Murray Klamkin Birthday Issue

There cannot be anyone involved in mathematical problem solving in any serious way who has not heard of Murray Klamkin.

This issue of ***Crux Mathematicorum with Mathematical Mayhem*** is dedicated to a celebration of Murray's 80th birthday.

We are delighted to take this opportunity to acknowledge the many, many contributions that Murray has made, and to recognize the profound influence that he has had on many people all over the world.

A few months ago, Murray underwent serious open heart surgery. Unfortunately, all did not go completely smoothly. But Murray's indominatable spirit helped to pull him through some very tough times. He made use of his love of Mathematics, and throughout this very trying time, he continued to solve problems, pose problems, and make shrewd comments on other people's proposals and solutions. The Editors of ***Crux Mathematicorum with Mathematical Mayhem*** wish Murray a continued good recovery, and look forward to his continuing valuable contributions.

The first time that Murray's name appears in ***Crux Mathematicorum*** (more correctly, in ***Eureka***), is on page 3 of volume 2 (1976), where the founding editor, Léo Sauvé, writing on the *Celebrated Butterfly Problem*, notes that

> Klamkin credits Cantab for the following extension [of the butterfly problem].

And he makes two further references to Murray's contributions to the problem.

Murray's first published solution occurs on page 78 of volume 2 (1976).

106. [1976 : 6] *Proposed by Viktors Linis, University of Ottawa.*
Prove that, for any quadrilateral with sides a, b, c, d,

$$a^2 + b^2 + c^2 \; > \; \tfrac{1}{3}d^2 \,.$$

Solution by M.S. Klamkin, University of Waterloo.

Consider any $(n+1)$–gon, coplanar or not, simple or not, with sides a_i, $i = 1, 2, \ldots, n+1$. Then by Hölder's Inequality (for $m > 1$) (or the Power Mean Inequality),

$$(a_1^m + a_2^m + \cdots + a_n^m)^{\frac{1}{m}}(1+1+\cdots+1)^{1-\frac{1}{m}} \; \geq \; a_1 + a_2 + \cdots + a_n \,,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. Also, by the triangle inequality,

$$a_1 + a_2 + \cdots + a_n \; \geq \; a_{n+1} \,;$$

thus

$$a_1^m + a_2^m + \cdots + a_n^m \;\ge\; a_{n+1}^m / n^{m-1} \,,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n = a_{n+1}/n$; that is, the polygon is degenerate.

The desired result corresponds to the special case $m = 2$, $n = 3$. Also Problem 74 [1975 : 71; 1976 : 10] corresponds to the special case $m = 2$, $n = 2$.

[There was also a solution by F.G.B. Maskell.]

So, Murray was mentioned only twice in 1976. But the following year, things changed profoundly. He was mentioned many more times — six problems proposed, five solutions published as well as three comments on problems. The first problem that he posed was

210. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

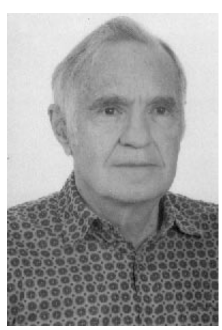
P, Q, R denote points on the sides BC, CA and AB, respectively, of a given triangle ABC. Determine all triangles ABC such that, if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \qquad (\neq 0,\ \tfrac{1}{2},\ 1),$$

then PQR (in some order) is similar to ABC.

This was only a taste of things to come! Murray had found ***Crux***, but, more importantly for us, ***Crux*** had found Murray!

Since then, Murray has served both United States and Canadian students well by coaching their IMO teams. At the 1995 IMO, held in Canada, Murray was the "senior man". An important postition that he held there was that of the adjudicator of any proposal for a special prize for a truly outstanding solution. The special prize had not been awarded for several years, but Murray judged the solution by Nikolay Nikolov (Bulgaria) to be outstanding, and had the pleasure of being the presentor of the special prize at the closing ceremonies in Toronto.



Murray Klamkin

Happy Birthday, Murray!

Andy Liu

March 5, 2001 marks the eightieth birthday of problemist *extraordinaire* **Murray S. Klamkin**. This is a cause for universal celebration in the problem world, especially in view of his remarkable recovery from serious heart surgeries just a year ago. I will take this opportunity to reflect on his influence on me.

I first met Murray when he came to the University of Alberta in 1976 as the new Chairman of the Department of Mathematics. I recognized his name as one which appeared frequently in problem sections, though I was unaware of his monumental contributions in S.I.A.M. Review, since I was not particularly interested in applied problems. I knew him mainly through his 1961 paper with D. J. Newman on the philosophy and applications of transform theory.

Since my personal knowledge of Murray came late in his life, I could only piece together some relevant information pertaining to his earlier days. He was an American citizen of Russian extraction and Jewish descent. He had a Bachelor Degree in Chemical Engineering from Cooper Union. Upon graduation, he served in the army during the Second World War. When he was demobilized, he continued his study at Polytechnic Institute of Brooklyn and obtained a Master's Degree in Physics. After a fellowship at Carnegie Institute of Technology, he returned to Brooklyn and taught mathematics from 1948 to 1956.

In 1957, he began a distinguished career as a research mathematician, first in Avco Research and Advanced Development Division and then at the Ford Motor Company. In between, he taught at the State University of New York in Buffalo from 1963 to 1964, and at the University of Minnesota from 1964 to 1965. He returned to academia in 1974 at the University of Waterloo. One of his responsibilities was the training of their Putnam team, which won first place that year.

When the Chairman's position opened up in Alberta, Murray saw an opportunity where he could make a bigger impact. He certainly stirred up the stoic atmosphere of the Department. Nowadays, many bemoan the overemphasis on business principles in university governance, to the detriment of academic pursuit. Murray brought with him the right dose of professional management, lighting a fire under a number of people who were at the time making no headway in their careers. Although he had to step on many toes, he did get the Department revitalized. Some people who were mad at him then were grateful to him later.

One of the first things Murray did was to institute the Freshmen and Undergraduate mathematics competitions, and I was roped into their administration. I had just completed my Doctorate and was engaged as a research associate. The competitions increased my contact with Murray, and soon, in very small ways, I was helping him edit his *Olympiad Corners* in ***Crux Mathematicorum***.

I remembered that at the time, I always approached his office with trepidation when summoned. Typically, he would say to me, "Here is a nice problem proposal but with a crummy solution. Give me a better one by Friday afternoon." With some chagrin, I began working on the problem, and surprised myself on many occasions that I was able to accomplish the mission. I had learned a lot during this period.

Another of Murray's reforms which had a big impact on me was his decision that our Department should emphasize geometry. In those days, Department Chairmen had no instructional duties, but Murray took it upon himself to teach one geometry class, and gave many talks to the university at large on the topic. I began to get involved too, and with a few colleagues, prepared a set of lecture notes that had metamorphisized over the years but are still in use today.

In 1979, I took up a sabbatical replacement appointment at the University of Regina. Murray had sufficient confidence in my ability that he pushed through my application for a position back at the University of Alberta. He came over to Regina and conducted the interview himself. I was hired on the tenure track in the fall of 1980.

The International Mathematical Olympiad was held in the United States in 1981. Sam Greitzer, the only leader the American team had ever had up till then, was kicked upstairs to manage the whole event. Murray, who had been his deputy since 1975, took over his position, and brought me along as the new deputy. I was completely out of my depth then, but the baptism under fire was a most valuable experience. I stayed on this post for four years, leaving with Murray after the 1984 IMO in former Czechoslovakia.

Murray had also put me on the Problem Committee for the USA Mathematical Olympiad and the Canadian Mathematical Olympiad. This gave me a broad perspective on the competition scene, as well as an opportunity to widen my network of contact. He spared no efforts in establishing me in the problem world.

Murray is the author of numerous problem books and editor of more problem sections than anyone else. He was one of the founding fathers of the Olympiad movement in America. He received the 1988 Distinguished Service Award from the Mathematical Association of America, which has established the Greitzer-Klamkin Prize for the top winner of the U.S.A. Mathematical Olympiad. Murray had been honoured with a David Hilbert International Award from the World Federation of National Mathematics Competitions. He holds an Honourary Doctorate from the University of Waterloo, and is a foreign member of the Belgian Royal Society. More recognitions are sure to come!

Quickies

One idea that originated with Murray Klamkin is that of the "Quickie". A Quickie is a problem with a quick neat solution. Here we present a selection of the best of Murray's Quickies from past issues. None of these are new, but they are all good problems. Thanks to Iliya Bluskov for preparing this section for us.

1992

1. Determine the extreme values of $r_1/h_1 + r_2/h_2 + r_3/h_3 + r_4/h_4$, where h_1, h_2, h_3, h_4 are the four altitudes of a given tetrahedron T, and r_1, r_2, r_3, r_4 are the corresponding signed perpendicular distances from any point in the space of T to the faces.

Solution. If the face areas and volume of the tetrahedron are F_1, F_2, F_3, F_4, and V respectively, then

$$r_1F_1 + r_2F_2 + r_3F_3 + r_4F_4 \;=\; 3V\,,$$

and $h_1F_1 = h_2F_2 = h_3F_3 = h_4F_4 = 3V$. Now eliminating the F_i's, we get

$$r_1/h_1 + r_2/h_2 + r_3/h_3 + r_4/h_4 \;=\; 1 \quad \text{(a constant)}\,.$$

2. Determine the minimum value of the product

$$P \;=\; (1+x_1+y_1)(1+x_2+y_2)\cdots(1+x_n+y_n)$$

where $x_i, y_i \geq 0$, and $x_1x_2\ldots x_n = y_1y_2\ldots y_n = a^n$.

Solution. More generally, consider

$$P \;=\; (1+x_1+y_1+\cdots+w_1)(1+x_2+y_2+\cdots+w_2)\ldots(1+x_n+y_n+\cdots+w_n)$$

where $x_1x_2\ldots x_n = \xi^n$, $y_1y_2\ldots y_n = \eta^n$, $\ldots$, $w_1w_2\ldots w_n = \omega^n$, and x_i, y_i, $\ldots$, $w_i \geq 0$. Then by Hölder's Inequality,

$$P^{1/n} \;\geq\; \left\{1 + \prod x_i^{1/n} + \prod y_i^{1/n} + \ldots + \prod w_i^{1/n}\right\}$$

or

$$P \;\geq\; (1+\xi+\eta+\ldots+\omega)^n\,.$$

In this case $\xi = \eta = a$, so

$$P \;\geq\; (1+2a)^n\,.$$

3. Prove that if $F(x, y, z)$ is a concave function of x, y, z, then $\{F(x, y, z)\}^{-2}$ is a convex function of x, y, z.

Solution. More generally $G(F)$ is a convex function where G is a convex decreasing function. By the convexity of G,

$$\lambda G\{F(x_1, y_1, z_1)\} + (1 - \lambda)G\{F(x_2, y_2, z_2)\} \geq G\{\lambda F(x_1, y_1, z_1) + (1 - \lambda)F(x_2, y_2, z_2)\} .$$

By the concavity of F,

$$\lambda F(x_1, y_1, z_1) + (1 - \lambda)F(x_2, y_2, z_2) \leq F([\lambda x_1 + (1 - \lambda)x_2], [\lambda y_1 + (1 - \lambda)y_2], [\lambda z_1 + (1 - \lambda)z_2]) .$$

Finally, since G is decreasing,

$$\lambda G\{F(x_1, y_1, z_1)\} + (1 - \lambda)G\{F(x_2, y_2, z_2)\} \geq G\{F([\lambda x_1 + (1 - \lambda)x_2], [\lambda y_1 + (1 - \lambda)y_2], [\lambda z_1 + (1 - \lambda)z_2])\} .$$

More generally and more precisely, we have the following known result: if $F(X)$ is a concave function of $X = (x_1, x_2, \ldots, x_n)$ and $G(y)$ is a convex decreasing function of y, where y is a real variable and the domain of G contains the range of F, then $G\{F(X)\}$ is a convex function of X.

4. If a, b, c are sides of a given triangle of perimeter p, determine the maximum values of

(i) $(a - b)^2 + (b - c)^2 + (c - a)^2$,

(ii) $|a - b| + |b - c| + |c - a|$,

(iii) $|a - b||b - c| + |b - c||c - a| + |c - a||a - b|$.

Solution. (i) $(a - b)^2 + (b - c)^2 + (c - a)^2 = 2(\sum a^2 - \sum bc) \leq kp^2$. Let $c = 0$, so that $k \geq 1/2$. We now show that $k = 1/2$ suffices. Here,

$$2\left(\sum a^2 - \sum bc\right) \leq \frac{1}{2}(a + b + c)^2$$

reduces to

$$2bc + 2ca + 2ab - a^2 - b^2 - c^2 \geq 0 .$$

The LHS is 16 times the square of the area of a triangle of sides $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ or

$$\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)\left(\sqrt{a} + \sqrt{b} - \sqrt{c}\right)\left(\sqrt{a} - \sqrt{b} + \sqrt{c}\right)\left(-\sqrt{a} + \sqrt{b} + \sqrt{c}\right) .$$

There is equality if and only if the triangle is degenerate with one side 0.

(ii) $|a-b|+|b-c|+|c-a| \leq kp$.

Letting $c=0$, $k \geq 1$. To show that $k=1$ suffices, assume that $a \geq b \geq c$, so that

$$|a-b|+|b-c|+|c-a| \;=\; 2a-2c \;\leq\; a+b+c$$

and there is equality if and only if $c=0$.

(iii) $|a-b|\,|b-c|+|b-c|\,|c-a|+|c-a|\,|a-b| \leq kp^2$.

Letting $c=0$, $k \geq 1/4$. To show that $k=1/4$ suffices, let $a=y+z$, $b=z+x$, $c=x+y$ where $z \geq y \geq x \geq 0$. Our inequality then becomes

$$|x-y|\,|z-y|+|y-z|\,|z-x|+|z-x|\,|x-y| \;\leq\; (x+y+z)^2$$

or

$$x^2-y^2+z^2+yz-3zx+xy \;\leq\; x^2+y^2+z^2+2yz+2zx+2xy$$

or

$$2y^2+5zx+1xy+1yz \;\geq\; 0\,.$$

There is equality if and only if $x=y=0$ or equivalently, $a=b$, and $c=0$.

5. If A, B, C are three dihedral angles of a trihedral angle, show that $\sin A$, $\sin B$, $\sin C$ satisfy the triangle inequality.

Solution. Let a, b, c be the face angles of the trihedral angle opposite to A, B, C respectively. Since

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} \;=\; \frac{\sin c}{\sin C}$$

by the Law of Sines for spherical triangles, it suffices to show that $\sin b+\sin c > \sin a$, or

$$2\sin\tfrac{1}{2}(b+c)\cos\tfrac{1}{2}(b-c) \;>\; 2\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a\,,$$

for any labelling of the angles. We now use the following properties of a, b and c:

(i) they satisfy the triangle inequality, (ii) $0<a+b+c<2\pi$.

Hence, $\cos\frac{1}{2}(b-c) > \cos\frac{1}{2}a$. To complete the proof, we show that

$$\sin\tfrac{1}{2}(b+c) \;>\; \sin\tfrac{1}{2}a\,.$$

This follows immediately if $b+c \leq \pi$; if $b+c > \pi$, then

$$\sin\tfrac{1}{2}(b+c) \;=\; \sin\left\{\pi-\tfrac{1}{2}(b+c)\right\} \;>\; \sin\tfrac{1}{2}a \quad \left(\text{since } \pi-\frac{b+c}{2} > \frac{a}{2}\right)\,.$$

Comment: More generally, if $a_1, a_2, \ldots, a_n$ are the sides of a spherical n-gon (convex), it then follows by induction over n that

$$\sin a_1 + \sin a_2 + \cdots + \sin a_n \;>\; 2\sin a_i\,, \quad i = 1, 2, \ldots, n\,.$$

It also follows by induction that

$$|\sin a_1| + |\sin a_2| + \cdots + |\sin a_n| \;>\; |\sin(a_1 + a_2 + \cdots + a_n)|$$

for any angles $a_1, a_2, \ldots, a_n$.

1995

1. Are there any integral solutions (x, y, z) of the Diophantine equation

$$(x - y - z)^3 \;=\; 27xyz$$

other than $(-a, a, a)$ or such that $xyz = 0$?

Solution. Let $x = u^3$, $y = v^3$, $z = w^3$, so that $u^3 - v^3 - w^3 = 3uvw$, or equivalently

$$(u - v - w)\left((u+v)^2 + (u+w)^2 + (v-w)^2\right) \;=\; 0\,.$$

Hence an infinite class of non-trivial solutions is given by

$$x = (v + w)^3\,, \quad y = v^3\,, \quad z = w^3\,.$$

Whether or not there are any other solutions is an open problem.

2. Does the Diophantine equation

$$(x - y - z)(x - y + z)(x + y - z) \;=\; 8xyz$$

have an infinite number of relatively prime solutions?

Solution. By inspection, we have the trivial solutions

$$(x, y, z) = (\pm 1, \pm 1, 0)$$

and permutations thereof. For other solutions, note that each of the three equations

$$x - y - z = 2\sqrt{yz}\,, \quad x - y + z = 2\sqrt{xz}\,, \quad x + y - z = 2\sqrt{xy}$$

is satisfied by $\sqrt{x} = \sqrt{y} + \sqrt{z}$. Consequently, we also have the infinite set of solutions

$$y = m^2\,, \quad z = n^2\,, \quad x = (m+n)^2 \quad \text{where } (m, n) = 1\,.$$

It is an open problem whether or not there are any other infinite sets of relatively prime solutions.

3. It is an easy result using calculus that if a polynomial $P(x)$ is divisible by its derivative $P'(x)$, then $P(x)$ must be of the form $a(x-r)^n$. Starting from the known result that

$$\frac{P'(x)}{P(x)} = \sum \frac{1}{x - r_i}$$

where the sum is over all the zeros r_i of $P(x)$, counting multiplicities, give a non-calculus proof of the above result.

Solution. Since $P'(x)$ is of degree one less than that of $P(x)$,

$$\frac{P'(x)}{P(x)} = \frac{1}{a(x-r)} = \sum \left(\frac{1}{x - r_i}\right) .$$

Now letting $x \to$ any r_i, it follows that $r = r_i$. Hence all the zeros of $P(x)$ must be the same.

4. Solve the simultaneous equations

$$x^2(y+z) = 1 , \quad y^2(z+x) = 8 , \quad z^2(x+y) = 13 .$$

Solution. More generally we can replace the constants 1, 8, 13 by a^3, b^3, c^3, respectively. Then by addition of the three equations and by multiplication of the three equations, we respectively get

$$\sum x^2 y = a^3 + b^3 + c^3 ,$$

$$(xyz)^2 \left[2xyz + \sum x^2 y\right] = (abc)^3 ,$$

where the sums are symmetric over x, y, z. Hence,

$$2t^3 + t^2\left(a^3 + b^3 + c^3\right) = (abc)^3 \tag{1}$$

where $t = xyz$. In terms of t, the original equations can be rewritten as

$$\frac{a^3}{tx} - \frac{1}{y} - \frac{1}{z} = 0 , \quad -\frac{1}{x} + \frac{b^3}{ty} - \frac{1}{z} = 0 , \quad -\frac{1}{x} - \frac{1}{y} + \frac{c^3}{tz} = 0 .$$

These latter homogeneous equations are consistent since the eliminant is equation (1). Solving the last two equations for y and z, we get

$$y = \frac{x(b_1 c_1 - 1)}{c_1 + 1} , \qquad z = \frac{x(b_1 c_1 - 1)}{b_1 + 1}$$

where $b_1 = b^3/t$, $c_1 = c^3/t$. On substituting back in $x^2(y+z) = a^3$, we obtain x^3 and then x, y, z.

5. Determine the area of a triangle of sides a, b, c and semi-perimeter s if

$$(s-b)(s-c) = a/h\,, \quad (s-c)(s-a) = b/k\,, \quad (s-a)(s-b) = c/l\,,$$

where h, k, l, are consistent given constants.

Solution.

$$\begin{aligned} h &= \frac{a}{(s-b)(s-c)} = \frac{1}{(s-b)} + \frac{1}{(s-c)}\,, \\ k &= \frac{1}{(s-c)} + \frac{1}{(s-a)}\,, \\ l &= \frac{1}{(s-a)} + \frac{1}{(s-b)}\,. \end{aligned}$$

Hence, h, k, l must satisfy the triangle inequality. Letting $2s' = h+k+l$, it follows by addition that

$$s' = \frac{1}{(s-a)} + \frac{1}{(s-b)} + \frac{1}{(s-c)}$$

and then

$$s-a = \frac{1}{(s'-h)}\,, \quad s-b = \frac{1}{(s'-k)}\,, \quad s-c = \frac{1}{(s'-l)}\,.$$

Adding the latter three equations, we get

$$s = \frac{1}{(s'-h)} + \frac{1}{(s'-k)} + \frac{1}{(s'-l)}\,.$$

Finally, the area of the triangle is given by

$$\begin{aligned} \Delta &= \{s(s-a)(s-b)(s-c)\}^{1/2} \\ &= \left\{ \frac{\frac{1}{(s'-h)} + \frac{1}{(s'-k)} + \frac{1}{(s'-l)}}{(s'-h)(s'-k)(s'-l)} \right\}^{1/2}\,. \end{aligned}$$

6. Prove that

$$3\left(x^2y + y^2z + z^2x\right)\left(xy^2 + yz^2 + zx^2\right) \geq xyz(x+y+z)^3$$

where x, y, $z \geq 0$.

Solution. By Cauchy's inequality

$$(x^2y + y^2z + z^2x)\,(zx^2 + xy^2 + yz^2) \;\geq\; \left(x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy}\right)^2 .$$

Hence it suffices to show that

$$\left\{\frac{(x^{3/2} + y^{3/2} + z^{3/2})}{3}\right\}^2 \;\geq\; \left\{\frac{(x + y + z)}{3}\right\}^3 .$$

But this follows immediately from the power mean inequality. There is equality **if and only if** $x = y = z$.

7. Determine all integral solutions of the Diophantine equation

$$\left(x^8 + y^8 + z^8\right) = 2\left(x^{16} + y^{16} + z^{16}\right) .$$

Solution. More generally one can find all integral solutions of

$$\left(x^{2n} + y^{2n} + z^{2n}\right)^2 \;=\; 2\left(x^{4n} + y^{4n} + z^{4n}\right) , \qquad (2)$$

where n is a positive integer, provided Fermat's equation $x^n + y^n = z^n$ does not have any integer solutions for particular values of $n > 2$ chosen.

Equation (2) can be rewritten as

$$(x^n + y^n + z^n)(y^n + z^n - x^n)(z^n + x^n - y^n)(x^n + y^n - z^n) \;=\; 0 . \qquad (3)$$

The trivial solutions occur for $(x, y, z) = (\pm a, a, 0)$ and permutations thereof.

For $n = 1$, any factor of the left hand side of (3) can be zero.

For $n = 2$, (x, y, z) can be $\pm$ the sides of any integral right triangle $(2mn,\, m^2 - n^2,\, m^2 + n^2)$ in any order.

Since Fermat's equation is at least known not to have any non-trivial solutions for all $n > 2$ and < 100 and integral multiples thereof, there are no non-trivial solutions for at least these cases.

8. Determine all the roots of the quintic equation

$$31x^5 + 165x^4 + 310x^3 + 330x^2 + 155x + 33 \;=\; 0 .$$

Solution. Since the equation can be rewritten as $(x-1)^5 = 32(x+1)^5$,

$$\frac{x - 1}{x + 1} \;=\; 2\omega^r , \quad r = 0, 1, 2, 3, 4,$$

where ω is a primitive 5^{th} root of unity. Hence,

$$x \;=\; \frac{1 + 2\omega^r}{1 - 2\omega^r} , \quad r = 0, 1, 2, 3, 4 .$$

More generally, the equation

$$ax^6 + 5bcx^4 + 10ac^2x^3 + 10bc^3x^2 + 5ac^4x + bc^5 \;=\; 0$$

is the same as

$$(b-a)(x-c)^5 \;=\; (b+a)(x+c)^5 \,.$$

9. If $F(x)$ and $G(x)$ are polynomials with integer coefficients such that $F(k)/G(k)$ is an integer for $k = 1, 2, 3, \ldots$, prove that $G(x)$ divides $F(x)$.

Solution. By taking k sufficiently large it follows that the degree of F is greater than or equal to the degree of G. Then by the Remainder Theorem,

$$\frac{F(x)}{G(x)} \;=\; \frac{Q(x)}{a} + \frac{R(x)}{G(x)}$$

where $Q(x)$ is an integral polynomial, a is an integer, and $R(x)$ is a polynomial whose degree is less than that of $G(x)$. Now $R(x)$ must identically vanish; otherwise, by taking k sufficiently large, we can make $R(k)/G(k)$ arbitrarily small, and this cannot add with $Q(k)/a$ to be an integer.

10. Given that $ABCDEF$ is a skew hexagon such that each pair of opposite sides are equal and parallel, prove that the mid-points of the six sides are coplanar.

Solution. Since each pair of opposite sides form a parallelogram whose diagonals bisect each other, all three different diagonals are concurrent say at point P. We now let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $-\mathbf{A}$, $-\mathbf{B}$, $-\mathbf{C}$ be vectors from P to A, B, C, D, E, F, respectively. The successive mid-points (multiplied by 2) are given by

$$\mathbf{A}+\mathbf{B}, \quad \mathbf{B}+\mathbf{C}, \quad \mathbf{C}-\mathbf{A}, \quad -\mathbf{A}-\mathbf{B}, \quad -\mathbf{B}-\mathbf{C}, \quad -\mathbf{C}+\mathbf{A}$$

and which, incidentally, form another centro-symmetric hexagon. It is enough now to note that $(\mathbf{A}+\mathbf{B}) - (\mathbf{B}+\mathbf{C}) + (\mathbf{C}-\mathbf{A}) = 0$.

