THE ACADEMY CORNER

No. 33

Bruce Shawyer

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We present more readers' solutions to some of the questions of the 1999 Atlantic Provinces Council on the Sciences Annual Mathematics Competition, which was held last year at Memorial University, St. John's, Newfoundland [1999 : 452].

1. Find the volume of the solid formed by one complete revolution about the x–axis of the area in common to the circles with equations $x^2 + y^2 - 4y + 3 = 0$ and $x^2 + y^2 = 3$.

   Solution by Richard Tod, The Royal Forest of Dean, Gloucestershire, England (adapted by the editor).

   Here we have the two circles:

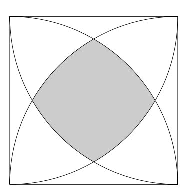
$$C_1 \ : \ x^2 + (y-2)^2 \ = 1 \qquad \text{and} \qquad C_2 \ : \ x^2 + y^2 \ = \ 3\,.$$

   These intersect when $6 - 4y = 0$; that is, when $y = \frac{3}{2}$. At this value, we have $x = \pm\frac{\sqrt{3}}{2}$.

   The volume is therefore

$$V \ = \ 4\pi \int_0^{\frac{\sqrt{3}}{2}} \left(2\sqrt{1-x^2} - 1\right)\, dx \ = \ \frac{4\pi^2}{3} - \pi\sqrt{3}\,.$$

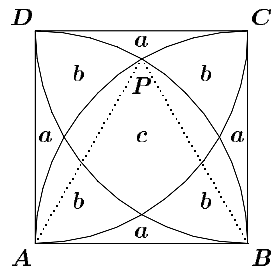
6. Inside a square of side r, four quarter circles are drawn, with radius r and centres at the corners of the square.



   Find the area of the shaded region.

   I. *Solution by Richard Tod, The Royal Forest of Dean, Gloucestershire, England*.

   The area of interest is the 'curved square region' in the centre of the given square, and is denoted by c. The other regions of interest are denoted by a and b.



Since the area of the square is r^2, we have

$$4a + 4b + c \;=\; r^2\,. \qquad (1)$$

Consider quarter circle ABC. This has an area of $\frac{\pi r^2}{4}$. Hence we have

$$2a + 3b + c \;=\; \frac{\pi r^2}{4}\,. \qquad (2)$$

Consider equilateral triangle ABP. This has area $\frac{\sqrt{3}r^2}{4}$.
Consider the sixth of a circle ABP. This has area $\frac{\pi r^2}{6}$.
The difference is $\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) r^2$. We need two of these, plus the equilateral triangle to get

$$a + 2b + c \;=\; \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) r^2\,. \qquad (3)$$

Solving (1), (2) and (3) simultaneously yields

$$c \;=\; \left(\frac{\pi + 3 - 3\sqrt{3}}{3}\right) r^2\,.$$

II. *Solution by Catherine Shevlin, Wallsend, England.*

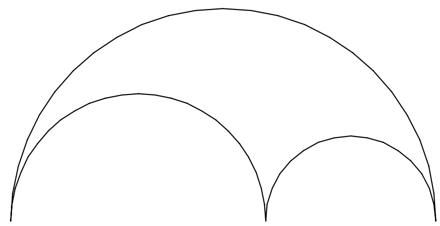
Let the origin be the centre of a square of side 1. Then the quarter circle DPB has equation $y = -\frac{1}{2} + \frac{\sqrt{-4x^2 - 4x + 3}}{2}$.

Thus, the general area is $4r^2 \displaystyle\int_0^{\frac{\sqrt{3}}{2} - \frac{1}{2}} \left(-\frac{1}{2} + \frac{\sqrt{-4x^2 - 4x + 3}}{2}\right) dx$.

This is $4r^2 \left(\frac{\sin^{-1}\left(\frac{2x+1}{2}\right)}{2} + \frac{(2x-1)\sqrt{-4x^2 - 4x + 3}}{8} - \frac{x}{2}\right)\Bigg|_0^{\frac{\sqrt{3}}{2} - \frac{1}{2}}$.

This simplifies to give $\left(\frac{\pi + 3 - 3\sqrt{3}}{3}\right) r^2$.

8. An arbelos consists of three semicircular arcs as shown:

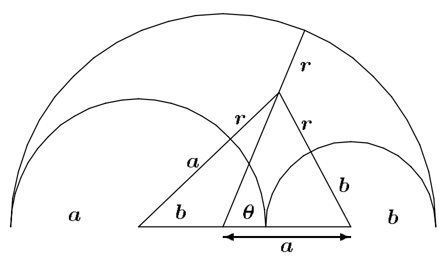


A circle is placed inside the arbelos so that it is tangent to all three semicircles.

Suppose that the radii of the two smaller semicircles are a and b, and that the radius of the circle is r.

Assuming that $a > b > r$ and that a, b and r are in arithmetic progression, calculate a/b.

Solution by Catherine Shevlin, Wallsend, England.



The radius of the large semi-circle is $a + b$. Thus, the centre of the large semi-circle is distances a and b from the centres of the smaller semi-circles as shown.

Applying the Cosine Rule to two triangles, we therefore have

$$\begin{aligned}(a+r)^2 - b^2 - (a+b-r)^2 &= 2b(a+b-r)\cos\theta\,, \\ (b+r)^2 - a^2 - (a+b-r)^2 &= -2a(a+b-r)\cos\theta\,.\end{aligned}$$

Eliminating $\cos\theta$, and solving for r, leads to

$$r = \frac{ab(a+b)}{a^2+ab+b^2}\,.$$

If a, b and r are in arithmetic progression, we have

$$2b - a = \frac{ab(a+b)}{a^2+ab+b^2} = \frac{ab(a^2-b^2)}{a^3-b^3}\,,$$

or $a^4 - a^3b - 2ab^3 + 2b^4 = 0$. Writing x for a/b, we have

$$0 = x^4 - x^3 - 2x + 2 = (x^3-2)(x-1)\,.$$

Thus, $a/b = x = \sqrt[3]{2}$.

THE OLYMPIAD CORNER

No. 206

R.E. Woodrow

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For your pleasure over the summer break for the *Olympiad Corner* we go on another quick tour of four Olympiad Contests. The first problem set we give consists of the problems of the First Round of the XXXIII Spanish Mathematical Olympiad 1996–97. My thanks for collecting these go to our regular contributor Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; and to Richard Nowakowski, Canadian Team Leader to the IMO at Buenos Aires.

XXXIII SPANISH MATHEMATICAL OLYMPIAD 1996–97

First Round, First Day

November 29, 1996 — Time: 4 hours

1. Show that any complex number $z \neq 0$ can be expressed as a sum of two complex numbers such that their difference and their quotient are purely imaginary (that is, with real part zero).

2. Consider a circle of centre O, radius r, and let P be an external point. We draw a chord AB parallel to OP.

(a) Show that $PA^2 + PB^2$ is constant.

(b) Find the length of the chord AB which maximizes the area of the triangle ABP.

3. Six musicians participate in a music festival. At each concert, some of them play music, and the others listen. What is the minimal number of concerts so that each musician listens to all the others?

4. The sum of two of the roots of the equation

$$x^3 - 503x^2 + (a+4)x - a = 0$$

is equal to 4. Determine the value of a.

First Round, Second Day

November 30, 1996 — Time: 4 hours

5. If a, b, c are positive real numbers, prove the inequality

$$a^2 + b^2 + c^2 - ab - bc - ca \;\geq\; 3(b-c)(a-b)\,.$$

When is the $=$ sign valid?

6. Find, with reasons, all the natural numbers n such that n^2 has only odd digits.

7. The triangle ABC has $\hat{A} = 90^\circ$, and AD is the altitude from A. The bisectors of the angles $\widehat{ABD}$ and $\widehat{ADB}$ intersect at I_1; the bisectors of the angles $\widehat{ACD}$ and $\widehat{ADC}$ intersect at I_2.

Find the acute angles of ABC, given that the sum of distances from I_1 and I_2 to AD is $BC/4$.

8. For each real number x, we denote by $\lfloor x \rfloor$ the largest integer which is less than or equal to x. We define

$$q(n) \;=\; \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor, \quad n = 1, 2, 3, \ldots\,.$$

(a) Forming a table with the values of $q(n)$ for $1 \leq n \leq 25$, make a conjecture about the numbers n for which $q(n) > q(n+1)$.

(b) Determine, with reasons, all the positive integers n such that

$$q(n) \;>\; q(n+1)\,.$$

Next we move to the problems of the 20th Austrian-Polish Mathematical Competition (1997). Thanks go to regular contributors Marcin E. Kuczma, Warszawa, Poland; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria, as well as to Richard Nowakowski.

20th AUSTRIAN-POLISH MATHEMATICAL COMPETITION (1997)

1. P is the common point of straight lines l_1 and l_2. Two circles S_1 and S_2 are externally tangent at P and l_1 is their common tangent line. Similarly, two circles T_1 and T_2 are externally tangent at P and l_2 is their common tangent line. The circles S_1 and T_1 have common points P and A, the circles S_1 and T_2 have common points P and B, the circles S_2 and T_2 have common points P and C, and the circles S_2 and T_1 have common points P and D. Prove that the points A, B, C, D lie on a circle if and only if the lines l_1 and l_2 are perpendicular.

2. Let m, n, p, q be positive integers. We have a rectangular chess-board of dimensions $m \times n$ divided into $m \cdot n$ equal squares. The squares are referred to by their coordinates (x, y), where $1 \leq x \leq m$, $1 \leq y \leq n$. There is a piece on each square. A piece can be moved from the square (x, y) to the square (x', y') if and only if $|x - x'| = p$ and $|y - y'| = q$. We want to move each piece simultaneously so as to get again one piece on each square. Find the number of ways in which such a multiple move can be made.

3. On the blackboard there are written the numbers 48, 24, 16, $\ldots$, $\frac{48}{97}$; that is, rational numbers $\frac{48}{k}$ with $k = 1, 2, \ldots, 97$. In each step, two arbitrarily chosen numbers a and b are cancelled and the number $2ab - a - b + 1$ is written on the blackboard. After 96 steps there is only one number on the blackboard. Determine the set of numbers which are possible outcomes of the procedure.

4. In a convex quadrilateral $ABCD$ the sides AB and CD are parallel, the diagonals AC and BD intersect at point E, and points F and G are the orthocentres of the triangles EBC and EAD, respectively. Prove that the mid-point of the segment GF lies on the line k perpendicular to AB such that $E \in k$.

5. Let p_1, p_2, p_3, and p_4 be four distinct prime numbers. Prove that there does not exist a cubic polynomial $Q(x) = ax^3 + bx^2 + cx + d$ with integer coefficients such that

$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3\,.$$

6. Prove that there does not exist a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x + f(y)) = f(x) - y$ for all integers x and y.

7. (a) Prove that for all real numbers p and q the inequality $p^2 + q^2 + 1 > p(q + 1)$ holds.

(b) Determine the greatest real number b such that for all real numbers p and q the inequality $p^2 + q^2 + 1 > bp(q + 1)$ holds.

(c) Determine the greatest real number c such that for all integers p and q the inequality $p^2 + q^2 + 1 > cp(q + 1)$ holds.

8. Let n be a natural number and let M be a set with n elements. Find the biggest integer k with the property: there exists a k-element family K of three-element subsets of M such that any two sets from K are non-disjoint.

9. Let P be a parallelepiped, let V be its volume, S its surface area, and L the sum of the lengths of the edges of P. For $t \geq 0$ let P_t be the solid consisting of points having distance from P not greater than t. Prove that the volume of P_t is equal to

$$V + St + \frac{\pi}{4}Lt^2 + \frac{4}{3}\pi t^3 .$$

Next we give Selected Problems from the Israel Mathematical Olympiads. My thanks go to Richard Nowakowski for obtaining them for the *Corner*.

SELECTED PROBLEMS FROM ISRAEL MATHEMATICAL OLYMPIADS

1. Prove that there are at most 3 primes between 10 and 10^{10} all of whose digits in base ten are 1 (for example, 11).

2. Is there a planar polygon whose vertices have integer coordinates, whose area is $\frac{1}{2}$, such that this polygon is

(a) a triangle with at least two sides longer than 1000?

(b) a triangle whose sides are all longer than 1000?

(c) a quadrangle?

3. Find all real solutions of

$$\sqrt[4]{13+x} + \sqrt[4]{4-x} = 3 .$$

4. Prove that if two altitudes of a tetrahedron intersect, then so do the other two altitudes.

5. Consider a partition of an $n \times n$ square (composed of n^2 1×1 squares) into rectangles where one side of each rectangle has an integer length and the other side is of length 1. What is the largest number of such partitions such that no two partitions have an identical rectangle at the same place?

6. A set of $n^2 + 1$ points in the plane, such that no three lie on a line, is given. Each line segment connecting a pair of these points is coloured by either red or blue. A path of length k is a sequence of k segments, where the end of each segment is the beginning of the next one (except for the first segment). A simple path is a path that does not intersect itself (that is, two segments meet if and only if they are consecutive segments of the path). Prove that there exists a monochromatic simple path of length n.

And as the 4^{th} set we give the problems of the 1997 Chinese Mathematical Olympiad. Once again thanks go to Richard Nowakowski, Canadian Team Leader to the IMO at Buenos Aires, for collecting this set.

THE 1997 CHINESE MATHEMATICAL OLYMPIAD

First Day — January 13, 1997

Time: 4.5 hours

1. Let $x_1, x_2, \ldots, x_{1997}$ be real numbers satisfying the following two conditions:

(a) $-\frac{1}{\sqrt{3}} \leq x_i \leq \sqrt{3} \quad (i = 1, 2, \ldots, 1997)$;

(b) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$.

Find the maximum value of $x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$, and give your reason.

2. Let $A_1B_1C_1D_1$ be an arbitrary convex quadrilateral and P a given point inside the quadrilateral. Suppose for vertex A_1, angles PA_1B_1 and PA_1D_1 are acute. Similarly, for vertices B_1, C_1 and D_1, angles PB_1A_1, PB_1C_1, PC_1B_1, PC_1D_1, PD_1A_1 and PD_1C_1 are all acute. Define inductively A_k, B_k, C_k and D_k to be the axisymmetric points of P about lines $A_{k-1}B_{k-1}$, $B_{k-1}C_{k-1}$, $C_{k-1}D_{k-1}$ and $D_{k-1}A_{k-1}$ respectively ($k = 2, 3, \ldots$). Consider the quadrilateral series

$$A_jB_jC_jD_j \qquad (j = 1, 2, \ldots)$$

and answer the following questions:

(a) Among the first 12 quadrilaterals, which ones must be similar to the 1997^{th} quadrilateral, and which ones are not necessarily so?

(b) Assume that the 1997^{th} quadrilateral is inscribed in a circle. Among the first 12 quadrilaterals, which ones are inscribed in a circle, and which ones are not necessarily so?

Give your proofs for the determinate answers of these questions, and give your examples for those indeterminate ones.

3. Show that there exist infinitely many positive integers n such that one can arrange

$$1, 2, \ldots, 3n$$

in the following table

$$\begin{array}{c} a_1, a_2, \ldots, a_n \\ b_1, b_2, \ldots, b_n \\ c_1, c_2, \ldots, c_n \end{array}$$

which satisfies the following two conditions:

(1) $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = \cdots = a_n + b_n + c_n$ is a multiple of 6;

and

(2) $a_1 + \cdots + a_n = b_1 + \cdots + b_n = c_1 + \cdots + c_n$ is also a multiple of 6.

Second Day — January 14, 1997

Time: 4.5 hours

4. Quadrilateral $ABCD$ is inscribed in a circle. Line AB meets DC at point P. Line AD meets BC at point Q. Tangent lines QE and QF touch the circle at points E and F respectively. Prove that points P, E and F are collinear.

5. Let $A = \{1, 2, 3, \ldots, 17\}$ and, for a function $f : A \longrightarrow A$, denote $f^{[1]}(x) = f(x)$ and $f^{[k+1]}(x) = f(f^{[k]}(x))$ $(k \in \mathbb{N})$.

Suppose that $f : A \longrightarrow A$ is a one-to-one function such that there exists a natural number M satisfying the following properties:

(a) if $m < M$ and $1 \le i \le 16$, then

$$\begin{aligned} f^{[m]}(i+1) - f^{[m]}(i) &\not\equiv \pm 1 \pmod{17}, \\ f^{[m]}(1) - f^{[m]}(17) &\not\equiv \pm 1 \pmod{17}; \end{aligned}$$

(b) for $1 \le i \le 16$,

$$\begin{aligned} f^{[M]}(i+1) - f^{[M]}(i) &\equiv 1 \text{ or } -1 \pmod{17}, \\ f^{[M]}(1) - f^{[M]}(17) &\equiv 1 \text{ or } -1 \pmod{17}. \end{aligned}$$

For all functions f satisfying the above conditions, find the maximum value of M, and prove your conclusion.

