Jessie Lei

Jessie Lei was a member of Canada's International Mathematical Olympiad team for the past two years She was involved in a major highway accident in California on Christmas Day, and later died of her injuries on New Year's Day. The accident, along with some biographical information about Jessie, is described in the article (part of which is below) taken from the December 28 issue of the Windsor Star, and is re-printed with permission. She was obviously a student with enormous potential. She was also a warm and attractive person whom you instinctively liked. She will be greatly missed by the many people in the Olympiad programs who knew her.

Math star clings to life after crash

California freeway accident kills Windsor woman, hurts six

By Dave Battagello Star Staff Reporter

Jessie Lei, a Windsor student who received national attention for her academic excellence is clinging to life after an accident on a California freeway that killed her mother.

Ya Wu, 45, died when the van she was riding in with six other family members rolled over on Interstate 15 about 100 km northeast of Los Angeles near San Bernardino on Christmas morning.

Her daughter Yin (Jessie) Lei, 19, suffered severe head injuries and Monday was in critical condition. The accident occurred around 3:30 a.m. in Yermo.

Lei, who graduated in June from Vincent Massey Secondary School, recently received national academic recognition and was the focus of a number of stories in The Star.

Last summer, for the second straight year, the young woman was selected as one of six high school students from across Canada to compete in Romania at the International Math Olympiad, which involved 83 nations.

Lei just completed her first semester at the University of Toronto, where she has studied computer science.

The tragedy occurred just hours after the seven family members flew from Detroit to Los Angeles for a one-week vacation in California. The family arrived around 2 a.m., boarded a rented Dodge van and headed north along I-15 toward Las Vegas.

According to the coroner's report in San Bernardino, the accident occurred when the driver, Zhi Wu, possibly fell asleep and the van left the road.

The accident has left the local Chinese community and hundreds of friends in Windsor stunned and saddened.

"My reaction was total shock," said Jack Zhao, president of the Chinese Association of Greater Windsor, whose twin daughters were high school classmates and best friends of Jessie.

"Everyone who has heard it has ... cried. It's very painful, especially when you think about the mother and daughter."

Her former teacher and math coach at Massey was shocked.

"I saw her two days before she left," said Bruce White, head of the math department at Massey. "She was so happy they were going on holiday as a family."

"It's just a real tragedy. This girl is so bright and so nice. I feel really bad. It's a tragic loss for her to lose her mother."

Twins Jaing and Ying Zhao, both 19, said the tragedy has been especially difficult since their good friend is thousands of kilometres away.

"We all feel helpless," said Jaing Zhao. "She is so far away, it seems unreal and not happening."

"Jessie and her mom were very close. She always called her mom and talked to her about a lot of things. I just can't believe it."

Zhao said the Chinese Association of Greater Windsor has set up a fund to help the family with medical costs. For more information, call 519-969-9655.

Ying Zhao added: "It's hard not to be there to support her ... to be by her side, see her, talk to her. She has always been a fighter, so hopefully she'll pull through."

Unfortunately, it was not to be.

Ed Barbeau attended a memorial service for Jessie Lei at the University of Toronto that was held on Monday, 10 January 2000. It was attended largely by her fellow students at the Innis College Residence and in the Engineering Science Program. One of her aunts read a poem in Mandarin, and one of the eulogies was delivered by Jimmy Chui; there was a PowerPoint presentation of pictures, several of them of her with the Olympiad team. Also present was Richard Hoshino, David Arthur and his father, Ian VanDerBurgh and June Collins.

Vincent Massey Secondary School is planning to establish a scholarship in Jessie's honour. The school's address is 1800 Liberty Avenue, Windsor, Ontario.

THE ACADEMY CORNER

No. 30

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

This month, we have the 1999 Putnam Competition, reprinted with permission of the Mathematical Association of America.

SIXTIETH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 4, 1999

Problem A1.
Find polynomials $f(x)$, $g(x)$ and $h(x)$, if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if} \quad x < -1 \\ 3x+2 & \text{if} \quad -1 \leq x \leq 0 \\ -2x+2 & \text{if} \quad x > 0\,. \end{cases}$$

Problem A2.
Let $p(x)$ be a polynomial that is non-negative for all x. Prove that, for some k, there are polynomials $f_1(x)$, $\ldots$, $f_k(x)$ such that $p(x) = \sum_{j=1}^{k} (f_j(x))^2$.

Problem A3.
Consider the power series expansion $\dfrac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n$.

Prove that, for each integer $n \geq 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$.

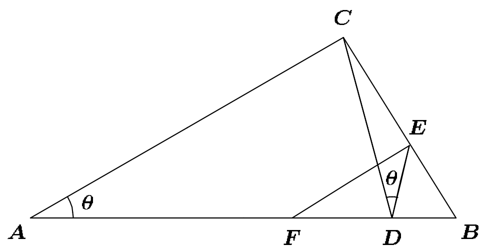
Problem A4.
Sum the series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \dfrac{m^2 n}{3^m\left(n3^m + m3^n\right)}$.

Problem A5.
Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then $|p(0)| \leq C \int_{-1}^{1} |p(x)|\,dx$.

Problem A6.
The sequence $\{a_n\}_{n\geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and, for $n \geq 4$, $a_n = \dfrac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$. Show that, for all n, a_n is an integer multiple of n.

Problem B1.

Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$. [Here, $|PQ|$ denotes the length of the line segment PQ.]



Problem B2.

Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots, then it must have n distinct roots. [The roots may be either real or complex.]

Problem B3.

Let $A = \{(x, y) : 0 \leq x, y < 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n ,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{\substack{(x,y) \to (1,1) \\ (x,y) \in A}} (1 - xy^2)(1 - x^2 y) S(x, y) .$$

Problem B4.

Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Problem B5.

For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{j,k})$ has entries $a_{j,k} = \cos(j\theta + k\theta)$ for all j, k.

Problem B6.

Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime. [Here, $\gcd(a, b)$ denotes the greatest common divisor of a and b.]

Send us your interesting solutions!

THE OLYMPIAD CORNER

No. 203

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

Here it is, the beginning of yet another year, decade, century and millennium(???). And we all survived the transition. It is time to recognize all the individual contributions which are such an integral part of the *Olympiad Corner*. First, I would like to thank Joanne Longworth, who somehow manages to compile my offerings into a readable LaTeX document, and to keep me nearly on schedule too. Secondly, I would like to acknowledge the excellent figures that are prepared by the Editorial Office.

The lifeblood of the *Corner*, however, is its readers, who provide me with challenging Olympiad materials, interesting solutions, and important generalizations and corrections. I would like to thank all these contributors and hope that I have not missed honourable mentions to anyone in the following compilation of 1999 *Corner* contributors.

Mohammed Aassila	Miguel Amengual Covas	Michel Bataille
Aart Blokhuis	Mansur Boase	Pierre Bornsztein
Christopher J. Bradley	Adrian Chan	Keon Choi
Jimmy Chui	J. Chris Fisher	Felipe Gago
J.P. Grossman	R.K. Guy	Yeo Keng Hee
Walther Janous	Masoud Kamgarpour	Geoffrey A. Kandall
Murray S. Klamkin	Marcin E. Kuczma	Pavlos Maragoudakis
David Nicholson	Bob Prielipp	Carl Johan Ragnarsson
Toshio Seimiya	Michael Selby	D.J. Smeenk
Daryl Tingley	Panos E. Tsaoussoglou	Ravi Vakil
	Edward T.H. Wang	

For our first problem set this issue we present the problems of the Final (selection) Round of the Estonian Mathematical contests 1995–96. My thanks go to J.P. Grossman for collecting them while he was Canadian Team Leader at the International Mathematical Olympiad at Mumbai, India.

ESTONIAN MATHEMATICAL CONTESTS 1995–96
Selected Problems
Final (selection) Round

1. The numbers x, y and $\frac{x^2+y^2+6}{xy}$ are positive integers. Prove that $\frac{x^2+y^2+6}{xy}$ is a perfect cube.

2. Let a, b, c be the sides of a triangle and α, β, γ the opposite angles of the sides respectively. Prove that if the inradius of the triangle is r then $a\sin\alpha + b\sin\beta + c\sin\gamma \geq 9r$.

3. Prove that the polynomial $P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$ has no zeros if n is even and has exactly one zero if n is odd.

4. Let H be the orthocentre of an obtuse triangle ABC and A_1, B_1, C_1 arbitrary points taken on the sides BC, AC, AB, respectively. Prove that the tangents drawn from the point H to the circles with diameters AA_1, BB_1, CC_1 are equal.

5. Find all functions $f : R \to R$ satisfying the following conditions for all $x \in R$:

(a) $f(x) = -f(-x)$;

(b) $f(x+1) = f(x) + 1$;

(c) $f(\frac{1}{x}) = \frac{1}{x^2}f(x)$, if $x \neq 0$.

6. Each face of a cube is divided into $n \times n$ equal squares. The vertices of the squares are called *nodes*, so each face of the cube has $(n+1)^2$ nodes.

(a) For $n = 2$, does there exist a closed broken line whose links are the edges of the squares which contains each node exactly once?

(b) For arbitrary n, prove that each such broken line divides the surface area of the cube into two equal parts.

To give you puzzling pleasure after our Christmas hiatus we next give the problems of the Final Round of the Japan Mathematical Olympiad, 1996. My thanks again go to J.P. Grossman, Canadian Team Leader at the IMO at Mumbai, India, for collecting them for our use.

JAPAN MATHEMATICAL OLYMPIAD
Final Round
11 February, 1996 (13:00 – 17:30)

1. We divide the plane by triangles. In other words, let $T = T^0 \cup T^1 \cup T^2$ be a simplicial decomposition of the plane. Let $\triangle ABC$ be a triangle with A, B, $C \in T^0$ ($=$ 0-skeleton of T), and let θ be the minimum value of $\angle A$, $\angle B$ and $\angle C$. We assume that no point of T^0 is contained inside of the circumcircle of $\triangle ABC$. Prove that there exists a triangle σ ($\in T^2$) of T such that $\sigma \cap \triangle ABC \neq \emptyset$ and that every inner angle of σ is greater than θ.

2. For positive integers m, n with $\gcd(m, n) = 1$, determine the value $\gcd(5^m + 7^m, 5^n + 7^n)$.

3. Let x be a real number with $x > 1$ and such that x is not an integer. Let $a_n = [x^{n+1}] - x[x^n]$ ($n = 1, 2, 3, \ldots$). Prove that the sequence of numbers $\{a_n\}$ is not periodic. (Here $[y]$ denotes, as usual, the largest integer $\leq y$.)

4. Let θ be the maximum of the six angles between six edges of a regular tetrahedron in space and a fixed horizontal plane. When we rotate the tetrahedron in space, determine the minimum value of θ.

5. Let q be a real number such that $\frac{1+\sqrt{5}}{2} < q < 2$. When we represent a positive integer n in binary expression as

$$n \;=\; 2^k + a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0$$

(here $a_i = 0$ or 1), we define p_n by

$$p_n \;=\; q^k + a_{k-1} q^{k-1} + \cdots + a_1 q + a_0 \,.$$

Prove that there exist infinitely many positive integers k which satisfy the following condition: There exists no positive integer l such that $p_{2k} < p_l < p_{2k+1}$.

We now turn to solutions from our readers to Final Round problems of the 8 th Korean Mathematical Olympiad [1998: 199].

8th KOREAN MATHEMATICAL OLYMPIAD
Final Round

1. For any positive integer m, show that there exist integers a, b satisfying

$$|a| \leq m \,, \quad |b| \leq m \,, \quad 0 < a + b\sqrt{2} \;\leq\; \frac{1+\sqrt{2}}{m+2} \,.$$

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's solution.

When a and b run through $0, 1, \dots, m$, then $a + b\sqrt{2}$ takes $(m+1)^2$ distinct values and the largest one is $m + m\sqrt{2}$. Dividing $[0, m + m\sqrt{2}]$ into $m^2 + 2m$ intervals of length $\frac{1+\sqrt{2}}{m+2}$ and using the Pigeonhole Principle, we deduce the existence of two elements (a_1, b_1), (a_2, b_2) with $a_1 + b_1\sqrt{2} > a_2 + b_2\sqrt{2} > 0$ and $\left(a_1 + b_1\sqrt{2}\right) - \left(a_2 + b_2\sqrt{2}\right) < \frac{1+\sqrt{2}}{m+2}$. Hence $a := a_1 - a_2$, $b := b_1 - b_2$ satisfy the required condition.

2. Let A be the set of all non-negative integers. Find all functions $f : A \to A$ satisfying the following two conditions:

(i) for any $m, n \in A$,

$$2f(m^2 + n^2) = \{f(m)\}^2 + \{f(n)\}^2 ;$$

(ii) for any $m, n \in A$ with $m \geq n$,

$$f(m^2) \geq f(n^2) .$$

Solution by Pierre Bornsztein, Courdimanche, France.

On résoud le problème en montrant que (ii) est inutile. Soit f une solution.

Pour $m = n = 0$, $f(0) = f^2(0)$, donc $f(0) = 0$ ou $f(0) = 1$.

Pour $m = 0$, $n = 1$, $2f(1) = f^2(1) + f^2(0)$. Trois possibilité se preésentent.

1^{er} **cas** : $f(0) = 0$ et $f(1) = 2$.

Pour tout $m \in A$,

$$2f(m^2) = f^2(m) \quad (1)$$
$$\text{et} \quad f(2m^2) = f^2(m) . \quad (2)$$

On calcule $f(m)$ pour $m \in \{0, 1, \dots, 11\}$. D'après (2)

$f(2) = f^2(1) = 4$,
donc d'après (1) $f(4) = 8$,
alors d'après (2) $f(8) = 16$.
Maintenant $2f(5) = 2f(1^2 + 2^2) = f^2(1) + f^2(2) = 20$,
donc d'après (i)
donc $f(5) = 10$ et $f(25) = 50$.

Puisque $2f(25) = 100 = 2f(4^2 + 3^2) = f^2(4) + f^2(3)$,
donc $f(3) = 6$,
et alors $f(9) = 18$.

Utilisant (i) $2f(10) = 2f(1^2 + 3^2) = f^2(1) + f^2(3)$
d'où $f(10) = 20$ et $f(100) = 200$.

En plus $2f(100) = 400 = 2f(6^2+8^2) = f^2(6)+f^2(8)$
donc $f(6) = 12$.

Vue $130 = 11^2+3^2 = 7^2+9^2$ et
$170 = 11^2+7^2 = 13^2+1^2$

nous avons d'après (i),

$$f^2(11)+f^2(3) = f^2(7)+f^2(9)$$
et $f^2(11)+f^2(7) = f^2(13)+f^2(1)$
et vue que $13 = 3^2+2^2$, donc
$f(13) = \frac{1}{2}(f^2(3)+f^2(2)) = 26$.

Par conséquent

$$f^2(11)+f^2(3) = f^2(11)+6^2 = f^2(7)+f^2(9) = f^2(7)+18^2,$$
$$f^2(11)+f^2(7) = (26)^2+f^2(1) = 26^2+2^2,$$

et donc $f(7)=14$, $f(11)=22$.

Finalement $f(m) = 2m$ pour $m \in \{0, \dots, 11\}$. Or, si $m = 2p+2$, $p \geq 4$ (et donc $m \geq 10$)

$$(2p+2)^2+(p-4)^2 = (2p-2)^2+(p+4)^2$$

avec $0 \leq p-4, p+4, 2p-2 < m$, donc, d'après (i)

$$f^2(2p+2) = f^2(2p-2)+f^2(p+4)-f^2(p-4).$$

Si $m = 2p+1$, $p \geq 2$ (et donc $m \geq 5$)

$$(2p+1)^2+(p-2)^2 = (2p-1)^2+(p+2)^2$$

avec $0 \leq p-1, p+2, 2p-1 < m$.
Donc, d'après (i),

$$f^2(2p+1) = f^2(2p-1)+f^2(p+2)-f^2(p-2).$$

Par suite, une recurrence immédiate conduit à $f(m) = 2m$ pour tout $m \in A$.

Réciproquement $f(m) = 2n$ vérifie bien (i) et (ii).

$2^{\text{ième}}$ **cas** : $f(0) = f(1) = 0$.

Exactement la même démarche conduit à $f \equiv 0$ (qui vérifie bien (i) et (ii)).

$3^{\text{ième}}$ **cas** : $f(0) = 1$ et alors $f(1) = 1$.

Pour tout $m \in A$

$$2f(m^2) = f^2(m)+1$$
$$\text{et} \quad f(2m^2) = f^2(m).$$

Exactement la même démarche conduit à $f \equiv 1$, (qui vérifie bien (i) et (ii)).

