THE ACADEMY CORNER

No. 29

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

Here are the hints and answers to the 1999 Bernoulli Trials [1999: 321]. Many thanks to Christopher Small for sending them to us.

The Bernoulli Trials 1999

Christopher G. Small & Ravindra Maharaj
University of Waterloo

Hints and Answers:

1. TRUE. We have

$$\frac{1}{1999} = \frac{1}{x} + \frac{1}{y} > \max\left(\frac{1}{x}, \frac{1}{y}\right) .$$

   Therefore x, $y \geq 2000$. Moreover, if

$$x > 3998000 = 2000 \times 1999$$

   then $1/x < 1/3998000$, so that

$$\frac{1}{1999} < \frac{1}{3998000} + \frac{1}{y} .$$

   This reduces to $y < 2000$, which is a contradiction.

2. TRUE. The equation implies that

$$\cos x = \pi/2 \pm \sin x \pmod{2\pi}$$

   As $|\cos x| \leq 1$ we must have $\cos x = \pi/2 \pm \sin x$, or

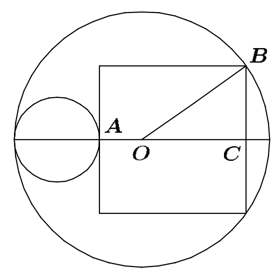
$$\cos^2 x \pm 2\cos x \sin x + \sin^2 x = \pi^2/4 .$$

   Therefore $|\sin 2x| = \pi^2/4 - 1 > 1$. But this cannot hold.

3. FALSE. As $n^3-n = n(n-1)(n+1)$, where n is odd, it follows that the right hand side is divisible by 9 and 16. As the sum of the digits must be $0 \pmod 9$ it follows that $A+B = 7 \pmod 9$. In addition, B must be even. The possibilities for AB are 70, 52, 34, 16 and 88. However, $48AB$ must be divisible by 16.

   So the answer is $AB = 16$. In fact, $CD = 76$, although this is not needed.

4. FALSE.



   We have $AO = 12 - 8 = 4$.

   Let x denote one-half of the side length of the square. Then $AC = 2x$. Therefore $OC = AC - AO = 2x - 4$. Thus:

$$OC = 2x - 4,\ BC = x, \text{ and } OB = 12.$$

   Therefore x is a solution to the quadratic equation $(2x-4)^2+x^2 = 12^2$. Solving this equation gives $2x = \dfrac{16+\sqrt{2816}}{5}$.

5. TRUE. The colouring of the squares in the first row and first column is arbitrary. There are 2^{15} choices for colouring these squares. Once the colours of these squares are determined, the rest of the board can only be coloured in one way to fulfil the condition.

6. TRUE. Consider polynomials. Let n be the degree of f. The equation implies that $2n = n^2$. This requires that f be constant or a quadratic. It can be checked that $f(x) = x^2$ works.

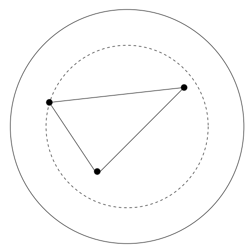
7. FALSE. Formally differentiating term by term gives us back the same series. Thus $\phi'(x) = \phi(x)$. This implies that $\phi(x) = A\,e^x$ for some constant A. We note that $f(x) = A$ works. Since $\phi(0) = 1999$, we find that $\phi(x) = 1999\,e^x$. Therefore $\phi(-1999) > 0$.

8. TRUE. Write

$$\begin{aligned}\int_0^1 x^{-x}\,dx &= \int_0^1 e^{x\ln(1/x)}\,dx \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 x^n\,[\ln(1/x)]^n\,dx\end{aligned}$$

Substituting $x^{n+1} = e^{-y}$ reduces the right-hand side to $\sum_{n=1}^{\infty} n^{-n}$. The result follows easily from this.

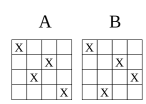
9. FALSE. First note that the event that a triangle is acute is independent of the choice of labels for the vertices.

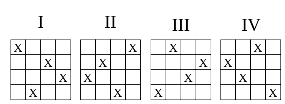


Shrink the circle around $\triangle ABX$ until one of the 3 points hits the boundary. Now relabel the triangle so that the two points in the interior of the new circle are A and B in random order, and the point on the boundary of the new circle is called X'.

This shows that with respect to events that are symmetric in the labels, $\triangle ABX$ and $\triangle ABX'$ have the same distribution. Thus $\Pi = \Pi'$.

10. TRUE. Since it takes 4 counters to block the lines in any 4×4 slice, any solution must have exactly 4 counters in each such horizontal slice. There are two basic patterns, A and B as shown, which will block the lines with 4 counters within a slice. Other configurations of four counters are obtained as rotations or reflections of these two basic cases.





Stacking I, II, III, and IV in order can be checked to block all lines parallel to a face or edge.

11. FALSE. Try

Day 1: M1W1 v M2W2 and M3W3 v M4W4

Day 2: M1W3 v M3W1 and M2W4 v M4W2

Day 3: M1W2 v M4W3 and M2W1 v M3W4

We now present the questions of the 1999 Atlantic Provinces Council on the Sciences Annual Mathematics Competition, held this year at Memorial University, St. John's, Newfoundland. The winning team consisted of Ian Caines and Jacky Pak Ki Li from Dalhousie University. The runners-up were the team of Shannon Sullivan and Jerome Terry from Memorial University of Newfoundland. You may view their pictures at

www.math.mun.ca/~apics/picture/shots.html

Send me your nice solutions!

APICS 1999 Mathematics Competition

1. Find the volume of the solid formed by one complete revolution about the x–axis of the area in common to the circles with equations $x^2 + y^2 - 4y + 3 = 0$ and $x^2 + y^2 = 3$.

2. The Memorial University Philosophers' Jockey Club has just received the bronze busts of the ten members of their hall of fame. Each will be placed in its designated place on a single shelf, above the gold plaque bearing the name of the member. The ten busts are drawn at random from the crate. What is the probability that at no time will there be an empty space between two busts already placed on the shelf?

3. Prove that $\sin^2(x+\alpha)+\sin^2(x+\beta)-2\cos(\alpha-\beta)\sin(x+\alpha)\sin(x+\beta)$ is a constant function of x.

4. In Scottish Dancing, there are three types of dances, two of which are fast rhythms, Jigs and Reels, and one is a slow rhythm, Strathspey.

   A Scottish Dance program always starts with a Jig. The following dances are selected (by type) according to the following rules:

   (i) the next dance is always of a different type from the previous one,

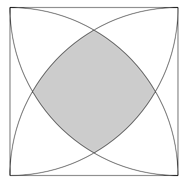
   (ii) no more than two fast dances can be consecutive.

   Find how many different arrangements of Jigs, Reels and Strathspeys are possible in a Scottish Dance list which has (a) seven dances, (b) fifteen dances.

5. Find all differentiable functions $f(x)$ which satisfy the integral equation

$$(f(x))^{2000} = \int_1^x (f(t))^{1999}\, dt \,.$$

6. Inside a square of side r, four quarter circles are drawn, with radius r and centres at the corners of the square.



Find the area of the shaded region.

7. Pat has a method for solving quadratic equations. For example, Pat solves $6x^2 + x - 2 = 0$ as follows:

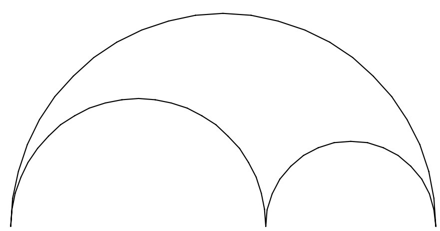
Step 1. Pat multiplies the leading coefficient by the constant coefficient and solved the simpler equation $x^2+x-12 = 0$ to get $(x+4)(x-3) = 0$.

Step 2. Pat then replaces each x by $6x$ (x times the leading coefficient) to get $(6x + 4)(6x - 3) = 0$.

Step 3. Pat then simplifies this equation to get $(3x+2)(2x-1) = 0$, which solves the original equation.

Prove or disprove that Pat's method always works.

8. An arbelos consists of three semicircular arcs as shown:



A circle is placed inside the arbelos so that it is tangent to all three semicircles.

Suppose that the radii of the two smaller semicircles are a and b, and that the radius of the circle is r.

Assuming that $a > b > r$ and that a, b and r are in arithmetic progression, calculate a/b.

THE OLYMPIAD CORNER

No. 202

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

In this issue, we give the problems of the 13^{th} Iranian Mathematical Olympiad, 1996 Second Round and the two exams of the Final Round. My thanks go to J.P. Grossman, Team Leader of the Canadian Mathematical Olympiad Team at Mumbai, India for collecting the problems for our use.

13^{th} IRANIAN MATHEMATICAL OLYMPIAD 1996

Second Round

Time: 2×4 hours

Problems are of equal value.

1. Prove that for every natural number $n \geq 3$ there exist two sets $A = \{x_1, x_2, \ldots, x_n\}$ and $B = \{y_1, y_2, \ldots, y_n\}$ such that

(a) $A \cap B = \emptyset$;

(b) $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$;

(c) $x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2$.

2. Let L be a line in the plane of an acute triangle ABC. Let the lines symmetric to L with respect to the sides of ABC intersect each other in the points A', B' and C'. Prove that the incentre of triangle $A'B'C'$ lies on the circumcircle of triangle ABC.

3. $12k$ persons have been invited to a party. Each person shakes hands with $3k+6$ persons. We know also that the number of persons who shake hands with any two persons is constant. Find the number of the persons invited.

4. Let n be a natural number. Prove that n can be written as a sum of some distinct numbers of the form 2^p3^q such that none of them divides any other. For example, $19 = 4 + 6 + 9$.

5. Prove that for any natural number n

$$\left\lceil \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \right\rceil = \left\lceil \sqrt{9n+8} \right\rceil .$$

6. In tetrahedron $ABCD$ let A', B', C', and D' be the circumcentres of faces BCD, ACD, ABD and ABC. We mean by $S(X, YZ)$, the plane perpendicular from point X to the line YZ. Prove that the planes $S(A, C'D')$, $S(B, D'A')$, $S(C, A'B')$, and $S(D, B'C')$ are concurrent.

Final Round – First Exam

Time: 2 × 4 hours

Each problem is worth seven points.

1. Prove the following inequality

$$(xy + xz + yz)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(x+z)^2}\right) \geq \frac{9}{4}$$

for positive real numbers x, y, z.

2. Prove that for every pair m, k of natural numbers, m can be expressed uniquely as

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t},$$

where

$$a_k > a_{k-1} > \cdots > a_t \geq t \geq 1.$$

3. In triangle ABC we have $\angle A = 60°$. Let O, H, I, and I' be the circumcentre, orthocentre, incentre, and the excentre with respect to A of the triangle ABC. Consider points B' and C' on AC and AB such that $AB = AB'$ and $AC = AC'$. Prove that

(a) eight points B, C, H, O, I, I', B', and C' are concyclic;

(b) if OH intersects AB and AC in E and F respectively, then triangle AEF has a perimeter equal to $AC + AB$;

(c) $OH = |AB - AC|$.

4. Let k be a positive integer. Prove that there are infinitely many perfect squares in the arithmetic progression $\{n \times 2^k - 7\}_{n \geq 1}$.

5. Let ABC be a non-isosceles triangle. Medians of the triangle ABC intersect the circumcircle in points L, M, N. If L lies on the median of BC and $LM = LN$, prove that $2a^2 = b^2 + c^2$.

6. We have attached the up and down, and left and right sides of an $n \times n$ chessboard (forming a torus) in order that a "tour" be constructed. Find the maximum number of knights which can be placed on this tour in such a way that none of them controls the square on which another lies.

Final Round – Second Exam

Time: 2 × 4 hours

Each problem is worth seven points.

1. Find all real numbers $a_1 \le a_2 \le \cdots \le a_n$ satisfying

$$\sum_{i=1}^{n} a_i = 96\,, \qquad \sum_{i=1}^{n} a_i^2 = 144\,, \qquad \sum_{i=1}^{n} a_i^3 = 216\,.$$

2. Points D and E are situated on the sides AB and AC of triangle ABC in such a way that $DE \| BC$. Let P be an arbitrary point inside the triangle ABC. Lines PB and PC intersect DE at F and G respectively. Let O_1 be the circumcentre of triangle PDG and let O_2 be that of PFE. Show that $AP \perp O_1O_2$.

3. Let $P(x)$ be a polynomial with rational coefficients such that $P^{-1}(\mathbb{Q}) \subseteq \mathbb{Q}$. Show that P is linear.

4. Let $S = \{x_1, x_2, \ldots, x_n\}$ be an n–element subset of $\{x \in R \mid x \ge 1\}$. Find the maximum number of the elements of the form

$$\sum_{i=1}^{n} \varepsilon_i x_i\,, \qquad \varepsilon_i = 0, 1$$

which belong to I, where I varies over all open intervals of length one and S over all n–element subsets.

5. Does there exist a function $f : R \to R$ that fulfils all of the following conditions:

(a) $f(1) = 1$;

(b) there exists $M > 0$ such that $-M < f(x) < M$;

(c) if $x \neq 0$ then

$$f\left(x + \frac{1}{x^2}\right) = f(x) + \left(f\left(\frac{1}{x}\right)\right)^2 .$$

6. Let a, b, c be positive real numbers. Find all real numbers x, y, z such that

$$x + y + z = a + b + c\,,$$

$$4xyz - (a^2x + b^2y + c^2z) = abc\,.$$

Next we turn to readers' comments and solutions for problems and solutions previously given in the *Corner*. First, a comment about a comment to a solution given in the October number of the *Corner*.

2. [1998: 196–197; 1999: 333] 44^{th} *Lithuanian Mathematical Olympiad*.
What is the least number of positive integers such that the sum of their squares is 1995?

Comment by R.K. Guy, University of Calgary.
In fact $1992 = 40^2 + 14^2 + 14^2$.

Here is a simpler solution to a problem discussed in the October *Corner*.

2. [1998: 133; 1999: 328] *VIII Nordic Mathematical Contest*.
A finite set S of points in the plane with integer coordinates is called a *two–neighbour set*, if for each (p, q) in S exactly two of the points $(p+1, q)$, $(p, q+1)$, $(p-1, q)$, $(p, q-1)$ are in S. For which n does there exist a two-neighbour set which contains exactly n points?

Solution by Carl Johan Ragnarsson, Lund, Sweden.
Put the points in the coordinate plane. Let (x, y) be even if $x + y = 0 \pmod 2$ and odd otherwise. Note that the even (odd) points have only odd (even) neighbours. So, if we have k even points, we will count $2k$ odd neighbours. But each odd point has two even neighbours, so each point was counted twice and there are k odd points as well. This means the number of points is even (it is equal to $2k$). 4 works by taking a 2 by 2 square $2k$, $k > 3$ works by starting with a 3 by 3 square with centre removed and then stretching it appropriately. Finally, it is evident that a set of six points does not work.

We now turn to solutions from our readers to the First Round problems of the 8th Korean Mathematical Olympiad [1998: 197–198].

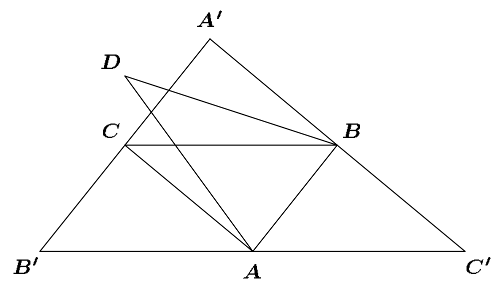
8th KOREAN MATHEMATICAL OLYMPIAD
First Round

1. Consider finitely many points in a plane such that, if we choose any three points A, B, C among them, the area of $\triangle ABC$ is always less than 1. Show that all of these finitely many points lie within the interior or on the boundary of a triangle with area less than 4.

Comments and solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's solution.
Let ABC be the triangle having the maximal area $[ABC]$. Then $[ABC] \leq 1$. Let $A'B'C'$ be the triangle whose medial triangle is ABC.

Then $[A'B'C'] \leq 4[ABC] \leq 4$. We will show that $A'B'C'$ contains all n of the points.

Assume, for a contradiction, that there is a point, D, outside $A'B'C'$. Then ABC and D lie on different sides of at least one of the lines $A'B'$, $B'C'$, $C'A'$.



Then $[ABD] \geq [ABC]$, contradicting the maximality of $[ABC]$.

Comment by P. Bornsztein: This problem and a solution appeared earlier [1993: 163].

2. For a given positive integer m, find all pairs (n, x, y) of positive integers such that m, n are relatively prime and $(x^2 + y^2)^m = (xy)^n$, where n, x, y can be represented by functions of m.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's solution.

Let p be a common prime divisor of x and y, and let α and β be the largest integers such that $p^\alpha \mid x$ and $p^\beta \mid y$.

Now $p^{2\alpha} \mid x^2$, $\quad p^{2\beta} \mid y^2$ and

$$p^{(\alpha+\beta)n} \mid (xy)^n \;=\; (x^2+y^2)^m \,.$$

We claim that $\alpha = \beta$ (and $x = y$). Indeed, if $\alpha < \beta$, then $p^{2\alpha} \mid x^2 + y^2$ and $p^{2\alpha m} \mid (x^2 + y^2)^m$; that is

$$2\alpha m \;=\; (\alpha+\beta)n \;>\; 2\alpha n\,,$$

and then $m > n$. But this is impossible since

$$(xy)^m < (2xy)^m \;\leq\; (x^2+y^2)^m \;=\; (xy)^n\,.$$

Similarly if $\alpha > \beta$ we obtain a contradiction. Hence $x = y$ and $(x^2+y^2)^m = (xy)^n$ reduces to $2^m x^{2m} = x^{2n}$. The solutions are of the form $(n, x, y) = (m+1, 2^{m/2}, 2^{m/2})$.

Comment by P. Bornsztein. This problem was posed on the 1992 William Lowell Putnam contest, and a solution was published in the American Math. Monthly (1993, pbA3, p. 760).

If m is odd there is no solution. For m even $n = m + 1$, $x = y = 2^m$.

Moreover, if we do not suppose $(n, m) = 1$ the solutions are those numbers of the form $m = 2a\alpha$, $n = m + \alpha$, $x = y = 2^a$.

3. Let A, B, C be three points lying on a circle, and let P, Q, R be the mid-points of arcs BC, CA, AB, respectively. AP, BQ, CR intersect BC, CA, AB at L, M, N, respectively. Show that

$$\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN} \geq 9 .$$

For which triangle ABC does equality hold?

Solutions and comments by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Amengual's solution.

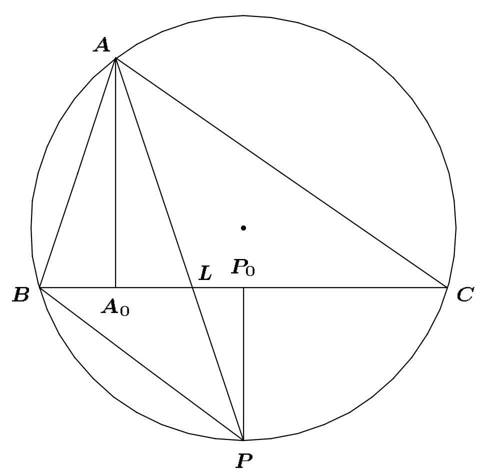
To be definite, we have taken P to lie on the arc BC that does not contain A.

Since P is the mid-point of BC, $\angle BAP = \angle PAC$ and AL is the internal bisector of the angle A of triangle ABC formed by joining the points A, B, C.

Let $AA_0 = h_a$ be the altitude from A and let P_0 be the foot of the perpendicular from P to the side BC.