11. If a, b, c, d are the lengths of sides of a quadrilateral, show that

$$\frac{\sqrt{a}}{(4+\sqrt{a})}\,, \quad \frac{\sqrt{b}}{\left(4+\sqrt{b}\right)}\,, \quad \frac{\sqrt{c}}{(4+\sqrt{c})}\,, \quad \frac{\sqrt{d}}{\left(4+\sqrt{d}\right)}\,,$$

are possible lengths of sides of another quadrilateral.

Solution. More generally one can show that if $a_1, a_2, \ldots, a_n$ are the lengths of sides of an n–gon, then $F(a_1)$, $F(a_2)$, $\ldots$, $F(a_n)$ are possible lengths of sides of another n–gon where $F(x)$ is an increasing concave function of x for $x \geq 0$ and $F(0) = 0$.

If a_1 is the largest of the a_i's, then it suffices to show that

$$F(a_2) + F(a_3) + \cdots + F(a_n) \;\geq\; F(a_1) \,.$$

By the majorization inequality, we have

$$F(a_2) + F(a_3) + \cdots + F(a_n) \;\geq\; F(a_2 + a_3 + \cdots + a_n) + (n-2)F(0)\,.$$

Finally, $F(a_2 + a_3 + \cdots + a_n) \geq F(a_1)$.

Some admissable functions are

$$F(x) \;=\; x^\alpha \quad \text{and} \quad \frac{x^\alpha}{k^2 + x^\alpha} \quad \text{for } 0 < 4\alpha < 1,\; \frac{x}{x+k^2},\; 1 - e^{-k^2 x},\, \tanh x\,.$$

12. Determine the maximum value of the sum of the cosines of the six dihedral angles of a tetrahedron.

Solution. Let A, B, C, D be unit outward vectors normal to the faces of a tetrahedron $ABCD$. Then

$$(x\mathrm{A} + y\mathrm{B} + z\mathrm{C} + w\mathrm{D})^2 \;\geq\; 0\,.$$

Expanding out and noting that $\mathrm{A}\cdot\mathrm{B} \;=\; -\cos CD$ (here CD denotes the dihedral angle of which the side CD is an edge), etc., we get

$$\begin{aligned} x^2 + y^2 + z^2 + w^2 \;\;&\geq\;\; 2xy\cos CD + 2xz\cos BD + 2xw\cos BC \\ &\quad +2yz\cos AD + 2yw\cos AC + 2zw\cos AB\,. \qquad (1) \end{aligned}$$

Setting $x \;=\; y \;=\; z \;=\; w$, we get that the sum of the cosines of the six dihedral angles is less than or equal to 2. There is equality *if and only if* $\mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} = 0$. Since, as known,

$$F_a\mathrm{A} + F_b\mathrm{B} + F_c\mathrm{C} + F_d\mathrm{D} \;=\; 0$$

where F_a denotes the area of the face of the tetrahedron opposite A, etc., it follows that there is equality *if and only if* the four faces have equal area or that the tetrahedron is isosceles.

Comment. In a similar fashion one can extend inequality (1) to n dimensions and then show that the sum of the cosines of the $n(n+1)/2$ dihedral angles of an n–dimensional simplex is less than or equal to $(n+1)/2$. Here the dihedral angles are the angles between pairs of $(n-1)$–dimensional faces and there is equality *if and only if* all the $(n-1)$–dimensional faces have the same volume.

1996

1. Which is larger

$$\left(\sqrt[3]{2}-1\right)^{1/3} \qquad \text{or} \qquad \sqrt[3]{1/9}-\sqrt[3]{2/9}+\sqrt[3]{4/9}\,?$$

Solution. That they are equal is an identity of Ramanujan.

Letting $x = \sqrt[3]{1/3}$ and $y = \sqrt[3]{2/3}$, it suffices to show that

$$(x+y)\left(\sqrt[3]{2}-1\right)^{1/3} \;=\; x^3+y^3 \;=\; 1\,,$$

or equivalently that

$$\left(\sqrt[3]{2}+1\right)^{3}\left(\sqrt[3]{2}-1\right) \;=\; 3\,,$$

which follows by expanding out the left hand side.

For other related radical identities of Ramanujan, see Susan Landau, *How to Tangle with a Nested Radical*, Math. Intelligencer, 16 (1994), pp. 49–54.

2. Prove that

$$3\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}, \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\} \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\,,$$

where a, b, c are sides of a triangle.

Solution. The inequalities

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \;\geq\; (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\,,$$

$$3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \;\geq\; (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\,,$$

follow from their equivalent forms (which follow by expansion):

$$(b+a-c)(c-a)^2+(c+b-a)(a-b)^2+(a+c-b)(b-c)^2 \;\geq\; 0\,,$$

$$(b+c-a)(c-a)^2+(c+a-b)(a-b)^2+(a+b-c)(b-c)^2 \;\geq\; 0\,.$$

3. Let $\omega = e^{i\pi/13}$. Express $\dfrac{1}{1-\omega}$ as a polynomial in ω with integral coefficients.

Solution. We have

$$\frac{2}{(1-\omega)} \;=\; \frac{(1-\omega^{13})}{(1-\omega)} = 1+\omega+\omega^2+\cdots+\omega^{12}\,,$$

$$0 \;=\; \frac{(1+\omega^{13})}{(1+\omega)} \;=\; 1-\omega+\omega^2-\cdots+\omega^{12}\,.$$

Adding or subtracting, we get

$$\begin{aligned} \frac{1}{(1-\omega)} &= 1+\omega^2+\omega^4+\cdots+\omega^{12} \\ &= \omega+\omega^3+\cdots+\omega^{11}. \end{aligned}$$

More generally, if $\omega = e^{i\pi/(2n+1)}$,

$$\frac{1}{(1-\omega)} = 1+\omega^2+\omega^4+\cdots+\omega^{2n}.$$

4. Determine all integral solutions of the simultaneous Diophantine equations $x^2+y^2+z^2=2w^2$ and $x^4+y^4+z^4=2w^4$.

Solution. Eliminating w we get

$$2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4 = 0$$

or

$$(x+y+z)(y+z-x)(z+x-y)(x+y-z) = 0,$$

so that in general we can take $z=x+y$. Note that if (x,y,z,w) is a solution, so is $(\pm x, \pm y, \pm z, \pm w)$ and permutations of the x, y, z. Substituting back, we get

$$x^2+xy+y^2 = w^2.$$

Since $(x,y,w)=(1,-1,1)$ is one solution, the general solution is obtained by the method of Desboves; that is, we set $x=r+p$, $y=-r+q$ and $w=r$. This gives $r=\frac{(p^2+pq+q^2)}{(q-p)}$. On rationalizing the solutions (since the equation is homogeneous), we get

$$\begin{aligned} x &= p^2+pq+q^2+p(q-p) = q^2+2pq, \\ -y &= p^2+pq+q^2-q(q-p) = p^2+2pq, \\ w &= p^2+pq+q^2, \\ z &= q^2-p^2. \end{aligned}$$

5. Prove that if the line joining the incentre to the centroid of a triangle is parallel to one of the sides of the triangle, then the sides are in arithmetic progression, and, conversely, if the sides of a triangle are in arithmetic progression then the line joining the incentre to the centroid is parallel to one of the sides of the triangle.

Solution. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ denote vectors to the respective vertices A, B, C of the triangle from a point outside the plane of the triangle. Then the incentre I and the centroid G have the respective vector representations $\mathbf{I}$ and $\mathbf{G}$, where

$$\mathbf{I} = \frac{(a\mathbf{A}+b\mathbf{B}+c\mathbf{C})}{(a+b+c)}, \qquad \mathbf{G} = \frac{(\mathbf{A}+\mathbf{B}+\mathbf{C})}{3},$$

(where a, b, c are sides of the triangle). If $\mathrm{G} - \mathrm{I} = k(\mathrm{A} - \mathrm{B})$, then by expanding out

$$(b + c - 2a - k')\mathrm{A} + (a + c - 2b + k')\mathrm{B} + (a + b - 2c)\mathrm{C} \ = \ 0\,,$$

where $k' = 3k(a + b + c)$. Since A, B, C are linearly independent, the coefficient of C must vanish so that the sides are in arithmetic progression. Also then $k' = b + c - 2a = 2b - a - c$.

Conversely, if $2c = a + b$, then $\mathrm{G} - \mathrm{I} = \frac{3(\mathrm{A}-\mathrm{B})(b-a)}{6(a+b+c)}$, so that GI is parallel to the side AB.

6. Determine integral solutions of the Diophantine equation

$$\frac{x-y}{x+y} + \frac{y-z}{y+z} + \frac{z-w}{z+w} + \frac{w-x}{w+x} \ = \ 0$$

(joint problem with Emeric Deutsch, Polytechnic University of Brooklyn).

Solution. It follows by inspection that $x = z$ and $y = w$ are two solutions. To find the remaining solution(s), we multiply the given equation by the least common denominator to give

$$P(x, y, z, w) \ = \ 0\,,$$

where P is the 4^{th} degree polynomial in x, y, z, w which is skew symmetric in x and z and also in y and w. Hence,

$$P(x, y, z, w) \ = \ (x - z)(y - w)Q(x, y, z, w)\,,$$

where Q is a quadratic polynomial. On calculating the coefficient of x^2 in P, we get $2z(y - w)$. Similarly the coefficient of y^2 is $-2w(x - z)$, so that

$$P(x, y, z, w) \ = \ 2(x - z)(y - w)(xz - yw)\,.$$

Hence, the third and remaining solution is given by $xz = yw$.

1997

1. For x, y, $z > 0$, prove that

(i) $1 + \dfrac{1}{(x+1)} \ \geq \ \left\{1 + \dfrac{1}{x(x+2)}\right\}^x$,

(ii) $[(x+y)(x+z)]^x[(y+z)(y+x)]^y[(z+x)(z+y)]^z \geq [4xy]^x[4yz]^y[4zx]^z$.

Solution. Both inequalities will follow by a judicious application of the Weighted Arithmetic-Geometric Mean Inequality (W–A.M.–G.M.) which for three weights is

$$u^a v^b w^c \;\leq\; \left[\frac{au+bv+cw}{a+b+c}\right]^{a+b+c}$$

where a, b, c, u, v, $w \geq 0$.

(i) The inequality can be rewritten in the more attractive form

$$\left[1+\frac{1}{x}\right]^x \;\leq\; \left[1+\frac{1}{x+1}\right]^{x+1},$$

which now follows by the W–A.M.–G.M.

$$\left[1+\frac{1}{x}\right]^x \;\leq\; \left\{\frac{1+x\left(1+\frac{1}{x}\right)}{1+x}\right\}^{x+1} \;=\; \left[1+\frac{1}{x+1}\right]^{x+1}.$$

(ii) Also, the inequality here can be rewritten in the more attractive form

$$\left[\frac{2x}{z+x}\right]^{z+x}\left[\frac{2y}{x+y}\right]^{x+y}\left[\frac{2z}{y+z}\right]^{y+z} \;\leq\; 1\,.$$

But this follows by applying the W–A.M.–G.M. to

$$1 \;=\; \sum[z+x]\left[\frac{2x}{z+x}\right] \div \sum[z+x]\,.$$

2. If $ABCD$ is a quadrilateral inscribed in a circle, prove that the four lines joining each vertex to the nine point centre of the triangle formed by the other three vertices are concurrent.

Solution. The given result still holds if we replace the nine point centres by either the orthocentres or the centroids.

A vector representation is particularly *à propos* here, since with the circumcentre O as an origin and $\mathbf{F}$ denoting the vector from O to any point F, the orthocentre H_a, the nine point centre N_a, the centroid G_a of $\triangle BCD$ are given simply by $\mathrm{H}_a = \mathrm{B}+\mathrm{C}+\mathrm{D}$, $\mathrm{N}_a = (\mathrm{B}+\mathrm{C}+\mathrm{D})/2$, $\mathrm{G}_a = (\mathrm{B}+\mathrm{C}+\mathrm{D})/3$, respectively, and similarly for the other three triangles. Since the proofs for each of the three cases are practically identical, we just give the one for the orthocentres. The vector equation of the line L_a joining A to H_a is given by $\mathrm{L}_a = \mathrm{A}+\lambda_a[\mathrm{B}+\mathrm{C}+\mathrm{D}-\mathrm{A}]$ where λ_a is a real parameter. By letting $\lambda_a = 1/2$, one point on the line is $[\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}]/2$ and similarly this point is on the other three lines. For the nine point centres, the point of concurrency will be $2[\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}]/3$, while for the centroids, the point of concurrency will be $3[\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}]/4$.

3. How many six-digit perfect squares are there each having the property that if each digit is increased by one, the resulting number is also a perfect square?

Solution. If the six-digit square is given by

$$m^2 = a \cdot 10^5 + b \cdot 10^4 + c \cdot 10^3 + d \cdot 10^2 + e \cdot 10 + f ,$$

then

$$n^2 = (a+1) \cdot 10^5 + (b+1) \cdot 10^4 + (c+1) \cdot 10^3 + (d+1) \cdot 10^2 + (e+1) \cdot 10 + (d+1) ,$$

so that

$$n^2 - m^2 = 111\,111 = (111)(1\,001) = (3 \cdot 37)(7 \cdot 11 \cdot 13) .$$

Hence,

$$n + m = d_i \quad \text{and} \quad n - m = 111\,111/d_i$$

where d_i is one of the divisors of $111\,111$. Since $111\,111$ is a product of five primes, it has 32 different divisors. But since we must have $d_i > 111\,111/d_i$, there are at most 16 solutions given by the form $m = \frac{1}{2}(d_i - 111\,111/d_i)$. Then since m^2 is a six-digit number, we must have

$$632.46 \approx 200\sqrt{10} < 2m < 2,000 .$$

On checking the various divisors, there are four solutions. One of them corresponds to $d_i = 3 \cdot 13 \cdot 37 = 1\,443$ so that $m = \frac{1}{2}(1\,443 - 7 \cdot 11) = 683$ and $m^2 = 466\,489$. Then, $466\,489 + 111\,111 = 577\,600 = 760^2$. The others are given by the table

d_i	m	m^2	n^2	n
$3 \cdot 7 \cdot 37 = 777$	317	100 489	211 600	460
$3 \cdot 11 \cdot 37 = 1\,221$	565	319 225	430 336	656
$7 \cdot 11 \cdot 13 = 1\,001$	445	198 025	309 136	556

4. Let $v_i w_i$, $i = 1, 2, 3, 4$, denote four cevians of a tetrahedron $v_1 v_2 v_3 v_4$ which are concurrent at an interior point p of the tetrahedron. Prove that

$$pw_1 + pw_2 + pw_3 + pw_4 \leq \max v_i w_i \leq \text{longest edge} .$$

Solution. We choose an origin o outside of the space of the tetrahedron and use the set of four linearly independent vectors $\mathbf{V}_i = ov_i$ as a basis. Also the vector from o to any point q will be denoted by $\mathbf{Q}$. The interior point p is then given by $\mathbf{P} = x_1 \mathbf{V}_1 + x_2 \mathbf{V}_2 + x_3 \mathbf{V}_3 + x_4 \mathbf{V}_4$ where $x_i > 0$ and $\sum_i x_i = 1$.

It now follows that $\mathbf{W}_i = \frac{\mathbf{P}-x_i\mathbf{V}_i}{1-x_i}$ (for other properties of concurrent cevians via vectors, see [1987 : 274–275]) and then that

$$pw_i = \left|\frac{\mathbf{P}-x_i\mathbf{V}_i}{1-x_i} - \mathbf{P}\right| = \left|\frac{x_i(\mathbf{P}-\mathbf{V}_i)}{1-x_i}\right| = \left|x_i\sum_j x_j\frac{\mathbf{V}_j-\mathbf{V}_i}{1-x_i}\right|,$$

$$v_iw_i = \left|\frac{\mathbf{P}-x_i\mathbf{V}_i}{1-x_i} - V_i\right| = \left|\frac{\mathbf{P}-\mathbf{V}_i}{1-x_i}\right| = \left|\sum_j x_j\frac{\mathbf{V}_j-\mathbf{V}_i}{1-x_i}\right|.$$

Summing

$$\sum_i pw_i = \sum_i \left|x_i\sum_j x_j\frac{\mathbf{V}_j-\mathbf{V}_i}{1-x_i}\right| = \sum_i x_i(v_iw_i) \leq \max_i v_iw_i\,,$$

and with equality only if v_iw_i is constant. Also,

$$v_iw_i \leq \sum_{j\neq i}\left[\frac{x_j}{1-x_i}\right]\max_r|\mathbf{V}_r-\mathbf{V}_i| = \max_r|\mathbf{V}_r-\mathbf{V}_i|\,.$$

Finally,

$$\sum_i pw_i \leq \max_i v_iw_i \leq \max_{r,s}|\mathbf{V}_r-\mathbf{V}_s|\,.$$

Comment: In a similar fashion, it can be shown that the result generalizes to n–dimensional simplexes. The results for triangles are due to Paul Erdős, Amer. Math. Monthly, Problem 3746, 1937, p. 400; Problem 3848, 1940, p. 575.

5. Determine the radius r of a circle inscribed in a given quadrilateral if the lengths of successive tangents from the vertices of the quadrilateral to the circle are a, a, b, b, c, c, d, d, respectively.

Solution. Let $2A$, $2B$, $2C$, $2D$ denote the angles between successive pairs of radii vectors to the points of tangency. Then

$$r = \frac{a}{\tan A} = \frac{b}{\tan B} = \frac{c}{\tan C} = \frac{d}{\tan D}\,.$$

Also, since $A+B+C+D=\pi$, we have $\tan(A+B)+\tan(C+D)=0$, or

$$\frac{\tan A+\tan B}{1-\tan A\tan B} + \frac{\tan C+\tan D}{1-\tan C\tan D} = 0\,,$$

so that

$$\frac{r(a+b)}{r^2-ab} + \frac{r(c+d)}{r^2-cd} = 0\,.$$

Finally,

$$r^2 = \frac{abc + bcd + cda + dab}{a+b+c+d}.$$

6. Determine the four roots of the equation $x^4 + 16x - 12 = 0$.

Solution. Since

$$x^4 + 16x - 12 = (x^2+2)^2 - 4(x-2)^2 = (x^2+2x-2)(x^2-2x+6) = 0,$$

the four roots are $-1 \pm \sqrt{3}$ and $1 \pm i\sqrt{5}$.

7. Prove that the smallest regular n–gon which can be inscribed in a given regular n–gon is one whose vertices are the mid-points of the sides of the given regular n–gon.

Solution. The circumcircle of the inscribed regular n–gon must intersect each side of the given regular n–gon. The smallest such a circle can be is the inscribed circle of the given n–gon and it touches each of its sides at its mid-points.

8. If 31^{1995} divides $a^2 + b^2$, prove that 31^{1996} divides ab.

Solution. If one calculates $1^2, 2^2, \ldots, 30^2 \pmod{31}$ one finds that the sum of no two of these equals $0 \pmod{31}$. Hence, $a = 31a_1$ and $b = 31b_1$, so that 31^{1993} divides $a_1^2 + b_1^2$. Then, $a_1 = 31a_2$ and $b_1 = 31b_2$. Continuing in this fashion (with $p = 31$), we must have $a = 31^{998}m$ and $b = 31^{998}n$, so that ab is divisible by 31^{1996}.

More generally, if a prime $p = 4k + 3$ divides $a^2 + b^2$, then both a and b must be divisible by p. This follows from the result that "*a natural number n is the sum of squares of two relatively prime natural numbers if and only if n is divisible neither by 4 nor by a natural number of the form* $4k + 3$" (see J.W. Sierpinski, *Elementary Theory of Numbers*, Hafner, N.Y., 1964, p. 170).

9. Determine the minimum value of

$$\begin{aligned} S &= \sqrt{(a+1)^2 + 2(b-2)^2 + (c+3)^2} + \sqrt{(b+1)^2 + 2(c-2)^2 + (d+3)^2} \\ &\quad + \sqrt{(c+1)^2 + 2(d-2)^2 + (a+3)^2} + \sqrt{(d+1)^2 + 2(a-2)^2 + (b+3)^2} \end{aligned}$$

where a, b, c, d are any real numbers.

Solution. Applying Minkowski's inequality,

$$S \geq \sqrt{(4+s)^2 + 2(s-8)^2 + (s+12)^2} = \sqrt{4s^2 + 288}$$

where $s = a + b + c + d$. Consequently, $\min S = 12\sqrt{2}$ and is taken on for $a + b + c + d = 0$.

10. A set of 500 real numbers is such that any number in the set is greater than one-fifth the sum of all the other numbers in the set. Determine the least number of negative numbers in the set.

Solution. Letting a_1, a_2, a_3, $\ldots$ denote the numbers of the set and S the sum of all the numbers in the set, we have

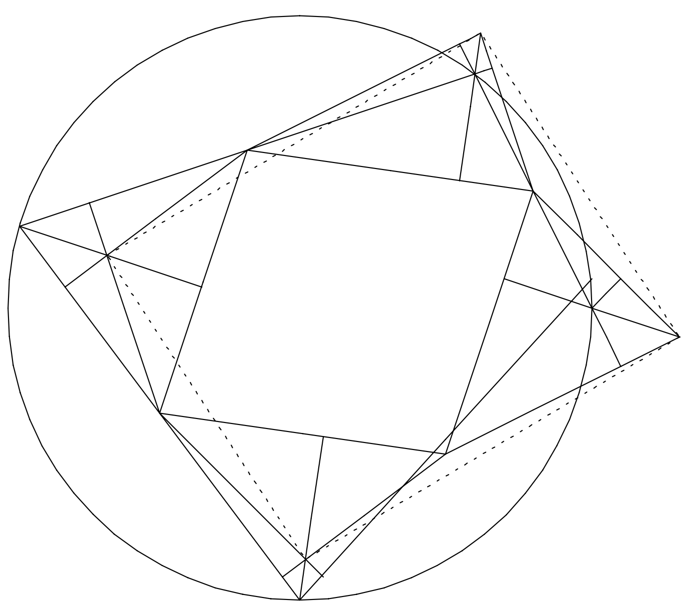
$$\begin{aligned} a_1 &> \frac{S - a_1}{5}, \\ a_2 &> \frac{S - a_2}{5}, \\ &\vdots \\ a_6 &> \frac{S - a_6}{5}. \end{aligned}$$

Adding, we get $0 > S - a_1 - a_2 - \cdots - a_6$ so that if there were six or less negative numbers in the set, the right hand side of the inequality could be positive. Hence, there must be at least 7 negative numbers.

Comment. This problem, where the 5 is replaced by 1, is due to Mark Kantrowitz, Carnegie–Mellon University.

A useful diagram!

Somewhere, in this issue, is a problem where the following diagram is useful.



There is a lot of very nice geometry in this figure!

THE ACADEMY CORNER

No. 39

Bruce Shawyer

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In this issue, we present solutions to problems of the 2000 APICS Math Competition problem [2000 : 450]. The competition is for university students.

2. Let $S = \{1, 2, 3, \ldots, 3n\}$. We define a sum–3–partition of S to be a collection of n disjoint 3–subsets of S, $A_i = \{a_i, b_i, c_i\}$, $i = 1, \ldots, n$, such that the union $A_1 \cup A_2 \cup \cdots \cup A_n$ is S, and within each triple A_i, some element is the sum of the other two. For example: $\{\{1, 5, 6\}, \{2, 9, 11\}, \{3, 7, 10\}, \{4, 8, 12\}\}$ is a sum–3–partition of $\{1, 2, 3, \ldots, 12\}$.

   (a) Find a sum–3–partition for $\{1, 2, 3, \ldots, 15\}$.

   (b) Prove that there exists no sum–3–partition for $n = 1999$.

Solution by Michael Watson, grade 12 student, Bishops College, St. John's, Newfoundland (*slightly adapted by the editor*).

(a) A sum–3–partition for $\{1, 2, 3, \ldots, 15\}$ is

$$\{1, 14, 15\}, \{5, 8, 13\}, \{2, 10, 12\}, \{4, 7, 11\}, \{3, 6, 9\}.$$

(b) Each triple is of the form $a + b = c$. Therefore, its sum is $2c$. The sum of all these triples is $2 \cdot \sum_{j=1}^{n} c_j = 2s$, say.

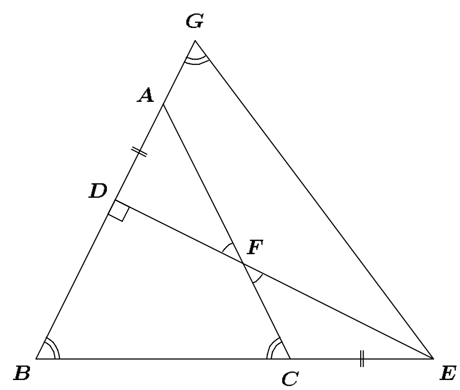
This sum is also the sum of the first $3n$ integers, giving $\frac{3n(3n+1)}{2} = 2s$.

Since n and s are both integers, it follows that $3n(3n + 1)$ must be divisible by 4.

If $n = 1999$, this is not true; hence, there is no sum–3–partition for $n = 1999$.

8. An isosceles triangle has vertex A and base BC. Through a point D on AB, we draw a perpendicular to meet BC extended at E, such that $AD = CE$. If DE meets AC at F, show that the area of triangle ADF is twice that of triangle CFE.

I. *Solution by Michael Watson, grade 12 student, Bishops College, St. John's, Newfoundland* (*slightly adapted by the editor*).



We have that $AD = CE$, $AB = AC$, $\angle ABC = \angle BCA$ and $\angle AFD = \angle EFC$.