Conclusion : il y a exactement trois solutions

$$f \equiv 0\,, \quad f \equiv 1\,, \quad f(m) \;=\; 2m\,,$$

et (ii) est inutile.

3. Let $\triangle ABC$ be an equilateral triangle of side length 1, D a point on BC, and let r_1, r_2 be inradii of triangles ABD, ADC, respectively. Express $r_1 r_2$ in terms of $p = BD$, and find the maximum of $r_1 r_2$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Pierre Bornsztein, Courdimanche, France. We give the solution by Amengual Covas.

By the Law of Cosines, applied to $\triangle ABD$,

$$\overline{AD}^{\,2} \;=\; 1^2 + p^2 - 2 \cdot 1 \cdot p \cdot \cos 60^\circ\,,$$

so that

$$\overline{AD} \;=\; \sqrt{p^2 - p + 1}\,.$$

The area of $\triangle ABD$ may be expressed as $\frac{1}{2} \cdot 1 \cdot p \cdot \sin 60^\circ = \frac{p\sqrt{3}}{4}$ and also as $\frac{1+p+\sqrt{p^2-p+1}}{2} \cdot r_1$. Equating these and solving for r_1, we get

$$r_1 \;=\; \frac{\sqrt{3}\left(1 + p - \sqrt{p^2 - p + 1}\right)}{6}\,.$$

In the same way, (or substituting $1 - p$ for p into the above result) we find that the inradius r_2 of triangle ADC satisfies

$$r_2 \;=\; \frac{\sqrt{3}\left(2 - p - \sqrt{p^2 - p + 1}\right)}{6}\,.$$

Therefore,

$$r_1 r_2 \;=\; \frac{1 - \sqrt{p^2 - p + 1}}{4} \;=\; \frac{1 - \sqrt{(p - \frac{1}{2})^2 + \frac{3}{4}}}{4}\,.$$

Maximizing $\frac{1-\sqrt{(p-\frac{1}{2})^2+\frac{3}{4}}}{4}$ is equivalent to minimizing $(p - \frac{1}{2})^2 + \frac{3}{4}$. Since $(p - \frac{1}{2})^2 + \frac{3}{4}$ is a minimum when the square is zero, the product $r_1 r_2$ takes on its maximum value

$$\frac{2 - \sqrt{3}}{8} \quad \text{for} \quad p \;=\; \frac{1}{2}\,.$$

5. Let p be a prime number such that

(i) p is the greatest common divisor of a and b;

(ii) p^2 is a divisor of a.

Prove that the polynomial $x^{n+2} + ax^{n+1} + bx^n + a + b$ cannot be decomposed into the product of two polynomials with integral coefficients, whose degrees are greater than one.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's remark.

This is a simple consequence of Eisenstein's criterion.

6. Let m, n be positive integers with $1 \le n \le m - 1$. A box is locked with several padlocks, all of which must be opened to open the box, and all of which have different keys. m people each have keys to some of the locks. No n people of them can open the box, but any $n + 1$ people can open the box. Find the smallest number l of locks, and in that case find the number of keys that each person has.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give the solution of Aassila.

Let us mark on each lock the fingerprints of the people who do not hold a key for that lock. Since any $n + 1$ people can open the box, no lock has more than $n + 1$ fingerprints on it. For any n people, there is at least one lock with the corresponding n fingerprints. Thus the smallest number l of locks is $\binom{m}{n}$ and each person has $\binom{m-1}{n}$ keys.

Now we turn to solutions by readers to selected problems from the Israel Mathematical Olympiads 1995 [1998: 200–201].

3. Two thieves stole an open chain with $2k$ white beads and $2m$ black beads. They want to share the loot equally, by cutting the chain to pieces in such a way that each one gets k white beads and m black beads. What is the minimal number of cuts that is always sufficient?

Solution by Mohammed Aassila, Strasbourg, France.

The minimal number of cuts is 2. Indeed, let the beads be 1, 2, $\dots$, $2k + 2m$ along the chain. Let $w(x)$ be the number of white beads among x, $x + 1$, $\dots$, $x + k + m$ for $0 \le x \le 2k + 2m - 1$. Since every white bead is counted exactly $k + m$ out of $2k + 2m$ times, the average value of $w(x)$ is k. Hence the value k must occur between the values x for which $w(x) \ge k$ and $w(x) \le k$ (since w changes by at most 1 at each step). We remove the corresponding segment with the cuts, give the segment to one thief and give the ends of the chain to the other thief.

5. Four points are given in space, in a general position (that is, they are not contained in a single plane). A plane π is called "an equalizing plane" if all four points have the same distance from π. Find the number of equalizing planes.

Solution by Pierre Bornsztein, Courdimanche, France.

Soit π un tel plan pour les points A, B, C, D. Comme A, B, C, D ne sont pas coplanaires, ils ne peuvent pas être du même côté de π.

Soit A, B, C d'un côté et D de l'autre.

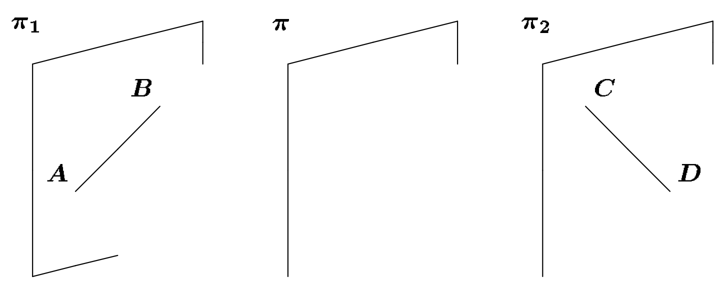
Alors π est parallèle au plan (ABC) et passe par le milieu de $[DH]$, où H est le projeté orthogonal de D sur (ABC). Or, il y a un unique plan vérifiant ces deux conditions. Réciproquement le plan défini ci-dessus convient de façon évidente. Il y a donc 4 plans possible selon le choix du point "isolé".

Soit A, B d'un côté, C, D de l'autre.

Comme $ABCD$ ne sont pas coplanaires, (AB) et (CD) ne sont pas non plus coplanaires.

Soient π_1 le plan contenant AB et parallèle à (CD) ; π_2 le plan contenant (CD) et parallèle à (AB).

On a alors nécessairement $\pi \| \pi_1$ et $\pi \| \pi_2$ et π équidistant de π_1 et π_2.



Or, il y a un unique plan vérifiant ces trois conditions (et réciproquement). Il y a trois plans possible, selon le choix du point couplé avec A.

Finalement : Il y a 7 "equalizing planes".

6. Let n be a given positive integer. A_n is the set of all points in the plane, whose x and y coordinates are positive integers between 0 and n. A point (i, j) is called "internal" if $0 < i, j < n$. A real function f, defined on A_n, is called a "good function" if it has the following property: for every internal point x, the value of $f(x)$ is the mean of its values on the four neighbouring points (the neighbouring points of x are the four points whose distance from x equals 1). f and g are two given good functions and $f(a) = g(a)$ for every point a in A_n which is not internal (that is, a boundary point). Prove that $f \equiv g$.

Solution by Mohammed Aassila, Strasbourg, France.

Let ∂A_n denote the boundary of A_n. By hypothesis $f = g$ on ∂A_n, and our goal is to prove that $f = g$ everywhere. To this end we will prove that $\min(f - g) = 0 = \max(f - g)$.

Assume that $\max(f - g)$ is achieved at a point m_0. If m_0 is an internal point, then the value of $f - g$ is the mean of its value on the four neighbouring points of m_0. But these have values at most equal to $\max(f - g)$. So, each of them achieves the maximum.

However, we conclude that the maximum is achieved at a point of ∂A_n, and then $\max(f - g) = 0$.

Similar arguments give $\min(f - g) = 0$.

Finally, we conclude $f \equiv g$.

7. Solve the system

$$\begin{aligned} x + \log(x + \sqrt{x^2 + 1}) &= y \,, \\ y + \log(y + \sqrt{y^2 + 1}) &= z \,, \\ z + \log(z + \sqrt{z^2 + 1}) &= x \,. \end{aligned}$$

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

It is clear that $x = y = z = 0$ is a solution and if any one of x, y, and z is zero, then they are all zeros. Assume then $xyz \neq 0$. Note that if t is a real number such that $t > 0$, then $t + \sqrt{t^2 + 1} > 1 \Longrightarrow \log(t + \sqrt{t^2 + 1}) > 0$. On the other hand, if $t < 0$, then

$$\begin{aligned} -2t > 0 \quad &\Longrightarrow \quad t^2 + 1 < (1 - t)^2 \\ &\Longrightarrow \quad \sqrt{t^2 + 1} < 1 - t \quad \Longrightarrow \quad t + \sqrt{t^2 + 1} < 1 \\ &\Longrightarrow \quad \log(t + \sqrt{t^2 + 1}) < 0 \,. \end{aligned}$$

Label the three given equations as (1), (2), and (3). If $x > 0$, then we get $y > x > 0$ by (1), $z > y > 0$ by (2) and $x > z$ by (3). Thus $x > z > y > x$, which is a contradiction. Similarly, if $x < 0$, then we get $y < x < 0$ by (1), $z < y < 0$ by (2) and $x < z$ by (3). Thus $x < z < y < x$, which is again a contradiction.

Therefore, $x = y = z = 0$ is the only solution.

Comment: A very cute problem indeed!

8. Prove the inequality

$$\frac{1}{kn} + \frac{1}{kn+1} + \frac{1}{kn+2} + \cdots + \frac{1}{(k+1)n - 1} \geq n \left(\sqrt[n]{\frac{k+1}{k}} - 1 \right)$$

for any positive integers k, n.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

Soient k, $n \in \mathbb{N}^*$. On a

$$\begin{aligned} &\frac{1}{n}\left(\frac{1}{kn}+\cdots+\frac{1}{kn+n-1}+n\right) \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{1}{kn+i}+1\right) = \frac{1}{n}\sum_{i=0}^{n-1}\frac{kn+i+1}{kn+i} \\ &\underset{(\text{AM/GM})}{\geq} \sqrt[n]{\prod_{i=0}^{n-1}\frac{kn+i+1}{kn+i}} = \sqrt[n]{\frac{kn+n}{kn}} = \sqrt[n]{\frac{k+1}{k}} \end{aligned}$$

c.à.d.

$$\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{kn+i} \geq \sqrt[n]{\frac{k+1}{k}}-1$$

d'où le résultat.

Remarque : On a égalité si et seulement si $n = 1$, car pour $n \geq 2$, il y a au moin deux termes distincts lors de l'utilisation de AM/GM.

10. α is a given real number. Find all functions $f : (0,\infty) \mapsto (0,\infty)$ such that the equality

$$\alpha x^2 f\left(\frac{1}{x}\right) + f(x) = \frac{x}{x+1}$$

holds for all real $x > 0$.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Aassila.

We have

$$\begin{aligned} \alpha x^2 f\left(\frac{1}{x}\right) + f(x) &= \frac{x}{x+1} \\ \alpha\frac{1}{x^2}f(x) + f\left(\frac{1}{x}\right) &= \frac{1}{x+1} \end{aligned}$$

(with $\frac{1}{x}$ instead of x) and hence

$$\begin{cases} \alpha x f(\frac{1}{x}) + \frac{1}{x}f(x) &= \frac{1}{x+1} \\ \alpha\frac{1}{x}f(x) + xf(\frac{1}{x}) &= \frac{x}{x+1}\,. \end{cases}$$

If $\alpha^2 \neq 1$, then we obtain

$$f(x) = \frac{x(1-\alpha x)}{(x+1)(1-\alpha^2)}$$

and if $\alpha^2 = 1$, there is no solution.

Since $f(x) > 0$, it is easy to see that α must belong to $(-1, 0)$.

We now turn to the September 1998 number of the *Corner* and solutions by our readers to problems of the 45th Latvian Mathematical Olympiad [1998: 263–264].

45th LATVIAN MATHEMATICAL OLYMPIAD, 1994
11th Grade

1. Prove for each choice of real non-zero numbers a_1, a_2, a_3, the "stars" can be replaced by "$<$" and "$>$" so that the system

$$\begin{cases} a_1x + b_1y + c_1 * 0 & (1) \\ a_2x + b_2y + c_2 * 0 & (2) \\ a_3x + b_3y + c_3 * 0 & (3) \end{cases}$$

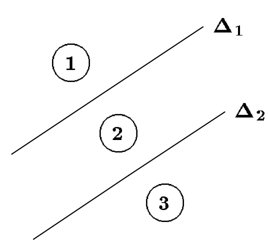
has no solution.

Solution by Pierre Bornsztein, Courdimanche, France.
Let Δ_i be the line with equation $a_ix + b_iy + c_i = 0$ (we have $(a_i, b_i) \neq (0, 0)$).

Let π_i^+ be the open half-plane with equation $a_ix + b_iy + c_i > 0$ and π_i^- be the open half-plane with equation $a_ix + b_iy + c_i < 0$.

Case 1. Two of the lines are parallel.

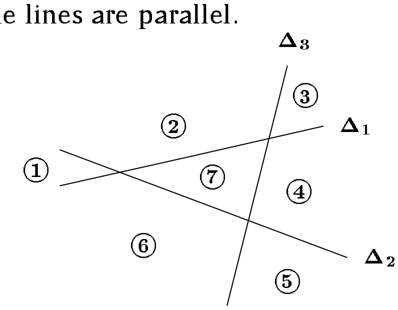
Without loss of generality, we may take Δ_1 and Δ_2 to be parallel.



We can select the $*$ for (1) so that (1) is the region and for (2) so that (3) is determined.

Then there is no point in the intersection and the system has no solution (even if $\Delta_1 = \Delta_2$ because the inequalities are strict).

Case 2. No two of the lines are parallel.



The intersections between $\pi_i^{\mp}$ and $\pi_j^{\pm}$ for $i \neq j$ determine 7 regions. We replace the stars in (1) and (3) to determine the region labelled ③. Then replace the star in (2) so that the solution must lie on the opposite side of Δ_2 (that is, ①, ⑥ or ⑤). Then the system has no solution.

2. Solve in natural numbers:

$$x(x+1) = y^7 .$$

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

We show more generally that if $n > 1$ is a natural number, then $x(x+1) = y^n$ has no solutions in natural numbers. Clearly, $y \neq 1$. Let p denote any prime divisor of y. Then we must have $p^n \mid x$ or $p^n \mid x+1$ as x and $x+1$ are coprime. Thus x and $x+1$ must each be an n^{th} power. That is, $x = b^n$, $x+1 = a^n$ for some natural numbers a and b with $a > b$. If $n > 1$, then from $a^n - b^n = 1$ we get $(a-b)(a^{n-1}+a^{n-2}b+\cdots+b^{n-1}) = 1$, which is clearly impossible since $a - b \geq 1$ and $a^{n-1} + a^{n-2}b + \cdots + b^{n-1} > 1$.

Comment by Amengual Covas. The following group of results closely related to this problem appears on page 86 of W. Sierpinski's *Elementary Theory of Numbers*.

1° The product of any three consecutive natural numbers cannot be a power with exponent greater than 1 of a natural number.

2° The product of k consecutive natural numbers with $k > 1$ cannot be the square of a natural number.

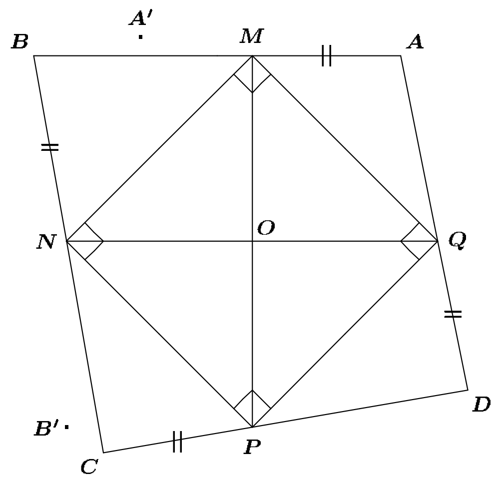
3° The product of k consecutive odd natural numbers with $k > 1$ cannot be a power with exponent > 1 of a natural number.

4° For natural numbers $m > 1$ and a sufficiently large k the product of k consecutive natural numbers cannot be the m^{th} power of a natural number.

5° For natural numbers $k \geq 3$ and $n \geq 2k$ the number $\binom{n}{k}$ cannot be a power with exponent greater than 1 of a natural number.

4. Let $ABCD$ be a convex quadrilateral, $M \in AB$, $N \in BC$, $P \in CD$, $Q \in DA$; $AM = BN = CP = DQ$, and $MNPQ$ is a square. Prove that $ABCD$ is a square, too.