The right triangles AA_0L and PP_0L are similar, so

$$\frac{AL}{PL} = \frac{AA_0}{PP_0} \quad \text{or} \quad \frac{AL}{PL} = \frac{h_a}{PP_0} . \tag{1}$$



If the lengths of the sides of $\triangle ABC$ are $BC = a$, $CA = b$ and $AB = c$, then the area of $\triangle ABC$ is $\frac{1}{2}a \cdot h_a$ and also $\frac{1}{2}bc \cdot \sin A$; hence

$$h_a = \frac{bc \sin A}{a} .$$

In $\triangle BP_0P$, we have $\angle P_0BP = \angle CBP = \frac{1}{2}\angle A$ (since both $\angle CBP$ and $\angle A/2$ are inscribed in the circular arc PC). Hence

$$PP_0 = BP_0 \cdot \tan\frac{A}{2} = \frac{a}{2} \cdot \tan\frac{A}{2}.$$

Substituting these expressions for h_a and PP_0 in (1) gives

$$\frac{AL}{PL} = \frac{4bc\cos^2\frac{A}{2}}{a^2}.$$

Since $\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$ where $s = \frac{a+b+c}{2}$, this is equivalent to

$$\begin{aligned}\frac{AL}{PL} &= \frac{2s}{a} \cdot \frac{2(s-a)}{a} \\ &= \left(1 + \frac{b}{a} + \frac{c}{a}\right)\left(-1 + \frac{b}{a} + \frac{c}{a}\right) \\ &= \left(\frac{b}{a} + \frac{c}{a}\right)^2 - 1.\end{aligned}$$

We have conducted our discussions with respect to the side BC of $\triangle ABC$. Applying the same reasoning to either of the other sides instead, we obtain

$$\begin{aligned}\frac{AL}{PL} &+ \frac{BM}{QM} + \frac{CN}{RN} \\ &= \left[\left(\frac{b}{a} + \frac{c}{a}\right)^2 - 1\right] + \left[\left(\frac{c}{b} + \frac{a}{b}\right)^2 - 1\right] + \left[\left(\frac{a}{c} + \frac{b}{c}\right)^2 - 1\right] \\ &= \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) + \left(\frac{b^2}{c^2} + \frac{c^2}{b^2}\right) + \left(\frac{c^2}{a^2} + \frac{a^2}{c^2}\right) \\ &\quad +2\left(\frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2}\right) - 3.\end{aligned}$$

According to the arithmetic mean–geometric mean inequality,

$$\frac{a^2}{b^2} + \frac{b^2}{a^2} \geq 2, \qquad \frac{b^2}{c^2} + \frac{c^2}{b^2} \geq 2,$$

$$\frac{c^2}{a^2} + \frac{a^2}{c^2} \geq 2, \qquad \frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2} \geq 3.$$

Therefore,

$$\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN} \geq 2 + 2 + 2 + 2 \cdot 3 - 3 = 9.$$

Equality occurs when $a = b = c$; that is, when $\triangle ABC$ is equilateral.

Aassila comments that this problem and a solution appeared in *Crux* [1989: 74; 1990: 158; 1991:48]. It was proposed by Mihaly Bencze, Brasov, Romania.

4. A partition of a positive integer n is a sequence $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Each λ_i is called a summand. For example, $(4, 3, 1)$ is a partition of 8 whose summands are distinct. Show that, for a positive integer m with $n > \frac{1}{2}m(m+1)$, the number of all partitions of n into distinct m summands is equal to the number of all partitions of $n - \frac{1}{2}m(m+1)$ into r summands $(r \leq m)$.

Solution by Mohammed Aassila, Strasbourg, France.

By the definition of a summand we have that $\lambda_m \geq 1$, $\lambda_{m-1} \geq 2, \dots$, $\lambda_1 \geq m$.

Define $\mu_k =: \lambda_k - (m - k + 1)$. Then $\mu_k \geq 0$ for all k and $\mu_1 + \mu_2 + \cdots + \mu_m = n - \frac{1}{2}m(m+1)$.

5. If we select at random three points on a given circle, find the probability that these three points lie on a semicircle.

Solutions by Mohammed Aassila, Strasbourg, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's solution and extensions.

It should have been stated that the random selection is one with a uniform distribution with respect to arc length; otherwise one could get varying answers. Also, this is a fairly widely known problem.

If θ_1, θ_2, θ_3 are measures of the three arcs between successive points, then their sum is 2π. So equivalently, we want the probability that θ_1, θ_2, θ_3 do not satisfy the triangle inequality; for example, each angle is less than π. Now consider an equilateral triangle $\triangle$ with altitude 2π. For any random point within or on the triangle the sum of the perpendiculars to the sides $= 2\pi$, so these perpendiculars can be taken as θ_1, θ_2, θ_3 (we can assume that the random point is uniformly distributed with respect to area). Then θ_1, θ_2, θ_3 will form a triangle if the point lies within the medial triangle, the one whose vertices are the mid-points of the sides of $\triangle$. Since the area of this triangle is $\frac{1}{4}$ of $\triangle$, the desired probability is $\frac{3}{4}$.

Comments: It may be of interest to give the analogous 3–dimensional problem from the Educational Times back in the 19th century:

"2621. (proposed by Rev. M.M.U. Wilkinson, M.A.) If four points be taken at random on the surface of a sphere, show that the chance of them all lying on some one hemisphere is $7/8$.

Solution by Stephan Watson. If great circles of the sphere be described through each pair of the three points, they will divide the surface of the sphere in eight portions, in *seven* of which the fourth point may lie so that the four points may all be in some one hemisphere. Moreover, when the three points take all positions on the surface of the sphere, the eight portions will all pass through the same magnitudes; hence the required chance is $7/8$."

As before, it is tacitly assumed that the random points are distributed uniformly with respect to surface area.

In "A problem in geometric probability", Math. Scand. 11 (1962), 109–111, J.G. Wendell gives an elegant proof that if one has N points scattered at random on the surface of a sphere in E^n, the probability that all the points lie in the same hemisphere is given by

$$p_{n,N} = 2^{-N+1} \sum_{k=0}^{n-1} \binom{N-1}{k},$$

and in particular, $p_{n,n+1} = 1 - 2^{-n}$. He also notes that $p_{n,N}$ equals the probability that in tossing a fair coin repeatedly, the nth "head" occurs on or after the Nth toss and that it does not seem possible to find an isomorphism between coin-tossing and the given problem that would make the result immediate.

6. Show that any positive integer $n > 1$ can be expressed by a finite sum of numbers satisfying the following conditions:

(i) they do not have factors except 2 or 3;

(ii) any two of them are neither a factor nor a multiple of each other.

That is,

$$n = \sum_{i=1}^{N} 2^{\alpha_i} 3^{\beta_i},$$

where α_i, β_i ($i = 1, 2, \ldots, N$) are non-negative integers and $(\alpha_i - \alpha_j)(\beta_i - \beta_j) < 0$ whenever $i \neq j$.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's solution.

We use induction on n. Assume that all integers $\leq n$ have the required representation, and let us prove that $n + 1$ has the same representation as well.

Case 1: n is odd. By the induction hypothesis, $\frac{n+1}{2}$ has the representation and hence $n + 1 = 2 \times \frac{n+1}{2}$ has the representation also. (Multiply the representation of $\frac{n+1}{2}$ by 2).

Case 2: n is even.

Case 2a: $n + 1$ is a power of 3.
Nothing to prove. The result is true.

Case 2b: $n + 1$ is not a power of 3.

In this case, there exists an integer m such that $3^m < n+1 < 3^{m+1}$. The integer $(n+1-3^m)$ is even, and $(n+1-3^m)/2$ is less than n. By the induction hypothesis it has a representation, whence $n+1-3^m$ does, and finally $n+1$ is the sum of 3^m and the representation of $n+1-3^m$.

7. Find all real-valued functions f defined on real numbers except 0 such that

$$\frac{1}{x}f(-x)+f\left(\frac{1}{x}\right) \;=\; x\,, \qquad x \neq 0\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's solution.

L'identité s'écrit : pour $t \neq 0$

$$f(-t)+tf\left(\frac{1}{t}\right) \;=\; t^2\,.$$

Pour $t = \frac{1}{x}$, l'identité s'éerit

$$f(t)+xf\left(-\frac{1}{t}\right) \;=\; \frac{1}{t} \qquad \text{(i)}$$

be même, pour $t = -x \neq 0$, on obtient

$$f(t)-tf\left(-\frac{1}{t}\right) \;=\; \frac{1}{t}\,. \qquad \text{(ii)}$$

On additionnant (i) et (ii), pour $x \neq 0$

$$2f(x) \;=\; x^2+\frac{1}{x}\,;$$

c.à.d.

$$f(x) \;=\; \frac{x^3+1}{2x}\,.$$

Réciproquement, si $x \neq 0$ et $f(x) = \frac{x^3+1}{2x}$,

$$\text{alors} \quad \frac{1}{x}f(-x)+f\left(\frac{1}{x}\right) \;=\; \frac{x^3-1}{2x^2}+\frac{x^3+1}{2x^2} \;=\; x\,.$$

Finalement, $f : \left(x \mapsto \frac{x^3+1}{2x}\right)$ est l'unique solution.

8. Two circles O_1, O_2 of radii r_1, r_2 $(r_1 < r_2)$, respectively, intersect at two points A and B. P is any point on circle O_1. Lines PA, PB and circle O_2 intersect at Q and R, respectively.

(i) Express $y = QR$ in terms of r_1, r_2, and $\theta = \angle APB$.

(ii) Show that $y = 2r_2$ is a necessary and sufficient condition that circle O_1 be orthogonal to circle O_2.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.
(i) In figure 1, $\angle PBA = \angle PQR$, both being supplements of $\angle ABR$.

Let W_1 and W_2 be the centres of O_1 and O_2. In Figure 2, the line W_1W_2, joining the centres of the intersecting circles, is perpendicular to their common chord AB which subtends at the centre W_1 (respectively W_2) twice the angle it subtends at P (respectively Q) on the circumference O_1 (respectively O_2), implying

$$\angle QPB = \angle W_2W_1B \qquad \text{and} \qquad \angle BQP = \angle BW_2W_1 .$$

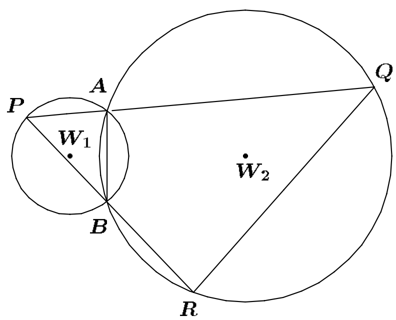


Figure 1.

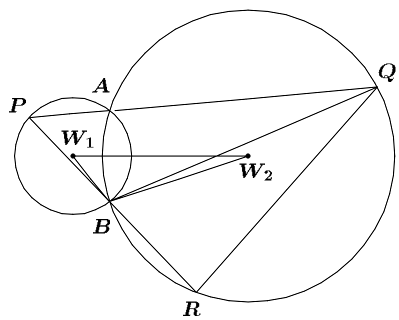


Figure 2.

The similar triangles $\triangle PAB \sim \triangle PRQ$ and $\triangle PBQ \sim \triangle W_1BW_2$ yield

$$\frac{QR}{AB} = \frac{PQ}{PB}, \qquad \frac{PQ}{PB} = \frac{W_1W_2}{W_1B},$$

whence

$$y = QR = \frac{AB}{W_1B} \cdot W_1W_2 \qquad (\overline{QR} \text{ is constant}) .$$

Since $\sin\theta = \dfrac{AB/2}{W_1B}$, we have $\dfrac{AB}{W_1B} = 2\sin\theta$, and hence

$$y = 2 \cdot W_1W_2 \cdot \sin\theta .$$

By the Law of Sines, applied to $\triangle W_1W_2B$ with $\angle BW_2W_1 = \alpha$,

$$\frac{W_2B}{\sin\theta} = \frac{W_1B}{\sin\alpha}; \qquad \text{that is,} \quad \sin\alpha = \frac{r_1}{r_2}\sin\theta .$$

Hence

$$\begin{aligned} W_1W_2 &= r_1\cos\theta + r_2\cos\alpha \\ &= r_1\cos\theta + r_2\sqrt{1 - \left(\frac{r_1}{r_2}\sin\theta\right)^2} \\ &= r_1\cos\theta + \sqrt{r_2^2 - r_1^2\sin^2\theta} . \end{aligned}$$

We conclude that

$$y \;=\; 2\left(r_1 \cos\theta + \sqrt{r_2^2 - r_1^2 \sin^2\theta}\right) \sin\theta \,.$$

(ii) In the equation $y = 2r_2$, we replace y by

$$2 \cdot \left(r_1 \cos\theta + \sqrt{r_2^2 - r_1^2 \sin^2\theta}\right) \sin\theta$$

and write it in the form

$$\left(\sqrt{r_2^2 - r_1^2 \sin^2\theta}\right) \sin\theta \;=\; r_2 - r_1 \sin\theta \cos\theta \,,$$

which is equivalent to the one obtained by squaring both sides; that is,

$$(r_2^2 - r_1^2 \sin^2\theta) \sin^2\theta \;=\; r_2^2 - 2r_1 r_2 \sin\theta\cos\theta + r_1^2 \sin^2\theta \cos^2\theta \,,$$

which is equivalent to

$$(r_2 \cos\theta - r_1 \sin\theta)^2 \;=\; 0 \,.$$

This, in turn, is equivalent to

$$\tan\theta \;=\; \frac{r_2}{r_1} \,,$$

or finally

$$\angle W_1 B W_2 \;=\; 90^\circ \,.$$

(Of course, this angle is the same at both intersections A and B).

Since the tangent line to a circle is perpendicular to the radius at the point of contact, the angle between the tangents at both intersections A and B is 90° and O_1, O_2 cut orthogonally.

This completes the *Olympiad Corner* for this issue. Please send me your nice solutions and Olympiad contest sets.

BOOK REVIEWS

ALAN LAW

Which way did the Bicycle Go? . . . and Other Intriguing Mathematical Mysteries by Joseph D.E. Konhauser, Dan Velleman, Stan Wagon,
published by the Mathematical Association of America, 1996 (Dolciani Mathematical Expositions, No. 18)
ISBN # 0-88385-325-6, softcover, 245+ pages.
Reviewed by ***Einar Rodland****, University of Oslo.*

"Which way did the bicycle go?" Such is the problem facing Sherlock Holmes having found the tracks in the snow. "It was undoubtably heading away from the school," he observes, presenting an incorrect argument about the hind wheel "obliterating the more shallow mark of the front one." Oh, no, my dear mister Holmes; this time, you should have turned to mathematics.

This problem sets the tone of the book: problems are formulated so as to be understandable and attractive without relying on a mathematical background. This was Joe Konhauser's style when he posted the "Problem of the Week" at Macalester College, a column later taken over by Stan Wagon. This reviewer believes them successful in doing so; though most of the problems may demand some mathematical interest from the students, there are some that may even appeal to those normally not attracted by the subject. The main target group of the problems is advanced high school pupils and beginning college students. However, there are plenty of problems that may be given to both younger and to older. As such, it may in particular be a valuable source for teachers trying to engage their students.

The problems, counting 191, are ordered by type—plane geometry, number theory, algebra, combinatorics and graph theory, three-dimensional geometry, and miscellaneous—which is handy when trying to recover a previously read problem. However, as the problems generally do not match any curriculum particularly well, I would expect a teacher seeking an inspiring problem to largely ignore this ordering, picking problems from throughout the book: it would not be easy to find problems to illustrate a particular technique.

Well over two thirds of the book is devoted to giving complete solutions to all the problems, and occasionally there are several alternative solutions. The solutions are clearly written and, though occasionally knowledge of some non-trivial results is needed, should be sufficiently detailed for even the less advanced students to read. Of course, the question "How did they think of that?" will occasionally arise: some solutions rely on finding the right trick, an art which is not easily explained. On the other hand, some of the problems

The complete set is ISBN 0-88385-300-0.

lead naturally to more traditional mathematical theory, or refer to research done by others, which helps indicate that mathematics goes beyond mere puzzle-solving.

It was said that the book contains 191 problems. This, however, is not fully true. In the solution of several problems, new problems are stated which allows one to test one's understanding of the solution: to try it out on a similar problem or proceed with a more advanced problem, perhaps a generalization. This may help coach students (and teachers) into not only trying out methods on new problems, but trying to pose new problems, a skill which is also useful: as useful as being able to solve the problems, many would claim.

The overall conclusion is that the book would have its greatest strength in the hands of a teacher; with this book at hand, a high-school or undergraduate teacher would have an ample supply of problems, and could, without too much of an effort (finding good problems can be quite hard), regularly challenge his or her students with problems of a different kind from what they traditionally meet. As such, it is highly recommended.

The Mathematical Olympiad Handbook; an Introduction to Problem Solving
by Tony Gardiner,
published by Oxford University Press, 1997,
ISBN # 0-19-850105-6, softcover, 229+ pages, US$29.95.
Reviewed by ***Catherine Shevlin****, Wallsend upon Tyne, England.*

Mathematical Olympiads began in Hungary in the nineteenth century. They are now held for high school students throughout the world. They feature problems which, though they require only high school mathematics, seem very difficult because they are unpredictable and have no obvious starting point. This book introduces readers to these delightful and challenging problems and aims to convince them that Olympiads are not just for a select minority. The book contains problems from the British Mathematical Olympiad (BMO) competitions between 1965 and 1996. It includes hints and solutions for each problem from 1975 on, a review of the basic mathematical skills needed, and a list of recommended reading, making it an ideal source for enriching one's experience in mathematics.

The World Championship Mathematical Olympiad, the International Mathematical Olympiad, began in Romania in 1959, with the participation of a few countries from eastern Europe. It has now grown to a worldwide competition with participation from all continents. Tony Gardiner has been the Leader of the United Kingdom team to the International Mathematical Olympiad, and so is well qualified to write a book such as this.

The contents are:

- Problems and Problem Solving
- How to Use this Book

The problems consist of all the BMO's from the first (1965) up to the 32nd (1996), together with hints and outline solutions for the 11th (1975) up to the 32nd (1996). These are given in sufficient detail to enable the interested reader to work out full solutions. For example, for question 5 of the 32nd BMO, he states:

> *If you have a clever idea about how to solve both parts at once, you will probably not need this outline solution. So I shall begin as though you have* not *yet solved the problem.*

He ends the solution to question 3 of the 31st BMO with the admonition:

> *There are many other correct ways, But beware: most arguments are flawed!*

The author states that the book is "unashamedly" for beginners. He aims to convince many people that solving Mathematical Olympiad problems is for them, and "not just for some bunch of freaks". There is this common misconception around that those students who are good at mathematics are abnormal. From my personal experience in coaching students for Mathematical Olympiads, nothing can be further from the truth. They are as normal a population as any other segment of the population.

What is sad is that they are not as highly regarded amongst their peers as are, say, those students who excel in games.

Gardiner's book covers the main topics required for Olympiad problems in a compact and readable form. As he says, those who want more must read more. His section entitled "Some Books for Your Bookshelf" is very comprehensive. It ends up with referring to this journal as "the Problem Solver's Bible"!

Tony Gardiner is well qualified to write such a book. He has been involved in the British Mathematical Olympiad for several years. He was Leader of the UK International Mathematical Olympiad team from 1990 to 1995, and has been Vice-President of the World Federation of National Mathematics Competitions. The Federation, in recognition of his work in Mathematical Olympiads, awarded him the 1995 Paul Erdős National Award.

This book is a must for any high school student desirous of a challenge in mathematical problem solving. It is also a must for anyone involved in the encouragement and training of potential Mathematical Olympiad participants. Problem solving is, after all, the essence of mathematics.

Mathematical Competitions in Croatia edited by Željko Hanjš,
published by the Croatian Mathematical Society, Zagreb, 1998.
Softcover, 31 pages.
Reviewed by ***Richard Hoshino****, University of Waterloo.*

There are many mathematical competitions administered in Croatia, for students in Grades 5 through 12. There are municipal contests in March, which are followed by regional contests in April, and the process culminates in a national olympiad contest in May, which only the very top students are invited to write. The Croatian contest system is quite similar to the American three-contest system, where students write the AHSME to qualify for the AIME, and outstanding performance in these contests qualifies a student for the USAMO.

In Croatia, students write different contests according to their grade. Thus a student in the I class (Grade 9) writes a different contest than does a fellow student in the II class (Grade 10). Each of the three contests (municipal, regional, and national) consists of four questions.

This book contains the 48 problems from the 1998 Croatian math contests, and solutions are given for each of the problems. Many of these problems are quite demanding for even the very best students, especially the problems intended for the IV class (Grade 12). To illustrate this, one of the questions on the IV class *municipal* contest is identical to Question 2 of the 1967 IMO!

The problems are both engaging and challenging, and will certainly be of benefit to anyone who is interested in problem-solving. We close off with one of the problems that appeared on the III class (Grade 11) regional contest.

> Prove that between every 79 successive natural numbers there exists at least one whose sum of digits is divisible by 13. Find a sequence of 78 successive natural numbers with the property that the sum of digits of any of its members is not divisible by 13.