Produce BA to G so that $DG = BD$. Join EG. Note that $\triangle BEG$ is isosceles, so that $\angle EGB = \angle GBE$ and $BE = GE$.

Applying Menelaus' Theorem to $\triangle ABC$ with transversal DFE, [Ed. since we deal with lengths, the unsigned version of the theorem is used.] we have

$$\frac{BD}{AD} \cdot \frac{AF}{FC} \cdot \frac{CE}{BE} = 1 .$$

Since $AD = CE$, this gives

$$\frac{BD}{BE} \cdot \frac{AF}{FC} = 1 . \qquad (1)$$

Similarly, applying Menelaus' Theorem to $\triangle BED$ with transversal CFA, we have

$$\frac{BC}{CE} \cdot \frac{EF}{FD} \cdot \frac{DA}{AB} = 1 .$$

Since $AD = CE$, this gives

$$\frac{BC}{AB} \cdot \frac{EF}{DF} = 1 . \qquad (2)$$

The area of $\triangle AFD$ is $\frac{1}{2}AF \cdot DF \sin\angle DFA$, and the area of $\triangle CFE$ is $\frac{1}{2}EF \cdot FC \sin\angle EFC$. In order to prove the result, we need to show that

$$2EF \cdot FC \;=\; AF \cdot DF\,.$$

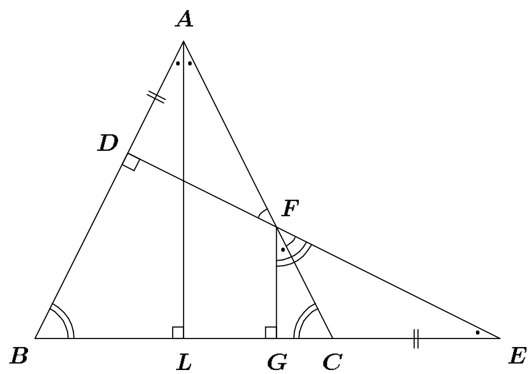
From (1) and (2), we have $\dfrac{BD}{BE} \cdot \dfrac{AF}{FC} = \dfrac{BC}{AB} \cdot \dfrac{EF}{DF}$, or

$$\frac{BD}{BE} \cdot AF \cdot DF \;=\; \frac{BC}{AB} \cdot CF \cdot EF\,.$$

It is sufficient to show that $\dfrac{2BD}{BE} = \dfrac{BC}{AB}$. Since $2BD = BG$, it now remains to show that $\dfrac{BG}{BE} = \dfrac{BC}{AB}$. Since $\triangle ABC$ is similar to $\triangle EGB$, and since $EG = BE$, the result follows.

II. *Solution by Catherine Shevlin, Wallsend upon Tyne, England.*

Drop a perpendicular from A to L on BC, and a perpendicular from F to G on BC.



It is sufficient to prove that $DF = 2FG$. Let $\theta = \angle BAL = \angle LAC = \angle GFC = \angle DEB$ (denoted in the diagram by $\cdot$), let $x = AD = CE$, and let $y = GC$. Then

$$\frac{DF}{AD} \;=\; \frac{DF}{x} \;=\; \tan(2\theta) \;=\; \frac{2\tan\theta}{1-\tan^2\theta}\,, \qquad \text{and}$$

$$\tan\theta \;=\; \frac{FG}{EG} \;=\; \frac{FG}{x+y} \;=\; \frac{y}{FG}\,, \qquad \text{giving} \qquad \frac{FG}{x+y} \;=\; \frac{y}{FG}\,.$$

Thus,
$$FG^2 \;=\; y(x+y) \;=\; xy+y^2\,.$$

Therefore,
$$DF \;=\; \frac{2xyFG}{FG^2-y^2} \;=\; \frac{2xyFG}{xy} \;=\; 2FG.$$

THE OLYMPIAD CORNER

No. 212

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin this number with the problems of the IV$^{\text{th}}$ class of the Croatian National Mathematical Competition, Novi Vinodolski, May 8–11, 1997. Thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina for collecting them.

CROATIAN NATIONAL MATHEMATICAL COMPETITION
Novi Vinodolski, May 8–11, 1997
IV$^{\text{th}}$ Class

1. Find the last four digits of the number 3^{1000} and the number 3^{1997}.

2. A circle k and the point K are on the same plane. For every two distinct points P and Q on k, the circle k' contains the points P, Q, and K. Let M be the intersection of the tangent to the circle k' at the point K and the line PQ. Find the locus of the points M when P and Q move over all points on k.

3. A function f is defined on the set of positive numbers, which has the following properties

$$f(1) = 1, \quad f(2) = 2,$$

$$f(n+2) = f(n+2-f(n+1)) + f(n+1-f(n)), \quad (n \geq 1).$$

(a) Show that $f(n+1) - f(n) \in \{0, 1\}$ for every $n \geq 1$.

(b) If $f(n)$ is odd, show that $f(n+1) = f(n) + 1$.

(c) For every natural number k determine all numbers n for which

$$f(n) = 2^{k-1} + 1.$$

4. Let k be a natural number. Determine the number of non-congruent triangles whose vertices are the vertices of the regular polygon with $6k$ sides.

We continue with the Additional Competition for selection of the 38^{th} IMO team of the Croatian National Mathematical Competition, also sent to us by Richard Nowakowski.

CROATIAN NATIONAL MATHEMATICAL COMPETITION
ADDITIONAL COMPETITION FOR SELECTION OF THE 38^{th} IMO TEAM
May 10, 1997

1. Three points A, B, C, are given on the same line, such that B is between A and C. Over the segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, as diameters, the semicircles are constructed on the same side of the line. The perpendicular from B to AC intersects the largest circle at point D. Prove that the common tangent of two smaller semicircles, different from BD, is parallel to the tangent on the largest semicircle through the point D.

2. Let a, b, c, d be real numbers such that at least one is different from zero. Prove that all roots of the polynomial

$$P(x) \;=\; x^6 + ax^3 + bx^2 + cx + d$$

cannot be real.

3. Squares $ABDE$ and $BCFG$ are constructed outside the acute triangle ABC in which $|AB| < |BC|$. Let M, N be the mid-points of the sides $\overline{BC}$, $\overline{AC}$, respectively, and S be the intersection of the lines BN and GM, $S \neq N$. Suppose that the points M, C, S and N are on the same circle. Prove that $|MD| = |MG|$.

4. Prove that every natural number can be represented in the unique form

$$a_0 + 2a_1 + 2^2 a_2 + \cdots + 2^n a_n$$

where $a_k \in \{-1, 0, 1\}$ and $a_k \cdot a_{k+1} = 0$ for every $0 \leq k \leq n-1$.

Next we turn to the Selection Round of the 1997 St. Petersburg City Mathematical Olympiad. Thanks go to Richard Nowakowski for collecting the problems while at the IMO in Argentina.

1997 ST. PETERSBURG CITY MATHEMATICAL OLYMPIAD

Selection Round – 10th Grade

March 10, 1997

1. Positive integers x, y, z satisfy the equation $2x^x + y^y = 3z^z$. Prove that they are equal.

2. The number N is the product of k different primes ($k \geq 3$). Two players play the following game: they in turn write on the blackboard **composite** divisors of N. One cannot write the number N. It is also not permitted that two coprime numbers or two numbers one of which divides the other are written on the blackboard. The player who cannot move loses. Which of them has a winning strategy: the player starting the game or his adversary?

3. K, L, M, N are the mid-points of sides AB, BC, CD, DA respectively of an inscribed quadrangle $ABCD$. Prove that the orthocentres of triangles AKN, BKL, CLM, DMN are vertices of a parallelogram.

4. A 100×100 checked square is folded several times along the lattice lines. Two straight cuts are made also going along the lattice lines. What is the maximum number of parts that the square can be cut into?

5. All sides of a convex polyhedron are triangles. At least 5 edges go from each of its vertices, and no two vertices of degree 5 are connected by an edge. Prove that this polyhedron has a face whose vertices have degrees 5, 6, and 6 respectively.

6. $2n + 1$ lines lie on the plane. Prove that there are not more than $n(n+1)(2n+1)/6$ different acute triangles with sides on these lines.

7. Prove that the collection of all 12-digit numbers cannot be divided into groups of four numbers in such a way that the digits of the four numbers in each group coincide in 11 positions, with the remaining position using four consecutive digits.

8. 360 points divide the circle into equal arcs. These points are connected by 180 non-intersecting chords. Consider another 180 chords obtained from these by rotation of the circle by the angle $38°$. Prove that the union of all these 360 chords cannot be a closed broken line.

11th Grade

1. Can a 75×75 table be partitioned into dominoes (that is, 1×2 rectangles) and crosses (that is, five-square figures consisting of a square and its four horizontal and vertical neighbours)?

2. Prove that for $x \geq 2$, $y \geq 2$, $z \geq 2$

$$(y^3 + x)(z^3 + y)(x^3 + z) \geq 125xyz .$$

3. Circles S_1 and S_2 intersect at points A and B. A point Q is chosen on S_1. The lines QA and QB meet S_2 at points C and D; the tangents to S_1 at A and B meet at point P. The point Q lies outside S_2, the points C and D lie outside S_1. Prove that the line QP goes through the mid-point of CD.

4. A convex 50-gon with vertices at integral points is drawn on a checked paper. What maximum number of its diagonals can lie on the lattice lines?

5. The number $99\ldots99$ (1997 nines) is written on a blackboard. Every minute one of the numbers written on the blackboard is factored into two factors, then wiped out, and these two factors, independently increased or diminished by 2, are written instead. Can it be that at last all the numbers on the blackboard will be equal to 9?

6. A device consists of $4n$ elements. Every pair of them are connected by a red or a blue wire. The numbers of red and blue wires are equal. The device is totally disabled when two wires of the same colour connecting four elements are removed. An agent of a supposed enemy found the number of ways to disable the device by removing two blue wires. Prove that there are as many ways to disable the device by removing two red wires.

7. See problem 7, 10$^{\text{th}}$ grade.

8. An Aztek diamond of rank n is a figure consisting of those cells of a checked coordinate plane that are wholly contained in the square $\{(x, y) : |x| + |y| \leq n + 1\}$. For any covering of an Aztek diamond by dominoes (1×2 rectangles) a "switch" operation is permitted: one can choose any 2×2 square covered by exactly two dominoes and rotate it by $90°$. Prove that not more than $n(n+1)(2n+1)/6$ such operations are required to transform an arbitrary covering into the covering consisting only of horizontal dominoes.

Next we give the problems of the 33$^{\text{rd}}$ Spanish Mathematical Olympiad, 2$^{\text{nd}}$ Round, March 1997, First Day. Thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina for collecting and forwarding this material to us.

33$^{\text{rd}}$ SPANISH MATHEMATICAL OLYMPIAD
Second Round — March 1997
First Day (Time: 4.5 hours)

1. Calculate the sum of the squares of the first 100 terms of an arithmetic progression, given that the sum of the first 100 terms of the progression equals -1, and that the sum of the even numbered terms equals $+1$.

2. Let A be the set of the 16 lattice points forming a square of side 4. Find, with reasons, the largest number of points of A such that any THREE of them do NOT form an isosceles right triangle.

3. Consider the parabolas

$$y = x^2 + px + q,$$

that intersect the coordinate axes at three distinct points. For these three points, a circle is drawn. Show that all the circles drawn when p and q vary over $\mathbb{R}$ pass through a fixed point, and determine the point.

Second Day (Time: 4.5 hours)

4. Let p be a prime number. Find all integers $k \in \mathbb{Z}$ such that $\sqrt{k^2 - pk}$ is a positive integer.

5. Show that for any convex quadrilateral with unit area, the sum of the sides and diagonals is not less than $2(2+\sqrt{2})$.

6. The exact quantity of gasoline necessary for ONE complete round of a circular circuit is distributed in n reservoirs, situated at any n fixed points of the circuit. At the beginning, the car has no gasoline.

Prove that, for any distribution of the gasoline in the reservoirs, there exists an initial point such that it is possible to make a complete circuit.

Notes: The consumption of gasoline is uniform and proportional to the distance. The car can contain all the gasoline.

As a final problem set this number we give the Final Round of the 7th Japan Mathematical Olympiad (1997). Again, my thanks go to Richard Nowakowski who collected the problem set while in Argentina.

7th JAPAN MATHEMATICAL OLYMPIAD
Final Round — February 1997
Time: 4.5 hours

1. Prove that, whenever we put ten points in any way on a circle whose diameter is 5, we can find two points whose distance is less than 2.

2. Let a, b, c be positive integers. Prove that the inequality

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}$$

holds. Determine also when the equality holds.

3. Let G be a graph with 9 vertices (with no self-looping edges). Assume that "given any five points of G, there are at least edges $e_1, \ldots,$ e_n, $(n \geq 2)$, forming a path between two of these five points".

What is the minimum number of edges such a graph G must have?

4. Let A, B, C, D be points in space in a general position. Assume that $AX + BX + CX + DX$ is a minimum at $X = X_0$ $(X_0 \neq A, B, C, D)$. Prove that $\angle AX_0B = \angle CX_0D$.

5. Let n be a positive integer. To each vertex of a regular 2^n–gon, we assign one of letters "A" or "B". Prove that we can do this in such a way that all possible sequences of n letters which appear in this 2^n–gon as an arc directed clockwise from some vertex are mutually distinct.

Next we turn to solutions by readers to the problems of the Dutch Mathematical Olympiad [1999 : 134–135].

1. A kangaroo jumps from lattice-point to lattice-point in the (x, y)–plane. She can make only two kinds of jumps:

Jump A: 1 to the right (in the positive x-direction) and 3 up (in the positive y-direction).
Jump B: 2 to the left and 4 down.

(a) The start position of the kangaroo is the origin $(0, 0)$. Show that the kangaroo can jump to the point $(19, 95)$ and determine the number of jumps she needs to reach that point.

(b) Take the start position to be the point $(1, 0)$. Show that it is impossible for her to reach the point $(19, 95)$.

(c) The start position of the kangaroo is once more the origin $(0, 0)$. Which points (m, n) with m, $n \geq 0$ can she reach, and which points can she not reach?

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

(a) and (c). Clearly, the point (m, n) is reachable if and only if the system

$$\begin{cases} x - 2y = m \\ 3x - 4y = n \end{cases}$$

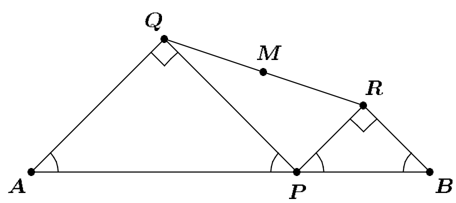
has solutions in non-negative integers x and y, where x and y denote the number of jumps of type A and type B, respectively. Since the solution of the system is $x = n - 2m$ and $y = \frac{n-3m}{2}$, we conclude that (m, n) is reachable if and only if m and n have the same parity such that $3m \leq n$. In particular, when $m = 19$, $n = 95$ we get $x = 57$ and $y = 19$; that is, she can reach $(19, 95)$ by making 57 jumps of type A and 19 jumps of type B.

(b) In this case, the system to be solved becomes

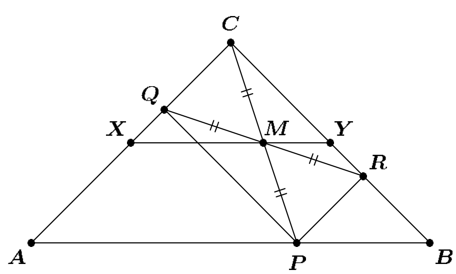
$$\begin{cases} x - 2y &= 18 \\ 3x - 4y &= 95\,. \end{cases}$$

Since $y = \frac{41}{2}$ is not an integer, $(19, 95)$ is no longer reachable.

2. On a segment AB a point P is chosen. On AP and PB, isosceles right-angled triangles AQP and PRB are constructed with Q and R on the same side of AB. M is the mid-point of QR. Determine the set of all points M for all points P on the segment AB.



Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.



We consider the problem on one side of AB. Let C be the intersection of AQ and BR. Since $\angle CAB = \angle QAP = 45^\circ$ and $\angle CBA = \angle RBP = 45^\circ$, we have that $\triangle CAB$ is isosceles and a right-angled triangle.

Since $\angle QPA = \angle RBP$ $(= 45^\circ)$, we have $PQ \| BR$. Similarly we have $PR \| AQ$, so that $CQPR$ is a parallelogram.

Since M is the mid-point of QR, M is also the mid-point of CP.

Let X and Y be the mid-points of CA and CB respectively.

Since X, M are mid-points of CA, CP respectively, we have $XM \| AP$.

Similarly we have $MY \| PB$. Hence, X, M, Y are collinear.

If P varies on the segment AB from A to B, then M varies on the segment XY from X to Y.

Thus, the set of all points M is the segment XY.

3. 101 marbles are numbered from 1 to 101. The marbles are divided over two baskets A and B. The marble numbered 40 is in basket A. This marble is removed from basket A and put in basket B. The average of all the numbers on the marbles in A increases by $\frac{1}{4}$. The average of all the numbers of the marbles in B increases by $\frac{1}{4}$ too. How many marbles were there originally in basket A?

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

Let $x_0, x_1, \dots, x_n$ (respectively $x_{n+1}, \dots, x_{100}$) denote the numbers on the marbles originally in basket A (respectively in basket B) with $x_0 = 40$. The original averages are

$$m_A = \frac{40 + x_1 + \cdots + x_n}{n+1} \quad \text{and} \quad m_B = \frac{x_{n+1} + \cdots + x_{100}}{100-n}.$$

After the marble numbered 40 has left basket A for basket B, the averages become

$$m'_A = \frac{x_1 + \cdots + x_n}{n} \quad \text{and} \quad m'_B = \frac{40 + x_{n+1} + \cdots + x_{100}}{101-n}.$$

Expressing $m'_A = m_A + \frac{1}{4}$ and $m'_B = m_B + \frac{1}{4}$, we easily get:

$$\frac{x_1 + \cdots + x_n}{n(n+1)} = \frac{40}{n+1} + \frac{1}{4}$$

and

$$\frac{x_{n+1} + \cdots + x_{100}}{(100-n)(101-n)} = \frac{40}{101-n} - \frac{1}{4}$$

which yield

$$x_1 + \cdots + x_n = \frac{n^2 + 161n}{4} \tag{1}$$

and

$$x_{n+1} + \cdots + x_{100} = \frac{5900 + 41n - n^2}{4}. \tag{2}$$

Since $x_1 + \cdots + x_n + x_{n+1} + \cdots + x_{100} = (1 + 2 + \cdots + 100 + 101) - 40 = \frac{101 \times 102}{2} - 40$, we obtain by addition of (1) and (2):

$$5900 + 202n = 204 \times 101 - 160.$$

Hence, $n = 72$ and there were initially 73 marbles in basket A.

4. A number of spheres, all with radius 1, are being placed in the form of a square pyramid. First, there is a layer in the form of a square with $n \times n$ spheres. On top of that layer comes the next layer with $(n-1) \times (n-1)$ spheres, and so on. The top layer consists of only one sphere. Determine the height of the pyramid.

Solutions by Mohammed Aassila, Strasbourg, France; and by Michel Bataille, Rouen, France. We give Bataille's solution.

We observe that there is a plane parallel to the base containing all the centres of the spheres of a layer, and we first determine the distance d between the planes so associated to two successive layers. Any sphere (S) of the $(k+1)^{\text{th}}$ layer rests on four spheres (S_1), (S_2), (S_3), (S_4) of the k^{th} layer so that (S) is tangent to these four spheres and (S_1), (S_2) [and (S_2), (S_3), and (S_3), (S_4) and (S_4), (S_1)] are tangent to each other

Let Ω, A, B, C, D be the centres of (S), (S_1), (S_2), (S_3), (S_4) respectively. Then d is the height of the regular pyramid $\Omega ABCD$ and we have

$$\Omega A \;=\; \Omega B \;=\; \Omega C \;=\; \Omega D \;=\; AB \;=\; BC \;=\; CD \;=\; DA \;=\; 2\,.$$

Let H be the projection of Ω on the plane $(ABCD)$ and K be the projection of H on, say, AB. Then K is also the projection of Ω on AB and Pythagoras' theorem gives:

$$d^2 \;=\; \Omega H^2 \;=\; \Omega K^2 - HK^2 \;=\; \left(2\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{2}{2}\right)^2 \;=\; 2\,.$$

Hence, $d = \sqrt{2}$.

Now, there are n planes associated to the different layers, so that the distance between the lowest (P_1) and the highest (P_n) of these planes is $(n-1)\sqrt{2}$.

Taking into account that the base is lower than (P_1) by a radius of the spheres and that the apex is higher than (P_n) by a radius as well, we finally obtain the height of the pyramid: $2 + (n-1)\sqrt{2}$.

5. We consider arrays $(a_1, a_2, \ldots, a_{13})$ containing 13 integers. An array is called "tame" when for each $i \in \{1,\, 2,\, \ldots,\, 13\}$ holds: if you leave a_i out, the remaining twelve integers can be divided in two groups in such a way that the sum of the numbers in one group is equal to the sum of the numbers in the other group. A "tame" array is called "turbo tame" if you can always divide the remaining twelve numbers in two groups of six numbers having the same sum.

(a) Give an example of an array of 13 integers (not all equal!) that is "tame". Show that your array is "tame".

(b) Prove that in a "tame" array all numbers are even or all numbers are odd.

(c) Prove that in a "turbo tame" array all numbers are equal.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's solution.

(a) For $i \in \{1, \ldots, 12\}$, let $a_i = 1$ and $a_{13} = 11$.

Leaving out a_{13}, then $a_1 + \cdots + a_6 = 6 = a_7 + \cdots + a_{12}$.

If a_{i_0}, for some $i_0 \in \{1, \ldots, 12\}$, then

$$a_{13} = 11 = \sum_{\substack{i \neq i_0 \\ i \leq 12}} a_i .$$

Thus, $(1, 1, \ldots, 1, 11)$ is "tame".

(b) Let $(a_1, \ldots, a_{13})$ be a "tame" array. Denote $S = a_1 + a_2 + \cdots + a_{13}$. Then there exist A_i and B_i, subsets of $\{1, \ldots, 13\} - \{i\}$ such that $A_i \cap B_i = \emptyset$, $A_i \cup B_i = \{1, \ldots, 13\} - \{i\}$ and

$$\sum_{a_j \in A_i} a_j = \sum_{a_j \in B_i} a_j .$$

Denote $S_i = \sum_{a_j \in A_i} a_j$. Thus,

$$S = a_i + 2S_i .$$

It follows, for every $i \in \{1, \ldots, 13\}$, that $a_i \equiv S \pmod 2$. Thus, all the a_i have the same parity, the parity of S.

(c) Let $(a_1, \ldots, a_{13})$ be "turbotame". Since $(a_1, \ldots, a_{13})$ is "tame", all the a_i have the same parity (from (b)). It follows that $a_j - a_1$ is even for $j = 2, \ldots, 13$. And it is easy to see that $(0, a_2 - a_1, \ldots, a_{13} - a_1)$ is also "turbotame". Suppose, for a contradiction, that there exists $i_0 \in \{2, \ldots, 13\}$ such that $a_{i_0} \neq a_1$. Then $a_{i_0} - a_1 \in \mathbb{Z}^*$ and $a_{i_0} - a_1$ is even.

Let p be the greatest power of 2 that divides $a_{i_0} - a_1$ (that is equivalent to, $a_{i_0} - a_1 \equiv 0 \pmod{2^p}$ and $a_{i_0} - a_1 \not\equiv 0 \pmod{2^{p+1}}$).

Denote, for $i \in \{1, \ldots, 13\}$,

$$\begin{aligned} U_0(i) &= a_i , \\ U_1(i) &= a_i - a_1 , \\ U_{n+1}(i) &= \frac{1}{2} U_n(i) \quad \text{for } n \geq 1 . \end{aligned}$$

Then, for all $n \geq 1$, $U_n(1) = 0$.

We know that $(U_1(1), U_1(2), \ldots, U_1(13))$ is "turbotame", and

$$U_1(i) \equiv 0 \pmod 2 \quad \text{for all } i \in \{1, \ldots, 13\} .$$

By an easy induction, for $n \geq 1$, $(U_n(1), U_n(2), \ldots, U_n(13))$ is "turbotame" and $U_n(i) \equiv 0 \pmod 2$ for all $i \in \{1, \ldots, 13\}$. (All the $U_n(i)$ are even since $(U_n(1), \ldots U_n(13))$ is "tame", and from (b), $U_n(i) \equiv U_n(1) \pmod 2$).

It follows that $U_{p+1}(i_0)$ is an even integer.

But $U_{p+1}(i_0) = \frac{1}{2}U_p(i_0) = \cdots = \frac{a_{i_0}-a_1}{2^p}$, an odd integer (from the maximality of p). This is a contradiction.

Then, for $i \in \{1, \ldots, 13\}$, $a_i = a_1$. It follows that all numbers are equal.

Remark. With the same reasoning, 13 can be replaced with any odd integer $k \geq 5$. I think that this general statement is a problem from the William Lowell Putnam Competition (but I have lost the reference ...).

Next we turn to the May 1999 number of the *Corner* and readers' solutions to problems of the XXXIX Republic Competition of Mathematics in Macedonia Class I [1999 : 196].

1. The sum of three integers a, b and c is 0. Prove that $2a^4 + 2b^4 + 2c^4$ is the square of an integer.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; by Andrei Simion, student, Brooklyn Technical High School, New York; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Simion, welcoming a new contributor to the Corner.

Let $p(x) = x^3 + sx^2 + qx + r$ be a third degree polynomial having a, b, and c as roots.