Solution by Pierre Bornsztein, Courdimanche, France.



Let O be the centre of $MNPQ$, and r the rotation with centre O and angle $\pm\frac{\pi}{2}$ such that $r(M) = N$. We have

$$\begin{array}{rclcrcl} M & \to & N & & & & \\ N & \to & P & & A & \to & A' \\ P & \to & Q & & B & \to & B' \\ Q & \to & M\,. & & & & \end{array}$$

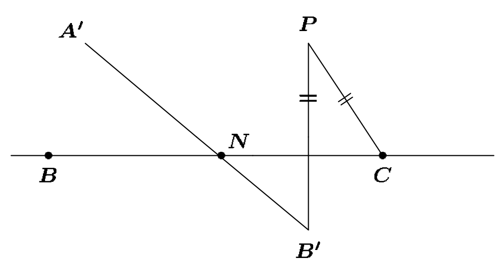
Then $(NB) \to (PB')$ so $(NB) \perp (PB')$. Thus

$$(NC) \;\perp\; (PB')\,. \tag{1}$$

Moreover

$$\begin{aligned} PB' &= NB \\ &= PC\,. \end{aligned} \tag{2}$$

From (1) and (2), we deduce that $B' \in (BC)$ or B' is on the other side of (BC) than P.



But $A' \in (B'N)$ and $A' \notin (BN)$ so that A' is on the same side of (BC) as M.

$$\text{So} \quad \angle MNA' \leq \angle MNB .$$

But rotations preserve angles, so that $\angle QMA = \angle MNA'$.

$$\text{Thus} \quad \angle QMA \leq \angle MNB ,$$

and by symmetry

$$\angle QMA \leq \angle MNB \leq \angle NPC \leq \angle PQD \leq \angle QMA .$$

Thus, they are all equal to a common value, θ, say. From this

$$\angle BMN = \frac{\pi}{2} - \angle QMA = \frac{\pi}{2} - \angle MNB = \angle CNP .$$

Similarly,

$$\angle BMN = \angle CNP = \angle DPQ = \angle AQM = \frac{\pi}{2} - \angle QMA .$$

Then $\angle A = \angle B = \angle C = \angle D = \frac{\pi}{2}$, and $ABCD$ is a rectangle. Further, by Pythagoras' Theorem,

$$BM^2 = MN^2 - BN^2 = MQ^2 - AM^2 = AQ^2 .$$

Thus $BM = AQ$ and $BA = MB + MA = AQ + QD = AD$. Then the rectangle is a square.

5. A square consists of $n \times n$ cells, $n \geq 2$. A letter is inserted into each cell. It is given that every two rows differ. Prove that there is a column which can be deleted from the square so that all rows are again different after this deletion.

Comment by Pierre Bornsztein, Courdimanche, France. See Ross Honsberger, "In Polya's Footsteps", pp. 138–140.

12 th Grade

1. Solve the equation $\cos x \cdot \cos 2x \cdot \cos 3x = 1$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Letting $y = \cos 2x$ and using the formula

$$
\begin{aligned}
2\cos A\cos B &= \cos(A+B) + \cos(A-B) \quad \text{we have} \\
& \cos x \cdot \cos 2x \cdot \cos 3x = 1 \\
\Longleftrightarrow\ & (\cos 2x)(\cos 4x + \cos 2x) = 2 \\
\Longleftrightarrow\ & (\cos 2x)(2\cos^2 2x - 1 + \cos 2x) = 2 \\
\Longleftrightarrow\ & 2y^3 + y^2 - y - 2 = 0 \\
\Longleftrightarrow\ & (y-1)(2y^2 + 3y + 2) = 0 \\
\Longleftrightarrow\ & y = 1 \quad (\because 2y^2 + 3y + 2 \text{ has no real roots}) \\
\Longleftrightarrow\ & \cos 2x = 1 \Longleftrightarrow 2x = 2k\pi \Longleftrightarrow x = k\pi\,,
\end{aligned}
$$

where k is any integer.

2. All faces of a convex polytope are triangles. What can be the number of the faces?

Solution by Pierre Bornsztein, Courdimanche, France.

Let f be the number of faces, and e the number of edges.

For each face there are 3 edges, but every edge belongs to 2 faces, so $2e = 3f$. So f is even. It is clear that $f \geq 4$.

For $f = 4$, the regular tetrahedron is a solution.

For $f = 2 \cdot n$, $n \geq 3$, choose n points, $A_1, \ldots, A_n$, forming a regular n–gon in a plane. Select two other points, P_1, P_2 one on each side of the plane of the n–gon lying on the perpendicular to the plane through O, the centre of $A_1 \ldots A_n$. This gives a convex polytope with $2n$ faces which are all triangles.

3. Does there exist a polynomial $P(x, y)$ in two variables such that

(a) $P(x, y) > 0$ for all x, y?

(b) for each $c > 0$ there exist x and y such that $P(x, y) = c$?

Solution by Pierre Bornsztein, Courdimanche, France.

Yes! Let $P(x, y) = (y^2+1)x^2+2xy+1$. For a fixed y, $P(x, y) = f_y(x)$ a polynomial in x of degree two with discriminant $\Delta = -4$. Hence for all x, y, $P(x, y) > 0$.

For a fixed y, $f_y(\mathbb{R}) = [M(y), +\infty)$ where $M(y) = \frac{1}{y^2+1}$. We have $\lim_{y \to +\infty} M(y) = 0$.

Then, for each $c > 0$ there is $y_0 \in \mathbb{R}$ such that $c \geq M(y_0)$, so there is x_0 with $P(x_0, y_0) = c$.

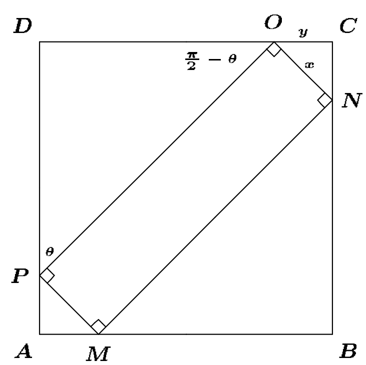
Comment by Mohammed Aassila, Strasbourg, France. This problem is similar to the one proposed by Finland but not used by the jury of the 28th IMO. A solution and comment appeared in ***CRUX*** [1989: 35].

To close this number of the *Corner*, we turn to solutions by the readers to problems of the Dutch Mathematical Olympiad 1994 [1998: 264–265].

DUTCH MATHEMATICAL OLYMPIAD
Second Round
16 September, 1994

1. A unit square is divided in two rectangles in such a way that the smallest rectangle can be put on the greatest rectangle with every vertex of the smallest on exactly one of the edges of the greatest. Calculate the dimensions of the smallest rectangle.

Solution by Pierre Bornsztein, Courdimanche, France.



The problem is to calculate $x = ON = PM$ (we have $OP = MN = 1$). Let $\theta = \angle NOC$, $y = OC$. Then $\angle DOP = \frac{\pi}{2} - \theta$, etc.

We deduce that

$$NOC \text{ and } PMA \text{ are isometric (congurent)} \qquad (1)$$

$$NOC\,,\ OPD \text{ are similar.} \qquad (2)$$

We have $NC^2 = ON^2 - OC^2$. Then $NC = PA = \sqrt{x^2 - y^2}$ (and $y < x$). So we get

$$\begin{aligned} OD &= 1 - x - y \\ DP &= 1 - PA = 1 - \sqrt{x^2 - y^2}\,, \end{aligned} \qquad (3)$$

and

$$DP^2 = OP^2 - OD^2 = 1 - (1 - x - y)^2\,. \qquad (4)$$

From (3) and (4) we get

$$2\sqrt{x^2 - y^2} = 2x^2 - 2x + 1 - 2y + 2xy\,. \qquad (5)$$

But from (2), $\frac{CN}{ON} = \frac{OD}{OP}$, which gives

$$\sqrt{x^2 - y^2} \;=\; x(1 - x - y)\,. \tag{6}$$

By (5) and (6)

$$2y(1 - 2x) \;=\; (1 - 2x)^2\,.$$

Then $x = \frac{1}{2}$ or $y = \frac{1}{2} - x$.

If $x = \frac{1}{2}$ then $y < \frac{1}{2}$ and from (6)

$$\sqrt{\left(\frac{1}{2} - y\right)\left(\frac{1}{2} + y\right)} \;=\; \frac{1}{2}\left(\frac{1}{2} - y\right)\,,$$

and then

$$\frac{1}{2} + y \;=\; \frac{1}{4}\left(\frac{1}{2} - y\right)\,,$$

so $y < 0$, a contradiction, so $x \neq \frac{1}{2}$.

Thus $y = \frac{1}{2} - x$, or $x + y = \frac{1}{2}$.

Using (6) again, we obtain

$$\left(2x - \frac{1}{2}\right)\frac{1}{2} \;=\; \frac{1}{4}x^2\,;$$

that is, $x^2 - 4x + 1 = 0$ with $x < 1$.

Thus $x = 2 - \sqrt{3}$.

3. (a) Prove that every multiple of 6 can be written as the sum of four third powers of integers.

(b) Prove that every integer can be written as the sum of five third powers of integers.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution and comments.

(a) For all $k \in \mathbb{Z}$, we have $6k = (k+1)^3 + (k-1)^3 + (-k)^3 + (-k)^3$.

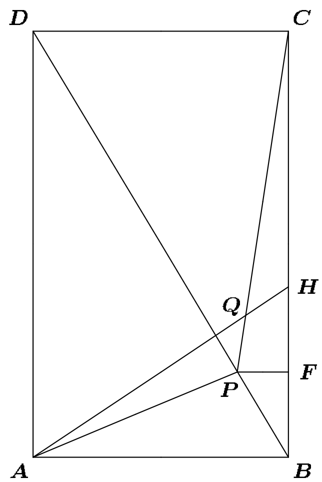
(b) For any $k \in \mathbb{Z}$, let $k = 6q + r$ where $0 \le r < 6$. Then for all $m \in \mathbb{Z}$, we have $(6m+r)^3 - k = (6m+r)^3 - (6q+r) \equiv r^3 - r = r(r-1)(r+1) \equiv 0 \pmod 6$ since r, $r-1$, and $r+1$ are three consecutive integers. Hence $k = (6m+r)^3 + t$ for some t such that $6 \mid t$, and the conclusion follows from (a).

Remarks. (1) In general, the representation given in (a) is not the only way of expressing a multiple of 6 as the sum of three cubes of integers; for example

$$18 \;=\; 4^3 + 2^3 + (-3)^3 + (-3)^3 \;=\; 3^3 + (-2)^3 + (-1)^3 + 0^3\,.$$

(2) Both (a) and (b) are well-known results and can be found in, for example §12, Chapter XI of *Elementary Theory of Numbers* by Sierpinski.

4. Let P be any point on the diagonal BD of a rectangle $ABCD$. F is the projection of P on BC. H lies on BC such that $BF = FH$. PC intersects AH in Q.



Prove: Area $\triangle APQ =$ Area $\triangle CHQ$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; by Vedula N. Murty, Dover, PA, USA; and by Challa K.S.N. Murali Sankar, Andbra Pradesh, India. We give the solution by Murali Sankar.

Join AC and PH.

Since F is the midpoint of BH, $\triangle PBH$ is an isosceles triangle, and

$$\angle PBH = \angle DBC = \angle PHB . \qquad (1)$$

Triangles $\triangle ACB$ and $\triangle DCB$ are congruent. So

$$\angle DBC = \angle ACB . \qquad (2)$$

From (1) and (2)

$$\angle PHB = \angle ACB . \qquad (3)$$

Thus PH is parallel to AC, and the area of $\triangle APC$ = area of $\triangle ACH$. $[APC] - [AQC] = [ACH] - [AQC]$. Thus $[APQ] = [CHQ]$, as required. ($[XYZ]$ means, as usual, the area of $\triangle XYZ$.)

5. Three real numbers a, b and c satisfy the inequality:

$$|ax^2 + bx + c| \leq 1 \qquad \text{for all } x \in [-1, +1] .$$

Prove: $|cx^2 + bx + a| \leq 2$ for all $x \in [-1, +1]$.

Comments and solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; by Vedula N. Murty, Dover PA, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's remarks.

This was problem No. 2153 in ***CRUX*** [1996 : 217] with solutions on page [1997 : 313]. As pointed out by many solvers in the published solutions, this problem has appeared many times in various mathematics contests in the past.

That is all for this number of the *Corner*. Send me Olympiad contests and your nice solutions and generalizations.

Awards of Subscriptions to *CRUX with MAYHEM*

Because of the generosity of a regular subscriber to ***CRUX with MAYHEM***, the Canadian Mathematical Society is very grateful to be able to award some complimentary one year subscriptions to assist students in some developing countries. The criterion for the award is good performance by a student from that country at the IMO.

For the year 2000, two subscriptions have been awarded to the schools attended by students from Indonesia and Thailand. We wish the teachers and students from these schools "Happy Problem Solving", and good luck for the 2000 IMO.

- SEKOLAH MENENGAH UMUM NEGERI NO. 1
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The Editorial Board of ***Crux Mathematicorum with Mathematical Mayhem*** congratulates the students involved, and welcomes these schools to the ***CRUX with MAYHEM*** family.

BOOK REVIEWS

ALAN LAW

IX and XIX International Mathematical Olympiads, edited by Vladimir Janković and Vladimir Mićić,
published by the Mathematicians Society of Serbia, Belgrade, 1997.
ISBN 86-81453-18-1, softcover, 85+ pages.
Reviewed by **Richard Hoshino**, *student, University of Waterloo, Waterloo, Ontario.*

This book commemorates the two International Mathematical Olympiads that took place in Yugoslavia. The names of all of the IMO team members and leaders are listed, as well as the summary of results from the two Olympiads that were held in Yugoslavia. Cetinje in Montenegro was the host of the 1967 IMO and Belgrade was the host of the 1977 IMO.

More importantly for problem-solving enthusiasts, there are 119 problems in this book, including the 12 problems which were on the 1967 and 1977 IMO's. All of the problems that were considered for either of the IMO's, but were not selected for the paper, are also in this commemorative publication. Some of the proposed problems are excellent questions which will keep students interested for hours. The following is just a small sample of some of these problems.

1. Let n be a positive integer. Find the maximal number of non-congruent triangles whose side lengths are integers less than or equal to n.

2. Suppose $\tan x = \frac{p}{q}$, where p and q are positive integers and $q \neq 0$. Prove that the number $\tan y$ for which $\tan 2y = \tan 3x$ is rational only when $p^2 + q^2$ is a perfect square.

3. Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7$, where n is an integer?

4. If x, y, and z are real numbers satisfying $x + y + z = 1$ and $\arctan x + \arctan y + \arctan z = \frac{\pi}{4}$, prove that $x^{2n+1} + y^{2n+1} + z^{2n+1} = 1$ holds for all positive integers n.

5. In a group of interpreters, each one speaks one or several foreign languages. 24 of them speak Malaysian, 24 Japanese, and 24 Farsi. Prove that it is possible to select a subgroup in which exactly 12 interpreters speak Malaysian, exactly 12 speak Japanese, and exactly 12 speak Farsi.

6. Prove that if $c_1, c_2, \ldots, c_n$ (with $n \geq 2$) are real numbers such that

   $$(n-1)(c_1^2 + c_2^2 + \cdots + c_n^2) = (c_1 + c_2 + \cdots + c_n)^2$$

   then either all these numbers are non-negative, or all these numbers are non-positive.

7. The numbers $1, 2, 3, \ldots, 64$ are placed on a chessboard, one number in each square. Consider all subsquares of size 2 by 2 of the chessboard. Prove that there are at least three such squares for which the sum of the 4 contained numbers exceeds 100.

The four mini-reviews below were provided by **Ed Barbeau**, *University of Toronto, Toronto, Ontario*. They are reviews of books he found in Bucharest in summer 1999, while he was attending the Romanian International Mathematical Olympiad (IMO).

1. *144 Problems of the Austrian-Polish Mathematics Competition* compiled and with solutions by Marcin E. Kuczma, University of Warsaw, published by the Academic Distribution Center, 1216 Walker Road, Freeland, MD 21053.
   ISBN 0-9640959-0-4.

   There are sixteen sets of nine problems from a major competition, complete with solutions by a master problemist. To order a copy, contact Walter Mientke at <walter@amc.unl.edu> or at American Mathematics Competitions Publications, PO Box 81606, Lincoln NE 68501-1606.

2. *Baltic Way, 1990–1996: Mathematical Team Competitions* edited by Marcus Better, Department of Mathematics, Stockholm University.

   There are seven papers of twenty questions each, all with solutions. This is an excellent source of olympiad-level training problems.