Women in Mathematics: Scaling the Heights edited by Deborah Nolan,
published by the Mathematical Association of America, 1997,
ISBN 0-88385-156-3, softcover, 121+ pages, $29.95 (U.S.)
Reviewed by ***Julia Johnson****, University of Regina, Regina, Saskatchewan.*

This book will engender emotion in any woman who has been driven by the joy of mathematics and met with obstacles to the fulfilment of that joy. It is #46 in the MAA Notes series which comprises a breadth of works in mathematics education. This particular volume is an augmented version of a report based on an NSF funded conference aimed at developing programs to advance women in mathematics.

Several introductory articles provide information about why "Somewhere between the beginning of graduate school and the tenure decision, a lot of things can and do go wrong for women, things that do not seem to go wrong as often for men." (Carol Wood, page 14). One of the factors is that women in mathematics are very often isolated. A special program for women is advocated, not because women have to learn mathematics in a special way, but because most successful male mathematicians have had such supportive environments. The strategies presented are therefore not gender specific. They can be used to create an exciting and challenging environment for both women and men to learn mathematics.

The conference centres on the program of the Mills College Summer Mathematics Institute (SMI). This augmented version of the conference report aims at giving insight into the type of women who attend the summer institute in terms of their math backgrounds and future career plans. An article entitled "A View of Mathematics from an Undergraduate Perspective" provides data which indicate that women are less confident in their ability to succeed in the field of mathematics than are men. This effect snowballs to make it less likely that women will succeed.

A number of modules complete with exercises and solutions for teaching various topics in upper level courses in undergraduate mathematics are provided. This section provides a terrific resource. The course designs were supplied by faculty of the SMI, who include advice on how to incorporate their techniques into the traditional classroom. A seminar that stands out is entitled "What are numbers", by Svetlana Katok, which uses Kirillov's general philosophy of consequent extensions to reveal the concept of *number* in modern mathematics.

The final section describes a variety of summer mathematics programs including the Carleton College and the St. Olaf College programs, the Program for Women in Mathematics at George Washington University and the Mills College SMI which has since moved to UC Berkeley and is now called the Summer Institute for the Mathematical Sciences (SIMS). Subjects range from "Where are the students now?" to "What do students and faculty think about the program?" At SIMS all instructors are women, not because men would not be effective teachers of women, but because women instructors serve as role models for students to break the stereotypical image of a mathematician being a man. Women generally come back from such a program with confidence in their ability to do mathematics and inspired to pursue graduate work in mathematics.

One is left with the realization that specific courses are critical to generating student interest in studying advanced mathematics. It is also apparent from this reading that an environment for learning mathematics which does not encourage working on math problems in teams provides a greater obstacle to those who are passionate about mathematics than to those who are less so.

After Math, Puzzles and Brainteasers by Ed Barbeau, published by Wall & Emerson, Inc., Toronto, Ontario/Dayton, Ohio. 1995.
ISBN # 0-921332-42-4, softcover, 198 pages.
Reviewed by **Mogens Esrom Larsen**, *University of Copenhagen, Copenhagen, the Netherlands.*

I love the book and heartily recommend it to all puzzle addicts.

It is a collection of challenging problems in elementary mathematics partly from the journal Alumni Magazine of the University of Toronto, from where it got the title.

Most of the problems are small gems with solutions varied from "Aha" to a systematic analysis of the inspired generalization. The problems are from most mathematical disciplines, but numbers and geometry are the favourites.

As an example, make a 9–digit number of the digits $1, \ldots, 9$ such that the number created by the first n digits is divisible by n. The solution is unique (381654729). Or, find the locus of the mid-point of a ladder sliding along the floor while leaning against a wall (a circular arc). Or, having 10 numbers less than 100, is it always possible to select two disjoint subsets having the same sum? Well, the total number of subsets is 1023, so, as the sums are always smaller than 1000, two of them must be equal.

Besides the solutions, there are hints before and explanations after to place the problems in the proper context. Perhaps too pedagogical to some but an understandable temptation for a university teacher, I agree.

THE SKOLIAD CORNER

No. 42

R.E. Woodrow

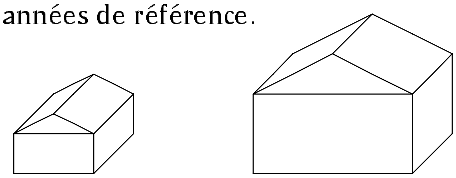
This number we give the thirty problems of Maxi éliminatoire 1996 of the 21$^{\text{ième}}$ Olympiade Belge organized by the Belgian Mathematics Teachers' Association. Twenty-six of the questions are multiple choice. For the remaining four the answer is an integer in the interval $[0, 999]$. Correct answers score 5 points, 2 points are given for no response and 0 for an incorrect answer. No calculators allowed! Time allowed, 90 minutes. My thanks go to Ravi Vakil for collecting this set when he was Canadian Team Deputy Leader to the International Mathematical Olympiad at Mumbai, India.

21$^{\text{ième}}$ OLYMPIADE MATHÉMATIQUE BELGE

Maxi éliminatoire 1996

Mercredi 17 janvier 1996

1. Dans un article de journal au sujet de la construction de logements, les schémas suivants sont supposés représenter le nombre d'habitations bâties durant deux années de référence.

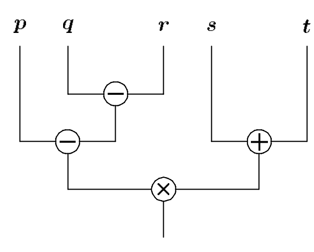


Quel est, selon ce graphique, le rapport du nombre de constructions de 1985 à celui de 1975 ?

(a) 2 (b) 4 (c) 8 (d) 12 (e) 16

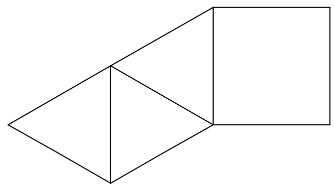
2. *Sans réponse préformulée* — Dans une pièce obscure, un tiroir contient des bougies de même forme et de même taille : 7 blanches, 5 bleues, 5 jaunes, 3 vertes et 3 rouges. Combien faut-il en prendre pour être sûr d'en avoir au moins deux de la même couleur ?

3. Laquelle des formules suivants traduit l'organigrame ci-contre ?

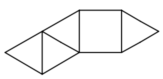


(a) $x = (p - q - r) \times (s + t)$
(b) $x = (p - (q - r)) \times (s + t)$
(c) $x = (p - (q - r)) \times s + t$
(d) $x = (p - q - r) \times s + t$
(e) $x = p - q - r \times s + t$

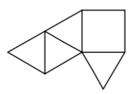
4. Voici une partie d'un développement d'une pyramide à base carrée dont les faces latérales sont des triangles équilatéraux. Laquelle des figures ci-dessous en donne un développement complet ?



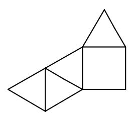
(a)



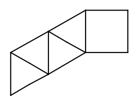
(b)



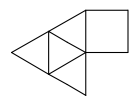
(c)



(d)



(e)



5. *Sans réponse préformulée* — Combien existe-t-il de nombres de 4 chiffres constitués de deux paires, distinctes de chiffres identiques, comme par exemple 1661, 1122 on 1414 (mais non 3333) ?

6. Si $p = 500\ 000^5$, $q = 200\ 000\ 000^4$, $r = 10^{25}$ et $s = (\frac{1}{10})^{1000}$, laquelle des chaines d'inégalités suivantes est exacte ?

(a) $p < q < r < s$ (b) $q < p < r < s$ (c) $s < r < p < q$ (d) $s < r < q < p$ (e) Aucune des précédentes

7. Pour tout nombre réel a différent de 0, de 1, de 2 et de 3, on pose $b = a - 1$, $c = b - 1$ et $d = c - 1$. La somme

$$\frac{a-1}{(a-2)(a-3)} + \frac{a-2}{(a-3)(a-1)} + \frac{a-3}{(a-1)(a-2)}$$

est toujours égale à l'une des expressions suivantes. Laquelle ?

(a) $\frac{3}{bcd}$ (b) $\frac{1}{bcd}$ (c) $\frac{b+c+d}{bcd}$ (d) $\frac{b^2+c^2+d^2}{bcd}$ (e) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

8. Quel est le reste de la division par 5 de l'entier n si $\frac{3n+4}{5} \in \mathbb{Z}$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

9. *Sans réponse préformulée* — Deux lignes de chemin de fer mesurent l'une 3672 m et l'autre 5472 m ; elles ont été construites avec des rails tous identiques, sans qu'il faille en recouper. Quelle est, en mètres, la longueur de ceux-ci, sachant que c'est un nombre entier et qu'il a été choisi aussi grand que possible ?

10. Quelle valeur faut-il donner au réel k pour que, quelle que soit la valeur du réel a, le polynôme

$$2x^4 - 5ax^3 + 2a^2x^2 - 5a^3x + 2k$$

soit divisible par $x + 2a$?

(a) $-90a^4$ (b) $90a^2$ (c) $-45a^4$ (d) $5a^4$ (e) 45

11. Un vol a été commis à l'école par une seule des quatre élèves ci-dessous, qui font successivement à la directrice les déclarations suivantes :

ALICE : "Barbara est coupable."

BARBARA : "Charlotte est coupable."

CHARLOTTE : "Barbara a menti."

DANIÈLE : "Je ne suis pas coupable."

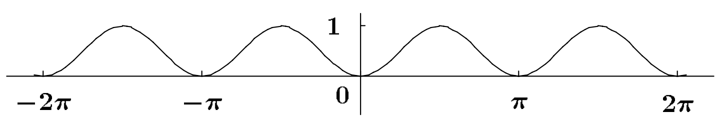
Si une et une seule de ces affirmations est vraie, qui a commis le vol ?

(a) Alice (b) Barbara (c) Charlotte (d) Danièle
(e) Les données ne permettent pas de la déterminer

12. Laquelle des transformations suivantes de $\mathbb{Z}$ est une bijection ?

(a) $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto 3x$
(b) $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto \frac{1}{2}x(x+1)$
(c) $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto 5 - x$
(d) $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto (3x-1)^2$
(e) $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto x - |x| + |x-1|$

13. Voici le graphe d'une fonction $f : \mathbb{R} \to \mathbb{R}$, périodique de période π.



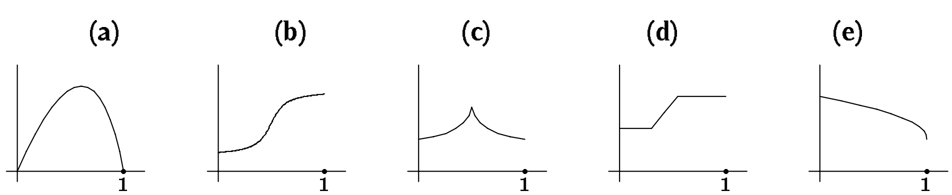
Sachant que $f(x)$ est donné par une des cinq expressions suivantes, de laquelle s'agit-il ?

(a) $\sin^2 x$ (b) $\cos^2 x$ (c) $|\sin x|$ (d) $\sin|x|$ (e) $1 + \sin x$

14. Par lequel des nombres suivants $2^{1996} - 1$ n'est-il *pas* divisible ?

(a) 1 (b) 3 (c) 5 (d) 7 (e) 15

15. Une fonction $f : \mathbb{R} \to \mathbb{R}$ est partout dérivable et sa dérivée s'annule en 0 ainsi qu'en 1. Lequel des graphes suivants est, de manière plausible, celui de f sur l'intervalle $[0, 1]$?

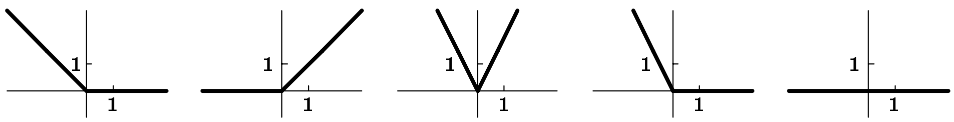


16. Une seule des inégalités ci-dessous est vraie pour tout, réel strictement positif a et tout réel x. Laquelle ?

(a) $a^x - a^{-x} \geq 0$ (b) $a^x - a^{-x} \leq 0$ (c) $a^x + a^{-x} \geq 2$
(d) $a^x + a^{-x} \leq 2$ (e) $a^x + a^{-x} \leq 1$

17. Étant donné $f : \mathbb{R} \to \mathbb{R} : x \mapsto -x$ et $g : \mathbb{R} \to \mathbb{R} : x \mapsto |x|$, lequel des graphes suivants est celui de $f + g$?

(a) (b) (c) (d) (e)



18. L'Algébristan est constitué de quatre provinces qui ont des densités de population de 20, 24, 36 et 84 habitants/km². Il résulte de ces informations que la densité de la population de l'Algébristan

(a) est de 30 habitants/km² ;
(b) est de 41 habitants/km² ;
(c) est inférieure à 11 habitants/km² ;
(d) est supérieure à 20 habitants/km² ;
(e) est comprise entre 22 et 60 habitants/km².

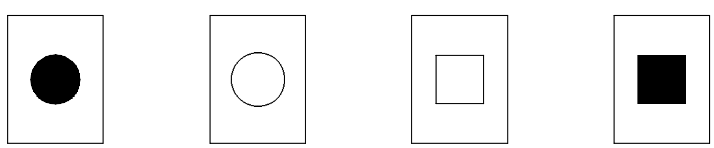
19. Si h est la composée $f \circ g$ des deux fonctions $g : \mathbb{R} \to \mathbb{R} : x \mapsto x + 2x^3$ et $f : \mathbb{R} \to \mathbb{R} : x \mapsto 1 + x^2$, alors, pour tout réel x, $h(x) =$

(a) $1 + x + x^2 + 2x^3$ **(b)** $x + 3x^3 + 2x^5$ **(c)** $1 + x^2 + 4x^6$
(d) $1 + x^2 + 4x^4 + 4x^6$ **(e)** $3 + 7x^2 + 6x^4 + 2x^6$

20. Soit a et b deux naturels, avec $a > b$. Soit x le reste de la division de a^{1996} par $a - b$ et y le reste de la division de b^{1996} par $a - b$. Alors, nécessairement,

(a) $x + y > 1995$ **(b)** $x - y = 0$ **(c)** $ax - by = 0$
(d) $x^{1996} - y^{1996} < 0$ **(e)** $x^{1996} - y^{1996} > 0$

21.



1 2 3 4

Claude a dessiné sur chacune des quatre fiches ci-dessus un disque d'un côté et un carré de l'autre, et m'affirme que, si le disque est noir, le carré l'est également. Pour m'assurer que cela est exact,

(a) je dois retourner les quatre fiches ;

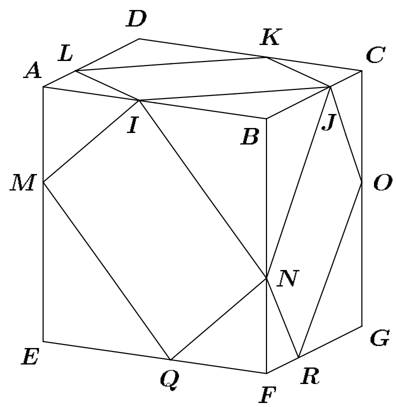
(b) il me suffit de retourner les fiches 1 et 2 ;

(c) il me suffit de retourner les fiches 1 et 3 ;

(d) il me suffit de retourner les fiches 1 et 4 ;

(e) il me suffit de retourner la fiche 1.

22. Du cube ci-contre, de volume V, sont enlevées les huit pyramides $AILM$, $BIJN$, etc., dont toutes les faces sont des triangles isocèles. Si $\overrightarrow{AI} = \frac{1}{3}\ \overrightarrow{AB}$, quel est le volume du solide restant ?



(a) $\frac{1}{2}V$ (b) $\frac{2}{3}V$ (c) $\frac{3}{4}V$ (d) $\frac{5}{6}V$ (e) $\frac{7}{9}V$

23. Un trapèze isocèle a sa petite base et les deux côtés adjacents de longueur fixée L. Pour quelle valeur (en radians) de l'angle entre la grande base et un côté adjacent l'aire de ce trapèze est-elle maximale ?

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{5}$ (e) $\frac{\pi}{6}$

24. La suite $(a_0, a_1, a_2, \dots)$ est définie par $a_0 = a_1 = 1$ et $(\forall n \in \mathbb{N})$ $a_{n+2} = a_{n+1} + a_n$. Sachant que $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ existe, que vaut cette limite ?

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}(\sqrt{5}-1)$ (e) $\frac{1}{2}(\sqrt{5}+1)$

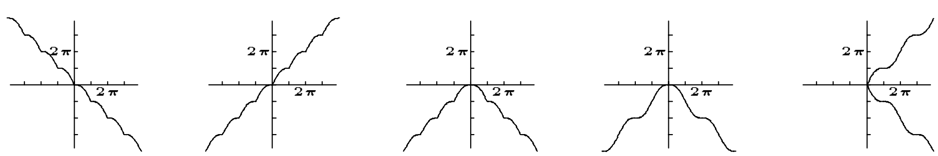
25. Quel est le reste de la division de $6^{83} + 8^{83}$ par 49 ?

(a) 0 (b) 2 (c) 28 (d) 35 (e) 42

26. Lequel des graphes ci-dessous est celui de la fonction

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto |\sin x| - |x| ?$$

(a) (b) (c) (d) (e)

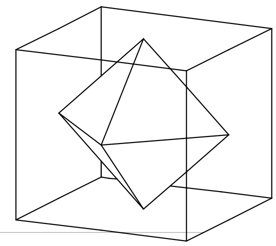


27. *Sans réponse préformulée* — Voici deux suites arithmétiques :

$$3, 7, 11, 15, \dots, 407$$
$$2, 9, 16, 23, \dots, 709$$

de raisons respectives 4 et 7. Combien ont-elles de termes en commun ?

28. Un octaèdre régulier est inscrit dans un cube, chacun de ses sommets étant le centre de l'une des faces du cube. Quel est le rapport de l'aire du cube à celle de l'octaèdre ?



(a) $\frac{\sqrt{3}}{6}$ (b) $\frac{3}{4}$ (c) $\sqrt{3}$ (d) $2\sqrt{3}$ (e) 6

29. Quels que soient les nombres réels a, b et x tels que $0 < a < b$ et $x \notin \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$, l'expression

$$\frac{1}{\sqrt{b-a}} \cdot \frac{\sqrt{\frac{b-a}{a}}\sin x}{\sqrt{1 + \left(\sqrt{\frac{b-a}{a}}\sin x\right)^2}} \cdot \sqrt{a + b\tan^2 x}$$

se simplifie en l'une des suivantes ; laquelle ?

(a) 1 (b) $\tan x$ (c) $\frac{\tan x}{a}$ (d) $|\tan x|$ (e) $\frac{\sin x}{|\cos x|}$

30. Quel est le nombre de solutions de l'équation $\sin(100x) = x$?

(a) 59 (b) 60 (c) 61 (d) 62 (e) 63

Last number we gave the problems of the British Columbia Senior High School Mathematics Contest, Final Round, Parts A and B. Next we give the "official" solutions. My thanks go to Jim Totten, one of the contest organizers, for furnishing them for our use.

BRITISH COLUMBIA COLLEGES SENIOR HIGH SCHOOL MATHEMATICS CONTEST

Final Round 1998

Part A

1. If $\left(r + \frac{1}{r}\right)^2 = 3$, then $r^3 + \frac{1}{r^3} =$

(a) 0 (b) 1 (c) 2 (d) $2\sqrt{3}$ (e) $4\sqrt{3}$

Answer: The correct answer is **(a)**.
If $(r + \frac{1}{r})^2 = 3$ then $r + \frac{1}{r} = \pm\sqrt{3}$, and

$$\begin{aligned} r^3 + \frac{1}{r^3} &= \left(r + \frac{1}{r}\right)^3 - 3r^2\frac{1}{r} - 3r\frac{1}{r^2} \\ &= \left(r + \frac{1}{r}\right)^3 - 3r\frac{1}{r}\left(r + \frac{1}{r}\right) = (\pm\sqrt{3})^3 \mp 3\sqrt{3} = 0. \end{aligned}$$

2. Kevin has five pairs of socks in his drawer, all of different colours and patterns and, being a typical teenage boy, they are not folded and have been thoroughly mixed up. On the first day of school Kevin reaches into his sock drawer without looking and pulls out three socks. What is the probability that two of the socks match?

(a) $\frac{3}{10}$ (b) $\frac{3}{5}$ (c) $\frac{1}{3}$ (d) $\frac{1}{24}$ (e) $\frac{1}{15}$

Answer: The correct answer is **(c)**.

