Choose an integer $K_\epsilon$ such that $f(a_{K_\epsilon}) > \frac{2}{\epsilon}$. If $n$ is sufficiently large, we have

$$\begin{aligned} F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} \;&=\; \left( F(a_0) + \sum_{k=0}^{K_\epsilon - 1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} \right) \\ &\qquad + \sum_{k=K_\epsilon}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} \\ &<\; \mathrm{O}_\epsilon(1) + \frac{1}{f(a_{K_\epsilon})} \sum_{k=K_\epsilon}^{n-1} (f(a_{k+1}) - f(a_k)) \\ &<\; \mathrm{O}_\epsilon(1) + \frac{\epsilon}{2}\,(f(a_n) - f(a_{K_\epsilon})) \\ &<\; \epsilon f(a_n)\,. \end{aligned} \tag{5}$$

Inequalities (4) and (5) together imply that for any positive $\epsilon$, if $n$ is sufficiently large, we have

$$F(a_n) - F(b_n) \ < \ \epsilon f(a_n)\,.$$

Again, by Lagrange's Theorem, there is a real number $\zeta \in (b_n, a_n)$ such that

$$F(a_n) - F(b_n) \ = \ f(\zeta)(a_n - b_n) \ > \ f(b_n)(a_n - b_n)\,. \qquad (6)$$

Thus,

$$f(b_n)(a_n - b_n) \ < \ \epsilon f(a_n)\,. \qquad (7)$$

Let $B$ be an upper bound for $f'$. Apply $f(a_n) < f(b_n) + B(a_n - b_n)$ in (7):

$$\begin{aligned} f(b_n)(a_n - b_n) &\ < \ \epsilon\left(f(b_n) + B(a_n - b_n)\right)\,, \\ \left(f(b_n) - \epsilon B\right)(a_n - b_n) &\ < \ \epsilon f(b_n)\,. \end{aligned}$$

Because of $\lim\limits_{n\to\infty} f(b_n) = \infty$, we see that the first factor is positive, and we have

$$a_n - b_n \ < \ \epsilon \frac{f(b_n)}{f(b_n) - \epsilon B} \ < \ 2\epsilon$$

for sufficiently large $n$.

Thus, for arbitrary positive $\epsilon$, we have proved that $0 < a_n - b_n < 2\epsilon$ if $n$ is sufficiently large, and the proof is complete.

## An IMO Training Problem

Given $\triangle A_0B_0C_0$, construct the First Gunther 3–Triangle $\triangle A_1B_1C_1$ as follows:

$A_1$ lies on $A_0B_0$ and is such that $A_0A_1 = \frac{1}{3}A_0B_0$, $B_1$ lies on $B_0C_0$ and is such that $B_0B_1 = \frac{1}{3}B_0C_0$ and $C_1$ lies on $C_0A_0$ and is such that $C_0C_1 = \frac{1}{3}C_0A_0$.

We can clearly iterate this construction.

Show that the Third Gunther 3–Triangle is similar to the First Gunther 3–Triangle.

In a similar way, we can define the First Gunther $n$–Triangle.

Is there a $K^{\text{th}}$ Gunther $n$–Triangle that is similar to the First Gunther $n$–Triangle?

# THE OLYMPIAD CORNER

## No. 214

R.E. Woodrow

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.*

Murray Klamkin has written to point out that "Quickies" did not originate with him. They first appeared in the March 1950 issue of *Mathematics Magazine*. Quickies originated with the late Charles W. Trigg, who was then Problems Editor. In those days, many Quickies were published by the late Leo Moser as well as by Murray. But most probably it is true that Murray has published more Quickies than anyone else.

We start this number with an additional set of five Klamkin Quickies. Thanks go to Murray Klamkin, University of Alberta. Try them before looking ahead to the solutions!

## FIVE MORE KLAMKIN QUICKIES

**1**. Determine the maximum value of

$$S = 4(a^4+b^4+c^4+d^4)-(a^2bc+b^2cd+c^2da+d^2ab)-(a^2b+b^2c+c^2d+d^2a),$$

where $1 \geq a,\ b,\ c,\ d \geq 0$.

**2**. If $a$, $b$, $c$, $d$ are $> 0$, prove or disprove the two inequalities:

(i) $\frac{ab}{c}+\frac{bc}{d}+\frac{cd}{a}+\frac{da}{b} \geq a+b+c+d$,

(ii) $a^2b+b^2c+c^2d+d^2a \geq abc+bcd+cda+dab$.

**3**. Determine all the points $P(x,y,z)$, if any, such that all the points of tangency of the enveloping (tangent) cone from $P$ to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $(a>b>c)$, are coplanar.

**4**. Determine whether or not there exists a set of $777$ distinct positive integers such that for every seven of them, their product is divisible by their sum.

**5**. If $R$ is any non-negative rational approximation to $\sqrt{5}$, determine an *always* better rational approximation.

Next we give the problems of the 28th Austrian Mathematics Olympiad 1997, Final Round Advanced Level. Thanks go to Walther Janous, Ursulinengymnasium, Innsbruck, Austria for sending them to us.

# $28^{\text{th}}$ AUSTRIAN MATHEMATICS OLYMPIAD 1997

## Final Round Advanced Level

**First Day — June 4, 1997 (Time: 4 hours)**

**1**. Let $a$ be a fixed whole number.

Determine all solutions $x$, $y$, $z$ in whole numbers to the system of equations

$$\begin{array}{rcrcrcl} 5x & + & (a+2)y & + & (a+2)z & = & a\,, \\ (2a+4)x & + & (a^2+3)y & + & (2a+2)z & = & 3a-1\,, \\ (2a+4)x & + & (2a+2)y & + & (a^2+3)z & = & a+1\,. \end{array}$$

**2**. Let $K$ be a positive whole number. The sequence $\{a_n : n \geq 1\}$ is defined by $a_1 = 1$ and $a_n$ is the $n^{\text{th}}$ natural number greater than $a_{n-1}$ which is congruent to $n$ modulo $K$.

(a) Determine an explicit formula for $a_n$.

(b) What is the result if $K = 2$?

**3**. We are given a triangle $ABC$. On side $AC$ a point $P$ is chosen. On the production of ray $CB$ (beyond $B$) there lies the point $Y$ which subtends equal angles with $AP$ and $PC$, respectively.

On side $BC$, point $Q$ is chosen. On the production of ray $AC$ (beyond $C$) there is point $X$, subtending equal angles with $BQ$ and $QC$, respectively.

Furthermore, $R$ is the point of intersection of lines $YP$ and $XB$, $S$ is the point of intersection of lines $XQ$ and $YA$, and $D$ is the point of intersection of lines $XB$ and $YA$.

Prove: $PQRS$ is a parallelogram if and only if $ACBD$ is inscribable.

**Second Day — June 5, 1997 (Time: 4 hours)**

**4**. Determine all quadruples $(a, b, c, d)$ of real numbers satisfying the equation

$$\begin{aligned} 256a^3b^3c^3d^3 \;=\; & (a^6+b^2+c^2+d^2)(a^2+b^6+c^2+d^2) \\ & \times(a^2+b^2+c^6+d^2)(a^2+b^2+c^2+d^6)\,. \end{aligned}$$

**5**. We define the following operation which will be applied to a row of bars being situated side-by-side on positions $1, \ldots, N$:

Each bar situated at an odd numbered position is left as is, while each bar at an even numbered position is replaced by two bars. After that, all bars will be put side-by-side in such a way that all bars form a new row and are situated (side-by-side) on positions $1, \ldots, M$.

From an initial number $a_0 > 0$ of bars there originates (by successive application of the above-defined operation) a sequence, $\{a_n : n \geq 0\}$ of natural numbers, where $a_n$ is the number of bars after having applied the operation $n$ times.

(a) Prove that for all $n > 0$ we have $a_n \neq 1997$.

(b) Determine the natural numbers that can only occur as $a_0$ or $a_1$.

**6**. Let $n$ be a fixed natural number. Determine all polynomials $x^2 + ax + b$, where $a^2 \geq 4b$, such that $x^2 + ax + b$ divides $x^{2n} + ax^n + b$.

Next we turn to the Icelandic Olympiad of 1995–1996. Thanks go to Mohammed Aassila, Strasbourg, France for sending this to us.

## ÍSLENZKA STAERŎFRÆŎIKEPPNI FRAMHALDSSKÓLANEMA 1995–1996

### Úrslitakeppni

**Laugardagur 23. mars 1996 kl. 10–14**

**1**. Calculate the area of the region in the plane determined by the inequality

$$|x| + |y| + |x + y| \;\leq\; 2\,.$$

**2**. Suppose that $a$, $b$ and $c$ are the three roots of the polynomial $p(x) = x^3 - 19x^2 + 26x - 2$. Calculate

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\,.$$

**3**. A collection of $52$ integers is given. Show that amongst these numbers it is possible to find two such that $100$ divides either their sum or their difference.

**4**. (i) Show that the sum of the digits of every integer multiple of $99$, from $1 \cdot 99$ up to and including $100 \cdot 99$, is $18$.

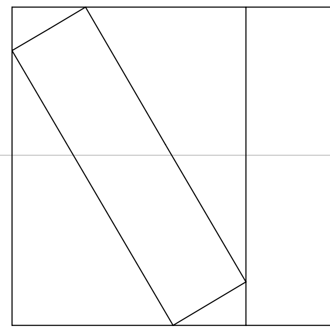
(ii) Show that the sum of the digits of every integer multiple of the number $10^n - 1$, from $1 \cdot (10^n - 1)$ up to and including $10^n \cdot (10^n - 1)$, is $n \cdot 9$.

**5**. The sequence $\{a_n\}$ is defined by $a_1 = 1$ and, for $n \geq 1$,

$$a_{n+1} \;=\; \frac{a_n}{1 + na_n}\,.$$

Find $a_{1996}$.

**6**. In a square bookcase two identical books are placed as shown in the figure. Suppose the height of the bookcase is $1$. How thick are the books?



As a third set of Olympiad problems we give the Second Round of the 1997 Iranian Mathematical Olympiad. Thanks go to Mohammed Aassila, Strasbourg, France.

## 1997 IRANIAN MATHEMATICAL OLYMPIAD
### Second Round
**Time: $2 \times 4$ hours**

**1**. Suppose that $S$ is a finite set of real numbers with the property that any two distinct elements of $S$ will form an arithmetic progression with another element of $S$. Give an example of such a set with $5$ elements and prove that no such set exists with at least $6$ elements.

**2**. Suppose that ten points are given in the plane such that any five of them contain four points which are concyclic. What is the largest number $N$ for which we can correctly say: "At least $N$ of the ten points lie on a circle"? ($4 \leq N \leq 10$.)

**3**. Suppose that $\Gamma$ is a semi-circle with centre $O$ and diameter $AB$. Assume that $M$ is a point on the extension of $AB$ such that $MA > MB$. A line through $M$ intersects $\Gamma$ at $C$ and $D$ such that $MC > MD$. Circumcircles of the triangles $AOC$ and $BOD$ will intersect at points $O$ and $K$. Prove that $OK \perp MK$.

**4**. Find all functions $f : \mathbb{N} \to \mathbb{N}\backslash\{1\}$ such that for all $n \in \mathbb{N}\backslash\{0\}$ we have,

$$f(n+1) + f(n+3) \;=\; f(n+5)f(n+7) - 1375\,.$$

**5**. Suppose that $ABC$ is an acute triangle with $AC < AB$ and the points $O$, $H$, and $P$ are circumcentre, orthocentre, and the foot of the altitude drawn, from $C$ on $AB$, respectively. The line perpendicular to $OP$ at $P$ intersects the line $AC$ at $Q$. Prove that $\angle PHQ = \angle BAC$.

**6**. Suppose that $A$ is a symmetric $(0,1)$-matrix such that all of its diagonal entries are $1$. Prove that there exist $0 \le i_1 < i_2 < \cdots < i_k \le n$ such that $A_{i_1} + A_{i_2} + \cdots + A_{i_k} \equiv (1, 1, \ldots, 1) \pmod{2}$, where $A_i$ is the $i^{\text{th}}$ row of $A$.

As a final set for this number we give the problems of the Final Round of the 1997 Iranian Mathematical Olympiad. Again, thanks go to Mohammed Aassila, Strasbourg, France.

## 1997 IRANIAN MATHEMATICAL OLYMPIAD
### Final Round
**Time: $2 \times 4$ hours**

**1**. Let $n$ be a positive integer. Prove that there exist polynomials $f(x)$ and $g(x)$ with integer coefficients such that,

$$f(x)(x+1)^{2^n} + g(x)(x^{2^n} + 1) \;=\; 2\,.$$

**2**. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has the following properties:

(a) $\forall x \in \mathbb{R},\ f(x) \le 1$

(b) $\forall x \in \mathbb{R},\ f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right)$.

Prove that $f$ is periodic; that is, there exists a non-zero real number $T$ such that for every real number $x$, we have $f(x+T) = f(x)$.

**3**. Suppose that $w_1, w_2, \ldots, w_k$ are distinct real numbers with a non-zero sum. Prove that there exist integer numbers $n_1, n_2, \ldots, n_k$ such that $\sum_{i=1}^{k} n_i w_i > 0$ and for any non-identity permutation $\pi$ on $\{1, 2, \ldots, k\}$ we have $\sum_{i=1}^{k} n_i w_{\pi(i)} < 0$.

**4**. Suppose that $P$ is a variable point on arc $BC$ of the circumcircle of triangle $ABC$, and let $I_1$, $I_2$ be the incentre of the triangles $PAB$ and $PAC$, respectively. Prove that,

(a) The circumcircle of $PI_1I_2$ passes through a fixed point.

(b) The circle with diameter $I_1I_2$ passes through a fixed point.

(c) The mid-point of $I_1I_2$ lies on a fixed circle.

**5**. Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing continuous function that fulfils the following condition for all $x, y \in \mathbb{R}^+$:

$$f(x+y) + f(f(x) + f(y)) \;=\; f\left(f(x + f(y)) + f(y + f(x))\right)\,.$$

Prove that $f(x) = f^{-1}(x)$.

**6**. A one story building consists of a finite number of rooms which have been separated by walls. There are one or more doors on some of these walls which can be used to travel in this building. Two of the rooms are marked by $S$ and $E$. An individual begins walking from $S$ and wants to reach to $E$.

By a program $P = (P_i)_{i \in I}$, we mean a sequence of $R$'s and $L$'s. The individual uses the program as follows: after passing through the $n^{\text{th}}$ door, he chooses the rightmost or the leftmost door, meaning that $P_n$ is $R$ or $L$. For the rooms with one door, any symbol means selecting the door that he had just passed. Notice that the person stops as soon as he reaches $E$.

Prove that there exists a program $P$ (possibly infinite) with the property that no matter how the structure of the building is, one can reach from $S$ to $E$ by following it. [Editor's comment: one has to assume that there is a way of getting from any room to any other room.]

Now we give Klamkin's solutions to the five Quickies given at the beginning of this *Corner*.

## SOLUTIONS TO FIVE MORE KLAMKIN QUICKIES

**1**. $$S \le 4(a^2+b^2+c^2+d^2) - (a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2) - (a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2).$$

Since the expression on the right hand side is linear in $a^2$, $b^2$, $c^2$, and $d^2$, it takes on its maximum at the endpoints $0$, $1$ for each variable. By inspection, $S_{\max} = 9$ and is taken on for $a = b = c = 1$ and $d = 0$.

**2**. Neither inequality is valid.

(i) Just consider the case: $b = 2$, $c = 5$, $d = 1$ and $a$ is very large.

(ii) Just consider the case: $a = 2$, $b = 1$, $c = 8$ and $d$ is very small.

**3**. Consider the affine transformation $x' = \frac{x}{a}$, $y' = \frac{y}{b}$, $z' = \frac{z}{c}$ which takes the ellipsoid into a sphere. Under this transformation, lines go to lines, planes go to planes, and tangency is preserved. Consequently, any enveloping cone of the ellipsoid goes into an enveloping cone of the sphere and which by symmetry is a right circular one and its points of tangency are a circle (coplanar) of the sphere. Thus, $P$ can be any exterior point of the ellipsoid.

**4**. Just take any $777$ distinct positive integers and multiply each one by the product of the sums of every $7$ of them.

**5**. Assuming the better approximation has the form $\frac{aR+b}{cR+d}$ where $a$, $b$, $c$, $d$ are rational, we must satisfy

$$\left|\frac{aR+b}{cR+d} - \sqrt{5}\right| < |R - \sqrt{5}|. \qquad (1)$$

If $R \to \sqrt{5}$, the left hand side must $\to 0$. Thus, we must have $\frac{a\sqrt{5}+b}{c\sqrt{5}+d} = \sqrt{5}$, so that $d = a$ and $b = 5c$. Then substituting these values in (1) and dividing both sides by the common factor $|R - \sqrt{5}|$, we get

$$|cR + a| \;>\; |c\sqrt{5} - a|$$

and which ~~can easily be satisfied by letting $a = 2$ and $c = 1$.~~ Finally, our better approximation is $\frac{2R+5}{R+2}$.

Before turning to readers' solutions, an apology. Somehow, my filing system misplaced a group of solutions sent in by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. He sent in solutions to problems 1, 3 of the XXXIX Republic Competition of Mathematics in Macedonia [1999 : 196, 2001 : 99], [1999 : 196, 2001 : 101], to problem 1, 2 of Class III [1999 : 197, 2001 : 105–106], and to problem 2 of Class IV [1999 : 197, 2001 : 107–108].

Next we turn to readers' solutions to problems from the September 1999 *Corner*, and the Georg Mohr Konkurrencen I Matematik 1996 [1999 : 261–262].