6. Let $a_1, a_2, \ldots$ be non-negative numbers which satisfy

$$a_{n+m} \le a_n + a_m \quad (m, n \in \mathbb{N}).$$

Prove that

$$a_n \le m a_1 + \left(\frac{n}{m} - 1\right) a_m$$

for all $n \ge m$.

Now we return to readers' comments and solutions to problems from 1998 numbers of the *Corner*, and particularly to solutions to problems of the 31[st] Spanish Mathematical Olympiad, First Round [1998: 452–453] for which we gave some solutions last number [2000: 141–146].

6. Consider the parabolas $y = cx^2 + d$, $x = ay^2 + b$, with $c > 0$, $d < 0$, $a > 0$, $b < 0$. These parabolas have four common points. Show that these four points are concyclic.

Solution by Michel Bataille, Rouen, France.

Solution I. Let $M_k(x_k, y_k)$ $(k = 1, 2, 3, 4)$ be the four common points. We suppose that they are distinct (otherwise there is nothing to prove). We have: $y_k = cx_k^2 + d$ and $x_k = ay_k^2 + b$. Multiplying the first equality by a and the second one by c, adding and rearranging, we obtain:

$$x_k^2 + y_k^2 - \frac{1}{a}x_k - \frac{1}{c}x_k + \frac{ad + bc}{ac} = 0 \qquad (k = 1, 2, 3, 4).$$

Now, the equation $x^2 + y^2 - \frac{1}{a}x - \frac{1}{c}y + \frac{ad+bc}{ac} = 0$ defines either $\emptyset$, or a singleton, or a circle. Since it is satisfied by (x_k, y_k) $(k = 1, 2, 3, 4)$, it does define a circle and the points M_k are concyclic.

Solution II. We introduce the complex representation of the points $M_k : z_k = x_k + iy_k$. We have to show that the cross-ratio

$$\frac{z_4 - z_1}{z_3 - z_1} \Big/ \frac{z_4 - z_2}{z_3 - z_2}$$

is a real number.

Since $y_1 = cx_1^2 + d$, $y_3 = cx_3^2 + d$, we obtain $z_3 - z_1 = (x_3 - x_1)$ $(1 + ic(x_3 + x_1))$ and analogous results for $z_4 - z_1$, $z_4 - z_2$, $z_3 - z_2$.

It is easy to see that we actually have to show that the product

$$P = (1 + ic(x_1 + x_4))(1 + ic(x_2 + x_3))(1 - ic(x_1 + x_3))(1 - ic(x_2 + x_4))$$

is a real number.

But, by eliminating y between $y = cx^2 + d$ and $x = ay^2 + b$, we see that x_1, x_2, x_3, x_4 are the roots of $ac^2x^4 + 2acdx^2 - x + ad^2 + b$, so that $x_1 + x_2 + x_3 + x_4 = 0$. By multiplying out the first two factors and the last two factors of P, it follows that P is a real number, as required.

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

The following similar problem appeared in Math Horizons, February 1997:

> "*Prove that if two ellipses whose axes are parallel intersect in four points, then the four points lie on a circle.*"

The solution is like the first above.

7. Show that there exists a polynomial $P(x)$, with integer coefficients, such that $\sin 1^\circ$ is a root of $P(x) = 0$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Geoffrey A. Kandall, Hamden, CT, USA; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Bornsztein.

We have

$$\begin{aligned} e^{i\pi/180} &= \cos(\frac{\pi}{180}) + i\sin(\frac{\pi}{180}) \\ &= \cos 1^\circ + i\sin 1^\circ \\ \text{and} \quad (e^{i\pi/180})^{180} &= e^{i\pi} = -1\,. \end{aligned}$$

Then

$$\left(\cos\frac{\pi}{180} + i\sin\frac{\pi}{180}\right)^{180} = -1\,.$$

Let $b = \sin\frac{\pi}{180}$, $a = \cos\frac{\pi}{180}$. Then $a^2 = 1 - b^2$ and we have

$$-1 = (a+ib)^{180} = \sum_{k=0}^{180}\binom{180}{k} a^k (ib)^{180-k}\,.$$

If we take the real parts

$$-1 = \sum_{k=0}^{90}\binom{180}{2k} a^{2k}(-1)^{90-k} b^{180-2k}\,;$$

that is,

$$1 + \sum_{k=0}^{90}(-1)^k\binom{180}{2k}(1-b^2)^{2k} b^{180-2k} = 0\,,$$

and $\sin 1^\circ$ is a root of a polynomial with integer coefficients.

Now we move to solutions of problems of the 31st Spanish Mathematical Olympiad, Second Round [1998: 453–454].

1. Consider sets A of 100 distinct natural numbers, such that the following property holds: "If a, b, c are elements of A (distinct or not), there exists a non-obtuse triangle of sides a, b, c." Let $S(A)$ be the sum of the perimeters of the triangles considered in the definition of A. Find the minimal value of $S(A)$.

Solution by Pierre Bornsztein, Courdimanche, France.

First we note that the problem is ambiguous: if we choose a, b, c, is it considered the same as if we choose b, c, a? For the rest of the solution we consider that the triangle (a, b, c) is "equal" to (b, c, a) and then it will only be counted one time in $S(A)$.

Let $A = \{a_1, \ldots, a_{100}\}$ be a set of 100 distinct natural numbers, with $a_1 < a_2 < \cdots < a_{100}$. Since the largest angle is opposite the longest side, and using the Law of Cosines, A satisfies the desired property if and only if for $1 \le i \le j \le k \le 100$, $a_i^2 + a_j^2 \ge a_k^2$. Moreover $a_1 \le a_i, a_j, a_k \le a_{100}$, so that it suffices to have $2a_1^2 \ge a_{100}^2$.

Since $a_{100} \ge a_1 + 99$ we must have

$$2a_1^2 \;\ge\; a_1^2 + 99^2 + 2 \times 99a_1\,;$$

that is,

$$a_1^2 - 2 \times 99a_1 - 99^2 \;\ge\; 0\,;$$

that is,

$$a_1 \;\ge\; 99(1+\sqrt{2}) \quad \text{with } a_1 \in \mathbb{N}\,.$$

Then $a_1 \ge 240$.

Then, if A has the property, we have

$$a_i \;\ge\; 239 + i\,. \tag{1}$$

It is easy to verify that $A = \{240, 241, \ldots, 339\}$ has the property, from the above. Consider $A = \{a_1, \ldots, a_{100}\}$ with $a_i = 239 + i$.

Moreover, if B has the property, then, from (1) with $B = \{b_1, \ldots, b_{100}\}$, $b_1 < b_2 < \cdots < b_{100}$ we have that $b_i \ge a_i$ for each i.

It is clear that $S(B) \ge S(A)$.

Thus, the minimal value is $S(A)$ with $A = \{240, \ldots, 339\}$. Now $S(A) = S_1 + S_2 + S_3$ where

$$S_1 \;=\; \sum_{a \in A} 3a\,, \qquad S_2 \;=\; \sum_{\substack{a,b \in A \\ a \ne b}} 2a + b\,,$$

$$S_3 \;=\; \sum_{\substack{a,b,c \in A \\ a \ne b \ne c \ne a}} a + b + c\,.$$

Let

$$s \;=\; \sum_{a \in A} a = \sum_{a=240}^{339} a \;=\; \frac{339 \times 340}{2} - \frac{239 \times 240}{2} \;=\; 28\,950\,.$$

Then $S_1 = 3s$.

For S_2, note that $x \in A$ appears in S_2,

(i) twice for every time, we can choose $b \in A\backslash\{x\}$; that is, 99 times,

(ii) once for every time, we can choose $a \in A\backslash\{x\}$.

Then each $x \in A$ appears in S_2 exactly 3×99 times. Thus, we have $S_2 = 3 \times 99 \times \sum_{a\in A} a = 297s$.

For S_3, each $x \in A$ appears in S_3 as many times as we can choose $(a,b) \in A\backslash\{x\}$ with $a \neq b$; that is, $\binom{99}{2}$ times.

Then $S_3 = \binom{99}{2}\sum_{a\in A} a = \binom{99}{2}s$.

Finally

$$S(A) \;=\; 3s + 297s + 4851s \;=\; 5151s \;=\; 149\,121\,450\,.$$

The minimal value is then $149\,121\,450$.

3. A line through the barycentre G of the triangle ABC intersects the side AB at P and the side AC at Q. Show that

$$\frac{PB}{PA}\cdot\frac{QC}{QA} \;\leq\; \frac{1}{4}\,.$$

Solutions by Pierre Bornsztein, Courdimanche, France; and by Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's generalization.

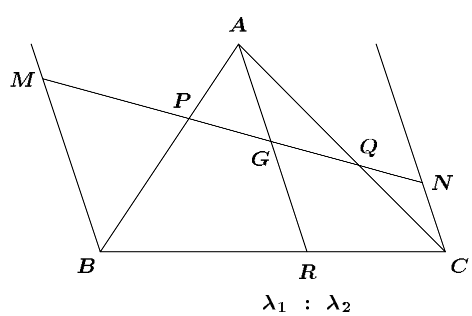
Let $u = \frac{PB}{PA}$, $v = \frac{QC}{QA}$. We will show that $u + v = 1$. It will then follow that $uv = u(1-u) = \frac{1}{4} - \left(\frac{1}{2} - u\right)^2 \leq \frac{1}{4}$.

That $u + v = 1$ is a special case of the theorem below.

Theorem. Suppose ABC is any triangle and P, Q, R are arbitrary points on the sides AB, AC, BC. Let G be the intersection of PQ and AR. Let $BR : RC = \lambda_1 : \lambda_2$, where $\lambda_1 + \lambda_2 = 1$. Then

$$\frac{GR}{GA} \;=\; \lambda_2\frac{PB}{PA} + \lambda_1\frac{QC}{QA}\,.$$

Proof. Draw lines through B and C that are parallel to AR, and extend PQ to meet these lines at M and N, as in the diagram.



Then $GR = \lambda_2 MB + \lambda_1 NC$, so that $\frac{GR}{GA} = \lambda_2 \frac{MB}{GA} + \lambda_1 \frac{NC}{GA}$. Therefore, $\frac{GR}{GA} = \lambda_2 \frac{PB}{PA} + \lambda_1 \frac{QC}{QA}$. Q.E.D.

When G is the barycentre (centroid), we have $\frac{GR}{GA} = \frac{1}{2}$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$. Consequently, $1 = \frac{PB}{PA} + \frac{QC}{QA}$, as asserted at the outset.

4. Find all the integer solutions of the equation

$$p(x+y) \;=\; xy$$

in which p is a prime number.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Geoffrey A. Kandall, Hamden, CT, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

$p(x+y) = xy$ is equivalent to $(x-p)(y-p) = p^2$.

Since p is a prime, the integral divisors of p^2 are $-p^2$, $-p$, -1, 1, p, p^2. It follows that (x,y) is a solution if and only if $(x,y) \in \{(p(1-p), p-1)$, $(0,0)$, $(p-1, p(1-p))$, $(p+1, p(p+1))$, $(2p, 2p)$, $(p(p+1), p+1)\}$.

5. Show that, if the equations

$$x^3 + mx - n \;=\; 0\,, \quad nx^3 - 2m^2x^2 - 5mnx - 2m^3 - n^2 \;=\; 0 \quad (m \neq 0\,, n \neq 0)$$

have a common root, then the first equation would have two equal roots, and determine in this case the roots of both equations in terms of n.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

Let a be a common root of the two equations. We have $a^3 = n - ma$. Thus, we also have

$$\begin{aligned} n(n-ma) - 2m^2a^2 - 5mna - 2m^3 - n^2 &= 0 \\ \text{and } a \text{ is a root of} \qquad mx^2 + 3nx + m^2 &= 0\,. \qquad (1) \end{aligned}$$

Substituting a for x in

$$x^3 + mx - n \;=\; \left(x^2 + 3\frac{n}{m}x + m\right)\left(x - \frac{3n}{m}\right) + \frac{9n^2}{m^2}x + 2n\,,$$

we obtain $a = -\frac{2m^2}{9n}$, and returning to (1), we deduce that $4m^3 + 27n^2 = 0$. From this, we easily get the following factorization:

$x^3 + mx - n = \left(x - \frac{3n}{2m}\right)^2 \left(x + \frac{3n}{m}\right)$, which shows that the first of the given equations has a simple root $-\frac{3n}{m}$ and a double root $\frac{3n}{2m}$.

We note in passing that $a = \frac{3n}{2m}$ (since $4m^3 + 27n^2 = 0$).

The second equation may be rewritten as $nx^3 - 2m^2x^2 - 5mnx + \frac{25}{2}n^2 = 0$, so that the sum of the roots is $\frac{2m^2}{n}$ and the product of the roots is $-\frac{25n}{2}$. If we denote by b and c the roots other than a, we thus have $b + c = \frac{2m^2}{n} - \frac{3n}{2m} = -\frac{15n}{m}$ and $bc = -\frac{25n}{2} \times \frac{2m}{3n} = -\frac{25m}{3}$, from which we easily obtain $b = c = -\frac{15n}{2m}$.

Using $4m^3 + 27n^2 = 0$, we see that $\frac{n}{m} = -\frac{4^{1/3}n^{1/3}}{3}$ (where $x \mapsto x^{1/3}$ denotes the inverse function of $x \mapsto x^3$ from $\mathbb{R}$ to $\mathbb{R}$) and we calculate:

$$\text{roots of the first equation: } (4n)^{1/3}; \quad -\left(\frac{n}{2}\right)^{1/3}; \quad -\left(\frac{n}{2}\right)^{1/3}$$

$$\text{roots of the second equation: } -\left(\frac{n}{2}\right)^{1/3}; \quad 5\left(\frac{n}{2}\right)^{1/3}; \quad 5\left(\frac{n}{2}\right)^{1/3}.$$

6. AB is a fixed segment and C a variable point, internal to AB. Equilateral triangles ACB' and CBA' are constructed, in the same half-plane defined by AB, and another equilateral triangle ABC' is constructed in the opposite half-plane. Show that:

(a) the lines AA', BB' and CC' are concurrent;

(b) if P is the common point of the lines of (a), find the locus of P when C varies on AB;

(c) the centres A'', B'', C'' of the three equilateral triangles also form an equilateral triangle;

(d) the points A'', B'', C'' and P are concyclic.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

(a) The rotation with centre C and angle 60° transforms B' into A and B into A'. Hence the lines AA' and BB' make an angle of 60° and, if we denote by P their point of intersection, we have: $\angle APB = 120°$, from which we deduce that the points A, P, B, C' are concyclic (with P and C' in different half-planes defined by AB). Therefore, $\angle C'PB = \angle C'AB = 60°$ and the line $C'P$ makes an angle of 60° with the line BB'.

Now, by the same reasoning as at the beginning, the line CC' also makes an angle of 60° with the line BB'. Thus, the lines CC' and $C'P$ coincide and AA', BB' and CC' are concurrent at P.

(b) From (a) P belongs to the arc AB subtending 120° on the segment AB. Conversely, if M is any point of this arc, let us first construct C' such that $\triangle ABC'$ is equilateral, and M and C' are not in the same half-plane defined by AB, and then C is the point of intersection of MC' and AB. The points A' and B' being obtained, the lines AA' and BB' intersect at the point P of CC' such that $\angle APB = 120°$; that is at M. Thus, $M = P$ and M is a point of the locus.

In conclusion, we find that the locus is the arc AB subtending 120° on the segment AB.

(c) The circles $(AB'C)$ and $(A'BC)$ intersect at P and C and have respective centres B'' and A''. Thus, $A''B'' \perp PC$. Similarly, $A''C'' \perp PB$. The lines $A''B''$ and $A''C''$ make the same angle as PC and PB; that is 60°. We have the same result for lines $A''B''$ and $B''C''$ and for lines $B''C''$ and $A''C''$ so that $\triangle A''B''C''$ is equilateral.

(d) $\triangle A''PC$ is isosceles at A'', so that

$$\begin{aligned}\angle A''PC &= \angle PCA'' = \left(\frac{1}{2}\right)(180^\circ - \angle PA''C) = 90^\circ - \angle CBP \\ &= 90^\circ - (120^\circ - \angle BCP) = \angle BCP - 30^\circ\,.\end{aligned}$$

Similarly, $\angle B''PC = \angle ACP - 30^\circ$. Hence

$$\begin{aligned}\angle B''PA'' &= \angle B''PC + \angle A''PC = \angle BCP + \angle ACP - 60^\circ \\ &= 180^\circ - 60^\circ = 120^\circ\end{aligned}$$

and P is on the circle $(A''B''C'')$.