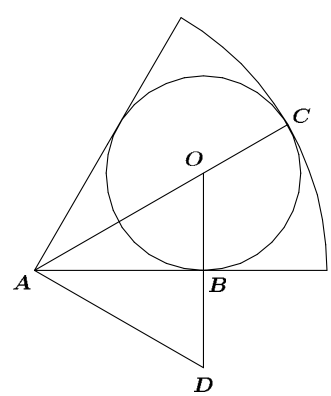
There are $120 = \binom{10}{3} = \frac{10\times9\times8}{1\times2\times3}$ different three socks subsets in the set of ten socks. On the other hand, every three sock subset with two matching socks contains one of the five matching pairs accompanied by one of the eight remaining socks. Therefore, there are $40 = 5 \times 8$ such subsets. This gives the probability of $\frac{40}{120} = \frac{1}{3}$.

3. A small circle is drawn within a $\frac{1}{6}$ sector of a circle of radius r, as shown. The small circle is tangent to the two radii and the arc of the sector. The radius of the small circle is:

(a) $\frac{r}{2}$ (b) $\frac{r}{3}$ (c) $\frac{2\sqrt{3}\,r}{3}$ (d) $\frac{\sqrt{2}\,r}{2}$ (e) none of these

Answer: The correct answer is **(b)**.

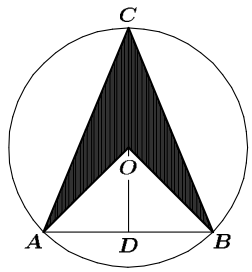
If ρ denotes the radius of the small circle then $AO = 2\rho$, since triangle AOB is a half of an equilateral triangle AOD. Furthermore, $OC = \rho$, and $r = AO + OC = 2\rho + \rho = 3\rho$. Thus, $\rho = \frac{r}{3}$.



4. In the accompanying diagram the circle has radius 1, the central angle AOB is a right angle and AC and BC are of equal length. The shaded area is:

(a) $\frac{\pi}{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\frac{\pi-\sqrt{2}}{2}$ (d) $\frac{\sqrt{2}+1}{2}$ (e) $\frac{1}{2}$

Answer: The correct answer is **(b)**.



The shaded area can be evaluated by subtracting the area of triangle AOB from the area of triangle ACB. The area of triangle AOB is $\frac{1}{2}(AO)(OB) = \frac{1}{2}$, since AOB is a right angle. For the same reason $AB = \sqrt{2}$, and consequently, $DO = \frac{1}{2}AB = \frac{\sqrt{2}}{2}$. The area of triangle ABC is $\frac{1}{2}(AB)(DC) = \frac{1}{2}(AB)(DO + OC) = \frac{1}{2}\sqrt{2}(\frac{\sqrt{2}}{2} + 1) = \frac{\sqrt{2}+1}{2}$. Consequently, the shaded area is $\frac{\sqrt{2}+1}{2} - \frac{1}{2} = \frac{\sqrt{2}}{2}$.

5. The side, front and bottom faces of a rectangular solid have areas $2x$, $\frac{y}{2}$, and xy square centimetres, respectively. The volume of the solid is:

(a) xy (b) $2xy$ (c) x^2y^2 (d) $4xy$ (e) impossible to determine from the given information

Answer: The correct answer is (a).

The volume of a rectangular solid is equal to the square root of the product of the areas of its nonparallel faces. For, if a, b, and c are the edges of the solid then ab, bc, and ca are the corresponding areas of its faces, and its volume is $abc = \sqrt{(ab)(bc)(ca)}$. Consequently, the volume of our solid is $\sqrt{(2x)(\frac{y}{2})(xy)} = \sqrt{(xy)^2}$. Obviously, x and y are non-negative, since $2x$ and $\frac{y}{2}$ represent areas. Therefore, $\sqrt{(xy)^2} = xy$.

6. The numbers from 1 to 25 are each written on separate slips of paper which are placed in a pile. You draw slips from the pile without replacing any slip you have chosen. You can continue drawing until the *product* of two numbers on any pair of slips you have chosen is a perfect square. The maximum number of slips you can choose before you will be forced to quit is:

(a) 13 (b) 14 (c) 15 (d) 16 (e) 17

Answer: The correct answer is (d).

Let a, b, and c be three numbers from our set. If ab and bc are perfect squares then ac is a perfect square too. For, if $ab = m^2$ and $bc = n^2$ then $ac = \frac{(ab)(bc)}{b^2} = (\frac{mn}{b})^2$ and, obviously, ac is an integer. This implies that the set $\{1, 2, 3, \ldots, 25\}$ can be partitioned into disjoint subsets with the following properties:

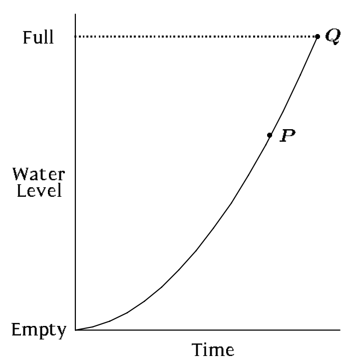
1. A product of any two elements from the same subset is a perfect square.

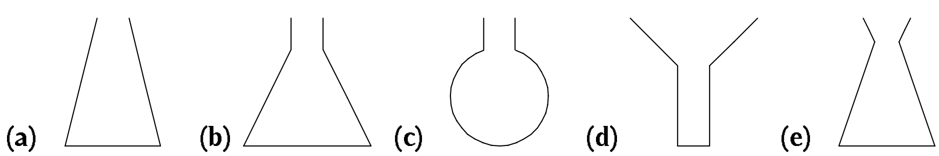
2. A product of any two elements selected from different subsets is not a perfect square.

In force of these properties we can easily find the partition, because in order to form a subset containing a number a, all we need is to collect all numbers b, such that ab is a perfect square. We start by finding the subset containing 1, then find the other subsets by successively completing the smallest number not yet selected to a subset. The partition is: $\{1, 4, 9, 16, 26\}$, $\{2, 8, 18\}$, $\{3, 12\}$, $\{5, 20\}$, $\{6, 24\}$, $\{7\}$, $\{10\}$, $\{11\}$, $\{13\}$, $\{14\}$, $\{15\}$, $\{17\}$, $\{19\}$, $\{21\}$, $\{22\}$, $\{23\}$. Clearly, we can pick at most one number slip from each subset. This gives a maximum of 16 slips.

Note: The reader familiar with the notion of an equivalence relation will notice that the relation $a \sim b$ *if and only if* ab *is a perfect square* is an equivalence relation and that the optimal choice of slips corresponds to a selector from the disjoint equivalence classes of this relation.

7. A container is completely filled from a tap running at a constant rate. The accompanying graph shows the level of the water in the container at any time while the container is being filled. The segment PQ is a straight line. The shape of the container which corresponds with the graph is:



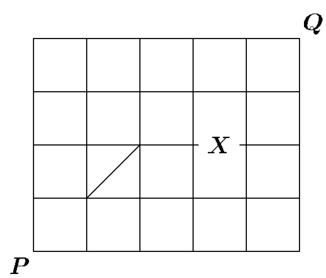


Answer: The correct answer is **(b)**.

It is clear that wider parts of a container will take more time to be filled with water than its narrower parts. More precisely, the rate of increase of the level of water is inversely proportional to the cross-sectional area of the container at that level. Thus, the constant slope of the last part of the graph, PQ, implies the constant width of the top of the container. This leaves us with two possible shapes: (b) or (c). However, the smaller slope of the first part of the graph indicates the wide bottom of the container. This agrees with the shape shown in (b).

Comment. As noted last issue, this problem is identical to problem 3 in part A of the British Columbia Colleges Junior High School Mathematics Contest Final Round 1998 [1999:341–343]. This solution completes the Junior Contest solutions which were given last issue.

8. The accompanying diagram is a road plan of a small city. All the roads go east-west or north-south, with the exception of the one short diagonal road shown. Due to repairs one road is impassable at the point X. Of all the possible routes from P to Q, there are several shortest routes. The total number of shortest routes is:

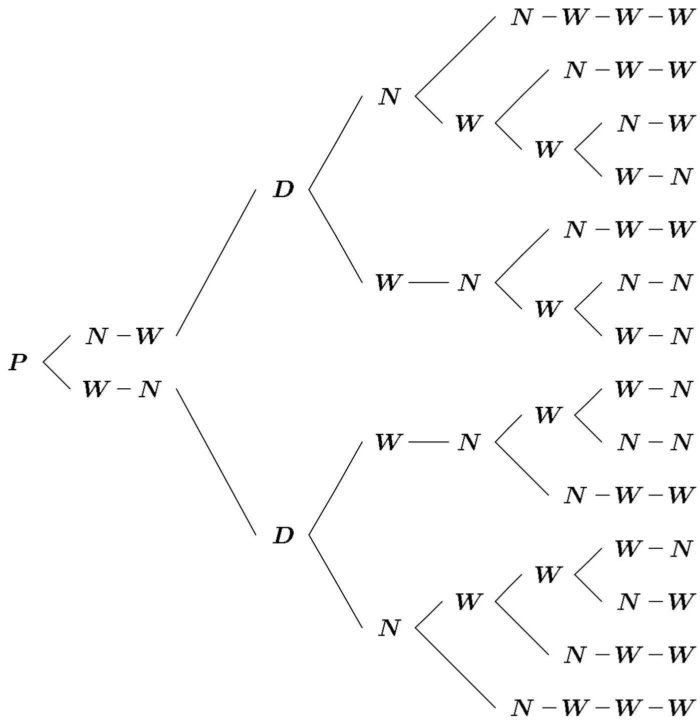


(a) 4 (b) 7 (c) 9 (d) 14 (e) 16

Answer: The correct answer is **(d)**.

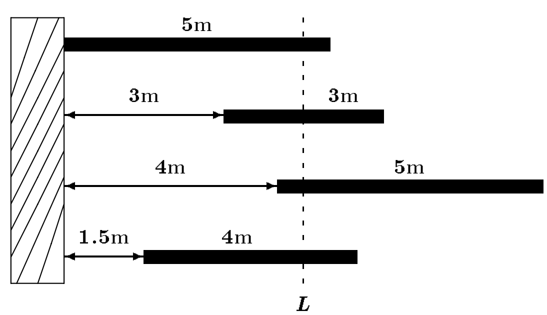
Method I. It is clear that every shortest route must consist of steps due west or north only, and that the diagonal road must be included. In the following we will consider only the shortest routes or their parts. Thus, the diagonal road D can be reached from P in two different ways, WN or NW, where W denotes a step due west and N a step due north. If all the roads were passable then in order to reach Q from the end of D we would need a five step route consisting of two Ns and three Ws performed in any order. Since the two Ns can be chosen in $\binom{5}{2} = 10$ ways in that sequence, there are 10 such roads. However, in order to count only passable routes, we need to remove all the routes with point X from the set of ten routes. The impassable routes begin by WW and then continue from the end of the impassable road to Q. There are three such routes, corresponding to three choices of W on the way to Q from the end of the impassable road. This gives $2 \times (10 - 3) = 14$ possibilities.

Method II. The number of shortest routes can be found by counting the number of last generation branches in a tree diagram:



Again, we find $2 \times 7 = 14$ last generation branches in the tree.

9. Four pieces of timber with the lengths shown are placed in the parallel positions shown. A single cut is made along the line L perpendicular to the lengths of timber so that the total length of timber on each side of L is the same. The length, in metres, of the longest piece of timber remaining is:



(a) 4.85 **(b)** 4.50 **(c)** 4.75 **(d)** 3.75 **(e)** none of the above

Answer: The correct answer is **(c)**.

Let x be the distance from the left-hand side to the line L. It is clear that the total length of timber on each side of L is the same for exactly one value of x. The expressions for the total length of timber on either side depend, however, on the value of x. If $4 < x < 5$ the cut passes through each of the four logs and $x + (x - 3) + (x - 4) + (x - 1.5) = (5 - x) + (3 + 3 - x) + (4 + 5 - x) + (1.5 + 4 - x)$. The solution of this equation, $x = 4.25$, is indeed between 4 and 5. Since the value of x is unique, we do not need to examine equations corresponding to $x \leq 4$ or $x \geq 5$. The longest piece of timber left to the cut has length x, while the longest piece right to the cut has length $9 - x$. The latter is the longest and its length is 4.75 m.

10. The positive integers are written in order with one appearing once, two appearing twice, three appearing three times, ..., ten appearing ten times, and so on, so that the beginning of the sequence looks like this:

$$1\,,\,2\,,\,2\,,\,3\,,\,3\,,\,3\,,\,4\,,\,4\,,\,4\,,\,4$$

The number of 9's appearing in the first 1998 *digits* of the sequence is:

(a) 57 (b) 96 (c) 113 (d) 145 (e) 204

Answer: The correct answer is **(b)**.

At first, we need to find the number in which the 1998th digit occurs. This is a two-digit number, since there are $\frac{(1+99)(99)}{2}$ numbers with one or two digits that appear in our sequence, which is greater than 1998. If x is the number with 1998th digit, then x is the smallest positive integer for which $\frac{9(1+9)}{2}(1) + \frac{(x-10+1)(10+x)}{2}(2) \geq 1998$. By solving this inequality, we get $x \geq \frac{-1+\sqrt{8172}}{2}$. Hence $x = 45$, since x is a positive integer. Now, we can count the 9s in the first 1998 digits: $9 + 19 + 29 + 39 = 96$.

Part B

1. A right triangle has an area of 5 and its hypotenuse has length 5. Determine the lengths of the other two sides.

Solution. The lengths of the other two sides are $\sqrt{5}$ and $2\sqrt{5}$.

Let a and b be the lengths of the sides forming the right angle of the triangle. Then $\frac{1}{2}ab = 5$, since the area of the triangle is 5, and $a^2 + b^2 = 5^2$, by the Pythagorean Theorem. By finding $b = \frac{10}{a}$, from the first equation, substituting this to the second and simplifying we get $a^4 - 25a^2 + 100 = 0$. This gives $a^2 = 5$ or $a^2 = 20$. Consequently, $a = \sqrt{5}$ and $b = 2\sqrt{5}$, or $a = 2\sqrt{5}$ and $b = \sqrt{5}$.

2. Find a set of three consecutive positive integers such that the smallest is a multiple of 5, the second is a multiple of 7 and the largest is a multiple of 9.

Solution. The set is of the form $\{160 + 315m,\ 161 + 315m,\ 162 + 315m\}$, where m is a non-negative integer.

Let $\{x, x+1, x+2\}$ be such a set. Then $x = 5k$, $x + 1 = 7s$, and $x + 2 = 9t$, where k, s, and t are positive integers. The first two conditions give $x + 1 = 5k + 1 = 7s$. The integer k can be expressed in the form $k = 7l + r$, where l is a non-negative integer and $0 \leq r < 7$. Hence,

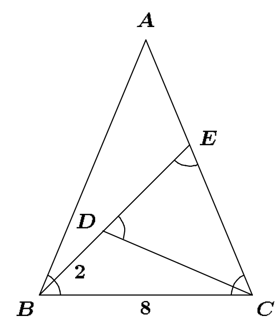
$5k + 1 = 5(7l + r) + 1 = 5l(7) + 5r + 1 = 7s$. From the last equation we find $r = 4$. The last condition, $x + 2 = 9t$, now implies $x + 2 = 5k + 2 = 35l + 22 = 9t$. The last part of this equation can be also written as $36l + 18 - l + 4 = 9t$. This is satisfied whenever $-l + 4$ is a multiple of 9; that is when l is of the form $9m + 4$, with $m \geq 0$. Finally, $x = 5k = 5(7l + 4) = 5(7(9m + 4) + 4) = 160 + 315m$. Thus, our set is $\{160 + 315m, 161 + 315m, 162 + 315m\}$, where m is a non-negative integer.

Note: The reader interested in this kind of problem may find it useful to familiarize oneself with the so called *Chinese Remainder Theorem* which can be found in any textbook of *Elementary Number Theory*.

3. In the diagram, $BD = 2$, $BC = 8$ and the marked angles are all equal; that is,

$$\angle ABC = \angle BCA = \angle CDE = \angle DEC .$$

Find AB.



Solution. $AB = \frac{16\sqrt{3}}{3}$.

We have: $BE = BC = 8$, since triangle BCE is isosceles; $DE = BE - BD = 8 - 2 = 6$; $CE : 6 = 8 : CE$, by similarity of triangles CBE and DCE. From the last equation we find $(CE)^2 = 48$. Hence, $CE = 4\sqrt{3}$. Finally, $AB : 8 = 4\sqrt{3} : 6$, by similarity of triangles ABC and CDE. This gives $AB = \frac{16\sqrt{3}}{3}$.

4. The ratio of male to female voters in an election was $a : b$. If c fewer men and d fewer women had voted, then the ratio would have been $e : f$. Determine the total number of voters who cast ballots in the election in terms of a, b, c, d, e and f.

Solution. The total number of voters is $\frac{(a+b)(cf-de)}{af-be}$.

Let M and F denote the number of male and female voters, respectively. Then $\frac{M}{F} = \frac{a}{b}$ and $\frac{M-c}{F-d} = \frac{e}{f}$. These equations can be transformed to the following system of linear equations:

$$\begin{cases} bM - aF = 0 , \\ fM - eF = cf - de . \end{cases}$$

The system has no solution if $be = af$, that is $a : b = e : f$, but $c : d \neq e : f$. If the three ratios, $a : b$, $e : f$, and $c : d$ are equal the system has infinitely many solutions. Finally, if $e : f \neq a : b$ the system has exactly one solution (M, F). The solution can be found by first finding M

or F from one of the equations and substituting this to the second equation, or by eliminating one of the unknowns in another way. After some algebra we get $M = \frac{acf-ade}{af-be}$ and $F = \frac{bcf-bde}{af-be}$. Thus, the total number of voters is $M + F = \frac{acf-ade+bcf-bde}{af-be} = \frac{(a+b)(cf-de)}{af-be}$.

5. Three neighbours named Penny, Quincy and Rosa took part in a local recycling drive. Each spent a Saturday afternoon collecting all the aluminum cans and glass bottles he or she could. At the end of the afternoon each person counted up what he or she had gathered, and they discovered that even though Penny had collected three times as many cans as Quincy, and Quincy had collected four times as many bottles as Rosa, each had collected exactly the same number of items, and the three as a group had collected exactly as many cans as bottles. In total, the three collected fewer than 200 items in all. Assuming that each person collected at least one can and one bottle, how many cans and bottles did each person collect?

Solution. Penny collected 18 cans and 16 bottles, Quincy collected 6 cans and 28 bottles, Rosa collected 27 cans and 7 bottles.

Let Cp, Cq and Cr denote the number of cans collected by Penny, Quincy and Rosa, respectively. Similarly, let Bp, Bq, and Br denote the corresponding numbers of bottles. Now, the given conditions can be translated to:

$$Cp = 3Cq\,, \qquad (1)$$

$$Bq = 4Br\,, \qquad (2)$$

$$Cp + Bp = Cq + Bq = Cr + Br\,, \qquad (3)$$

$$Cp + Cq + Cr = Bp + Bq + Br\,, \qquad (4)$$

$$Cp + Cq + Cr + Bp + Bq + Br < 200\,. \qquad (5)$$

The third equation represents a pair of single equations:

$$Cp + Bp = Cq + Bq \qquad (3a)$$

and

$$Cq + Bq = Cr + Br\,. \qquad (3b)$$

Now, we use the equations (1), (2), (3a), (3b), in the given order, to successively eliminate the unknowns. We get: $Cp = 3Cq$, $Bq = 4Br$, $Bp = 4Br - 2Cq$, $Cr = Cq + 3Br$. By substituting this into the equation (4) and simplifying, we get $7Cq = 6Br$. Hence, $Cq = 6k$, $Br = 7k$. and consequently, $Cp = 18k$, $Cr = 27k$, $Bp = 16k$, $Bq = 28k$, where k is a positive integer. This gives a total of $100k$ items. Thus $k = 1$, since the total is less than 200.

That completes the *Skoliad Corner* for this issue. My bank of contests at a suitable level is getting low. Please send me your contest materials, together with comments and suggestions about the *Skoliad Corner*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (University of Toronto), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from the previous issue be submitted in time for issue 8 of 2000.