3. *The German Teams at the International Mathematical Olympiads, 1959–1998* by Wolfgang Engel, Hans-Dietrich Gronau, Hans-Heinrich Langmann and Horst Sewerin, published by Verlag Heinrich Bock, 53604 Bad Honnef, Germany
   ISBN 3-87066-755-9.

   While there is a lot of specific information about the German IMO teams, this volume is of general interest as it provides results of all participating countries, statistics about awards received and participation in each year, names of particularly successful students and recipients of World Federation of National Mathematics Competitions Awards, logos for each IMO and regulations for the IMO.

4. *Notes on Geometry* by Dan Barnzei, published by Editura Paralela 45, str. Fratii Golesti 29, Pitesti 0300 <ep45@pitesti.ro>

   This is an extensive collection of geometry results and examples, with many problems left for the reader, sorted under forty headings. The bibliography includes 288 books in Romanian, 371 books in other languages, and 206 selected papers.

THE SKOLIAD CORNER

No. 43

R.E. Woodrow

We begin the first issue of the new Millennium with the problems from Part I of the Alberta High School Mathematics Competition written in November, 1999. Thanks go to Ted Lewis, University of Alberta, for forwarding this contest for use in the *Corner*.

THE ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION
PART I
November 16, 1999

1. Subtracting 99% of 19 from 19% of 99, the difference d satisfies

(a) $d < -1$ **(b)** $d = -1$ **(c)** $-1 < d < 1$ **(d)** $d = 1$ **(e)** $d > 1$

2. Suppose you multiply three different positive prime numbers together and get a product which is greater than 1999. The smallest possible size of the largest of your primes is

(a) 11 **(b)** 13 **(c)** 17 **(d)** 19 **(e)** undefined

3. Suppose you multiply three different positive prime numbers together and get a product which is greater than 1999. The largest possible size of the smallest of your primes is

(a) 3 **(b)** 5 **(c)** 7 **(d)** 11 **(e)** undefined

4. The number of two-digit positive integers such that the difference between the integer and the product of its digits is 12 is

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** none of these

5. The non-zero slope of a certain straight line is equal to its y–intercept if and only if the x–intercept a satisfies

(a) $a = 1$ **(b)** $a = -1$ **(c)** $a > 0$ **(d)** $a < 0$ **(e)** none of these

6. A and B are positive integers. The sum of the digits of A is 19. The sum of the digits of B is 99. The smallest possible sum of the digits of the number $A + B$ is

(a) 1 **(b)** 19 **(c)** 20 **(d)** 118 **(e)** none of these

7. O is the origin of the coordinate plane. A, B and C are points on the x–axis such that $OA = AB = BC = 1$. D, E and F are points on the y–axis such that $OD = DE = EF \geq 1$. If $CD \cdot AF = BE^2$, then OD is

(a) 1 (b) $\sqrt{7}$ (c) 7 (d) 49 (e) none of these

8. The integer closest to $100(12 - \sqrt{143})$ is

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

9. A bag contains four balls numbered -2, -1, 1 and 2. Two balls are drawn at random from the bag, and the numbers on them are multiplied together. The probability that this product is either odd or negative (or both) is

(a) $\frac{1}{6}$ (b) $\frac{1}{2}$ (c) $\frac{9}{16}$ (d) $\frac{2}{3}$ (e) $\frac{3}{4}$

10. The number of positive perfect cubes which divide 9^9 is

(a) 6 (b) 9 (c) 18 (d) 27 (e) none of these

11. In the quadratic equation $x^2 - 14x + k = 0$, k is a positive integer. The roots of the equation are two different prime numbers p and q. The value of $\frac{p}{q} + \frac{q}{p}$ is

(a) 2 (b) $\frac{106}{45}$ (c) $\frac{130}{33}$ (d) $\frac{170}{13}$ (e) none of these

12. In the quadrilateral $ABCD$, AB is parallel to CD, $AB = 4$ and $BC = CD = 9$. X is on BC and Y is on DA such that XY is parallel to AB. If the quadrilaterals $ABXY$ and $YXCD$ are similar, distance BX is

(a) 3 (b) 3.6 (c) 5.4 (d) 6 (e) none of these

13. The country of Magyaria has three kinds of coins, each worth a different integral number of dollars. Matthew collected four Magyarian coins with a total worth of 28 dollars, while Daniel collected five with a total worth of 21 dollars. Each had at least one Magyarian coin of each kind. In dollars, the total worth of the three kinds of Magyarian coins is

(a) 16 (b) 17 (c) 18 (d) 19 (e) none of these

14. Colin wants a function f which satisfies $f(f(x)) = f(x + 2) - 3$ for all integers x. If he chooses $f(1)$ to be 4 and $f(4)$ to be 3, then he must choose $f(5)$ to be

(a) 3 (b) 6 (c) 9 (d) 12 (e) 15

15. Lindsay summed all the integers from a to b, including a and b. She chose these numbers so that $1 \leq a \leq 10$ and $11 \leq b \leq 20$. This sum cannot be equal to

(a) 91 (b) 92 (c) 95 (d) 98 (e) 99

16. A set of points in the plane is such that each of the numbers 1, 2, 4, 8, 16 and 32 is a distance between two of the points in the set. The minimum number of points in this set is

(a) 4 (b) 5 (c) 6 (d) 7 (e) more than 7

Last issue we gave the thirty problems of Maxi éliminatoire 1996 of the 21[ième] Olympiade Belge organized by the Belgian Mathematics Teachers' Association. Thanks go to Ravi Vakil for collecting this set when he was Canadian Team Deputy Leader to the International Mathematical Olympiad at Mumbai, India. Here are the answers.

1.	c	**2**.	Six	**3**.	b	**4**.	c	**5**.	243
6.	c	**7**.	d	**8**.	c	**9**.	72m	**10**.	c
11.	d	**12**.	c	**13**.	b	**14**.	d	**15**.	b
16.	c	**17**.	a	**18**.	d	**19**.	d	**20**.	b
21.	a	**22**.	e	**23**.	b	**24**.	e	**25**.	d
26.	c	**27**.	Fourteen	**28**.	e	**29**.	e	**30**.	a

Editor's note: One arrives at **29**. (e) by elimination, (a) is not allowed because of $x = 0$, none of (b), (c) (d) work because of $x = \pi$, since the expression is not identically zero. This leaves (e).

Articles

Readers will have noticed a lack of articles in recent issues. There are two reasons for this:

1. a lack of submitted articles;
2. most of those submitted have been at too high a level.

We do want to publish a ***CRUX*** article in each issue. So we warmly invite submissions. But please read the mandate of **CRUX with MAYHEM**, which is printed on the inside back cover:

> **Crux Mathematicorum with Mathematical Mayhem** *is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics.*

So, please keep the level appropriate.

We welcome the new Articles Editor, Bruce Gilligan. Please send him your articles now.

Also, at this time we welcome Iliya Bluskov to the Board as a Problems Editor. Iliya has been a participant to ***CRUX*** since his student days, and has also helped out with editing problems for the past two years.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), and David Savitt (Harvard University)

Shreds and Slices

A Fifth Way to Count

Mrs. Luyun Zhong-Qiao adds a fifth way to solve the problem posed in Jimmy Chui's article "Four Ways to Count" [1999 : 235-237]:

Evaluate

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}.$$

Solution 5. Let the given sum be equal to S. Consider the following array:

$$\begin{array}{ccccc}
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & & \binom{n}{n}
\end{array}$$

The sum of all entries is simply $(n+1)2^n$.

The sum of the diagonal elements is

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n .$$

The sum of the numbers in the upper-right triangle is

$$\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = S .$$

The sum of the numbers in the lower-left triangle is

$$\begin{aligned} & n\binom{n}{0} + (n-1)\binom{n}{1} + \cdots + \binom{n}{n-1} \\ = \; & \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} \\ = \; & S . \end{aligned}$$

Therefore,

$$2S + 2^n = (n+1)2^n \implies S = n2^{n-1} .$$

Our apologies to Qiao, first for misspelling her name, and second for referring to her as "he".

Errata

Several errors crept into the article "An Identity of a Tetrahedron", Murat Aygen [1999 : 422-425]. The three middle equations on p. 424 should read:

Then the equations above become

$$\begin{aligned} (B'C')^2 &= a^2[1 + \tan^2\phi \, \sin^2(B-\theta)] \\ &= \frac{a^2[h^2 + (r_1 - r_2)^2 + 4r_1 r_2 \sin^2(B-\theta)]}{h^2 + (r_1 - r_2)^2} \\ &= \frac{a^2 a_1^2}{h^2 + (r_1 - r_2)^2} , \\ (AC')^2 &= b^2[1 + \tan^2\phi \, \sin^2(A+\theta)] \\ &= \frac{b^2 b_1^2}{h^2 + (r_1 - r_2)^2} , \\ (BC')^2 &= c^2[1 + \tan^2\phi \, \sin^2\theta] \\ &= \frac{c^2 c_1^2}{h^2 + (r_1 - r_2)^2} . \end{aligned}$$

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 2 of 2001.

High School Problems

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H265. Find all triangular numbers that can be expressed as the sum of 18 consecutive integers, with the smallest of these integers being a perfect square.

H266. Consider a polynomial $f(x)$ with integer coefficients such that $f(0) = p$, where p is a prime.

(a) What is the maximum number of lattice points lying on the line $y = x$ that the graph of $f(x)$ can pass through?

(b) What are the specific lattice points in (a)?

H267. Find all solutions (a, b, c) to the following system:

$$\frac{b-c}{a} = \frac{c-a}{b} = \frac{a-b}{c}.$$

H268. Let ABC be a triangle such that $\angle ACB = 3\angle ABC$ and $AB = \frac{10}{3}BC$. Evaluate $\cos A + \cos B + \cos C$.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A241. *Proposed by Mohammed Aassila, CRM, Montréal, Québec.*

Prove that every power of 2 has a multiple whose decimal representation contains only the digits 1 and 2.

A242. Find all solutions of the equation $a^m + b^m = (a + b)^n$ in positive integers.

A243. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Suppose that the integers a and $a+1997$ are roots of $P(x)$ and that $Q(1998) = 2000$. Prove that the equation $Q(P(x)) = 1$ has no integer solutions.

(1997 Baltic Way)

A244. *Proposed by Ravi Vakil.*

Prove that $x^{2000} + y^{2000} = z^{2000}$ has no solution in positive integers (x, y, z). (See "A Do-It-Yourself Proof of the $n = 4$ case of Fermat's Last Theorem", [1999 : 502–504].)

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C91. Let P_n be the set of all partitions of the positive integer n, and let $P(n)$ be the size of the set P_n. (That is, P_n consists of all unordered collections of positive integers $(m_1, \ldots, m_k)$ such that $m_1 + \cdots + m_k = n$. For example, P_4 is the set $\{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}$, and so $P(4) = 5$.) Fix a one-to-one correspondence between the elements of P_n and the integers from 1 to $P(n)$. For a non-negative integer N, define v_N to be the vector (of length $P(n)$) whose j^{th} entry is equal to the number of solutions in non-negative integers $x_1, \ldots, x_k$ to the equation $m_1x_1 + \cdots + m_kx_k = N$, where $(m_1, \ldots, m_k)$ is the partition corresponding to j under our one-to-one correspondence. Let W_n be the sub-vector space of $\mathbb{Q}^{P(n)}$ spanned by all the v_N, $N \geq 0$. Prove that

$$\dim W_n \;\leq\; \frac{n(n-1)}{2} + 1\,.$$

(Note that $\dim W_n$ is independent of the choice of one-to-one correspondence, as changing the correspondence merely permutes the entries of the v_N's.)

C92. Do there exist arbitrarily long finite arithmetic progressions which contain only square-free integers? (An integer n is said to be square-free if 1 is the only perfect square which divides n.)

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Solve the equation for x:

$$(x^2 - 3x + 1)^2 - 3(x^2 - 3x + 1) + 1 = x .$$

(1996 Euclid, Problem 8)

Solution 1. Expanding and rearranging the given equation yields

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = 0 .$$

Let us try to factor the left side quartic into two quadratics of the form

$$(x^2 + ax + 1)(x^2 + bx - 1) = x^4 + (a + b)x^3 + abx^2 + (b - a)x - 1 .$$

Equating coefficients yields the three equations

$$\begin{aligned} a + b &= -6 , \\ ab &= 8 , \\ b - a &= 2 . \end{aligned}$$

The first and third equations together have the solution $a = -4$ and $b = -2$, and this solution does satisfy the second equation. So the given equality is

$$(x^2 - 4x + 1)(x^2 - 2x - 1) = 0$$

which has the solutions $x = 2 \pm \sqrt{3}$ and $x = 1 \pm \sqrt{2}$.

Solution 2. Adding $x^2 - 3x + 1$ to both sides of the given equation gives

$$\begin{aligned} & (x^2 - 3x + 1)^2 - 2(x^2 - 3x + 1) + 1 &= x^2 - 2x + 1 \\ \Longleftrightarrow\quad & [(x^2 - 3x + 1) - 1]^2 &= (x - 1)^2 \\ \Longleftrightarrow\quad & (x^2 - 3x)^2 - (x - 1)^2 &= 0 \\ \Longleftrightarrow\quad & (x^2 - 4x + 1)(x^2 - 2x - 1) &= 0 , \end{aligned}$$

and the solution follows as above.

J.I.R. McKnight Problems Contest 1993

1. If an octahedron has vertices at $A(1,0,0)$, $B(0,1,0)$, $C(-1,0,0)$, $D(0,-1,0)$, $E(0,0,1)$, and $F(0,0,-1)$, determine the following:

   (a) its surface area;

   (b) its volume;

   (c) the equation in scalar form of the plane containing points A, B, and C;

   (d) the distance from D to the plane containing A, B, and C.

2. Lines are drawn on a square so that no two of the lines are parallel and no three of the lines meet at a point.

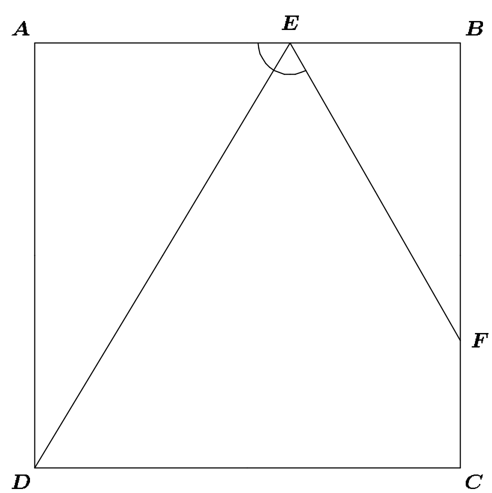
   (a) Conjecture a formula for the maximum number of regions of the square formed by n of these lines.

   (b) Prove your conjecture using the Principle of Mathematical Induction.

3. A straight line, $y = mx + k$, intersects a parabola, $y = ax^2 + bx + c$, at two distinct points. Show that the average of the slopes of the parabola at the 2 points of intersection is equal to the slope of the straight line.

4. Three positive integers form a geometric sequence whose common ratio is an integer. When 9 is added to the middle term of these three integers an arithmetic sequence is formed. Find all possible sets of three integers which satisfy these conditions.

5. $ABCD$ is a square with point E on AB and F on BC so that $\angle AED = \angle DEF$. Prove $EF = AE + FC$.



6. If x and y are real numbers such that $x + y = 1$ and $x^3 + y^3 = 4$ determine

   (a) $x^2 + y^2$;

   (b) $x^5 + y^5$.

7. Consider the function defined for $0 \leq x \leq 1$ by the equation $f(x) = -kx(x-1)$, where $0 < k \leq 4$.

   (a) Consider the curves $y_1 = f(x)$ and $y_2 = f(f(x))$. For what k, $0 \leq k \leq 4$, will these 2 curves have:

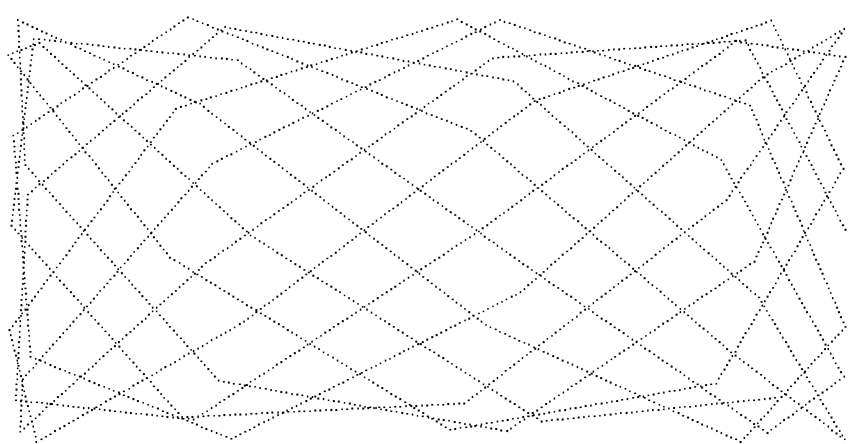
      (1) 2 points of intersection, for the range $0 \leq x \leq 1$?