Remark. For the 'degenerate' triangle ABC, P is the Fermat point and (c) is Napoleon's theorem.

To complete this number of the *Corner* and solutions for problems given in 1998 numbers of the *Corner*, we turn to solutions of problems of the 46$^{\text{th}}$ Polish Mathematical Olympiad, 1994–95 given [1998: 455].

1. Find the number of those subsets of $\{1, 2, \ldots, 2n\}$ in which the equation $x + y = 2n + 1$ has no solutions.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Aassila's solution, though they were all very similar.

That the equation $x + y = 2n + 1$ has no solution means that x and $2n + 1 - x$ are not both chosen from $x = 1, 2, \ldots, n$. Hence we have three possibilities: to choose one, the other, or neither. The answer is 3^n subsets (counting the empty set).

3. Let $p \geq 3$ be a given prime number. Define a sequence (a_n) by

$$\begin{aligned}a_n &= n && \text{for } n = 0, 1, 2, \ldots, p-1, \\ a_n &= a_{n-1} + a_{n-p} && \text{for } n \geq p.\end{aligned}$$

Determine the remainder left by a_{p^3} on division by p.

Solution by Pierre Bornsztein, Courdimanche, France.

Lemma. Let n, k be integers such that $n \geq kp \geq p$. Then

$$a_n \;=\; \sum_{i=0}^{k} \binom{k}{i} a_{n-ip-k+i}\,. \tag{1}$$

Proof. We use induction on k.

For $k = 1$, let n ($\geq p$) be an integer. Then, from the definition of $\{a_n\}$

$$a_n \;=\; a_{n-1} + a_{n-p}$$

which is (1) for $k = 1$.

Let k be a given positive integer. Suppose that (1) holds for each $n \geq kp$.

We show (1) holds for $n \geq (k+1)p$. The minimum value of $n-ip-k+i$ for $i = 0, \ldots, k$ is obtained for $i = k$, where it is $n - kp$.

Thus, for $n \geq (k+1)p$ we have $n - kp \geq p$.

Thus, in the following sum, we can use the definition of (a_n) for each term.

$$\begin{aligned}
a_n &= \sum_{i=0}^{k} \binom{k}{i} a_{n-ip-k+i} \\
&= \sum_{i=0}^{k} \binom{k}{i} (a_{n-ip-k+i-1} + a_{n-(i+1)p-k+i}) \\
&= \binom{k}{0} a_{n-k-1} + \sum_{i=1}^{k} \binom{k}{i} a_{n-ip-k+i-1} \\
&\qquad + \sum_{i=0}^{k-1} \binom{k}{i} a_{n-(i+1)p-k+i} + \binom{k}{k} a_{n-(k+1)p} \\
&= \binom{k}{0} a_{n-(k+1)} + \sum_{i=0}^{k-1} \binom{k}{i+1} a_{n-(i+1)p-k+i} \\
&\qquad + \sum_{i=0}^{k-1} \binom{k}{i} a_{n-(i+1)p-k+i} + \binom{k}{k} a_{n-(k+1)p}\,.
\end{aligned}$$

Of course $\binom{k}{0} = 1 = \binom{k+1}{0}$, $\binom{k}{k} = 1 = \binom{k+1}{k+1}$ and $\binom{k+1}{i+1} = \binom{k}{i+1} + \binom{k}{i}$, for $i = 0, 1, \ldots, k-1$. Thus,

$$\begin{aligned} a_n &= \binom{k+1}{0} a_{n-(k+1)} + \sum_{i=0}^{k-1} \binom{k+1}{i+1} a_{n-(i+1)p-k+i} \\ &\qquad + \binom{k+1}{k+1} a_{n-(k+1)p} \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} a_{n-ip-(k+1)+i} \,. \end{aligned}$$

Thus, (1) holds for $k+1$, when $n \geq (k+1)p$.

We deduce from the lemma, that for $n \geq p^2$

$$a_n = \sum_{i=0}^{p} \binom{p}{i} a_{n-ip-p+i}$$

(using (1) with $k = p$).

But, it is well known that, since p is a prime,

$$\binom{p}{i} \equiv 0 \pmod{p}\,, \quad \text{for } i = 1, 2, \ldots, p-1\,.$$

Then $a_n \equiv a_{n-p} + a_{n-p^2} \pmod{p}$. But $a_n = a_{n-1} + a_{n-p}$.

Then, for $n \geq p^2$, $a_{n-1} \equiv a_{n-p^2} \pmod{p}$, or equivalently, for $t \geq 0$

$$a_{t+p^2-1} \equiv a_t \pmod{p}\,.$$

The sequence formed by the remainders left by division of a_n by p is therefore periodic, with period $p^2 - 1$.

Moreover, $p^3 = p(p^2-1) + p$.

Then $a_{p^3} \equiv a_p \pmod{p}$. But $a_p = a_{p-1} + a_0 = p - 1$, so that we have

$$a_{p^3} \equiv p - 1 \pmod{p}\,.$$

4. For a fixed integer $n \geq 1$ compute the minimum value of the sum

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n}\,,$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers satisfying the condition

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = n\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the solution of Bornsztein, but all were similar.

Denote

$$\begin{aligned} S &= x_1 + \frac{x_2^2}{2} + \cdots + \frac{x_n^n}{n}, \\ H &= 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \\ w_i &= \frac{1}{iH}, \quad \text{for } i = 1, 2, \ldots, n. \end{aligned}$$

Then $w_i > 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. Thus,

$$\frac{S}{H} = \sum_{i=1}^{n} w_i x_i^i \geq \prod_{i=1}^{n} (x_i^i)^{w_i}$$

from the weighted AM-GM. (Note that we have used $n!$ copies of $\frac{x_1}{n!}$, $\frac{n!}{2}$ copies of $\frac{x_2^2}{n!}$, $\frac{n!}{3}$ copies of $\frac{x_3^3}{n!}$, etc.).

Thus,

$$\frac{S}{H} \geq \left(\prod_{i=1}^{n} x_i\right)^{1/H}.$$

Now after using the AM-GM inequality, we have

$$\prod_{i=1}^{n} x_i \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = 1,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$ $(= 1)$. Then $\frac{S}{H} \geq 1$.

Thus $S \geq H$ with equality if and only if $x_1 = \cdots = x_n = 1$. Then the minimal value of

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n}$$

is

$$1 + \frac{1}{2} + \cdots + \frac{1}{n},$$

and it is obtained if and only if $x_1 = x_2 = \cdots = x_n = 1$.

That completes the *Corner* for this issue. Send me your nice solutions and Olympiad materials.

BOOK REVIEWS

ALAN LAW

Conjecture and Proof by Miklós Laczkovich,
published by Typotex, Budapest, 1998.
ISBN # 963-7546-88X, softcover, 100+ pages, $24.00 (US).
Reviewed by ***Andy Liu****, University of Alberta, Edmonton, Alberta.*

This is one of a few books Typotex publishes in English. It is essentially the textbook of a course of the same name in the famed Budapest Mathematics Semesters, a program in advanced mathematics for North American undergraduate students, taught in English by top-notch Hungarian mathematicians.

The book deals with the existence or non-existence of various mathematical objects, constructions or procedures. It is divided into two parts. The first part, which consists of seven chapters, deals with non-existence. Chapter 1 contains five different proofs of the irrationality of $\sqrt{2}$; that is, the non-existence of positive integers p and q such that $\frac{p}{q} = \sqrt{2}$. Then come the harder proofs of the irrationality of e and π.

Chapter 2 makes a switch from number theory to geometry, and considers the problem of what lengths are constructible by ruler and compass given a segment of length 1. It is shown that $\sqrt[3]{2}$ and $\cos\frac{\pi}{9}$ are not constructible. This proves that two of the three classical problems, namely, the duplication of the cube and the trisection of an arbitrary angle, cannot be solved.

The second part of the book consists of ten chapters and deals with existence. A proof of existence can be graded according to how directly it leads to the existence of a given object. It may consist of an explicit construction of the object or, alternatively, it may establish the existence of the object without giving the slightest hint of how to find it. The purest form of proof of existence uses *reductio ad absurdum* by assuming the non-existence of the object and arriving at a contradiction.

Below is the table of contents of the second part:

8 The Pigeonhole Principle
9 Liouville numbers
10 Countable and uncountable sets
11 Isometries of $\mathbb{R}^n$
12 The problem of invariant measures
13 The Banach-Tarski paradox
14 Open and closed sets in $\mathbb{R}$ — The Cantor set
15 The Peano curve
16 Borel sets
17 The diagonal method.

This is one of the most fascinating books I have ever read. Many gems from various fields in mathematics are brought together in this compact package. The exposition is succinct and smooth, and leads the reader from the basics to the frontiers of mathematical research. It is recommended without any reservation.

Note: The email address of Typotex is <typotex@euroweb.hu>, for prices and ordering information. My understanding is that one copy costs US$24, two copies in the same order cost US$18 each, three in an order cost US$16 each, and four or more are US$15 each.

Winning Solutions by Edward Lozansky and Cecil Rousseau,
published by Springer-Verlag (Problem Books in Mathematics Series), 1996.
ISBN # 0-387-94743-4, softcover, 244+ pages, $34.95 (US).
Reviewed by ***Andy Liu****, University of Alberta, Edmonton, Alberta.*

This book is in the Springer-Verlag *Problem Books in Mathematics* series and is similar in format to another title in the series, *Problem Solving through Problems* by Loren Larson. As the authors acknowledge in the Preface, however, the present volume is more limited in scope. For example, Euclidean geometry is omitted entirely. The sparseness in material on graph theory is particularly surprising, since one of the authors is an expert in the field.

The book is organized into three chapters titled Numbers, Algebra and Combinatorics, with 8, 4 and 8 sections respectively. The basic concepts are covered in concise manner, supported by examples of two types. The first type may be described as standard bookwork, constructed primarily to illustrate the concepts. The second type consists of problems loosely based on the textual material, and they come primarily from various national and international contests, on which many problem books of this kind draw. The exercises, of which there are plenty, are similarly divided into two types along these lines.

The following problem is taken from page 194 of the book. "Suppose that the squares of an $n \times n$ chessboard are labeled arbitrarily with the numbers 1 through n^2. Prove that there are two adjacent squares (not diagonally) whose labels differ in absolute value by at least n." After Erdős mentioned it in a talk in Texas, Ben Walter, then a student, came up with a solution which is essentially the same as the one given in this book, and far from easy. There is a related but simpler problem, A2, in the 1981 Putnam Competition.

This book is a welcome addition to the shelves of anyone associated with mathematics problem solving in general and mathematics competitions in particular.

A Panorama of Harmonic Analysis by Steven G. Krantz,
published by The Mathematical Association of America, 1999,
The Carus Mathematical Monographs, Number 27.
ISBN # 0-88385-031-3, hardcover, 357+xi pages, $41.50 (US).
Reviewed by **Alan Law**, *University of Waterloo, Waterloo, Ontario.*

This is the most recent volume in the MAA Carus monograph series. These monographs are intended to provide expositions of mathematical subjects that are comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields. Steven Krantz's contribution is a first-class addition to this outstanding series.

The book assumes that the reader has some acquaintance with elementary real analysis. A short overview of rudiments from measure theory and elementary functional analysis precedes the chapter on basics of Fourier series: interweaving of historical developments along with the author's easy style bring the reader, simply, through summability, kernels, pointwise and norm convergence, and the Hilbert transform.

Chapter 2 provides a readable introduction to the Fourier transform and properties, including the Uncertainty Principle, in E_n. Chapter 3 addresses multiple Fourier series and some of the accompanying complexities — problems are illustrated through discussions in an E_2 setting. The next two chapters touch on a substantial amount of mathematical structure, including spherical harmonics, fractional and singular integrals, and Hardy spaces (a real-variable approach is outlined, too). Chapter 6 gives a clear introduction of substance to modern theories of integral operators, including motivation for and discussion about the celebrated $T(1)$ Theorem.

This reviewer particularly liked Chapter 7: "Wavelets". It illustrates some restrictions in applications of classical Fourier analysis and how wavelets are providing a powerful extension of harmonic analysis. The chapter supplies a nice introduction to the one-dimensional Haar wavelet basis, gives a mathematical description for a multi-resolution analysis, and shows the construction of a continuous wavelet.

A Panorama of Harmonic Analysis is well-titled and well-placed in the Carus monograph series. The author provides a most-readable tour of harmonic analysis from its beginnings in the 1800's to its current stage of evolution. I recommend the book to professionals, graduate students and interested senior undergraduates.

The Devil's Dartboard

Trevor Lipscombe and Arturo Sangalli

Abstract

To produce the most-difficult dartboard possible, we devise a polynomial-time algorithm. The problem, and the algorithm, are shown to be closely related to the Travelling Salesman Problem.

AMS Subject Classification: 68Q25 (Analysis of algorithms and problem complexity), 05A99 (Classical combinatorial problems, and 00A08 (Recreational mathematics).

Introduction

Students of mathematics can often be found playing darts in bars. The reason, no doubt, is their love of the subject. For example, recent analyses such as [1] and [2] have shown that the mathematically adept, yet physically inept, darts players should aim at the bull's eye in order to maximize their potential score. This raises the question: what could dartboard manufacturers do to overcome this strategy? The answer could be to renumber the board completely. One mathematician to consider the benefits and strategies of possible renumbering was Ivars Peterson, in a popular article [3].

The numbers in a dartboard are (sort of) arranged so that there is a penalty for error. For instance, the number 20 at the top of the board is sandwiched between the far smaller numbers 1 and 5. Aim at the 20 with three darts and, if you are not very good, you risk a score of $20+5+1=26$. Likewise, 19 is flanked by 3 and 7 (for a possible score of 29), and the 17 has a 2 and 3 on either side ($=22$).

There are more than 1.2×10^{17} circular arrangements of the numbers 1, 2, ..., 20 — the exact number being $19!$ (that is, factorial 19). The challenge is to find the one arrangement among them that gives the least reward for throwing error. More mathematically, what is the arrangement for which the standard deviation of the three-sums (that is, the sums of three consecutive numbers) is the smallest? We call this arrangement "The Devil's Dartboard."

We shall consider n-number dartboards ($n>3$) and ask: Is there a polynomial-time[4] algorithm that will generate the Devil's Dartboard for each n? A Devil's Dartboard Problem (DDP) consists of finding such an algorithm or proving that it does not exist. For the record, the actual dartboard, found in drinking establishments across the globe, is the permutation: 20, 1, 18, 4, 13, 6, 10, 15, 2, 17, 3, 19, 7, 16, 8, 11, 14, 9, 12, 5.

A Decent Algorithm

Here is a polynomial-time algorithm that, if it does not always produce a Devil's Dartboard, seems to come pretty close, especially for large n.

The permutation p is defined by induction. Choose $p(k+1)$ such that the 3–sum $p(k-1)+p(k)+p(k+1)$ is as close as possible to the mean $m = 3(n+1)/2$ of the 3–sums. In the case of a tie, select $p(k+1)$ so that consecutive 3–sums (different from m) fall on opposite sides of m, beginning with a 3–sum greater than m. To get the induction started, set $p(1) = n$ and $p(2) = 1$.

For example, if $n = 6$, we have $m = 3(6+1)/2 = 10.5$. The algorithm then produces the Devil's Dartboard 6, 1, 4, 5, 2, 3. The sequence of 3–sums is 11, 10, 11, 10, 11, 10, and the standard deviation $\sigma = 0.5$, clearly the smallest one possible. It is not, however, the only Devil's Dartboard for 6 numbers. Every Devil's Dartboard has its 'opposite'; that is, the permutation obtained by reading the numbers right to left, which is another Devil's Dartboard. These two permutations correspond to the two ways of writing the numbers on a circular board: clockwise or counterclockwise.

For $n = 20$, the algorithm generates the permutation 20, 1, 11, 19, 2, 10, 18, 4, 9, 17, 6, 8, 16, 7, 12, 13, 5, 14, 15, 3, having $\sigma = 2.54$. It is not a Devil's Dartboard, though, because the permutation that ends ..., 3, 15, 14, 5 — and coincides with the previous permutation elsewhere — has $\sigma = 2.52$.