High School Solutions

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <all238@sprint.com>

H245. Determine how many distinct integers there are in the set

$$\left\{ \left\lfloor \frac{1^2}{1998} \right\rfloor, \ \left\lfloor \frac{2^2}{1998} \right\rfloor, \ \left\lfloor \frac{3^2}{1998} \right\rfloor, \ \ldots, \ \left\lfloor \frac{1998^2}{1998} \right\rfloor \right\}.$$

Solution. If $a \le 44$, then $a^2 \le 1936 < 1998$, and $\lfloor a^2/1998 \rfloor = 0$. If $a = 45$, then $\lfloor a^2/1998 \rfloor = 1$, and if $a = 999$, then $\lfloor a^2/1998 \rfloor = \lfloor 999/2 \rfloor = 499$. Now for k with $45 \le k \le 999$, we have that

$$\frac{k^2}{1998} - \frac{(k-1)^2}{1998} \;=\; \frac{2k-1}{1998} \;<\; 1\,,$$

so

$$\left\lfloor \frac{k^2}{1998} \right\rfloor - \left\lfloor \frac{(k-1)^2}{1998} \right\rfloor \;=\; 0 \quad \text{or} \quad 1\,,$$

so that we do not skip any integers from 1 to 499 when we list out the values of $\lfloor k^2/1998 \rfloor$, from $k = 45$ to $k = 999$. Thus, each of the 500 integers from 0 to 499 appears in our list.

Now for $1000 \le k \le 1998$,

$$\frac{k^2}{1998} - \frac{(k-1)^2}{1998} \;=\; \frac{2k-1}{1998} \;>\; 1\,,$$

so

$$\left\lfloor \frac{k^2}{1998} \right\rfloor - \left\lfloor \frac{(k-1)^2}{1998} \right\rfloor \;\ge\; 1\,.$$

In other words, no integer in the set $\{\lfloor 1000^2/1998 \rfloor, \lfloor 1001^2/1998 \rfloor, \ldots, \lfloor 1998^2/1998 \rfloor\}$ appears more than once, since consecutive terms differ by at least one. So there are 999 integers in this set.

Hence, in total, there are $500 + 999 = 1499$ integers in the set.

H246. Let $S(n)$ denote the sum of the first n positive integers. We say that an integer n is *fantastic* if both n and $S(n)$ are perfect squares. For example, 49 is fantastic, because $49 = 7^2$ and $S(49) = 1+2+3+\cdots+49 = 1225 = 35^2$ are both perfect squares. Find another integer $n > 49$ that is fantastic.

Solution. Let $n = k^2$. We wish to find a positive integer k such that $S(n) = n(n+1)/2 = k^2(k^2+1)/2$ is a perfect square. Since k^2 is a perfect square, we need $(k^2+1)/2$ to be a perfect square.

So, let $(k^2+1)/2 = m^2$, for some positive integer m. Then $k^2 - 2m^2 = (k + m\sqrt{2})(k - m\sqrt{2}) = -1$. Note that $(k,m) = (7,5)$ satisfies this equation. Now, $(3+2\sqrt{2})(3-2\sqrt{2}) = 1$, so multiplying these two equations together, we obtain

$$\begin{aligned}
&(7+5\sqrt{2})(7-5\sqrt{2})(3+2\sqrt{2})(3-2\sqrt{2}) \\
&\quad= [(7+5\sqrt{2})(3+2\sqrt{2})][(7-5\sqrt{2})(3-2\sqrt{2})] \\
&\quad= (41+29\sqrt{2})(41-29\sqrt{2}) \\
&\quad= 41^2 - 2\cdot 29^2 = -1\,.
\end{aligned}$$

The last equation above tells us that $(k,m) = (41,29)$ is a solution to $k^2 - 2m^2 = -1$. Therefore, $n = 41^2 = 1681$ is another fantastic number.

H247. Say that the integers a, b, c, d, p, and r form a *cyclic sextuple* (a, b, c, d, p, r) if there exists a cyclic quadrilateral with circumradius r, sides a, b, c, and d, and diagonals p and $2r$.

(a) Show that if $r < 25$, then no cyclic sextuple exists.

(b) Find a cyclic sextuple (a, b, c, d, p, r) for $r = 25$.

Solution. Since one of the diagonals of the cyclic quadrilateral is $2r$, it must be a diameter of the circle. Suppose that the sides of the quadrilateral are a, b, c, and d (in that order). Then $a^2 + b^2 = c^2 + d^2 = 4r^2$, since an angle subtended by a diameter is a right angle.

(a) Let us attempt to find two distinct Pythagorean triples (a, b, t) and (c, d, t), where $t = 2r < 50$. Listing all Pythagorean triples with hypotenuse less than 50, we find that only one value of t, $t = 25$, appears in two *different* Pythagorean triples as the hypotenuse. These triples are $(15, 20, 25)$ and $(7, 24, 25)$. But then $2r = 25$, for which r is not an integer. Thus, we conclude that there is no cyclic sextuple for $r < 25$.

(b) If $r = 25$, we can have $a = 14$, $b = 48$, $c = 40$, and $d = 30$, since $(14, 48, 50)$ and $(40, 30, 50)$ are both Pythagorean triples with $t = 2r = 50$. Then by Ptolemy's Theorem, $2pr = ac+bd$, or $50p = 2000$, so $p = 40$, which is an integer. So a cyclical sextuple with $r = 25$ is $(14, 48, 40, 30, 40, 25)$.

H248. Consider a tetrahedral die that has the four integers 1, 2, 3, and 4 written on its faces. Roll the die 2000 times. For each i, $1 \le i \le 4$, let $f(i)$ represent the number of times that i turned up. (So, $f(1) + f(2) + f(3) + f(4) = 2000$). Also, let S denote the total sum of the 2000 rolls.

If $S^4 = 6144 \cdot f(1)f(2)f(3)f(4)$, determine the values of $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

Solution. We have $f(1) + f(2) + f(3) + f(4) = 2000$ and $S = f(1) + 2f(2) + 3f(3) + 4f(4)$. Also, $f(i) \ge 0$ for $i = 1$, 2, 3, and 4. By the AM-GM Inequality,

$$\begin{aligned} \frac{S}{4} &= \frac{f(1) + 2f(2) + 3f(3) + 4f(4)}{4} \ge \sqrt[4]{24f(1)f(2)f(3)f(4)} \\ \Longrightarrow \quad S &\ge 4\sqrt[4]{24f(1)f(2)f(3)f(4)} \\ \Longrightarrow \quad S^4 &\ge 4^4 \cdot 24f(1)f(2)f(3)f(4) \\ &= 6144f(1)f(2)f(3)f(4). \end{aligned}$$

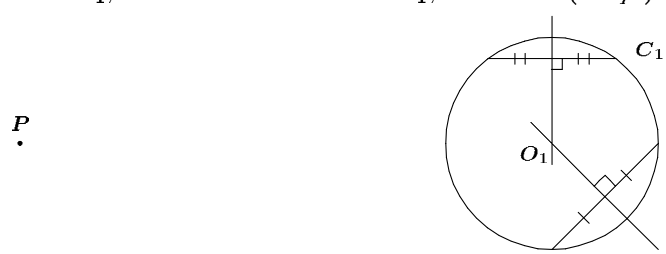
Equality occurs if and only if $f(1) = 2f(2) = 3f(3) = 4f(4)$, and since $f(1) + f(2) + f(3) + f(4) = 2000$, we can quickly solve to get $f(1) = 960$, $f(2) = 480$, $f(3) = 320$, and $f(4) = 240$.

Advanced Solutions

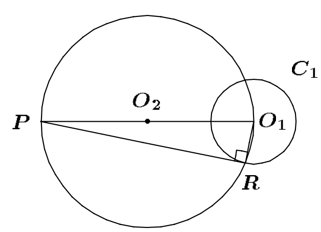
Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A221. Construct, using ruler and compass only, the common tangents of two non-intersecting circles.

Solution. First, we show how to construct a tangent from a given point P to a given circle C_1. Find the centre of C_1 by taking any two non-parallel chords in the circle and drawing their perpendicular bisectors. These bisectors intersect at O_1, the centre of circle of C_1, as shown (Why?).



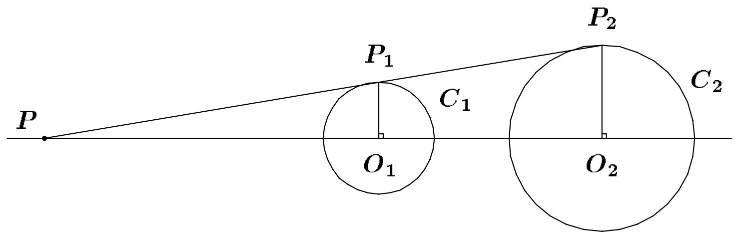
Next, construct the mid-point of PO_1 and label it O_2. Construct the circle with radius PO_2 centred at O_2. Call this circle C_2. Let one of the intersection points of C_1 and C_2 be R. See the following diagram.



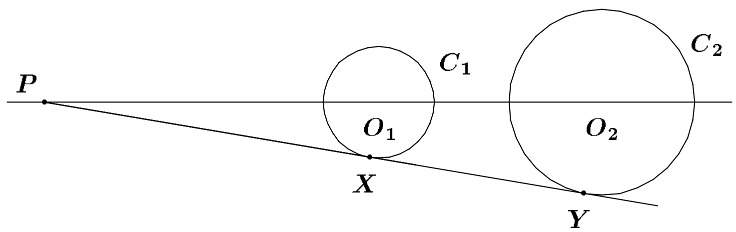
Line PR is the tangent from the point P to the given circle C_1. This follows from the fact that PO_1 is the diameter of circle C_2 so $\angle O_1RP = 90^\circ$ and that O_1R is the radius of C_1.

Now we are ready to construct the common tangents of two non-intersecting circles. We give the construction for the outer common tangents of two circles with radii of different lengths. The cases of inner common tangents and common tangents of circles with equal radii are left to the reader.

Let the circles C_1 and C_2 have centres O_1 and O_2 respectively. Construct two points P_1 and P_2 as follows. Draw the line ℓ connecting the two centres O_1 and O_2. Draw the lines perpendicular to ℓ through O_1 and O_2. Let P_1 and P_2 be the intersection points of these perpendicular lines with the circles C_1 and C_2 respectively on the same side of the line as shown. Extend the line of centres and the line P_1P_2 to intersect at P.



Now, we use the result shown at the very beginning to construct the tangent from point P to circle C_1. Let the point of tangency be X. Extend PX. We claim that PX is tangent to circle C_2, say at Y. Then XY is the common outer tangent to C_1 and C_2.



The point P is a centre of homothety for circles C_1 and C_2; that is, we can dilate circle C_1 with respect to P to obtain circle C_2. In particular, point P_1 is taken to point P_2 under this dilatation — this is how we found point P. Therefore, the point of tangency X gets taken to the point of tangency Y.

A222. Does there exist a set of n consecutive positive integers such that for every positive integer $k < n$, it is possible to pick k of these numbers whose mean is still in the set?

Solution. We claim that the answer is yes. Consider the set of n integers $\{0, 1, \ldots, n-1\}$.

If k is odd, then take the first k integers $0, 1, \ldots, k-1$. The mean is $[(k-1)k/2]/k = (k-1)/2$, which is an integer in the set.

If k is even, then take the k integers $0, 1, \ldots, k/2-1, k/2+1, \ldots, k$. The sum of these numbers is $k(k+1)/2 - k/2 = k^2/2$. The mean is then $k/2$.

Now to obtain a set of n consecutive positive integers, just add any positive constant, say c, to each number. The means will also be increased by c.

A223. *Proposed by Mohammed Aassila, Strasbourg, France.*

Suppose p is a prime with $p \equiv 3 \pmod 4$. Show that for any set of $p-1$ consecutive integers, the set cannot be divided into two subsets so that the product of the members of the one set is equal to the product of the members of the other set.

(Generalization of Question 4, IMO 1970)

Solution by the Proposer.

We will show that the two products will not even be congruent to each other modulo p. Henceforth, we will be taking congruence modulo p. If one of the numbers is congruent to 0 modulo p, then the product in which that term appears is congruent to 0 as well, whereas the other is not. Thus we may assume that the numbers we are given are congruent to $1, 2, \ldots, p-1$ modulo p. Suppose that the two products are α and β. Then α and β are congruent modulo p and hence so are $\alpha\beta$ and β^2. But $\alpha\beta \equiv (p-1)! \equiv -1$ modulo p by Wilson's Theorem. Hence so is β^2.

To complete the proof, it suffices to show that the congruence $\beta^2 \equiv -1$ has no solution. Suppose on the contrary that it does. Let $p = 4n+3$. Then $\beta^{p-1} \equiv \beta^{4n+2} \equiv \beta^{2(2n+1)} \equiv (-1)^{2n+1} \equiv -1$. This contradicts Fermat's Little Theorem.

A224. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

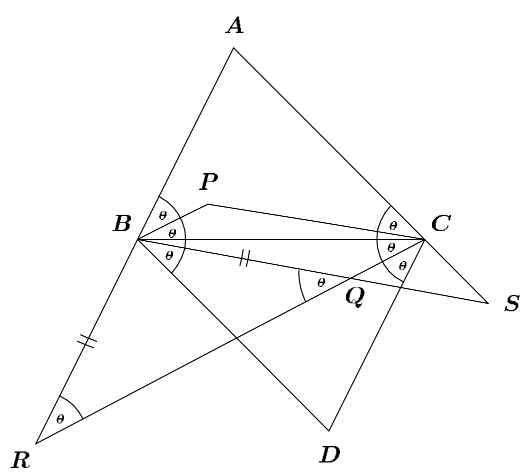
Let P be an interior point of triangle ABC such that $\angle PBA = \angle PCA = (\angle ABC + \angle ACB)/3$. Prove that

$$\frac{AC}{AB+PC} = \frac{AB}{AC+PB}.$$

Solution

Let D be the point on the opposite side of BC to A such that $BACD$ is a parallelogram. Let $\theta = (B+C)/3$ so $A + 3\theta = 180°$.

Let Q be the point in triangle BDC such that $BPCQ$ is a parallelogram. Extend CQ to meet AB extended at R and likewise extend BQ to meet AC extended at S as shown.



Now $\angle ARC = \angle ABP = \theta$ and $\angle ASB = \angle ACP = \theta$. Then $\angle PCR = 180° - (A+2\theta) = \theta$. Similarly, $\angle PBS = 180° - (A+2\theta) = \theta$. Further, $\angle BQR$ and $\angle CQS$ both equal θ, since $BQ \| PC$ and $CQ \| PB$.

Thus $AR = AB + BR = AB + BQ = AB + PC$. Similarly $AS = AC + PB$. Triangles ARC and ASB are similar with angles A, θ, and 2θ. Therefore, $AC/AR = AB/AS$, or $AC/(AB+PC) = AB/(AC+PB)$.

Also solved by Andrei Simion, Brooklyn Technical High School, New York, NY, USA.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C81. Let $\{a_n\}$ be the sequence defined as follows: $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 4a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$.

(a) Prove that $a_n^2 - a_{n-1}a_{n+1} = 1$ for all $n \geq 1$.

(b) Evaluate $\displaystyle\sum_{k=1}^{\infty} \arctan\left(\frac{1}{4a_k^2}\right)$.

Solution. (a) We will proceed by induction. The statement holds for $n = 1$. Assume it holds for some $n = k - 1$, so $a_{k-1}^2 - a_{k-2}a_k = 1$. Then

$$\begin{aligned} a_k^2 - a_{k-1}a_{k+1} &= a_k(4a_{k-1} - a_{k-2}) - a_{k-1}(4a_k - a_{k-1}) \\ &= a_{k-1}^2 - a_{k-2}a_k \\ &= 1\,. \end{aligned}$$

Hence, the statement holds for $n = k$, and by induction, for all $n \geq 1$.

(b) Note that the first few terms of the sequence are $a_0 = 0$, $a_1 = 1$, $a_2 = 4$, $a_3 = 15$, $a_4 = 56$, etc. Let S_n denote the n^{th} partial sum

$$\sum_{k=1}^{n} \arctan\left(\frac{1}{4a_k^2}\right)\,.$$

We will compute the first few S_n using the following result: let a and b be positive reals such that $ab < 1$. Then

$$\arctan a + \arctan b \;=\; \arctan\left(\frac{a+b}{1-ab}\right)\,,$$

where the arctangents are chosen to lie in the interval $(0, \pi/2)$. To see this, let $\alpha = \arctan a$ and $\beta = \arctan b$. Then

$$\begin{aligned} \tan(\arctan a + \arctan b) &= \tan(\alpha + \beta) \\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ &= \frac{a+b}{1-ab} \;>\; 0\,. \end{aligned}$$

Hence, $0 < \arctan a + \arctan b < \pi/2$, and the result follows.

The first few S_n are then

$$\begin{aligned} S_1 &= \arctan\left(\frac{1}{4}\right), \\ S_2 &= \arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{64}\right) = \arctan\left(\frac{\frac{1}{4}+\frac{1}{64}}{1-\frac{1}{4}\cdot\frac{1}{64}}\right) \\ &= \arctan\left(\frac{4}{15}\right), \\ S_3 &= \arctan\left(\frac{4}{15}\right) + \arctan\left(\frac{1}{900}\right) = \arctan\left(\frac{\frac{4}{15}+\frac{1}{900}}{1-\frac{4}{15}\cdot\frac{1}{900}}\right) \\ &= \arctan\left(\frac{15}{56}\right), \textit{etc.} \end{aligned}$$

By induction, we will prove that $S_n = \arctan(a_n/a_{n+1})$. The statement holds for $n = 1$. Assume that it holds for some $n = k - 1$, so $S_{k-1} = \arctan(a_{k-1}/a_k)$. Then

$$\begin{aligned} S_k &= \arctan\left(\frac{a_{k-1}}{a_k}\right) + \arctan\left(\frac{1}{4a_k^2}\right) \\ &= \arctan\left(\frac{\frac{a_{k-1}}{a_k}+\frac{1}{4a_k^2}}{1-\frac{a_{k-1}}{a_k}\cdot\frac{1}{4a_k^2}}\right) \\ &= \arctan\left(\frac{a_k(4a_{k-1}a_k+1)}{4a_k^3-a_{k-1}}\right). \end{aligned}$$

Now

$$\begin{aligned} 4a_k^3 - a_{k-1} &= (4a_k)(a_k^2) - a_{k-1} = 4a_k(a_{k-1}a_{k+1}+1) - a_{k-1} \\ &= 4a_{k-1}a_ka_{k+1} + 4a_k - a_{k-1} = 4a_{k-1}a_ka_{k+1} + a_{k+1} \\ &= a_{k+1}(4a_{k-1}a_k + 1), \end{aligned}$$

so

$$S_k = \arctan\left(\frac{a_k(4a_{k-1}a_k+1)}{a_{k+1}(4a_{k-1}a_k+1)}\right) = \arctan\left(\frac{a_k}{a_{k+1}}\right).$$

Hence, the statement holds for $n = k$, and by induction, for all $n \geq 1$.

Finally, solving for $\{a_n\}$ using its characteristic equation, we obtain

$$a_n = \frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}}.$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} \arctan\left(\frac{1}{4a_k^2}\right) &= \lim_{n\to\infty} \arctan\left(\frac{a_n}{a_{n+1}}\right) \\ &= \arctan\left(\frac{1}{2+\sqrt{3}}\right) \\ &= \arctan(2-\sqrt{3}) \;=\; \frac{\pi}{12}\,.\end{aligned}$$

C82. Find the smallest multiple of 1998 which appears as a partial sum of the increasing sequence

$$1\,,\quad 1\,,\quad 2\,,\quad 2\,,\quad 2\,,\quad 4\,,\quad 4\,,\quad 4\,,\quad 4\,,\quad 8\,,\quad \ldots\,,$$

in which the number 2^k appears $k+2$ times (for k a non-negative integer).

Solution by Ivailo Dimov, freshman, M.I.T., MA, USA.

We will prove the following lemma: the partial sums of this sequence are the integers of the form $m \cdot 2^n$, where n is a non-negative integer and m runs from $n+1$ to $2(n+1)$. To solve the problem using the lemma, if an integer $m \cdot 2^n$ is divisible by 1998, then m must be divisible by 999. So, we are looking for the smallest positive integer n such that a multiple of 999 lies between $n+1$ and $2(n+1)$. Evidently the smallest such n is 499, and so the smallest partial sum divisible by 1998 is $999 \cdot 2^{499} = 1998 \cdot 2^{498}$.