According to Viète's Theorem $s = -(a+b+c) = 0$. Then

$$\begin{aligned} a^3 + qa + r &= 0 \\ b^3 + qb + r &= 0 \\ c^3 + qc + r &= 0\,. \end{aligned}$$

Upon multiplying each equation by $2a$, $2b$ and $2c$, respectively, and adding we have

$$2a^4 + 2b^4 + 2c^4 + 2q\left(a^2 + b^2 + c^2\right) = 0$$

(the term with r vanishing since $a + b + c = 0$). However,

$$\begin{aligned} a^2 + b^2 + c^2 &= (a+b+c)^2 - 2(ab+bc+ac) \\ &= -2q\,. \end{aligned}$$

Thus,

$$2a^4 + 2b^4 + 2c^4 - 4q^2 = 0$$

and

$$2a^4 + 2b^4 + 2c^4 = (2q)^2$$

and we are done.

2. Prove that if

$$a_0^{a_1} = a_1^{a_2} = \cdots = a_{1995}^{a_{1996}} = a_{1996}^{a_0}, \quad a_1 \in \mathbb{R}^*,$$

then

$$a_0 = a_1 = \cdots = a_{1996}.$$

Solution by Pierre Bornsztein, Courdimanche, France.

I think that the a_i are supposed to be non-negative real numbers, and give that solution.

If $a_0, \ldots, a_{1996}$ are non-negative real numbers such that

$$a_0^{a_1} = \cdots = a_{1996}^{a_0}, \quad \text{then } a_0 = a_1 = \cdots = a_{1996}.$$

We first prove that $a_i \neq 0$ for all i.

Indeed, we are given $a_1 \neq 0$. Then, $a_1^{a_2} = a_0^{a_1} \neq 0$. Thus, $a_0 \neq 0$. An easy induction yields $a_i \neq 0$ for all i.

Next we show that either $a_i = 1$ for all i, or $0 < a_i < 1$ for all i or $1 < a_i$ for all i.

First suppose $a_i = 1$. Then $a_i^{a_{i+1}} = 1 = a_{i+2}^{a_{i+3}}$ (indices read modulo 1997). Now $a_{i+2} \neq 1$ entails $a_{i+3} = 0$, contrary to the last claim. It follows then that $a_i = 1$ for some i implies that $a_j = 1$ for all $j = 0, 1, \ldots, 1996$. Thus, we may suppose that $a_i \neq 0, 1$ for all $i = 0, 1, \ldots, 1996$. Suppose, for a contradiction, that $0 < a_i < 1$ for some i and $1 < a_j$ for some value of j. Then, we may suppose, without loss of generality, that $0 < a_i < 1$ and $1 < a_{i+1}$. But then $1 > a_i^{a_{i+1}} = a_{i+1}^{a_{i+2}}$ gives $a_{i+2} < 0$, contrary to the hypothesis. The claim now follows.

To complete the proof we distinguish two cases:

Case 1. $0 < a_i < 1$ for all $i = 0, 1, \ldots, 1996$.

Now first suppose $a_0 < a_1$. From $a_1^{a_2} = a_0^{a_1} < a_1^{a_1}$ and the fact that a_1^x is monotone decreasing, we obtain $a_1 < a_2$.

From this we get $a_0 < a_1 < a_i < \cdots < a_{1996} < a_{1997} = a_0$, a contradiction. The assumption that $a_1 < a_0$ similarly leads to a contradiction, completing this case.

Case 2. $1 < a_i$ for all i.

Suppose for a contradiction, that $a_0 < a_1$. Then $a_1^{a_2} = a_0^{a_1} < a_1^{a_1}$. It follows that $a_2 < a_1$. Thus, $a_1^{a_2} = a_2^{a_3} < a_1^{a_3}$, and we obtain $a_2 < a_3$.

By an easy induction we get $a_{2p} < a_{2p+1}$ for all p (subscripts are read modulo 1997).

But $a_{1998} = a_1$ and $a_{1999} = a_2$, giving $a_2 > a_1$, a contradiction. Thus, we have $a_0 \geq a_1$.

Next suppose $a_0 > a_1$. By similar reasoning we obtain $a_{2p} > a_{2p+1}$ for all p, leading to a similar contradiction.

Thus, $a_0 = a_1$.

From the cyclic symmetry of the assumptions we get $a_i = a_{i+1}$ for all i; that is, $a_0 = a_1 = a_2 = \cdots = a_{1996}$.

3. Let h_a, h_b and h_c be the altitudes of the triangle with edges a, b and c, and r be the radius of the inscribed circle in the triangle. Prove that the triangle is equilateral if and only if $h_a + h_b + h_c = 9r$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; by Toshio Seimiya, Kawasaki, Japan; and by Andrei Simion, student, Brooklyn Technical High School, New York. We give Bataille's write-up.

Clearly $h_a + h_b + h_c = 9r$ when the triangle is equilateral since in that case, we have

$$h_a \;=\; h_b \;=\; h_c \;=\; a\frac{\sqrt{3}}{2} \quad \text{and} \quad r \;=\; \frac{1}{3}\cdot a\frac{\sqrt{3}}{2}\,.$$

Conversely, let S denote the area of a $\triangle ABC$ in which $h_a + h_b + h_c = 9r$. Then, on the one hand,

$$2S = ah_a = bh_b = ch_c \quad \text{so that } h_a + h_b + h_c = 2S\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

and on the other hand

$$2S = ar + br + cr\,, \quad \text{so that} \quad r = \frac{2S}{a+b+c}\,.$$

From the hypothesis, we get

$$\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \;=\; \frac{3}{a+b+c}\,.$$

In other words, the harmonic mean of a, b, c equals their arithmetic mean. This implies $a = b = c$ and the triangle is equilateral.

4. Prove that each square can be cut into n ($n \geq 6$) squares.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Let S denote a given square. For any integer $k \geq 2$, we can divide S into k^2 equal (smaller) squares (as on a $k \times k$ chessboard). If we erase all the internal line segments forming all the $(k-1)^2$ smaller squares situated at the southeast corner of S, then we obtain a square T of dimension $(k-1) \times (k-1)$ which, together with the $2k-1$ smaller squares in the first row and first column of S, constitute a decomposition of S into $2k$ smaller squares. (Figures 1 and 2 below depict the cases when $k = 3$ and 4, respectively). If we further divide T into four equal squares as shown in Figure 3, then we have a decomposition of S into $2k + 3$ squares. Since $k \geq 2$ is arbitrary we can always divide S into n squares for any $n \geq 6$ (as well as $n = 4$, of course).

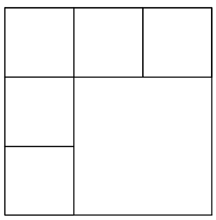


Figure 1

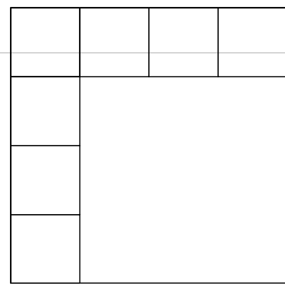


Figure 2

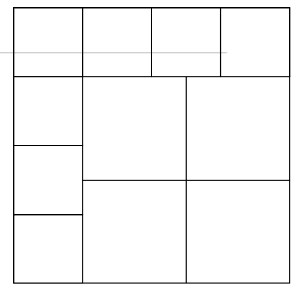


Figure 3

Remarks. (1) If we require that all the smaller squares must have different dimensions, then we have the famous "Squaring the Square" problem first studied by Brooks, Smith, Stone, and Tutte in 1936. (See, for example, *Ingenuity in Mathematics* by Ross Honsberger, New Mathematical Library, pp. 46–60).

(2) A similar problem about dissecting an equilateral triangle into n smaller equilateral triangles was proposed jointly by (the late) Helen Sturtevant and E.T.H. Wang in [1986 : 27; Solution 1987: 189]. It was shown that such dissection is possible for all n except when $n = 2, 3, 5$. It seems that the same conclusion is also true for dissecting a square. It would be interesting to see a proof (or a counterexample) of this.

Next are solutions to Class II problems of the XXXIX Republic Competition of Mathematics in Macedonia [1999 : 196–197]

1. Prove that for positive real numbers a and b

$$2 \cdot \sqrt{a} + 3 \cdot \sqrt[3]{b} \geq 5 \cdot \sqrt[5]{ab}\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's solution.

Set $a_1 = \sqrt{a}$, $a_2 = \sqrt{a}$, $a_3 = \sqrt[3]{b}$, $a_4 = \sqrt[3]{b}$, $a_5 = \sqrt[3]{b}$. Then, by the A.M.–G.M. inequality, we have

$$\sqrt[5]{ab} = \sqrt[5]{a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5} \leq \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = \frac{2}{5}\sqrt{a} + \frac{3}{5}\sqrt[3]{b}$$

and the result follows.

Remarks. The generalization is immediate: if m and n are integers ≥ 2, then we have

$$m\sqrt[m]{a}+n\sqrt[n]{b} \;\geq\; (m+n)\sqrt[m+n]{ab} \quad \text{for all positive real numbers } a \text{ and } b\,.$$

This also results from Hölder's Inequality:

$$uv \;\leq\; \frac{u^p}{p}+\frac{v^q}{q} \quad \text{when } u, v > 0 \text{ and } \frac{1}{p}+\frac{1}{q} \;=\; 1\,,$$

by taking

$$u \;=\; a^{1/(m+n)}\,, \quad v \;=\; b^{1/(m+n)}\,, \quad p \;=\; \frac{m+n}{m}\,, \quad q \;=\; \frac{m+n}{n}\,.$$

3. Let $A = \{z_1, z_2, \ldots, z_{1996}\}$ be a set of complex numbers and for each $i \in \{1, 2, \ldots, 1996\}$ suppose $\{z_iz_1, z_iz_2, \ldots, z_iz_{1996}\} = A$.

(a) Prove that $|z_i| = 1$ for each i.

(b) Prove that $z \in A$ implies $\overline{z} \in A$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

We note the obvious misprint in the original statement [1999 : 197] and solve the general problem by assuming that $A = \{z_1, z_2, \ldots, z_n\}$ is a set of n (distinct) complex numbers, $n \geq 2$, such that for each $i \in \{1, 2, \ldots, n\}$ we have

$$\{z_iz_1, z_iz_2, \ldots, z_iz_n\} \;=\; A\,, \tag{1}$$

(a) Let i be fixed, $1 \leq i \leq n$. Then by (1) we have

$$\prod_{j=1}^{n} z_iz_j \;=\; \prod_{j=1}^{n} z_j, \quad \text{or} \quad z_i^n\left(\prod_{j=1}^{n} z_j\right) \;=\; \prod_{j=1}^{n} z_j\,.$$

If $z_j = 0$ for any j, then $\{z_jz_1, z_jz_2, \ldots, z_jz_n\} = \{0\} \neq A$, since A has at least two elements. Thus, $\prod_{j=1}^{n} z_j \neq 0$, and we get $z_i^n = 1$. Hence, $|z_i|^n = |z_i^n| = 1$ from which $|z_i| = 1$ follows.

(b) Let $z \in A$ be fixed. Then by (1), $\{zz_1, zz_2, \ldots, zz_n\} = A$, and thus, $z = zz_i$ or $z_i = 1$ for some i, $1 \leq i \leq n$. It follows that $zz_k = 1$ for some k, $1 \leq k \leq n$. Since $z\overline{z} = |z|^2 = 1$, we conclude that $\overline{z} = z_k \in A$.

Remark. An example of a set A which satisfies the assumption (and hence, the conclusion) of the problem would be the set of all n of the n^{th} roots of unity.

4. Find the biggest value of the difference $x - y$ if $2 \cdot (x^2 + y^2) = x + y$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Heinz-Jürgen Seiffert, Berlin, Germany; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Seiffert's generalization and solution.

More generally: Let $p > 0$ and $q > 1$. If the real numbers x and y satisfy the equation

$$p(|x - y|^q + |x + y|^q) \;=\; x + y\,, \tag{1}$$

then there holds the sharp inequality

$$x - y \;\le\; (pq)^{1/(1-q)}(q-1)^{1/q}\,. \tag{2}$$

Proof. The function $f(t) = t/p - t^q$, $t \ge 0$, has first derivative $f'(t) = \frac{1}{p} - qt^{q-1}$, $t > 0$. Let $t_0 = (pq)^{1/(1-q)}$. From $f'(t) > 0$ if $0 < t < t_0$, and $f'(t) < 0$ if $t > t_0$, it follows that $f(t) \le f(t_0)$ for all $t \ge 0$; that is,

$$\frac{t}{p} - t^q \;\le\; (pq)^{q/(1-q)}(q-1), \quad t \;\ge\; 0\,.$$

Suppose that the real numbers x and y satisfy (1). Then

$$\begin{aligned} |x-y|^q \;&=\; \frac{x+y}{p} - |x+y|^q \;\le\; \frac{|x+y|}{p} - |x+y|^q \\ &\le\; (pq)^{q/(1-q)}(q-1)\,. \end{aligned}$$

Hence,

$$x - y \;\le\; |x-y| \;\le\; (pq)^{1/(1-q)}(q-1)^{1/q}\,,$$

proving (2). It is easily verified that

$$x \;=\; \frac{1}{2}(pq)^{1/(1-q)}(1 + (q-1)^{1/q})$$

and

$$y \;=\; \frac{1}{2}(pq)^{1/(1-q)}(1 - (q-1)^{1/q})$$

satisfy (1), and that there is equality in (2) for these values. Thus, (2) is sharp.

With $p = 1$ and $q = 2$, (1) becomes $2(x^2 + y^2) = x + y$. Then, by (2), there holds the sharp inequality $x - y \le \frac{1}{2}$. Hence, $\frac{1}{2}$ is the biggest value asked for.

Now we look at readers' solutions to Class III of the XXXIX Republic Competition of Mathematics in Macedonia [1999 : 197].

1. Solve the equation $x^{1996} - 1996x^{1995} + \cdots + 1 = 0$ (the coefficients in front of x, ..., x^{1994} are unknown), if it is known that its roots are positive real numbers.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Andrei Simion, student, Brooklyn Technical High School, New York. We give Bataille's solution.

Let $x_1, x_2, \ldots, x_{1996}$ be the roots of the given polynomial. By hypothesis, $x_k > 0$ for all k in $\{1, 2, \ldots, 1996\}$ and from the known coefficients: $x_1 + x_2 + \cdots + x_{1996} = 1996$ and $x_1 \cdot x_2 \cdots x_{1996} = 1$. It follows that:

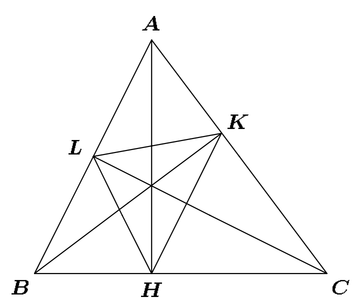
$$\frac{x_1 + x_2 + \cdots + x_{1996}}{1996} = 1 = \sqrt[1996]{x_1 \cdot x_2 \cdots x_{1996}}\,.$$

Therefore, we are in the case where the A.M.–G.M. inequality is actually an equality. As it is well known, this means that $x_1 = x_2 = \cdots = x_{1996}$ from which $x_1 = x_2 = \cdots = x_{1996} = 1$ immediately follows. Thus, the solution of the given equation is 1 with multiplicity 1996.

2. Let AH, BK and CL be the altitudes of arbitrary triangle ABC. Prove that

$$\overline{AK} \cdot \overline{BL} \cdot \overline{CH} = \overline{AL} \cdot \overline{BH} \cdot \overline{CK} = \overline{HK} \cdot \overline{KL} \cdot \overline{LH}\,.$$

Solutions by Toshio Seimiya, Kawasaki, Japan; and by Andrei Simion, student, Brooklyn Technical High School, New York. We give Seimiya's answer.



Since $\angle BKC = \angle BLC = 90^\circ$, B, C, K, L are concyclic. Hence, $\angle AKL = \angle ABC$, so that we have

$$\triangle AKL \sim \triangle ABC\,.$$

Thus,

$$\frac{\overline{AK}}{\overline{KL}} = \frac{\overline{AB}}{\overline{BC}}\,. \qquad (1)$$

Similarly we have

$$\frac{\overline{BL}}{\overline{LH}} = \frac{\overline{BC}}{\overline{CA}}, \tag{2}$$

and

$$\frac{\overline{CH}}{\overline{HK}} = \frac{\overline{CA}}{\overline{AB}}. \tag{3}$$

From (1), (2), (3) we get

$$\frac{\overline{AK}}{\overline{KL}} \cdot \frac{\overline{BL}}{\overline{LH}} \cdot \frac{\overline{CH}}{\overline{HK}} = \frac{\overline{AB}}{\overline{BC}} \cdot \frac{\overline{BC}}{\overline{CA}} \cdot \frac{\overline{CA}}{\overline{AB}} = 1.$$

It follows that

$$\overline{AK} \cdot \overline{BL} \cdot \overline{CH} = \overline{HK} \cdot \overline{KL} \cdot \overline{LH}.$$

Similarly we have

$$\overline{AL} \cdot \overline{BH} \cdot \overline{CK} = \overline{HK} \cdot \overline{KL} \cdot \overline{LH}.$$

3. An initial triple of numbers 2, $\sqrt{2}$, $\frac{1}{\sqrt{2}}$ is given. It is admitted to obtain a new triple from an old one as follows: two numbers a and b of the triple are changed to $\frac{(a+b)}{\sqrt{2}}$ and $\frac{(a-b)}{\sqrt{2}}$ and the third number is unchanged. Is it possible after a finite number of such steps to obtain the triple $(1, \sqrt{2}, 1+\sqrt{2})$?

Solution by Pierre Bornsztein, Courdimanche, France.

The answer is no.

Let T be the transformation described in the statement of the problem. If

$$T(a, b, c) = (x, y, z),$$

it is clear that

$$a^2 + b^2 + c^2 = x^2 + y^2 + z^2.$$

Then, from an initial triple (a, b, c), the number $S = x^2 + y^2 + z^2$ is invariant by the use of T. Starting from $\left(2, \sqrt{2}, \frac{1}{\sqrt{2}}\right)$ with $2^2 + \sqrt{2}^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 6.5$ we cannot obtain $\left(1, \sqrt{2}, 1+\sqrt{2}\right)$ after any number of steps because

$$1^2 + \left(\sqrt{2}\right)^2 + \left(1+\sqrt{2}\right)^2 = 6 + 2\sqrt{2} \neq 6.5.$$

4. A finite number of points in the plane are given such that not all of them are collinear. A real number is assigned to each point. The sum of the numbers for each line containing at least two of the given points is zero. Prove that all numbers are zeros.

Solution by Pierre Bornsztein, Courdimanche, France.

Let M_1, M_2, ..., M_n be the points. Then $n \geq 3$. Denote by a_i, the real number assigned to M_i, and let $S = a_1 + a_2 + \cdots + a_n$.

For a fixed $i_0 \in \{1, \ldots, n\}$, denote by n_{i_0} the number of lines containing M_{i_0} and at least one of the M_j for $j \neq i_0$.

Since not all the M_i are collinear, we have $n_i \geq 2$ for each i. Let Δ be one of the lines containing M_{i_0} and another of the M_j.

We then have

$$S(\Delta) \;=\; \sum_{M_j \in \Delta} a_j \;=\; 0\,.$$

Adding all these equalities, for all such Δ, we obtain

$$\sum_{M_{i_0} \in \Delta} S(\Delta) \;=\; 0 \;=\; S + (n_{i_0} - 1)a_{i_0}\,.$$

Since M_{i_0} is arbitrary, we then have

$$S + (n_i - 1)a_i \quad \text{for each } i\,. \tag{1}$$

Suppose that $S \neq 0$. Then there exists $i_0 \in \{1, \ldots, n\}$ such that a_{i_0} has the same sign as S.

For this choice of i_0, we cannot have $S + (n_{i_0} - 1)a_{i_0} = 0$. This is a contradiction.

Then $S = 0$, and, from (1), we have $a_i = 0$ for each i.

Next we look at solutions to Class IV, XXXIX Republic Competition of Mathematics in Macedonia, [1999 : 197].

1. Let a_1, a_2, ..., a_n be real numbers which satisfy:

There exists a real number M such that $|a_i| \leq M$ for each $i \in \{1, \ldots, n\}$.

Prove that $a_1 + 2a_2 + \cdots + na_n \leq \frac{Mn^2}{4}$.

Comments and solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Aassila's remarks.

This problem is similar to the one proposed by Morocco but not used by the jury at the 28$^{\text{th}}$ IMO in Cuba. A solution appeared in [1998 : 38]. The problem as stated is false: If $a_i = M$ for $i = 1, 2, \ldots, n$, then $\frac{Mn(n+1)}{2} \not\leq \frac{Mn^2}{4}$. One has to assume that $a_1 + a_2 + \cdots + a_n = 0$ as in [1998 : 38].

2. Two circles with radii R and r touch from inside. Find the side of an equilateral triangle having one vertex at the common point of the circles and the other two vertices lying on the two circles.

Solution by Michel Bataille, Rouen, France.

Let Γ, γ denote the two given circles (with centres Ω, ω and radii R, r respectively) and let A be their point of tangency.

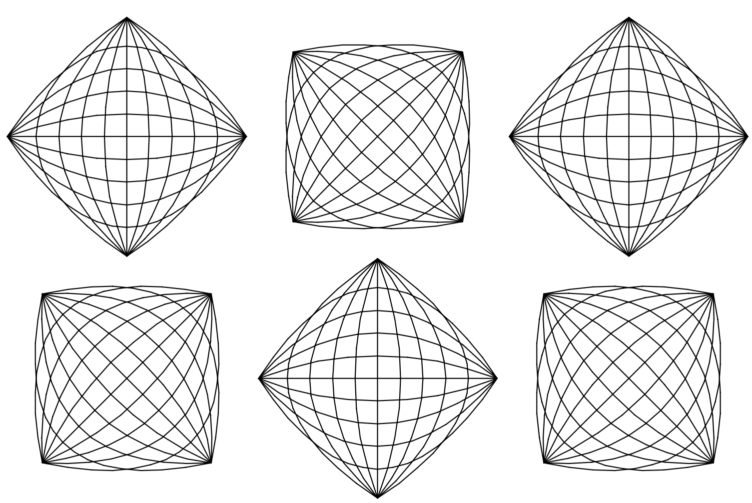
An equilateral triangle ABC having B on γ and C on Γ is easily obtained by drawing the circle Γ' image of Γ under a rotation ρ with centre A and angle 60°. B is the point other than A common to γ and Γ', and C is the image of B under ρ^{-1}. [There are two such triangles — because we can use either a direct or an indirect rotation; they are symmetrical about the line $A\Omega$ and, consequently, have the same length of sides].

Let Ω' be the centre of Γ' (so that $A\Omega = A\Omega' = R$). Then $AB = 2h$ where h is the length of the altitude from A in $\Delta A\omega\Omega'$. Since $A\omega = r$, $A\Omega' = R$ and $\angle\omega A\Omega' = 60^\circ$, the Cosine Law gives: $\omega\Omega'^2 = r^2 + R^2 - rR$.

Now, the area of $\triangle A\omega\Omega'$ is $\frac{1}{2}h \times \omega\Omega'$ as well as $\frac{1}{2} \times rR\sin(\angle\omega A\Omega') = \frac{\sqrt{3}}{4}rR$. This provides immediately:

$$h = \frac{\sqrt{3}}{2}\,\frac{rR}{\sqrt{r^2+R^2-rR}} \quad \text{and} \quad AB = \sqrt{3}\frac{rR}{\sqrt{r^2+R^2-rR}}\,.$$

That completes the *Corner* for this issue. Olympiad Season is coming up! Send me your nice contests and your nice solutions to problems from the *Corner* for future use.



BOOK REVIEWS

ALAN LAW

Geometry Turned On! Dynamic Software In Learning, Teaching, and Research,

edited by James R. King and Doris Schattschneider,
published by the Mathematical Association of America, 1997.
ISBN 0-88385-099-0, softcover, 206+ pages, $38.95 (U.S.).
Reviewed by **Murray S. Klamkin**, *University of Alberta, Edmonton, Alberta.*

Since I do not believe that geometry has been adequately treated in our schools in Canada and the USA, I am hopeful that my opening remarks, some of which have been written before, will be of some interest and helpful to teachers and students of geometry.

It is some forty years ago since Dieudonné's Royaumont Seminar address [1] in which he proclaimed "Euclid Must Go" and that it should be replaced by two-dimensional linear algebra. Dieudonné's quarrel was not with the purpose of geometry, which he considered important, but with the method of teaching geometry. Dieudonné's reasons were that Euclid as taught was a "Process Fantastically Laborious" as well as having "the affine and metric properties hopelessly mixed up". While there are many who agreed with the latter reasons, there were many who did not agree with the linear algebra replacement as proposed. Freudenthal [2] in his chapter on "The Case of Geometry" gave a profound critique of Dieudonné and recommended a piecemeal school geometry program with a strong intuitive spatial content. In his book, *The Foundation of Science* [3], Poincaré addressed the downgrading of geometry, and I quote:

> "It looks as if geometry could contain nothing which is not already included in algebra or analysis; that geometrical facts are only algebraic or analytic facts expressed in another language. It then may be thought that after our review there would be nothing more for us to say relating specially to geometry. This would fail to recognize the importance of well-constructed language, not to comprehend what is added to the things themselves by the method of expressing these things and consequently of grouping them. First the geometric considerations lead us to set ourselves new problems; these may be, if you choose, analytic problems, but such as we never would have set ourselves in connection with analysis. Analysis profits by them however, as it profits by those it has to solve and satisfy the need of physics. A great advantage of geometry lies in the fact that in it the senses can come to the aid of thought, and help find the path to follow, and many minds prefer to put the problems of analysis into geometric form. Unhappily our senses cannot carry us very far, and they desert us when we wish to soar beyond the classical three dimensions.