Devil's Dartboards and Travelling Salesmen

The Devil's Dartboard Problem, which concerns permutations and minimization, is reminiscent of a famous optimization puzzle, the so-called Travelling Salesman Problem (TSP) [5]: Given a set $\{1, 2, \ldots, n\}$ of n "cities" and the set of distances $d(i,j)$ between cities i and j, find an ordering of cities — that is, a tour — of minimum length (the length of a tour is the sum of the distances between consecutive cities.) The "cities" may be arbitrary objects, and the "distances" $d(i,j)$ any non-negative real-valued function.

The so-called NP problems (for Nondeterministic Polynomial Time Verifiable) [6] are, roughly speaking, those problems whose solutions might be difficult to find but are easy to check. For example, it is straightforward to verify whether a given tour of the cities in a TSP has length less than, say, 1,000. The TSP is therefore an NP problem, but it is actually more than that: it has the property that any other NP problem may be transformed into an instance of the TSP in polynomial time. In other words, the TSP is "as hard as" any other NP problem. This implies that any polynomial-time algorithm for the TSP could be used to solve every other NP problem also in polynomial time. Problems that have this property are known as NP complete. In practical terms, it is considered highly unlikely that NP-complete problems could be "efficiently" solved — that is, within polynomial time — although strictly speaking the question is still open. Strategies to find near-optimal tours rather than exact solutions are known as heuristics. The nearest-neighbour heuristic for the TSP is the common-sense rule of travelling to the nearest city not yet called upon, with the initial city being chosen at random. In other words, a tour p is constructed step by step such that each new city $p(k+1)$ adds the minimal length to the partial tour $p(1), p(2), \ldots, p(k)$.

Consider a generalized travelling salesman problem (GTSP): There is a (reasonably simple) function L which assigns to every permutation p of the n cities and to each k $(= 1, 2, \ldots, n)$, a non-negative real number $L_p(k)$. Find the permutation p of the cities for which the sum $L_p(1)+L_p(2)+\ldots+L_p(n)$ is a minimum.

The standard TSP is obtained by taking $L_p(k) = d(p(k), p(k+1))$; the Devil's Dartboard Problem results from taking $L_p(k) = (p(k) + p(k+1) + p(k+2) - 3(n+1)/2)^2$. Moreover, in this general setting, our algorithm is the equivalent of the nearest-neighbour heuristic.

Mathematically sophisticated salespersons, therefore, will be able to spend more time in the bar, and less on the road, and be far better at darts than their colleagues.

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THE SKOLIAD CORNER

No. 46

R.E. Woodrow

For your summer problem pleasure, we are giving the first rounds of two contests. I hope you enjoy the problems. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Buenos Aires for collecting them.

BUNDESWETTBEWERB MATHEMATIK 1997
Federal Contest in Mathematics (Germany)
First Round

1. Can you always choose 15 from 100 arbitrary integers so that the difference of any two of the chosen integers is divisible by 7?

What is the answer if 15 is replaced by 16? (Proof!)

2. Determine all primes p for which the system

$$\begin{aligned} p+1 &= 2x^2 \\ p^2+1 &= 2y^2 \end{aligned}$$

has a solution in integers x, y.

3. A square S_a is inscribed in an acute-angled triangle ABC by placing two corners on the side BC and one corner on AC and AB, respectively. In a similar way, squares S_b and S_c are inscribed in ABC.

For which kind of triangle ABC do the sides of S_a, S_b and S_c have equal length?

4. In a park there are 10 000 trees, placed in a square lattice of 100 rows and 100 columns. Determine the maximum number of trees that can be cut down satisfying the condition: sitting on a stump, you cannot see any other stump.

MACEDONIAN MATHEMATICAL COMPETITIONS 1997
Round 1 — Part 1

1. Find a number with three different digits, five times smaller than the sum of all the other numbers with the same digits. Determine all solutions!

2. Calculate $\sqrt{7-\sqrt{48}}+\sqrt{5-\sqrt{24}}+\sqrt{3-\sqrt{8}}$.

3. Prove that, if $a>0$, $b>0$, $c>0$, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Prove that the equality is valid if and only if $a=b=c$.

4. Determine all the ordered pairs (x,y), $x\in\mathbb{N}$, $y\in\mathbb{N}$, for which three of the stated properties are true, and only one is false:

(i) $y\mid(x+1)$,

(ii) $x=2y+5$,

(iii) $3\mid(x+y)$,

(iv) $x+7y$ is a prime number.

5. For which natural numbers n is the sum

$$n^2+(n+1)^2+(n+2)^2+(n+3)^2$$

divisible by 10?

6. Three tired travellers arrived at an inn and asked for a meal. The innkeeper did not have anything else to offer them but potatoes. While the potatoes were being baked the travellers fell asleep. After a while, when the potatoes were done, the first traveller woke up, ate $\frac{1}{3}$ of the potatoes and went back to sleep without waking up the others. Then the second traveller woke up, ate $\frac{1}{3}$ of the rest of the potatoes and went back to sleep. At last, the third traveller woke up, ate $\frac{1}{3}$ of the rest of the potatoes and went back to sleep. The innkeeper watched all of these carefully and, in the end, when he counted the rest of the potatoes, there were 8. How many did the innkeeper bake?

Last issue we gave the problems of the Kangarou des Mathématiques, Épreuve EUROPÉENNE Cadets, 1997. Here are the solutions.

1.	c	**2**.	c	**3**.	c	**4**.	b	**5**.	b
6.	c	**7**.	b	**8**.	e	**9**.	e	**10**.	a
11.	d	**12**.	b	**13**.	c	**14**.	c	**15**.	c
16.	c	**17**.	a	**18**.	d	**19**.	c	**20**.	b
21.	b	**22**.	b	**23**.	d	**24**.	c	**25**.	d
26.	e	**27**.	b	**28**.	a	**29**.	e	**30**.	e

That completes the *Skoliad Corner* for this number. Send me your comments and suggestions as well as suitable contest materials.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), and David Savitt (Harvard University)

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from the previous issue be submitted in time for issue 4 of 2001.

High School Solutions

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H253. Find all real solutions to the equation

$$\sqrt{3x^2-18x+52}+\sqrt{2x^2-12x+162} \;=\; \sqrt{-x^2+6x+280}\,.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and Lino Demasi, student, St. Ignatius High School, Thunder Bay, Ontario.

Notice that by completing the square,

$$\begin{aligned} 3x^2 - 18x + 52 &= 3(x-3)^2 + 25\,, \\ 2x^2 - 12x + 162 &= 2(x-3)^2 + 144\,, \\ -x^2 + 6x + 280 &= -(x-3)^2 + 289\,. \end{aligned}$$

Therefore, the LHS is at least $\sqrt{25} + \sqrt{144} = 5 + 12 = 17$ and the RHS is at most $\sqrt{289} = 17$, with equality for both when $x = 3$. Therefore, $x = 3$ is the only solution.

Also solved by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

H254. *Proposed by Alexandre Trichtchenko, 1st year, Carleton University, Ottawa, Ontario.*

Let p and q be relatively prime positive integers, and n a multiple of pq. Find all ordered pairs (a, b) of non-negative integers that satisfy the diophantine equation $n = ap + bq$.

Solution by Masoud Kamgarpour, student, Carson Graham Secondary School, North Vancouver, BC; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

If n is negative, then there are no non-negative solutions in a and b. So, assume that n is non-negative. Since n is a multiple of pq, let $n = kpq$ for some non-negative integer k. Then $kpq = ap + bq \Longrightarrow p|bq \Longrightarrow p|b$, since $\gcd(p, q) = 1$. Therefore, $b = sp$ for some non-negative integer s. Similarly, $q|ap \Longrightarrow q|a$, since $\gcd(p, q) = 1$. Therefore, $a = tq$ for some non-negative integer t. So, we have $kpq = tpq + spq \Longrightarrow k = t + s$, and $(a, b) = (sp, tq)$ where $k = t + s$ for non-negative integers t, s.

H255. We have a set of tiles which contains an infinite number of regular n–gons, for each $n = 3, 4, \ldots$. Which subsets of tiles can be chosen, so that they fit around a common vertex? For example, we can choose four squares, or four triangles and a hexagon.

Solution. Let $n_1 \le n_2 \le \ldots \le n_t$ be the number of sides of the n–gons. Recall that the interior angle of a regular n–gon measures $180(n-2)/n = 180 - 360/n$ degrees. Then

$$\sum_{i=1}^{t} \left(180 - \frac{360}{n_i}\right) = 180t - 360 \sum_{i=1}^{t} \frac{1}{n_i} = 360 \qquad (1)$$

$$\Longrightarrow \sum_{i=1}^{t} \frac{1}{n_i} = \frac{180t - 360}{360} = \frac{t-2}{2}\,. \qquad (2)$$

Since $\sum_{i=1}^{t} 1/n_i > 0$, $t \ge 3$. Also, $n_i \ge 3$ for all i, yielding $180 - 360/n_i \ge 60$ for all i, so

$$360 = \sum_{i=1}^{t} \left(180 - \frac{360}{n_i}\right) \ge 60t\,,$$

which implies that $t \leq 6$.

Case 1: $t = 3$.
Equation (2) becomes

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2}.$$

Since $n_1 \leq n_2 \leq n_3$, $1/n_1 \geq (1/2)/3 = 1/6$, so $n_1 \leq 6$.

If $n_1 = 3$, then $1/n_2 + 1/n_3 = 1/6$, and the solutions are $(n_1, n_2, n_3) = (3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, $(3, 12, 12)$.

If $n_1 = 4$, then $1/n_2 + 1/n_3 = 1/4$, and the solutions are $(n_1, n_2, n_3) = (4, 5, 20)$, $(4, 6, 12)$, $(4, 8, 8)$.

If $n_1 = 5$, then $1/n_2 + 1/n_3 = 3/10$, and the only solution is $(n_1, n_2, n_3) = (5, 5, 10)$.

If $n_1 = 6$, then $1/n_2 + 1/n_3 = 1/3$, and the only solution is $(n_1, n_2, n_3) = (6, 6, 6)$.

Case 2: $t = 4$.
Equation (2) becomes

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = 1.$$

Since $n_1 \leq n_2 \leq n_3 \leq n_4$, $1/n_1 \geq 1/4$, so $n_1 \leq 4$.

If $n_1 = n_2 = 3$, then $1/n_3 + 1/n_4 = 1/3$, and the solutions are $(n_1, n_2, n_3, n_4) = (3, 3, 4, 12)$, $(3, 3, 6, 6)$.

If $n_1 = 3$ and $n_2 \geq 4$, then $1/n_2 + 1/n_3 + 1/n_4 = 2/3$, so that $1/n_2 \geq (2/3)/3 = 2/9$, or $n_2 \leq 9/2$, so $n_2 = 4$.

Hence, $1/n_3 + 1/n_4 = 5/12$, and the only solution is $(n_1, n_2, n_3, n_4) = (3, 4, 4, 6)$.

If $n_1 \geq 4$, then $n_i \geq 4$ for all i, and the only solution is $(n_1, n_2, n_3, n_4) = (4, 4, 4, 4)$.

Case 3: $t = 5$.
Equation (2) becomes

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5} = \frac{3}{2}.$$

Since $n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5$, $1/n_1 \geq (3/2)/5 = 3/10$, so $n_1 \leq 10/3$, or $n_1 = 3$.

Then

$$\frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5} = \frac{3}{2} - \frac{1}{3} = \frac{7}{6},$$

so $1/n_2 \geq (7/6)/4 = 7/24$, or $n_2 \leq 24/7$, so $n_2 = 3$.

Then again,

$$\frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5} = \frac{7}{6} - \frac{1}{3} = \frac{5}{6},$$

so $1/n_3 \geq (5/6)/3 = 5/18$, or $n_3 \leq 18/5$, so $n_3 = 3$.

Finally,

$$\frac{1}{n_4} + \frac{1}{n_5} = \frac{5}{6} - \frac{1}{3} = \frac{1}{2},$$

so the solutions are $(n_1, n_2, n_3, n_4, n_5) = (3,3,3,3,6)$, $(3,3,3,4,4)$.

Case 4: $t = 6$.
Equation (2) becomes

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \frac{1}{n_5} + \frac{1}{n_6} = 2.$$

Since $1/n_i \leq 1/3$ for all i, we must have equality for all i, so the only solution is $(n_1, n_2, n_3, n_4, n_5, n_6) = (3,3,3,3,3,3)$.

H256. Let $A = 2^a p_1^b p_2^c$, where p_1 and p_2 are primes, possibly equal to each other and to 2, and a, b, and c are positive integers. It is known that $p_1 \equiv p_2 \pmod 4$, $b \equiv c \pmod 2$, and that 2^a, p_1^b, p_2^c are three consecutive terms of an arithmetic sequence, not necessarily in that order. Find all possible values for A.

Solution. Since 2^a, p_1^b, and p_2^c form an arithmetic sequence, p_1 is even if and only p_2 is even.

Case 1: p_1, p_2 are both even.
Since p_1, p_2 are prime, $p_1 = p_2 = 2$, and 2^a, 2^b, 2^c form an arithmetic sequence in some order. Without loss of generality, assume that $1 \leq a \leq b \leq c$. Then $2^a + 2^c = 2^{b+1} \implies 1 + 2^{c-a} = 2^{b+1-a}$. The only solution is $c - a = 0$ and $b + 1 - a = 1$, which implies that $a = b = c$, so $A = (2^a)(2^b)(2^c) = 2^{3a} = 8^a$.

Case 2: $p_1 \equiv p_2 \equiv 1 \pmod 4$.
Without loss of generality, assume that $p_1^b < p_2^c$. Notice that the order of the arithmetic sequence must be p_1^b, 2^a, p_2^c, which implies that $p_1^b + p_2^c = 2^{a+1}$. But $p_1^b \equiv p_2^c \equiv 1 \pmod 4$. Thus, $p_1^b + p_2^c \equiv 2 \pmod 4$. But, $2^2 | 2^{a+1} \implies 2^{a+1} \equiv 0 \pmod 4$. There are no solutions for this case.

Case 3: $p_1 \equiv p_2 \equiv -1 \pmod 4$.
Without loss of generality, assume that $p_1^b < p_2^c$. Notice that the order of the arithmetic sequence must be p_1^b, 2^a, p_2^c, which implies that $p_1^b + p_2^c = 2^{a+1}$. But $p_1^b + p_2^c \equiv 2 \pmod 4$ if $b \equiv c \equiv 0 \pmod 2$ and $-2 \pmod 4$ if $b \equiv c \equiv 1 \pmod 2$. Again, $2^{a+1} \equiv 0 \pmod 4$. There are no solutions for this case either.

The only solution is $A = 8^a$ for some positive integer A.

Advanced Solutions

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A229. In tetrahedron $SABC$, the medians of the faces SAB, SBC, and SCA, taken from the vertex S, make equal angles with the edges that they lead to. Prove that $|SA| = |SB| = |SC|$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let the vectors SA, SB, SC be denoted by $2\mathbf{A}$, $2\mathbf{B}$, $2\mathbf{C}$, respectively. Then the medians to the faces SAB, SBC, and SCA are given by $\mathbf{A}+\mathbf{B}$, $\mathbf{B}+\mathbf{C}$, and $\mathbf{C}+\mathbf{A}$, respectively. The cosines of the equal angles made are given by

$$\frac{(\mathbf{B}-\mathbf{C})\cdot(\mathbf{B}+\mathbf{C})}{|\mathbf{B}-\mathbf{C}|\,|\mathbf{B}+\mathbf{C}|} = \frac{(\mathbf{C}-\mathbf{A})\cdot(\mathbf{C}+\mathbf{A})}{|\mathbf{C}-\mathbf{A}|\,|\mathbf{C}+\mathbf{A}|} = \frac{(\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})}{|\mathbf{A}-\mathbf{B}|\,|\mathbf{A}+\mathbf{B}|}\,.$$

These three fractions are equal to a fraction whose numerator is the sum of the three numerators and whose denominator is the sum of the three denominators. Since this latter fraction equals 0, each of the other three numerators must also be 0; that is, $|\mathbf{B}| = |\mathbf{C}| = |\mathbf{A}|$.

A230. *Proposed by Naoki Sato.*

For non-negative integers n and k, let $P_{n,k}(x)$ denote the rational function

$$\frac{(x^n-1)(x^n-x)\cdots(x^n-x^{k-1})}{(x^k-1)(x^k-x)\cdots(x^k-x^{k-1})}\,.$$

Show that $P_{n,k}(x)$ is actually a polynomial for all n, k.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Factoring out all the x's, we get

$$\frac{(x^n-1)(x^{n-1}-1)\cdots(x^{n-k+1}-1)}{(x^k-1)(x^{k-1}-1)\cdots(x-1)}\,. \tag{1}$$

If $k \geq n-k+1$, then, we can cancel out like factors in the numerator and denominator. So we can assume that $k < n-k+1$.