To prove the lemma, we proceed by induction on n. To begin with, the first two partial sums are $1\cdot 2^0$ and $2\cdot 2^0$. Next, assume that the lemma is true for $n = k-1$, which entails that $1+1+2+\cdots+2^{k-1} = 2(k-1+1)\cdot 2^{k-1}$, where the left-hand side contains every term up to and including the last of the $k+1$ terms which are equal to 2^{k-1}. Then the next $k+2$ partial sums are $2k \cdot 2^{k-1} + i \cdot 2^k = (k+i) \cdot 2^k$ with i running from 1 to $k+2$, exactly as desired, and so the lemma is true for $n = k$.

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. If p_1 and p_2 are distinct odd primes and $A = (p_1p_2+1)^4-1$, show that A has at least 4 distinct prime divisors.

(1999 Descartes, problem D1)

Solution. Let $p_3 = p_1p_2 + 2$. Observe that

$$\begin{aligned} A &= (p_1^2p_2^2 + 2p_1p_2 + 1)^2 - 1^2 \\ &= (p_1^2p_2^2 + 2p_1p_2 + 1 - 1)(p_1^2p_2^2 + 2p_1p_2 + 1 + 1) \\ &= p_1p_2(p_1p_2 + 2)(p_1^2p_2^2 + 2p_1p_2 + 2) \\ &= p_1p_2p_3(p_1p_2p_3 + 2)\,. \end{aligned}$$

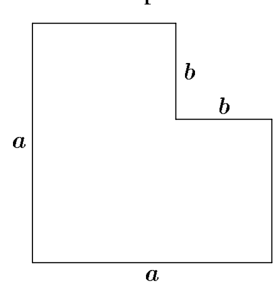
Now, $p_3 \equiv 2 \pmod{p_1}$ and $p_3 \equiv 2 \pmod{p_2}$. Since p_1, $p_2 \geq 3$, we know that $p_1 \nmid p_3$ and $p_2 \nmid p_3$. Furthermore, note that $p_3 > p_1$, p_2.

Thus, two cases exist. If p_3 is composite, then A has at least four distinct prime factors, and we are done. However, if p_3 is prime, then let $p_4 = p_1p_2p_3 + 2$.

Then, $p_4 \equiv 2 \pmod{p_1}$, $p_4 \equiv 2 \pmod{p_2}$, and $p_4 \equiv 2 \pmod{p_3}$. This means that $p_1 \nmid p_4$, $p_2 \nmid p_4$, and $p_3 \nmid p_4$, which leads us to the conclusion that A contains at least four distinct prime factors, QED.

J.I.R. McKnight Problems Contest 1992

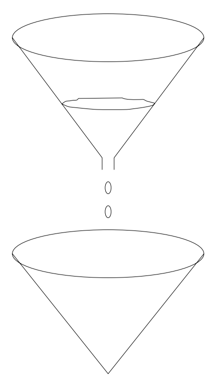
1. (a) If p and q are the roots of $2x^2 - 5x + 1 = 0$, what is the value of $\log_2 p + \log_2 q$?
   (b) Solve for x:
   $$\frac{x-7}{x-9} - \frac{x-9}{x-11} = \frac{x-13}{x-15} - \frac{x-15}{x-17}\,.$$
2. (a) Given $x^2 - x + 1 = 0$, evaluate $x^9 - 3x^6 + 4x^3$.
   (b) The L-shaped room has a floor made with 231 identical square tiles, each of side 1 m. Find the least perimeter of the room.



3. Determine the rational values of x, y, and k that satisfy the system:

$$\begin{aligned} x + y &= 2k\,, \\ x^2 + y^2 &= 5k\,, \\ x^3 + y^3 &= 14k\,. \end{aligned}$$

4. Two identical cones of radius 8 cm and height 8 cm are shown. At the start of the problem the upper is full of water and the lower one is empty. Water drains from the upper cone into the lower one. At the time the depth of the water in the upper cone is 4 cm and falling at 0.1 cm per second, how deep is the water in the lower cone, and how quickly is it rising?



5. Show that $2^n \cdot 3^{2n} - 1$ is always divisible by 17 for $n \in \mathbb{N}$.

6. (a) Three urns are arranged in a row. The left urn contains two red balls and one white ball. The centre urn contains one red ball and one white ball. The right urn contains one red ball and two white balls. A ball is randomly selected from the appropriate urn and then discarded. If the chosen ball is white, then one moves one urn to the left; however, if the chosen ball is red, then one moves one urn to the right. [Ed. It is the person selecting the ball that moves one urn over and proceeds to select another ball if it is possible.] The game continues until either an empty urn is reached or a move is made past an end urn. Assuming that the game commences with a selection from the centre urn, what is the probability that the game ends because an empty urn was reached?

   (b) Every time John Olerud gets a hit, his confidence increases. The next time he bats, he has a 42 per cent probability of getting a hit. But when he does not get a hit, the probability of his getting a hit in his next at bat drops to 23 per cent. Determine John Olerud's batting average for the season (that is, in the long term).

7. The parabola $y = 4x^2 - 24x + 31$ crosses the x–axis at $(y, 0)$ and $(g, 0)$, both to the right of the origin. A circle also passes through these two points. Find the length of the tangent from the origin to any such circle.

8. Triangle ABC has sides 8, 15 and 17. A point P is inside the triangle. Find the minimum value of $PA^2 + PB^2 + PC^2$.

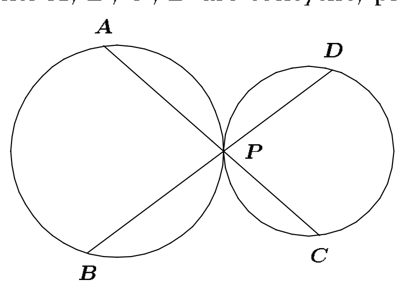
9. (a) Consider the system of equations:

$$\begin{aligned} x_1 + x_2 &= 2\,, \\ x_2 + x_3 &= 4\,, \\ x_3 + x_4 &= 8\,, \\ &\vdots \\ x_{k-1} + x_k &= 2^{k-1}\,, \\ &\vdots \\ x_{2000} + x_{2001} &= 2^{2000}\,, \\ x_{2001} + x_1 &= 2^{2001}\,. \end{aligned}$$

Evaluate x_{2001}.

(b) In the original system, replace 2001 by n. Find the solution for x_n, for all values of n.

10. Two circles are tangent at P as shown. Lines AC and BD are drawn through P. If points A, B, C, D are concyclic, prove $AC = BD$.



Three Gems in Geometry

Naoki Sato

Geometry may well be the most elegant branch of mathematics, and unfortunately the most under-appreciated, with respect to the high school curriculum. Here we present three pretty results, which come in the form of do-it-yourself exercises. Get ready to roll up your sleeves, it will be well worth it!

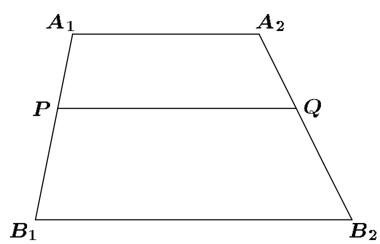
The QM-AM-GM-HM Inequality

First, we introduce some definitions. For non-negative reals a and b, let

$$\sqrt{\frac{a^2+b^2}{2}}, \quad \frac{a+b}{2}, \quad \sqrt{ab}, \quad \text{and} \quad \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}$$

denote the quadratic mean (QM), arithmetic mean (AM), geometric mean (GM), and harmonic mean (HM) of a and b, respectively.

Now, let $A_1A_2B_2B_1$ be a trapezoid as shown, with $a = A_1A_2$ and $b = B_1B_2$. Let P and Q be points on A_1B_1 and A_2B_2 respectively such that PQ is parallel to the bases.



(a) Show that the line PQ dividing the trapezoid into two trapezoids of equal area has length $QM(a,b)$.

(b) Show that the line PQ at equal distance to both bases has length $AM(a,b)$.

(c) Show that the line PQ dividing the trapezoid into two similar trapezoids has length $GM(a,b)$.

(d) Show that the line PQ passing through the intersection of A_1B_2 and A_2B_1 has length $HM(a,b)$. Furthermore, show that the intersection is the mid-point of PQ.

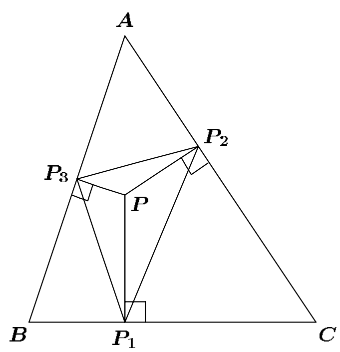
(e) Show that for all $a,\ b > 0$,

$$HM(a,b) \ \le\ GM(a,b) \ \le\ AM(a,b) \ \le\ QM(a,b)\,,$$

with equality occurring in any of the above inequalities if and only if $a = b$.

The Area of a Pedal Triangle

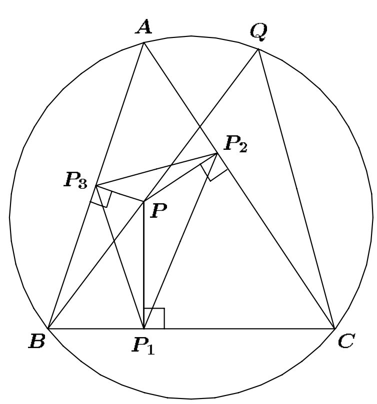
Let ABC be a triangle and P a point in the plane, and let P_1, P_2, and P_3 be the feet of the perpendiculars from P to sides BC, AC, and AB respectively. Then triangle $P_1P_2P_3$ is called the *pedal triangle* of P. We will derive a simple formula for K', the area of this triangle.



(a) We have $$K' \ =\ \frac{1}{2}\, P_1P_2 \cdot P_1P_3 \sin\left(\angle P_2P_1P_3\right)\ .$$

(b) Show that $P_1P_2 = PC\sin C$ and $P_1P_3 = PB\sin B$.

(c) Extend BP to Q on the circumcircle of triangle ABC. Show that $\angle QCP = \angle P_1P_2P_3$.



(d) Show that $$\frac{\sin \angle QCP}{\sin \angle BQC} = \frac{PQ}{PC}.$$

(e) Show that $$K' = \frac{1}{2} \cdot PB \cdot PQ \sin A \sin B \sin C.$$

(f) Recall by power of a point that $PB \cdot PQ = R^2 - OP^2$, where R and O are the circumradius and circumcentre of triangle ABC respectively, and that
$$\sin A \sin B \sin C = \frac{K}{2R^2}.$$
Conclude that $$K' = \frac{R^2 - OP^2}{4R^2} \cdot K.$$

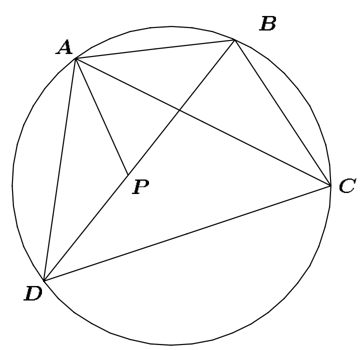
Note that the formula depends only on OP, namely the distance from O to P. There are two interesting cases:

(i) First, K' is maximized when $P = O$ (for P in triangle ABC), in which case $P_1P_2P_3$ is the medial triangle of ABC, and $K' = K/4$.

(ii) Second, we see that $K' = 0$ when $OP = R$; that is, when P lies on the circumcircle of triangle ABC. In this case, triangle $P_1P_2P_3$ degenerates into a line, called the *Simson line*.

Ptolemy's Theorem

Let $ABCD$ be a cyclic quadrilateral.



(a) Let P be the point on BD such that $\angle PAB = \angle DAC$. Show that triangles PAB and DAC are similar, and that triangles PAD and BAC are similar.

(b) Show that $$\frac{BP}{CD} = \frac{AB}{AC} \quad \text{and} \quad \frac{DP}{BC} = \frac{AD}{AC}.$$

(c) Prove Ptolemy's Theorem: $$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

Hints

The QM-AM-GM-HM Inequality

(a) Let h_1 and h_2 be the heights of the upper and lower trapezoids respectively. Show that, in general, $\dfrac{h_1}{h_2} = \dfrac{PQ - A_1A_2}{B_1B_2 - PQ}$.

(c) Show that the ratios a/PQ and PQ/b must be equal.

(d) Let M be the intersection of A_1B_2 and A_2B_1. Show that triangles A_1A_2M and B_2B_1M are similar, so $A_1M : MB_2 = a/b$. Compute PM.

(e) Assume without loss of generality that $a \leq b$. Let A be the area of the whole trapezoid. The length of PQ determines its position, and hence the area of the upper trapezoid, say U. Let $f(PQ) = U/A$. Thus, $f(a) = 0$, $f(b) = 1$, and by part (a), $f(QM(a,b)) = 1/2$. It is clear that f is an increasing function. The idea of this problem is to compare f at different values. For example, $f(AM(a,b)) \leq 1/2 = f(QM(a,b))$, so $AM(a,b) \leq QM(a,b)$.

To prove that $GM(a,b) \leq AM(a,b)$, let $g(PQ) = h_1 - h_2$, where h_1 and h_2 are the heights of the upper and lower trapezoid respectively. Is g increasing or decreasing? Compute $g(GM(a,b))$ and $g(GM(a,b))$, and compare.

The Area of a Pedal Triangle

(b) Quadrilateral PP_1CP_2 is cyclic, and furthermore, PC is a diameter of the circumcircle of PP_1CP_2. Use the Sine Law.

(c) Show that $\angle PP_1P_3 = \angle PBP_3$ and $\angle PP_1P_2 = \angle PCP_2$. The rest is an angle chase; use $\angle P_2P_1P_3 = \angle PP_1P_3 + \angle PP_1P_2 = \angle PBP_3 + \angle PCP_2$, etc.

(d) Use the Sine Law in triangle QCP.

(e) Put (a), (b), (c), and (d) together.

Ptolemy's Theorem

(a) Since $ABCD$ is cyclic, $\angle ABP = \angle ABD = \angle ACD$. Similarly, $\angle ADP = \angle ADB = \angle ACB$.

(b) Use similar triangles.

(c) Expand $BD = BP + PD$.

Naoki Sato
403 Renforth Drive
Etobicoke, ON
M9C 2M3
mayhem@math.toronto.edu

A Do-It-Yourself Proof of the $n = 4$ case of Fermat's Last Theorem

Ravi Vakil

Introduction

The impact of Fermat's Last theorem on the development of mathematics is immeasurable. This is not simply because it resisted humankind's best efforts for centuries — there are many other such problems that are long-forgotten. But the mathematics that has come up in uncounted attempts to solve it has proved to be fundamental in number theory (and in math in general).

Here, for the record, is Fermat's Last Theorem.

Theorem (Wiles, Wiles-Taylor). If n, x, y, and z are integers with $n \geq 3$ and $x^n + y^n = z^n$, then $xyz = 0$.

We will refer to Fermat's Last Theorem for a particular n by "FLTn".

Even individual cases are difficult, and proofs of special cases shed light on advanced mathematical ideas. In this article, you the reader will prove the $n = 4$ case of Fermat's Last Theorem, and in the sequel (to appear next issue), you will prove the $n = 3$ case. (FLT3 is often referred to in the literature as the easiest case, for example [D] p. 96–104, but this is not true.)

Although Fermat's Last Theorem turned out to follow from Wiles's proof of the more important Taniyama-Shimura-Weil conjecture, problems in the vicinity of FLT remain on the cutting edge of current research. For example, a powerful generalization of Fermat, the ABC Conjecture (due to Masser and Oesterlé) is considered an important and fundamental open problem.

Warning: This is a very interactive article! You will really have to try all of the problems. If you get stuck, then skip to the next one. Even getting stuck is a good thing (much better than not trying at all), because the seeds of ideas that come up in one problem invariably turn up again in a later one. You will see that no advanced background is required; ambitious high school students should be able to tackle it.

Warming Up: Primitive Pythagorean Triples

The proof of FLT4 will be similar in character to one method for generating primitive Pythagorean triples, and these triples will come up in the proof, so we will start there.

A primitive Pythagorean triple is an ordered triple of positive integers (a, b, c), pairwise relatively prime, that are the sides of a right-angled triangle, that is $a^2 + b^2 = c^2$. Familiar examples are $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, and $(8, 15, 17)$.

1. Show that if (a, b, c) is a primitive Pythagorean triple, then exactly one of a, b is odd.

   (Hint: Check modulo 4.) Without loss of generality, say a is odd.

2. Then $a^2 = c^2 - b^2 = (c - b)(c + b)$. Show that $c - b$ and $c + b$ have no common factor.

3. Two relatively prime odd numbers multiplying to a perfect square must both be odd perfect squares. In the previous problem, show that $c - b$ and $c + b$ can be taken to be $(m - n)^2$ and $(m + n)^2$ respectively, where m and n are positive integers, $m > n$, and exactly one of m and n is even. (Remember that we are assuming that a is odd!)

4. In the previous problem, show that m and n are relatively prime.

   (Hint: Show that if they have a common factor d, then d is a factor of both b and c, which are relatively prime.)

5. Solve for b and c to get $b = 2mn$, $c = m^2 + n^2$. Then show that $a = m^2 - n^2$.

In conclusion, any primitive Pythagorean triple is of the form $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$ or $(2mn, m^2 - n^2, m^2 + n^2)$, where m and n are relatively prime positive integers, one of which is even, with $m > n$. With this result, you can now do lots of interesting things involving Pythagorean triples. (You will not need to do them to tackle FLT4.) For example:

6. Plug in some large values of m and n to get ridiculously huge Pythagorean triples.

7. Show that any Pythagorean triple (a, b, c) can be written as a multiple k of a primitive Pythagorean triple.

8. $319^2 + 459^2 = 555^2$. Which k, m, and n give this triple?

9. How many Pythagorean triangles are there with hypotenuse 60?

10. Suppose that (a, b, c) is a primitive Pythagorean triple, and a is odd. Show that $(c - a)/2$, $(c + a)/2$, $c + b$, and $c - b$ are all perfect squares.

11. If you know some trigonometry, try "breaking the rules" and substituting $m = \cos\theta$, $n = \cos\theta$ in the formula for primitive Pythagorean triples. What formula do you get? (Remember the double angle formulas: $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.)

    (Many of these examples appear on [V] p. 130.)

Are you all warmed up? Then let's get to it ...

The proof of FLT4

12. Show that if there is a solution of FLT4, then there is a solution where x, y, and z are pairwise relatively prime.

13. Show that if there is a solution of FLT4, then there is a solution of

$$x^4 + y^4 = z^2, \tag{1}$$

where x, y, and z are pairwise relatively prime positive integers.

14. (**This is the big one!**) Assume that there is a solution $(x, y, z) = (a, b, c)$ to equation (1). Then (a^2, b^2, c) is a primitive Pythagorean triple, so you can use what you know about such triples. Play around with the algebra. (Another primitive Pythagorean triple may come up.) You will hopefully end up with another solution to equation (1) that is in some sense smaller than the solution $(x, y, z) = (a, b, c)$. (Make that precise.) Then the argument by contradiction will go as follows: Suppose (x, y, z) is the "smallest" solution of equation (1). Then this method produces a smaller solution — contradiction.

If you prove FLT4, then please send it in — it will be problem **A241** next issue. And congratulations — reach over your shoulder and pat yourself on the back! If you are able to tackle other advanced-level problems but think this one must be too hard, you're wrong — just be ambitious and try it; you might surprise yourself!

Acknowledgements. Suggestions by David Savitt of Harvard University on an earlier version of this article have greatly improved the exposition. He also suggested a couple of fun-related problems.

1. Show that if a, b, and c are integers that are the sides of a right-angled triangle, then 60 divides abc.

2. Find a right-angled triangle with rational sides and area 5. (Hint: Try to scale well-known Pythagorean triples.) One such triangle was discovered by Fibonacci, among others. In fact, 5 is the smallest integer which is the area of a right-angled triangle with rational sides.

It is a classical unsolved problem to determine all of the integers which are areas of right triangles with rational sides. In 1983, it was shown that a solution to one of the most important conjectures in number theory, the Birch-Swinnerton-Dyer conjecture, would give a solution to this problem as well. For more on this fascinating connection between classical diophantine equations and the frontier of mathematics, see an upcoming ***CRUX with MAYHEM*** article by David Savitt.

References

[D] H. Dörrie, *100 Great Problems of Elementary Mathematics: Their History and Solution*, Dover: New York, 1965.