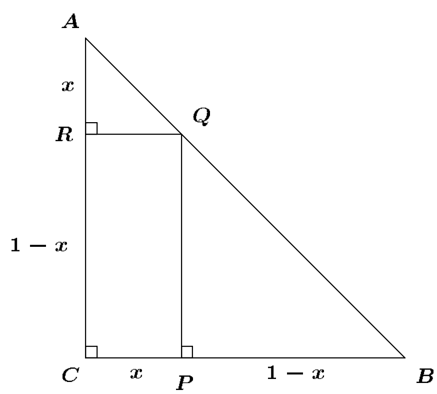
## GEORG MOHR KONKURRENCEN I MATEMATIK 1996
### January 11, 9–13

*Only writing and drawing materials are allowed.*

**1**. $\angle C$ in $\triangle ABC$ is a right angle and the legs $BC$ and $AC$ are both of length 1. For an arbitrary point $P$ on the leg $BC$ construct points $Q$, respectively, $R$, on the hypotenuse, respectively, on the other leg, such that $PQ$ is parallel to $AC$ and $QR$ is parallel to $BC$. This divides the triangle into three parts.

Determine positions of the point $P$ on $BC$ such that the rectangular part has greater area than each of the other two parts.

*Solution by Pierre Bornsztein, Pontoise, France.*




Denote $x = CP$, $x \in (0,1)$. Then $PB = 1 - x$, and, from Thales' Theorem,

$$QB = (1-x)\sqrt{2}\,, \quad QA = x\sqrt{2}\,, \quad RC = 1-x, \quad AR = x\,.$$

Thus,

$$[RQPC] = x(1-x)\,, \quad [PBQ] = \tfrac{1}{2}(1-x)^2\,, \quad [AQR] = \tfrac{1}{2}x^2\,.$$

It remains to solve

$$\begin{cases} x(1-x) > \frac{1}{2}(1-x)^2\,, \\ x(1-x) > \frac{1}{2}x^2\,, \end{cases}$$

which is equivalent to

$$\begin{cases} x > \frac{1}{3}\,, \\ x < \frac{2}{3}\,. \end{cases}$$

Thus, we will have the desired result if and only if $PC = x$ with $x \in \left(\frac{1}{3}, \frac{2}{3}\right)$.

**2**. Determine all triples $(x, y, z)$, satisfying

$$\begin{aligned} xy &= z\,, & (1)\\ xz &= y\,, & (2)\\ yz &= x\,. & (3) \end{aligned}$$

*Solution by Pierre Bornsztein, Pontoise, France.*

If $(x, y, z)$ is a solution, then, multiplying, we have $(xyz)^2 = xyz$. Thus, $xyz = 0$ or $xyz = 1$.

*Case 1.* If $xyz = 0$ then, for example, $z = 0$.

From (2) we get $y = 0$, and from (3) we obtain $x = 0$. Conversely, it is easy to see that $(0, 0, 0)$ is a solution.

*Case 2.* If $xyz = 1$ then $z = \frac{1}{xy}$.

From (1), we deduce $z^2 = 1$. Thus, $z \in \{-1,\, 1\}$.

In the same way, $x \in \{-1,\, 1\}$ and $y \in \{-1,\, 1\}$. Moreover, since $xyz = 1$, the number of $-1$'s in $(x, y, z)$ is even. This leads to

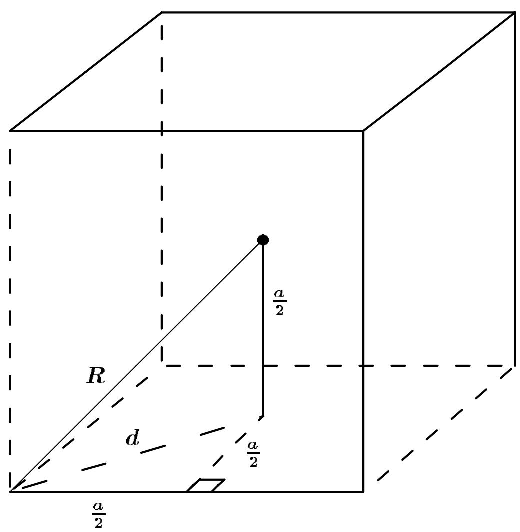
$$(-1,-1,1)\,,\quad (-1,1,-1)\,,\quad (1,-1,-1)\,,\quad (1,1,1)\,.$$

Conversely, it is easy to see that these triples are solutions. Then,

$$S \;=\; \{(0,0,0),\,(1,1,1),\,(-1,-1,1),\,(-1,1,-1),\,(1,-1,-1)\}\,.$$

**3**. This year's idea for a gift is from "BabyMath", namely a series of 9 coloured plastic figures of decreasing sizes, alternating cube, sphere, cube, sphere, etc. Each figure may be opened and the succeeding one may be placed inside, fitting exactly. The largest and the smallest figures are both cubes. Determine the ratio between their side-lengths.

*Solution by Pierre Bornsztein, Pontoise, France.*




If a sphere with radius $R$ is circumscribed to a cube with edge $a$ then the sphere and the cube have the same centre, and the vertices of the cube are points of the sphere.

From Pythagoras' Theorem:

$$R^2 \;=\; d^2 + \frac{a^2}{4} \;=\; \frac{3a^2}{4}\,.$$

Thus,

$$R \;=\; \frac{\sqrt{3}}{2}a\,. \tag{1}$$

If a sphere with radius $R$ is inscribed in a cube with edge $b$, then the sphere and the cube have the same centre, and the centres of the sides of the cube are points of the sphere. Then

$$R = \frac{b}{2}. \qquad (2)$$

From (1) and (2), it follows that the ratio between the side-lengths of the "outside cube" and the "inside cube" is

$$\frac{b}{a} = \sqrt{3}.$$

Since there are $5$ cubes, the ratios between the side-lengths of the largest and the smallest figures is $\left(\sqrt{3}\right)^4 = 9$.

**4**. $n$ is a positive integer. It is known that the last but one digit in the decimal expression of $n^2$ is $7$. What is the last digit?

*Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give Bataille's solution.*

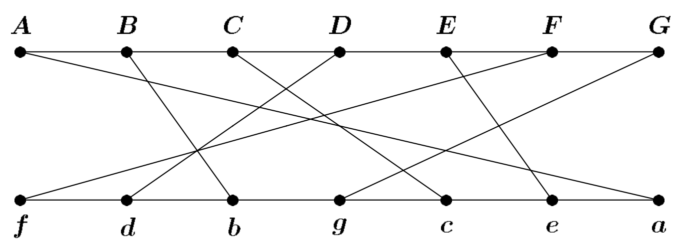
We prove that this last digit is $6$.

Write $n$ as $10a + b$ where $a$ is a non-negative integer and $b \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then $n^2 = 100a^2 + 20ab + b^2$ and the last two digits of $n^2$ are also those of $20ab + b^2$.

But $20ab = 10 \times 2ab$ ends with the digits $00$ or $20$ or $40$ or $60$ or $80$. Since it is given that the penultimate digit of $20ab + b^2$ is $7$, we see that the penultimate digit of $b^2$ must be odd; this can occur only when $b = 4$ or $b = 6$, and then $b^2 = 16$ or $b^2 = 36$. Adding these two values to any of the integers $00$, $20$, $40$, $60$, $80$, we obtain only one result whose last but one digit is $7$, and it is $76$. This completes the proof.

**5**. In a ballroom $7$ gentlemen, $A$, $B$, $C$, $D$, $E$, $F$ and $G$ are sitting opposite $7$ ladies $a$, $b$, $c$, $d$, $e$, $f$ and $g$ in arbitrary order. When the gentlemen walk across the dance floor to ask each of their ladies for a dance, one observes that at least two gentlemen walk distances of equal length. Is that always the case?

The figure shows an example. In this example $Bb = Ee$ and $Dd = Cc$.



*Solutions by Pierre Bornsztein, Pontoise, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.*

Note first that the observation is only correct if we assume that the $7$ gentlemen are "evenly" spaced.

We show that in general, if there are $n$ gentlemen and $n$ ladies, then the same conclusion holds when $n \equiv 2,\ 3 \pmod 4$.

Suppose the $n$ ladies are situated at $(k, 0)$ and the $n$ gentlemen, at $(k, 1)$; $k = 1,\ 2,\ \ldots,\ n$. Suppose that the gentleman at $(k, 1)$ walks a distance of $d_k$ to the lady at $(a_k, 0)$, $a_k \in \{1,\ 2,\ \ldots,\ n\}$. Then $(a_1, a_2, \ldots, a_n)$ is a permutation of $(1, 2, \ldots, n)$ and thus, $\sum_{k=1}^{n}(a_k - k) = 0$. Since $d_k = (1 + (a_k - k)^2)^{1/2}$, we have $d_k^2 = 1 + (a_k - k)^2$. We show that the values of the $d_k$'s cannot all be distinct.

Note that $a_k - k \in \{0,\ \pm 1,\ \pm 2,\ \ldots,\ \pm(n-1)\}$. Suppose to the contrary that $(a_k - k)^2 \neq (a_j - j)^2$ for all $j \neq k$, $j$, $k = 1,\ 2,\ \ldots,\ n$. Then we have $\{|a_k - k| : k = 1,\ 2,\ \ldots,\ n\} = \{0,\ 1,\ 2,\ \ldots,\ n-1\}$ and thus,

$$\sum_{k=1}^{n} |a_k - k| \;=\; \sum_{k=0}^{n-1} i \;=\; \frac{n(n-1)}{2}\,. \tag{1}$$

On the other hand, since $|t| - t$ must be even for all integers $t$, we have

$$\sum_{k=1}^{n} |a_k - k| \;=\; 2d + \sum_{k=1}^{n}(a_k - k) \;=\; 2d\,, \quad \text{for some integer } d\,. \tag{2}$$

Comparing (1) and (2) we get $n(n-1) = 4d$ which implies that $n \equiv 0,\ 1 \pmod 4$. Therefore, if $n \equiv 2,\ 3 \pmod 4$ then we must have $d_j = d_k$ for some $j \neq k$.

*Editor's question.* If $n$ is congruent to either $0$ or $1$, is it always possible to arrange the $n$ gentlemen and the $n$ ladies in a way such that the distance are all different?

Next we turn to solutions to problems of the St. Petersburg City Mathematical Olympiad, Third Round, 1996 [1999 : 262].

# ST. PETERSBURG CITY MATHEMATICAL OLYMPIAD

## Third Round – February 25, 1996

### 11$^{\text{th}}$ Grade (Time: 4 hours)

**1**. Serge was solving the equation $f(19x - 96/x) = 0$ and found $11$ different solutions. Prove that if he tried hard he would be able to find at least one more solution.

*Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's write-up.*

Note first that $x = 0$ is not a solution. If $r \neq 0$ is a solution, then so is $t = -\frac{96}{19r}$, since

$$f\left(19t - \frac{96}{t}\right) \;=\; f\left(-\frac{96}{r} + 19r\right) \;=\; 0\,.$$

Since $r = \frac{-96}{19r}$ is impossible, different $r$'s will correspond to different $t$'s. Therefore, the number of solutions must be even (if it is finite) and the conclusion easily follows.

**2**. The numbers $1, 2, \ldots, 2n$ are divided into two groups of $n$ numbers. Prove that pairwise sums of numbers in each group (sums of the form $a + a$ included) have identical sets of remainders on division by $2n$.

*Solution by Pierre Bornsztein, Pontoise, France.*

Let $G_1$, $G_2$ be the two groups, and let $S_1 = \{p \in \{1,\, 2,\, \ldots,\, 2n\} :$ there exist $a,\, b \in G_1$ for which $a + b \equiv p \pmod{2n}\}$. $S_2$ is defined in the same way. We must prove that $S_1 = S_2$.

Let $p \in \{1, \ldots, 2n\}$.

We will have $a+b \equiv p \pmod{2n}$, with $a,\, b \in \{1, \ldots, 2n\}$, if and only if $a + b = p$ or $a + b = 2n + p$.

*Case 1*. If $p$ is odd, then $p = 2k + 1$ for some integer $k$ such that $0 \leq k < n$.

Let $a,\, b \in \{1, \ldots, 2n\}$, with $a \leq b$.

We will have $a + b = p$ if and only if $(a, b)$ is one of the $k$ pairs

$$(1, 2k)\,, \quad (2, 2k-1)\,, \quad \ldots\,, \quad (k, k+1)\,.$$

We will have $a + b = 2n + p$ if and only if $(a, b)$ is one of the $n - k$ pairs

$$(2k+1, 2n)\,, \quad (2k+2, 2n-1)\,, \quad \ldots\,, \quad (n+k, n+k+1)\,.$$

Then the numbers $1,\, 2,\, \ldots,\, 2n$ are divided into $n$ pairs to give the remainder $p$.

If $p \in S_1$, then $G_1$ contains at least one of these pairs. We have to choose at most $n - 2$ numbers to complete the group $G_1$. It cannot be done if we want $G_1$ to contain at least one of the members of each pair. Then at least one pair is included in $G_2$. Thus, $p \in S_2$.

*Case 2.* If $p$ is even, then $p = 2k$ for some integer $k$ such that $1 \leq k \leq n$.

As above the numbers $1, 2, \ldots, 2n$ are divided into $n+1$ pairs to give the remainder $p$. The pairs are

$$(1, 2k-1)\,, \quad (2, 2k-2)\,, \quad \ldots\,, \quad (k-1, k+1)\,, \quad (k, k)$$

and

$$(2k, 2n)\,, \quad (2k+1, 2n-1)\,, \quad \ldots\,, \quad (n+k-1, n+k+1)\,, \quad (n+k, n+k)\,.$$

Then we have $n-1$ pairs and two "isolated" numbers ($k$ and $n+k$).

If $p \in S_1$ and if at least one of the two "isolated" numbers is not in $G_1$, then $k \in G_2$ or $n+k \in G_2$. Thus, $p \in S_2$.

If $p \in S_1$ and if both of the two "isolated" numbers are in $G_1$, then $G_1$ contains $n-2$ other numbers. Thus, $G_1$ cannot contain at least one of the numbers from each of the $n-1$ pairs.

It follows that at least one of the pairs is included in $G_2$. So, we deduce that $p \in S_2$.

Then, in each case, if $p \in S_1$ then $p \in S_2$. That is,

$$S_1 \subset S_2\,.$$

By symmetry, we have $S_2 \subset S_1$. Thus,

$$S_1 = S_2\,.$$

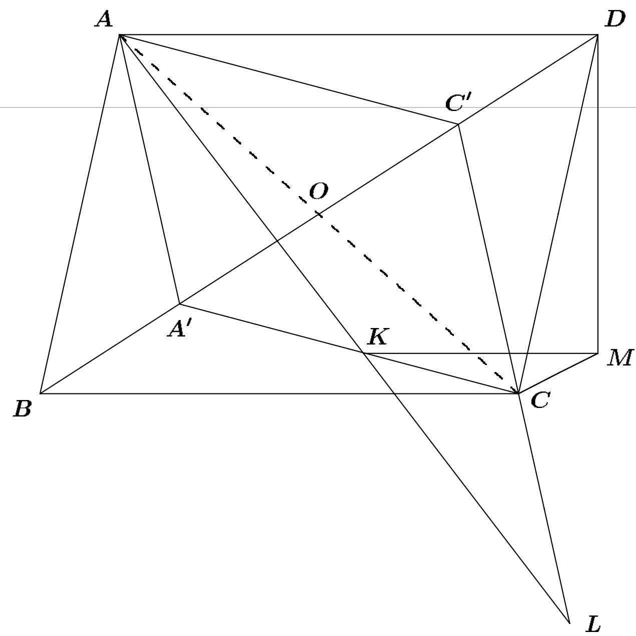
**3**. No three diagonals of a convex $1996$–gon meet in one point. Prove that the number of the triangles lying in the interior of the $1996$–gon and having sides on its diagonals is divisible by $11$.

*Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Pontoise, France. We give the solution of Aassila.*

If $ABCDEF$ is a 6–gon such that no three diagonals meet in one point, then the triangle formed by $AD$, $BE$ and $CF$ is the **only one** having sides on its diagonals. Hence, the number of such triangles for a $1996$–gon is $\binom{1996}{6}$, but since $1991 = 11 \times 181$, we deduce the desired result.

**4**. Points $A'$ and $C'$ are taken on the diagonal $BD$ of a parallelogram $ABCD$ so that $AA' \| CC'$. Point $K$ lies on the segment $A'C$, the line $AK$ meets the line $C'C$ at the point $L$. A line parallel to $BC$ is drawn through $K$, and a line parallel to $BD$ is drawn through $C$. These two lines meet at point $M$. Prove that the points $D$, $M$, $L$ are collinear.

*Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.*




Let $O$ be the intersection of $AC$ and $BD$. Since $ABCD$ is a parallelogram we have $AO = OC$.

Since $AA' \| CC'$, we obtain $A'O : OC' = AO : OC = 1 : 1$, so that $A'O = OC'$. Thus, $AA'CC'$ is a parallelogram, and further, $AC' \| A'C$. Since $AD \| KM$, $C'D \| CM$, $AC' \| KC$, and $AC' \neq KC$, we have $AK$, $C'C$, and $DM$ are concurrent at $L$.

Therefore, $D$, $M$, $L$ are collinear.

That completes this issue of the *Corner*. We will continue with these problems in the next number. Please send me Olympiad Contests and your nice solutions and generalizations.