> Does this mean that, beyond the restricted domain wherein they seem to imprison us, we should rely only on pure analysis and that all geometry of more than three dimensions is vain and object-less? The greatest masters of a preceding generation would have answered 'yes'; today we are so familiarized with this notion that we can speak of it, even in a university course, without arousing too much astonishment.
>
> But what good is it? That is easy to see: First it gives us a very convenient terminology, which expresses concisely what the ordinary analytic language would say in prolix phrases. Moreover, this language makes us call like things by the same name and emphasize analogies it will never again let us forget. It enables us therefore still to find our way in this space, which is too big for us and which we cannot see, always recalling visible space, which is only an imperfect image of it doubtless, but which is nevertheless an image. Here, again, as in all the preceding examples, it is analogy with the simple which enables us to comprehend the complex."

In 1967 [4], Steenrod gave a lecture at a geometry conference which is highly in tune with Poincaré above. Again, I quote in part:

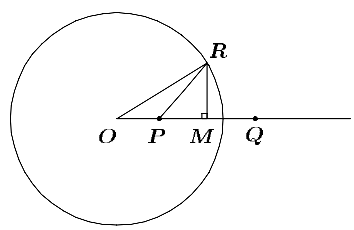
> "The pervasiveness of geometry is an idea that goes back to Descartes, for a coordinate system in the plane or in space sets up an equivalence between geometry and algebra-analysis. Every geometric proposition can be translated into its algebraic-analytic analog and vice versa. I am not proposing that we lead the student through the details of the formal isomorphism between these two systems, but I am trying to remind you that the geometry is always there and to keep in mind that the geometric language for the conversion is always at hand. For example, here are a few geometric terms and their algebraic-analytic counterparts.

Geometric language	*Algebraic-Analytic language*
point, vector	number triple (x, y, z)
projection	coordinate, variable
surface	equation
plane	linear equation
region	system of inequalities
mapping, transformation	function
neighbourhood	ϵ, δ
limit (using deleted neighbourhoods)	limit (using ϵ, δ)
tangent	derivative "

Although Dieudonné's message was not heeded in Canada and the USA, geometry as a subject was highly downgraded. The texts used took great pains to prove very obvious theorems but not un-obvious attractive theorems and insisted that proofs be done in "two columns".

With the advent of good dynamic-geometry software, geometry teaching has improved but not everywhere (Alberta being at least one exception).

This software allows students to obtain quick accurate constructions of various geometric configurations and to see what happens with almost immediate visual feedback when varying some of the points or figures of the configuration. It has been remarked many times that "A good figure is worth a thousand words". And in geometry, figures seem to be essential for most descriptions and proofs. Yet there is also the danger of relying on figures, since the sketch may suggest extra assumptions; special cases are missed; or absurd results are derived from the figure being topologically inaccurate. One widely known case of the latter is the "theorem and proof" that all triangles are isosceles. Another not so well known case is the "theorem and proof" that two distinct points can be zero distance apart: Here let P be an interior point of a circle of radius r and centre O with $OP = a$ as in the figure. Q (the inverse point of P) is on the line OP such that $OQ = r^2/a$. M is the midpoint of PQ and RM is perpendicular to line OP. Then $PQ = (r^2 - a^2)/a$, $PM = (r^2 - a^2)/2a$, $OM = (r^2 + a^2)/2a$, so that $PR^2 = PM^2 + MR^2 = PM^2 + OR^2 - OM^2 = [(r^2 - a^2)/2a]^2 + r^2 - [(r^2 + a^2)/2a]^2 = 0$.



Nevertheless, this software helps students learn geometry by visualizing, exploring, problem solving, making and testing conjectures, and assessing and verifying the conjectures. Also, importantly, it helps build up the students' geometric intuition.

Unfortunately, this software can be used in a horrible way, as in Alberta. Consider the elementary result that a central angle of a circle has twice the measure of an inscribed angle with the same arc. Here it suffices for the students to just take two different examples and determine their measures numerically with the software whose programming is even given to them. Having this software does not eliminate the need for proof, especially in the light of the following two quotations:

> "*A good proof is one that makes us wiser*". Yu. I. Manin
> "*Proofs really aren't there to convince you something is true—they're there to show you why it is true*". Andrew Gleason

The compilation being reviewed is #41 in the well edited MAA Note Series. The articles and references therein indicate the present resurgence of geometry. The papers in this volume are quite interesting and give a good idea of the ways in which the software can be used and some of the effects it can have. It will be clear that the software raises questions for teaching and research.

Geometer's Sketchpad, *Cabri II*, and *Geometric SuperSupposer* are incredible interactive geometry computer software that allow students to explore the properties of polygons, circles, and other geometric configurations very easily. Students no longer have to laboriously make constructions by hand, which may not be very accurate, to first verify a geometric theorem for themselves. It allows them to dynamically transform their figures easily with the mouse while preserving all dependent relationships and constraints. For example, consider a triangle and its three medians which will appear to be concurrent. As we move one of the vertices with the mouse, we will get other triangles very quickly with their three medians which will still appear to be concurrent. This encourages a process of discovery where students increase their geometric intuition and first visualize a problem and make conjectures before trying to give a proof why their observations are true. This also leads them to make and check generalizations and hopefully prove them (many examples of this are given in the papers of this volume).

There are even two articles on physical applications; "teaching and research in optometry and vision" and "creating airfoils from circles".

Unfortunately, there is not much on projective, non-Euclidean, and three dimensional geometry. When we first start learning any geometry, our 3–dimensional intuitive world suddenly becomes a flat 2–dimensional world. I recall that the first time I had any 3–D geometry was when I took a course in solid geometry in my last semester in high school. The text for the course had the same failings mentioned previously for plane geometry.The treatment of geometry in our schools in the USA and Canada should be contrasted with those in Bulgaria, Hungary, Romania, and the former USSR where plane and solid geometry is covered repeatedly in the different grades. They have many good books with challenging problems, unlike most of the sterile ones in the USA and Canada. As a challenging problem (from an old British text), show how to construct the radius of a solid sphere (by Euclidean methods). One construction leads to an instrument called a spherometer for measuring the radius of curvature of glass spherical lenses. There is also another construction which is simpler.

Continuing from before about the disadvantage of only first studying solid geometry in my last semester in high school, this almost got me to flunk a course in descriptive geometry in a freshman engineering course. I just could not visualize intersections of various 3–D bodies, e.g., two cones, the 2–D layout for making a ventilator cowling from a cone. Fortunately, after about 6 weeks of hard work, I then was able to do so. It would have been very helpful if I had some 3–D geometry throughout the lower grades and/or I had the translation of a Hungarian descriptive geometry book [5] which was published much later. This book had many stereograms so it was easy to see 3–D intersections and by moving one's eyes, one could even obtain projective transformations of the figures.

At the end of the volume there is information as to where to find interactive geometric sketches on the World-Wide Web as well as to where to purchase the available software. In particular, one can download "Geomview", which is a 3–D object viewer, free from the Geometry Center at the University of Minnesota.

In view of the above, I highly recommend this volume to teachers and students of geometry.

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Finally, I append a list of recommended books which may be of interest and useful to teachers and students of geometry. Even though I am not familiar with the recent publication of Cinderella, I am including the reference to it by virtue of the review of it by Ed Sandifer in the MAA Online book review column who notes that "Cinderella is more. Besides all the familiar constructions of Sketchpad, Cinderella supports constructions in spherical and hyperbolic geometry, includes a theorem prover (more about that later), has more general animation features, and generates applets that paste easily onto web pages. You can also generate applets to create self-checking construction exercises. The creators of the software have a web site, http://www.cinderella.de that demonstrates many of these features".

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Announcement

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The Editors

Forum Geometricorum

Here are the abstracts of the first papers published:

Forum Geometricorum, 1 (2001) 1 – 6.

Friendship Among Triangle Centers.

Floor van Lamoen

Abstract: If we erect on the sides of a scalene triangle three squares, then at the vertices of the triangle we find new triangles, the flanks. We study pairs of triangle centers X and Y such that the triangle of X's in the three flanks is perspective with ABC at Y, and vice versa. These centers X and Y, we call friends. Some examples of friendship among triangle centers are given.

Forum Geometricorum, 1 (2001) 7 – 8.

Another Proof of the Erdős-Mordell Theorem.

Hojoo Lee

Abstract: We give a proof of the famous Erdős-Mordell inequality using Ptolemy's Theorem.

Tangent circles in the ratio $2 : 1$

Hiroshi Okumura and Masayuki Watanabe

In this article we consider the following old Japanese geometry problem (see Figure 1), whose statement in [1, p. 39] is missing the condition that two of the vertices are the opposite ends of a diameter. (The authors implicitly correct the omission in the proof they provide on page 118.) We denote by $O(r)$ the circle with centre O, radius r.

Problem [1, Example 3.2]. The squares $ACB'D'$ and $ABC'D$ have a common vertex A, and the vertices C and B', C' and B lie on the circle $O(R)$ whose diameter is $B'C'$, A lying within the circle. The circle $O_1(r_1)$ touches AB and AC and also internally touches $O(R)$, and $O_2(r_2)$ is the incircle of triangle ABC. Show that

$$r_1 = 2r_2 . \tag{1}$$

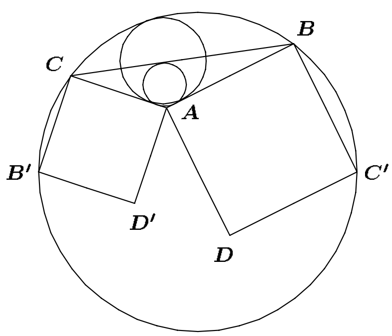

Figure 1.

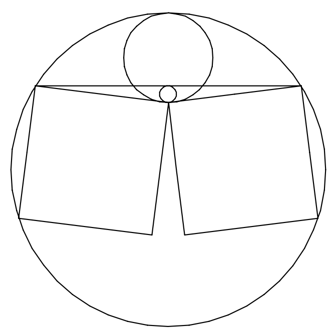

Figure 2.

We shall see that $B'C'$, being a diameter, is merely a sufficient condition. Figure 2 shows that some additional property involving B' and C' is required for deducing (1). In Theorem 1 we give a simple condition that implies (1).

Theorem 1. For any triangle ABC, if A' is the reflection in BC of a point A, then the radius of one of the circles internally touching the circumcircle of $A'BC$ and also touching AB and AC is twice the size of the radius of the incircle of ABC.

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Proof. Let AA' intersect BC at P and the circumcircle γ (say) of $A'BC$ again at D, and let Q be the foot of the perpendicular from C to BA' (see Figure 3). Then $\angle BA'D = \angle QCB$ since the right triangles $BA'P$ and BCQ share a common angle $\angle A'BC$. Moreover $\angle BA'D = \angle BCD$. Hence, $\angle QCB = \angle BCD$. This implies that H and D are symmetric in BC, where H is the orthocentre of $A'BC$. Thus, D is the orthocentre of ABC, and therefore, DBC and ABC share a common nine-point circle β (say), and β touches the incircle α (say) of ABC internally by Feuerbach's Theorem. Therefore, the dilatation of magnification 2 with centre A carries β into the circumcircle of DBC, which is γ, and α into one of the circles touching AB, AC and γ internally. This implies that the last circle is twice the size of α, and the proof is complete.

We mentioned "one of the circles" in the theorem, but if we introduce orientations of lines and circles, the circle is determined uniquely. Let us assume that γ and ABC have counter-clockwise orientations. Then the circle of twice the size of α (illustrated by a dotted line in Figure 3) is the one which touches γ, $\overrightarrow{AB}$ and $\overrightarrow{CA}$ so that the orientations at the points of tangency are the same (see the arrows in Figure 3), and the point of tangency of the circle with γ is the reflection of A in the point of tangency of α and β.

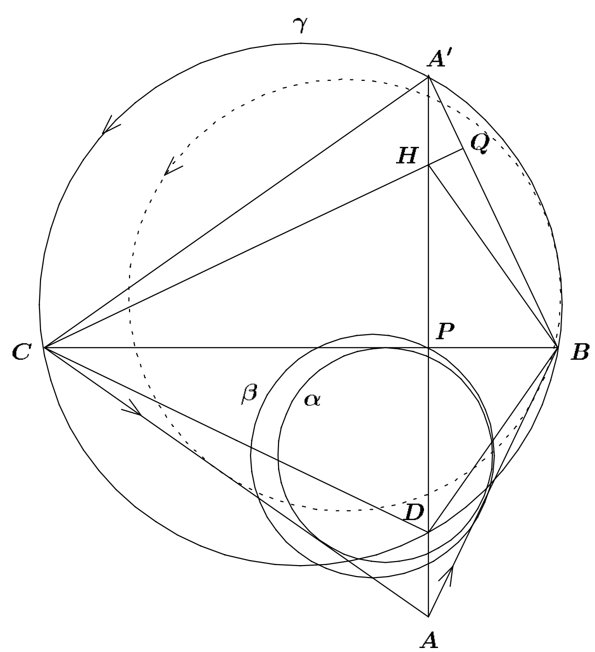

Figure 3.

The following two properties follow immediately (see Figures 4 and 5). They can be found without a proof in [1, p. 29].

Corollary 1. If ABC is a right triangle with right angle at A, then the circle touching the circumcircle of ABC internally and also touching AB and AC is twice the size of the incircle of ABC.

Corollary 2. If $CB'A$ is an isosceles triangle with $CB' = CA$, and B is a point lying on the line $B'A$, then one of the circles touching the circumcircle of BCB' internally and also touching AB and AC is twice the size of the incircle of ABC.

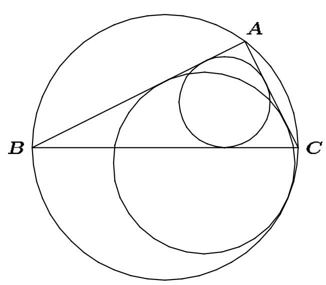

Figure 4.

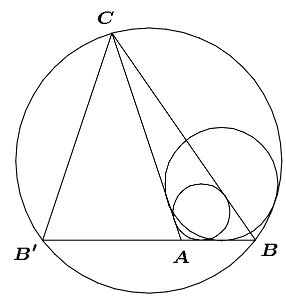

Figure 5.

The last corollary holds since the reflection in BC of A lies on the circumcircle of BCB'. Though Figure 5 illustrates only the case where A lies on the segment BB', and this is the case stated in [1], the corollary does not need this condition (see Figure 6b).

Conversely, for a triangle ABC, let A' be the reflection in BC of A, and let B' be the intersection of the line AB and the circumcircle of $A'BC$. Then $CA = CB'$ holds. Hence, with the two sides AB and AC and their intersections with the circumcircle, we can construct two similar isosceles triangles (see Figures 6a and 6b, where the ratio of the two smaller circles is $2 : 1$). A lies within the circumcircle if and only if $\angle CAB > 90°$, and Figure 1 can be obtained by letting $\angle CAB = 135°$.

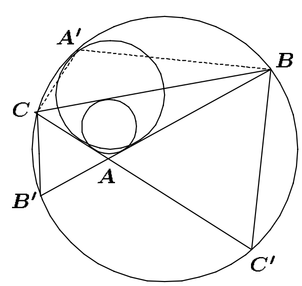

Figure 6a.

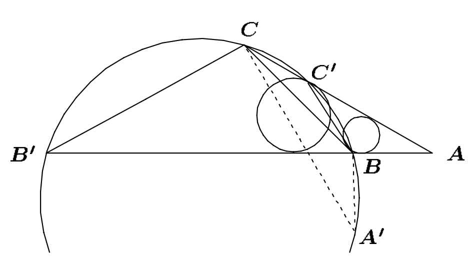

Figure 6b.

The following property with a proof using trigonometric functions can be found in [2, p. 75] (see Figure 7).

Corollary 3. If $ACB'DE$ and $ABC'D'E$ are two regular pentagons sharing the side AE, then the circle internally touching the circumscribed circle of $CB'C'B$ and the sides AB and AC is twice the size of the incircle of DED'.

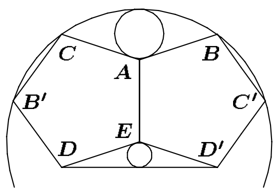

Figure 7.

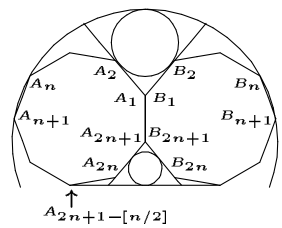

Figure 8.

The corollary can be generalized yet further (see Figure 8).

Theorem 2. If $A_1,\ A_2,\ \cdots,\ A_{2n+1}$ are vertices of a regular $(2n+1)$–gon lying in this order, and B_i is the reflection of A_i in the line A_1A_{2n+1}, and γ is the circle passing through $A_n,\ A_{n+1},\ B_{n+1},\ B_n$, then one of the circles touching the lines A_1A_2, B_1B_2 and γ internally is twice the size of the incircle of the triangle made by the lines $A_{2n}A_{2n+1}$, $B_{2n}B_{2n+1}$ and $A_{2n+1-[n/2]}B_{2n+1-[n/2]}$, where $[x]$ is the largest integer which does not exceed x.

Proof. A_{2n+1} is the centre of γ, and the reflection of $A_{2n+1-[n/2]}$ in the line through the centres of the two regular polygons is $A_{1+[n/2]}$. Let us produce $A_{2n+1}A_1$ to P, where P lies on γ and A_1 lies on the segment $A_{2n+1}P$, and let Q be the intersection of A_1P and $A_{1+[n/2]}B_{1+[n/2]}$. By simple calculation we have

$$A_{2n+1}P \;=\; 2r\cos\frac{\pi}{2(2n+1)}\,, \qquad A_{2n+1}A_1 \;=\; 2r\sin\frac{\pi}{2n+1}\,,$$

where r is the circumradius of $A_1A_2\cdots A_{2n+1}$. Now let us suppose that n is odd; then $[n/2]=(n-1)/2$ and we have

$$\begin{aligned}
A_1Q &= r\sin\Big(\frac{2\pi}{2n+1}\cdot\frac{n-1}{2}+\frac{\pi}{2n+1}\Big)-\frac{A_{2n+1}A_1}{2}\\
&= r\Big(\sin\frac{n}{2n+1}\pi-\sin\frac{1}{2n+1}\pi\Big)\,,\\
A_{2n+1}Q &= A_{2n+1}A_1+A_1Q\\
&= r\Big(\sin\frac{n}{2n+1}\pi+\sin\frac{1}{2n+1}\pi\Big)\,,
\end{aligned}$$

$$A_1Q + A_{2n+1}Q = 2r\sin\frac{n}{2n+1}\pi = 2r\sin\left(\frac{1}{2}\pi - \frac{1}{2(2n+1)}\pi\right)$$
$$= 2r\cos\frac{1}{2(2n+1)}\pi\,.$$

Therefore, we get $A_1Q + A_{2n+1}Q = A_{2n+1}P$, and this implies that Q is the mid-point of A_1P. Similarly, we can prove the same fact in the case of n being even. Thus, the ratio of the two similar isosceles triangles made by the lines A_1A_2, B_1B_2, $A_{1+[n/2]}B_{1+[n/2]}$, and A_1A_2, B_1B_2, the tangent of γ at P, is $1:2$ and the theorem is proved.

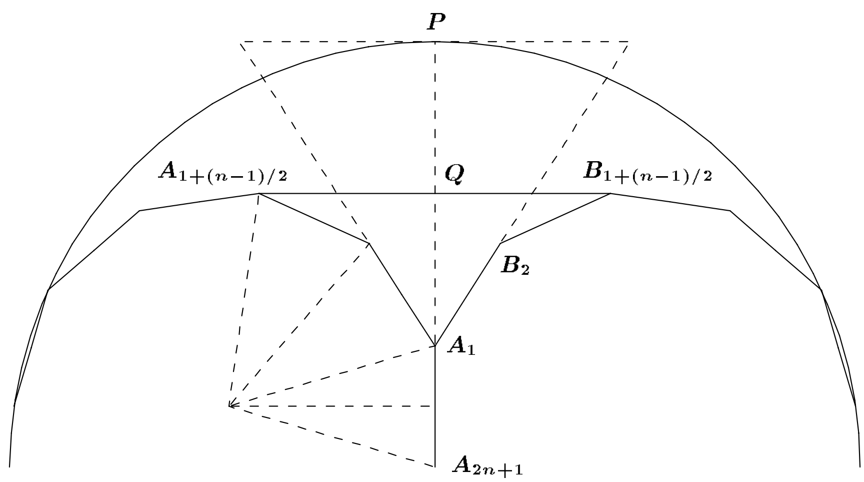

Figure 9 (when n is odd).

The authors would like to thank the referee for suggesting helpful comments.

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THE SKOLIAD CORNER

No. 52

R.E. Woodrow

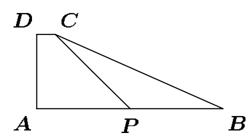
This number we feature the problems of the Final Round of the Senior Mathematics Contest of the British Columbia Colleges. My thanks go to Jim Totten, the University College of the Cariboo, one of the contest organizers, for sending them to us. We will discuss the official solutions next issue.

BRITISH COLUMBIA COLLEGES

Senior High School Mathematics Contest

Part A — Final Round — May 5, 2000

1. In the diagram, DC is parallel to AB, and DA is perpendicular to AB. If $DC = 1$, $DA = 4$, $AB = 10$, and the area of quadrilateral $APCD$ equals the area of triangle CPB, then PB equals:



(a) 3 (b) $3\frac{1}{2}$ (c) 4 (d) 5 (e) $5\frac{1}{2}$

2. Label the vertices of a *regular* pentagon with A, B, C, D, and E, so that edges of the pentagon are line segments AB, BC, CD, DE, and EA. One of the angles formed at the intersection of AC and BD has measure:

(a) 72° (b) 135° (c) 36° (d) 54° (e) 120°

3. Three children are all under the age of 15. If I tell you that the product of their ages is 90, you do not have enough information to determine their ages. If I also tell you the sum of their ages, you still do not have enough information to determine their ages. Which of the following is *not* a possible age for one of the children?

(a) 2 (b) 3 (c) 5 (d) 6 (e) 9

4. I know I can fill my bathtub in 10 minutes if I put the hot water tap on full, and that it takes 8 minutes if I put the cold water on full. I was in a hurry so that I put both on full. Unfortunately, I forgot to put in the plug. A full tub empties in 5 minutes. How long, in minutes, will it take for the tub to fill?

(a) 15 (b) 24 (c) 40 (d) 60 (e) it will never fill

5. The smallest positive integer k such that

$$(k+1)+(k+2)+\cdots+(k+19)$$

is a perfect square is:

(a) 4 (b) 9 (c) 19 (d) 28 (e) impossible to find

6. A six-digit number begins with 1. If this digit is moved from the extreme left to the extreme right without changing the order of the other digits, the new number is three times the original. The sum of the digits in either number is:

(a) 6 (b) 9 (c) 18 (d) 27 (e) 51

7. A cube of edge 5 cm is cut into smaller cubes, not all the same size, in such a way that the smallest possible number of cubes is formed. If the edge of each of the smaller cubes is a whole number of centimetres, how many cubes with edge 2 cm are formed?

(a) 0 (b) 3 (c) 5 (d) 7 (e) 8

8. The nine councillors on the student council are not all on speaking terms. The table below shows the current relationship between each pair of councillors, where '1' means 'is on speaking terms', '0' means 'is not on speaking terms', and the letters stand for the councillors' names.

	A	B	C	D	E	F	G	H	I
A	–	0	0	1	0	0	1	0	0
B	0	–	1	1	1	1	1	1	1
C	0	1	–	0	0	0	1	1	0
D	1	1	0	–	1	0	1	0	1
E	0	1	0	1	–	0	1	0	0
F	0	1	0	0	0	–	0	0	1
G	1	1	1	1	1	0	–	0	0
H	0	1	1	0	0	0	0	–	0
I	0	1	0	1	0	1	0	0	–

Councillor A recently started a rumor. It was heard by each councillor once and only once. Each councillor heard it from and passed it to a councillor with whom he or she was on speaking terms. Counting councillor A as zero, councillor E was the eighth and last to hear it. Who was the fourth councillor to hear the rumor?

(a) B (b) C (c) F (d) H (e) I

9. A, B, and C are thermometers with different scales. When A reads $10°$ and $34°$, B reads $15°$ and $31°$, respectively. When B reads $30°$ and $42°$, C reads $5°$ and $77°$, respectively. If the temperature drops $18°$ using A's scale, how many degrees does it drop using C's scale?

(a) 12 (b) 24 (c) 48 (d) 54 (e) 72

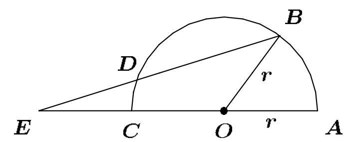
10. The number of positive integers between 200 and 2000 that are multiples of 6 or 7 but not both is:

(a) 469 (b) 471 (c) 513 (d) 514 (e) 557

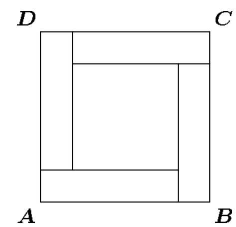
Part B — Final Round — May 5, 2000

1. In the diagram O is the centre of a circle with radius r, and $ED = r$.

The angle $\angle DEC = k\angle BOA$. Find k.



2. The square $ABCD$, whose area is 180 square units, is divided into five rectangular regions of equal area, four of which are congruent as shown. What are the dimensions of one of the rectangular regions which is not a square?