[V] R. Vakil, *A Mathematical Mosaic: Patterns and Problem Solving*, Brendan Kelly Publ.: Toronto, 1996.

Ravi Vakil
MIT Dept. of Mathematics
77 Massachusetts Ave. Rm. 2-248
Cambridge MA USA 02139
vakil@math.mit.edu

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 May 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2489 *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

The set of twelve vertices of a regular icosahedron can be partitioned into three sets of four vertices, each being such that none of the sets have their four vertices forming a golden rectangle. In how many different ways can this be done?

2490 *Proposed by Mihály Bencze, Brasov, Romania.*

Let $\alpha > 1$. Denote by x_n the only positive root of the equation:

$$(x+n^2)(2x+n^2)(3x+n^2)\ldots(nx+n^2) \;=\; \alpha\, n^{2n}\,.$$

Find $\lim\limits_{n\to\infty} x_n$.

2491 *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and that $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are two geometric sequences for which

$$\sum_{k=1}^n f(a_k) \;<\; 0 \;<\; \sum_{k=1}^n f(b_k)\,.$$

Prove that there exists a geometric sequence $\{c_k\}_{k=1}^n$ for which

$$\sum_{k=1}^n f(c_k) \;=\; 0\,.$$

2492 *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, suppose that $\angle BAC$ is a right angle. Let I be the incentre of $\triangle ABC$, and that D and E are the intersections of BI and CI with AC and AB respectively. Let points P and Q be on BC such that $IP \| AB$ and $IQ \| AC$.

Prove that $BE + CD = 2PQ$.

2493 *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $ABCD$ is a convex cyclic quadrilateral, that $\angle ACB = 2\angle CAD$, and that $\angle ACD = 2\angle BAC$.

Prove that $BC + CD = AC$.

2494 *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given $\triangle ABC$ with $AB < AC$, let I be the incentre and M be the mid-point of BC. The line MI meets AB and AC at P and Q respectively. A tangent to the incircle meets sides AB and AC at D and E respectively.

Prove that $\dfrac{AP}{BD} + \dfrac{AQ}{CE} = \dfrac{PQ}{2MI}$.

2495 *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let P be the interior isodynamic point of $\triangle ABC$; that is, $\dfrac{AP}{bc} = \dfrac{BP}{ca} = \dfrac{CP}{ab}$ (a, b, c are the side lengths, BC, CA, AB, of $\triangle ABC$).

Prove that the pedal triangle of P has area $\dfrac{\sqrt{3}}{d^2}F$, where F is the area of $\triangle ABC$ and $d = \dfrac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}F$.

2496 *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a triangle ABC, let C_A be the circle tangent to the sides AB, AC, and to the circumcircle internally. Define C_B and C_C analogously. Find the triangle, unique up to similarity, for which the inradius and the radii of the three circles C_A, C_B, and C_C are in arithmetic progression.

2497 *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Given $\triangle ABC$ and a point D on AC, let $\angle ABD = \delta$ and $\angle DBC = \gamma$. Find all values of $\angle BAC$ for which $\frac{\delta}{\gamma} > \frac{AD}{DC}$.

2498 *Proposed by K.R.S. Sastry, Dodballapur, India.*

A Gergonne cevian is the line segment from a vertex of a triangle to the point of contact, on the opposite side, of the incircle. The Gergonne point is the point of concurrency of the Gergonne cevians.

In an integer triangle ABC, prove that the Gergonne point Γ bisects the Gergonne cevian AD if and only if b, c, $\frac{1}{2}|3a - b - c|$ form a triangle where the measure of the angle between b and c is $\frac{\pi}{3}$.

2499 *Proposed by K.R.S. Sastry, Dodballapur, India.*

A Gergonne cevian is the line segment from a vertex of a triangle to the point of contact, on the opposite side, of the incircle. The Gergonne point is the point of concurrency of the Gergonne cevians.

Prove or disprove:

two Gergonne cevians may be perpendicular to each other.

2500 *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

In the lattice plane, the unit circle is the incircle of $\triangle ABC$.

Determine all possible triangles ABC.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1492. [1989: 297; 1991: 50] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$. Suppose also that $BA' = CB' = AC'$.

1. If either the centroids G, G' or the circumcentres O, O' of the triangles coincide, prove that $\triangle ABC$ is equilateral.

2.* If either the incentres I, I' or the orthocentres H, H' of the triangles coincide, characterize $\triangle ABC$.

III. *Solution to part 2* by C.R. Pranesachar and B.J. Venkatachala, Department of Mathematics, Indian Institute of Science, Bangalore 560 012 India.*

1. The case in which triangles ABC and $A'B'C'$ have the same orthocentre.

We first prove that *when A', B', C' lie in the interior of the sides BC, CA, AB, and $H = H'$, then $\triangle ABC$ is equilateral.*

Let triangles ABC and $A'B'C'$ have H as their common orthocentre. Taking H as the origin, we have

$$\overrightarrow{HA} \cdot \overrightarrow{HB} = \overrightarrow{HB} \cdot \overrightarrow{HC} = \overrightarrow{HC} \cdot \overrightarrow{HA} = T\,.$$

Since $HA = 2R\cos A$, $HB = 2R\cos B$ and $\angle AHB = \pi - C$, it follows that $T = -4R^2 \cos A \cos B \cos C$ (for all triangles ABC). Further, as $BA' = CB' = AC' = x$ (say), we obtain

$$\begin{aligned} \overrightarrow{HA'} &= \frac{x\overrightarrow{HC} + (a-x)\overrightarrow{HB}}{a}\,, \\ \overrightarrow{HB'} &= \frac{x\overrightarrow{HA} + (b-x)\overrightarrow{HC}}{b}\,, \\ \text{and } \overrightarrow{HC'} &= \frac{x\overrightarrow{HB} + (c-x)\overrightarrow{HA}}{c}\,. \end{aligned}$$

We also have

$$\overrightarrow{HA'} \cdot \overrightarrow{HB'} = \overrightarrow{HB'} \cdot \overrightarrow{HC'} = \overrightarrow{HC'} \cdot \overrightarrow{HA'}\,.$$

Here the first term $\overrightarrow{HA'} \cdot \overrightarrow{HB'}$ is equal to

$$\begin{aligned}
&\frac{1}{ab}\left(x\overrightarrow{HC} + (a-x)\overrightarrow{HB}\right) \cdot \left(x\overrightarrow{HA} + (b-x)\overrightarrow{HC}\right) \\
&\quad = \frac{1}{ab}\left(x^2T + x(b-x)4R^2\cos^2 C + x(a-x)T + (a-x)(b-x)T\right) \\
&\quad = \frac{1}{ab}((x^2 - bx + ab)T + x(b-x)4R^2\cos^2 C) \\
&\quad = \frac{1}{ab}x(b-x)(4R^2\cos^2 C + 4R^2\cos A\cos B\cos C) + T \\
&\quad = \frac{1}{ab}x(b-x)4R^2\cos C \cdot \sin A \sin B + T \\
&\quad = x(b-x)\cos C + T\,.
\end{aligned}$$

Similarly $\overrightarrow{HB'} \cdot \overrightarrow{HC'} = x(c-x)\cos A + T$, and $\overrightarrow{HC'} \cdot \overrightarrow{HA'} = x(a-x)\cos B + T$. Hence we have

$$(c-x)\cos A = (a-x)\cos B = (b-x)\cos C\,. \qquad (1)$$

Because $0 < x < a$, $0 < x < b$ and $0 < x < c$, we see that $\cos A$, $\cos B$ and $\cos C$ are all of the same sign and hence are all positive. Thus ABC is an acute triangle. If some two sides were equal, say $a = b$, then we get $\cos B = \cos C$ from (1) and so $b = c$. Thus triangle ABC is equilateral. Next, suppose to the contrary that $a < b < c$. Then

$$0 < a-x < b-x < c-x$$

and

$$0 < \cos C < \cos B < \cos A\,,$$

giving

$$(a-x)\cos B < (c-x)\cos A\,,$$

a contradiction.

Finally, suppose $a < c < b$. Then

$$0 < a-x < c-x < b-x$$

and

$$0 < \cos B < \cos C < \cos A\,,$$

giving

$$(a-x)\cos B < (c-x)\cos A\,,$$

once again a contradiction.

By the cyclic symmetry of the relations (1), it is enough to consider these cases. Hence we conclude that triangle ABC is necessarily equilateral.

Remark: If we allow x to exceed one of the sides, say $x > a$, then it is possible that triangle ABC be non-equilateral and still triangles ABC and $A'B'C'$ have a common orthocentre. We proceed as follows:

Eliminating x from the relations (1), we get

$$(a-b)\cos B\cos C + (b-c)\cos C\cos A + (c-a)\cos A\cos B \;=\; 0\,.$$

Multiplying this by $a^2b^2c^2$, [setting $\cos A \;=\; (b^2+c^2-a^2)/2bc$, etc.], and removing the factor $(a+b+c)$, we obtain

$$\begin{aligned} 0 \;=\; & a^5b+b^5c+c^5a+a^2b^4+b^2c^4+c^2a^4 \\ & -a^4b^2-b^4c^2-c^4a^2-a^3b^3-b^3c^3-c^3a^3 \\ & -2abc(a^3+b^3+c^3)+2abc(a^2b+b^2c+c^2a) \\ & +abc(ab^2+bc^2+ca^2)-3a^2b^2c^2\,. \end{aligned}$$

Setting $a \;=\; 8,\;\; b \;=\; 27$ and solving the last equation for c (using MAPLE), we get $c \;=\; 24.159993$. Hence from (1), $x \;=\; 21.248900$. If we take $B=(0,0)$ and $C=(8,0)$ in the xy–plane, we obtain

$$\begin{aligned} A &= (-5.0809203,\; 23.619685)\,, \\ A' &= (21.248900,\; 0)\,, \\ B' &= (-2.2946361,\; 18.588605)\,, \\ C' &= (-0.61221170,\; 2.8459897)\,, \end{aligned}$$

and the common orthocentre

$$H=(-5.0809203,\; -2.8138865)\,.$$

2. The case in which triangles ABC and $A'B'C'$ have the same orthocentre.

Editor's comment. The authors prove that again in this case, *when A', B', C' lie in the interior of the sides BC, CA, AB, and $I=I'$ then $\triangle ABC$ is equilateral*. They further provide an example of nonequilateral triangles for which the incentre of $\triangle ABC$ coincides with an excentre of $\triangle A'B'C'$. Their treatment relies heavily on the use of MAPLE for their algebraic manipulations. While their proof of this interesting result is certainly valid (and rather clever), computer calculations seem out of place here. Readers who would like to see the 5-page proof can apply to the solvers.

2206. [1997: 46; 1998: 61, 311] *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let a and b denote distinct positive real numbers.

(a) Show that if $0 < p < 1$, $p \neq \frac{1}{2}$, then

$$\frac{1}{2}(a^p b^{1-p} + a^{1-p} b^p) < 4p(1-p)\sqrt{ab} + \Big(1 - 4p(1-p)\Big)\,\frac{a+b}{2}\,.$$

(b) Use (a) to deduce Pólya's Inequality:

$$\frac{a-b}{\log a - \log b} < \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right)\,.$$

Note: "$\log$" is, of course, the natural logarithm.

Comment on part (a) *by the proposer, slightly adapted by the editor.*

The proposer writes that, in the displayed equation on page 312, the term x^{2k+1} should be x^{2k+2}.

2373. [1998: 365] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $AB > AC$. Let M be the mid-point of BC. Suppose that D is the reflection of M across the bisector of $\angle BAC$, and that A, B, C and D are concyclic.

Determine the value of $\dfrac{AB - AC}{BC}$.

Solution by Michel Bataille, Rouen, France.

The answer is : $\dfrac{AB - AC}{BC} = \dfrac{1}{\sqrt{2}}$.

Suppose the bisector of $\angle BAC$ meets the segment BC at K. Since $KC/AC = KB/AB$ and $AB > AC$, we have $BK > KC$ and M belongs to segment KB. Hence $\angle KAM < \angle KAB$, so that $\angle KAD < \angle KAC$ (by reflection across AK). From this, we see that B and D are on the same side of the line AC, and, since A, B, C and D are concyclic, we have $\angle CDA = \angle CBA$.

Furthermore $\angle CAD = \angle BAM$, so that $\triangle ACD$ and $\triangle AMB$ are similar. Therefore we have: $AB/AD = AM/AC$ and, since $AD = AM$, we obtain

$$AM^2 \;=\; AB \cdot AC\,. \qquad (1)$$

But it is well known that the median AM of $\triangle ABC$ satifies

$$4AM^2 + BC^2 \;=\; 2AB^2 + 2AC^2\,. \qquad (2)$$

From (1) and (2), we immediately get $2(AB - AC)^2 = BC^2$, and the result follows (since $AB > AC$).

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (2 solutions); D.J. SMEENK, Zaltbommel, the Netherlands; PARAYIOU THEOKLITOS, Limassol, Crete; and the proposer.

2374. [1998: 365, 424] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $\angle BAC > 60°$. Let M be the mid-point of BC. Let P be any point in the plane of $\triangle ABC$.

Prove that $AP + BP + CP \geq 2AM$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

It is sufficient to prove the stated inequality for the point P for which $PA + PB + PC$ is a minimum. It is well known that if $\angle A < 120°$ then P is the intersection of the concurrent lines AD, BE, CF, where D, E, F are the three vertices of the external equilateral triangles on sides BC, CA, AB, respectively, but if $\angle A \geq 120°$, then P coincides with A (see for example Courant & Robbins, *What is Mathematics*, the discussion of Steiner's problem).

If $\angle A \geq 120°$ then $\min(PA+PB+PC) = AB+AC$. But $AB+AC > 2AM$, as is easily seen by completing the parallelogram of BAC (the median AM is half the diagonal of this parallelogram and the triangle inequality applies). So this case is dealt with and we may assume $\angle A < 120°$.

In this case, it is well known and easy to see that $AD = BE = CF =$ (the stated minimum). Thus we are reduced to showing $CF \geq 2AM$; that is, $CF^2 \geq 4AM^2$. By the Cosine Rule on $\triangle AFC$ we have

$$\begin{aligned} CF^2 &= AF^2 + AC^2 - 2AF \cdot AC \cos(\angle A + 60°) \\ &= c^2 + b^2 - 2bc\cos(\angle A + 60°)\,. \end{aligned}$$

But AM is a median of ABC so that

$$\begin{aligned} 4AM^2 &= 2b^2 + 2c^2 - a^2 \;=\; b^2 + c^2 + (b^2 + c^2 - a^2) \\ &= b^2 + c^2 + 2bc\cos\angle A\,. \end{aligned}$$

In other words we are to show $-\cos(\angle A + 60°) \geq \cos\angle A$. Recall now the problem has the restriction $\angle A > 60°$; that is, $\angle A + 30° > 90°$, and so

$$\begin{aligned} \cos\angle A + \cos(\angle A + 60°) &= 2\cos(\angle A + 30°)\cos 30° \\ &= \sqrt{3}\cos(\angle A + 30°) < 0\,, \end{aligned}$$

showing that $\cos\angle A < -\cos(\angle A + 60^\circ)$, as required. This completes the proof. In fact, we have shown the strict inequality $PA+PB+PC > 2AM$.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; VEDULA N. MURTY, Dover, PA, USA; VICTOR OXMAN, University of Haifa, Haifa, Israel; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer. There was one partial solution.

Konečný suggests that readers might be interested in the article, "An Advanced Calculus Approach to Finding the Fermat Point" by Mowaffaq Hajja in Math Magazine *67, no. 1 (1994).*

2378. [1998: 425] *Proposed by David Doster, Choate Rosemary Hall, Wallingford, CT, USA.*

Find the exact value of: $\cot\left(\frac{\pi}{22}\right) - 4\cos\left(\frac{3\pi}{22}\right)$.

I. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let $x = \frac{\pi}{22}$ and $z = \cos x + i\sin x$; then $11x = \frac{\pi}{2}$ and $z^2 + 1 \neq 0$. We shall prove that

$$\cot x - 4\cos(3x) = \sqrt{11} . \qquad (1)$$

We have

$$z^{22} = \cos(22x) + i\sin(22x) = -1 . \qquad \text{(De Moivre)}$$

This is equivalent to each of the following:

$$z^{22} + 1 = 0 ,$$

$$(z^2+1)(z^{20} - z^{18} + z^{16} - \cdots - z^2 + 1) = 0 ,$$

$$z^{10} + \frac{1}{z^{10}} - \left(z^8 + \frac{1}{z^8}\right) + \left(z^6 + \frac{1}{z^6}\right) - \left(z^4 + \frac{1}{z^4}\right) + \left(z^2 + \frac{1}{z^2}\right) - 1 = 0 ,$$

$$2\cos(10x) - 2\cos(8x) + 2\cos(6x) - 2\cos(4x) + 2\cos(2x) - 1 = 0 . \qquad (2)$$

Squaring both sides of (1), we get

$$\left(\cos x - 4\sin x \cdot \cos(3x)\right)^2 = 11\sin^2 x .$$

This is equivalent to each of the following:

$$\cos^2 x - 8\cos x \cdot \sin x \cdot \cos(3x) + 16\sin^2 x \cdot \cos^2(3x) = 11\sin^2 x ,$$

$$1 + \cos(2x) - 8\sin(2x)\cdot\cos(3x) + 8[1 - \cos(2x)][1 + \cos(6x)] = 11[1 - \cos(2x)] ,$$

$$-2 + 4\cos(2x) - 4[\sin(5x) - \sin x] + 8\cos(6x) - 8\cos(2x)\cdot\cos(6x) = 0 ,$$

$$-2 + 4\cos(2x) - 4[\cos(6x) - \cos(10x)] + 8\cos(6x) - 4[\cos(4x) + \cos(8x)] = 0 ,$$

$$2\cos(10x) - 2\cos(8x) + 2\cos(6x) - 2\cos(4x) + 2\cos(2x) - 1 = 0 ,$$

which is true by (2).

Editor's Remark. We now have that $\cot x - 4\cos(3x) = \pm\sqrt{11}$. The expression is clearly positive (by calculator, for example). Diminnie and White make it clearer: since $0 < \sin\left(\frac{\pi}{22}\right) < \frac{\pi}{22} < \frac{1}{4}$, it follows that $\cot\left(\frac{\pi}{22}\right) - 4\cos\left(\frac{3\pi}{22}\right) > 4\cos\left(\frac{\pi}{22}\right) - 4\cos\left(\frac{3\pi}{22}\right) > 0$.