      (2) 3 points of intersection, for the range $0 \leq x \leq 1$?

      (3) 4 points of intersection, for the range $0 \leq x \leq 1$?

   (b) Find the value of k for which some of these points of intersection are orthogonal intersections. (That is, the curves intersect at right angles.)

8. Let $ABCD$ be a rectangle with $AB = a$ and $BC = b$. Two circles are drawn, the first circle of radius r_1 passes through the points C and D and is tangent to AB and the second circle of radius r_2 is tangent to BC and passes through A and D. Show that $r_1 + r_2 \geq \frac{5}{8}(a+b)$.



A Do-It-Yourself Proof of the $n = 3$ case of Fermat's Last Theorem

Ravi Vakil

Introduction

As explained in [FLT4] (see [1999 : 502]), Fermat's Last Theorem has had a lasting impact on mathematics because the tools and concepts developed in various attempts have turned out to be extraordinarily effective elsewhere in number theory. The culmination of these efforts, Wiles's wonderful proof of the Taniyama-Shimura-Weil conjecture, has not just dispatched the most famous conjecture of all time, but also fundamentally changed the landscape of number theory.

For the record, on the off chance that the reader has spent the last several centuries on a desert island, here is Fermat's Last Theorem.

Theorem (Wiles, Wiles-Taylor). If n, x, y, and z are integers with $n \geq 3$ and $x^n + y^n = z^n$, then $xyz = 0$.

When referring to Fermat's Last Theorem for a fixed n, we will use the short-form "FLTn".

In [FLT4], a Do-It-Yourself proof of FLT4 was given. The second-easiest case, FLT3, is also reasonable enough for the ambitious reader to figure out, although more hints may be helpful (and this article is admittedly harder than most in ***MATHEMATICAL MAYHEM***). Many important ideas come up that figure prominently in number theory, and mathematics in general. We will summarize these ideas at the very end.

Warning: This is a very interactive article! You will really have to try all of the problems. If you get stuck, then skip to the next one. Even getting stuck is a good thing (much better than not trying at all), because the seeds of ideas that come up in one problem invariably turn up again in a later one. If you successfully solve most of the problems, congratulations — you have figured out part of one of the most important problems in the history of humankind! (And if you have also proved the $n = 4$ case, [FLT4], then you will have proved "half" of Fermat's Last Theorem — you will have proofs for "half" the positive integers n!)

The only advanced background you will need is some experience with complex numbers. Ambitious high school students should be able to tackle it. Warmup questions are marked with a "W"; many of them will be used later, so do not skip them!

The questions giving the actual proof are marked with a "P". The argument is not as long as it looks, as you will see from the summary at the end.

Unique Factorization Revisited

Recall the Fundamental Theorem of Arithmetic, also known as the "Unique Factorization Theorem": *Every positive integer can be uniquely factored into prime numbers.* Let us rephrase this in unusual, more general terms, so it works for negative integers as well. We will need to extend some well-known definitions.

If n is a non-zero integer, then another integer m is *divisible* by n if m/n is also an integer. So, for example, 18 is divisible by -3.

Call 1 and -1 *units* of the integers. These are the integers that every other integer is divisible by; they are the integers of absolute value 1.

Let us define *primes* to be integers p that are not units, and are divisible only by ± 1 and $\pm p$. Thus -3 is a prime. But a number is divisible by 3 if and only if it is divisible by -3, so we will consider them to be "almost" the same thing. Define two primes to be *associates* if one is a unit times the other. So the primes 3 and -3 are associates. In general, we say that two integers are *associates* if one is a unit times the other.

Then the Unique Factorization Theorem can be written as follows: Any non-zero integer n can be uniquely factored into primes, up to units (so $(-2) \times 3$ and $2 \times (-3)$ are considered to be the same factorization).

W1. Suppose that x, y, and z are pairwise relatively prime integers, and that $xyz = w^2$ for some non-zero integer w. Show that $x = \alpha a^2$, $y = \beta b^2$, and $z = \gamma c^2$ for some units α, β, and γ, and integers a, b, and c. (This may seem an awkward way to phrase this result, but bear with it . . . .) There was no real reason to require w to be non-zero, except to save you the hassle of dealing with the special case $w = 0$ in your proof.

W2. Suppose that x, y, and z are pairwise relatively prime integers, and that $xyz = w^3$ for some non-zero integer w. Show that $x = \alpha a^3$, $y = \beta b^3$, and $z = \gamma c^3$ for some units α, β, and γ, and integers a, b, and c.

Because we are dealing with cubes, we can do a little better: The α, β, and γ can be incorporated into the a^3, b^3, and c^3:

W3. Suppose that x, y, and z are pairwise relatively prime integers, and that $xyz = w^3$ for some non-zero integer w. Show that $x = a^3$, $y = b^3$, and $z = c^3$ for some integers a, b, and c.

A Twist on the Integers: F–integers

We will define a new set of numbers that behaves a lot like the integers. They will be complex numbers. You will have to recall how to add, subtract, multiply, and divide complex numbers, and take complex conjugates: $\overline{a+bi} = a - bi$. Recall also the definition of the norm $N(z)$ of a complex number z (named after a character on the popular television show "Cheers"):

$$N(a+bi) \;=\; (a+bi)\overline{(a+bi)} = a^2 + b^2\,.$$

You can check that $N(z_1z_2) = N(z_1)N(z_2)$. The absolute value of a complex number z is $|z| = \sqrt{N(z)}$: $|a+bi| = \sqrt{a^2+b^2}$.

Let $\omega = (-1+i\sqrt{3})/2$. ($\omega$ is the Greek letter *omega*.)

W4. Check that $\omega^3 = 1$ and $\omega^2 = -1-\omega$. Calculate $(1-\omega)(1-\omega^2)$. Show that $(1-\omega^2) = -\omega^2(1-\omega)$, $\overline{\omega} = \omega^2$, and $3 = -\omega^2(1-\omega)^2$. Expand $(x+y)(x+\omega y)(x+\omega^2 y)$. (We will use all of these later, so make sure to do them!)

Let the set of complex numbers of the form $a + b\omega$ (where a and b are integers) be called F–integers. (This is a non-standard name, so do not use it in public.)

You already know how to add and subtract and multiply them, because they are just complex numbers:

$$\begin{aligned}(a+b\omega)+(c+d\omega) &= (a+c)+(b+d)\omega\,,\\ (a+b\omega)-(c+d\omega) &= (a-c)+(b-d)\omega\,.\end{aligned}$$

You can multiply them without translating back to i's, using $\omega^2 = -1-\omega$ (W4). For example, $(a+b\omega)(c+d\omega)$ can be calculated by just expanding out and simplifying:

$$\begin{aligned}(a+b\omega)(c+d\omega) &= ac + ad\omega + bc\omega + bd\omega^2\\ &= ac + (ad+bc)\omega + bd(-1-\omega)\\ &= (ac-bd) + (ad+bc-bd)\omega\,.\end{aligned}$$

You can find norms too:

$$\begin{aligned}N(a+b\omega) &= (a+b\omega)\overline{(a+b\omega)} = (a+b\omega)(a+b\omega^2)\\ &= a^2 + ab\omega + ab\omega^2 + b^2 = a^2 + b^2 - ab\,.\end{aligned}$$

(Aside: Does this remind you of the Cosine Law with a triangle with sides a and b, and enclosed angle 60 degrees?)

You can picture the F–integers as a triangular grid in the complex plane (see Figure 1).

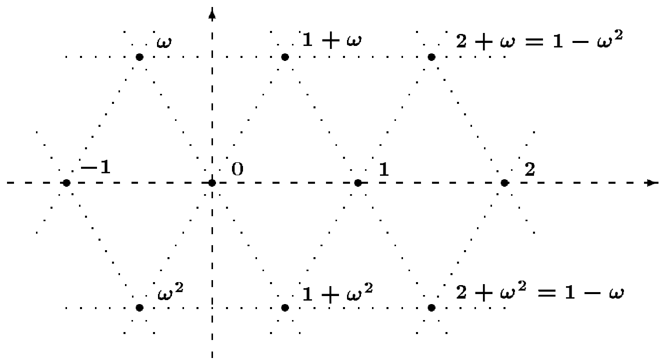


Figure 1.

Most things that you can do with integers, you can also do with F–integers. For example, while there are two units in the integers, there are actually six units in the F–integers (the F–integers with norm 1). Two F–integers are *associates* if one is a unit times the other.

W5. Find the six units of the F–integers (Hint: See Figure 1.)

An F–integer is *prime* if it is not a unit, and it is only divisible by its associates and the units. Some of these primes might look surprising to you. For example, 2 is prime, but 3 is not a prime F–integer, because it factors:

$$3 = (1-\omega)(1-\omega^2)\,.$$

The F–integers $1-\omega$ and $1-\omega^2$ are both prime. In fact, they are associates — they differ by the unit ω^2 (see W4). The prime number $1-\omega$ will be a big player in the proof of FLT3.

W6. Show that $1+2\omega = i\sqrt{3}$ is a prime number that is an associate of $1-\omega$. (Hint: Find the unit that associates them.)

W7. Show that $1-\omega$ is prime as follows. If it is composite, and thus factors into two non-units x and y, then $N(x)N(y) = N(1-\omega)$. What are the possible values of $N(x)$ and $N(y)$? Why does one of x and y have to be a unit?

W8. 7 is not a prime F–integer. Can you factor it?

We can use modular arithmetic on F–integers as well. Recall that for two ordinary integers a and b to be congruent modulo n, there must exist an integer r such that $a-b = nr$. Congruences for F–integers work exactly the same way: If α and β are two F–integers, then they are congruent modulo the F–integer ν if and only if there exists an F–integer ρ such that $\alpha-\beta = \nu\rho$.

W9. For an ordinary integer n, check that the F–integers $a+b\omega$ and $c+d\omega$ are congruent modulo n if and only if $a \equiv c \pmod{n}$ and $b \equiv d \pmod{n}$ as ordinary integers.

W10. Confirm that the two notions of congruence coincide: That is, if a, b, and n are ordinary integers, then $a \equiv b \pmod{n}$ as F–integers if and only if $a \equiv b \pmod{n}$ by the usual notion of congruence.

Therefore, we can speak about congruence without having to specify whether we are talking about congruence as F–integers or about regular congruence: They mean the same thing!

W11. Show that the cube of an F–integer is congruent to an integer modulo 3. (Hint: Expand $(a + b\omega)^3$ and see what you get.)

In other words, $\rho^3 \equiv 0$, 1, or $2 \pmod{3}$ (and not ω, $1 + \omega$, $2 + \omega$, 2ω, $1 + 2\omega$, $2 + 2\omega \pmod{3}$). (Aside: More generally, if ρ is an F–integer not divisible by $1 - \omega$, then $\rho^3 \equiv \pm 1 \pmod{9}$, which can be rewritten $\rho^3 \equiv \pm 1$ $(\mathrm{mod}\ (1-\omega)^4)$, although we will not need it, [D] p. 104. This should be seen as similar to the fact that if ρ is an integer not divisible by 2, then $\rho^2 \equiv \pm 1$ $(\mathrm{mod}\ 2^3)$.)

F–integers and Unique Factorization

F–integers also have unique factorization. In other words, given any F–integer, you can uniquely factor it into primes (up to the 6 units). (This is the only part of the argument that will not be done here. It is not hard, but it relates to other fun problems, so it will be discussed in [UF].) We say two F–integers are *relatively prime* if they have no common prime factor.

We now have results about factoring cubes into relatively prime terms that are similar to W2 and W3 for integers.

W12. Suppose that x, y, and z are pairwise relatively prime F–integers, and that $xyz = w^3$ for some non-zero F–integer w. Show that $x = \alpha a^3$, $y = \beta b^3$, and $z = \gamma c^3$ for some integers a, b, and c, and units α, β, and γ (that is, α, β, $\gamma \in \{1, \omega, \omega^2, -1, -\omega, -\omega^2\}$ — oops, I just gave away the answer to W5!). Remember, use unique factorization.

W13. Suppose that x, y, and z are pairwise relatively prime F–integers, and that $xyz = w^3$ for some non-zero F–integer w. Show that $x = \alpha a^3$, $y = \beta b^3$, and $z = \gamma c^3$ for some integers a, b, c, where α, β, and γ are each in the set $\{1, \omega, \omega^2\}$.

Fermat's Last Theorem ($n = 3$ case)

You are now ready to begin figuring out the proof! You will use the warm-ups, but the main ideas about F–integers that you will need are: Comfort with multiplying them, the cubic factorization problem W13, and the prime $1 - \omega$.

Notice that to show FLT3, we need only show that $\pm a^3 \pm b^3 \pm c^3 = 0$ has no solutions in integers a, b, and c unless $abc = 0$. (We can pick our signs in any way we wish. Choosing $+$, $+$, and $-$ gives FLT3 the way it is usually seen.)

P1. Show that if there is a solution, then there is a solution where a, b, and c are pairwise relatively prime. From here on in, we will assume that this is the case.

P2. Show that there are no solutions when none of a, b, or c is divisible by 3. (Hint: Look modulo some number. You will find that 3 will not work. Numbers like 2 and 5 will not help. So what should you try?)

So we can assume that one of them is divisible by 3. So we want to show that there are no solutions to

$$x^3 + y^3 \;=\; (3^m z)^3\,, \tag{1}$$

where x, y, and z are integers, pairwise relatively prime, and not divisible by 3, and m is a non-negative integer. We have already taken care of the case $m = 0$ (P2), so suppose from now on that $m > 0$.

Now $x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y)$ (see W4), which is a factorization in the F–integers! For convenience, let $A = x + y$, $B = x + \omega y$, and $C = x + \omega^2 y$. Then A, B, and C multiply to a cube! Could they be pairwise relatively prime? Sadly, they are not.

P3. Show that the prime number $1 - \omega$ divides A, B, and C as follows. $1 - \omega$ divides 3, which divides A. $1 - \omega$ divides $x + y$ and $(\omega - 1)y$, so it divides their sum B. Do something similar with C.

So they are not pairwise relatively prime. But they are almost as good. Aside from the common factor of $1 - \omega$, they *are* relatively prime.

P4. Show that $x = -\omega(A - \omega^2 B)/(1 - \omega)$ and $y = (A - B)/(1 - \omega)$. Hence, if A and B have a common factor on top of $1 - \omega$, then x and y would too. Show the analogous fact for the pairs (B, C) and (C, A). This means that $\gcd(A, B) = \gcd(B, C) = \gcd(C, A) = 1 - \omega$.

P5. We have seen that $x + y = A$ is divisible by 3. Using $ABC = (3^m z)^3$ and the previous problem P4, show that

$$\left\{\begin{array}{rcl} x + y &=& \eta_0 3^{3m-1} \rho_0^3\,, \\ x + \omega y &=& \eta_1 (1 - \omega) \rho_1^3\,, \\ x + \omega^2 y &=& \eta_2 (1 - \omega) \rho_2^3\,, \end{array}\right. \tag{2}$$

where each η_i is one of the units 1, ω, or ω^2, and ρ_0, ρ_1, and ρ_2 are relatively prime F–integers not divisible by the prime $1 - \omega$. (Hint: $x + y$ is divisible by two factors of $(1 - \omega)^2$ from $3 = -\omega^2(1 - \omega)^2$, and $x + \omega y$ and $x + \omega^2 y$ have one factor of $1 - \omega$ from P4. Thus $x + y$ gets the rest of the factors of $1 - \omega$, which gives 3^{3m-1}. You will need the same trick as in W13 to show that η_i can be taken to be one of the units 1, ω, or ω^2.)

We will deal with these three equations separately. The third one we can deal with quickly.

P6. Ignore the third equation. (That is the easy part!)

We next deal with the first equation.

P7. Check that $\eta_0 = 1$ by checking that $\eta_0/\overline{\eta_0}$ is a cube, in particular $\overline{\rho_0^3}/\rho_0^3$. Then check that the only units that are cubes are 1 and -1.