# BOOK REVIEWS

## ALAN LAW

*Mathematical Olympiads: Problems and Solutions from Around the World 1998–1999*,
edited by Titu Andreescu and Zuming Feng,
published by the Mathematical Association of America, 1999,
ISBN 0-88385-803-7, softcover, 280 pages, $28.50 (U.S.).
*Reviewed by* **Christopher G. Small**, *University of Waterloo, Waterloo, Ontario.*

This book represents a continuation of the book *Mathematical Contests 1997-1998: Olympiad Problems and Solutions from around the World*. The authors have collected olympiad problems from the national contests of 18 different countries (including Canada), together with seven regional contests from 1998 and the national contests of 22 countries and eight regional contests from 1999. Problems from 1998 are published with solutions, but solutions for 1999 problems are not included. The volume comes with a brief glossary of basic mathematical identities and definitions, and an index of problems classified lexicographically by subject area, country of origin, and year. Both problems and solutions are presented with a unified notation.

There are many reasons why people involved in mathematics contests should want a book of this kind. Having schmoozed at the IMO several times, I have found that the task of gathering national contest problems is not easy. Hauling my mathematical loot back home, I am forced to contemplate my linguistic inadequacies. (Perhaps you know that "cung" is Vietnamese for "arc," but I did not. Unfortunately, more or less knowing — as I do now — that "cung" means "arc" provides no assistance with the host of other words I need to know.) Fortunately, many countries translate their problems into English, as they know full well that their English is better than our Vietnamese/Farsi/whatever. However, the translation process is often imperfect. Consider the following recent problem translated from a contest in a Spanish-speaking country:

> A sequence is defined as $a_1 = 3$, y $a_{n+1} = a_n + a_n^2$. Determine the last two digits of $a_{2000}$.

A student could be forgiven for asking what the value of $y$ is. Occasionally, the translation into English suffers the dubious honour of being too good. For example, another recent contest used the English word "wherefrom," which is perfectly good English, but should probably be avoided in contest problems in our postliterate world.

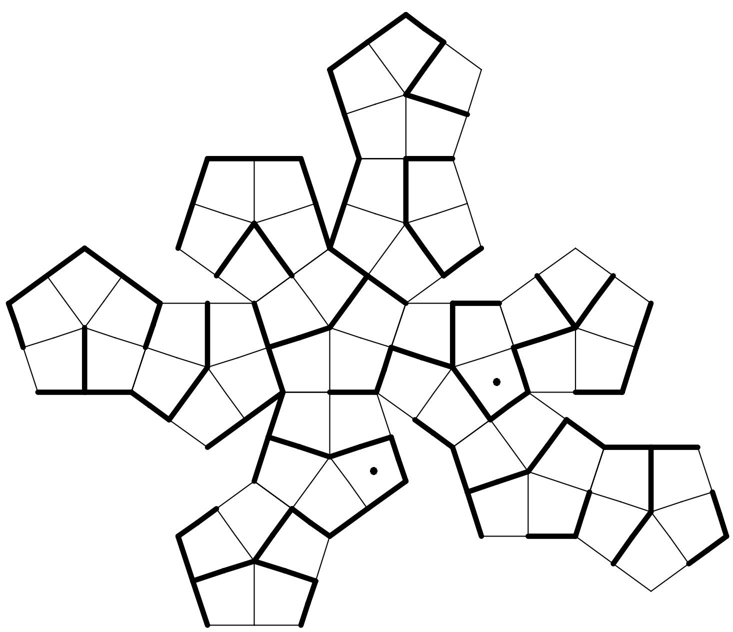
On reading through the problem collections and the various addenda, I had only one criticism. The authors have chosen to continue the common tendency to describe convex functions as "concave up" and concave functions as "concave down." If it is unnecessary for us to use such terms in research papers, it is surely also unnecessary in the school system.

Anyway, hats off to Titu Andreescu and Zuming Feng for putting together this interesting collection of the best from around the world. The problems are written in plain and simple English without any of the translation effects that I have mentioned above. From each volume of problems we can celebrate the vigour of mathematical culture in many different countries. Now where's that Russian problem I was working on . . . ?

## Another Maze in Three Dimensions

### Izador Hafner

Here is another maze, this time, on a dodecahedron, given as an unfolded plane plan. Can you solve it?



Izidor Hafner
Faculty of Electrical Engineering
University of Ljubljana
Tržaška 25
1000 Ljubljana, Slovenia

# Another Look at the Volume of a Tetrahedron

Nairi M. Sedrakyan

In November 1999's ***CRUX with MAYHEM*** [1999 : 422-425], Murat Aygen proved that for the tetrahedron $ABCD$ with

$$a = BC,\ b = CA,\ c = AB,\ a_1 = AD,\ b_1 = BD,\ c_1 = CD,$$

the volume $V$ and the circumradius $R$ are related by

$$6VR = K,$$

where $K$ is the area of the associated triangle with sides $aa_1$, $bb_1$, and $cc_1$. Here is an alternative proof.

*Solution*. First, we will prove a lemma.

*Lemma*. Given a triangle $DMN$, choose on the rays $DM$ and $DN$ points $M_1$ and $N_1$ such that $DM_1 = \dfrac{1}{DM}$ and $DN_1 = \dfrac{1}{DN}$.

Then, we claim that

$$\angle DM_1N_1 = \angle DNM, \quad \angle DN_1M_1 = \angle DMN \qquad (1)$$

and

$$M_1N_1 = \frac{MN}{DM \cdot DN}.$$

*Proof*. Indeed, we have $\dfrac{DM_1}{DN} = \dfrac{DN_1}{DM} = \dfrac{1}{DM \cdot DN}$. From this, it follows that $\triangle M_1DN_1 \sim \triangle NDM$, from which we find $\dfrac{M_1N_1}{MN} = \dfrac{1}{DM \cdot DN}$. This implies (1). ■

Draw a diameter $DS$ of the sphere and consider points $A_1$, $B_1$, $C_1$, $S_1$ contained by the rays $DA$, $DB$, $DC$ and $DS$, correspondingly, where

$$DA_1 = \frac{1}{DA}, \quad DB_1 = \frac{1}{DB}, \quad DC_1 = \frac{1}{DC}, \quad DS_1 = \frac{1}{DS}.$$

According to the lemma, we have $\angle DS_1A_1 = \angle DAS = 90^0$, $\angle DS_1B_1 = \angle DBS = 90^0$ and $\angle DS_1C_1 = \angle DCS = 90^0$. Hence, $DS_1$ is the altitude of the pyramid $DA_1B_1C_1$.

Let $V_1$ denote the volume of the pyramid $DA_1B_1C_1$. Denote the area of the triangle $XYZ$ by $[XYZ]$. Denote by $\rho(X, MNK)$, the distance of the point $X$ from the plane $(MNK)$. Now, according to the lemma, we have

$$A_1B_1 = \frac{AB}{DA \cdot DB} = \frac{cc_1}{a_1b_1c_1},$$
$$B_1C_1 = \frac{aa_1}{a_1b_1c_1}, \quad A_1C_1 = \frac{bb_1}{a_1b_1c_1}.$$

Hence,

$$[A_1B_1C_1] = K\frac{1}{a_1^2b_1^2c_1^2},$$

where $K$ is the area of the triangle with sides $aa_1$, $bb_1$, $cc_1$, and

$$V_1 = \frac{1}{3}[A_1B_1C_1] \cdot DS_1 = \frac{1}{6R} \cdot \frac{K}{a_1^2b_1^2c_1^2}. \tag{2}$$

On the other hand,

$$\frac{V_1}{V} = \frac{\frac{1}{3}[DB_1C_1]\rho(A_1, DB_1C_1)}{\frac{1}{3}[DBC]\rho(A, DBC)} = \frac{DB_1 \cdot DC_1}{DB \cdot DC} \cdot \frac{DA_1}{DA} = \frac{1}{a_1^2b_1^2c_1^2}. \tag{3}$$

From (2) and (3), we find $6VR = K$.

See [1] and [2] for other versions of the formula for a tetrahedron's volume in terms of its edges. In [1], the authors traced their formula back to Euler, and provided the references [3] pages 285-289, and [4] pages 124-125. Reference [2] attributes the determinant version to Cayley and Menger, but provides no reference. See also the discussion of Crux problem 930 [1985 : 162] for further references and related information.

## References


1. Keven L. Dove and John S. Sumner, *Tetrahedra with integer edges and integer volume*, Math. Mag. 65:2 (April, 1992) 104-111.
2. Ira Rosenholtz, **Problem Q890**, Math. Mag. 72:2 (April, 1999) 149.
3. H. Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, New York, 1965.
4. D.M.Y. Sommerville, *An Introduction to the Geometry of $N$-Dimensions*, Dover, New York, 1958.



Nairi M. Sedrakyan
A. Shahinyan High School of Physics and Mathematics,
Azatutyan av, 2-nd side street, 9
375037, Yerevan, Armenia
gagri@lx2.yerphi.am

# THE SKOLIAD CORNER

## No. 54

R.E. Woodrow

Starting with the September number of ***Crux Mathematicorum with Mathematical Mayhem***, the *Skoliad Corner* will be moving to the care of Shawn Godin. He has some excellent ideas for involving school-age students and their teachers more fully.

To close my role with the *Skoliad*, we give the problems and solutions of the first round of the Alberta High School Mathematics Competition Prize Examination written in November, 2000.

## ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION 2000-2001

### Part I

**November 20, 2000**

**1**. Amy has 58 coins totalling one dollar. They are all pennies, nickels and dimes. The number of nickels she has is

**(a)** 2 **(b)** 3 **(c)** 4 **(d)** 5 **(e)** 6

**2**. Nima runs for 5 kilometres at 10 kilometres per hour, followed by 10 kilometres at 5 kilometres per hour. Her average speed in kilometres per hour for the whole trip is

**(a)** 6 **(b)** 6.5 **(c)** 7 **(d)** 7.5 **(e)** 8

**3**. The sum of the digits of the number $2^{2000}5^{2004}$ is

**(a)** 8 **(b)** 9 **(c)** 11 **(d)** 13 **(e)** 14

**4**. A bug crawls 1 centimetre north, 2 centimetres west, 3 centimetres south, 4 centimetres east, 5 centimetres north and so on, at 1 centimetre per second. Each segment is 1 centimetre longer than the preceding one, and at the end of a segment, the bug makes a left turn. The direction in which the bug is facing 1 minute after the start is

**(a)** north **(b)** west **(c)** south **(d)** east **(e)** changing

**5**. The average of fifteen different positive integers is 13. The maximum value for the second largest of these integers is

**(a)** 28 **(b)** 46 **(c)** 52 **(d)** 90 **(e)** none of these

**6**. A polynomial $p(x)$ has four terms, so that it is of the form $p(x) = ax^k + bx^\ell + cx^m + dx^n$, where $k$, $\ell$, $m$ and $n$ are four different non-negative integers and $a$, $b$, $c$ and $d$ are non-zero real numbers. The largest number of terms the polynomial $(p(x))^2$ can have is

(a) 6 (b) 8 (c) 10 (d) 12 (e) 16

**7**. If $x^2 - x - 1 = 0$, then the value of $x^3 - 2x + 1$ is

(a) $\frac{1-\sqrt{5}}{2}$ (b) 0 (c) $\frac{1+\sqrt{5}}{2}$ (d) 2 (e) not uniquely determined

**8**. A thin rod 6 metres long has one end on the level ground and is leaning against a sphere on the ground, 4 metres in diameter. The rod touches the sphere at a point 1 metre from the top end of the rod. The distance, in metres, between the bottom of the sphere and the bottom of the rod is

(a) 3 (b) $2\sqrt{5}$ (c) 5 (d) 6 (e) none of these

**9**. David went to a casino with \$64 and played a certain game six times. Each time, he bet half the amount of money he had at the time. The chance of winning this game was $\frac{1}{2}$, and David won three of the six games, though it was not recorded which three they were. In each of those three games, he won an amount of money equal to his bet. After these six games, the amount of money David had was

(a) \$27 (b) \$32 (c) \$64 (d) \$128 (e) dependent on which games David lost

**10**. The number of integers $n$ which satisfy $(n^2 - n - 1)^{n+2} = 1$ is

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

**11**. In triangle $ABC$, $\angle BCA = 90°$. Points $E$ and $F$ lie on the hypotenuse $AB$ such that $AE = AC$ and $BF = BC$. If $\angle ECF = x$, then

(a) $x = 30°$ (b) $30° < x < 45°$ (c) $x = 45°$ (d) $45° < x < 60°$ (e) $60°$

**12**. The integer part of a number is the largest integer not exceeding the number, and the fractional part of a number is the difference between it and its integer part. For instance, the integer part of $\frac{5}{3}$ is 1 and the fractional part is $\frac{2}{3}$. Suppose that the product of the integer parts of two positive rational numbers is 5, and the product of their fractional parts is $\frac{1}{4}$. Their product is

(a) $\frac{5}{4}$ (b) $\frac{21}{4}$ (c) $\frac{13}{2}$ (d) $\frac{33}{4}$ (e) not uniquely determined

**13**. The sum of the positive divisors of 120 is 360. The sum of the reciprocals of the positive divisors of 120 is

(a) 2 (b) 3 (c) 4 (d) 6 (e) not uniquely determined

**14**. The expression $\dfrac{a-b}{a+b}+\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$ simplifies to

(a) $1$ (b) $a+b+c$ (c) $\frac{1}{a+b+c}$ (d) $\frac{2(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$ (e) none of these

**15**. A certain positive integer has $m$ digits when written in base $3$ and $m+1$ digits when written in base $2$. The largest possible value of $m$ is

(a) $1$ (b) $2$ (c) $3$ (d) $4$ (e) $5$

**16**. $ABCD$ is a square. A line through $B$ intersects the extension of $CD$ at $E$, the side $AD$ at $F$ and the diagonal $AC$ at $G$. If $BG=3$ and $GF=1$, then the length of $FE$ is

(a) $4$ (b) $6$ (c) $8$ (d) $10$ (e) $12$

Now for the solutions. Hope you did not peek!

## ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION 2000-2001

### Solutions to Part I

**1**. (e) The number of pennies Amy has must be a multiple of $5$. If she has $55$ pennies, then the maximum value of the other $3$ coins is $30$ cents, and the total is at most $85$ cents. If she has $45$ pennies, then the minimum value of the other $13$ coins is $65$ cents, and the total is at least $120$ cents. If she has fewer pennies, the total will be higher. Hence, she has $50$ pennies. If the other $8$ coins are all nickels, the total will be $10$ cents short. Hence, two of the nickels must be changed into dimes.

**2**. (a) Nima covers $15$ kilometres in $2.5$ hours.

**3**. (d) The number is equal to $10^{2000}$ times $5^4=625$.

**4**. (c) Since $1+2+\cdots+10=55$ and $1+2+\cdots+11=66$, the bug is on its eleventh segment $1$ minute after the start.

**5**. (e) The minimum sum of the smallest thirteen of the numbers is $1+2+\cdots+13=91$. Hence, the maximum sum of the largest two is $13\cdot 15-91=104$. It follows that the maximum value of the second largest number is $51$.

**6**. (c) When expanded, $(p(x))^2$ becomes

$$a^2x^{2k}+b^2x^{2\ell}+c^2x^{2m}+d^2x^{2n}$$
$$+2abx^{k+\ell}+2acx^{k+m}+2adx^{k+n}+2bcx^{\ell+m}+2bdx^{\ell+n}+2cdx^{m+n}\,.$$

By choosing $k=1$, $\ell=10$, $m=100$ and $n=1000$, all the terms in the above expansion are distinct.

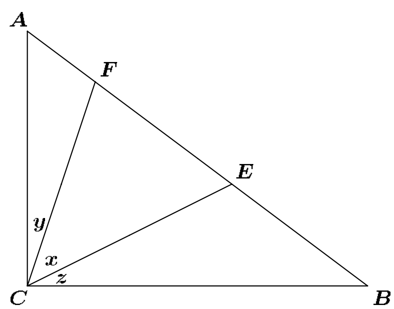
**7**. **(d)** Long division yields $x^3 - 2x + 1 = (x^2 - x - 1)(x + 1) + 2$. Since $x^2 - x - 1 = 0$, $x^3 - 2x + 1 = 2$.

**8**. **(c)** In the vertical plane containing the rod and the centre of the sphere, the cross-section of the sphere is a circle. From the bottom of the rod, two equal tangents can be drawn to this circle, one running along the rod and the other connecting the bottom of the sphere and the bottom of the rod.

**9**. **(a)** After each game in which he lost, David had half as much money as he had before. After each game in which he won, David had one and a half times as much money as he had before. No matter which three games he won, he would have $\$64(\frac{1}{2})^3(\frac{3}{2})^3 = \$27$ at the end.

**10**. **(d)** There are three cases. First, $n + 2 = 0$, while $n^2 - n - 1 \neq 0$. This yields $n = -2$. Second, $n^2 - n - 1 = -1$ while $n + 2$ is a non-zero even integer. This yields $n = 0$. Finally, $n^2 - n - 1 = 1$ while $n + 2$ is any non-zero integer. This yields $n = -1$ and $n = 2$.

**11**. **(c)** Let $\angle ACF = y$ and $\angle BCE = z$, as illustrated in the diagram. Then $x + y + z = 90°$. Since $AC = AE$, $\angle AEC = \angle FEC = x + y$. Similarly, $\angle EFC = x + z$. It follows that $x + (x + y) + (x + z) = 180°$, so that $2x = 90°$.




**12**. **(e)** The integer parts of these two rational numbers must be 5 and 1, but there are many possible choices for their fractional parts, such as $\frac{1}{3}$ and $\frac{3}{4}$. Now $(5 + \frac{1}{3})(1 + \frac{3}{4}) = \frac{28}{3}$, while $(5 + \frac{3}{4})(1 + \frac{1}{3}) = \frac{23}{3}$.

**13**. **(b)** When the reciprocal of a divisor of 120 is multiplied by 120, the product is also a divisor of 120. When the sum of the reciprocals of all the divisors of 120 is multiplied by 120, the product is the sum of all the divisors of 120, which is given to be 360.