3. An integer i *evenly divides* an integer j if there exists an integer k such that $j = ik$; that is, if j is an integer multiple of i.

(a) Recall $n! = (n)(n-1)(n-2)\cdots(2)(1)$. Find the largest value of n such that 25 evenly divides $n! + 1$.

(b) Show that if 3 evenly divides $x + 2y$, then 3 evenly divides $y + 2x$.

4. A circular coin is placed on a table. Then identical coins are placed around it so that each coin touches the first coin and its other two neighbours. It is known that exactly 6 coins can be so placed.

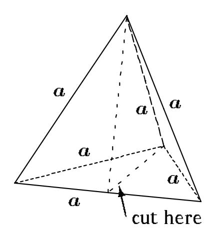
(a) If the radius of all 7 coins is 1, find the total area of the spaces between the inner coin and the 6 outer coins.

(b) If the inner coin has radius 1, find the radius of a larger coin, so that exactly 4 such larger coins fit around the outside of the coin of radius 1.

5. The 6 edges of a regular tetrahedron are of length a. The tetrahedron is sliced along one of its edges to form two identical solids.

(a) Find the perimeter of the slice.

(b) Find the area of the slice.



That concludes the *Skoliad Corner* for this issue. As always we welcome contest materials for school level problem solving pleasure.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5 (NEW!)**. The electronic address is **mayhem-editors@cms.math.ca** NEW!

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University).

Editorial

Shawn Godin

When the new members of the Mayhem board got together with the previously existing board we discussed new items that we could add to ***MAYHEM***. Since ***MAYHEM*** was primarily fashioned as a problem solving magazine, we readily agreed on a column that stayed with that theme. The Problem of the Month is already a regular feature of ***MAYHEM*** that highlights ingenious problems and their solutions. We sought a column that had a different focus but would be complementary to the the problem of the month.

As a companion to the problem of the month we decided on a column that looked at specific problem solving techniques. So the "technique of the month" was born.

The focus of the column will be specific insights, or techniques that can be used on a wide variety of problems. ***MAYHEM*** is in the process of aiming towards material that is accessible to high school students, and that should be reflected in the column. Submissions should address that audience and attempt to be as broad as possible.

As for the name, we thought that "technique of the month" was a little blasé but nobody had any great insight. It is now the night before my "final deadline" with my material a bit on the late side (it is fun trying to figure out all the details of a new job; I am sure I will be good at it sometime before I retire, or am retired). I thought it might be good to include Polya in the title after the great problem solver George Polya. Being a fan of alliteration I went searching for a "p" word that means treasure or gem. After consulting my thesaurus in my wordprocessor I came across paragon. Since paragon did not belong to my limited vocabulary I searched for enlightenment. My dictionary defines paragon as *a model of excellence or perfection*. I think that I have hit on the essence of the column as we envisioned it. I just hope that my co-workers will agree on my choice!

Later in the issue you will find the first submission to the **Polya's Paragon** by new MAYHEM Assistant Editor Chris Cappadocia. Enjoy, and send any columns to mayhem-editors@cms.math.ca.

Polya's Paragon

Chris Cappadocia, student, University of Waterloo

The following simple problem illustrates a very useful idea that you should always keep in mind when tackling a system of equations.

Problem. If $x + y = 5$ and $xy = 7$, what is the value of $x^2 + y^2$?

If your plan is to try to find the values of x and y straightaway, and then compute $x^2 + y^2$ you will find that x and y are not real numbers at all. This is fine if you know how to solve the general quadratic $ax^2 + bx + c = 0$ but if you have not learned that yet (I am not sure anymore when that is taught, but you can probably look forward to learning it in grade ten or eleven), you will instead find the values of x and y to be quite elusive. But none of this matters! The easiest way to solve the problem is to remain ignorant of the actual values of x and y. A few straightforward manipulations will bring the solution home. Here goes:

Solution.

$$\begin{array}{rrcl} & x + y & = & 5\,, \\ \text{so that} & (x+y)^2 & = & 5^2\,, \quad \text{or} \quad x^2 + 2xy + y^2 \;=\; 25\,. \\ \text{Thus} & x^2 + y^2 & = & 25 - 2xy \;=\; 25 - 2(7) \;=\; 25 - 14 \;=\; 11\,. \end{array}$$

The first time I saw this I was quite impressed. It is not a very deep concept, but it pops up time and again on contests. Either it is the basis for a simple problem, or it is an especially tricky variation on the theme, or it is a necessary step in the solution of a larger problem. So learn the technique, become comfortable with it, and it will be second nature in no time.

Here are a couple of problems that you can practice your technique on.

1. If $x + y = \sqrt{3}$ and $xy = 8$, what is the value of $\frac{1}{x^2} + \frac{1}{y^2}$?

2. If $x + y = a$, $xy = b$ and $x > y$, what is the value of $x^2 - y^2$ in terms of a and b?

3. If $x - y = 20$ and $xy = 12$, find the value of $x^3 + y^3$.
4. What is the sum of the squares of the roots of the cubic $P(x) = x^3 + 7x^2 - 18x + 4$?

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Given that $\sum\limits_{k=1}^{35} \sin 5k = \tan \frac{m}{n}$, where angles are measured in degrees, and m and n are relatively prime positive integers that satisfy $\frac{m}{n} < 90$, find $m + n$.

(1999 AIME, Problem 11)

Solution. We will use the trigonometric identities

$\sin\alpha \sin\beta = -\frac{1}{2}\left(\cos(\alpha+\beta) - \cos(\alpha-\beta)\right)$ and
$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$.

$$\begin{aligned}
\sum_{k=1}^{35} \sin 5k &= \sum_{k=1}^{35} \frac{\sin 5k \sin 2.5}{\sin 2.5} \\
&= \sum_{k=1}^{35} \frac{-\frac{1}{2}\left(\cos(5k+2.5) - \cos((5k-2.5))\right)}{\sin 2.5} \\
&= \frac{-\frac{1}{2}(\cos 177.5 - \cos 2.5)}{\sin 2.5} = \frac{\sin 90 \sin 87.5}{\sin 2.5} \\
&= \frac{\sin 87.5}{\cos 87.5} = \tan 87.5\,.
\end{aligned}$$

(All angles are in degrees.)

Hence, our value for m/n must be $87.5 = 175/2$. We then have $m = 175$ and $n = 2$, yielding an answer of $m + n = 177$.

Note: A general formula for sums of sines is

$$\sum_{k=1}^{N} \sin k\theta = \frac{\sin\frac{(N+1)\theta}{2}\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}\,.$$

This formula can be derived in a similar manner as in the above solution.

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from the previous issue be submitted in time for issue 2 of 2002.

High School Solutions

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H265. Find all triangular numbers that can be expressed as the sum of 18 consecutive integers, with the smallest of these integers being a perfect square.

Solution.

The n^{th} triangular number can be represented by $\dfrac{n(n+1)}{2}$, where n is a positive integer. Let the smallest integer of the 18 consecutive be k^2, where k is a non-negative integer. Hence, the sum of the 18 consecutive numbers is $9[2(k^2)+17]$. Setting the two equal to each other, and simplifying, we are left with the equation, $n^2+n-(36k^2+306)=0$. Since n must be a positive integer, the discriminant of the quadratic must be a perfect square.

Therefore, $144k^2+1225=t^2$ for some positive integer t. This implies that $(t-12k)(t+12k)=(5^2)(7^2)$, giving $(t+12k, t-12k)=(1225,1)$, $(245,5)$, $(175,7)$, $(49,25)$, $(35,35)$.

Thus, $(t,k)=(613,51)$, $(125,10)$, $(91,7)$, $(37,1)$, $(35,0)$, yielding, $n=306$, 62, 45, 18, 17.

Also solved by David Loeffler, student, Cotham School, Bristol, UK.

H266. Consider a polynomial $f(x)$ with integer coefficients such that $f(0)=p$, where p is a prime.

(a) What is the maximum number of lattice points lying on the line $y=x$ that the graph of $f(x)$ can pass through?

(b) What are the specific lattice points in (a)?

Solution by David Loeffler, student, Cotham School, Bristol, UK.

Let $g(x) = f(x) - x$. Our question becomes looking for the maximum number of integral roots of $g(x)$. Let a be one such integral root of $g(x)$. By the Factor Theorem, $g(x) = (x - a)Q(x)$ where $Q(x)$ is a polynomial with integer coefficients. Since $f(0) = p$, then $g(0) = p = (-a)Q(0)$.

It follows that $a|p$, so that $a = 1, p, -1, -p$.

Therefore, there are at most 4 lattice points on the line $y = x$ that $f(x)$ can pass through, and they are (p, p), $(-p, -p)$, $(1, 1)$, $(-1, -1)$.

H267. Find all solutions (a, b, c) to the following system:

$$\frac{b-c}{a} = \frac{c-a}{b} = \frac{a-b}{c}.$$

Solution by Andrei Simion, student, Brooklyn Technical HS, Brooklyn, NY, USA.

NOTE: a, b, c are **real** numbers.

Recall that if $x = \frac{a}{b} = \frac{c}{d}$, then $x = \frac{a+b}{c+d}$.

Thus, $\frac{b-c}{a} = \frac{c-a}{b} = \frac{a-b}{c} = \frac{(b-c)+(c-a)+(a-b)}{a+b+c} = 0$.

Hence, $a = b = c$.

This implies that $(a, b, c) = (t, t, t)$, where t is a real number.

Also solved by DAVID LOEFFLER, student, Cotham School, Bristol, UK; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario.

H268. Let ABC be a triangle such that $\angle ACB = 3\angle ABC$ and $AB = \frac{10}{3}BC$. Evaluate $cosA + cosB + cosC$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

We have that $C = 3B$. Hence, $A = 180° - 4B$. Let the circumradius of ABC, $R = 1$. Therefore, $a = 2\sin(4B)$ and $c = 2\sin(3B)$. But $c = \frac{10}{3}a$.

Thus, $10\sin(4B) = 3\sin(3B)$, yielding $40\sin(B)\cos(B)\cos(2B) = -12(\sin(B))^3 + 9\sin B$. Since $\sin(B) \neq 0$, letting $x = \cos(B)$, and simplifying the equation above to get $80x^3 - 12x^2 - 40x + 3 = 0$, we obtain $(4x-3)(20x^2+12x-1) = 0$. This implies that $x = \frac{3}{4}$ or $20x^2+12x-1 = 0$, which yields extraneous solutions since $180° - 4B > 0$ or $B < 45°$.

Since $x = \cos(B) = \frac{3}{4}$, we have $\cos(C) = \cos(3B) = -\frac{9}{16}$ and $\cos(A) = -\cos(4B) = \frac{31}{32}$. Therefore $\cos A + \cos B + \cos C = \frac{37}{32}$.

Also solved by VEDULA N. MURTY, Dover, PA, USA.

Advanced Solutions

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A241. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Prove that every power of 2 has a multiple whose decimal representation contains only the digits 1 and 2.

Solution by Román Fresneda, student, Universidad de la Habana, Cuba.

We show by induction that for an integer $n \geq 1$, there is an n-digit number in base-10 representation, A_n, containing only the digits 1 and 2 which is a multiple of 2^n.

If $n = 1$, take $A_1 = 2$. Now assume that A_n is an n-digit number containing only the digits 1 and 2 which is a multiple of 2^n. Then we have either $A_n \equiv 2^n \pmod{2^{n+1}}$ or $A_n \equiv 0 \pmod{2^{n+1}}$. In the first case, we let $A_{n+1} = 10^n + A_n \equiv 2^n + 2^n \equiv 0 \pmod{2^{n+1}}$, and in the second case, $A_{n+1} = 2 \cdot 10^n + A_n \equiv 0 + 0 \equiv 0 \pmod{2^{n+1}}$. Since A_n is an n-digit number, clearly A_{n+1} will be an $(n+1)$-digit number, containing only the digits 1 and 2, as desired, and our result is established by mathematical induction.

Also solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

A242. Find all solutions of the equation $a^m + b^m = (a+b)^n$ in positive integers.

Comment. At time of press we do not have a solution for this problem.

A243. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Suppose that the integers a and $a + 1997$ are roots of $P(x)$ and that $Q(1998) = 2000$. Prove that the equation $Q(P(x)) = 1$ has no integer solutions.

Solution.

Consider the polynomials $P(x)$ and $Q(x)$ reduced modulo 2. We have

$$Q(0) \equiv Q(1998) \equiv 2000 \equiv 0 \pmod{2},$$

and

$$P(a+1) \equiv P(a+1997) \equiv P(a) \equiv 0 \pmod{2}$$

for some value of a. However, any integer choice of a yields

$$P(0) \equiv P(1) \equiv 0 \pmod{2},$$

so that

$$Q(P(x)) \equiv Q(0) \equiv 0 \not\equiv 1 \pmod{2},$$

and there are no solutions to the equation $Q(P(x)) \equiv 1 \pmod 2$.

If $Q(P(x)) = 1$ has integer solutions, then the equation $Q(P(x)) \equiv 1 \pmod 2$ must also have solutions, so we conclude that $Q(P(x)) = 1$ has no integer solutions.

A244. *Proposed by Ravi Vakil, MIT, Cambridge, MA, USA.*

Prove that $x^{2000} + y^{2000} = z^{2000}$ has no solution in positive integers (x, y, z).

Solution.

This result follows from the similar result for the equation $x^4 + y^4 = z^4$. (See "A Do-It-Yourself Proof of the $n = 4$ case of Fermat's Last Theorem", [1999 : 502]).

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C91. Let P_n be the set of all partitions of the positive integer n, and let $P(n)$ be the size of the set P_n. (That is, P_n consists of all unordered collections of positive integers $(m_1, \ldots, m_k)$ such that $m_1 + \cdots + m_k = n$. For example, P_4 is the set $\{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}$, and so $P(4) = 5$.) Fix a one-to-one correspondence between the elements of P_n and the integers from 1 to $P(n)$. For a non-negative integer N, define $\mathrm{v_N}$ to be the vector (of length $P(n)$) whose j^{th} entry is equal to the number of solutions in non-negative integers $x_1, \ldots, x_k$ to the equation $m_1x_1 + \cdots + m_kx_k = N$, where $(m_1, \ldots, m_k)$ is the partition corresponding to j under our one-to-one correspondence. Let W_n be the sub-vector space of $\mathbb{Q}^{P(n)}$ spanned by all the $\mathrm{v_N}$, $N \geq 0$. Prove that

$$\dim W_n \;\leq\; \frac{n(n-1)}{2} + 1\,.$$

(Note that $\dim W_n$ is independent of the choice of one-to-one correspondence, since changing the correspondence merely permutes the entries of the $\mathrm{v_N}$'s.)

Solution by Florian Herzig, student, University of Cambridge, Cambridge, UK.

Consider the array A with countably many rows and $P(n)$ columns, whose (N, j)–entry $a_{N,j}$ is the number of solutions in non-negative integers $x_1, \ldots, x_k$ to the equation $m_1x_1 + \cdots + m_kx_k = N$. We wish to prove that the row-span of this matrix has dimension at most $n(n-1)/2 + 1$, and so we may, equivalently, show that the column-span of A has dimension at most $n(n-1)/2 + 1$.

Defining the generating function

$$f_j(x) = \sum_{N=0}^{\infty} a_{N,j} x^N ,$$

the reader may verify from the definition of the $a_{N,j}$ that

$$f_j(x) = \frac{1}{(1-x^{m_1})\cdots(1-x^{m_k})} .$$

Let $q_j(x)$ be the denominator in this expression for $f_j(x)$. We are interested in the dimension of the space spanned by the $f_j(x)$.

We recall that the d^{th} cyclotomic polynomial $\Phi_d(x)$ is a polynomial whose roots are precisely the primitive d^{th} roots of unity; then

$$1 - x^m = \Pi_{d\,|m} \Phi_d(x) .$$

It follows that $q_j(x)$ is a product of cyclotomic polynomials, where the polynomial $\Phi_d(x)$ appears once for each m_i which is divisible by d. In particular, the power of $\Phi_d(x)$ dividing $q_j(x)$ is certainly at most $\lfloor n/d \rfloor$. Since

$$(1-x)(1-x^2)\cdots(1-x^n) = \Pi_{d=1}^{n} \Phi_d(x)^{\lfloor n/d \rfloor} ,$$

we find that $q_j(x)$ divides $(1-x)(1-x^2)\cdots(1-x^n)$. We may therefore write

$$\begin{aligned} f_j(x) &= \frac{1}{q_j(x)} \\ &= \frac{p_j(x)}{(1-x)(1-x^2)\cdots(1-x^n)} , \end{aligned}$$

and so we have written the $f_j(x)$ as rational functions with a common denominator.

The degree of $q_j(x)$ is n, while the degree of $(1-x)(1-x^2)\cdots(1-x^n)$ is $n(n+1)/2$, and so the degree of $p_j(x)$ is $n(n-1)/2$. Since the space spanned by the $f_j(x)$ has the same dimension as the space spanned by the $p_j(x)$, and since the space of polynomials of degree $n(n-1)/2$ has dimension $n(n-1)/2+1$, this latter number is indeed an upper bound for the desired dimension.

Also solved by Christopher Long, graduate student, Rutgers University, NJ, USA.

C92. Do there exist arbitrarily long finite arithmetic progressions which contain only square-free integers? (An integer n is said to be square-free if 1 is the only perfect square which divides n.)

Solution by Christopher Long, graduate student, Rutgers University, NJ, USA.

We provide a solution which we find somewhat unsatisfying, in that it relies completely on a famous, easily statable, but remarkably difficult to prove theorem — we hope the reader will take this as a challenge to supply for us a more elementary solution! A subset S of the positive integers is said to have positive upper density if

$$\limsup_{n\to\infty} \frac{\#\{x \in S \mid x \leq n\}}{n} \; > \; 0\,.$$

We then have

Szemeredi's Theorem: If a subset of the positive integers has positive density then it contains arbitrarily long finite arithmetic progressions.

This result was originally conjectured by Erdős, and was proved by Szemerédi in 1975 [3]. While other proofs have been found (notably, in 1977 by H. Furstenberg using ergodic theory [2], and in 1998 by Gowers using Fourier analysis [1]) none are particularly simple or accessible.

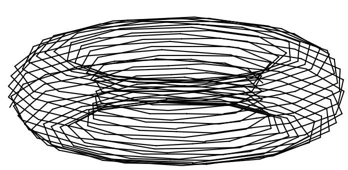
In any case, Szemerédi's Theorem immediately resolves the problem at hand, since it is well known that the square-free numbers have positive density. Indeed, there is a short argument which shows that they have density exactly $6/\pi^2$, and any reader who has not seen this fact before may find it rewarding to try to find the proof.

References

1. H. Furstenberg, *Ergodic behaviour of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. Anal. Math. 31 (1997), 204-256.

2. W.T Gowers, *Fourier Analysis and Szemerédi's Theorem*, Doc. Math. 1998, Extra Vol. I, (Electronic), 617-629.

3. E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. 27 (1975), 199-245.

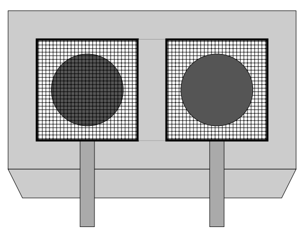


When Do We Eat?

Edward Sitarski

Although we scarcely think about it, we are continuously planning and scheduling. From the moment we wake up, we do things that we hope will affect our future outcome. For example, we put on shoes to protect our feet — not from the warmth of our beds but from the weather and pavement outside. But, before we can put on shoes we must put on socks, etc. It all seems so simple. Surprisingly, scheduling can get very hard very fast.

Let's say we need to cook some hamburgers on a barbecue. Each hamburger has two sides (just in case there was any doubt!), and each side takes 5 minutes to cook. Only one side of the hamburger can be cooked at a time (no "George Foreman" appliances here). Our barbecue is small, and only has two grills to cook the hamburgers as seen in the following diagram:



Let's say we have three hamburgers to cook. The goal is to cook all of the hamburgers in the shortest possible time. Simple, right? Because I am so nice (or maybe not, as you shall see later), I am going to give you a solution to the problem:

	5	**10**	**15**	**20**	**25**	**30**
G1	H1	H1	H3	H3		
G2	H2	H2				

Let me explain my solution. This is a Gantt chart (Gantt was an engineer who invented this kind of chart over 100 years ago). There are two rows: one for each grill (appropriately called G1 and G2). The columns show the cooking time in minutes. The table shows when each hamburger is cooking on each grill. Each hamburger has the name H1, H2 and H3. The names must appear twice because each hamburger must be cooked on two sides. Since we cannot cook both sides of a hamburger at the same time, a hamburger cannot appear twice in the same column. My solution shows a total cooking time of 20 minutes. Can you do better?

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It turns out that we can. In fact, we can cook all the hamburgers in 15 minutes, but it requires a trick (this is math, after all). Take a look at:

	5	10	15	20	25	30
G1	H1	H3	H3			
G2	H2	H2	H1			

What is going on here? It turns out that we can cook all the hamburgers faster if we take one off for 5 minutes and then put it on again (in my solution, this is done with H1). Presumably, we could temporarily put the hamburger on a plate for 5 minutes. Amazingly, this does not violate any of the rules about not cooking both sides at once.

Is a better solution possible? Let's see — all of the grills are being fully used for the entire 15 minutes. Therefore we conclude that there is no better solution.

We were able to reduce the total cooking time by 5 minutes — a reduction of 25 percent. No big deal, right? In this case we have a hungry guest who has to wait an extra 5 minutes. It hardly seems worth all this cleverness just for 5 minutes.

In the real world this is a very big deal. Many factories have 100's of assembly steps to make a product like a cell phone or a kidney dialysis machine. Imagine if those factories could increase their output by 25 percent with a better schedule. Without any new investment, they could have 25 percent more product to sell. Perhaps they would make 25 percent less pollution. Alternatively, they could reduce their prices so that more people could afford the product. They may also be able to make more returns for their shareholders and drive up their stock price. A whole fascinating branch of mathematics called Operations Research is devoted to coming up with ways to make things run more efficiently.

Sometimes 25 percent does not sound like a lot. But - what if you could get a 25 percent higher mark in math without any extra studying? Maybe 25 percent is more than it sounds ...

This trick we used before was clever, but what if we have more hamburgers we need to cook (say N)? Can we always cook N hamburgers in $5N$ minutes?

This is another trick question (aren't I nice?). Because we cannot cook both sides of a hamburger at the same time, it is impossible to cook one hamburger in 5 minutes. So, when $N = 1$, it still takes 10 minutes to cook the hamburger, but this is a special case.

So far we know the fastest cooking time for 0, 1 and 3 hamburgers. Let's try to solve the problem when $N = 2$:

	5	10	15	20	25	30
G1	H1	H1				
G2	H2	H2				

That was pretty easy, was it not? In fact, it does not take much to see that it is easy to schedule any even number of hamburgers as follows:

	5	10	15	20	25	30	$\cdots$
G1	H1	H1	H3	H3	H5	H5	$\cdots$
G2	H2	H2	H4	H4	H6	H6	$\cdots$

This covers a big piece of our problem. We now know that the cooking time is $5N$ when N is 0, 2, 3, 4, 6, 8, 10, ... Notice that 3 is also there to include the work we did before. But what about when N is an odd number and $N > 3$?

Remember, we already know the answer for 0, 2, 3, 4, 6, 8, 10, ..., and we know that 1 is a special case. This leaves us to solve for 5, 7, 9, 11, 13, ...

Think of what happens when you subtract any odd number from any other odd number — you get an even number. Remember our initial solution with $N = 3$? This becomes really important now. Watch what happens when we re-write the numbers we need to solve as follows:

$$5, 7, 9, 11, 13, \ldots \quad = \quad 2+3, 4+3, 6+3, 8+3, 10+3, \ldots .$$

In general, when N is odd and $N > 3$, we can always rewrite it as $N = (N-3) + 3$, and we know for sure that $(N-3)$ is an even number. Because $N - 3$ is even, we know its cooking time is $5(N-3) = 5N - 15$ (we just showed this previously). We also know the cooking time for 3 hamburgers is $5 \times 3 = 15$ (from the original problem). Thus, the total cooking time when N is odd and $N > 3$ is the sum of the cooking time of its components, which is $(5N - 15) + 15 = 5N$. We conclude that it is always possible to cook N hamburgers in $5N$ minutes, except when $N = 1$, which has to take 10 minutes. This is a really nice result!

I hope that you get some idea that even simple looking scheduling problems can get very complex in the real world. To get even more realistic, imagine if you had to cook S–sided hamburgers on K grills, where each hamburger was of T different thicknesses that each required different cooking time? By the way, did I mention that each hamburger requires different ingredients and each side has to be cooked in a specific order on different grills?

Edward Sitarski
Vice President of Advanced Planning
J.D. Edwards
Toronto

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 October 2001**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

In this special Murray Klamkin Birthday issue, we are pleased to present some problems posed by Murray, and some problems dedicated to him.

2613. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* Dedicated to Toshio Seimiya, and based on his problem 2515.

In $\triangle ABC$, the three cevians AD, BE and CF through a non-exterior point P are such that $AF + BD + CE = s$ (the semi-perimeter). Characterize $\triangle ABC$ for each of the cases when P is (i) the orthocentre, and (ii) the Lemoine point.
[Ed. The Lemoine point is also known as the symmedian point. See, for example, James R. Smart, *Modern Geometries*, 4^{th} Edition, 1994, Brooks/Cole, California, USA. p. 161.]

2614 *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* Dedicated to Toshio Seimiya, and suggested by his problem 2514.

In $\triangle ABC$, the two cevians through a non-exterior point P meet AC and AB at D and E respectively. Suppose that $AE = BD$ and $AD = CE$. Characterize $\triangle ABC$ for the cases when P is (i) the orthocentre, (ii) the centroid, and (iii) the Lemoine point.