Since $\frac{(n)(n-1)\cdots(n-k+1)}{(k)(k-1)\cdots(1)}$ is always an integer, all the factors of the $(k-i)$'s, $i = 0, 1, \ldots, k-1$, divide into the factors of the $(n-i)$'s, $i = 0, 1, \ldots, k-1$. Hence all the factors of the denominator of (1) which are of the form $(x-\omega)$ where ω is a root of unity are cancelled by like factors in the numerator. Hence, $P_{n,k}(x)$ is actually a polynomial for all n, k.

Also solved by Vishaal Kapoor, Simon Fraser University, Burnaby, BC.

A231. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

For the sides of a triangle a, b, and c, prove that

$$\frac{13}{27} \le \frac{(a+b+c)(a^2+b^2+c^2)+4abc}{(a+b+c)^3} \le \frac{1}{2}\,.$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

We can eliminate the triangle constraints by letting, as usual, $a = y+z$, $b = z+x$, and $c = x+y$ where x, y, $z \ge 0$. Then $a+b+c = 2T_1$, $a^2+b^2+c^2 = 2(T_1^2 - T_2)$, and $abc = T_1T_2 - T_3$, where $T_1 = x+y+z$, $T_2 = yz+zx+xy$, and $T_3 = xyz$. The left hand inequality now becomes, after simplification, $T_1^3 \ge 27T_3$, which follows by the AM-GM inequality. The right hand inequality becomes $4T_3 \ge 0$. There is equality in the left hand inequality if and only if the triangle is equilateral. There is equality in the right hand inequality if and only if the triangle is degenerate with sides p, p, 0.

Also solved by Vedula N. Murty, Dover, PA, USA; Vishaal Kapoor, Simon Fraser University, Burnaby, BC; and Masoud Kamgarpour, student, Carson Graham Secondary School, North Vancouver, BC.

A232. Five distinct points, A, B, C, D, and E lie on a line (in this order) and $|AB| = |BC| = |CD| = |DE|$. The point F lies outside the line. Let G be the circumcentre of triangle ADF and H the circumcentre of triangle BEF. Show that the lines GH and FC are perpendicular.

(1997 Baltic Way)

Comment. At time of press, we do not have a solution for this problem.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C85. *Proposed by Christopher Long, graduate student, Rutgers University, NJ, USA. From a set of course notes on analytic number theory.*

Let $C = (c_{i,j})$ be an $m \times n$ matrix with complex entries, and let D be a real number. Show that the following statements are equivalent:

(i) For any complex numbers a_j, $j = 1, \ldots, n$,

$$\sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_j c_{i,j} \right|^2 \le D \sum_{j=1}^{n} |a_j|^2\,.$$

(ii) For any complex numbers b_i, $i = 1, \ldots, m$,

$$\sum_{j=1}^{n} \left| \sum_{i=1}^{m} b_i c_{i,j} \right|^2 \leq D \sum_{i=1}^{m} |b_i|^2 .$$

Solution by Michael Lambrou, University of Crete, Crete, Greece.
It suffices to show (i) implies (ii). Put

$$x_j \;=\; \sum_{i=1}^{m} b_i c_{ij} .$$

Then the left-hand side of (ii) is

$$\begin{aligned} \sum_{j=1}^{n} |x_j|^2 \;&=\; \sum_{j=1}^{n} \left(\bar{x}_j \sum_{i=1}^{m} b_i c_{ij} \right) \;=\; \sum_{i=1}^{m} \left(b_i \sum_{j=1}^{n} \bar{x}_j c_{ij} \right) \\ &\leq\; \left(\sum_{i=1}^{m} |b_i|^2 \right)^{1/2} \left(\sum_{i=1}^{m} \left| \sum_{j=1}^{n} \bar{x}_j c_{ij} \right|^2 \right)^{1/2} \\ &\leq\; \left(\sum_{i=1}^{m} |b_i|^2 \right)^{1/2} D^{1/2} \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2} , \end{aligned}$$

by Cauchy's inequality and by (i). This easily becomes

$$\sum_{j=1}^{n} |x_j|^2 \leq D \sum_{i=1}^{m} |b_m|^2 ,$$

which is the statement (ii).

To rephrase this using highbrow terminology, define the **norm** of an $m \times n$ matrix A to be the maximum value of the length of the vector Av, with the maximum taken over all vectors $v \in \mathbb{C}^n$ of length 1. Observe then that (i) says simply that the norm of C is at most $\sqrt{D}$, and (ii) says that the norm of the adjoint C^* of C (the complex conjugate of the transpose of C) is at most $\sqrt{D}$. Therefore the equivalence of (i) and (ii) is just the statement that the norm of C is equal to the norm of its adjoint, which is a standard fact.

C86. Let K_n denote the complete graph on n vertices; that is, the graph on n vertices with all possible edges present. Show that K_n can be decomposed into $n-1$ disjoint paths of length $1, 2, \ldots, n-1$. For example, for $n = 4$, the graph K_4, with vertices A, B, C, and D, decomposes into the paths $\{AC\}$, $\{BD, DA\}$, and $\{AB, BC, CD\}$.

Can we require that the paths in the decomposition be simple? (We say that a path is simple if it passes through each vertex at most once; that is, if no vertex is the end-point of more than two edges along the path.)

I. *Solution*. If n is odd, then at any vertex of K_n the number of incident edges is even. Hence, by a famous theorem due to Euler, it is possible to trace an unbroken path which traverses each edge of K_n exactly once. Break such a path into pieces of length $1, 2, \dots, n-1$ in any way we choose, and we obtain a solution for n odd.

However, we must try something different when n is even, as Euler's theorem does not apply. Instead, draw the vertices of K_n as if they were evenly spaced around a circle, and label the vertices $A_1, \dots, A_n$. Now, draw the zigzag path

$$A_1 \to A_2 \to A_n \to A_3 \to A_{n-1} \to \cdots \to A_{n/2} \to A_{n/2+2} \to A_{n/2+1} .$$

One may easily check that drawing the same path rotated by $2\pi j/n$ (in other words, with the subscripts of the A's all augmented by j modulo n) for integers $j < n/2$, yields a decomposition of K_n into $n/2$ disjoint paths of length $n-1$. Leaving one such path intact, splitting the second such path into a path of length 1 and a path of length $n-2$, and so on, gives the desired decomposition of K_n, and in fact this decomposition is simple.

It remains to answer whether or not our paths can be chosen to be simple in the case where n is odd. In this case, select any $n-1$ of the vertices of our graph, and call the remaining vertex A. Using the method of the preceding paragraph, construct the $(n-1)/2$ zigzags on the chosen $n-1$ vertices. Join A to the endpoints of each zigzag, and we obtain a decomposition of our K_n into $(n-1)/2$ disjoint loops of length n, each passing through every vertex of the graph. Finally, split these loops into paths of length 1 and $n-1$, of length 2 and $n-2$, and so on, which shows that indeed these paths can be taken to be simple when n is odd.

II. *Solution by Roman Muchnik, Yale University, New Haven, CT, USA.*

We prove the result using induction. The result is easily checked for $n = 2$ and $n = 3$. Assume the result is true for $n = t$ for some positive integer t.

Let $A_1, A_2, \dots, A_t$ denote the vertices of K_t. By assumption, K_t can be decomposed into paths of length $1, 2, \dots, t-1$. We add two vertices P and Q, and the appropriate edges to form K_{t+2}. Consider the path $PA_1QA_2PA_3\dots$, alternating between P and Q, passing through every vertex of K_t, and terminating on P or Q after A_t. This path has length $2n$, and adding PQ, forms a path of length $2t+1$. Divide this path into one of length t and the other of length $t+1$. Thus, K_{t+2} is decomposed into paths of length $1, 2, \dots, t-1, t, t+1$.

Therefore, by induction, the result is true for all positive integers n.

Problem of the Month

Jimmy Chui, student, University of Toronto

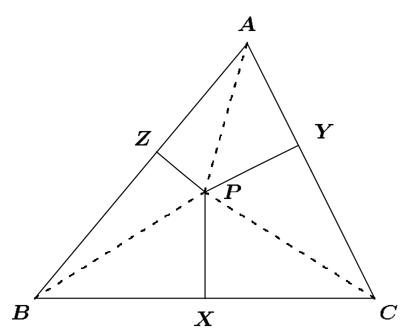
Problem. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC .$$

Let D, E be the incentres of triangles APB, APC, respectively. Show that AP, BD, and CE are concurrent.

(1996 IMO, Problem 2. Proposed by Canada)

We present two solutions to this problem. One solution involves pedal triangles, and the other solution uses inversion. The solutions start off identically.



Let X, Y, and Z be the feet of the perpendiculars from P to the sides BC, CA, and AB respectively. Let $\theta = \angle APB - \angle ACB = \angle APC - \angle ABC$. So, $\angle APB = C + \theta$ and $\angle APC = B + \theta$.

Note that BD is the angle bisector of $\angle ABP$, from properties of incentres. Let the intersection of BD and AP be Q; then, from the angle bisector theorem, we have

$$\frac{AB}{BP} = \frac{AQ}{QP} .$$

Similarly, if R is the intersection of CE and AP, then

$$\frac{AC}{CP} = \frac{AR}{RP} .$$

Now, AP, BD, and CE concur if and only if Q and R coincide, or equivalently

$$\frac{AQ}{QP} = \frac{AR}{RP} \iff \frac{AB}{BP} = \frac{AQ}{QP} = \frac{AR}{RP} = \frac{AC}{CP} .$$

I. *Solution — Pedal Triangles.*

Consider the pedal triangle XYZ. From properties of pedal triangles, $\angle YZX = \angle APB - \angle ACB = \theta$ and $\angle ZYX = \angle APC - \angle ABC = \theta$. So triangle YZX is isosceles with $XY = XZ$.

Now, $XY = CP\sin C$ and $XZ = BP\sin B$, from pedal triangle properties. So $BP\sin B = CP\sin C$. Then,

$$\frac{BP}{CP} = \frac{\sin C}{\sin B} = \frac{AB}{AC},$$

with the last equality coming from the Sine Law. Thus

$$\frac{AB}{BP} = \frac{AC}{CP},$$

and so AP, BD, and CE are concurrent.

II. *Solution — Inversion.*

Invert about A with a radius of 1, and let B, C, and P map onto B', C', and P' respectively. Now, from inversion properties of angles, $\angle AC'B' = \angle ABC = B$, and $AC'P' = \angle APC = B + \theta$. Hence, $\angle B'C'P' = \angle P'C'A - \angle B'C'A = \theta$. Similarly, $\angle C'B'P' = \theta$. Therefore, triangle $B'C'P'$ is isosceles with $B'P' = C'P'$.

From inversion properties of line segments,

$$B'P' = \frac{BP}{AB\cdot AP} \iff \frac{AB}{BP} = \frac{1}{AP\cdot B'P'},$$

and

$$C'P' = \frac{CP}{AC\cdot AP} \iff \frac{AC}{CP} = \frac{1}{AP\cdot C'P'}.$$

Thus,

$$\frac{AB}{BP} = \frac{AC}{CP},$$

and so AP, BD, and CE are concurrent.

J.I.R. McKnight Problems Contest 1996

1. (a) Prove $\log_2 x + \log_4 x + \log_{16} x + \log_{256} x + \ldots = 2\log_2 x$.

   (b) If $60^a = 3$ and $60^b = 5$, find $12^{\frac{1-a-b}{2(1-b)}}$.

2. (a) Given that $f(x+1) - f(x) = 4x + 5$ and $f(0) = 6$, find $f(x)$.

   (b) Given $g(x/2) - 3g(2/x) = 16x$, find $g(x)$.

3. A regular 18-sided polygon is inscribed in a circle, and triangles are formed by joining any 3 of the 18 vertices.

   (a) How many triangles are there?

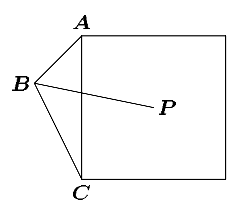
   (b) How many right angled triangles are there?

   (c) How many obtuse triangles are there?

   (d) How many acute triangles are there?

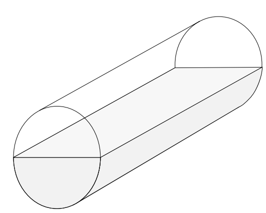
4. Let P be the centre of the square constructed on the hypotenuse AC of the right-angled triangle ABC.

   Prove that BP bisects angle ABC.



5. If three vertices of a cube in 3-dimensional space have coordinates $(4, 6, 3)$, $(0, 9, 3)$ and $(-3, 5, 8)$, find the coordinates of the other five vertices.

6. An oil drum is lying on its side. It has a radius of 50 cm and a length of 150 cm, and oil is leaking from it at a rate of 45 litres per minute. How fast is the oil level changing, when the oil is still 25 cm deep? (Note: 1 litre = 1000 cubic cm.)



7. Prove that $\sin^2\theta\cos^6\theta \le 3^3/4^4$.

8. Prove that

   (a) $\cos A \sin B = [\sin(A+B) - \sin(A-B)]/2$.

   (b)
$$\sum_{r=1}^{n} \cos^2 r\theta = \frac{\cos(n+1)\theta \sin n\theta}{2\sin\theta} + \frac{n}{2}.$$

Constructive Geometry, Part I

Cyrus C. Hsia
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Introduction

Geometric constructions have played an important role in the development of classical mathematics. Euclid used various constructions in his proofs in the Elements. In retrospect, a construction may seem straightforward and trivial, but coming up with the construction in the first place can be rather difficult. In this part, we will give the rules for construction in Euclidean geometry and begin the journey into constructing some of the most basic figures and shapes. In the second and third parts of this article, we will delve into some olympiad problems and some extensions.

Tools

There are two basic fundamental tools used by the Greeks in their constructions of geometric figures. The limitations of these two objects give rise to many obstacles and creative solutions in many problems. So, for any given problem in Euclidean geometry that we will discuss later on, we are allowed to use only the following:

1. A Straight Edge

   This is simply any straight edge with no markings on it to determine length. What can you do with this? For any two points in the plane, you may draw a straight line through them.

2. A Collapsible Compass

   This compass consists of two arms joined together at one of their ends. When the compass is lifted from the paper, the two arms collapse together. What can you do with this? For any two points in the plane, you may draw a circular arc (or complete circle) with one arm pivoted at one of the points, allowing the compass to rotate about this point.

Strategy

Here is some simple advice that the author suggests following in solving a geometric construction problem:

1. Work the problem backwards.

   If you are asked to construct a point, then assume the point exists and see how its properties are related to the given information.

2. Give the construction.

   Once you have the solution, give the construction step-by-step according to the rules of Euclidean geometry.

3. Prove that your construction does satisfy the given problem.

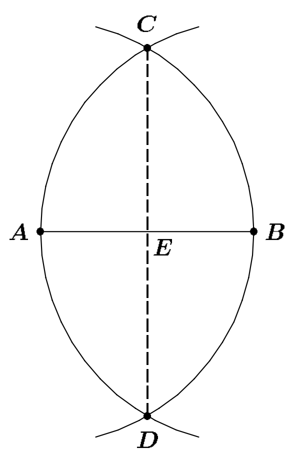
   Show that the construction is indeed the desired one, and that all the conditions of the problem are satisfied.

Basic Constructions

The best way to learn how to solve these types of problems is to do a lot of them. So, let us get right into some construction problems. We will give the solutions to a few of them and leave the remainder for the reader to complete. Also, a later problem can be solved more readily by using previously known constructions.

Problem 1A. Construct the mid-point of a given line segment.

Solution. First, attempt to find a solution on your own with only a straight-edge and compass. Once you have the solution, compare it with the construction given here.



(a) Label the end-points A and B as shown in the diagram.

(b) Draw a circular arc centred at A with radius AB as shown.

(c) Draw a circular arc centred at B with radius AB as shown.

(d) Let the two intersection points be C and D.

(e) Draw a straight line through C and D.

(f) The intersection of the two lines, E, is the required mid-point.

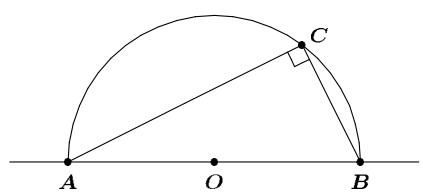
Now we show that E is indeed the mid-point. To see this, note that it is sufficient to show that $ABCD$ is a parallelogram; it is well known that the diagonals of a parallelogram bisect each other. One way of showing that $ABCD$ is a parallelogram is to show that the opposite sides are equal in length. Now, by construction, $AC = AB = BD$, since these lengths are the radii of the two circles. Similarly, $AD = BC$. Thus, $ABCD$ is a parallelogram and the diagonals bisect each other at E.