**14**. **(e)** We have $\dfrac{a-b}{a+b} + \dfrac{b-c}{b+c} = \dfrac{2b(a-c)}{(a+b)(b+c)}$, while

$$\frac{c-a}{c+a}\left(1 + \frac{(a-b)(b-c)}{(a+b)(b+c)}\right) = \frac{c-a}{c+a} \cdot \frac{2b(c+a)}{(a+b)(b+c)} = \frac{2b(c-a)}{(a+b)(b+c)}.$$

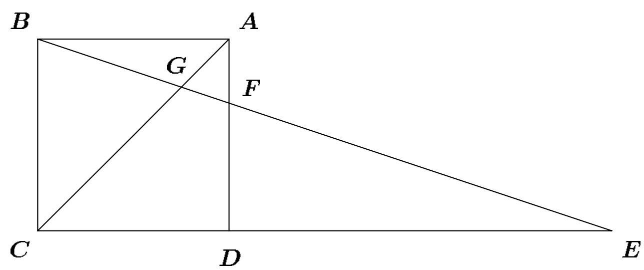
Hence, the original expression simplifies to 0.

**15**. **(d)** The chart shows the numbers of digits of a positive integer $n$ in bases $2$ and $3$. Since the number of digits in base $2$ increases every second integer, while that in base $3$ every third, the largest number with one digit more in base $2$ than in base $3$ is $9$.

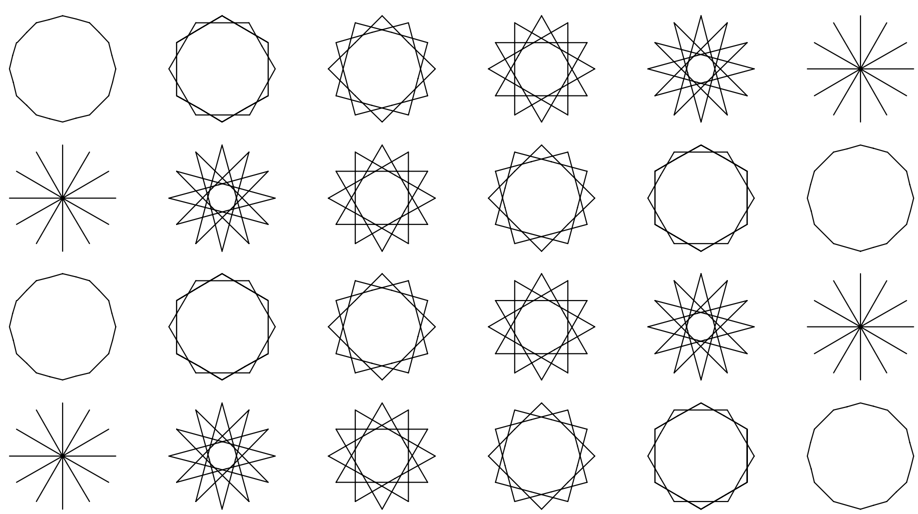
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Base 2 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 |
| Base 3 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 |

**16**. **(c)** In the diagram, triangles $BCG$ and $FAG$ are similar, as are triangles $ABF$ and $DEF$.

Hence, $\dfrac{EF}{BF} = \dfrac{DF}{AF} = \dfrac{BC - AF}{AF} = \dfrac{BG}{FG} - 1 = 2$.




I will be forwarding my *Skoliad* files to Shawn. Please send him your school contest materials and ideas.



# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5 (NEW!)**. The electronic address is
**NEW! mayhem-editors@cms.math.ca NEW!**



## Polya's Paragon

### Shawn Godin

Sometimes a very simple observation can go a long way. Let us start with a problem that, hopefully by the end of the article, you will be set to solve.

**Problem**. Given any $52$ integers, show that there exist two of them whose sum, or difference, is divisible by $100$.

If you have not encountered a problem like this before, chances are you have no idea of how to start. Worry not! The Dirichlet Box Principle (or as it is more colourfully called, the Pigeonhole Principle) will give us the necessary method.

**The Pigeonhole Principle**. If you have $n$ pigeonholes and $n + 1$ pigeons to go into them, then one of the pigeonholes must contain at least $2$ pigeons.

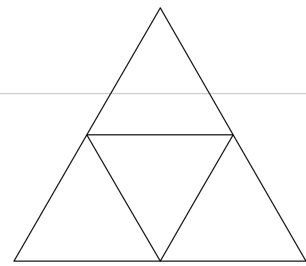
The idea behind the Principle is simple and obvious, but very powerful. If we set up our pigeonholes and pigeons carefully we can go a long way.

**Example 1**. Show that if $13$ people are gathered together, at least two are born on the same month.

*Solution*. Here the months are our pigeonholes and the people are the pigeons, and the result follows directly from the Principle.

**Example 2**. Show that, of any $5$ points chosen inside an equilateral triangle with side $2$, there are two whose distance apart is at most $1$ unit.

*Solution*. Here we can break the original triangle up into $4$ smaller equilateral triangles of side length $1$ by joining the mid-points of the sides of the original triangle (see figure). Now the 4 triangles are our pigeonholes and the points are the pigeons. So at least two of the points are in the same triangle. Since the furthest apart that these two points can be is 1 unit (if the points are at the vertices), the result follows.



**Example 3**. In a class of $25$ students, show that there is a group of at least $4$ who were born on the same day of the week.

*Solution*. To solve this we need the generalized version of the Principle.

**The Pigeonhole Principle**. If you have $n$ pigeonholes and $mn+1$ pigeons to go into them, then one of the pigeonholes must contain at least $m+1$ pigeons.

Again, the idea is simple but powerful. Now if we use the days of the week as our $7$ pigeonholes, then we have $25 > 3 \times 7 + 1$ students (pigeons), and our result follows again.

Now, hopefully, you are ready to attack the original problem. Choose your pigeonholes wisely! Below are a couple of other problems to try.

**1.** Prove that if any number of people are gathered together, at least $2$ people know the same number of people. (We can assume that if $A$ knows $B$, then $B$ knows $A$.)

**2.** Prove that, of any $5$ points chosen within a square of side length $1$, there are two whose distance apart is at most $\frac{1}{\sqrt{2}}$.

**3.** Prove that, in any group of $6$ people, there is a group of $3$ who know each other or a group of $3$ who are mutual strangers. (We can assume that if $A$ knows $B$, then $B$ knows $A$.)

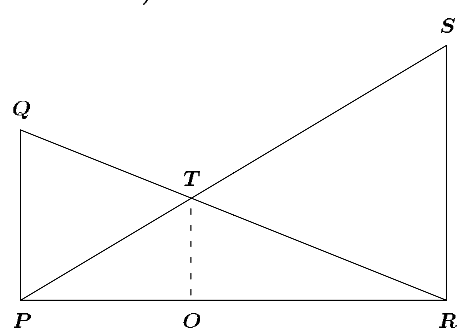
**4.** Seventeen people correspond by mail with one another — each one with all the rest. In their letters, only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic. (IMO 1964)

# Problem of the Month

Jimmy Chui, student, University of Toronto

**Problem**. A pair of telephone poles $d$ metres apart is supported by two cables which run from the top of each pole to the bottom of the other. The poles are 4 m and 6 m tall. Determine the height above the ground of the point, $T$, where the two cables intersect. What happens to this height as $d$ increases?

(1997 Euclid, Problem 7b)




**Solution.** Let the height of the poles be $a$ and $b$. Let the distance $PO$ be $c$. Let $h$ be the height above ground of the point, $T$.

Since $\triangle QPR$ and $\triangle TOR$ are similar, $a : d = h : (d - c)$, or equivalently, $d - c = \frac{dh}{a}$.

Since $\triangle SRP$ and $\triangle TOP$ are similar, $b : d = h : c$. This may be written as $c = \frac{dh}{b}$.

Adding these equations (eliminating $c$), we have $d = dh\left(\frac{1}{a} + \frac{1}{b}\right)$.

Solving for $h$ yields $h = \frac{ab}{a+b}$.

Hence, if $a = 4$ and $b = 6$, then $h = \frac{12}{5}$ (all values in metres).

Thus, the height of $T$ is $\frac{12}{5}$ metres, regardless of $d$.

Note: The fact that $h$ does not change when $d$ changes does make sense intuitively. View the collection of lines as two rigid poles and three elastic strings, and imagine pulling outward on the two poles (and increasing $d$ in the process). The horizontal measurements change, but the vertical measurements stay constant, due to the rigidness of the two poles.

# Mayhem Problems

The Mayhem Problems editors are:

| | |
|---|---|
| ***Adrian Chan*** | *Mayhem High School Problems Editor,* |
| ***Donny Cheung*** | *Mayhem Advanced Problems Editor,* |
| ***David Savitt*** | *Mayhem Challenge Board Problems Editor.* |

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from the previous issue be submitted in time for issue 4 of 2002.

# High School Solutions

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

**H281**. **Correction**. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Suppose the monic polynomial $A(z) = \sum_{k=0}^{n} a_k z^k$ can be factored into $(z - z_1)(z - z_2) \cdots (z - z_n)$, where $z_1, z_2, \ldots, z_n$ are positive real numbers and $n$ is an even number. Prove that $a_1 a_{n-1} \geq n^2 a_0$.

**H267**. *Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, and Ravi Vakil, MIT, Cambridge, MA, USA.*

Without the additional note $a$, $b$, $c$ are real numbers, you get an additional pretty solution, if $a$, $b$, $c$ are vertices of an equilateral triangle centered at the origin in the Argand plane.

*Editors note*: Two typos, introduced by the rookie editor, appear in the original solution: $x = \frac{a+b}{c+d}$ should be $x = \frac{a+c}{b+d}$ and the conclusion should have $t$ is a non-zero real number.

**H269**. Find the lengths of the sides of a triangle with $20$, $28$, and $35$ as the lengths of its altitudes.

*Solution by Michel Bataille, Rouen, France; and Andrei Simion, student, Brooklyn Technical High School, Brooklyn NY, USA.*

Let $a$, $b$, $c$ be the sides corresponding to the altitudes $20$, $28$, $35$, respectively, and let $A$ be the area of the triangle.

Thus, $2A = 20a = 28b = 35c$ and we get, $b = \frac{5a}{7}$ and $c = \frac{4a}{7}$.

If $s$ is the semi-perimeter of the triangle, then $s = \frac{1}{2}(a+b+c) = \frac{8a}{7}$.

By Heron's Formula, we have

$$A \;=\; \sqrt{s(s-a)(s-b)(s-c)} \;=\; \sqrt{\left(\frac{8a}{7}\right)\left(\frac{a}{7}\right)\left(\frac{3a}{7}\right)\left(\frac{4a}{7}\right)} \;=\; \frac{4a^2\sqrt{6}}{49}\,.$$

But, recall that $A = 10a$.

Hence, $a = \dfrac{245\sqrt{6}}{12}$, $b = \dfrac{175\sqrt{6}}{12}$, $c = \dfrac{35\sqrt{6}}{3}$.

**H270**. Find all triangles $ABC$ that satisfy

$$\sin(A-B) + \sin(B-C) + \sin(C-A) \;=\; 0\,.$$

*Solution by Michel Bataille, Rouen, France; Andrei Simion, student, Brooklyn Technical HS, Brooklyn NY, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Using elementary trigonometric formulae, we get:

$$\begin{aligned}
&\sin(A-B) + \sin(B-C) + \sin(C-A) \\
&\quad=\; 2\sin\left(\frac{A-C}{2}\right)\cos\left(\frac{A-2B+C}{2}\right) + 2\sin\left(\frac{C-A}{2}\right)\cos\left(\frac{C-A}{2}\right) \\
&\quad=\; 2\sin\left(\frac{A-C}{2}\right)\left\{\cos\left(\frac{A-2B+C}{2}\right) + \cos\left(\frac{C-A}{2}\right)\right\} \\
&\quad=\; -4\sin\left(\frac{A-C}{2}\right)\sin\left(\frac{C-B}{2}\right)\sin\left(\frac{A-B}{2}\right)\,.
\end{aligned}$$

Hence, our given equation becomes,

$$\sin\left(\frac{A-C}{2}\right)\sin\left(\frac{C-B}{2}\right)\sin\left(\frac{A-B}{2}\right) \;=\; 0\,.$$

From this, we can conclude that $A = B$, or $B = C$, or $C = A$. This implies that $ABC$ must be an isosceles (or equilateral) triangle.

*Also solved by HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea.*

**H271**. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to $x$. Let $n = \lfloor 1/(a - \lfloor a \rfloor) \rfloor$ for some positive real number $a$.

Show that $\lfloor (n+1)a \rfloor \equiv 1 \pmod{(n+1)}$.

*Solution by Michel Bataille, Rouen, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

For $n$ to be well defined, we assume that $a$ is not an integer. From the properties of the floor function, we get:

$\lfloor a \rfloor < a < \lfloor a \rfloor + 1$ and $n \le \frac{1}{a - \lfloor a \rfloor} < n + 1$. Since $a - \lfloor a \rfloor > 0$, this yields $n(a - \lfloor a \rfloor) \le 1 < (n+1)(a - \lfloor a \rfloor)$ and $1 + (n+1)\lfloor a \rfloor < (n+1)a \le 1 + n\lfloor a \rfloor + a$. Using the fact that $a < \lfloor a \rfloor + 1$, we get $1 + (n+1)\lfloor a \rfloor < (n+1)a < 2 + (n+1)\lfloor a \rfloor$.

It follows that $\lfloor (n+1)a \rfloor = 1 + (n+1)\lfloor a \rfloor$, meaning that $\lfloor (n+1)a \rfloor \equiv 1 \pmod{(n+1)}$.

**H272**. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Let $\{a_1, a_2, a_3, \ldots, a_n\}$ be a set of real numbers. Let $s = a_1 + a_2 + \cdots + a_n$. Show that $\sum_{i=1}^{n}\sum_{k=1}^{n}(a_k - a_i)(s - a_i) \ \ge \ 0$.

*Solution by Michel Bataille, Rouen, France; Vedula N. Murty, Dover, PA, USA; Andrei Simion, Brooklyn Technical HS, Brooklyn NY, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{k=1}^{n}(a_k - a_i)(s - a_i) &= \sum_{i=1}^{n}(s - a_i)\sum_{k=1}^{n}(a_k - a_i) \\
&= \sum_{i=1}^{n}(s - a_i)(s - na_i) \\
&= \sum_{i=1}^{n}\left(s^2 - (n+1)(a_i)(s) + n(a_i)^2\right) \\
&= n\left(s^2\right) - (n+1)\left(s^2\right) + n\sum_{i=1}^{n}(a_i)^2 \\
&= n\sum_{i=1}^{n}(a_i)^2 - \left(\sum_{i=1}^{n} a_i\right)^2 \\
&= \left(\sum_{i=1}^{n} 1^2\right)\left(\sum_{i=1}^{n}(a_i)^2\right) - \left(\sum_{i=1}^{n} a_i\right)^2 \ \ge \ 0
\end{aligned}$$

by the Cauchy-Schwarz Inequality.

# Advanced Solutions

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

**A245**. Show that a polygon with fixed side lengths has maximal area when it can be inscribed in a circle.

*Solution.*

Consider the polygon with the given fixed side lengths which is inscribed in a circle, and attach to each side the corresponding segment of the circumcircle (the region bounded by the side and the arc of the circle between the two endpoints of the side). Note that the area of the polygon is exactly the area of the segments subtracted from the area of the circle.

Consider the polygon with its sides in a different configuration, but still with the segments from before attached. The area of the polygon is still at most the area bounded by the circular arcs minus the fixed area of the segments, but the area bounded by the circular arcs has the same perimeter as the original circumscribing circle, and thus cannot have a larger area. This means that the polygon's area is also bounded by the area of the original polygon, inscribed in the circle.

**A246**. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Given a triangle with angles $A$, $B$, $C$, circumradius $R$, and inradius $r$, prove that

$$1+\frac{r}{R}\leq \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} \;\leq\; \frac{17}{12}+\frac{r}{6R}\,.$$

*Solution by Vedula N. Murty, Dover, PA, USA.*

The inequality $\sum\sin\frac{A}{2}\geq 1+\frac{r}{R}$ is well known, and a simple proof of this is $2\sum\cos A = \sum(\cos B+\cos C) = \sum 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) \leq \sum 2\sin\frac{A}{2}$ (where each sum is cyclic). Thus, $\sum\sin\frac{A}{2}\geq\sum\cos A = 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1+\frac{r}{R}$.

For $\sum\sin\frac{A}{2}\leq\frac{17}{12}+\frac{r}{6R}$, we use the known inequality $2\sum\sin\frac{B}{2}\sin\frac{C}{2}\leq\sum\cos A$ and the equation $\sum\sin^2\frac{A}{2}=\frac{1}{2}\sum(1-\cos A)=\frac{3}{2}-\frac{1}{2}\sum\cos A$ to get

$$\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2 \;\leq\; \frac{3}{2}+\frac{1}{2}\sum\cos A\,.$$

Using the formula $\sum\cos A = 1+\frac{r}{R}$ gives us the inequality

$$\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2 \;\leq\; \frac{3}{2}+\frac{1}{2}\left(1+\frac{r}{R}\right) \;=\; 2+\frac{x}{2}\,,$$

where $0<x=\frac{r}{R}\leq\frac{1}{2}$.

Thus, $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq\sqrt{2+\frac{x}{2}}$.

Finally, $0\leq\frac{(2x-1)^2}{144}$, so that $2+\frac{x}{2}\leq\left(\frac{17}{12}+\frac{1}{6}x\right)^2$, and the result follows.