2615. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Suppose that x_1, x_2, ..., x_n, are non-negative numbers such that

$$\sum x_1^2 + \sum (x_1 x_2)^2 = \frac{n(n+1)}{2},$$

where the sums here and subsequently are symmetric over the subscripts 1, 2, ..., n.

(a) Determine the maximum of $\sum x_1$.

(b)⋆ Prove or disprove that the minimum of $\sum x_1$ is $\sqrt{\frac{n(n+1)}{2}}$.

2616⋆. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

The following are three known properties of parabolas:

1. The area of the parabolic segment upon any chord as base is equal to $\frac{4}{3}$ times the area of the triangle having the same base and height (the tangent at a vertex of the triangle is parallel to the chord). [Due to Archimedes.]

2. The area of the parabolic segment cut off by any chord is $\frac{2}{3}$ times the area of the triangle formed by the chord and the tangents at its extremities.

3. The area of a triangle formed by three tangents to a parabola is $\frac{1}{2}$ times the area of the triangle whose vertices are the points of contact.

Are there any other smooth curves having any one of the above properties?

2617. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

A problem in one book was to prove that each edge of an isosceles tetrahedron is equally inclined to its opposite edge. A problem in another book was to prove that the three angles formed by the opposite edges of a tetrahedron cannot be equal unless they are at right angles.

1. Show that only the second result is valid.

2. Show that a tetrahedron which is both isosceles and orthocentric must be regular.

2618 *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine a geometric problem whose solution is given by the positive solution of the equation

$$3x^2\left(\frac{1}{\sqrt{4R^2+x^2-a^2}}+\frac{1}{\sqrt{4R^2+x^2-b^2}}+\frac{1}{\sqrt{4R^2+x^2-c^2}}\right)$$
$$= \left(\sqrt{4R^2+x^2-a^2}+\sqrt{4R^2+x^2-b^2}+\sqrt{4R^2+x^2-c^2}+a+b+c\right),$$

where a, b, c and R are the sides and circumradius of a given triangle ABC.

2619. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, dedicated to Murray S. Klamkin, on his 80th birthday.*

For natural numbers n, define functions f and g by $f(n) = \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rfloor$ and $g(n) = \left\lceil \frac{n}{\lceil\sqrt{n}\rceil} \right\rceil$. Determine all possible values of $f(n)-g(n)$, and characterize all those n for which $f(n)=g(n)$. [See [2000 : 197], Q. 8.]

2620. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta, dedicated to Murray S. Klamkin, on his 80th birthday.*

Three cards are handed to you. On each card are three non-negative real numbers, written one below the other, so that the sum of the numbers on each card is 1. You are permitted to put the three cards in any order you like, then write down the first number from the first card, the second number from the second card, and the third number from the third card. You add these three numbers together.

Prove that you can always arrange the three cards so that your sum lies in the interval $\left[\frac{1}{3},\frac{2}{3}\right]$.

2621. *Proposed by J. Chris Fisher, University of Regina, Regina, Saskatchewan, and Bruce Shawyer, Memorial University of Newfoundland, St. John's, Newfoundland, dedicated to Murray S. Klamkin, on his 80th birthday.*

You are given:

(a) fixed real numbers λ and μ in the open interval $(0,1)$;

(b) circle ABC with fixed chord AB, variable point C, and points L and M on BC and CA, respectively, such that $BL : LC = \lambda : (1-\lambda)$ and $CM : MA = \mu : (1-\mu)$;

(c) P is the intersection of AL and BM.

Find the locus of P as C varies around the circle ABC. (If $\lambda=\mu=\frac{1}{2}$, it is known that the locus of P is a circle.)

2622. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Find the exact value of $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}}{(2n+1)\binom{2n}{n}}$.

2623★. *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that $x_1, x_2, \ldots, x_n > 0$. Let $x_{n+1} = x_1$, $x_{n+2} = x_2$, etc. For $k = 0, 1, \ldots, n-1$, let

$$S_k = \sum_{j=1}^{n} \left(\frac{\sum_{i=0}^{k} x_{j+i}}{\sum_{i=0}^{k} x_{j+1+i}} \right).$$

Prove or disprove that $S_k \geq S_{k+1}$.

2624. *Proposed by H.A. Shah Ali, Tehran, Iran.*

Let n black objects and n white objects be placed on the circumference of a circle, and define any set of m consecutive objects from this cyclic sequence to be an m–chain.

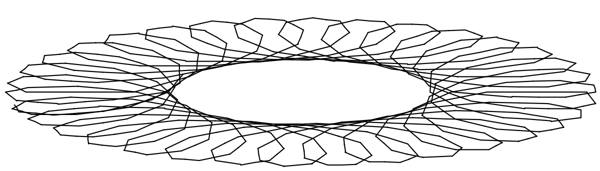
(a) Prove that, for each natural number $k \leq n$, there exists at least one $2k$–chain consisting of k black objects and k white objects.

(b) Prove that, for each natural number $k \leq \sqrt{2n+5} - 2$, there exist at least two such disjoint $2k$–chains.

2625. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

If R denotes the circumradius of triangle ABC, prove that

$$18R^3 \geq \left(a^2 + b^2 + c^2\right) R + \sqrt{3}abc.$$



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2505. [2000 : 46] *Proposed by Hayo Ahlburg, Benidorm, Spain, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that A, B and C are the angles of a triangle. Determine the best lower and upper bounds of $\prod_{\text{cyclic}} \sin(B-C)$.

Combination of solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, and Kee-Wai Lau, Hong Kong.

We show that

$$-\frac{3\sqrt{3}}{8} \;\le\; \prod_{\text{cyclic}} \sin(B-C) \;\le\; \frac{3\sqrt{3}}{8}\,.$$

First, for the best upper bound we consider two cases.

Case 1: $\sin(B-C) \ge 0$. Using the identity

$$\sin X \sin Y \;=\; \frac{1}{2}\Big(\cos(X-Y) - \cos(X+Y)\Big)\,,$$

we have

$$\begin{aligned}\prod_{\text{cyclic}} \sin(B-C) &= \frac{1}{2}\sin(B-C)\Big(\cos(B+C-2A) - \cos(C-B)\Big)\\ &\le \frac{1}{2}\sin(B-C)\Big(1-\cos(B-C)\Big)\,.\end{aligned}$$

The maximum of $\sin(B-C)(1-\cos(B-C))$ occurs at the same value as for its square

$$\sin^2(B-C)\Big(1-\cos(B-C)\Big)^2 \;=\; (1+c)(1-c)^3\,,$$

where $c = \cos(B-C)$. By the AM–GM inequality,

$$\left[(1+c)\left(\frac{1-c}{3}\right)^3\right]^{1/4} \le \frac{1}{4}\left[(1+c) + 3\left(\frac{1-c}{3}\right)\right] \;=\; \frac{1}{2}\,,$$

with equality only when all four numbers are equal; that is, when $(1-c)/3 = 1+c$ or $c = -1/2$. Therefore,

$$(1+c)(1-c)^3 \;\le\; \left(\frac{1}{2}\right)^4 \cdot 27 \;=\; \frac{27}{16}\,,$$

with equality only when $\cos(B - C) = c = -1/2$. Thus, the upper bound will be

$$\prod_{\text{cyclic}} \sin(B - C) \;\le\; \frac{1}{2}\sqrt{\frac{27}{16}} \;=\; \frac{3\sqrt{3}}{8}\,.$$

The degenerate triangle with $A = \pi/3$, $B = 2\pi/3$, $C = 0$ shows that the upper bound cannot be improved.

Case 2: $\sin(B - C) < 0$. Similarly we have

$$\begin{aligned}\prod_{\text{cyclic}} \sin(B - C) &= -\frac{1}{2}\sin(B - C)\Big(\cos(C - B) - \cos(B + C - 2A)\Big) \\ &\le -\frac{1}{2}\sin(B - C)\Big(\cos(B - C) + 1\Big) \\ &= \frac{1}{2}\sqrt{(1 - c)(1 + c)^3}\,,\end{aligned}$$

with c as above. The maximum occurs when $(1 + c)/3 = 1 - c$ or when $\cos(B - C) = c = 1/2$. This gives the same upper bound as before, but here there are no corresponding angles of a triangle.

For the lower bound, we get

$$\prod_{\text{cyclic}} \sin(B - C) \;\ge\; -\frac{3\sqrt{3}}{8}\,.$$

[*Editorial note*: The easiest way to see this, as some readers noted, is because

$$\prod_{\text{cyclic}} \sin(B - C) \;=\; -\prod_{\text{cyclic}} \sin(C - B) \;\ge\; -\frac{3\sqrt{3}}{8}\,,$$

using the upper bound.] The degenerate triangle with $A = 2\pi/3$, $B = \pi/3$, $C = 0$ shows that the lower bound cannot be improved either.

Editorial note: Klamkin observes that these bounds hold for ***arbitrary*** angles A, B, C, as the above proof (which is largely Klamkin's, but uses some elements of Lau's solution as well) shows. Thus readers might like to investigate the best bounds, in terms of n, for

$$\prod_{\text{cyclic}} \sin(A_1 - A_2),$$

where A_1, A_2, ..., A_n are arbitrary angles.

Also solved by AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HENRY LIU, student, University of Cambridge, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, *UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One other reader misread the product for a sum.*

Most readers obtained strict inequality in both bounds, which is correct if degenerate triangles are not allowed.

2512. [2000 : 46] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

In $\triangle ABC$, the sides satisfy $a \geq b \geq c$. Let R and r be the circumradius and the inradius respectively. Prove that

$$bc \;\leq\; 6Rr \;\leq\; a^2\,,$$

with equality if and only if $a = b = c$.

Solution by Henri Liu, graduate student, University of Cambridge, UK.

Let A be the area of $\triangle ABC$. Combining the well-known identities $4AR = abc$ and $2A = r(a+b+c)$, we have

$$6Rr \;=\; \frac{3abc}{a+b+c}\,.$$

Using $a + b + c \leq 3a$, we obtain

$$6Rr \;\geq\; \frac{3abc}{3a} \;=\; bc\,.$$

Equality holds if and only if $a = b = c$. Using $a + b + c \geq 3c$, we get

$$6Rr \;\leq\; \frac{3abc}{3c} \;=\; ab \;\leq\; a^2$$

with equality if and only if $a = b = c$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JONATHAN CAMPBELL, student, Chapel Hill High School, North Carolina, USA; GORAN CONAR, student, University of Zagreb, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; ROMÁN FRESNEDA, Universidad de la Habana, Cuba; KARTHIK GOPALRATNAM, student, Angelo State University, Texas, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY PAN, student, East York C.I., Toronto, Ontario; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. Most of the submitted solutions are similar to the one given above.

2514. [2000 :] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle ABC$ and $\angle BCA$ meet CA and AB at D and E respectively. Suppose that $AE = BD$ and that $AD = CE$. Characterize $\triangle ABC$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

As is known,

$$AE^2 = \left(\frac{bc}{b+a}\right)^2, \qquad AD^2 = \left(\frac{bc}{c+a}\right)^2,$$
$$BD^2 = ca\left(1 - \frac{b^2}{(c+a)^2}\right), \qquad CE^2 = ba\left(1 - \frac{c^2}{(b+a)^2}\right),$$

so that $\left(\dfrac{AE}{AD}\right)^2 = \left(\dfrac{BD}{CE}\right)^2$ reduces to

$$b(c+a)^4\left((b+a)^2 - c^2\right) = c(b+a)^4\left((c+a)^2 - b^2\right),$$

or successively to

$$(c+a)^2(b+a)^2\left((b(c+a)^2 - c(b+a)^2\right) = bc\left(c(c+a)^4 - b(b+a)^4\right),$$

or

$$\begin{aligned} 0 &= (b-c)\left((c+a)^2(b+a)^2a^2\right) \\ &\quad + bc\left(b(b+a)^4 - c(c+a)^4 - (b-c)(c+a)^2(b+a)^2\right). \end{aligned}$$

The second term also has a factor $(b-c)$. When this is factored out, there remains a polynomial all of whose coefficients are positive. [Ed. I checked — it has 19 terms!] Hence, $b = c$, and the triangle is isosceles.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

All of the other solvers, with one exception, also showed that the base angles of the triangle are 36°.

2515. [2000 : 114] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle BAC$, $\angle ABC$ and $\angle BCA$ meet BC, AC and AB at D, E and F respectively. Let p be the perimeter of $\triangle ABC$. Suppose that $AF + BD + CE = \frac{1}{2}p$. Characterize $\triangle ABC$.

I. *Solution by David Loeffler, student, Cotham School, Bristol, UK.*

We show first that the angle bisector of a triangle divides the opposite side in the ratio of the other two sides. By the Law of Sines on $\triangle ABD$, we have

$$BD \;=\; c\,\frac{\sin\left(\frac{1}{2}\angle BAC\right)}{\sin(\angle ADB)}\,.$$

Similarly, from $\triangle ADC$ we get

$$DC \;=\; b\,\frac{\sin\left(\frac{1}{2}BAC\right)}{\sin(\angle ADC)}\,.$$

Thus,

$$\frac{BD}{DC} \;=\; \frac{c}{b}\left(\frac{\sin(\angle ADC)}{\sin(180^\circ - \angle ADC)}\right) \;=\; \frac{c}{b}\,.$$

Likewise, we have $\dfrac{CE}{EA} = \dfrac{a}{c}$ and $\dfrac{AF}{FB} = \dfrac{b}{a}$. Thus, we have

$$AF \;=\; \frac{bc}{a+b}\,, \qquad BD \;=\; \frac{ca}{b+c}\,, \qquad CE \;=\; \frac{ab}{c+a}\,.$$

The condition that $AF + BD + CE = \frac{1}{2}p$ is now equivalent to

$$\frac{2bc}{a+b} + \frac{2ca}{b+c} + \frac{2ab}{c+a} \;=\; a+b+c\,,$$

or, upon multiplying by $(a+b)(b+c)(c+a)$,

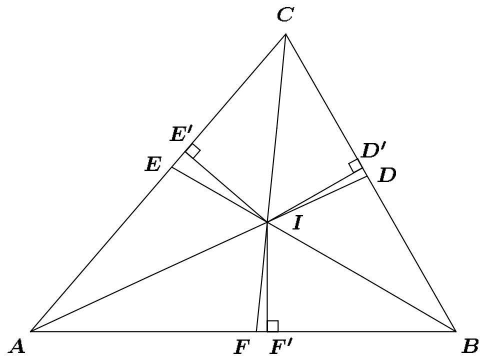
$$\begin{aligned} &2bc(b+c)(c+a) + 2ca(c+a)(a+b) + 2ab(a+b)(b+c) \\ &\quad=\; (a+b+c)(a+b)(b+c)(c+a)\,. \end{aligned}$$

After expanding, most of the terms cancel and the equation becomes

$$ab^3 + bc^3 + ca^3 \;=\; a^3b + b^3c + c^3a\,,$$

which factors as $(a+b+c)(a-b)(b-c)(c-a) = 0$. Thus $a = b$, or $b = c$, or $c = a$; that is, the triangle is isosceles.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editors.*



Without loss of generality, assume $A \le B \le C$. Let r be the inradius, and I the incentre. Let the projections of I upon BC, CA, AB be D', E', F' respectively as shown in the diagram above. Let $s = \frac{1}{2}(a+b+c)$ where $a = BC$, $b = CA$, $c = AB$. Then

$$\angle DID' \;=\; \angle ADB - \frac{\pi}{2} \;=\; \frac{A}{2} + C - \frac{A+B+C}{2} \;=\; \frac{C-B}{2}\,.$$

Similarly, $\angle EIE' = (C-A)/2$ and $\angle FIF' = (B-A)/2$.

Notice that $AF' + BD' + CE' = (s-a)+(s-b)+(s-c) = s = AF + BD + CE$. Hence $EE' = FF' + DD'$; that is,

$$r\tan\frac{1}{2}(B-A) + r\tan\frac{1}{2}(C-B) \;=\; r\tan\frac{1}{2}(C-A)\,.$$

Let $\alpha = (B-A)/2$ and $\beta = (C-B)/2$. The above equation becomes:

$$\tan\alpha + \tan\beta \;=\; \tan(\alpha+\beta)\,;$$

$$\text{that is,}\quad \tan\alpha + \tan\beta \;=\; \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\,.$$

Therefore, we have $\tan\alpha = -\tan\beta$ or $\tan\alpha\tan\beta = 0$. The first case leads to $\alpha = -\beta$ and thus $A = C$; the second case yields either $A = B$ or $B = C$. Therefore, $\triangle ABC$ is isosceles. Conversely, if $\triangle ABC$ is isosceles, it is trivial to see that $AF + BD + CE = s$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were two incomplete and two incorrect solutions.

2516. [2000 : 115] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In isosceles $\triangle ABC$ (with $AB = AC$), let D and E be points on sides AB and AC respectively such that $AD < AE$. Suppose that BE and CD meet at P. Prove that $AE + EP < AD + DP$.

I. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let Ax and Ay be the half-lines starting at A and containing B and C respectively, and let Az be the bisector of $\angle xAy$. The bisector of $\angle DCy$ meets Az at K, which is therefore an excentre of triangle ADC; the bisector of $\angle EBx$ meets Az at L, an excentre of triangle ABE. We have

$$\angle LBx \;=\; \frac{1}{2}\angle EBx \;<\; \frac{1}{2}\angle DCy \;=\; \angle KCy\,,$$

and hence,

$$AL \;>\; AK\,.$$

If F on AB is the symmetric point of P with respect to the angle bisector DK, and G on AC is the symmetric point of P with respect to the angle bisector EL, then we have

$$DP \;=\; DF\,, \quad \text{or} \quad AF \;=\; AD + DP\,, \quad \text{and}$$

$$EP \;=\; EG\,, \quad \text{or} \quad AG \;=\; AE + EP\,.$$

Since K is on the same side of EL as P, then $KG > KP = KF$. Finally, from the fact that K and A lie on opposite sides of the line EF, we have that $AF > AG$, or $AD + DP > AE + EP$.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

We shall generalize the problem using Green's Theorem with some basic calculus. I hope that the solution has a sufficient degree of elegance to befit the celebration of the 90th birthday of Prof. Seimiya.

Theorem. Let CD, BE be cevians of a triangle ABC where $BD \geq CD$. Let CD intersect BE at P. Then $AD + DP > AE + EP$.

Remark. The given problem is a corollary: Let A' be on AC such that $AA' = AD$. Let BA' intersect CD at D'. Then

$$AD + DD' \;=\; AA' + A'D'$$

by symmetry, and

$$A'D' + D'P \;>\; A'E + EP$$

by the theorem. Adding the two expressions then gives the result.

Proof of the theorem. Let $x_1 = \angle BCD$, $x_2 = \angle BCA$, $y_1 = \angle CBE$, $y_2 = \angle CBA$. Assume $BC = 1$ unit. For any point X in the plane of $\triangle ABC$ let $\angle XCB = x$ and $\angle CBX = y$. The point is inside quadrangle $ADPE$ when both $x_1 < x < x_2$ and $y_1 < y < y_2$. When X is on AD, if x is increased by dx, X would move along AD from D to A through ds. From the Law of Sines,

$$BX = \frac{BC \sin x}{\sin(x+y)}.$$

Thus $$\frac{\partial s}{\partial x} = \frac{\sin(x+y)\cos x - \sin x \cos(x+y)}{\sin^2(x+y)} = \frac{\sin y}{\sin^2(x+y)}.$$

Consequently,

$$\begin{aligned} DA &= \int_{x_1}^{x_2} \frac{\sin y_2}{\sin^2(x+y_2)} dx \\ &= \text{line integral } \int_{DA} \frac{\sin y dx + \sin x dy}{\sin^2(x+y)}. \end{aligned}$$

Similar results hold for DP, PE, EA, so that

$$AE + EP - PD - DA = \text{line integral } \int_{ADPE} \frac{\sin y dx + \sin x dy}{\sin^2(x+y)}.$$

By Green's Theorem, this equals

$$\int\int \left[\frac{\partial}{\partial x}\frac{\sin x}{\sin^2(x+y)} - \frac{\partial}{\partial y}\frac{\sin y}{\sin^2(x+y)}\right] dxdy$$

over the interior of $ADPE$.

Let $\alpha = y + x$ and $\beta = y - x$. Then the integrand simplifies to

$$\begin{aligned} & \sin^{-3}\alpha[\sin\alpha\cos x - 2\sin x\cos\alpha - \sin\alpha\cos y + 2\sin y\cos\alpha] \\ = \; & \sin^{-3}\alpha[\sin\alpha(\cos x - \cos y) + 2\cos\alpha(\sin y - \sin x)] \\ = \; & \sin^{-3}\alpha\left[\sin\alpha\left(2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\right) + 4\cos\alpha\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\right] \\ = \; & \sin^{-3}\alpha\sin\left(\frac{\beta}{2}\right)\left[\cos\left(\frac{\alpha}{2}\right) - \cos\left(\frac{3\alpha}{2}\right) + 2\cos\left(\frac{3\alpha}{2}\right) + 2\cos\left(\frac{\alpha}{2}\right)\right] \\ = \; & \sin^{-3}\alpha\sin\left(\frac{\beta}{2}\right)\left[3\cos\left(\frac{\alpha}{2}\right) + \cos\left(\frac{3\alpha}{2}\right)\right]. \end{aligned}$$

With a bit of calculus, it can been seen that $3\cos\left(\frac{\alpha}{2}\right) + \cos\left(\frac{3\alpha}{2}\right)$ is a decreasing function for $0 \le \alpha \le \pi$, reaching 0 only when $\alpha = \pi$. This proves that the integrand is positive or negative according as β (that is, $y - x$) is positive or negative; so too with the double integral and therefore also the line integral. The theorem specifies that $y_2 \le x_1$, so that $y \le x$ and the line integral is negative. Thus $AE + EP - PD - DA < 0$. ■

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain (two solutions); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HENRY LIU, student, *Trinity College Cambridge; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA (a second solution); and the proposer.*

Both Yiu and the proposer showed that the result follows quickly from an 1846 result of Steiner:

> *Given a convex quadrangle $ABCD$ whose sides AB and CD meet at E while AD and BC meet at F, labeled so that B is between A and E, C is between E and D and also between F and B, while D is between F and A, then $AB + BC = AD + DC$ if and only if $AE + EC = AF + FC$, if and only if the productions of the four sides of $ABCD$ are tangent to a circle.*

Seimiya provides the reference F. G.-M., Exercices de géométrie, *p. 318, Théorème 157. Liu calls the result "Urquhart's Theorem," but provides no reference, presumably because it was easier to give a proof than to find a reference. The story of how M.L. Urquhart (1902-1966) got his name on the result is told by Dan Pedoe in "The Most 'Elementary' Theorem of Euclidean Geometry," Math. Magazine* **49** *: 1 (January 1976), 40–42.*

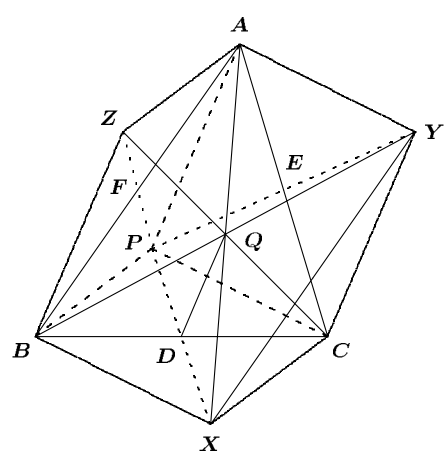
2517. [2000 : 115] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Suppose that D, E, F are the mid-points of the sides BC, CA, AB, respectively, of $\triangle ABC$. Let P be any point in the plane of the triangle, distinct from A, B and C.

1. Show that the lines parallel to AP, BP, CP, through D, E, F, respectively, are concurrent (at Q, say).
2. If X, Y, Z are the symmetrics of P with respect to D, E, F, respectively, show that AX, BY, CZ are concurrent at Q.

I. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

[Ed.: The most elementary solution was given by Nikolaos Dergiades.]



Since $PBXC$ and $PCYA$ are parallelograms, it follows that $ABXY$ is a parallelogram, and hence, BY passes through the mid-point Q of AX. Similarly, CZ passes through Q. Hence, AX, BY, CZ are concurrent at Q. In $\triangle APX$, note that D is the mid-point of PX, and that Q is the mid-point of AX. Therefore, DQ is parallel to AP. Likewise, EQ is parallel to BP, and FQ is parallel to CP.

II. *Solution by Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland.*

[Ed.: A number of solvers used a transformational argument based on properties of homotheties. However, no-one else saw Gunther's short solution presented below.]

Denote by $H(R, \lambda)$ the homothety (central dilatation) with centre R and ratio λ. Note that the composition $H(R, \lambda) \circ H(S, \mu) = H(T, \lambda\mu)$, where the point T lies on the line RS (in the case where $\lambda\mu = 1$, this is a translation when T lies at infinity and the identity when $R = S = T$).