Problem 1A – **Generalization**. Divide a given line segment into n equal line segments, $n \geq 1$.

Problem 1B. Construct the perpendicular bisector of a given line segment.

Solution. Left as an exercise for the reader.

Problem 1C. Construct a right-angled triangle with hypotenuse on a given line segment.



Solution.

(a) Take two points, A and B, on the line and construct the mid-point O.

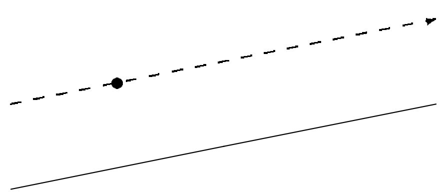
(b) Construct the circle centred at O with radius OA.

(c) Pick any point, C, on the circle not at A or B.

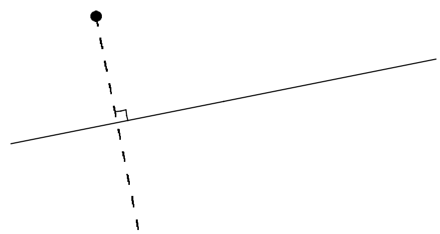
(d) Triangle ABC is a right triangle with the right angle at vertex C.

The proof that this is the required triangle is left to the reader.

Problem 2. Construct a line through a point parallel to a given line.

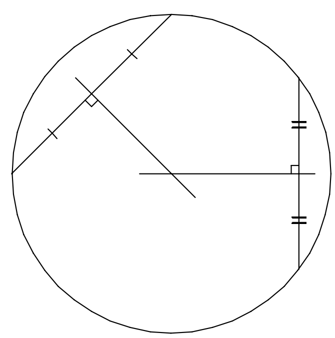


Problem 3. Construct a line through a point perpendicular to a given line.



Problem 4. Construct the centre of a circle, given the circle.

Solution. Work this problem backwards to see what properties are common to all chords of a circle and the centre of the circle. Once you have played around with this a bit, we may come up with the following construction.



(a) Draw any two distinct chords of the circle which are not parallel.

(b) Construct the perpendicular bisectors of both chords.

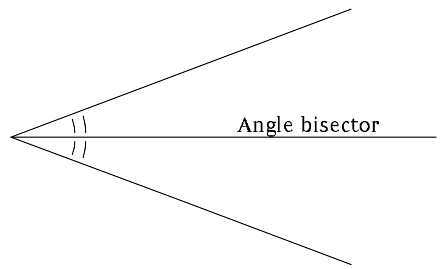
(c) The intersection of these perpendicular bisectors is the centre of the given circle.

The proof that this gives the centre of the circle is left to the reader.

Problem 5. Construct the tangent from a given point outside a given circle to that circle.

Problem 6. Construct an equilateral triangle.

Problem 7. Construct the angle bisector of a given angle. That is, draw a line that divides an angle into two equal halves as shown.



Exercises

All of the following constructions should be performed using straight edge and compass only.

1. Give constructions for the problems above whose solutions are not provided.
2. Construct two circles tangent to each other.
3. Construct the common tangents to these circles.
4. Construct the common tangents for two non-intersecting circles. **A221**, [1998 : 411, 1999 : 489]
5. Construct a 15° angle. How many ways can you do this?

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 December 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

For the information of proposers who submitted problems in 1998, we have either used them, or will not be using them.

2539. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Let $ABCD$ be a given convex quadrilateral. Let X and Y be points on the line segments BC and AD respectively, and P be an arbitrary point on XY and inside $ABCD$. Suppose that $\angle APB < \angle PXB$, and that $\angle DPC < \angle PXC$. Prove that $AD \cdot BC \geq 4PX \cdot PY$.

2540. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given an equilateral triangle ABC, let P and Z be points on the incircle such that P is the mid-point of AZ and $BZ < CZ$. The segment CZ and the extension of BZ meet the incircle again at X and Y respectively. Show that:

1. triangle XYZ is equilateral;
2. the points A, X and Y are collinear; and
3. each of the segments XA, YB and ZC is divided in the golden ratio by the incircle.

2541. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Show that, for all natural numbers n and $k \geq 2$,

$$\sum_{i} \binom{n}{2i} k^{n-2i} \left(k^2 - 4\right)^i$$

is divisible by 2^{n-1}.

2542. *Proposed by Hassan A. Shah Ali, Tehran, Iran.*

Suppose that k is a natural number and $\alpha_i \geq 0$, $i = 1, \ldots, n$, and $\alpha_{n+1} = \alpha_1$. Prove that

$$\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} \alpha_i^{k-j} \alpha_{i+1}^{j-1} \;\geq\; \frac{k}{n^{k-2}} \left(\sum_{1 \leq i \leq n} \alpha_i \right)^{k-1} .$$

Determine the necessary and sufficient conditions for equality.

2543. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

In quadrilateral $ABCD$, we have $\angle ABD = \angle ADB = \angle BDC = 40°$, $\angle DBC = 10°$ and AC and BD meet at P. Show that $BP = AP + PD$.

2544. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

For any triangle ABC, find the exact value of

$$\sum_{\text{cyclic}} \frac{\cos A + \cos B}{1 + \cos A + \cos B - \cos C} .$$

2545. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

For any triangle ABC, prove that

$$\sum_{\text{cyclic}} \frac{\sin^3 A}{\sin A \left(-\cos^2 A + \cos^2 B + \cos^2 C\right) + \sin B \cos(A - B) + \sin C \cos(A - C)} \;=\; 1 .$$

2546. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Prove that triangle ABC is equilateral if and only if

$$a \cos(B - C) + b \cos(C - A) + c \cos(A - B) \;=\; \frac{a^4 + b^4 + c^4}{abc} .$$

2547. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In triangle ABC with angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$, and area Δ, prove that

$$\frac{a^2+b^2+c^2}{\Delta} = 4\sqrt{7}.$$

2548★. *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana, USA.*

Let $a(1)=1$ and, for $n \geq 2$, define $a(n) = \lfloor a(n-1)/2 \rfloor$, if this is not in $\{0, a(1), \ldots, a(n-1)\}$, and $a(n) = 3a(n-1)$ otherwise.

(a) Does any positive integer occur more than once in this sequence?

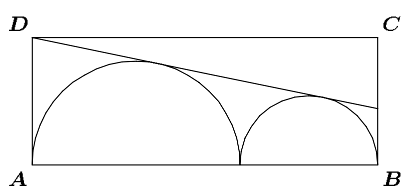
(b) Does every positive integer occur in this sequence?

2549★. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Is it possible to choose four points in the plane such that all the distances that they determine are odd integers?

2550. *Proposed by Catherine Shevlin, Wallsend upon Tyne, England.*

Given are two semi-circles, C_a and C_b of different radii a and b, and a rectangle $ABCD$ such that the diameters of the semi-circles lie contiguously on the side AB as shown, and the common tangent to the semicircles passes through the vertex D of the rectangle.



Find, in terms of a and b, the ratio in which the common tangent divides the side BC.

Correction

2495 *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let P be the interior isodynamic point of $\triangle ABC$; that is, $\frac{AP}{bc} = \frac{BP}{ca} = \frac{CP}{ab}$ (a, b, c are the side lengths, BC, CA, AB, of $\triangle ABC$).

Prove that the pedal triangle of P has area $\frac{\sqrt{3}}{d^2}F^2$, where F is the area of $\triangle ABC$ and $d^2 = \frac{a^2+b^2+c^2}{2} + 2\sqrt{3}F$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2431. [1999: 172] *Proposed by Jill Taylor, student, Mount Allison University, Sackville, New Brunswick.*

Let $n \in \mathbb{N}$. Prove that there exist triangles with integer area, integer side lengths, one side n and perimeter $4n$, where n is not necessarily prime.

For a given n, are such triangles uniquely determined?

[Compare problem **2331**.]

Combination of solutions by Gerry Leversha, St. Paul's School, London, England, and Kenneth M. Wilke, Topeka, KS, USA.

Let n, b and c be the sides of the triangle. Since the perimeter $n+b+c=4n$, if $b<n$ then $c>2n$, contradicting the triangle inequality. Hence we can take $n<b<c<2n$. Furthermore we can assume that n, b and c are coprime; for, if $(n,b)=d>1$ say, then $d|c$ since $c=3n-b$, and the triangle would be similar to a smaller triangle whose sides are coprime [and whose area is still an integer, since it is rational (A/d^2); by Heron's formula, any triangle with integer sides and semiperimeter and rational area has integer area. — *Ed.*]

Let $b=n+k$ for some integer k, so that $c=2n-k$. Then by Heron's formula, the area A of the triangle satisfies

$$A^2 \;=\; s(s-n)(s-b)(s-c) \;=\; 2n^2k(n-k)\,,$$

where $s=2n$ is the semiperimeter. Since $(n,b)=(n,c)=(b,c)=1$ there are positive integers x and y, such that either $n-k=x^2$ and $k=2y^2$, or $n-k=2y^2$ and $k=x^2$, so that either way $n=x^2+2y^2$.

Conversely if $n=x^2+2y^2$ for positive integers x and y, we can take $k=x^2$ to produce the triangle n, $n+x^2$, $n+2y^2$ where the perimeter is $3n+x^2+2y^2=4n$ and $A^2=2n(n)(2y^2)(x^2)=(2nxy)^2$ by Heron.

[*Editorial comment*. Therefore it has been proved that for a positive integer n there is a triangle satisfying the conditions of the problem if and only if n is of the form $j(x^2+2y^2)$ for positive integers j, x, y (where we may as well assume that $(x,y)=1$). The solution now goes on to identify these numbers as precisely those n having at least one prime factor of the form $8t+1$ or $8t+3$.]

It is known (for example, [1], pp. 188–191) that $p=x^2+2y^2$ for positive integers x and y whenever p is a prime of the form $8t+1$ or $8t+3$. Conversely, if $n=x^2+2y^2$ for relatively prime x and y, and p is a prime factor of n, then $x^2+2y^2\equiv 0 \pmod p$. So, multiplying by the inverse of y in the group of residues [that is, the unique element $z\in\{1,2,\ldots,p-1\}$ such that $yz\equiv 1 \pmod p$] we obtain $(xz)^2+2\equiv 0 \pmod p$. Thus -2 is a quadratic residue of p. It is now a standard number-theoretic result (for example, [1], p. 139) that $p \equiv 1$ or $3 \pmod 8$.

If n is a product of more than one such prime, one can construct the necessary representation of n in the form $x^2 + 2y^2$ from the corresponding representations for each prime factor by using the identity

$$(A^2 + 2B^2)(C^2 + 2D^2) \;=\; (AC \pm 2BD)^2 + 2(AD \mp BC)^2\,,$$

and the resulting triangles are not uniquely determined. For example, for $n = 33$ we have $3 = 1^2 + 2 \cdot 1^2$ and $11 = 3^2 + 2 \cdot 1^2$, so that

$$33 \;=\; (3 \pm 2 \cdot 1)^2 + 2(3 \mp 1)^2 \qquad \text{or} \qquad 33 \;=\; 5^2 + 2 \cdot 2^2 = 1^2 + 2 \cdot 4^2\,,$$

and the corresponding triangles are 33, 41, 58 with perimeter 132 and area 660, and 33, 34, 65 with perimeter 132 and area 264.

Reference:

[1] T. Nagell, *Introduction to Number Theory*, second edition, Chelsea Publishing Company, 1964.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

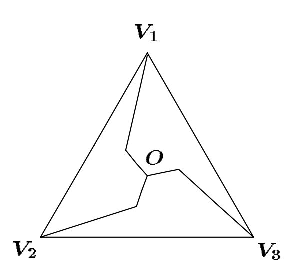
The characterization of those n for which the problem has a solution was essentially found by most solvers, including the proposer, though nobody included every detail of the argument. Some solvers seemed uncertain whether the proposer wished the problem solved for every *positive integer n. In fact she did not: the problem as published was the editor's (misleading) restatement of the original proposal.*

2438. [1999: 173] *Proposed by Peter Hurthig, Columbia College, Burnaby, BC.*

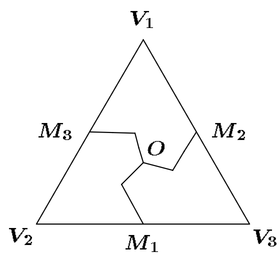
Show how to tile an equilateral triangle with congruent pentagons. Reflections are allowed. (Compare problem **1988**.)

Solution by Mark Lyon and Max Shkarayev, students, University of Arizona, Tuscon, AZ, USA.

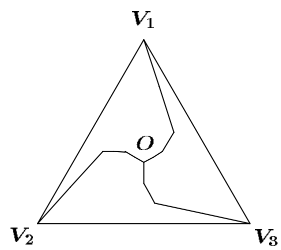
Define O as the circumcentre of the triangle. If we connect the vertices V_1, V_2, V_3 of the triangle to O we get a partition of the triangle into congruent triangles. If we now "bend" each of the segments V_iO in the middle, we will end up with a partitioning of the triangle by congruent pentagons, as required.



This solution could be extended to a partition of the equilateral triangle into three congruent n–gons, for any integer $n \geq 3$. In the case of odd n the procedure is very similar; we just have to bend each V_iO $(n-3)/2$ times. In the case of even n, we first define M_1, M_2, M_3 to be the mid-points of the sides. Drawing the segments M_iO will partition the triangle into congruent quadrilaterals. We now bend the M_iO's (similar to the odd case) $(n-4)/2$ times to obtain a partition of the triangle into congruent n–gons.

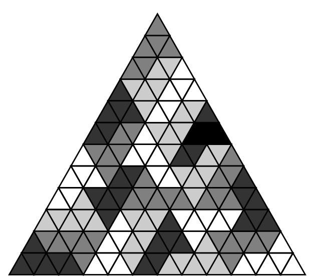


n even



n odd

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; J. SUCK, Essen, Germany; and the proposer. All solutions were the same, except for the proposer's, whose solution used 24 *congruent pentagons:*



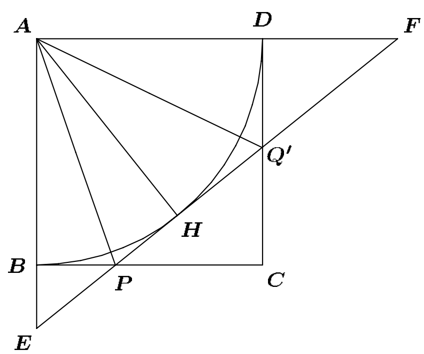
Lambrou gave the same construction as above to divide the triangle into three congruent n*–gons, for any integer* $n \geq 3$*. Suck went further to similarly divide the regular* n*–gon into congruent* k*–gons, for any integers* n*,* $k \geq 3$*.*

The proposer would like to see a solution (perhaps with more pentagons) in which the congruent pentagons are ***convex****. (Or a proof that it is impossible?)*

2439. [1999: 238] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $ABCD$ is a square with side a. Let P and Q be points on sides BC and CD respectively, such that $\angle PAQ = 45^\circ$. Let E and F be the intersections of PQ with AB and AD respectively. Prove that $AE + AF \geq 2\sqrt{2}a$.

I *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*



The circle centred at A with radius a is tangent at B, D to BC and CD respectively. Let PQ' be the tangent at H to the above circle. Then AP is the bisector of $\angle BAH = \omega$, AQ' is the bisector of $\angle HAD = \phi$, and hence we have $\angle PAQ' = 45^\circ$ because $\omega + \phi = 90^\circ$. From this we conclude that $Q' \equiv Q$. Thus

$$\begin{aligned} AE + AF &= \frac{AH}{\cos\omega} + \frac{AH}{\cos\phi} = a\left(\frac{1}{\cos\omega} + \frac{1}{\cos\phi}\right) \\ &\geq a \cdot 2\frac{1}{\cos\dfrac{\omega+\phi}{2}} = 2\sqrt{2}a \end{aligned}$$

by Jensen's inequality since the function $f(x) = 1/\cos x$ is convex for $0 < x < \pi/2$ because $f'(x) = \dfrac{1 + 2\tan^2 x}{\cos x} > 0$. The equality holds when $\omega = \phi$ or $\angle BAP = 22.5^\circ$.