*Also solved by Kee-Wai Lau, Hong Kong, China.*

**A248.**

(a) Prove that in every sequence of $79$ consecutive positive integers written in the decimal system, there is a positive integer whose sum of digits is divisible by $13$.

(1997 Baltic Way)

(b) Give a sequence of $78$ consecutive positive integers each with a sum of digits not divisible by $13$.

*Solution by Michel Bataille, Rouen, France.*

(a) We will denote by $[a_ka_{k-1}\ldots a_1a_0]$ the integer $a_0+10a_1+\cdots+10^{k-1}a_{k-1}+10^ka_k$. We will say that such an integer is *good* when $a_k+a_{k-1}+\cdots+a_1+a_0$ is divisible by $13$.

Consider a list $L$ of ten consecutive integers of the form:

$$[a_ka_{k-1}\cdots a_10],[a_ka_{k-1}\cdots a_11],\ldots,[a_ka_{k-1}\cdots a_19].$$

If the sum $a_k+a_{k-1}+\cdots+a_1$ is congruent modulo $13$ to one of the integers $13$, $12$, $\ldots$, $5$, $4$, then clearly $L$ contains a good integer.

Now, we observe that a sequence of $79$ consecutive positive integers necessarily includes $7$ lists such as $L$; moreover, there must be $4$ consecutive lists out of these $7$ which are between two integers of the form $[b_kb_{k-1}\cdots b_200]$ and $[b_kb_{k-1}\cdots b_299]$, say:

$L_1$ $[b_kb_{k-1}\cdots b_2b_10],\ldots,[b_kb_{k-1}\cdots b_2b_19]$
$L_2$ $[b_kb_{k-1}\cdots b_2(b_1+1)0],\ldots,[b_kb_{k-1}\cdots b_2(b_1+1)9]$
$L_3$ $[b_kb_{k-1}\cdots b_2(b_1+2)0],\ldots,[b_kb_{k-1}\cdots b_2(b_1+2)9]$
$L_4$ $[b_kb_{k-1}\cdots b_2(b_1+3)0],\ldots,[b_kb_{k-1}\cdots b_2(b_1+3)9]$
where $b_1\in\{0,1,2,3,4,5,6\}$.

Then, either $L_1$ contains a good integer, or $b_k+b_{b-1}+\cdots+b_2+b_1\equiv 3$, $2$, or $1\pmod{13}$, in which case $L_2$, $L_3$, or $L_4$ contains a good integer, respectively.

(b) When there are only $78$ integers, we have only $6$ lists such as $L$ with $3$ on either side of a multiple of $100$ in order for the above to fail. For example, the $78$ positive integers from $9,999,999,961$ to $10,000,000,038$ inclusively do not have a sum of digits divisible by $13$.

# Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

**C93**. Let $H$ be a subset of the positive integers with the property that if $x$, $y \in H$ then $x + y \in H$. Define the *gap sequence* $G_H$ of $H$ to be the set of positive integers not contained in $H$.

(a) If $G_H$ is a finite set, prove that the arithmetic mean of the integers in $G_H$ is less than or equal to the number of elements in $G_H$.

(b) Determine all sets $H$ for which equality holds in part (a).

*Solution*. We remark, before we begin, that a subset of the positive integers with the property that if $x$, $y \in H$ then $x + y \in H$ is called a *semigroup*.

(a) We proceed by induction on the size $g$ of $G_H$. Since $G_H$ is non-empty if and only if $1 \in G_H$, the case $g = 1$ is clear. For larger $g$, the key observation is:

**Lemma**. The largest element of $G_H$ is at most $2g - 1$.

**Proof:** For an integer $n \geq 2g$, consider the pairs of integers $(1, n-1), \ldots, (g, n-g)$. If $n \in G_H$, then $H$ has exactly $g - 1$ other gaps, and so for at least one of the pairs of integers $(1, n-1)$, ..., $(g, n-g)$, both integers in the pair are contained in $H$. Since $n$ is the sum of this pair of integers, $n$ is in $H$, and so is not in $G_H$.

Next, we observe that if $x$ is the largest element of $G_H$, then $H' = H \cup \{x\}$ is also a semigroup, and $G_{H'}$ has size $g - 1$. By the induction hypothesis,

$$\frac{1}{g-1} \sum_{y \in G_{H'}} y \;\leq\; g - 1\,.$$

Since $x \leq 2g - 1$, we get

$$\sum_{y \in G_H} y \leq (g-1)^2 + 2g - 1 \;=\; g^2\,,$$

and the desired result follows by induction.

(b) Equality holds in (a) exactly when $x = 2g - 1$ and the elements of $G_{H'}$ have average size $g - 1$. Inductively, it follows that equality holds whenever $G_H$ has the form $\{1, 3, \ldots, 2g-1\}$; that is, whenever $H$ has the form

$$\{\text{even integers less than } 2g \text{ and all integers at least } 2g\}\,.$$

**C94**. *Proposed by Edward Crane and Russell Mann, graduate students, Harvard University Cambridge, MA, USA.*

Suppose that $V$ is a $k$–dimensional vector subspace of the Euclidean space $\mathbb{R}^n$ which is defined by linear equations with coefficients in $\mathbb{Q}$. Let $\Lambda$ be the lattice in $V$ given by the intersection of $V$ with the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, and let $\Lambda^\perp$ be the lattice given by the intersection of the perpendicular vector space $V^\perp$ with $\mathbb{Z}^n$. Show that the ($k$–dimensional) volume of $\Lambda$ is equal to the (($n-k$)–dimensional) volume of $\Lambda^\perp$.

*Solution by Mark Dickinson, graduate student, Harvard University, Cambridge, MA, USA.*

Let us say that a sublattice of $\mathbb{Z}^n$ is *primitive* if it arises as the intersection with $\mathbb{Z}^n$ of some vector subspace of $\mathbb{R}^n$. Any sublattice of $\mathbb{Z}^n$ can be represented by a matrix with integer entries whose rows or columns give a basis for the lattice. The following lemma uses this to give a characterization of the primitive lattices.

We recall the following terminology: If $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are vectors (of the same length), then their $\mathbb{Z}$–span (respectively, their $\mathbb{R}$–span) is the collection of all vectors of the form

$$\sum_i c_i \cdot \mathbf{a}_i$$

with $c_i \in \mathbb{Z}$ (respectively, with $c_i \in \mathbb{R}$).

**Lemma**. Suppose that $A$ is an $m \times n$ matrix with integer entries, and write $\Lambda$ for the $\mathbb{Z}$–span of the rows of $A$. Then $\Lambda$ is a primitive sublattice of $\mathbb{Z}^n$ having the rows of $A$ as a $\mathbb{Z}$–basis, if and only if $A$ can be completed, by adding extra rows, to an $n \times n$ matrix $B$ with integer entries such that $B$ is invertible and $B^{-1}$ has integer entries.

**Proof**. First suppose that $A$ can indeed be completed to some such $B$. Take $V$ to be the $\mathbb{R}$–span of the rows of $A$. We claim that $\Lambda$ is the intersection of $V$ with $\mathbb{Z}^n$. Since $B^{-1}$ exists and has integer entries, every element of $\mathbb{Z}^n$ is an element of the $\mathbb{Z}$–span of the rows of $B$, namely, $\mathbf{v} = (\mathbf{v}B^{-1})B$. In fact, since the rows of $B$ are linearly independent, every element of $\mathbb{Z}^n$ can be written uniquely as a combination of the rows of $B$. But every element of $V$ is an element of the $\mathbb{R}$–span of the rows of $A$, and so by the above uniqueness, an element in both $V$ and $\mathbb{Z}^n$ is in the $\mathbb{Z}$–span of the rows of $A$. Hence $V \cap \mathbb{Z}^n \supset \Lambda$, and since the reverse inclusion is clear, this half of the lemma follows.

Conversely, suppose that $\Lambda$ is primitive and that the rows of $A$ form a basis for $\Lambda$, so are linearly independent. Then by elementary row and column operations we can reduce $A$ to a matrix of the form $(D\ 0)$ where $D$ is a diagonal $m \times m$ matrix with positive entries $d_1, \ldots, d_m$. It follows that by row operations alone, we can reduce $A$ to a matrix $A'$ in which all entries of the $i^{\text{th}}$ row are divisible by $d_i$. (To see this, one may easily check that the property of being reducible by elementary row operations to a matrix in which all the entries of the $i^{\text{th}}$ row are divisible by $d_i$ is preserved under both elementary row and column operations. Since $(D\ 0)$ has this property, so does $A$.)

The rows of $A'$ give a new basis of $\Lambda$. Now if any of the $d_i$ were larger than 1 we would be able to divide the corresponding row of $A'$ by $d_i$ and take the $\mathbb{Z}$–span of the rows to obtain a sublattice of $\mathbb{Z}^n$ which is contained in $V$ and strictly contains $\Lambda$; but this would violate the assumption that $\Lambda$ is primitive. Hence all the $d_i$ are equal to 1, and by augmenting the matrix $(D\ 0)$ to the identity matrix and then undoing the row and column operations that gave $(D\ 0)$ from $A$ we obtain the desired matrix $B$ extending $A$. ∎

With this lemma in hand, we now suppose that the rows of $A$ are the basis of a primitive lattice $\Lambda$, and extend $A$ as above to obtain the matrix $B$. Consider the rightmost $n-m$ columns of the inverse of $B$. By the lemma, these columns give a basis of a primitive lattice of rank $n-m$, and from the equation $B \cdot B^{-1} = I_n$ we see that this lattice is perpendicular to $\Lambda$, hence must be the lattice $\Lambda^\perp$.

Recall that the square of the volume of a lattice $\Lambda$ is equal to the determinant of $A \cdot A^T$ for any matrix $A$ whose rows give a basis of $\Lambda$. Therefore, since $B$ has determinant $\pm 1$, in order to complete the solution it is enough to prove the following:

**Proposition**. Suppose $B$ is an invertible $n \times n$ real matrix, and suppose that $A$ is the $m \times n$ given by the first $m$ rows of $B$ while $C$ is the $n \times (n-m)$ matrix given by the last $n-m$ columns of $B^{-1}$. Then

$$\det(A \cdot A^T) \;=\; \det(C^T \cdot C) \;=\; \det(B)^2 \,.$$

**Proof**. We can always write $B = L \cdot Q$ where $L$ is a lower-triangular matrix and $Q$ is an orthogonal matrix (that is, $Q^T \cdot Q = Q \cdot Q^T = I_n$). Writing $L$ in the form

$$L \;=\; \begin{pmatrix} L_0 & 0 \\ L_2 & L_1 \end{pmatrix}$$

with $L_0$ an $m \times m$ matrix and $L_1$ an $(n-m) \times (n-m)$ matrix, we have

$$A \;=\; (L_0\ 0)Q \quad \text{and} \quad C = Q^T \begin{pmatrix} 0 \\ L_1^{-1} \end{pmatrix} \,,$$

from which

$$\det(A \cdot A^T) \;=\; \det(L_0 \cdot L_0^T) \;=\; \det(L_0)^2$$

and similarly $\det(C^T C) = \det(L_1)^{-2}$. Since we also know that $\det(B) = \det(L_0)\det(L_1)$, the proposition follows.

# Real Roots of Cubic Polynomials

Keith A. Brandt and Joseph C. Roma

**Abstract**: Elementary calculus is used to find conditions that indicate when a cubic polynomial will have all real roots. This note serves as an example of the power of calculus and gives some insight into a famous problem from classical algebra.

In this note, we study the roots (counting multiplicities) of polynomials with real coefficients. The following statement is an immediate consequence of the quadratic formula: $f(x) = ax^2 + bx + c$ has two real roots if and only if $b^2 - 4ac \geq 0$. We use elementary concepts from calculus to derive an analogous statement for cubic polynomials. We have not found this problem in standard calculus texts, although specific cases are studied in [2] (see pages 130-131). The results in this note are often presented along with Cardan's formulas for the cubic ([1], p. 9). Our approach shows that some important information regarding the nature of the roots can be found without deriving Cardan's formulas.

Let $f(x)$ be a cubic polynomial. We can use a substitution ([1], p. 9) to eliminate the square term, so write $f(x) = x^3 + bx + c$. First note that for $f(x)$ to have more than one real root, it is necessary for $f'(x) = 3x^2 + b$ to have a real root. This happens if and only if $b \leq 0$. This condition ensures that the graph of $y = f(x)$ "bends" sufficiently.

**Condition 1**: If $f(x)$ has three real roots, then $b \leq 0$.

Condition 1 above is not, however, sufficient for $f(x)$ to have three real roots. The size of the coefficient $c$ will determine whether the graph of $y = f(x)$ will touch the $x$–axis more than once. To find the allowable values for $c$, assume $b \leq 0$ and consider the function $g(x) = x^3 + bx$. If $b = 0$, then necessarily, $c = 0$. If $b < 0$, the maximum and minimum values of $g(x)$ determine the values of $c$ that guarantee $f(x)$ has three real roots. They are easy to calculate:

$$\begin{aligned} \max(g) &= g\left(-\sqrt{\tfrac{-b}{3}}\right) = \tfrac{-2b}{3}\sqrt{\tfrac{-b}{3}}\,, \\ \min(g) &= g\left(\sqrt{\tfrac{-b}{3}}\right) = \tfrac{2b}{3}\sqrt{\tfrac{-b}{3}}\,. \end{aligned}$$

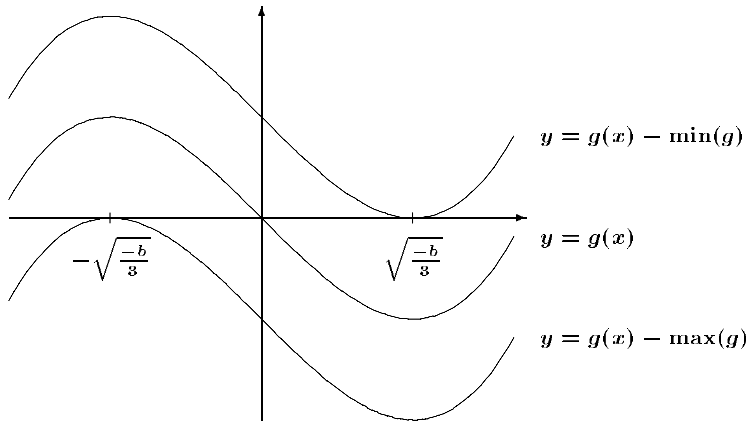



Figure 1. Finding the allowable values for $c$.

We must then have $-\max(g) \leq c \leq -\min(g)$ (see figure 1) which simplifies to $\frac{b^3}{27} + \frac{c^2}{4} \leq 0$.

**Condition 2**: If $f(x)$ has three real roots, then $\frac{b^3}{27} + \frac{c^2}{4} \leq 0$.

Note that Condition 2 implies Condition 1. Furthermore, it is easy to see that Condition 2 implies that $f(x)$ has three real roots. Hence we have the following ([1], p. 9):

**Theorem**: $f(x) = x^3 + bx + c$ has three real roots if and only if $\frac{b^3}{27} + \frac{c^2}{4} \leq 0$.

When the inequality is strict, $f(x)$ has three distinct roots. If $b = 0$, then $c = 0$ so $f(x)$ has a root of multiplicity three at $x = 0$. If $b < 0$ and $\frac{b^3}{27} + \frac{c^2}{4} = 0$,then $f(x)$ has a double root at one of $x = \pm\sqrt{\frac{-b}{3}}$.

We give the corresponding statements for the more general cubic polynomial $f(x) = x^3 + ax^2 + bx + c$. They can be derived using the approach above — with more technical calculations — or via the substitution $x = y - \frac{a}{3}$.

**Condition 1**: If $f(x)$ has three real roots, then $3b - a^2 \leq 0$.

**Condition 2**: If $f(x)$ has three real roots, then

$\left(\frac{2}{27}\right)^2 (3b - a^2)^3 + \left(c - \frac{1}{3}ab + \frac{2}{27}a^3\right)^2 \leq 0$.

Not surprisingly, the expressions in these conditions have a striking resemblance to those in Cardan's formulas.

**Example 1**: Let $f(x) = x^3 + 2x^2 + 3x + 4$. Since $3b - a^2 = 5$, $f(x)$ fails Condition 1 and thus has a non-real root.

**Example 2**: Let $f(x) = x^3 + 3x^2 + 2x + 1$. Since $3b - a^2 = -3$, $f(x)$ passes Condition 1; however, the quantity in Condition 2 equals $\frac{23}{27}$. Thus $f(x)$ fails Condition 2 and has a non-real root.

**Example 3**: Let $f(x) = x^3 - 3x^2 - 3x + 1$. The quantity in Condition 2 equals $-16$. Hence $f(x)$ has three real roots.

Conditions 1 and 2 can be used to gain some information about higher degree polynomials. We use a lemma (implicitly used for Condition 1), whose proof follows from Rolle's Theorem and basic properties of repeated roots.

**Lemma**: Let $f(x)$ be a polynomial of degree $n$. If $f(x)$ has $n$ real roots, then $f'(x)$ has $n-1$ real roots.

To use the lemma, let $f(x) = x^n + d_{n-1}x^{n-1} + d_{n-2}x^{n-2} + \cdots + d_1x + d_0$, and consider the cubic polynomial $k(x) = f^{(n-3)}(x)$. If $k(x)$ has a non-real root, then $f(x)$ has a non-real root. Therefore, by applying Conditions 1 and 2 to $k(x)$, we can find conditions necessary for $f(x)$ to have all real roots. For example, Condition 1 applied to $k(x)$ yields the following:

**Condition 3**: If $f(x)$ has all real roots, then $2nd_{n-2} - (n-1)d_{n-1}^2 \leq 0$.

Loosely speaking, Condition 3 says that if $d_{n-2}$ is large enough (relative to $d_{n-1}$), then $f(x)$ has a non-real root.