In $\triangle ABC$, let G be the centroid. Observe first that $H(G, -\frac{1}{2})$ takes $\triangle ABC$ to $\triangle DEF$. Since a homothety takes a line to a parallel line, it follows that AP, BP, CP are taken to the respectively parallel lines DQ, EQ, FQ, where Q is the image of P under $H(G, -\frac{1}{2})$. Next note that $H(P, 2)$ takes $\triangle DEF$ to $\triangle XYZ$. Hence, the composition $H(P, 2) \circ H(G, -\frac{1}{2})$ is a homothety which takes $\triangle ABC$ to $\triangle XYZ$, and whose centre lies on the line PG. It is easy to verify that this composition has Q as a fixed point, and hence $H(P, 2) \circ H(G, -\frac{1}{2}) = H(Q, -1)$. Thus AX, BY, CZ are all concurrent with Q.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL COVAS, Mallorca, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

A number of solutions employed vectors; others used coordinates (both affine and areal). Klamkin noted that a generalization of part (1) *is given in Problem 39, Math. Horizons, April 1996, by T.S. Bolis and M. Klamkin:* "Let $n + 1$ points be given on a sphere. From the centroid of any n of these points a line is drawn normal to the tangent plane to the sphere at the remaining point. Prove that all of these $n + 1$ lines are concurrent." *In the solution to this problem, it was noted that* (1) *had appeared on a Cambridge Scholarship Examination. As well, Benito and Fernández noted that a problem similar to* (2) *is exercise #196 of the Leçons de Géométrie Élémentaire, vol. 1 by J. Hadamard.*

2518. [2000 : 115] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

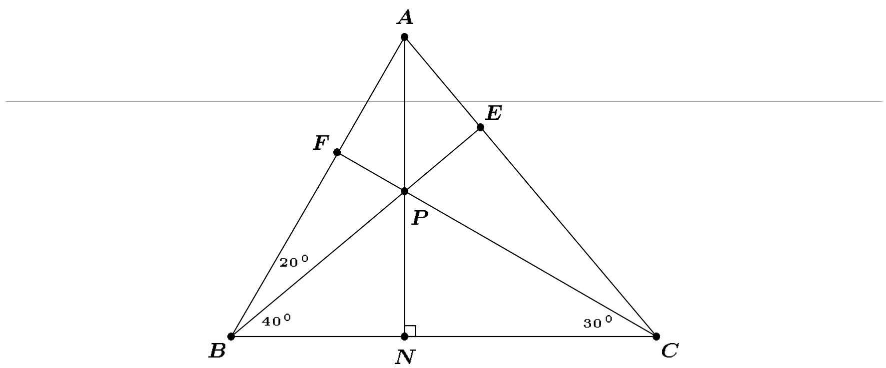
If P is a point on the altitude AN of $\triangle ABC$, if $\angle PBA = 20°$, if $\angle PBC = 40°$ and if $\angle PCB = 30°$, without using trigonometry, find $\angle PCA$.

Solution by Henri Liu, student, Trinity College Cambridge, England.

See figure on page 150

Let BP and CP meet AC and AB at the points E and F, respectively. Then $\angle BFC = 180° - 20° - 40° - 30° = 90°$, so CF is an altitude of $\triangle ABC$. Hence P is the orthocentre of $\triangle ABC$. Then $BCEF$ is a cyclic quadrilateral and we obtain $\angle PCA = \angle FCE = \angle FBE = 20°$.

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO-TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta,



Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JONATHAN CAMPBELL, student, Chapel Hill High School, North Carolina, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; MICHAEL SHIAW-TIAN, Biola University, La Mirada, CA, USA; SKIDMORE COLLEGE PROBLEM GROUP, Skidmore College, New York, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; JENS WINDELBAND, Hegel-Gymnasium, Magdeburg, Germany; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Most of the submitted solutions are similar to the one given above.

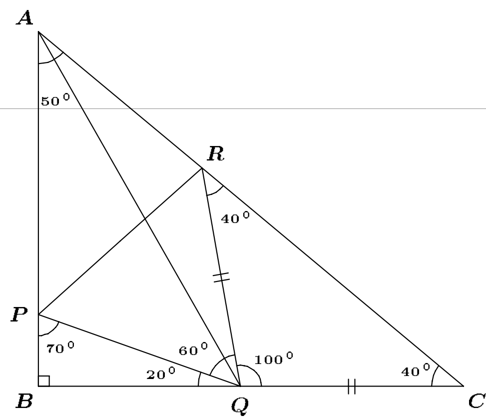
2519. [2000 : 115] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

In $\triangle ABC$, $\angle ACB = 40°$, $AB \perp BC$, P and Q are points on AB and BC respectively with $\angle PQB = 20°$. Without using trigonometry, prove that $AQ = 2BQ$ if and only if $PQ = CQ$.

Solution by Toshio Seimiya, Kawasaki, Japan. Since $\angle ABC = 90°$ and $\angle ACB = 40°$, we have $\angle BAC = 50°$. Let R be the point on the ray CA for which $\angle QRC = \angle QCA = 40°$. Then $QR = QC$ and $\angle RQC = 100°$. Hence $\angle PQR = 60°$. See figure on page 151.

(1) If $PQ = CQ$, then $PQ = CQ = QR$. Since $\angle PQR = 60°$, $\triangle PQR$ is equilateral, so that $\angle RPQ = 60°$. Since $\angle BPQ = 70°$, we have $\angle APR = 180° - \angle BPQ - \angle RPQ = 180° - 70° - 60° = 50° = \angle PAR$. Thus, we have $RA = RP = RQ$. Consequently, $\angle RAQ = \angle RQA = \frac{1}{2}\angle QRC = 20°$. Then, $\angle BAQ = \angle BAC - \angle QAC = 50° - 20° = 30°$. Therefore, $AQ = 2BQ$. Thus,

$$PQ = QC \Longrightarrow AQ = 2BQ\,.$$



(2) If $AQ = 2BQ$, then $\angle BAQ = 30°$. As a consequence, $\angle QAC = \angle BAC - \angle BAQ = 50° - 30° = 20°$. Also, $\angle RQA = \angle QRC - \angle QAR = 40° - 20° = 20°$. Hence, $RA = RQ$.

Let O be the circumcentre of $\triangle APQ$. Since $\angle APQ = 180° - \angle BPQ = 110°$, the major arc AQ is $220°$. The minor arc AQ is then $140°$, so that $\angle AOQ = 140°$. However, points P and R are in different half-planes with respect to the line AQ. Also, $\angle ARQ = 180° - \angle QRC = 140°$ and $RA = RQ$. Therefore, point R coincides with O. Thus, R is the circumcentre of $\triangle APQ$ and $RP = RQ$. Since $\angle PQR = 60°$, $\triangle PQR$ is equilateral. Hence, $PQ = QR$, and since $QR = QC$, we obtain $PQ = QC$. Thus,

$$AQ = 2BQ \Longrightarrow PQ = QC\,.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRI LIU, student, Trinity College Cambridge, England; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

2521. [2000 : 116] *Proposed by Eric Postpschil, Nashua, New Hampshire, USA.*

Given a permutation τ, determine all pairs of permutations α and β, such that $\tau = \beta \circ \alpha$ and $\alpha^2 = \beta^2 = \imath$ (the identity permutation). That is, determine all factorizations of τ into two permutations, each composed of **disjoint** transpositions.

Solution by the proposer.

Define the **order** of an element i in τ to be the least positive integer m such that $\tau^m(i) = i$; that is, the length of the cycle containing i.

Construct α by selecting any two elements of the permutation, i and j, of the same order in τ and not necessarily distinct, and make these assignments to α:

For each integer t, let α transpose $\tau^t(i)$ and $\tau^{-t}(j)$. (1)

In particular, for $t = 0$, this gives $\alpha(i) = j$. A number of other elements are assigned transpositions with i and j, but many elements may remain unassigned. Continue the construction of α by repeating the above selection (from unassigned elements) and assignment until α is completely defined. Because the i and j of each selection are of the same order, the transpositions specified in (1) are consistent even though they are redundant when t exceeds the order of i and j. It is also clear that the transpositions defined for each selection use elements distinct from those used for other selections, so, when the construction of α is complete, its defintion is consistent and composed solely of disjoint transpositions.

When α is complete, β is determined:

For each element i, $\beta(i) = \tau(\alpha(i))$. (2)

We will see that each such construction of an α and a β satisfies the conditions of the problem and that each solution takes the form of such a construction.

In (2), substitute $\alpha(i)$ for i to get

$$\beta(\alpha(i)) \;=\; \tau(\alpha(\alpha(i))) \;=\; \tau(i)\,,$$

showing that $\tau = \beta \circ \alpha$, so it remains only to show that β is composed solely of transpositions.

In (1) we may substitute 1 for t and $\alpha(i)$ for j to obtain the property $\alpha(\tau(i)) = \tau^{-1}(\alpha(i))$. We can apply $\alpha \circ \tau$ to both sides and use $\alpha^2(i) = i$ to obtain:

$$\alpha(\tau(\alpha(\tau(i)))) \;=\; i\,. \tag{3}$$

To see that β is composed of disjoint transpositions, we consider whether $\beta(\beta(i)) = i$:

$$\begin{aligned}
\beta(\beta(i)) &= \tau(\alpha(\tau(\alpha(i)))), \quad \text{by two applications of (2)}\\
\alpha(\beta(\beta(i))) &= \alpha(\tau(\alpha(\tau(\alpha(i)))))\\
\alpha(\beta(\beta(i))) &= \alpha(i)\,, \quad \text{by (3)}\\
\beta(\beta(i)) &= i\,.
\end{aligned}$$

Thus any α and β constructed as described give a desired factorization of τ. To see that all such factorizations are the results of such constructions, we show that $\alpha(i) = j$ requires i and j to be of the same order and determines the factorization.

Suppose $\alpha(i) = j$. This is $\alpha(\tau^t(i)) = \tau^{-1}(j)$ for $t = 0$. We assume this holds for some t and prove it holds for $t+1$. Since $\tau = \beta\circ\alpha$, $\tau(x) = \beta(\alpha(x))$. Applying β to both sides yields:

$$\beta(\tau(x)) \;=\; \alpha(x)\,. \tag{4}$$

Alternately, substituting $\alpha(\tau(\tau^t(i)))$ for x yields:

$$\begin{aligned}
\tau(\alpha(\tau(\tau^t(i)))) &= \beta(\alpha(\alpha(\tau(\tau^t(i))))) \\
\tau(\alpha(\tau(\tau^t(i)))) &= \beta(\tau(\tau^t(i))) \\
\tau(\alpha(\tau(\tau^t(i)))) &= \alpha(\tau^t(i)), \quad \text{by (4)} \\
\alpha(\tau(\tau^t(i))) &= \tau^{-1}(\alpha(\tau^t(i))), \quad \text{applying } \tau^{-1} \text{ to both sides} \\
\alpha(\tau(\tau^t(i))) &= \tau^{-1}(\tau^{-t}(j)), \quad \text{by induction hypothesis} \\
\alpha(\tau^{t+1}(i)) &= \tau^{-(t+1)}(j).
\end{aligned}$$

So induction holds, and $\alpha(\tau^t(i)) = \tau^{-t}(j)$ for all non-negative integers t. The negative integers follow similarly from $\alpha(j) = i$, itself a consequence of α's composition of transpositions. This property also demonstrates i and j have the same order: since $i = \tau^m(i)$ implies

$$j \;=\; \alpha(i) \;=\; \alpha(\tau^m(i)) \;=\; \tau^{-m}(j),$$

which implies $\tau^m(j) = j$, and the converse holds similarly.

Also solved by MICHEL BATAILLE, Rouen, France; and HENRY LIU, student, Trinity College Cambridge, UK. There was one incomplete solution.

2522*. [2000 : 116] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that a, b and c are positive real numbers. Prove that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right) \;\geq\; \frac{9}{1+abc}\,.$$

Solution by Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark.

Applying the Arithmetic-Geometric-Mean Inequality twice, we have

$$\begin{aligned}
\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} &= \frac{\frac{1}{a}}{1+\frac{1}{a}}+\frac{\frac{1}{b}}{1+\frac{1}{b}}+\frac{\frac{1}{c}}{1+\frac{1}{c}} \\
&\geq 3\cdot\sqrt[3]{\frac{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}{(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})}} \\
&= \frac{3}{\sqrt[3]{abc}}\cdot\frac{1}{\sqrt[3]{(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})}} \\
&\geq \frac{3}{\sqrt[3]{abc}}\cdot\frac{3}{3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\,,
\end{aligned}$$

and so by the Geometric-Harmonic-Mean Inequality we have

$$\begin{aligned}
\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right) &\geq \frac{9}{\sqrt[3]{abc}\left(\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}+1\right)} \\
&\geq \frac{9}{\sqrt[3]{abc}\ \left(\sqrt[3]{abc}+1\right)}\,. \qquad (1)
\end{aligned}$$

We also have, for $x\in\mathbb{R}^+$, that

$$(1+x^3)-x(x+1) \;=\; (x-1)(x^2-1) \;=\; (x-1)^2(x+1) \;\geq\; 0\,,$$

whence $x(x+1)\leq 1+x^3$ with equality if and only if $x=1$. (2)

The result now follows immediately from (1) and (2) with $x=\sqrt[3]{abc}$. It is easily seen by the conditions on the preceding inequalities that equality holds if and only if $a=b=c=1$.

Also solved by the AUSTRIAN IMO–TEAM 2000. GEORGE BALOGLOU, SUNY at Oswego, NY, USA; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; G.P. HENDERSON, Garden Hill, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong; and PETER Y. WOO, Biola University, La Mirada, CA, USA. Partial or incomplete solution were submited by MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; and VEDULA N. MURTY, Visakhapatnam, India. There was also one incorrect solution.

The solution given by Woo was quite similar to the one above and was based on the following interesting lemma which can be proved easily by the AM-GM Inequality:

Lemma: For any finite set S of positive real numbers, let $\mathrm{AM}(S)$ and $\mathrm{GM}(S)$ denote the arithmetic mean and the geometric mean of the elements in S, respectively. Then for any p, q, $r>0$, we have

$$\mathrm{GM}(\{p,\,q,\,r\}) \;\leq\; \mathrm{GM}(\{1+p,\,1+q,\,1+r\})-1 \;\leq\; \mathrm{AM}(\{p,\,q,\,r\})\,.$$

2523. [2000 : 116] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove that, if $t \geq 1$, then

$$\ln t \;\leq\; \frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right) .$$

Also, prove that, if $0 < t \leq 1$, then

$$\ln t \;\geq\; \frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right) .$$

I. *Solution by Michel Bataille, Rouen, France.*

First we remark that

$$\frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right)$$

is changed into its negative when t is replaced by $1/t$ (just as $\ln t$ is). The second inequality of the problem is thus an immediate consequence of the first one which we will only consider. Thus we suppose $t > 1$ (there is equality for $t = 1$).

Letting $x = \ln t > 0$, we readily see that our inequality is successively equivalent to

$$x \;\leq\; \frac{1}{2}\sqrt{\frac{\cosh x - 1}{\cosh x + 1}}\left(1+\sqrt{4\cosh x + 5}\right) ,$$

$$x\left(\sqrt{4\cosh x + 5} - 1\right) \;\leq\; 2\sinh x ,$$

$$\sqrt{4\cosh x + 5} \;\leq\; 1 + \frac{2\sinh x}{x} ,$$

and finally,

$$\cosh x + 1 \leq \frac{\sinh x}{x} + \frac{\sinh^2 x}{x^2} . \qquad (1)$$

Now, from the usual expressions of $\cosh x$ and $\sinh x$ as power series, we get

$$\cosh x + 1 \;=\; 2 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$$

and (using $\sinh^2 x = (\cosh 2x - 1)/2$)

$$\frac{\sinh x}{x} + \frac{\sinh^2 x}{x^2} = 2 + \sum_{n=1}^{\infty} \left(\frac{2^{2n+1}}{(2n+2)!} + \frac{1}{(2n+1)!} \right) x^{2n} .$$

Hence, to obtain (1), it suffices to prove that, for all integers $n \geq 1$:

$$\frac{1}{(2n)!} \leq \frac{2^{2n+1}}{(2n+2)!} + \frac{1}{(2n+1)!} .$$

But this is equivalent to $n(n+1)/2 \leq 2^{2n-2}$, which is clearly true for $n = 1$ and, for $n \geq 2$, results from

$$1 + 2 + 3 + \cdots + n \leq 1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1 < 2^n \leq 2^{2n-2} .$$

The proof is now complete.

II. *Solution by Christopher J. Bradley, Clifton College, Bristol, UK.*
Consider

$$f(t) = \frac{t-1}{2(t+1)} \left(1 + \sqrt{\frac{2t^2 + 5t + 2}{t}} \right) - \ln t .$$

Clearly $f(1) = 0$. Thus, to prove the proposition, it is sufficient to prove that $f'(t) \geq 0$ for $t > 0$. Now,

$$f'(t) = \frac{(t^4 + 4t^3 + 8t^2 + 4t + 1) - 2(t^2 + t + 1)\sqrt{t(2t^2 + 5t + 2)}}{2t(t+1)^2\sqrt{t(2t^2 + 5t + 2)}} ,$$

and

$$(t^4 + 4t^3 + 8t^2 + 4t + 1)^2 - 4(t^2 + t + 1)^2 t(2t^2 + 5t + 2) = (t^2 - 1)^4 \geq 0$$

for $t > 0$, so that the result follows.

A curious and rather magnificent result, that I could not have contemplated solving without **DERIVE**.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Bracken's proof also used power series of hyperbolic functions. Most other solutions were similar to Solution II.

2524. [2000 : 116] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

What conditions must the real numbers x, y and z satisfy so that

$$\cot x \cot y \cot z \;=\; \cot x + \cot y + \cot z\,,$$

where x, y, $z \neq n\pi$, with n being an integer?

Solution by Eckard Specht, Otto-von-Guericke University Magdeburg, Germany.

By the angle sum relationships for trigonometric functions we obtain

$$\begin{aligned}\cot(x+y+z) &= \frac{\cos(x+y+z)}{\sin(x+y+z)} \\ &= \frac{\cot x\cot y\cot z - \cot x - \cot y - \cot z}{\cot y\cot z + \cot z\cot x + \cot x\cot y - 1}\,. \qquad (1)\end{aligned}$$

Hence the equation $\cot x \cot y \cot z = \cot x + \cot y + \cot z$ is satisfied if and only if the numerator of (1) vanishes. This leads to the zeros of the cotangent function (or cosine function):

$$x+y+z \;=\; (2k+1)\frac{\pi}{2}\,, \qquad k \in \mathbb{Z}\,. \qquad (2)$$

Clearly the denominator in (1) is non-zero in this case, so that (2) is the condition we are looking for.

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; JOSÉ LUIS DIAZ, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Most of the solutions were similar to the one given above.

2525 March [2000 : 116] . *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Consider the recursions: $x_{n+1} = 2x_n + 3y_n$, $y_{n+1} = x_n + 2y_n$, with $x_1 = 2$, $y_1 = 1$. Show that, for each integer $n \geq 1$, there is a positive integer K_n such that

$$x_{2n+1} \;=\; 2\left(K_n^2 + (K_n+1)^2\right)\,.$$

I. *Solution by Michel Bataille, Rouen, France.*
First, for all n we have:

$$\begin{aligned} x_{n+2} &= 2x_{n+1} + 3y_{n+1} \\ &= 2x_{n+1} + 3\left(x_n + 2y_n\right) \\ &= 2x_{n+1} + 3x_n + 2\left(x_{n+1} - 2x_n\right) . \end{aligned}$$

Hence

$$x_{n+2} = 4x_{n+1} - x_n . \qquad (1)$$

With the help of $x_1 = 2$ and $x_2 = 7$, we classically obtain $x_n = \frac{1}{2}\left(u^n + v^n\right)$, where $u = 2 + \sqrt{3}$ and $v = 2 - \sqrt{3}$. Noticing $uv = 1$, we deduce that, for all n

$$2x_n^2 = x_{2n} + 1 \qquad \text{and} \qquad x_n x_{n+1} = \tfrac{1}{2}x_{2n+1} + 1 . \qquad (2)$$

Now $x_{n+1} + x_n = 3\left(x_n + y_n\right) = 3\left(y_{n+1} - y_n\right)$, so that (by addition)

$$\begin{aligned} x_1 + 2\left(x_2 + x_3 + \cdots + x_n\right) + x_{n+1} &= 3y_{n+1} - 3y_1 \\ &= 3y_{n+1} - 3 \\ &= x_{n+2} - 2x_{n+1} - 3 , \end{aligned}$$

which, using (1), easily yields

$$x_1 + x_2 + \cdots + x_n = \tfrac{1}{2}\left(x_{n+1} - x_n - 1\right) .$$

Thus, denoting by K_n the positive integer $x_1 + x_2 + \cdots + x_n$, we get

$$\begin{aligned} K_n^2 + \left(K_n + 1\right)^2 &= \tfrac{1}{4}\left[\left(x_{n+1} - x_n - 1\right)^2 + \left(x_{n+1} - x_n + 1\right)^2\right] \\ &= \tfrac{1}{4}\left(2x_{n+1}^2 + 2x_n^2 + 2 - 4x_n x_{n+1}\right) \\ &= \tfrac{1}{4}\left(x_{2n+2} + x_{2n} - 2x_{2n+1}\right) \qquad \text{using (2)} \\ &= \tfrac{1}{2}x_{2n+1} \qquad \text{using (1).} \end{aligned}$$

The result follows.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
[*Editor's remark:* As above the solver first established that $x_n = \frac{1}{2}\left(u^n + v^n\right)$ where $u = 2 + \sqrt{3}$ and $v = 2 - \sqrt{3}$.]
Then

$$x_n = \frac{1}{2}\left[\left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right)^{2n} + \left(\frac{1 - \sqrt{3}}{\sqrt{2}}\right)^{2n}\right]$$

for $n \in \mathbb{N}^*$. Thus, we have to verify the existence of $K_n \in \mathbb{N}^*$ such that $x_{2n+1} = 2\left(2K_n^2 + 2K_n + 1\right)$; that is

$$K_n = \frac{1}{2}\left(-1 + \sqrt{x_{2n+1} - 1}\right) .$$

Clearly, $K_n \neq 0$ for $n \in \mathbb{N}^*$; otherwise, we have $x_{2n+1} = 2$, in contradiction to the fact that x_n, $n = 1, 2, 3, \ldots$, is strictly increasing (as an easy induction will show). Now,

$$\begin{aligned} x_{2n+1} - 1 &= \frac{1}{2}\left[\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{4n+2} - 2 + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{4n+2}\right] \\ &= \left(\frac{1}{\sqrt{2}}\left[\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{2n+1} + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{2n+1}\right]\right)^2 \\ &= \left[\frac{1+\sqrt{3}}{2}\left(2+\sqrt{3}\right)^n + \frac{1-\sqrt{3}}{2}\left(2-\sqrt{3}\right)^n\right]^2 \\ &= w_n^2\,, \end{aligned}$$

where we have used the last line to define the number w_n. Thus, $K_n = \frac{1}{2}(-1 + w_n)$. Thus, it remains to show only that w_n is an odd integer for all $n \in \mathbb{N}^*$. Looking at the structure of w_n we obtain (note $2 \pm \sqrt{3}$ are the characteristic roots!) its recursion $w_{n+2} = 4w_{n+1} - w_n$ where $w_1 = 5$ and $w_2 = 19$. Because $w_{n+2} \equiv w_n \pmod{2}$, the proof is complete.

III. *Solution by David R. Stone, Georgia Southern University, Statesboro, GA, USA.*

The defining conditions and initial values characterize (x_n, y_n) as the solutions to the Pell equation $x^2 - 3y^2 = 1$. If we set

$$X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix},$$

the given recursions can be written as $X_{n+1} = PX_n$. Then the odd-subscripted terms can be obtained by left multiplication by

$$P^2 = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}.$$

That is,

$$\begin{aligned} x_{2n+1} &= 7x_{2n-1} + 12y_{2n-1} \\ y_{2n+1} &= 4x_{2n-1} + 7y_{2n-1}\,. \end{aligned} \qquad (1)$$

The first few examples are:

$$\begin{aligned} x_1 &= 2 \\ x_3 &= 26 = 2(2^2+3^2) \\ x_5 &= 362 = 2(9^2+10^2) \\ x_7 &= 5042 = 2(35^2+36^2) \\ x_9 &= 70226 = 2(132^2+133^2) \\ x_{11} &= 978122 = 2(494^2+495^2)\,. \end{aligned}$$

Finding K_n so that $x_{2n+1} = 2\left(K_n^2 + (K_n+1)^2\right) = 2\left(2K_n^2 + 2K_n + 1\right)$ is equivalent to solving:

$$4K_n^2 + 4K_n + (2 - x_{2n+1}) \;=\; 0\,,$$

which has the solution $K_n = \frac{1}{2}\left(-1 \pm \sqrt{x_{2n+1} - 1}\right)$. Thus, if we let $K_n = \frac{1}{2}\left(\sqrt{x_{2n+1} - 1}\right)$, we have $x_{2n+1} = 2\left(K_n^2 + (K_n+1)^2\right)$. It remains to show that a thus-defined K_n is an integer. From (1) it is clear by induction that x_{2n+1} is even and $x_{2n+1} + 1$ is a multiple of 3. Thus, the consecutive integers $x_{2n+1} - 1$ and $x_{2n+1} + 1$ are relatively prime. Therefore, the square y_{2n+1}^2 factors into relatively prime factors:

$$y_{2n+1}^2 \;=\; \frac{x_{2n+1}^2 - 1}{3} \;=\; (x_{2n+1} - 1)\,\frac{x_{2n+1} + 1}{3}\,.$$

Hence $x_{2n+1} - 1$ is an (odd) perfect square, so that K_n as defined is an integer.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; the AUSTRIAN IMO–TEAM 2000; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, KS, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Most of the solvers used an approach similar to II *above. Diminnie actually solved the recurrence relation for y_n also, and then showed that*

$$K_n \;=\; \frac{y_n + y_{n+1} - 1}{2}\,.$$

He also pointed out other interesting properties of x_n and y_n, namely:

1. $\displaystyle\lim_{n\to\infty} \frac{x_n}{y_n} = \sqrt{3}$.
2. $x_{2n} = 6y_n^2 + 1$.
3. $y_{n+1}^2 - y_n^2 = y_{2n+1}$.

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