II *Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let $BP = x$, $DQ = y$, $BE = u$, $DF = v$ and $\angle PAB = \alpha$. (See figure on page 243.) Thus $x = a\tan\alpha$. Then $\angle QAD = \pi/4 - \alpha$ and

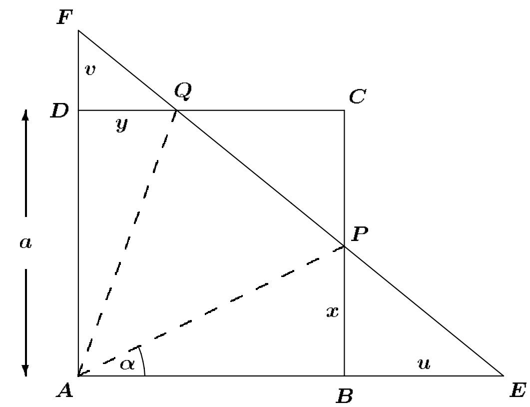
$$y = a\tan\left(\frac{\pi}{4} - \alpha\right) = a\frac{\tan\frac{\pi}{4} - \tan\alpha}{1 + \tan\frac{\pi}{4}\tan\alpha} = a\frac{1 - \tan\alpha}{1 + \tan\alpha} = \frac{a(a-x)}{a+x}.$$

Evidently $\triangle PBE \sim \triangle FDQ$, so $uv = xy = \dfrac{a(ax - x^2)}{a+x} \equiv f(x)$. By the AM-GM inequality, we have

$$u + v \geq 2\sqrt{uv}. \tag{1}$$

It is clear that the lower bound of the sum $u + v$ can be found by maximizing the product uv. Hence

$$f'(x) = a\frac{a^2 - 2ax - x^2}{(a+x)^2} = 0 \qquad \text{for} \qquad \overline{x} = a(\sqrt{2} - 1),$$



yielding

$$uv \;=\; f(\overline{x}) \;=\; a\overline{x}\frac{a-\overline{x}}{a+\overline{x}} \;=\; a^2(\sqrt{2}-1)^2\,, \quad 2\sqrt{uv} \;=\; 2a(\sqrt{2}-1)\,. \qquad (2)$$

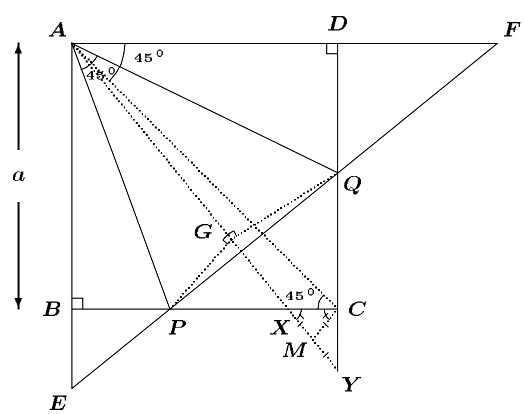
The second derivative is

$$f''(x) \;=\; -2a\frac{(a+\overline{x})^2+(a^2-2a\overline{x}-\overline{x}^2)}{(a+x)^3} \;=\; -\frac{2a}{a+\overline{x}} \;<\; 0\,,$$

which verifies the maximum. Thus we have from (1) and (2)

$$AE+AF \;=\; 2a+(u+v) \;\geq\; 2a+(2\sqrt{2}a-2a) \;=\; 2\sqrt{2}a\,.$$

III *Solution by the proposer.*



We may assume that $\angle BAP \leq \angle DAQ$. Let G be the point of reflection of B across AP. Then $\angle PAG = \angle PAB$, $\angle AGP = \angle ABP = 90^\circ$ and $AG = AB$. As $\angle PAQ = 45^\circ$ we get $\angle BAP + \angle DAQ = 45^\circ$, so that $\angle GAQ = \angle DAQ$. Since we have $AG = AB = AD$, we get

$\triangle AGQ \equiv \triangle ADQ$. Thus $\angle AGQ = \angle ADQ = 90^\circ$. Hence P, G, Q are collinear, so that $AG \perp PQ$.

Let X, Y be the intersections of AG with the lines BC, CD, respectively. By the assumption that $\angle BAP \leq \angle DAQ$ we get

$$\angle PAG = \angle PAB \leq \angle DAQ = 45^\circ - \angle CAQ = \angle PAC,$$

so that X is a point on the side BC. Since $AX \perp PQ$, we have $\angle AXP = \angle AEP$. As $\angle EAP = \angle BAP = \angle GAP = \angle XAP$, we get $\triangle AEP \equiv \triangle AXP$, whence $AE = AX$. Similarly, we have $\triangle AFQ \equiv \triangle AYQ$, and $AF = AY$.

Suppose X does not coincide with the point C. Then $X \neq Y$. Let M be the mid-point of XY. As $\angle XCY = 90^\circ$, we have $XM = CM = MY$. Hence we have $\angle MCX = \angle MXC > \angle ACX = 45^\circ$. Thus $\angle ACM = \angle ACX + \angle MCX > 90^\circ$. Therefore, $AM > AC = \sqrt{2}a$. Since M is the mid-point of XY, we get

$$AX + AY = 2AM > 2\sqrt{2}a.$$

Since $AX = AE$ and $AF = AY$, we have

$$AE + AF \geq 2\sqrt{2}a.$$

We include equality in the above since equality holds when X coincides with C; that is, when $\angle BAP = \angle DAQ$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; IAN JUNE L. GARCES, and RICHARD B. EDEN, student, Ateneo de Manila University, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MASOUD KAMGARPOUR, student, Carson Graham Secondary School, North Vancouver, BC; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; R. LAUMEN, Antwerp, Belgium; HO-JOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; and PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece.

2440. [1999: 238] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given: triangle ABC with $\angle BAC = 90^\circ$. The incircle of triangle ABC touches BC at D. Let E and F be the feet of the perpendiculars from D to AB and AC respectively. Let H be the foot of the perpendicular from A to BC.

Prove that the area of the rectangle $AEDF$ is equal to $\dfrac{AH^2}{2}$.

I. *Solution by Dimitar Mitkov Kunchev, student, Baba Tonka School of Mathematics, Rousse, Bulgaria.*

Let $AB = c$, $AC = b$, and $BC = a$. Then if s is the semiperimeter of $\triangle ABC$ we have

$$CD = s - c = \frac{a+b-c}{2} \quad \text{and} \quad DB = s - b = \frac{a+c-b}{2}.$$

Since $\triangle EBD$ and $\triangle ABC$ are similar, we have

$$\frac{ED}{AC} = \frac{BD}{BC} \implies ED = \frac{a+c-b}{2} \cdot \frac{b}{a},$$

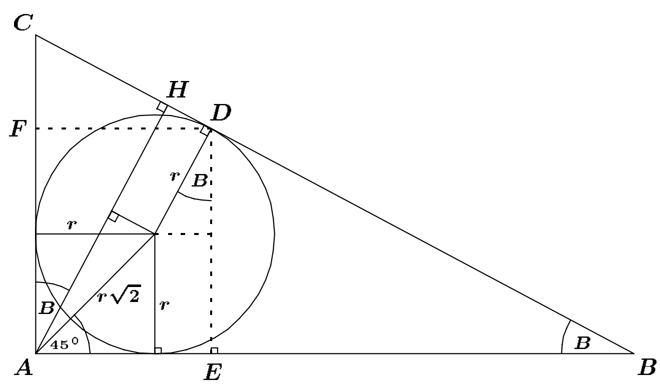
and using $\triangle FDC \sim \triangle ABC$, we similarly obtain:

$$FD = \frac{a+b-c}{2} \cdot \frac{c}{a}.$$

If we denote by S the area of the rectangle $AEDF$, then

$$\begin{aligned} S &= ED \cdot FD = \frac{(a+c-b)(a+b-c)bc}{4a^2} = \frac{a^2-(c-b)^2}{4a^2}bc \\ &= \frac{a^2-c^2+2bc-b^2}{4a^2}bc = \frac{2b^2c^2}{4a^2} = \frac{1}{2}\left(\frac{bc}{a}\right)^2 = \frac{AH^2}{2}. \end{aligned}$$

We used the Theorem of Pythagoras and the formula for the altitude in a right triangle: $AH = (bc/a)$.



II. *Solution by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Let r be the inradius of triangle ABC. From the figure:

$$\begin{aligned} \frac{(AH)^2}{2} &= \frac{(r + r\sqrt{2}\cos(45^\circ - B))^2}{2} = \frac{r^2(1+\cos B+\sin B)^2}{2} \\ &= \frac{r^2}{2}(1+\cos^2 B+\sin^2 B+2\cos B+2\sin B+2\cos B\sin B) \\ &= r^2(1+\cos B+\sin B+\cos B\sin B) \\ &= (r+r\cos B)(r+r\sin B) = ED \cdot AE. \end{aligned}$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ATENEO PROBLEM-SOLVING GROUP, Ateneo de Manila University, The Philippines; SAM BAETHGE, Nordheim, TX, USA; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MASOUD KAMGARPOUR, student, Carson Graham Secondary School, North Vancouver, BC; MICHAEL LAMBROU, University of Crete, Crete, Greece; R. LAUMEN, Antwerp, Belgium; HO-JOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ANDREI SIMION, student, Brooklyn Technical High School, New York; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; APLAKIDES YIANNIS, Veria, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2441. [1999: 239] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Suppose that D, E, F are the mid-points of the sides BC, CA, AB of $\triangle ABC$. The incircle of $\triangle AEF$ touches EF at X, the incircle of $\triangle BFD$ touches FD at Y, and the incircle of $\triangle CDE$ touches DE at Z.

Show that DX, EY, FZ are concurrent. What is the intersection point?

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

We shall prove that DX, EY, and FZ all pass through the incentre of $\triangle ABC$. Because it is clear that DX passes through the incentre when $AB = AC$, assume that $AB > AC$ and let L denote the point where the incircle of $\triangle ABC$ touches BC. [Note that the dilatation with centre A and ratio 2 takes the incircle of $\triangle AFE$ and its contact point X to the incircle of $\triangle ABC$ and the point L.] Then X lies on AL and $AX = XL$. Let AH be the bisector of $\angle A$ with H on BC, and call P the point where AH meets DX.

Claim: P is the incentre.

To prove the claim, let s be the semiperimeter of $\triangle ABC$; then

$$BL = s - AC, \qquad LC = s - AB,$$

$$2DL = AB - AC, \quad \text{and} \quad 2DH = BH - HC.$$

Menelaus's theorem for D, P, X on $\triangle AHL$ gives $\left|\frac{HP}{PA} \cdot \frac{AX}{XL} \cdot \frac{LD}{DH}\right| = 1$, or

$$\frac{HP}{PA} = \frac{2HD}{2DL} = \frac{BH - HC}{AB - AC} = \frac{BH}{AB} \cdot \frac{1 - \frac{HC}{BH}}{1 - \frac{AC}{AB}} = \frac{BH}{AB},$$

where the last equality holds because AH is the angle bisector (so that $HC : BH = AC : AB$). From this we conclude that in $\triangle ABH$, BP is the bisector of $\angle B$, and therefore, P is the incentre of $\triangle ABC$. In the same way we prove that EY and FZ also pass through the incentre, as promised.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain (2 solutions); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MASOUD KAMGARPOUR, student, Carson Graham Secondary School, North Vancouver, BC (only the concurrence part); MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; MICHAEL LAMBROU, University of Crete, Crete, Greece (2 solutions); GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ANDREI SIMION, New York, NY, USA; ECKARD SPECHT, Magdeburg, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

The Nagel point *of a triangle is defined to be the intersection point of the lines from a vertex to the point of the opposite side where that side touches its excircle. Many readers provided accessible references — too many to list — and showed how the solution to* **2441** *follows easily from familiar properties of the Nagel point. In particular, the intersection point of our problem is the Nagel point of $\triangle DEF$ (because X, Y, and Z are precisely the points where the excircles of $\triangle DEF$ touch the sides). A standard theorem states that the incentre of a triangle is the Nagel point of its mid-point triangle. Other approaches treat the Nagel point as the isotomic conjugate of the Gergonne point, which is almost easier to prove than to say! The story can be found in the many references.*

2442. [1999: 239] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let $\{a_n\}_1^\infty$, $\{x_n\}_1^\infty$, $\{y_n\}_1^\infty$, ..., $\{z_n\}_1^\infty$, be a finite number of given sequences of non-negative numbers, where all $a_n > 0$. Suppose that $\sum a_n$ is divergent and all the other infinite series, $\sum x_n$, $\sum y_n$, ..., $\sum z_n$, are convergent. Let $A_n = \sum_{k=1}^n a_k$.

(a) Show that, for every $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that, simultaneously,

$$0 \le \frac{A_n x_n}{a_n} < \epsilon\,, \quad 0 \le \frac{A_n y_n}{a_n} < \epsilon\,, \quad \ldots\,, \quad 0 \le \frac{A_n z_n}{a_n} < \epsilon\,.$$

(b) From part (a), it is clear that if $\lim\limits_{n\to\infty} \dfrac{A_n x_n}{a_n}$ exists, it must have value zero. Construct an example of sequences as above such that the stated limit does not exist.

Solution by Achilleas Sinefakopoulos, student, University of Athens, Greece.

(a) Let $\epsilon > 0$ be given and let $X = \{n \in \mathbb{N} : 0 \le \frac{A_n x_n}{a_n} < \epsilon\}$, $Y = \{n \in \mathbb{N} : 0 \le \frac{A_n y_n}{a_n} < \epsilon\}$, ..., $Z = \{n \in \mathbb{N} : 0 \le \frac{A_n z_n}{a_n} < \epsilon\}$.

Suppose to the contrary that $X \cap Y \cap \ldots \cap Z = \emptyset$. Now, if $n \in \mathbb{N}$, then $n \notin X$ or $n \notin Y$, ..., or $n \notin Z$. Since $x_n \ge 0$, $y_n \ge 0$, ..., $z_n \ge 0$, in any case, we have

$$0 < \frac{a_n}{A_n} \le \frac{x_n + y_n + \cdots + z_n}{\epsilon}.$$

It is clear that the series $\sum \dfrac{x_n + y_n + \cdots + z_n}{\epsilon}$ converges. This, together with use of the comparison test, contradicts a theorem of Abel, which implies that the series $\sum(a_n/A_n)$ diverges. Thus, $X \cap Y \cap \ldots \cap Z \neq \emptyset$, and the proof of (a) is complete.

(b) For all positive $n \in \mathbb{N}$, set $x_n = \frac{1}{n}$ if n is a perfect square, and $x_n = \frac{1}{n^2}$ if n is not a perfect square.

Also, let $a_n = 1$ for all $n \in \mathbb{N}$. Then $\sum a_n$ is divergent, $\sum x_n$ is convergent (because its partial sums are bounded above by $2\sum_{n=1}^{\infty} \frac{1}{n^2}$), but $\lim_{n\to\infty} \frac{A_n x_n}{a_n}$ does not exist because $\liminf_{n\to\infty} \frac{A_n x_n}{a_n} = \liminf_{n\to\infty} n x_n = 0$ and $\limsup_{n\to\infty} \frac{A_n x_n}{a_n} = \limsup_{n\to\infty} n x_n = 1$. We are done.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA (part (a) only); and the proposer.

2443. [1999: 239] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Without the use of any calculating device, find an explicit example of an integer, M, such that $\sin(M) > \sin(33)(\approx 0.99991)$. (Of course, M and 33 are in radians.)

Solution by Václav Konečný, Ferris State University, Big Rapids, MI, USA.

$M = -11$ will do. Recall that $\sin(3A) = 3\sin(A) - 4\sin^3(A)$. Thus, $\sin(33) - \sin(-11) = 4\sin(11) - 4\sin^3(11) = 4\sin(11)\cos^2(11) < 0$, since the angle 11 is in the 4^{th} quadrant. [Ed: $\frac{7\pi}{2} < 11 < 4\pi$.] Thus $\sin(11) < 0$, and hence, $\sin(-11) > \sin(33)$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE and TREY SMITH (jointly), Angelo State University, San Angelo, TX, USA; SHAWN GODIN, Cairine Wilson S.S., Orleans, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL JOSEPHY, Universidad de Costa Rica, San José, Costa Rica; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; DIGBY SMITH, Mount Royal College, Calgary, Alberta; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

The answer $M = -11$ was also given by Diminnie and T. Smith, Josephy, and D. Smith. Other answers included $M = 322$, 366, 699, and 156522 (each given by two solvers); $M = -677$, 1032 (each given by one solver). Stone gave a 36-digit "monster" for his M, while the proposer's answer is another "monster": $M = \lfloor 10^{38}\pi \rfloor + 40$. Konečný commented that there are an infinite number of solutions and stated that for any given N, the number of *solutions M satisfying $0 < M < N$ is approximately $\frac{66-21\pi}{2\pi}N$, or $0.004226N$; for example, for $N = 10,000$, we expect 42 positive solutions, and he provided a computer print-out which actually lists all 42 solutions in the range $0 < M < 10,000$. From this list, one can see that the answers $M = 322$ and 366 given by some solvers are indeed the smallest positive solutions. Using a computer, he also found that the next solution less than -11 is $M = -344$.*

Both Godin and Josephy commented that, by examining the continued fraction expansion of π, one can get an infinite sequence of solutions M such that $\sin(M)$ tends to 1.