P8. Now $x + y = 3^{3m-1}\rho_0^3$. Show that we can assume that ρ_0 is an integer (not just an F–integer). Hint: $\rho_0^3 = 3^{1-3m}(x + y)$ is a real number. If $\rho_0 = s + t\omega$, show — by expanding out — that if ρ_0^3 is real, then $s = t$, $s = 0$, or $t = 0$. If $t = 0$, then ρ_0 is real. If $s = 0$, then we can replace ρ_0 by ρ_0/ω, which is real (and has the same cube as ρ_0). If $s = t$, then we can replace ρ_0 by $\rho_0/\omega^2 = -\rho_0/(1+\omega)$, which is also real (and also has the same cube as ρ_0).

We have now massaged the equations into

$$\begin{aligned} x + y &= 3^{3m-1}\rho_0^3\,, \\ x + \omega y &= \eta_1(1-\omega)\rho_1^3\,, \end{aligned}$$

where η_1 is 1, ω, or ω^2, ρ_1 is an F–integer, and ρ_0 is an honest-to-goodness integer.

We next deal with the second equation.

P9. Show that $\eta_1 = 1$ as follows. Reduce the equation

$$x + y + (\omega - 1)y \;=\; \eta_1(1-\omega)\rho_1^3 \pmod{3(1-\omega)}\,.$$

Now, $A = x + y$ is divisible by $9 = 3(1-\omega)(1-\omega^2)$, so that it is divisible by $3(1-\omega)$. Use this to show that $y \equiv -\eta_1\rho_1^3 \pmod 3$. But $\rho_1^3 \equiv 1$ or $2 \pmod 3$ (from W11; why not 0?), and $y \equiv 1$ or $2 \pmod 3$, so η_1 can not be ω or ω^2.

Let $\rho_1 = a + b\omega$, where a and b are integers.

P10. Show that $x+\omega y = (a^3+b^3-6ab^2+3a^2b)+(-a^3-b^3+6a^2b-3ab^2)\omega$. Hence, $x = a^3 + b^3 - 6ab^2 + 3a^2b$ and $y = -a^3 - b^3 + 6a^2b - 3ab^2$.

P11. Show that a, b, and $a-b$ are pairwise relatively prime. Hint: Anything dividing two of them would also have to divide both x and y, which are supposed to be relatively prime.

P12. Show that there are no solutions with $a = 0$, $b = 0$, or $a = b$. (Just plug in for x and y, and see that they do not satisfy equation (1)!)

P13. Show that $9ab(a-b) = x+y = 9\cdot 3^{3(m-1)}\rho_0^3$ from earlier results.

P14. So a, b, and $a-b$ are pairwise relatively prime integers multiplying to the cube of a number $3^{m-1}\rho_0$! Show that this somehow gives a solution to the equation $x_0^3+y_0^3 = (3^{m-1}z_0)^3$, where x_0, y_0, and z_0 are pairwise relatively prime, and not divisible by 3.

P15. You have now shown that if equation (1) has a solution for some positive integer m, then it has a solution for $m-1$ as well. You have already shown that it has no solution for $m=0$ (P2), so finish off the proof!

Summary

We summarize the argument to show that it is not as complicated as it first appears. The first key idea is to generalize the integers to F–integers, and to see that the F–integers have unique factorization. (F–integers in turn can be generalized greatly. Unfortunately, these generalizations rarely have unique factorization — so the F–integers are quite special — and much of modern number theory has been developed to try to deal with this lack of unique factorization.)

Then we return to the problem at hand. Look at equation (1), $x^3+y^3 = (3^m z)^3$ where m is a non-negative integer, and x, y, and z are relatively prime and not divisible by 3. This has no solutions when $m=0$ (P2). If $m>0$, then we factor the left side in F–integers $(x+y)(x+\omega y)(x+\omega^2 y)$. They are not relatively prime, but they only have the common factor $1-\omega$, from which we get the important equations (2):

$$\begin{aligned} x+y &= \eta_0 3^{3m-1}\rho_0^3, \\ x+\omega y &= \eta_1(1-\omega)\rho_1^3, \\ x+\omega^2 y &= \eta_2(1-\omega)\rho_2^3. \end{aligned}$$

We forget the third. In the first, we show that $\eta_0 = 1$ (by showing that $\eta_0/\overline{\eta_0} = \overline{\rho_0^3}/\rho_0^3$ is a cube (P7), and then show that we can take ρ_0 to be an integer. In the second, we show that $\eta_1 = 1$ by considering the equation modulo $3(1-\omega)$. Finally, we let $\rho_1 = a+b\omega$, work out x and y in terms of a and b, and then substitute in the equation for $x+y$, getting a solution to (1) with m one less.

Many ideas introduced here turn out be fundamentally important in number theory, in particular the ideas of F–integers (which are generalized greatly), unique factorization, the complex numbers (and the geometry of F–integers in the complex numbers), the importance of the prime $1-\omega$, and the idea of "descent" (in this case, taking a hypothetical solution to (1) for some m, and producing a solution to (1) with m one less).

If you have got this far, and successfully proved the $n=3$ case of Fermat's Last Theorem, well done! Reach over your shoulder and pat yourself on the back.

Acknowledgements. Suggestions from Dave Savitt of Harvard University greatly improved the exposition. The form of this proof comes from the excellent book [Wa], especially Ex. 9.5 p. 183, although the exposition has been changed significantly. Another proof is given in [D] (pp. 96–104), where Gauss's original proof that FLT3 holds even when x, y, and z are F–integers is given. The term "F–integers" is non-standard; F–integers are usually called Eisenstein numbers.

References

[D] H. Dörrie, *100 Great Problems of Elementary Mathematics: Their History and Solution*, Dover: New York, 1965.

[FLT4] R. Vakil, *A Do-It-Yourself Proof of the $n = 4$ Case of Fermat's Last Theorem*, ***Crux Mathematicorum with Mathematical Mayhem***, 1999 (26), pp. 502–504.

[UF] R. Vakil, *Unique Factorization, Funny Dice, the 2–Square Theorem, and FLT3*, to appear in ***Crux Mathematicorum with Mathematical Mayhem***. In this article, we will discuss unique factorization in variants of the integers, and use it to (1) fill in the last proof of FLT3, (2) find a funny pair of dice that gives the same outcome as a usual pair of dice, and (3) solve the famous problem: find all integers that can be expressed as a sum of 2 squares.

[Wa] L. Washington, *Cyclotomic Fields*, Graduate Texts in Mathematics vol. 83, Springer-Verlag: New York, 1982.

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 September 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2501. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle BAC$ and $\angle ABC$ meet BC and AC at D and E respectively. Suppose that $AB + BD = AE + EB$. Characterize $\triangle ABC$.

2502. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle BAC$, $\angle ABC$ and $\angle BCA$ meet BC, AC and AB at D, E and F respectively. Let p and q be the perimeters of $\triangle ABC$ and $\triangle DEF$ respectively.

Prove that $p \geq 2q$, and that equality holds if and only if $\triangle ABC$ is equilateral.

2503. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The incircle of $\triangle ABC$ touches BC at D, and the excircle opposite to B touches BC at E. Suppose that $AD = AE$. Prove that

$$2\angle BCA - \angle ABC \;=\; 180^\circ\,.$$

2504. *Proposed by Hayo Ahlburg, Benidorm, Spain, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that A, B and C are the angles of a triangle. Determine the best lower and upper bounds of $\prod\limits_{\text{cyclic}} \cos(B - C)$.

2505. *Proposed by Hayo Ahlburg, Benidorm, Spain, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that A, B and C are the angles of a triangle. Determine the best lower and upper bounds of $\prod_{\text{cyclic}} \sin(B-C)$.

2506. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

In the lattice plane, determine all lattice straight lines that are tangent to the unit circle. (A lattice straight line is a straight line containing two lattice points.)

2507. *Proposed by Ice B. Risteski, Skopje, Macedonia.*

Show that there are infinitely many pairs of distinct natural numbers, n and k such that $\gcd(n!+1, k!+1) > 1$;

2508. *Proposed by J. Chris Fisher, University of Regina, Regina, Saskatchewan.*

In problem 2408 [1999: 49; 2000: 55] we defined a point P to be *Cevic* with respect to $\triangle ABC$ if the vertices D, E, F of its pedal triangle determine concurrent cevians; more precisely, D, E, F are the feet of perpendiculars from P to the respective sides BC, CA, AB, while AD, BE, CF are concurrent.

1. Show that for every point D on the line BC there is a unique point E on the line AC for which P is Cevic.

2.$\star$ Describe the location of E if D divides the segment BC in the ratio $\lambda : (1-\lambda)$ (when P is Cevic and λ is an arbitrary real number).

2509. *Proposed by Ice B. Risteski, Skopje, Macedonia.*

Show that there are infinitely many pairs of distinct natural numbers, n and k such that $\gcd(n!-1, k!-1) > 1$.

2510. *Proposed by Ho-joo Lee, student, Kwangwoon University, South Korea.*

In $\triangle ABC$, $\angle ABC = \angle ACB = 80^\circ$ and P is on the line segment AB such that $AP = BC$. Find $\angle BPC$.

2511. *Proposed by Ho-joo Lee, student, Kwangwoon University, South Korea.*

In $\triangle ABC$, $\angle ABC = 60^\circ$ and $\angle ACB = 70^\circ$. Point D is on the line segment BC such that $\angle BAD = 20^\circ$. Prove that $AB + BD = AD + DC$.

2512. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

In $\triangle ABC$, the sides satisfy $a \geq b \geq c$. Let R and r be the circumradius and the inradius respectively. Prove that

$$bc \leq 6Rr \leq a^2,$$

with equality if and only if $a = b = c$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Acknowledgements of solutions to **2392** by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina, and to **2381** and **2382** by ANDREI SIMION, student, Brooklyn Technical H.S., New York, NY, USA were inadvertantly omitted when these solutions were published last year.

2401. [1999: 47] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In triangle ABC, CD is the altitude from C to AB. E and F are the mid-points of AB and CD respectively. P and Q are points on line segments BC and AC respectively, and are such that $PQ \| BA$. The projection of Q onto AB is R. PR and EF intersect at S.

Prove that

(a) S is the mid-point of line segment PR,

(b) $\dfrac{1}{PR^2} \leq \dfrac{1}{AB^2} + \dfrac{1}{CD^2}$.

Solution by Florian Herzig, student, Cambridge, UK.

(a) More generally suppose λ is an arbitrary real number, D is *any* point on line AB and R is the point of intersection of AB with the line parallel CD through Q. The other points are defined as before. Denote vectors $\overrightarrow{DX}$ by $\vec{x}$ for brevity; that is, take D as origin. Then $\vec{p} = \lambda\vec{b} + (1-\lambda)\vec{c}$, $\vec{r} = \lambda\vec{a}$ so that

$$\vec{s} = \lambda\frac{\vec{a}+\vec{b}}{2} + (1-\lambda)\frac{\vec{c}}{2}.$$

Thus S is a point on the line joining the mid-point E of AB to the mid-point F of CD, and moreover S divides EF in the same ratio as Q divides AC.

(b) Suppose Q divides segment AC in the ratio $(1-\lambda) : \lambda$ with $0 < \lambda < 1$. Then $PR^2 = PQ^2 + QR^2 = \lambda^2 AB^2 + (1-\lambda)^2 CD^2$ and hence

$$\begin{aligned}\frac{1}{PR^2} &= \frac{1}{\lambda(\lambda AB^2) + (1-\lambda)((1-\lambda)CD^2)} \\ &\leq \frac{\lambda}{\lambda AB^2} + \frac{1-\lambda}{(1-\lambda)CD^2} = \frac{1}{AB^2} + \frac{1}{CD^2}\end{aligned}$$

by the weighted arithmetic-harmonic mean inequality. Equality hence holds if and only if $\lambda AB^2 = (1-\lambda)CD^2$; that is, if and only if $PQ : QR = CD : AB$.

Note that if $\lambda \le 0$ then $PR \ge CD$ and if $\lambda \ge 1$ then $PR \ge AB$ (just consider QR and PQ respectively) so that in these cases the inequality holds as well and equality is impossible.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; PARAYIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Miranda, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Bataille also made use of vectors whereas the other solvers were equally split between pure geometric methods and coordinates.

2402. [1999: 47] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Find all sets of positive integers a, b, c, d, such that

$$bd > ad+bc \quad \text{and} \quad (9ac+bd)(ad+bc) = a^2d^2+10abcd+b^2c^2 .$$

Solution by Kee-Wai Lau, Hong Kong.

Since $bd > ad + bc$, it follows that $b > a$ and $d > \dfrac{bc}{b-a}$. For real numbers x, let

$$\begin{aligned} F(x) &= (9ac+bx)(ax+bc) - a^2x^2 - 10abcx - b^2c^2 \\ &= a(b-a)x^2 + c(9a-b)(a-b)x + bc^2(9a-b) . \end{aligned}$$

Then $F(d) = 0$. It is easy to check, since $a(b-a) > 0$, that the quadratic polynomial $F(x)$ attains its unique minimum $\dfrac{(9a-b)(b-3a)^2c^2}{4a}$ at $x = x_0$ where $x_0 = \dfrac{c(9a-b)}{2a}$.

If $b < 9a$ and $b \ne 3a$, then $F(x) > 0$ for all real numbers x so that $F(d) > 0$, a contradiction.

If $b \ge 9a$, then $x_0 \le 0$. Since $F(x)$ is increasing for all $x \ge x_0$ and since $d > \dfrac{bc}{b-a} > 0$, therefore

$$\begin{aligned} F(d) &> F\left(\frac{bc}{b-a}\right) \\ &= \frac{ab^2c^2}{b-a} > 0 , \end{aligned}$$

which is, again, a contradiction.

Hence we must have $b = 3a$, in which case $F(d) = 0$ is the minimum value of $F(x)$. Furthermore,

$$\begin{aligned} d &= x_0 \\ &= \frac{c(6a)}{2a} = 3c\,. \end{aligned}$$

Therefore, all solutions are given by $(a, b, c, d) = (a, 3a, c, 3c)$ where a and c are only positive integers.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

2403. [1999: 48] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

The positive integers a, b, c have the following properties:

1. a is odd;
2. the greatest common divisor of a, b, c is 1;
3. they satisfy the Diophantine equation:

$$\frac{2}{a} + \frac{1}{b} = \frac{1}{c}\,.$$

Prove that abc is a perfect square.

I. *Solution and generalization by Heinz-Jürgen Seiffert, Berlin, Germany.*

We shall prove the following more general result: *if a, b, c and x, y, z are positive integers such that*

(a) $\gcd(a, x) = \gcd(b, y) = \gcd(c, z) = 1$*;*

(b) $\gcd(a, b, c) = 1$*;*

(c) they satisfy the Diophantine equation $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{z}{c}$*;*

then abc is a perfect square.

From (c), we have $bcx + cay = abz$. Now (a) implies that $a|bc$, $b|ca$, and $c|ab$. Hence there exist positive integers u, v, w such that $bc = au$, $ca = bv$, and $ab = cw$. It follows that

$$a^2 = vw\,, \qquad b^2 = wu\,, \qquad c^2 = uv\,. \tag{1}$$

We claim that

$$\gcd(u, v) = \gcd(v, w) = \gcd(w, u) = 1\,. \tag{2}$$

Let p be any prime divisor of u. From (1), it follows that $p|b$ and $p|c$, so that in view of (b), p cannot divide a. Hence by (1) again, p does not divide vw. This proves that $\gcd(u,v) = \gcd(w,u) = 1$. Quite similarly one shows that $\gcd(v,w) = 1$, so that (2) is established. Clearly, (1) and (2) imply that each of the positive integers u, v, w is a perfect square. Multiplying the equations of (1), we obtain $abc = uvw$. The desired result follows.

To solve the present proposal, take $x = 2$ and $y = z = 1$, and note that the condition (a) is satisfied if a is odd.

II. *Solution by Michel Bataille, Rouen, France.*

Condition 3 is equivalent to $2bc = a(b-c)$ which may be written

$$a^2 + (b+c)^2 \;=\; (a+b-c)^2\,. \tag{3}$$

Note also, by condition 3, that $b > c$ and therefore, $a + b - c$ is a positive integer. By (3), $(a, b+c, a+b-c)$ is a Pythagorean triple. We show that it is necessarily a primitive one; that is, the positive integers a, $b+c$, $a+b-c$ are relatively prime. Let us assume that a prime number p divides a, $b+c$ and $a+b-c$. Then p also divides $b-c = (a+b-c)-a$. Since p divides $b+c$ as well, p divides $2b$ and $2c$. But a is odd, so $p \neq 2$ and therefore p divides b and c. We might then conclude that p divides a, b, c, a contradiction with condition 2.