**Example 4**: Let $f(x) = x^4 + 4x^3 + 8x^2 + 3x - 1$. The expression in Condition 3 equals 16, so $f(x)$ has a non-real root. Note that the constant term can be chosen so that $f(x)$ has no real roots. The point is that the coefficients $d_3 = 4$ and $d_2 = 8$ do not allow $y = f(x)$ to bend enough to have four real roots.

**Example 5**: Let $f(x) = x^5 + 2x^3 - 4x^2 + 1$. Since $d_4 = 0$ and $d_3$ is positive, Condition 3 is not satisfied. Thus $f(x)$ has a non-real root.

In the spirit of the Rational Roots Theorem and Descartes' Rule of Signs, the conditions studied here give more insight into the connections between the roots of a polynomial and its coefficients. Moreover, these conditions follow from elementary calculus and are easily programmed into a computer or programmable calculator.

**Exercises**

1. Derive Condition 2 for the general case.
2. Prove the lemma.
3. Derive Condition 3.
4. Derive a weaker version of Condition 3 that does not involve $n$.
5. Apply Condition 2 to $k(x)$ to derive Condition 4 for higher degree polynomials.

## References

[1] Beyer, William H., ed., *CRC Standard Mathematical Tables*, CRC Press, Boca Raton, FL, USA (1981).

[2] Kodaira, Kunihiko, ed., *Basic Analysis: Japanese Grade 11*, American Mathematical Society, Providence, RI, USA (1996).

Keith A. Brandt — Joseph C. Roma

Department of Mathematics, Computer Science, and Physics
Rockhurst University
Kansas City, MO 64110, USA

Keith.Brandt@Rockhurst.edu — joecool17@earthlink.net

# PROBLEMS

*Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.*

*In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.*

*To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 December 2001**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in LaTeX format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

**Addendum — 2338**. [1998 : 234, 1999 : 243] Peter Y. Woo, Biola University, La Mirada, CA, USA, points out that problem 2338 was originated by Wu WeiChao, Honan Normal University, China, in 1995. Janous had stated that he remembered the problem, but could not give a reference. Perhaps this is it?

**Correction — 2598**. [2000 : 499] D.J. Smeenk, Zaltbommel, the Netherlands, points out that part (d) should read $AD < a$.

**Clarification — 2602$\star$**. [2001 : 48] In response to a query, Walther Janous, Ursulinengymnasium, Innsbruck, Austria, states that his intention was that it should be shown that no member of $Q(6, 3, -2)$ is a square number.

**Correction — 2620**. [2001 : 138, 213] The Editor regrets that he has to state that the interval in this problem is, in fact, $[1/2, 3/2]$, not $[1/3, 3/2]$, as stated in the April 2001 correction.

**2637**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $ABC$ is an isosceles triangle with $AB = AC$. Let $D$ be a point on side $AB$, and let $E$ be a point on $AC$ produced beyond $C$. The line $DE$ meets $BC$ at $P$. The incircle of $\triangle ADE$ touches $DE$ at $Q$.

Prove that $BP \cdot PC \leq DQ \cdot QE$, and that equality holds if and only if $BD = CE$.

**2638**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $ABC$ is an acute angled triangle with $AB \neq AC$, and that $H$ and $G$ are the orthocentre and centroid of $\triangle ABC$ respectively. Suppose further that $\dfrac{1}{[HAB]} + \dfrac{1}{[HAC]} = \dfrac{2}{[HBC]}$, where $[PQR]$ denotes the area of $\triangle PQR$.

Prove that $\angle AGH = 90^\circ$.

**2639**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $A$ is a point outside a circle $\Gamma$. The two tangents through $A$ to $\Gamma$ touch $\Gamma$ at $B$ and $C$. A variable tangent to $\Gamma$ meets $AB$ and $AC$ at $P$ and $Q$ respectively. The line through $P$ parallel to $AC$ meets $BC$ at $R$.

Prove that the line $QR$ passes through a fixed point.

**2640**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $ABC$ is an acute angled triangle with $\angle BAC = 45^\circ$. Let $O$ be the circumcentre of $\triangle ABC$, and let $D$ and $E$ be the intersections of $BO$ and $CO$ with $AC$ and $AB$ respectively. Suppose that $P$ and $Q$ are points on $BC$ such that $OP \| AB$ and $OQ \| AC$.

Prove that $OD + OE = \sqrt{2} PQ$.

**2641**. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $H$ be a centrosymmetric convex hexagon, with area $h$, and let $P$ be its minimal circumscribed parallelogram, with area $p$.

Prove that $3p \leq 4h$.

**2642**. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that $k$ is a positive integer. Prove that $\displaystyle\sum_{n=0}^{\infty} \frac{n+2k}{2^{n+1}\binom{n+k+1}{k}} = 1$.

**2643**. *Proposed by Mª Jesús Villar Rubio, Instituto Torres Quevedo Santander, Spain.*

It is known that if a quadrilateral has sides $a$, $b$, $c$ and $d$, then its area is less than or equal to $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $s$ is the semi-perimeter ($2s = a+b+c+d$). What happens for polygons with more than four sides?

**2644**. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find a closed form expression for the sum of the first $n$ terms of the series

$$1+2+4+4+8+8+8+8+16+16+16+16+16+16+16+16+\ldots,$$

where $1$ occurs once and, for $k \geq 1$, $2^k$ occurs $2^{k-1}$ times.

**2645**. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that $a$, $b$ and $c$ are positive real numbers. Prove that

$$\frac{2\left(a^3+b^3+c^3\right)}{abc}+\frac{9(a+b+c)^2}{a^2+b^2+c^2} \;\geq\; 33\,.$$

**2646**. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that $p \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Show that there exists a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that, for all $n \in \mathbb{N}_0$, we have $f(f(n)) + f(n) = 2n + p$ if and only if $3|p$.

**2647**. *Proposed by Michel Bataille, Rouen, France.*

Let $O$, $H$ and $R$ denote the circumcentre, the orthocentre and the circumradius of $\triangle ABC$, and let $\Gamma$ be the circle with centre $O$ and radius $\rho = OH$. The tangents to $\Gamma$ at its points of intersection with the rays $[OA)$, $[OB)$ and $[OC)$ form a triangle.

Express the circumradius of this triangle as a function of $R$ and $\rho$.

**2648**. *Proposed by Michel Bataille, Rouen, France.*

Find the smallest positive integer $n$ such that the rightmost digits of $5^{2001+n}$ reproduce the digits of $5^{2001}$. (Here "digit" means "decimal digit", and the order of the digits in $5^{2001+n}$ and $5^{2001}$ must be the same.)

**2649** *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Solve the equations:

$$\begin{aligned} \sin(2x)\sin(4x)\sin(8x) &= \frac{\sqrt{3}}{8}\,, & (1)\\ \cos(2x)\cos(4x)\cos(8x) &= \frac{1}{8}\,. & (2) \end{aligned}$$

See problem 2486 [1999 : 431, 2000 : 510].

**2650** *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

In $\triangle ABC$, let $a$ denote the side $BC$, and $h_a$, the corresponding altitude. Let $r$ and $R$ be the radii of the inscribed and circumscribed circles, respectively. Prove that $ra < h_a R$.

# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

**2525** April [2000 : 177] . *Proposed by Antreas P. Hatzipolakis, Athens, Greece, and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

In $\triangle ABC$, we have $B = 135^\circ - \dfrac{A}{2}$ and $C = 45^\circ - \dfrac{A}{2}$. Show that

(a) the centre $V$ of the nine-point circle of $\triangle ABC$ lies on the side $BC$;

(b) if $A = 60^\circ$, then $AV$ bisects angle $A$.

I. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let $R$ be the circumradius of $\triangle ABC$, $M$ be the mid-point of $BC$, and $D$ be the foot of the altitude from $A$ to $BC$.

(a) Since the nine-point circle passes through $D$ and $M$ and its radius is $\dfrac{R}{2}$, to prove that $V$ lies on $BC$, it is sufficient to prove that $DM = R$. Since $B = 135^\circ - \dfrac{A}{2}$ and $C = 45^\circ - \dfrac{A}{2}$, then $2C = 90^\circ - A$ and $B = 90^\circ + C$. Hence, $\angle BAD = C$ and

$$\begin{aligned} DB &= AB\sin C = 2R\sin C\sin C = R(1-\cos(2C)) \\ &= R - R\sin A = R - \frac{a}{2}. \end{aligned}$$

Thus, $DM = DB + BM = R - \dfrac{a}{2} + \dfrac{a}{2} = R$, and $V$ lies on $BC$.

(b) If $A = 60^\circ$ then $C = 15^\circ$, and from triangle $ABD$

$$AD = AB\cos C = 2R\sin C\cos C = R\sin(2C) = \frac{R}{2} = DV,$$

which means that $\angle DAV = 45^\circ$, so that $\angle BAV = 30^\circ$ and $AV$ is the bisector of angle $A$.

II. *Solution by David Loeffler, student, Cotham School, Bristol, UK.*

Kimberling tells us that the trilinear coordinates of the nine-point centre has coordinates $\cos(B-C) : \cos(C-A) : \cos(A-B)$. Substituting the given values, we have $V$ represented by

$$\begin{aligned} &\cos 90^\circ : \cos\left(45^\circ - \frac{3A}{2}\right) : \cos\left(\frac{3A}{2} - 135^\circ\right) \\ &\qquad = 0 : \cos\left(45^\circ - \frac{3A}{2}\right) : \cos\left(\frac{3A}{2} - 135^\circ\right). \end{aligned}$$

Since the trilinear coordinates of a point are proportional to the distances of the point from the corresponding sides $BC$, $CA$, $AB$ of the triangle, it is immediate that the nine-point centre must lie on $BC$ (as the first coordinate is zero).

Furthermore, if $A = 60^\circ$ then $V$ is represented by $0 : \cos(-45)^\circ : \cos(-45)^\circ$; therefore $V$ is equidistant from $AB$ and $AC$, which implies that it lies on the bisector of angle $A$, as claimed.

*Editor's comments.* Several solvers pointed out that the condition in part (a) is both necessary and sufficient. Although Loeffler did not say so, his proof is clearly reversible: when $V$ lies on $BC$, its first coordinate is $0$, so that $B - C = 90^\circ$ and, therefore, $B = 135^\circ - \frac{A}{2}$ and $C = 45^\circ - \frac{A}{2}$. Janous, whose solution is essentially the same as Loeffler's, suggested that we all agree to some simple way of referring to Clark Kimberling's encyclopedic list of important points, somewhat like the way we refer to the "Bottema Bible." I propose that we simply say "Kimberling says so" (like our Pythagorean colleagues use of *ipse dixit*). I feared that "Kimberling's Koran" might offend some readers, and this editor does not know any suitable $k$-words. Readers may have better ideas. For those who have not seen the list, one can access it electronically,

http://cedar.evansville.edu/~ck6/encyclopedia/

(Loeffler referred to the Mathworld site, http://Mathworld.wolfram.com/, but it is presently shut down due to an ugly copyright dispute.)

or hard-copily,

Central Points and Central Lines in the Plane of a Triangle, *Mathematics Magazine*, **67**:3 (1994), 163-187. (This is an early version with 101 points listed.)

Triangle Centers and Central Triangles, *Congressus Numerantium* 129, Utilitas Mathematica Publishing Inc., University of Manitoba, 1998. (Expanded version with many further points listed.)

*Also solved by AUSTRIAN IMO TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece (2 solutions); D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposers.*

**2526**. [2000 : 177] *Proposed by K.R.S. Sastry, Bangalore, India.*

In a triangle, prove that an internal angle bisector trisects an altitude if and only if the bisected angle has the measure $\pi/3$ or $2\pi/3$.

*Preliminary comments.* To make sense of the possibility that the bisected angle be obtuse, it is necessary to rephrase the problem:

*Prove that the internal angle bisector of one angle of a triangle divides an altitude in the ratio* $1 : 2$ *if and only if the bisected angle measures* $\dfrac{\pi}{3}$ *or* $\dfrac{2\pi}{3}$.

Several readers point out that the word *trisect* (in the original statement) implies division into three equal parts, which eliminates the obtuse case.

*Solution by Jeremy Young, student, University of Cambridge, Cambridge, UK.*

In triangle $ABC$ let $BN$ be the altitude from $B$ to $AC$, and let $P$ be the point where the bisector of $\angle CAB$ meets $BN$. We consider two cases.

(i) $A \leq \dfrac{\pi}{2}$.

$$BN = AN \tan A = \frac{2AN \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} \qquad \text{and} \qquad PN = AN \tan \frac{A}{2}$$

imply that $\dfrac{PN}{BN} = \dfrac{1 - \tan^2 \frac{A}{2}}{2}$.

$PN$ is divided internally in the ratio $1 : 2$ if and only if $\dfrac{PN}{BN} = \dfrac{2}{3}$ or $\dfrac{1}{3}$. In the former case $1 - \tan^2\left(\dfrac{A}{2}\right) = \dfrac{4}{3}$ has no solution. In the latter case $1 - \tan^2\left(\dfrac{A}{2}\right) = \dfrac{2}{3}$ implies $\dfrac{A}{2} = \dfrac{\pi}{6}$ (since $A > 0$), so that $A = \dfrac{\pi}{3}$. Conversely, if $A = \dfrac{\pi}{3}$ then $\dfrac{PN}{BN} = \dfrac{1}{3}$.

(ii) $A > \dfrac{\pi}{2}$.

$$BN = AN \tan(\pi - A) = -AN \tan A \qquad \text{and} \qquad PN = AN \tan \frac{A}{2}$$

imply that $\dfrac{PN}{BN} = \dfrac{\tan^2 \frac{A}{2} - 1}{2}$.

Since $BP \geq PN$, we know that $BN$ is divided externally in the ratio $1 : 2$ if and only if $\dfrac{BP}{PN} = 2$. In this case $PN = BN$, so that $\tan^2\left(\dfrac{A}{2}\right) - 1 = 2$ and (since $\dfrac{A}{2} < \dfrac{\pi}{2}$), therefore $\dfrac{A}{2} = \dfrac{\pi}{3}$. Thus, $A = \dfrac{2\pi}{3}$. Finally, it is clear that the argument can again be reversed so that $A = \dfrac{2\pi}{3}$ is also sufficient.

*Also solved by the AUSTRIAN IMO TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JONATHAN CAMPBELL, student,* *Chapel Hill High School, Chapel Hill, NC; NIKOLAOS DERGIADES, Thessaloniki, Greece; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; OLEG IVRII, student, Cummer Valley Middle School, Toronto; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; GERRY LEVERSHA, St. Paul's School, London, England; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

**2527**. [2000 : 177] *Proposed by K.R.S. Sastry, Bangalore, India.*

Let $AD$, $BE$ and $CF$ be concurrent cevians of $\triangle ABC$. Assume that:

(a) $AD$ is a median; (b) $BE$ bisects $\angle ABC$; (c) $BE$ bisects $AD$.

Prove that $CF > BE$.

I. *Solution by the Austrian IMO team 2000.*

Let $P$ be the point of intersection of lines $AD$, $BE$, $CF$. Since $BP$ is both median and angle bisector in $\triangle ABD$, the triangle is isosceles with $BD = BA$, and $\angle BDP$ must be acute. In $\triangle CDP$ and $\triangle BDP$, we have $CD = BD$; they share the common side $DP$, and $\angle BDP$ being acute implies its supplement $\angle CDP$ is obtuse. Therefore

$$CP > BP \,. \tag{1}$$

Since $AD$, $BE$, and $CF$ are concurrent, Ceva's Theorem says that $\frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CD}{DB} = 1$; since $\frac{CD}{DB} = 1$, $\frac{AE}{EC} = \frac{FA}{BF}$ so that $EF$ and $BC$ are parallel. This implies (because $P$ is the common point of $BE$ and $CF$) that $\triangle CBP \sim \triangle FEP$.

Hence from (1), $FP > EP$, and $CF = CP + PF > BP + PE = BE$, as desired.

II. *Independent identical solutions by Manuel Benito and Emilio Fernández, I.B. Praxedes Mateo Sagasta, Logroño, Spain and Kee-Wai Lau, Hong Kong.*

In a Cartesian system assign (without loss of generality) the coordinates $A(-1,0)$ and $D(1,0)$. The line $BE$ bisects $\angle ABD$ by condition (b) and passes through $O(0,0)$ by $(c)$, so that by the bisector theorem applied to $\triangle ABD$, $BE$ is the $y$-axis and $B$ has coordinates $B(0,b)$. Because $D(1,0)$ is the mid-point of $BC$, we have $C = (2,-b)$. Moreover $E$, the intersection of $AC$ with the $y$-axis, is $E\left(0, -\frac{b}{3}\right)$, and $F$ (where $CO$ intersects $AB$) is $F\left(-\frac{2}{3}, \frac{b}{3}\right)$. We have, finally,

$$CF \;=\; \frac{4}{3}\sqrt{4+b^2} \;>\; \frac{4}{3}|b| \;=\; BE \,.$$

*Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD B. EDEN, Ateneo de Manila University, Manila, the Philippines; OLEG IVRII, student, Cummer Valley Middle School, Toronto; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College Cambridge, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA (2 solutions); and the proposer.*

*As a by-product of his solution Seiffert obtained* $CF > \frac{4}{3}AB > BE$. *These inequalities are clear from solution II above.*

**2531**. [2000 : 178] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $F$ be a convex plane set and $AB$ its diameter. The points $A$ and $B$ divide the perimeter of $F$ into two parts, $L_1$ and $L_2$, say. Prove that

$$\frac{1}{\pi - 1} < \frac{L_1}{L_2} < \pi - 1 .$$

While a number of solvers proved the inequality stated in the problem, others established a sharper set of bounds. We present two solutions.

I. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

It is known [1] that $\pi D \geq L \geq 2D$, where $L$ and $D$ are the perimeter and diameter of a convex plane set. Hence, for each of the two regions divided by $AB$

$$\pi D \geq L_1 + D \geq 2D \quad \text{and} \quad \pi D \geq L_2 + D \geq 2D , \quad \text{so that}$$

$$\pi - 1 \geq \frac{L_1}{D} \geq 1 \quad \text{and} \quad 1 \geq \frac{D}{L_2} \geq \frac{1}{\pi - 1} .$$

Then, by multiplying the latter inequalities, we get

$$\pi - 1 > \frac{L_1}{L_2} > \frac{1}{\pi - 1}$$

(the equality signs do not hold here).

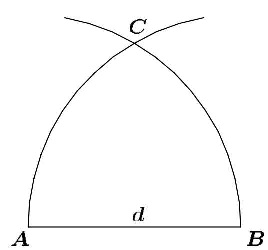
1. T. Bonnesen, W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, Idaho, 1987, p. 88.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

In what follows, we show the sharper inequality

$$\frac{3}{2\pi} \leq \frac{L_1}{L_2} \leq \frac{2\pi}{3} .$$

It is enough to show the right-hand inequality, for the other side then follows by taking reciprocals of the first one. We thus let $L_1 \geq L_2$. We have $d := \overline{AB} \leq L_2$.



Then the "$L_1$-part" of the boundary $\partial F$ has to be contained in the region determined by the $d$-circles about $A$ and $B$. Because of the convexity of $F$, it follows that:

$$L_1 \;\leq\; \overline{AC} + \overline{CB} \;=\; \frac{2d\pi}{6} + \frac{2d\pi}{6} \;=\; \frac{2d\pi}{3}\,.$$

Therefore, $\dfrac{L_1}{L_2} \;\leq\; \dfrac{2d\pi}{3d} \;=\; \dfrac{2\pi}{3}$ (which is less than $\pi - 1$).

*Also solved by AUSTRIAN IMO TEAM 2000; MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; J. SUCK, Essen, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

**2533**. [2000 : 178] *Proposed by K.R.S. Sastry, Bangalore, India.*

In the integer sided $\triangle ABC$, let $e$ denote the length of the segment of the Euler line between the orthocentre and the circumcentre.

Prove that $\triangle ABC$ is right-angled if and only if $e$ equals one half of the length of one of the sides of $\triangle ABC$.

Compare problem 2433. [1999 : 173, 2000 : 187]

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

To show, without loss of generality: $OH = \dfrac{c}{2}$ if and only if $\triangle ABC$ is right-angled.

One direction is clear — if $C$ is a right angle then $OH = R = \dfrac{c}{2}$.

For the other direction, we assume that $OH = \dfrac{c}{2}$. Using a familiar formula [found in books that deal with the Euler line, such as Nathan Altshiller Court's *College Geometry*, formula 203, page 102, or Coxeter and Greitzer, *Geometry Revisited*, exercise 1.7.2, pages 20 and 157], we see that

$$OH^2 \;=\; 9R^2 - a^2 - b^2 - c^2 \;=\; \frac{c^2}{4}\,.$$

Next use $R^2 = \dfrac{a^2b^2c^2}{16F^2}$ (where $F$ is the area of $\triangle ABC$), and multiply through by $4 \cdot 16F^2$, where (by Heron's formula)

$$16F^2 \;=\; 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\,,$$

and factor to get

$$(c^2 - a^2 - b^2) \cdot \left(5c^4 - (a^2 + b^2)\,c^2 - 4\,(a^2 - b^2)^2\right) \;=\; 0\,.$$

If the first factor is zero, then the triangle has a right angle at $C$, as desired. If not, then

$$c^2 \;=\; \frac{1}{10}\left(a^2 + b^2 \pm \sqrt{(a^2+b^2)^2 + 80\,(a^2-b^2)^2}\right)\,. \qquad (1)$$

For $c$ to be an integer, we must have $81(a^2 - b^2)^2 + 4a^2b^2 = z^2$ for some integer $z$; that is,

$$\left[(3a)^2 - (3b)^2\right]^2 + [2ab]^2 \;=\; z^2\,.$$

Comparing this to the formula for generating Pythagorean triples (namely $[a^2 - b^2]^2 + [2ab]^2 = [a^2 + b^2]^2$), we infer $a = b$. When $a = b$, formula (1) gives $c^2 = (2a^2 \pm 2a^2)/10$. Since $c$ is non-zero (in a non-degenerate triangle), we conclude that if the triangle were not a right triangle then

$$c^2 \;=\; \frac{2a^2}{5}\,,$$

which is impossible for integers $a$ and $c$.

*Also solved by the proposer. There was one incomplete submission.*

**2534**. [2000 : 178] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that $a$ is an integer and $x$ and $y$ natural numbers. Define $z_a(x,y) \;=\; \dfrac{x^2+y^2+a}{xy}$.

1. Show that there exist infinitely many values of $a$ such that $z_a(x,y)$ is an integer for infinitely many pairs $(x,y) \in \mathbb{N}^2$.

2.$\star$ Is the set $E(a)$ of integers $z_a(x,y)$ as obtained above necessarily infinite? If the answer is "no", determine those $a$'s which determine finite sets $E(a)$.

I. *Solution by Manuel Benito and Emilio Fernández, I.B. Praxedes Mateo Sagasta, Logroño, Spain.*

1. For example, for any pair of natural numbers $\lambda$ and $d$, we have

$$z_{-d^2}(\lambda d, d) \;=\; \frac{\lambda^2 d^2 + d^2 - d^2}{\lambda d^2} \;=\; \lambda\,.$$

2. For the first part, the answer is "no", since for $a = 0$, for example, we have

$$z_0(x, y) = \frac{x^2 + y^2}{xy}.$$

Let $\gcd(x, y) = d$, and thus $x = x_1 d$ and $y = y_1 d$ with $x_1$, $y_1$ relatively prime natural numbers. We then have that

$$z_0(x, y) = \frac{x_1^2 + y_1^2}{x_1 y_1}.$$

But $\gcd(x_1^2+y_1^2, x_1) = \gcd(x_1^2+y_1^2, y_1) = 1$, so that for $z_0(x, y)$ to be a natural number it must necessarily be true that $x_1 = y_1 = 1$. Thus $z_0(x, y) = 2$, and we have shown that $E(0) = \{2\}$.

For the second part, we have already shown in 1, that if $a = -d^2$, where $d$ is any positive integer, then $E(a)$ is an infinite set; in fact $E(a) = \mathbb{N}$. We will prove now that, for every other integer value of $a$, $E(a)$ shall be a finite set, with cardinality greater than or equal to $1$.

In the following proposition we prove that $|E(a)| \leq a + 2$ when $a \in \mathbb{N} \cup \{0\}$.

**Proposition 1**. Let $a \in \mathbb{N} \cup \{0\}$ be given. If $\frac{x^2+y^2+a}{xy} = \beta$ where $x, y$ and $\beta$ are natural numbers, then $\beta \leq a + 2$.

*Proof.* Suppose that, for a certain fixed value of $y \in \mathbb{N}$, there are $x$ and $\beta$, natural numbers, such that $\frac{x^2+y^2+a}{xy} = \beta$ or, what is the same, $y^2 + a = x(\beta y - x)$. Both $x$ and $\beta y - x$ being necessarily positive, the maximum value of the sum $x + (\beta y - x) = \beta y$ is attained when $x = 1$ and $\beta y - x = y^2 + a$, or when $x = y^2 + a$ and $\beta y - x = 1$. The said maximum value of $\beta y$ is thus $y^2 + a + 1$. Since $y$ was fixed, the maximum value of $\beta$ is

$$\beta_m(y) = \frac{y^2 + a + 1}{y} = y + \frac{a+1}{y}.$$

But the maximum integer value of the function $B : \mathbb{N} \to \mathbb{R}$ given by $B(y) = y + \frac{a+1}{y}$ is $B(1) = B(a+1) = a+2$, as is easily seen (if $y \in \mathbb{N}$ and $y > a + 1$ then $y + \frac{a+1}{y} \not\in \mathbb{N}$), and this completes the proof. ∎

Lastly, the following proposition proves that $E(a)$ is also a finite set when $a < 0$ and $a$ is not of the form $-d^2$ for $d \in \mathbb{N}$.

**Proposition 2**. If the positive integer $a$ is not a perfect square, then $E(-a)$ is a finite set.

*Proof.* We suppose that, for some natural numbers $x$ and $y$, the number $\beta = (x^2 + y^2 - a)/(xy)$ is an integer.

Consider first the possibility that $y^2 - a < 0$. This gives a finite number of possible values for $y$ (namely $1, 2, \ldots, \lfloor\sqrt{a}\rfloor$), and since $x$ must divide

$y^2 - a \neq 0$, $x$ can take on only a finite number of values for each particular value of $y$. Consequently, the number of values that $\beta$ takes on is finite.

It remains only to examine the possibility that $y^2 - a > 0$. By reasoning as in the proof of Proposition 1, for each fixed value of $y$, the maximum value of $\beta y$ shall be $y^2 - a + 1$, and the maximum value of $\beta$ is

$$\beta^*(y) = \frac{y^2 - a + 1}{y} = y - \frac{a-1}{y}. \qquad (1)$$

But the maximum integer value of the function (1), since $y$ goes over all natural numbers between $\lceil \sqrt{a} \rceil$ and $a - 1$, is $\beta^*(a-1) = a - 2$ [since $\beta^*(y)$ is increasing]. Consequently, the number of values possibly taken by $\beta$ is also finite in this case. ∎

II. *Partial solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

1. Let $S_a$ denote the set of all $(x, y) \in \mathbb{N}^2$ such that $z_a(x, y)$ is an integer. We shall show that $S_a$ is infinite for *all* non-negative integers $a$. This is obvious for $a = 0$ since $z_0(x, x) = 2$ for all $x \in \mathbb{N}$. Hence, we assume that $a > 0$. Note first that $(1, 1) \in S_a$ since $z_a(1, 1) = a + 2$ is an integer. Next suppose $(b, c) \in S_a$. Then $b^2 + c^2 + a = kbc$ for some integer $k = z_a(b, c)$. Hence [since $b^2 + a = c(kb - c)$ means that $kb - c > 0$],

$$\begin{aligned} z_a(kb - c, b) &= \frac{(kb-c)^2 + b^2 + a}{b(kb-c)} = \frac{k^2b^2 - 2kbc + (b^2 + c^2 + a)}{b(kb-c)} \\ &= \frac{k^2b^2 - kbc}{b(kb-c)} = k. \end{aligned}$$

This shows that $(kb - c, b) \in S_a$ whenever $(b, c) \in S_a$; in fact $z_a(kb - c, b) = z_a(b, c) = k$.

Using this observation, we can construct, for all $a > 0$, an infinite sequence of solutions $(x_n, y_n) \in S_a$ as follows. Let $(x_1, y_1) = (1, 1)$ [for which $z_a(1, 1) = a + 2$], and for $n \geq 1$ define

$$(x_{n+1}, y_{n+1}) = ((a+2)x_n - y_n, x_n).$$

Then $(x_n, y_n) \in S_a$ for all $n$ as above, provided that $x_n > 0$ for all $n$. We now show by induction that the sequence $\{x_n\}$ is strictly increasing. It would then follow that $x_n > 0$ and all the $(x_n, y_n)$'s are distinct. Clearly $x_2 = (a+2)x_1 - y_1 = a + 1 > 1 = x_1$. Suppose that $x_{n+1} > x_n$ for some $n \geq 1$. Then

$$x_{n+2} = (a+2)x_{n+1} - y_{n+1} = (a+2)x_{n+1} - x_n \geq 2x_{n+1} - x_n > x_{n+1},$$

completing the induction and the proof.

*Editorial note.* This shows that any non-negative integer value of $a$ will answer part 1. In Solution I it was shown that any $a$ which is the negative of a perfect square will work too, but that other negative values of $a$ will not.

2. (Partial solution) No, the set $E(a)$ could be finite. In fact we shall show that $E(6) =$ [redacted].

[*Editorial comment.* Here Wang specified a certain finite set of positive integers. We withhold this information at this time because it forms part of a solution by Wang of Problem 1 from the Estonian Math Contest 1995–1996, which appeared in the *Olympiad Corner* [2000 : 6] and which readers may still want to work on. This problem and 2534 are of course closely related. In fact Wang states his "hunch" that 2534 and the Estonian Math Olympiad problem have the same "origin".]

*Part 1 was also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; and by HEINZ-JÜRGEN SEIFFERT, Berlin, Germany. Bradley gave the same general result as in Solution* **II**. *Seiffert solved this part as in Solution* **I**.

**2535**$^\star$. [2000 : 179] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

1. Prove that neither of the integers $a(n) = 3n^2 + 3n + 1$ and $b(n) = n^2 + 3n + 3$ $(n \geq 1)$ has a divisor $k$ such that $k \equiv 2 \pmod 3$.

2. Prove or disprove that both of the sequences $\{a(n)\}$ and $\{b(n)\}$ $(n \geq 1)$ contain infinitely many primes.

*Solution of part* 1 *compiled from received solutions.*

Supposing that $a(n)$ or $b(n)$ has a divisor $k \equiv 2 \pmod 3$, then it has a prime divisor $p \equiv 2 \pmod 3$. Since $a(n)$ and $b(n)$ are odd, $p \equiv 5 \pmod 6$. Now

$$12a(n) \;=\; 12(3n^2+3n+1) \;=\; (6n+3)^2+3$$

and

$$4b(n) \;=\; 4(n^2+3n+3) \;=\; (2n+3)^2+3\,,$$

so that $p|a(n)$ or $p|b(n)$ implies that $p$ divides an integer of the form $N^2+3$; that is, $N^2 \equiv -3 \pmod p$; that is, $-3$ is a quadratic residue of $p$. But it is known that $-3$ is a quadratic residue of a prime $p > 3$ if and only if $p \equiv 1 \pmod 6$ (see for instance #5 of Problems 9.3 in David Burton's *Elementary Number Theory*), which is a contradiction.

*Part 1 solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; KENNETH M. WILKE, Topeka, KS, USA; and JEREMY YOUNG, student, University of Cambridge, Cambridge, UK.*

*Most solvers used quadratic reciprocity to prove that $-3$ is a quadratic nonresidue of any prime $p \equiv 5 \pmod 6$.*

*Part 2 was not attempted by any solvers. In fact this part should probably not have been included, as there is not a single quadratic polynomial which has been proven to generate infinitely many primes. Although most everyone believes that there are infinitely many primes of the form $n^2+1$, for instance, nobody has the slightest idea how to prove it.*

**2537**. [2000 : 179] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Find the exact value of $\cot\left(\dfrac{\pi}{7}\right)+\cot\left(\dfrac{2\pi}{7}\right)-\cot\left(\dfrac{3\pi}{7}\right)$ .

I. Solution by *David Doster, Choate Rosemary Hall, Wallingford, CT, USA* (*slighty modified by the editor*).

Let $S$ denote the given expression. We show that $S=\sqrt{7}$. We let $\theta=\dfrac{\pi}{7}$ and use the following facts freely throughout: for all integers $k$, $\cos(k\theta)=-\cos\left((7-k)\theta\right)$, $\sin(k\theta)=\sin\left((7-k)\theta\right)$, and in particular, $\sin 3\theta=\sin 4\theta=2\sin 2\theta\cos 2\theta$.

We have

$$\begin{aligned}
S &= \frac{\cos\theta}{\sin\theta}-\frac{\cos 3\theta}{\sin 3\theta}+\frac{\cos 2\theta}{\sin 2\theta} = \frac{\sin 3\theta\cos\theta-\cos 3\theta\sin\theta}{\sin\theta\sin 3\theta}+\frac{\cos 2\theta}{\sin 2\theta}\\
&= \frac{\sin 2\theta}{\sin\theta\sin 3\theta}+\frac{\cos 2\theta}{\sin 2\theta} = \frac{1}{\sin\theta}\left(\frac{\sin 2\theta}{\sin 3\theta}+\frac{\cos 2\theta}{2\cos\theta}\right)\\
&= \frac{1}{\sin\theta}\left(\frac{1}{2\cos 2\theta}+\frac{\cos 2\theta}{2\cos\theta}\right) = \frac{1}{2\sin\theta}\left(\frac{\cos\theta+\cos^2 2\theta}{\cos\theta\cos 2\theta}\right)\\
&= \frac{2\cos\theta+2\cos^2 2\theta}{2\sin 2\theta\cos 2\theta} = \frac{2\cos\theta+(1+\cos 4\theta)}{\sin 4\theta}\\
&= \frac{2\cos\theta-\cos 3\theta+1}{\sin 3\theta} = T\,.
\end{aligned}$$

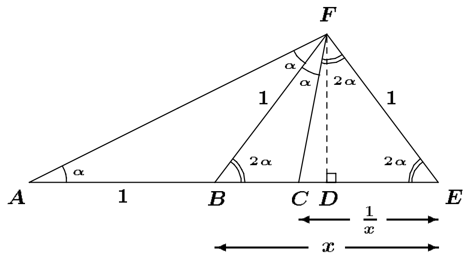
Now,

$$\begin{aligned}
T^2 &= \frac{4\cos^2\theta+\cos^2 3\theta+1-4\cos\theta\cos 3\theta+4\cos\theta-2\cos 3\theta}{\sin^2 3\theta}\\
&= \frac{(4+4\cos 2\theta)+(1+\cos 6\theta)+2-4(\cos 4\theta+\cos 2\theta)+8\cos\theta-4\cos 3\theta}{2\sin^2 3\theta}\\
&= \frac{7+(\cos 6\theta+8\cos\theta)-4(\cos 4\theta+\cos 3\theta)}{1-\cos 6\theta}\\
&= \frac{7-7\cos 6\theta}{1-\cos 6\theta} = 7\,.
\end{aligned}$$

Since $0<\theta<3\theta<\dfrac{\pi}{2}$, clearly $T>0$, and thus, it follows that $S=\sqrt{7}$.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $\alpha=\frac{\pi}{7}$ and let $Y$ denote the answer. Let $ABCDE$ be a side of a triangle $AEF$ in which $\angle BAF=\alpha$, $AB=BF=FE=1, CE=CF$ and $FD\perp AE$. Let $x=BE$. Then $BD=DE=\frac{x}{2}$. (See figure.)