2444. [1999: 239] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Determine $\displaystyle\lim_{n\to\infty}\left(\frac{\ln(n!)}{n} - \frac{1}{n}\sum_{k=1}^{n}\ln(k)\left(\sum_{j=k}^{n}\frac{1}{j}\right)\right)$.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

Based on the identity $\displaystyle\sum_{k=1}^{n}\ln(k) = \ln(n!)$, a simple induction argument yields

$$\sum_{k=1}^{n}\ln(k)\left(\sum_{j=k}^{n}\frac{1}{j}\right) = \sum_{k=1}^{n}\frac{\ln(k!)}{k}.$$

It is easily verified that

$$\frac{\ln(n!)}{n} - \frac{1}{n}\sum_{k=1}^{n}\ln(k)\left(\sum_{j=k}^{n}\frac{1}{j}\right) = \frac{1}{n}\sum_{k=1}^{n}\left(\ln(k) - \frac{\ln(k!)}{k}\right). \qquad (1)$$

Stirling's Formula gives

$$\lim_{n\to\infty}\left(\ln(n) - \frac{\ln(n!)}{n}\right) = 1.$$

Hence, by a well-known result due to Cauchy, the right hand expression of (1) tends to 1 as n tends to infinity. From (1), it now follows that the required limit exists and equals 1.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; and the proposer.

The Editor was surprised at how many different ways this problem could be solved!

2445. [1999: 240] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let A, B be a partition of the set $C = \{q \in \mathbb{Q} : 0 < q < 1\}$ (so that A, B are disjoint sets whose union is C).

Show that there exist sequences $\{a_n\}$, $\{b_n\}$ of elements of A and B respectively such that $(a_n - b_n) \to 0$ as $n \to \infty$.

Solution by Jeremy Young, student, Nottingham High School, Nottingham, UK.

We assume that elements may recur in a sequence, since this is necessary if one of A, B is of finite size.

The question assumes A, B are nonempty. Therefore let a_1, b_1 be any elements of A, B respectively, and construct the sequences $\{a_n\}$, $\{b_n\}$ recursively. Given a_i, b_i, consider $(a_i + b_i)/2$:

- if $(a_i + b_i)/2 \in A$, let $a_{i+1} = (a_i + b_i)/2$ and $b_{i+1} = b_i$;
- if $(a_i + b_i)/2 \in B$, let $a_{i+1} = a_i$ and $b_{i+1} = (a_i + b_i)/2$.

Then $|a_{i+1} - b_{i+1}| = \frac{1}{2}|a_i - b_i|$. By induction,

$$|a_{n+1} - b_{n+1}| = \frac{1}{2^n}|a_1 - b_1| < \frac{1}{2^n},$$

which goes to 0 as $n \to \infty$.

Also solved by MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL JOSEPHY, Universidad de Costa Rica, San José, Costa Rica; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; TREY SMITH and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Josephy and Parmenter gave the same solution as Young.

2446. [1999: 240, 306] *Proposed by Catherine Shevlin, Wallsend upon Tyne, England.*

A sequence of integers, $\{a_n\}$ with $a_1 > 0$, is defined by

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \equiv 0 \pmod 4, \\ 3a_n + 1 & \text{if } a_n \equiv 1 \pmod 4, \\ 2a_n - 1 & \text{if } a_n \equiv 2 \pmod 4, \\ \frac{a_n+1}{4} & \text{if } a_n \equiv 3 \pmod 4. \end{cases}$$

Prove that there is an integer m such that $a_m = 1$.

(Compare **OQ**.117 in *OCTOGON*, vol 5, No. 2, p. 108.)

Solution by Nikolaos Dergiades, Thessaloniki, Greece, slightly modified by the editor.

The sequence $\{a_n\}$ is clearly bounded below, since no term can be negative or zero. Suppose that there is no natural number m for which $a_m = 1$. We will derive a contradiction by showing that for every term a_m of the sequence, there is a term $a_{m+j} < a_m$. This follows from the following observations:

If $a_m \equiv 0 \pmod 4$, then $a_m = 4k$ for some $k \neq 0$ and it follows that $a_{m+1} = 2k < a_m$.

If $a_m \equiv 3 \pmod 4$, then $a_m = 4k+3$ for some $k \geq 0$, so that we have $a_{m+1} = k + 1 < a_m$.

If $a_m \equiv 2 \pmod 4$, then $a_m = 4k+2$ for some $k \geq 0$, so that we have $a_{m+1} = 8k + 3$, giving $a_{m+2} = 2k + 1 < a_m$.

If $a_m \equiv 1 \pmod 4$, then $a_m = 4k+1$ for some $k > 0$, so that we have $a_{m+1} = 12k+4$ giving $a_{m+2} = 6k+2$. Hence, either $a_{m+3} = 3k+1 < a_m$ if $a_{m+2} \equiv 0 \pmod 4$, or $a_{m+3} = 12k + 3$ if $a_{m+2} \equiv 2 \pmod 4$. In the latter case, we have $a_{m+4} = 3k + 1 < a_m$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, NY, USA: DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; and the proposer. There was one partially incorrect solution.

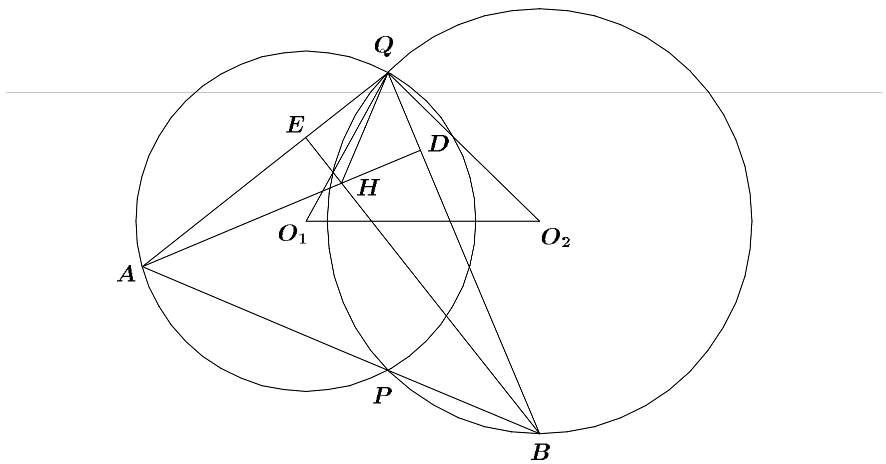
Most of the submitted solutions are similar to the one given. The Con Amore Problem Group also submitted a proof for the "incorrect" version [1999: 240].

2447. [1999: 240] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect at P and Q. A variable line through P meets the circles again at A and B. Find the locus of the orthocentre of triangle ABQ.

Solution by John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta.

Let O_1 and O_2 be the centres of the two circles.



Join Q to A, B, O_1 and O_2. Triangle QO_1O_2 is similar to triangle QAB (because $\angle QO_1O_2 = \frac{1}{2}\angle QO_1P = \angle QAB$ and $\angle QO_2O_1 = \frac{1}{2}\angle QO_2P = \angle QBA$). Consequently, triangle QAB is obtained from triangle QO_1O_2 by spiral similarity with centre Q and a rotation by an angle α equal to $\angle AQO_1$ and a similarity coefficient $k = QA/QO_1$.

In triangle ABQ, draw the altitudes from A and B, which intersect in H and meet QA and QB at E and D respectively. Consider triangle EQH, and let triangle $E'QH'$ be its equivalent in triangle QO_1O_2. We have $\angle EQH = \frac{\pi}{2} - \angle BAQ$, $QE = QB\cos(\angle AQB)$, $QH = 2R\cos(\angle AQB)$, where R is the circumradius of triangle ABQ.

It follows that triangle EQH is obtained by spiral similarity from triangle $E'QH'$ with centre Q, a rotation by angle α and similarity coefficient k.

The locus of H is then a circle passing through Q with the exclusion of the point Q itself.

Also solved by MICHEL BATAILLE, Rouen, France; NIELS BEJLEGAARD, Stavanger, Norway; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Some solvers did not explicity mention the exclusion of the point Q, but the editor has been generous.

Janous and Konečný both calculated the radius of the locus circle as

$$\frac{d(a^2+b^2-d^2)}{\sqrt{(a+b+d)(a+b-d)(b-a+d)(a-b+d)}},$$

where a and b are the radii of the given circles [containing A and B respectively], and $d = a\cos A + b\cos B$.

2448. [1999: 240] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Suppose that S is a circle, centre O, and P is a point outside S. The tangents from P to S meet the circle at A and B. Through any point Q on S, the line perpendicular to PQ intersects OA at T and OB at U. Prove that $OT \times OU = OP^2$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

First we note that points T and U exist only if $Q \neq A$ and $Q \neq B$. Let us consider the case when Q is on the major arc AB. The solution is similar when Q is on the minor arc AB.

We prove that $\triangle OPU$ is similar to $\triangle OTP$. Since PA and PB are tangents to the circle S, $\angle POA = \angle POB$. Also, $\angle AOU = \angle BOT$, so that

$$\angle UOP \;=\; \angle POT\,. \qquad (1)$$

Now, $APTQ$ is a cyclic quadrilateral, so

$$\angle OTP \;=\; \angle AQP \;=\; \angle AQB - \angle BQP\,.$$

Since $BPUQ$ is a cyclic quadrilateral, $\angle BQP = \angle BUP$. Also,

$$\angle AQB \;=\; \tfrac{1}{2}\angle AOB \;=\; \angle BOP\,.$$

Hence

$$\angle OTP \;=\; \angle BOP - \angle BUP \;=\; \angle OPU\,.$$

Thus

$$\angle OTP \;=\; \angle OPU\,. \qquad (2)$$

From (1) and (2), $\triangle OPU$ is similar to $\triangle OTP$. Therefore,

$$\frac{OT}{OP} \;=\; \frac{OP}{OU}\,,$$

which completes the proof.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (two solutions); MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2449. [1999: 240] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect at D and E. They are tangent to the sides AB and AC of $\triangle ABC$ at B and C respectively. If D is the mid-point of BC, prove that $DA \times DE = DC^2$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Since $\angle BED = \angle ABC$ and $\angle DEC = \angle ACB$, then

$$\angle BAC + \angle BEC \;=\; 180^\circ\,.$$

Therefore, the points A, B, E and C lie on a circle. Let AD meet this circle at the point A'. We have

$$\angle A'CB \;=\; \angle A'EB \;=\; \angle ABC$$

and

$$\angle A'BC \;=\; \angle A'EC \;=\; \angle ACB\,,$$

so that the triangles ABC and $A'BC$ are equal and then $DA' = DA$. By the Intersecting Chords Theorem, $DA' \cdot DE = DB \cdot DC$, or, $DA \cdot DE = DC^2$, as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Most of the solvers found the same or quite similar solution. There were also several solutions by analytic geometry and one by inversion. Woo proved that DB bisects the angle ADE. This implies $\triangle ADC$ and $\triangle CDE$ are similar, and the claim follows immediately.

2450. [1999: 240] *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Find the exact value of $$\frac{\displaystyle\sum_{k=0}^{\infty}\frac{(2k)!}{(k!)^3}}{\displaystyle\sum_{k=0}^{\infty}\frac{1}{(k!)^2}}\,.$$

I. *Solution by Michel Bataille, Rouen, France.*

Clearly, both of the given series have positive terms and are convergent (by ratio test).

Let $u_n = \dfrac{(2n)!}{(n!)^3}$ and $A = \sum\limits_{n=0}^{\infty} u_n$. Using the well-known result

$$\binom{2n}{n} \;=\; \sum_{k=0}^{n}\binom{n}{k}^2\,,$$

we have

$$u_n = \frac{1}{n!}\binom{2n}{n} = \sum_{k=0}^{n} \frac{1}{n!}\binom{n}{k}^2 = \sum_{k=0}^{n} \frac{1}{(k!)^2}\frac{n!}{((n-k)!)^2},$$

and so

$$A = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\frac{n!}{((n-k)!)^2}\right) = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\sum_{n=k}^{\infty}\frac{n!}{((n-k)!)^2}\right)$$

by rearrangement.

Let us compute $a_k = \sum_{n=k}^{\infty}\frac{n!}{((n-k)!)^2}$. We have $a_0 = \sum_{n=0}^{\infty}\frac{1}{n!} = e$, and for $k \geq 1$, $a_k = \sum_{n=k}^{\infty}\frac{n(n-1)\cdots(n-k+1)}{(n-k)!} = f^{(k)}(1)$, where $f^{(k)}$ denotes the k^{th} derivative of the function f given by

$$f(x) = x^k \cdot e^x = \sum_{n=0}^{\infty}\frac{x^{n+k}}{n!} = \sum_{n=k}^{\infty}\frac{x^n}{(n-k)!}.$$

Now, by Leibniz's formula, we have

$$\begin{aligned} f^{(k)}(x) &= \sum_{p=0}^{k}\binom{k}{p}(x^k)^{(p)}(e^x)^{(k-p)} = \sum_{p=0}^{k}\binom{k}{p}\frac{k!}{(k-p)!}x^{k-p}\cdot e^x \\ &= (k!)^2\, e^x \sum_{p=0}^{k}\frac{1}{p!((k-p)!)^2}\, x^{k-p}, \end{aligned}$$

and so $a_k = e(k!)^2\sum_{p=0}^{k}\frac{1}{p!((k-p)!)^2}$. Substituting in $A = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}a_k$ gives $\frac{A}{e} = \sum_{k=0}^{\infty}\left(\sum_{p=0}^{\infty}\frac{1}{p!((k-p)!)^2}\right)$. But this series is just the Cauchy product of the series $\sum_{k=0}^{\infty}\frac{1}{(k!)^2}$ and $\sum_{k=0}^{\infty}\frac{1}{k!}$, so its value is $e\sum_{k=0}^{\infty}\frac{1}{(k!)^2}$.

[Ed: Convergence of the product series is not a problem, since both series are absolutely convergent.]

It follows that $A = e^2\sum_{k=0}^{\infty}\frac{1}{(k!)^2}$ and hence the required value is e^2.

II. *Solution by the proposer.*

We make use of the identity $e^{\left(x+\frac{1}{x}\right)^2} \equiv e^2 \cdot e^{x^2} \cdot e^{1/x^2}$. The usual series expansion yields

$$\sum_{k=0}^{\infty} \frac{1}{k!}\left(x+\frac{1}{x}\right)^{2k} \equiv e^2\left(\sum_{k=0}^{\infty}\frac{x^{2k}}{k!}\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!x^{2k}}\right).$$

Equating the constant terms of both sides of the identity above then yields $\sum_{k=0}^{\infty}\frac{1}{k!}\binom{2k}{k} = e^2\left(\sum_{k=0}^{\infty}\frac{1}{k!}\right)^2$ and hence the required value is e^2. [Ed: If any reader feels that this "proof" is not rigorous, the following comments by the proposer himself might help you to accept a "compromise" attitude:

"*This argument shows a cavalier disregard for the finer points of rigorous analysis but is surely in the spirit of Euler and Gauss, and that is good enough for me!*"]

Also solved by CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany. Two other readers conjectured the correct answer, but were unable to supply a proof.

Both Janous and Lau obtained the answer as immediate consequence of the known results that $\sum_{k=0}^{\infty}\frac{x^{2k}}{(k!)^2} = I_0(2x)$ *and* $\sum_{k=0}^{\infty}\frac{(2k)!}{(k!)^3}x^k = e^{2x}I_0(2x)$, *where* $I_0(x)$ *denotes the modified Bessel function of the first kind. The answer* e^2 *is obtained by setting* $x = 1$ *in the identities above. They gave the following references in which these identities can be found.*

[1] *A.P. Prudnikov et al.*, Integrals and Series (Elementary Functions) [in Russian], *Moscow 1981.*

[2] *Eldon R. Hansen*, A Table of Series and Products, *Prentice Hall Inc. Englewood Cliffs, N.J. 1975.*

Lambrou gave a direct proof by using "term-by-term" differentiation of some power series. Seiffert's solution is based on Legendre's Doubling Formula for Beta functions and the known identity $\sum_{k=0}^{\infty}\frac{1}{(k!)^2} = \frac{1}{\pi}\int_0^{\pi} e^{-2\cos u}\,du$.