Primitive Pythagorean triples are well-known: since a is odd, there exist positive integers m and n, relatively prime and of opposite parity, such that

$$a = m^2 - n^2\,, \qquad b + c = 2mn\,, \qquad a + b - c = m^2 + n^2\,.$$

It follows that $b = n(m+n)$ and $c = n(m-n)$. Hence

$$abc \;=\; (m^2 - n^2)n(m+n)n(m-n) \;=\; [n(m^2-n^2)]^2\,,$$

a perfect square as required.

Also solved by JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CARL JOHAN RAGNARSSON, Lund, Sweden; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; PARAYIOU THEOKLITOS, Limassol, Cyprus, Greece; KENNETH M. WILKE, Topeka, KS, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

All solvers noticed, and Heuver explicitly noted, that "denominator" in the original statement should have been "divisor". This has been corrected above.

Stone gave a weaker generalization than Seiffert in Solution I. Herzig, Ragnarsson and Wilke noted that the original question has the general solution

$$a = x(x+2y)\,, \qquad b = y(x+2y)\,, \qquad c = xy\,,$$

where $\gcd(x,y) = 1$ *and* x *is odd.*

2404★. [1999: 48] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Let $f(x) = \dfrac{(1-2x)^p + 2(x+1)^p}{2\left((x+1)(1-2x)\right)^p + (x+1)^{2p}}$, where $x \in \left(0, \frac{1}{2}\right)$ and $p \geq 2$ is an integer.

The relative maximum of $f(x)$ on $\left(0, \frac{1}{2}\right)$ has been found by computer to be at the point x_p for several values of p as shown:

p	2	3	4	5	6	7
x_p	0.2538	0.1133	0.0633	0.0403	0.0279	0.0205

Explain why x_p is "near" $\frac{1}{p^2}$.

Solution by Kee-Wai Lau, Hong Kong.

We show that for $p \geq 2$,

$$\frac{1}{p^2} \;<\; x_p \;<\; \frac{1}{p^2-1}\,. \tag{1}$$

Let $t = \frac{1-2x}{1+x}$. Then $0 < t < 1$ and $x = \frac{1-t}{2+t}$. We have

$$f(x) \;=\; \frac{(t^p+2)(t+2)^p}{3^p(2t^p+1)}$$

and

$$\frac{df}{dt} \;=\; \frac{2p(t+2)^{p-1}F(t)}{3^p(2t^p+1)^2}\,,$$

where $F(t) = t^{2p} + t^p - 3t^{p-1} + 1$.

Note that if $f(t)$ is a local maximum for some $t \in (0,1)$ then $F(t) = 0$. By the Descartes' Rule of Signs, the polynomial equation $F(t) = 0$ has at most two positive roots. In fact, it has exactly two positive roots, because $F(1) = 0$ (and the number of sign changes of $F(t)$ is two). We show that the other positive root t_p satisfies

$$1 - \frac{3}{p^2} \;<\; t_p \;<\; 1 - \frac{3}{p^2+1} \tag{2}$$

for $p \geq 2$, and (1) will follow. To prove (2), we need only to prove that for $p \geq 2$,

$$F\left(1 - \frac{3}{p^2}\right) \;>\; 0 \tag{3}$$

and

$$F\left(1 - \frac{3}{p^2+1}\right) \;<\; 0\,. \tag{4}$$

For $p = 2, 3, 4, 5$, or 6, (3) and (4) can be checked directly. We can therefore assume that $p > 6$. Consider the binomial expansion

$$\left(1-\frac{3}{p^2}\right)^m = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\frac{3}{p^2}\right)^k,$$

where m is a positive integer. The absolute value of the ratio of the consecutive terms

$$\frac{\binom{m}{k+1}\left(\frac{3}{p^2}\right)^{k+1}}{\binom{m}{k}\left(\frac{3}{p^2}\right)^k} = \frac{3(m-k)}{(k+1)p^2}$$

is less than 1 for $0 \le k \le m \le 2p$ and $p > 6$. Hence

$$\begin{aligned}
F\left(1-\frac{3}{p^2}\right) &= \left(1-\frac{3}{p^2}\right)^{2p} + \left(1-\frac{3}{p^2}\right)^{p} - 3\left(1-\frac{3}{p^2}\right)^{p-1} + 1 \\
&> \sum_{k=0}^{5}(-1)^k\binom{2p}{k}\left(\frac{3}{p^2}\right)^k + \sum_{k=0}^{5}(-1)^k\binom{p}{k}\left(\frac{3}{p^2}\right)^k \\
&\quad -3\sum_{k=0}^{4}(-1)^k\binom{p-1}{k}\left(\frac{3}{p^2}\right)^k + 1 \\
&= \frac{27(10p^5 - 39p^4 + 140p^3 - 285p^2 + 390p - 216)}{40p^9} \\
&> 0
\end{aligned}$$

for $p > 6$. This proves (3). Similarly,

$$\begin{aligned}
F\left(1-\frac{3}{p^2+1}\right) &= \left(1-\frac{3}{p^2}\right)^{2p} + \left(1-\frac{3}{p^2}\right)^{p} - 3\left(1-\frac{3}{p^2}\right)^{p-1} + 1 \\
&< \sum_{k=0}^{6}(-1)^k\binom{2p}{k}\left(\frac{3}{p^2+1}\right)^k \\
&\quad + \sum_{k=0}^{6}(-1)^k\binom{p}{k}\left(\frac{3}{p^2+1}\right)^k \\
&\quad -3\sum_{k=0}^{5}(-1)^k\binom{p-1}{k}\left(\frac{3}{p^2+1}\right)^k + 1 \\
&= \frac{-9}{16(p^2+1)^6}(4p^8 - 12p^7 + 37p^6 - 81p^5 \\
&\quad +315p^4 - 1071p^3 + 2236p^2 - 3108p + 1936) \\
&< 0
\end{aligned}$$

for $p > 6$. This proves (4) and completes the solution.

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and MICHAEL LAMBROU, University of Crete, Crete, Greece.

2405★. [1999: 48, 238] *Proposed by the editors, based on a submission by G.P. Henderson, Garden Hill, Campbellcroft, Ontario.*

Given two n-sided dice, one with a_k sides with k dots $(1 \leq k \leq n)$ such that $\sum_{k=1}^{n} a_k = n$ and the other with b_k sides with k dots $(1 \leq k \leq n)$ such that $\sum_{k=1}^{n} b_k = n$. Both are rolled. Let r_k be the probability that the sum of the two faces showing is k.

How should the a_k and the b_k be chosen to minimize $\displaystyle\sum_{k=1}^{n}\left(r_k - \frac{1}{n-1}\right)^2$

(a) with $n = 6$?

(b) with general n?

No solution was submitted for part (b). There was one submission for part (a), from Walter Janous, Ursulinengymnasium, Innsbruck, Austria, who ran a computer program showing that there are four ways to choose a_k and b_k so as to minimize $\displaystyle\sum_{k=1}^{n}\left(r_k - \frac{1}{n-1}\right)^2$, where $n = 6$:

(a) $a_1 = 3$ (meaning that for the first die there are 3 sides with one dot), $a_2 = 2$, $a_3 = 1$; $b_1 = 2$, $b_2 = 1$ (meaning that for the second die there is one side with 2 dots), $b_3 = 1$, $b_4 = 1$, $b_5 = 1$

(b) $a_1 = 3$, $a_3 = 2$, $a_4 = 1$; $b_1 = 2$, $b_2 = 2$, $b_3 = 1$, $b_5 = 1$

(c) $a_1 = 2$, $a_2 = 2$, $a_3 = 1$, $a_5 = 1$; $b_1 = 3$, $b_3 = 2$, $b_4 = 1$

(d) $a_1 = 2$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$; $b_1 = 3$, $b_2 = 2$, $b_3 = 1$.

The sum given in each case is 4.399999.

2406. [1999: 48] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Suppose that an integer-sided triangle contains a $120°$ angle, with the two containing arms differing by 1. Prove that the length of the longest side is the sum of two consecutive squares.

Solution by Jeremy Young, student, Nottingham High School, Nottingham, UK (slightly adapted by the editors).

Let b be the length of the side opposite the $120°$ angle, and let a and $a + 1$ be the lengths of the two containing sides. Applying the Law of Cosines we get: $b^2 = 3a^2 + 3a + 1$, or $3(2b-1)(2b+1) = (6a+3)^2$. Since $2b-1$ and $2b+1$ are coprime, there are integers m, n such that either

$$\begin{aligned} \text{(i)}\quad 2b+1 &= m^2\,, \quad 2b-1 = 3n^2\,; \\ \text{or (ii)}\quad 2b+1 &= 3m^2\,, \quad 2b-1 = n^2\,. \end{aligned}$$

If (i), then $m^2 - 3n^2 = 2$, which is impossible modulo 3. Hence we have (ii). So $b = \dfrac{n^2+1}{2}$ (since b is an integer, n is odd). Thus we have

$$b = \left(\frac{n-1}{2}\right)^2 + \left(\frac{n+1}{2}\right)^2\,,$$

which is the sum of consecutive squares, since n is odd.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; J. SUCK, Essen, Germany; KENNETH M. WILKE, Topeka, KS, USA; and the proposer. There was one incomplete solution.

The proposer comments "This problem combines an observation of E. P. Starke and a pattern conjectured by T. Running in their solutions to Problem E702, *American Mathematical Monthly*, vol. 53 (1946), June-July issue." *He further adds that the conjecture can be confirmed by studying the solutions of a Pellian equation through second order linear recurrence relations. Indeed, most solvers used that very approach with the Pell equation $x^2 - 3y^2 = 1$. Many of these solvers not only solved the problem posed, but also found formulas to generate all such triangles.*

2407. [1999: 49] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Triangle ABC is given with $\angle BAC = 72°$. The perpendicular from B to CA meets the internal bisector of $\angle BCA$ at P. The perpendicular from C to AB meets the internal bisector of $\angle ABC$ at Q.

If A, P and Q are collinear, determine $\angle ABC$ and $\angle BCA$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

By assumption AP, AQ coincide in a line that cuts BC at D, say; further, suppose BP, BQ cut AC at E and G respectively, and that CP, CQ cut AB at F and H respectively. By Ceva's theorem on the cevians AD, BE, CF through P we get

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1\,.$$

Similarly, using the cevians AD, BG, CH through Q we have

$$\frac{BD}{DC}\cdot\frac{CG}{GA}\cdot\frac{AH}{HB} = 1\,.$$

Equating the two products, using $CE = a\cos C$, $EA = c\cos A$, $AH = b\cos A$, $HB = a\cos B$, and also using the bisector theorem that says $\frac{AF}{FB} = \frac{b}{a}$, $\frac{CG}{GA} = \frac{a}{c}$, we find

$$\cos^2 A = \cos B\cos C\,.$$

But we are given $A = 72°$, so $2\cos^2 72° = \cos(B+C) + \cos(B-C) = \cos 108° + \cos(B-C)$, whence $\cos(B-C) = 2\sin^2 18° + \sin 18°$. Since $\sin 18° = \cos 72° = \frac{\sqrt{5}-1}{4}$ [see any book that deals with the regular pentagon, such as Coxeter's *Introduction to Geometry*, or see ***CRUX with MAYHEM*** **2348** [1999: 312], or derive it yourself using complex numbers], $\cos(B-C) = 2\left(\frac{\sqrt{5}-1}{4}\right)^2 + \frac{\sqrt{5}-1}{4} = \frac{1}{2}$. We get $|B-C| = 60°$, which together with $B + C = 108°$ says that our angles are $84°$ and $24°$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany;TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Yiu commented that from $\cos(B-C) = 2\cos^2 A + \cos A$ it follows that for every angle A between $60°$ and $90°$ there exists a unique triangle ABC with A, P, Q collinear. Of course $A = 72°$ is the only familiar angle in that interval for which the resulting equation permits an elementary solution.

2408. [1999: 49] *Proposed by Mansur Boase, student, St. Paul's School, London, England.*

Perpendiculars are dropped from a point P inside an acute-angled triangle ABC to the sides BC, CA, AB, meeting them at D, E, F respectively.

(a) Prove that the perpendiculars from A to EF, B to FD, C to DE are concurrent (at K, say).

(b) A point, P, is called "Cevic" with respect to $\triangle ABC$ if AD, BE and CF are concurrent. Prove that K is Cevic with respect to $\triangle DEF$ if and only if P is Cevic with respect to $\triangle ABC$.

Preliminary comments. Part (a) is a special case of several familiar theorems.

Theorem *If $\triangle ABC$ has its sides BC, CA, and AB respectively perpendicular to PD, PE and PF, where P, D, E, F are four coplanar points, then there is a point K such that KA, KB, KC are respectively perpendicular* *to* EF*,* FD*, and* DE. The triangles ABC and DEF are called *orthologic triangles* if this reciprocal relationship exists between them. [Dan Pedoe, *Geometry: A Comprehensive Course*, Dover (1988), p. 42.] There are many simple proofs; Pedoe provides two (p. 37 exercise 6.1) and § 8.3, pp. 42-43). Note that the result of part (a) holds if P is a point *anywhere* in the plane of an *arbitrary* $\triangle ABC$, although if P is on the circumcircle then the points D, E, F would be collinear and the perpendiculars to that line would be parallel (sending K out to infinity). Here are three further proofs of our special case.

I. *Solution to* (a) *by Michael Lambrou, University of Crete, Crete, Greece.*

As the perpendiculars at D, E, F meet at P, Carnot's theorem gives

$$AF^2 - FB^2 + BD^2 - DC^2 + CE^2 - EA^2 \;=\; 0\,. \qquad (1)$$

If the perpendiculars from A, B, C to EF, FD, DE meet them at X, Y, Z respectively, then $DY^2 - YF^2 = (DY^2 + BY^2) - (YF^2 + BY^2) = BD^2 - BF^2$, and cyclically for the other two. Adding these and using (1) we find $(DY^2 - YF^2) + (FX^2 - XE^2) + (EZ^2 - ZD^2) = 0$, so AX, BY, CZ concur by the converse of Carnot's theorem applied to $\triangle DEF$.

II. *Solution to* (a) *by Gerry Leversha, St. Paul's School, London, England.*

$AEPF$ is cyclic (since it has two opposite right angles), and so $\angle PAE = \angle PFE$. However, $\angle PFE = 90^\circ - \angle AFE = 90^\circ - \angle AFX = \angle FAX$. Hence $\angle PAC = \angle KAB$ for any point K on the line AX. The same is true for the angles at B and C; hence the three lines AX, BY, CZ are concurrent at K, the *isogonal conjugate* of P.

III. *Solution to* (a) *by Michel Bataille, Rouen, France.*

Let P_A, P_B, P_C, be the images of P reflected in the lines BC, CA, AB respectively. E and F are the mid-points of PP_B and PP_C, so that $EF || P_BP_C$. The perpendicular from A to EF is thus also perpendicular to P_BP_C; moreover, since $AP_B = AP_C (= AP)$ this line is the perpendicular bisector of P_BP_C. Similarly, the perpendiculars from B to FD and from C to DE are the perpendicular bisectors of P_CP_A and of P_AP_B, respectively. Hence, the three lines are concurrent at K, the circumcentre of $\triangle P_AP_BP_C$.

Solution to (b) *by Gerry Leversha, St. Paul's School, London, England.*

$$\frac{AE}{EC} \;=\; \frac{AE}{PE}\cdot\frac{PE}{EC} \;=\; \frac{\cot PAE}{\cot PCE} \;=\; \frac{\cot XAF}{\cot ZCD} \;=\; \frac{AX\cdot DZ}{XF\cdot CZ}\,.$$

Hence

$$\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA} \;=\; \frac{AX\cdot DZ}{XF\cdot CZ}\cdot\frac{CZ\cdot FY}{ZE\cdot BY}\cdot\frac{BY\cdot EX}{YD\cdot AX} \;=\; \frac{DZ}{ZE}\cdot\frac{EX}{XF}\cdot\frac{FY}{YD}\,.$$

These are, of course, the Ceva ratio products, and it follows immediately that P is Cevic with respect to ABC if and only if K is Cevic with respect to DEF.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (a) only); MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PARAYIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Miranda, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

It is my Cevic-minded duty to remark that all submitted solutions merely verified part (b). Nobody explained how to find a Cevic point, or even if such a point existed. Could it be that Boase's result provides yet another property of the empty set? See problem **2508**.

2409. [1999: 49] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC with sides a, b, c, has O as the centre of its circumcircle (with radius R) and H as its orthocentre. Suppose that OH intersects CB and CA at P and Q respectively.

(a) Prove that quadrilateral $ABPQ$ is cyclic if and only if $a^2 + b^2 = 6R^2$.

(b) If quadrilateral $ABPQ$ is cyclic, find a formula for the length of PQ in terms of a, b and c alone.