Since $\angle FBE = \angle FEB = \angle CFE = 2\alpha$, $\triangle FBE$ is similar to $\triangle CFE$. Hence, $\angle FCE = 3\alpha$ and $CE = \dfrac{CE}{FE} = \dfrac{FE}{BE} = \dfrac{1}{x}$, which implies $CD = \dfrac{1}{x} - \dfrac{x}{2}$.

Let $h = FD$. Then $h^2 + \dfrac{x^2}{4} = 1$ and

$$Y \;=\; \cot(\alpha) + \cot(2\alpha) - \cot(3\alpha) \;=\; \frac{AD}{h} + \frac{DE}{h} - \frac{CD}{h}\,,$$

or

$$hY \;=\; 1 + x - \left(\frac{1}{x} - \frac{x}{2}\right) \;=\; 1 + \frac{3x}{2} - \frac{1}{x}\,. \tag{1}$$

Since $\triangle FBC$ is similar to $\triangle AEF$, we have

$$BC \;=\; \frac{BC}{BF} \;=\; \frac{FE}{AE} \;=\; \frac{1}{AE} \;=\; \frac{1}{1+x}\,.$$

Hence, $x = BE = BC + CE = \dfrac{1}{1+x} + \dfrac{1}{x}$, which simplifies to

$$x^3 + x^2 \;=\; 2x + 1\,. \tag{2}$$

From (2), we have $\dfrac{1}{x} = x^2 + x - 2$. Substituting into (1), we get

$$hY \;=\; 1 + \frac{3x}{2} - (x^2 + x - 2) \;=\; 3 + \frac{x}{2} - x^2 \quad \text{or} \quad 2hY \;=\; 6 + x - 2x^2\,.$$

Hence,

$$4h^2Y^2 \;=\; \left(6 + x - 2x^2\right)^2 \;=\; 36 + 12x - 23x^2 - 4x^3 + 4x^4\,. \tag{3}$$

From (2) we have $x^4 = -x^3 + 2x^2 + x$. Substituting into (3), we get

$$4h^2Y^2 \;=\; 36 + 16x - 15x^2 - 8x^3\,. \tag{4}$$

Again, from (2), we have $x^3 = -x^2 + 2x + 1$. Substituting into (4), we get

(surprise!!) $4h^2Y^2 = 28 - 7x^2 = 28\left(1 - \frac{x^2}{4}\right) = 28h^2$ or $Y^2 = 7$.

Since, clearly, $Y > 0$, we conclude that $Y = \sqrt{7}$.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; HENRY LIU, student, Trinity College, Cambridge, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; RICHARD TOD, Forest of Dean, England, UK; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer. There was also one partially incorrect solution.*

*Most of the submitted solutions use either the expansion formulae for $\sin 7\theta$, $\cos 7\theta$ and/or $\tan 7\theta$, with or without derivations; or De Moivre's Theorem and Euler's formula; or various other known results. Solution I above is one of the very few exceptions and is self-contained. On the other hand, the proof given in Solution II is the only one based on geometry.*

*Sinefakopoulos pointed out that this problem is equivalent to problem #2547. Tod commented that it was part of a Cambridge University Scholarship and Exhibitions Examination in Mathematics which took place in the nineteen fifties.*

**2538**. [2000 : 179] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

On a recent calculus test, students were asked to compute the arc length of a curve represented by a certain function $f(x)$, for $x = a$ to $x = b$, $a < b$. One of the students, a Mr. Fluke, simply calculated $f'(b) - f'(a)$, and obtained the correct answer.

Determine all real functions, $f(x)$, differentiable on some open interval $I$, such that, for all $a$, $b$ satisfying $(a, b) \subset I$, the arc length of the curve $y = f(x)$, from $x = a$ to $x = b$ is equal to $f'(b) - f'(a)$.

*Solution by David Loeffler, student, Cotham School, Bristol, UK.*

We must have

$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx = f'(x)\Big|_a^b .$$

Since this must be true for all $a$, $b$ in $I$, and the left hand side possesses a derivative, the right hand side must also. Then, we may differentiate with respect to $b$:

$$\sqrt{1 + (f'(b))^2} = f''(b) .$$

Suppose for simplicity of notation, that $g(x) = f'(x)$. Thus, we must solve $g'(x) = \sqrt{1 + g(x)^2}$, which we do as follows:

$$\frac{dg}{dx} = \sqrt{1+g^2}\,, \qquad \frac{dg}{\sqrt{1+g^2}} = dx\,,$$
$$\int \frac{dg}{\sqrt{1+g^2}} = \int dx\,,$$
$$\sinh^{-1}(g) = x + c\,, \qquad g(x) = \sinh(x+c)\,.$$

Since $g(x) = f'(x)$, it follows that $f(x) = \cosh(x + c) + d$. (No special integrals exist, since $\sqrt{1+g^2}$ cannot be zero; thus, we have not "lost" any solutions by dividing by it.)

*Also solved by MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; DAVID VELLA, Skidmore College, Saratoga Springs, New York; ALBERT WHITE, St. Bonaventure University, St. Bonaventure, New York; KENNETH M. WILKE, Topeka, KS, USA; and the proposer. There was one incomplete and one incorrect solution.*

Note: *Álvarez also makes the following observation:*

*The hyperbolic cosine is known as the 'catenary' in mechanics because it is the shape adopted by an inextensible string ('catena' is Latin for 'chain') of constant density per unit length, with fixed endpoints and subject only to its own weight. The equilibrium condition for a string is that the tension at any two points cancel the weight of the string between the two points. Since the tensions at points $x = a, b$ are parallel to $(-1, -f'(a))$ and $(1, f'(b))$ respectively and the weight of the string is proportional to the length of the string and directed along $(0, -1)$, the equilibrium condition is equivalent to the condition that the length of the string is proportional to $f'(b) - f'(a)$.*

**2539**. [2000 : 236, 372] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea, adapted by the editor.*

Let $ABCD$ be a convex quadrilateral with vertices oriented in the clockwise sense. Let $X$ and $Y$ be interior points on $AD$ and $BC$, respectively. Suppose that $P$ is a point between $X$ and $Y$ such that $\angle AXP = \angle BYP = \angle APB = \theta$ and $\angle CPD = \pi - \theta$ for some $\theta$.

(a) Prove that $AD \cdot BC \geq 4\,PX \cdot PY$.

(b)$\star$ Find the case(s) of equality.

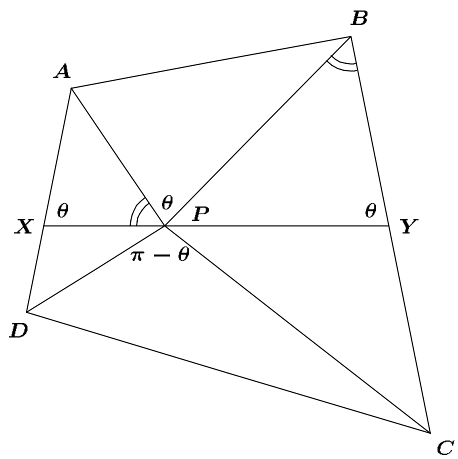
*Solution by Toshio Seimiya, Kawasaki, Japan.*

(a) The triangles $APX$ and $PBY$ are similar, because

$$\angle AXP = \angle PYB = \theta$$

and

$$\angle APX = \angle BPX - \angle APB = (\angle PBY + \angle PYB) - \angle APB = \angle PBY\,.$$




Then $AX : PY = PX : BY$, so that

$$AX \cdot BY \ = \ PX \cdot PY \ . \tag{1}$$

Since $\angle DXP = \angle PYC = \angle DPC = \pi - \theta$, we similarly obtain

$$DX \cdot CY \ = \ PX \cdot PY \ . \tag{2}$$

By the AM–GM Inequality,

$$AD \ = \ AX + DX \ \geq \ 2\sqrt{AX \cdot DX}$$

and

$$BC \ = \ BY + CY \ \geq \ 2\sqrt{BY \cdot CY} \ .$$

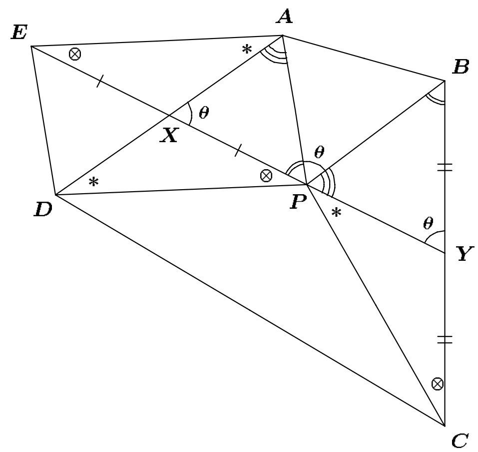
Multiplying these two inequalities, we get

$$AD \cdot BC \ \geq \ 4\sqrt{AX \cdot BY \cdot DX \cdot CY} \ ,$$

and from (1) and (2),

$$AD \cdot BC \ \geq \ 4\sqrt{PX^2 \cdot PY^2} \ = \ 4PX \cdot PY \ .$$

(b) Equality holds when $AX = DX$ and $BY = CY$. We can show that there is a quadrilateral $ABCD$ having all the required properties and such that $AX = DX$ and $BY = CY$.




Start with any triangle $PCB$. Let $Y$ be the mid-point of the side $CB$. Extend $YP$ and construct the point $E$ on the line $YP$ (so that the order of points is $Y$, $P$, $E$) and a point $A$ such that

$$\angle APE = \angle PBC$$

and

$$\angle AEP = \angle PCB .$$

Thus, $\triangle AEP$ is similar to $\triangle PCB$. Therefore, if $X$ is the mid-point of $PE$, we have

$$\angle PXA = \angle BYP$$

and

$$\angle PAX = \angle BPY .$$

Consequently,

$$\begin{aligned} \angle APB &= \pi - \angle APE - \angle BPY \\ &= \pi - \angle PBC - \angle BPY \\ &= \angle BYP . \end{aligned}$$

Thus, $\angle APB = \angle BYP = \angle PXA$. Let $\angle PXA = \theta$, and let $D$ be the point symmetric to point $A$ with respect to point $X$. We can now show that the quadrilateral $ABCD$ has all of the required properties. We only need a proof that $\angle DPC = \pi - \theta$. Since $\triangle AEP$ is similar to $\triangle PCB$, we obtain

$$\angle XAE = \angle YPC .$$

Since $\triangle AXE$ and $\triangle DXP$ are congruent, we have

$$\angle XEA = \angle XPD .$$

Consequently,

$$\begin{aligned}\angle DPC &= \pi - \angle YPC - \angle XPD \\ &= \pi - \angle XAE - \angle XEA \\ &= \angle AXE \\ &= \pi - \angle PXA \\ &= \pi - \theta\,,\end{aligned}$$

which completes the proof.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and MIHAI CIPU, IMAR, Bucharest, Romania.*

**2540**. [2000 : 236] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given an equilateral triangle $ABC$, let $P$ and $Z$ be points on the incircle such that $P$ is the mid-point of $AZ$ and $BZ < CZ$. The segment $CZ$ and the extension of $BZ$ meet the incircle again at $X$ and $Y$ respectively. Show that:

1. triangle $XYZ$ is equilateral;
2. the points $A$, $X$ and $Y$ are collinear; and
3. each of the segments $XA$, $YB$ and $ZC$ is divided in the golden ratio by the incircle.

*Solution by Toshio Seimiya, Kawasaki, Japan.*

1. Let $I$ be the incentre of $\triangle ABC$. The incircle of $\triangle ABC$ touches $BC$, $CA$, and $AB$ at $D$, $E$, and $F$, respectively. Then $D$, $E$, and $F$ are mid-points of $BC$, $CA$, and $AB$, respectively. (See the diagram below.) Since $P$ is the mid-point of $AZ$, we have $FP \parallel BZ$ and $PE \parallel ZC$. Hence,

$$\angle BZC = \angle FPE = \angle FEC = 180^\circ - \angle AEF = 120^\circ\,.$$

Since $I$ is the incentre of the equilateral triangle $ABC$, we have

$$\angle BIC = 120^\circ = \angle BZC\,.$$

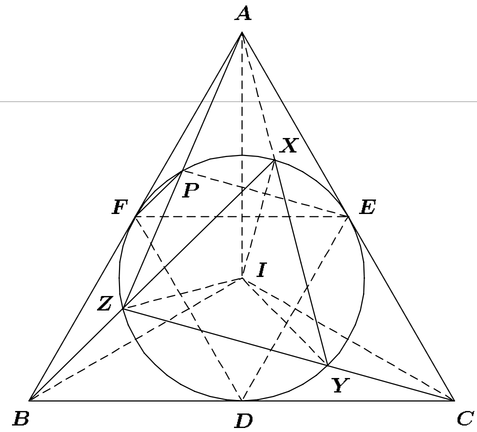
Therefore, $B$, $C$, $I$, $Z$ are concyclic, implying that

$$\angle IZX = \angle ICB = 30^\circ\,, \qquad \angle IZC = \angle IBC = 30^\circ\,.$$

Since $IX = IZ = IY$, we get

$$\angle IXZ = \angle IZX = 30^\circ\,, \qquad \angle IYZ = \angle IZY = 30^\circ\,.$$

Thus, we have $\triangle IZX \cong \triangle IZY$, so that $ZX = ZY$. Since $\angle BZC = 120^\circ$, we also have that $\angle XZY = 60^\circ$. Hence, $\triangle XYZ$ is equilateral.




2. Since $\angle IXB = 30^\circ = \angle IAB$, we see that $A$, $B$, $I$, and $X$ are concyclic. Hence, $\angle AXB = \angle AIB = 120^\circ$.

Thus, $\angle AXB + \angle BXY = 120^\circ + 60^\circ = 180^\circ$. Therefore, $A$, $X$, and $Y$ are collinear.

3. Since $\triangle XYZ$ and $\triangle DEF$ are equilateral triangles which are inscribed in the same circle, they are congruent. Hence,

$$XY = YZ = ZX = DE = EF = FD = \tfrac{1}{2}BC = \tfrac{1}{2}CA = \tfrac{1}{2}AB\,.$$

Thus, $XY^2 = AE^2 = AX \cdot AY$. Similarly, $YZ^2 = CD^2 = CY \cdot CZ$, and $ZX^2 = BF^2 = BZ \cdot BX$. Therefore, each of the segments $YA$, $XB$, and $ZC$ is divided in a golden ratio by the incircle.

*Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College Cambridge, UK; D.J. SMEENK, Zaltbommel, the Netherlands; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There was one incorrect solution.*

**2541**. [2000 : 237] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Show that, for all natural numbers $n$ and $k \geq 2$,

$$\sum_i \binom{n}{2i} k^{n-2i} \left(k^2 - 4\right)^i$$

is divisible by $2^{n-1}$.

I. *Solution by Gerry Leversha, St. Paul's School, London, England.*

Let $a_n$ denote the given sum. Note first, using the Binomial Expansion, that $a_n = \frac{1}{2}\left[\left(k+\sqrt{k^2-4}\right)^n + \left(k-\sqrt{k^2-4}\right)^n\right]$. Now this expression is recognisable as the solution to a second-degree linear recurrence relation. The auxiliary equation is $x^2 - 2kx + 4 = 0$, so that the relation is $a_{n+2} = 2ka_{n+1} - 4a_n$, and the initial conditions are $a_0 = 1$ and $a_1 = k$. To prove that $2^{n-1}|a_n$ for all $n \geq 1$, we proceed by induction. The case $n = 1$ is trivial and $a_2 = 2k^2 - 4$, which is divisible by $2$. Suppose $2^{r-1}|a_r$ and $2^r|a_{r+1}$ for some $r \geq 1$. Then $a_{r+2} = 2k\cdot A\cdot 2^r - 4B\cdot 2^{r-1} = (Ak-B)2^{r+1}$ for some integers $A$ and $B$, completing the induction.

II. *Solution by the proposer.*

Let $x = \frac{1}{2}\left(k+\sqrt{k^2-4}\right)$. Then $x^{-1} = \frac{1}{2}\left(k-\sqrt{k^2-4}\right)$, and, as in I, $x^n + x^{-n} = \dfrac{a_n}{2^{n-1}}$.

It therefore sufficies to show that $x^n + x^{-n}$ is an integer for all natural numbers $n$. This is clearly true for $n = 1$. Suppose $x^m + x^{-m}$ is an integer for all $m = 1, 2, \ldots, n-1$ for some $n \geq 2$. Then

$$x^{n+1} + x^{-(n+1)} = (x + x^{-1})(x^n + x^{-n}) - \left(x^{n-1} + x^{-(n-1)}\right),$$

which is an integer by the induction hypothesis.

*Also solved by MICHEL BATAILLE, Rouen, France; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and PETER Y. WOO, Biola University, La Mirada, CA, USA.*

*Most of the submitted solutions are virtually the same as solution* I *above.*