Combination of solutions by Nikolaos Dergiades, Thessaloniki, Greece and Toshio Seimiya, Kawasaki, Japan.

(a) The proof is in three steps.

Step 1. If the Euler line of $\triangle ABC$ (which is OH) *meets* BC *in* P *and* AC *in* Q*, then it is perpendicular to* CO *if and only if* $ABPQ$ *is cyclic.*

In this problem it is tacitly assumed that $CA \neq CB$, so that without loss of generality we may take $A < B$. Let the tangent at C to the circumcircle of $\triangle ABC$ intersect AB at T (with B between A and T). Then $\angle TCB = \angle CAB$ and $OC \perp CT$. It follows that $ABPQ$ is cyclic if and only if $\angle CPQ = \angle CAB = \angle TCB$, which is equivalent to $CT \| PQ$, and therefore to $CO \perp OH$.

Step 2. For any $\triangle ABC$*, we have* $CH = 2R\cos C$ *and* $|\angle OCH| = B - A$.

These standard formulas are valid for all triangles with vertex angles A, B, C, orthocentre H and circumcentre O. See, for example, Roger A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960, pp. 162-163.

Proof of (a).

$CO \perp OH$ is equivalent to each of the following statements:

$$\begin{aligned} CO &= CH \cos \angle HCO\,, \\ R &= 2R \cos C \cos(B - A)\,, \\ -1 &= 2\cos(A + B)\cos(A - B)\,, \\ -1 &= \cos 2A + \cos 2B\,, \\ -1 &= 1 - 2\sin^2 A + 1 - 2\sin^2 B\,, \\ 4R^2 \sin^2 A + 4R^2 \sin^2 B &= 6R^2\,, \\ a^2 + b^2 &= 6R^2\,. \end{aligned} \tag{1}$$

From step 1 we therefore conclude that quadrilateral $ABPQ$ is cyclic if and only if $a^2 + b^2 = 6R^2$.

Proof of (b)

We begin with $ABPQ$ cyclic and will derive two formulas for the length of PQ. Because their angles are equal, $\triangle CPQ \sim \triangle CAB$ and therefore $PQ : AB = CP : CA$. Thus,

$$PQ = \frac{c}{b} CP\,. \tag{2}$$

Since $CO \perp PO$ and $\angle OPC = A$, we have $CO = CP \sin A = CP \frac{a}{2R}$, so that

$$CP = \frac{2R^2}{a}\,. \tag{3}$$

From (1), (2), and (3) we obtain

$$PQ = \frac{(a^2 + b^2)c}{3ab}\,.$$

Alternatively, since the area of $\triangle ABC$ satisfies $\triangle = \frac{abc}{4R}$, we deduce from (2) and (3)

$$\begin{aligned} PQ &= \frac{2R^2 c}{ab} \cdot \frac{c^2}{ab} \cdot \frac{ab}{c^2} \cdot \frac{8}{8} = \frac{abc^3}{8\triangle^2} \\ &= \frac{2abc^3}{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}\,. \end{aligned}$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; PETER Y. WOO, Biola University, La Miranda, CA, USA; and the proposer.

2410. [1999: 49] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

For $n \geq 1$, define $v_n = [1, 2, 3, \ldots, n-1, n]$, where the square bracket denotes the least common multiple. Let $p_1 < p_2$ be twin primes.

Prove or disprove that $v_{p_2} = p_2\, v_{p_1}$ for $p_1 > 3$.

Solution by David R. Stone, Georgia Southern University, Statesboro, GA, USA.

Note that v_n will "jump" exactly when n is a power of a prime, including a first power. That is,

$$v_n = \begin{cases} p\, v_{n-1}\,, & \text{if } n = p^i\,,\ i \geq 1\,; \\ v_{n-1}\,, & \text{otherwise.} \end{cases}$$

If p_1 and p_2 are twin primes with $3 < p_1 < p_2$, then $p_1 \not\equiv 0 \pmod 3$ and $p_1 + 2 = p_2 \not\equiv 0 \pmod 3$ imply that $p_1 \equiv 2 \pmod 3$. Hence $p_1 + 1$ is a multiple of 6 and cannot be a power of a prime. Therefore, $v_{p_1+1} = v_{p_1}$. It then follows that $v_{p_2} = p_2 v_{p_2-1} = p_2 v_{p_1+1} = p_2 v_{p_1}$.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, KS, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

This result is actually contained in a problem proposed by Sydney Bulman-Fleming and Edward T.H. Wang in Mathematics Magazine *(#1252, Least Common Multiple of* $\{1, 2, \ldots, n\}$*; 59 (1986), p. 297; Solution in 61 (1988), pp. 47-48.) The observation made by Stone and most of the solvers was explicitly mentioned in the published solution of that problem.*

2411. [1999: 49] *Proposed by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is well known and easy to show that the product of four consecutive positive integers plus one is always a perfect square. It is also easy to show that the product of any two consecutive positive integers plus one is never a perfect square. Also, note that

$$2 \times 3 \times 4 + 1 = 5^2 \qquad \text{and} \qquad 4 \times 5 \times 6 + 1 = 11^2\,.$$

(a) Find another natural number n such that $n(n+1)(n+2)+1$ is a perfect square.

(b)* Are there further examples?

Solution by Michael Lambrou, University of Crete, Crete, Greece.

In the article by D. W. Boyd and H. H. Kisilevsky: *The Diophantine equation* $u(u+1)(u+2)(u+3) = v(v+1)(v+2)$, Pacific Journal of Mathematics, vol. 40, 1972, pp. 23-32, there is a complete and non-elementary solution of the quoted equation. Now, in the present problem it is required to solve $v(v+1)(v+2)+1 = m^2$. Setting $m-1 = u(u+3)$, so that $m+1 = (u+1)(u+2)$, we are reduced to the equation solved in Boyd and Kisilevsky. Thus, from that paper, the only other solution is $(v, m) = (55, 419)$. This answers both part (a) and part (b).

LAMBROU was the only one to solve part (b). *Part* (a) *was also solved by FEDERICO ARBOLEDA, student, Bogotá, Columbia; CHARLES ASHBACHER, Cedar Rapids, IA, USA; JIM BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; STEWART METCHETTE, Gardena, CA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE and MIKE DOWELL, Georgia Southern University, Statesboro, GA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; and the proposers.*

2412. [1999: 50] *Proposed by Darko Veljan, University of Zagreb, Zagreb, Croatia.*

Suppose that A_1, A_2, A_3, A_4 are the vertices of a tetrahedron $\mathcal{T}$. On the faces opposite A_1, A_2, A_3, construct tetrahedra outside $\mathcal{T}$ with apexes A_1', A_2', A_3', and volumes V_1, V_2, V_3, respectively.

Let A_4' be the point such that $\overrightarrow{A_1A_4'} = \overrightarrow{BA_4}$, where B is the point of intersection of the planes through A_i' parallel to the respective bases $(i = 1, 2, 3)$.

Let V_4 be the volume of tetrahedron $A_1A_2A_3A_4'$.

Prove that $V_4 = V_1 + V_2 + V_3$.

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

As B and A_1' lie in a plane parallel to the base $A_2A_3A_4$, we have $V_1 = (A_1'A_2A_3A_4) = (BA_2A_3A_4)$. Similarly, $V_2 = (BA_1A_3A_4)$, and $V_3 = (BA_1A_2A_4)$. So if V denotes the volume $(A_4A_1A_2A_3)$ we have

$$V_1 + V_2 + V_3 + V \;=\; (BA_1A_2A_3)\,. \tag{1}$$

If BA_4 cuts the plane $A_1A_2A_3$ at B' we have

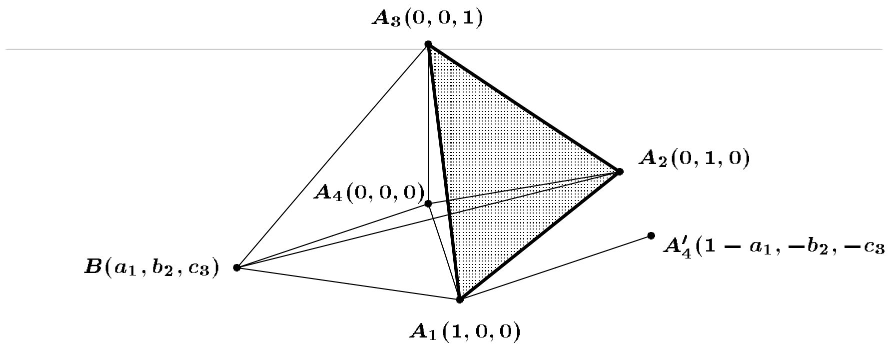
$$\overrightarrow{BB'} \;=\; \overrightarrow{BA_4} + \overrightarrow{A_4B'} \;=\; \overrightarrow{A_1A_4'} + \overrightarrow{A_4B'}\,.$$

The tetrahedra $BA_1A_2A_3$, $A_4A_1A_2A_3$, $A_4'A_1A_2A_3$ all have base $A_1A_2A_3$, so we have

$$(BA_1A_2A_3) \;=\; (A_4'A_1A_2A_3) + (A_4A_1A_2A_3) \;=\; V_4 + V\,.$$

This combined with (1) gives us $V_1 + V_2 + V_3 = V$, as required.

II. *Solution by Michel Bataille, Rouen, France.*



Let us adopt the coordinate system with origin A_4, and assign to the vertices of the given tetrahedron the coordinates $A_4(0,0,0)$, $A_1(1,0,0)$, $A_2(0,1,0)$, $A_3(0,0,1)$. We denote by $(a_i,\ b_i,\ c_i)$ the coordinates of A_i' $(i = 1, 2, 3)$ and remark that, by construction, a_1, b_2, c_3 are negative. Moreover, the planes intersecting at B satisfy the equations, $x = a_1$, $y = b_2$, and $z = c_3$. Thus B is the point (a_1, b_2, c_3), and so A_4' is $(1 - a_1, -b_2, -c_3)$. Recall that the volume V of the tetrahedron $MNPQ$ is given by

$$6V = \left|\det(\overrightarrow{MN}, \overrightarrow{MP}, \overrightarrow{MQ})\right| .$$

Thus $6V_1 = \left|\det(\overrightarrow{A_4A_1'}, \overrightarrow{A_4A_2}, \overrightarrow{A_4A_3})\right| = |a_1| = -a_1$, and similarly, $6V_2 = -b_2$, and $6V_3 = -c_3$. As for V_4, we get

$$\begin{aligned} 6V_4 &= \left|\det(\overrightarrow{A_1A_4'}, \overrightarrow{A_1A_2}, \overrightarrow{A_1A_3})\right| = \left|\det\begin{pmatrix} a_1 & -1 & -1 \\ b_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix}\right| \\ &= |a_1 + b_2 + c_3| , \end{aligned}$$

and finally,

$$V_4 = -\frac{(a_1 + b_2 + c_3)}{6} = V_1 + V_2 + V_3 .$$

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece(a second solution); GERRY LEVERSHA, St. Paul's School, London, England; PARAYIOU THEOKLITOS, Limassol, Cyprus, Greece; and the proposer.

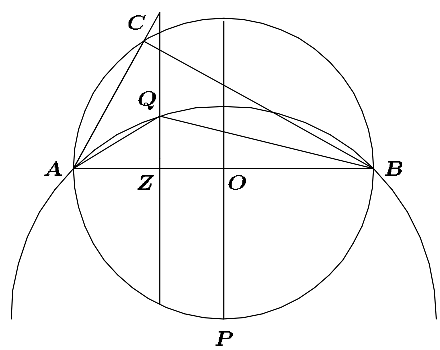
Most solvers used very neat vector arguments, but in each case after carefully interpreting the notation one ends up with our featured solution II. *After his particularly slick solution Leversha added the comment,* "Like most vector proofs, this leaves me thinking: So what? The algebra all turns out very nicely but there is no feeling of 'Ah! Now I see why!'" *To this we add,* "Amen."

2415. [1999: 110] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a point Z on a line segment AB, find a Euclidean construction of a right-angled triangle ABC whose incircle touches hypotenuse AB at Z.

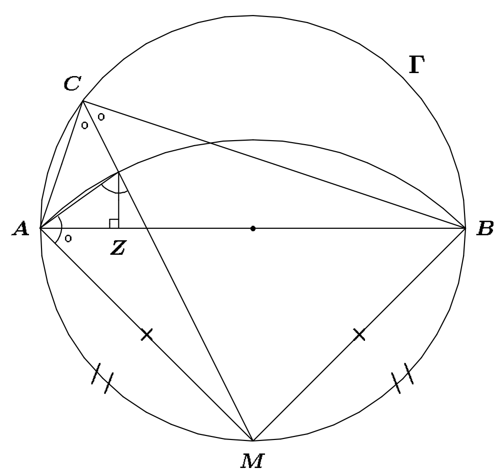
I. *Solution by Gerry Leversha, St. Paul's School, London, England.*

Construct a circle with centre O on diameter AB, a perpendicular radius OP, and a chord perpendicular to AB at Z (see diagram below). Draw the circle with centre P and radius PA; let this meet the perpendicular chord through Z at Q. Since APB is a right angle, the angle AQB is 135°. Now reflect AB in AQ and in BQ to obtain two lines which meet at a point C. Since $\angle QAB + \angle QBA = 45^\circ$, we have $\angle CAB + \angle CBA = 90^\circ$; thus the angle at C is a right angle and C will lie on the circle with diameter AB. Also Q is clearly the incentre of the triangle ABC and Z is therefore the point of contact of the incircle.



II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Construction: Describe circle Γ with diameter AB, and let M be a mid-point of arc AB of Γ. Draw minor arc AB with centre M and radius MA $(= MB)$. Construct the perpendicular through Z to AB meeting the arc AB at I. Let C be the second intersection of MI with Γ. Then $\triangle ABC$ is a triangle we are looking for.



Proof. Since AB is a diameter of Γ we have $\angle ACB = 90^\circ$. So $\triangle ABC$ is a right-angled triangle with hypotenuse AB. Since $\text{arc } AB = \text{arc } BM$ we have

$$\angle ACM = \angle BCM, \quad \text{so} \quad \angle ACI = \angle BCI. \tag{1}$$

As $CAMB$ is concyclic we get

$$\angle ACM = \angle BCM = \angle MAB. \tag{2}$$

Since $MA = MI$ it follows that $\angle MAI = \angle MIA$, so $\angle MAB + \angle IAB = \angle ACM + \angle CAI$. Thus we have from (2):

$$\angle IAB = \angle CAI. \tag{3}$$

From (1) and (3) I is the incentre of $\triangle ABC$. As $IZ \perp AB$, the incircle of $\triangle ABC$ touches AB at Z. Thus $\triangle ABC$ is a triangle we are looking for.

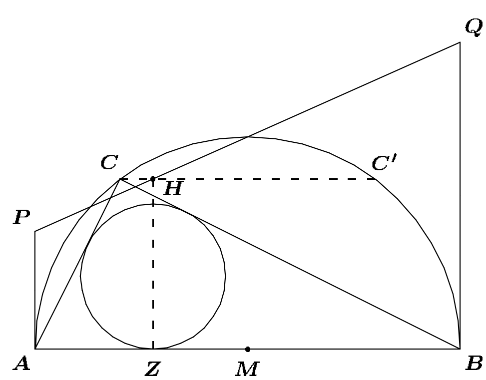
III. *Solution by the proposer.*
With standard notation for the sides of the right triangle, the inradius is $r = s - c$. Suppose $AZ = u$ and $BZ = v$. Then $c = u + v$, and $a = r + v$, $b = r + u$. Since $(r+u)^2 + (r+v)^2 = (u+v)^2$, we have $(r+u)(r+v) = 2uv$. This means that the area of the triangle is $\frac{1}{2}(r+u)(r+v) = uv$. If h is the height on the hypotenuse, then $\frac{1}{2}(u+v) = uv$, and h is the harmonic mean of u and v. This leads to the following construction:

1. Erect segments AP and BQ on the same side of and perpendicular to AB so that $AP = u$ and $BQ = v$.

2. Construct the segment through Z parallel to AP to meet the segment PQ at H.

3. Construct the line through H parallel to AB, to intersect the circle with diameter AB at two points.

The one nearer to Z is the vertex of the right angle of the required triangle.



Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; NIELS BEJLEGAARD, Stavanger, Norway; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; J. SUCK, Essen, Germany; PARAYIOU THEOKLITOS, Limassol, Cyprus, Greece; and PETER Y. WOO, Biola University, La Miranda, CA, USA. There was one incorrect solution.

Perz remarks that this "problem appeared in the Austrian Mathematical Olympiad 1996 as Problem 4 in the 2nd Round (Regional Competition) for beginners (Grade 9)".

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