II. *Solution by Allen Herman, Regina, Canada*.

The quadratic Gauss sum for p prime and $p \equiv 3 \pmod 4$ has the value

$$\sum_{k=0}^{p-1} \left(\frac{k}{p}\right) \zeta^k \;=\; i\sqrt{p}\,,$$

where $\zeta = e^{\frac{2\pi i}{p}}$ is a primitive pth root of unity and $\left(\frac{k}{p}\right)$ is the Legendre symbol (equal to 1 when k is a square modulo p, and to -1 otherwise). See, for example, §6.3 of Kenneth Ireland and Michael Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Springer 1990, p. 75. In particular, when $p = 11$ then $\zeta = e^{\frac{2\pi i}{11}}$ and 1, 3, 4, 5, 9 are the squares modulo 11. Therefore,

$$\zeta - \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 - \zeta^6 - \zeta^7 - \zeta^8 + \zeta^9 - \zeta^{10} \;=\; i\sqrt{11}\,.$$

As in solution I let $z = e^{\frac{\pi i}{22}}$. Then $z = -i\zeta^3$, and so (making ample use of $\zeta^{11} = 1$)

$$\begin{aligned} \cos\frac{\pi}{22} &= \frac{z+z^{-1}}{2} = \frac{i}{2}(\zeta^8 - \zeta^3)\,, \qquad \text{and} \\ \sin\frac{\pi}{22} &= \frac{z-z^{-1}}{2i} = -\frac{1}{2}(\zeta^3 + \zeta^8)\,. \end{aligned}$$

Thus

$$\cot\frac{\pi}{22} \;=\; i\left(\frac{\zeta^3 - \zeta^8}{\zeta^3 + \zeta^8}\right) \;=\; i\left(\frac{\zeta^6 - 1}{\zeta^6 + 1}\right) \;=\; i\left(1 - \frac{2}{1+\zeta^6}\right)\,.$$

Since $(\zeta^6)^{11} = 1$, we have

$$(1+\zeta^6)(1 - (\zeta^6) + (\zeta^6)^2 - (\zeta^6)^3 + \cdots + (\zeta^6)^{10}) \;=\; 2\,,$$

so that

$$\begin{aligned} \cot\frac{\pi}{22} &= i(1 - (1 - \zeta^6 + \zeta - \zeta^7 + \zeta^2 - \zeta^8 + \zeta^3 - \zeta^9 + \zeta^4 - \zeta^{10} + \zeta^5)) \\ &= i(-\zeta - \zeta^2 - \zeta^3 - \zeta^4 - \zeta^5 + \zeta^6 + \zeta^7 + \zeta^8 + \zeta^9 + \zeta^{10})\,. \end{aligned}$$

Finally, $\cos\frac{3\pi}{22} = \frac{z^3+z^{-3}}{2} = \frac{i}{2}(\zeta^9-\zeta^2)$, and thus

$$\begin{aligned}
&\cot\frac{\pi}{22} - 4\cos\frac{3\pi}{22} \\
&= -i((\zeta+\zeta^2+\zeta^3+\zeta^4+\zeta^5-\zeta^6-\zeta^7-\zeta^8-\zeta^9-\zeta^{10})+2(\zeta^9-\zeta^2)) \\
&= -i(\zeta-\zeta^2+\zeta^3+\zeta^4+\zeta^5-\zeta^6-\zeta^7-\zeta^8+\zeta^9-\zeta^{10}) \\
&= -i(i\sqrt{11}) \\
&= \sqrt{11}\,.
\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; CHARLES DIMINNIE and LARRY WHITE, San Angelo, TX, USA; C. FESTRAETS-HAMOIR, Brussels, Belgium; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; MICHAEL LAMBROU, University of Crete, Crete, Greece; M. PERISASTRY, Vizianagaram, and VEDULA N. MURTY, Visakhapatnam, India; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Perisastry and Murty deduce the result as the special case $m = 2$ of the identity

$$\tan\frac{3m\pi}{11} + 4\sin\frac{2m\pi}{11} = \pm\sqrt{11}\,,$$

where "+" is used for $m = 1, 3, 4, 5, 9$ — the perfect squares modulo 11 as in solution **II** *— and "−" for the non-squares.* (*Compare the recent proposal* **2463*** [1999: 366].) *An instance of this identity is problem 218* (*posed by Murty!*) *in the* College Mathematics Journal 14:4 (Sept. 1983) *358-359. References there trace it back to the Math. Tripos of 1895. Murty adds that Murray Klamkin found it as problem #29 in Hobson's* Treatise on Plane Trigonometry, 7th ed. *p. 123. Most solvers used trigonometric identities as in solution* **I**; *Doster and Herzig both noticed a connection with the squares modulo 11, with Doster using the Gauss sum much as in solution* **II**.

2379. [1998: 425] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that M_1, M_2 and M_3 are the mid-points of the altitudes from A to BC, from B to CA and from C to AB in $\triangle ABC$. Suppose that T_1, T_2 and T_3 are the points where the excircles to $\triangle ABC$ opposite A, B and C, touch BC, CA and AB.

Prove that M_1T_1, M_2T_2 and M_3T_3 are concurrent.

Determine the point of concurrency.

Nearly identical solutions by J.F. Rigby, Cardiff, Wales, and Peter Y. Woo, Biola University, La Miranda, CA, USA.

Let D be the foot of the altitude from A to BC, P be the point where the incircle touches BC, and Q be the point of the incircle diametrically opposite P. There is a dilatation with centre A mapping the incircle and its centre I to the excircle opposite A and its centre I_1. This dilatation maps Q to T_1, so that A, Q, and T_1 are collinear. Because M_1 is the mid-point of AD while I is the mid-point of the parallel segment PQ, it follows that

M_1, I, and T_1 are also collinear. Similarly I lies on M_2T_2 and M_3T_3, so the three lines are concurrent at the incentre I.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; HIDETOSHI FUKAGAWA, Gifu, Japan; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; G. TSINTSIFAS, Thessaloniki, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

2386*. [1998: 426] *Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA.*

Write

$$\begin{matrix}1\\ \end{matrix} \to \begin{matrix}1\\1\end{matrix} \to \begin{matrix}3\\1\end{matrix} \to \begin{matrix}4 & 1\\1 & 3\end{matrix} \to \begin{matrix}6 & 2 & 1\\1 & 3 & 4\end{matrix} \to \begin{matrix}8 & 1 & 3 & 2 & 1\\1 & 2 & 3 & 4 & 6\end{matrix} \to$$

(The last ten numbers shown indicate that up to this point, eight 1's, one 2, three 3's, two 4's and one 6 have been written.)

(a) If this is continued indefinitely, will 5 eventually appear?

(b) Will every positive integer eventually be written?

Note: 11 is a number and not two 1's.

Solution.

All solvers pointed out that 5 appears in the very next iteration. So the answer to part (a) is trivially "yes". No solver was able to solve part (b), but all seemed to believe the answer here was also "yes". So part (b) remains open.

Solved by CHARLES ASHBACHER, Cedar Rapids, IA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and J.A. McCALLUM, Medicine Hat, Alberta.

2387. [1998: 426] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For fixed $p \in \mathbb{N}$, consider the power sums

$$S_p(n) := \sum_{k=1}^{n} (2k-1)^p, \qquad \text{where } n \geq 1,$$

so that $S_p(n)$ is a polynomial in n of degree $p+1$ with rational coefficients.

Prove that

(a)* If all coefficients of $S_p(n)$ are integers, then $p = 2^m - 1$ for some $m \in \mathbb{N}$.

(b)* The only values of p yielding such polynomials are $p = 1$ and $p = 3$ (with $S_1(n) = n^2$ and $S_3(n) = 2n^4 - n^2$).

Solution by Florian Herzig, student, Cambridge, UK.

(a) Let $R_p(n) = \sum_{k=1}^{n} k^p$. To explicitly find the coefficients in the polynomial $R_p(n)$, we can use the following approach using generating functions:

$$\begin{aligned} R_0(n) - R_1(n)x + \frac{R_2(n)x^2}{2!} - \frac{R_3(n)x^3}{3!} + - \cdots \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p x^p}{p!} \sum_{k=1}^{n} k^p \;=\; \sum_{k=1}^{n} \sum_{p=0}^{\infty} \frac{(-kx)^p}{p!} \\ &= e^{-x} + e^{-2x} + \cdots + e^{-nx} \;=\; e^{-x} \frac{1 - e^{-nx}}{1 - e^{-x}} \\ &= \frac{x}{e^x - 1} \cdot \frac{1 - e^{-nx}}{x} \,. \end{aligned} \tag{1}$$

Now the first factor in the last expression has power series $\sum_{i \geq 0} B_i x^i / i!$ where the B_i are the Bernoulli numbers (by definition) [see equation 6.81 of [1]].

So the above expression becomes

$$\left(\sum_{i=0}^{\infty} \frac{B_i}{i!} x^i \right) \left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{n^k x^{k-1}}{k!} \right) .$$

Comparing the coefficient of x^p on both sides of (1) we get

$$\frac{(-1)^p R_p(n)}{p!} = \sum_{k=1}^{p+1} \frac{(-1)^{k-1} n^k}{k!} \frac{B_{p+1-k}}{(p+1-k)!} \,,$$

and so

$$\begin{aligned} R_p(n) &= (-1)^p p! \sum_{k=1}^{p+1} \frac{1}{k!(p+1-k)!} n^k (-1)^{1-k} B_{p+1-k} \\ &= \frac{1}{p+1} \sum_{k=1}^{p+1} \binom{p+1}{k} n^k (-1)^{p+1-k} B_{p+1-k} \,. \end{aligned} \tag{2}$$

Therefore from $S_p(n) = R_p(2n) - 2^p R_p(n)$,

$$S_p(n) = \frac{1}{p+1} \sum_{k=1}^{p+1} \binom{p+1}{k} (2^k - 2^p) n^k (-1)^{p+1-k} B_{p+1-k} \,. \tag{3}$$

Note that we get a recursion for the B_i from $(e^x-1)\sum_{i\geq 0} B_i x^i/i! = x$ by equating coefficients, namely $B_0 = 1$ and $\sum_{k=1}^{n-1}\binom{n}{k}B_k = 0$ for all $n \geq 2$ [see equation 6.79 of [1]]. From this we easily find that $B_1 = -1/2$, $B_2 = 1/6$, etc. Since

$$\frac{x}{e^x - 1} + \frac{x}{2}$$

is an even function of x, it also follows that $B_n = 0$ for odd $n > 1$. [See 6.84 of [1].]

If we look back at equation 3 we see that the coefficient of n^{p+1} equals $2^p/(p+1)$. Thus if we want $S_p(n)$ to have integer coefficients, it is necessary that $p + 1 = 2^m$ for some integer m, as required.

(b) I will show that 1 and 3 are the only positive integers p such that $S_p(n)$ has only integer coefficients. Assume for the proof that $p > 3$ is an integer with this property ($p = 2$ is clearly not possible, by (a)). The strategy is to show that p has to be divisible by arbitrarily large powers of 3. To simplify the notation introduce the function v from the non-zero rationals into the integers, defined by $r = 3^{v(r)}q$ where q is a rational number with numerator and denominator not divisible by 3 (that is, "$v(r)$ is the exponent of the power of 3 contained in r"). This is clearly well-defined and a homomorphism of the multiplicative group of non-zero rationals into the additive group of integers; that is, $v(rs) = v(r) + v(s)$ for all non-zero rationals r, s. We will need a few lemmas now.

Lemma 1 $v(B_{2n}) = -1$ for all $n \geq 1$.

Proof. The proof is by induction on n.

First it is true for $n = 1$. Assume then that $v(B_{2k}) = -1$ for all $1 \leq k < n$. Then $R_{2n}(3) = 1 + 2^{2n} + 3^{2n} \equiv 2 \pmod 3$ and

$$\begin{aligned} R_{2n}(3) &= 3B_{2n} + \frac{1}{2n+1}\sum_{k=2}^{2n+1}\binom{2n+1}{k}3^k(-1)^{2n+1-k}B_{2n+1-k} \\ &= 3B_{2n} + \sum_{k=2}^{2n+1}\binom{2n}{k-1}\frac{1}{k}\,3^k(-1)^{2n+1-k}B_{2n+1-k} \end{aligned}$$

from equation (2). Note now that $3^{k-1} = (1+2)^{k-1} \geq 1 + 2(k-1) > k$ for all $k > 1$, and so $v(3^{k-1}/k) \geq 0$ for $k > 1$ (as the power of 3 in k is completely cancelled). Also $v(3B_{2n+1-k}) \geq 0$ when $k > 1$ is odd, by the induction hypothesis. Thus (using also that $B_{2n+1-k} = 0$ for even $k > 0$) the whole sum in the above displayed equation has denominator not divisible by 3. Thus

$$\frac{1}{3}\left(R_{2n}(3) + 1\right) \;=\; B_{2n} + \frac{1}{3} + s\,,$$

where s is a rational number with denominator not divisible by 3. Since the left-hand side is an integer and $v(\frac{1}{3}+s) = -1$, it follows that $v(B_{2n}) = -1$ as claimed. ∎

Actually a much more general result is true: the denominator of B_{2n} is the product of all primes p such that $(p-1)|2n$; compare [1], p. 315, problem 54.

Lemma 2 $v(2^k - 2^p) = 0$ whenever $k - p$ is odd.

Proof. First $v(2^k - 2^p) = v(1 - 2^{|p-k|})$. The claim follows because

$$2^n \equiv 1 \pmod 3 \iff n \equiv 0 \pmod 2 .$$ ∎

Lemma 3 p does not equal a power of 3 (if p is as assumed).

Proof. By (a) we know that $p = 2^m - 1$ for some natural number m. But $p > 3$ and $3^n \not\equiv -1 \pmod 8$ for all natural numbers n so that the lemma follows. ∎

We can finish the proof now. By induction we show that $v(p) \geq a$ for all natural numbers a. For the induction basis consider the coefficient of n^{p-1} in $S_p(n)$. From (3) it follows that it equals

$$\alpha = \frac{1}{p+1}\binom{p+1}{2} B_2(-2^{p-1}) = \frac{p}{12}(-2^{p-1}) .$$

So $v(\alpha) = v(p) - 1 \geq 0$ since we assumed that $S_p(n)$ has integer coefficients. Hence $v(p) \geq 1$ as we wanted to show.

For the induction step suppose that $v(p) \geq a$ for some $a \geq 1$. By Lemma 3 we know that $p > 3^a$. So we can consider the coefficient of n^{p-3^a} in $S_p(n)$, which is the *integer*

$$\begin{aligned}\beta &= \frac{1}{p+1}\binom{p+1}{3^a+1} B_{3^a+1}(2^{p-3^a} - 2^p) \\ &= \frac{p}{3^a(3^a+1)} \cdot \frac{(p-1)(p-2)\cdots(p-3^a+1)}{(3^a-1)(3^a-2)\cdots 1} B_{3^a+1}(2^{p-3^a} - 2^p) .\end{aligned}$$

As p is divisible by 3^a, the big fraction has v–value 0 (the powers of 3 cancel in corresponding terms of the numerator and the denominator). Thus, using Lemmas 1 and 2, $v(\beta) \geq v(p) - a + 0 - 1 + 0 \geq 0$, whence $v(p) \geq a + 1$. By induction this completes the proof of (b).

Reference

[1] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, 2nd Ed., Addison-Wesley, 1994.

Both parts also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and MICHAEL LAMBROU, University of Crete, Crete, Greece. Part (a) *only solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; and HEINZ-JÜRGEN SEIFFERT, Berlin, Germany.*

"Late" solutions to **2388** [1998: 503; 1999: 171; 1999: 445] were received from MICHEL BATAILLE, Rouen, France and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

2389. [1998: 503; 1999, 171] *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Suppose that f is continuous on $\mathbb{R}^n$ and satisfies the condition that when any two of its variables are replaced by their arithmetic mean, the value of the function increases; for example:

$$f(a_1, a_2, a_3, \ldots, a_n) \le f\left(\frac{a_1+a_3}{2}, a_2, \frac{a_1+a_3}{2}, a_4, \ldots, a_n\right).$$

Let $m = \dfrac{a_1+a_2+\ldots+a_n}{n}$. Prove that

$$f(a_1, a_2, a_3, \ldots, a_n) \le f(m, m, m, \ldots, m).$$

Solution by Michael Lambrou, University of Crete, Crete, Greece.

We show how to construct a sequence of closed nested intervals $J_1 \supseteq J_2 \supseteq \ldots$ and a sequence of n-tuples $(a_{N,1}, a_{N,2} \ldots, a_{N,n}) \in \mathbb{R}^n$ for $N = 0, 1, 2, \ldots$ with $(a_{0,1}, a_{0,2}, \ldots, a_{0,n}) = (a_1, a_2, \ldots, a_n)$ such that

(i) $\sum_{i=1}^{n} a_{N,i} = \sum_{i=1}^{n} a_i$,

(ii) $a_{N,i} \in J_N$ for $i = 1, 2, \ldots, n$,

(iii) $\lim_{N\to\infty} |J_N| = 0$, where $|J|$ denotes the length of the interval J, and

(iv) the sequence $t_N = f(a_{N,1}, a_{N,2}, \ldots, a_{N,n})$ is increasing.

Assuming this for the moment, we complete the proof as follows: By the nested interval theorem the intersection $\bigcap_{N=1}^{\infty} J_N$ is non-empty, consisting of a unique number (because of (iii)) which we call m. By (ii) and (iii) we have $\lim_{N\to\infty} a_{N,i} \to m$ for each $i = 1, 2, \ldots, n$ and so by (i) we have $\sum_{i=1}^{n} a_i = nm$. Finally by the continuity of f at $(m, m, \ldots, m) \in \mathbb{R}^n$ we have

$$t_N = f(a_{N,1}, a_{N,2}, \ldots, a_{N,n}) \to f(m, m, \ldots, m).$$

Then by (iv) we have

$$f(a_1, a_2, \ldots, a_n) = f(a_{0,1}, a_{0,2}, \ldots, a_{0,n}) \le f(m, m, \ldots, m),$$

as required.

Now for the details. Set $J_1 = [a, b]$ where $a = \min\{a_1, a_2, \ldots, a_n\}$ and $b = \max\{a_1, a_2, \ldots, a_n\}$. Consider the intervals:

$$I_1 = \left[a, \frac{2a+b}{3}\right), \quad I_2 = \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right], \quad I_3 = \left(\frac{a+2b}{3}, b\right].$$

As long as each of I_1, I_3 contains at least one of the a_k's, say $a_i \in I_1$ and $a_j \in I_3$, repeat the following procedure: Replace a_i, a_j by their average $x = \frac{1}{2}(a_i + a_j)$. Notice that by assumption the value of f increases since

$$f(a_1, a_2, \ldots, a_n) \le f(a_1, \ldots, x, \ldots, x, \ldots, a_n),$$

and that the sum

$$\begin{aligned} & a_1 + a_2 + \cdots + a_n \\ & = a_1 + \cdots + a_{i-1} + x + a_{i+1} + \cdots + a_{j-1} + x + a_{j+1} + \cdots + a_n \qquad (1) \end{aligned}$$

does not change value. Also, as $a \le a_i < \frac{2a+b}{3}$, $\frac{a+2b}{3} < a_j \le b$ we have

$$\frac{2a+b}{3} < \frac{a_i + a_j}{2} < \frac{a+2b}{3}.$$

In other words $x \in I_2$. After a finite number of repetitions (at most $\frac{1}{2}n$) of this replacing procedure, at least one of I_1, I_3 will not contain any of the a_k's or their replacements (they move to I_2). Thus we end up with numbers $a_{1,1}$, $a_{1,2}$, ..., $a_{1,n}$, none of which is in I_1 or none of which is in I_3. That is, they are all either in $I_2 \cup I_3$ or in $I_1 \cup I_2$. Call J_2 that one of $I_2 \cup I_3$ or $I_1 \cup I_2$ that contains all the $a_{i,k}$'s. Observe that (i) holds (because of (1)), that $J_1 \supseteq J_2$ and that $|J_2| = \frac{2}{3}(b-a) = \frac{2}{3}|J_1|$.

Repeating the whole process with J_2 in place of J_1, etc. we find gradually $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$ such that $|J_{N+1}| = \frac{2}{3}|J_N| = \left(\frac{2}{3}\right)^N (b-a) \to 0$ such that conditions (i), (ii), (iii), and (iv) are satisfied. This completes the construction and the proof.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARK LYON and MAX SHKARAYEV, students, University of Arizona, Tuscon, AZ, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2390. [1998: 504] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $\lambda \ge 0$ and $p, q \ge 1$, let $S_n(\lambda, p, q) := \sum_{i=1}^{n-\lambda} \sum_{j=i+\lambda}^{n} i^p j^q$, where $n > \lambda$.

Given the statement: "$S_n(\lambda, p, q)$, understood as a polynomial in $\mathbb{Q}[n]$, is always divisible by $(n-\lambda)(n-\lambda+1)(n-\lambda+2)$",

(a) give examples for $\lambda = 0, 1, 2, 3, 4$;

(b)* prove the statement in general.

Solution to both parts by G.P. Henderson, Garden Hill, Campbellcroft, Ontario.

We will show that the statement is true for $0 \leq \lambda < n$, $p \geq 1$, and $q \geq 0$.

First consider the case $\lambda = 0$. We have

$$S_n(0,p,q) \;=\; \sum_{i=1}^{n}\sum_{j=i}^{n} i^p j^q \;=\; \sum_{j=1}^{n}\sum_{i=1}^{j} i^p j^q .$$

Using Newton's interpolation formula we can write

$$i^p \;=\; \sum_{r=1}^{p} a_{pr}\binom{i}{r}, \qquad p \geq 1 ,$$

where the a's are independent of i. Then

$$S_n(0,p,q) \;=\; \sum_{j=1}^{n} j^q \sum_{r=1}^{p} a_{pr} \sum_{i=1}^{j}\binom{i}{r} .$$

The inner sum is

$$\sum_{i=1}^{j}\left[\binom{i+1}{r+1} - \binom{i}{r+1}\right] \;=\; \binom{j+1}{r+1},$$

and so

$$S_n(0,p,q) \;=\; \sum_{j=1}^{n}\sum_{r=1}^{p} a_{pr} j^q \binom{j+1}{r+1} . \qquad (1)$$

The following is easily proved by induction on q.

Lemma. If $P(j)$ is a polynomial in j with $0 \leq \deg(P) \leq q$ then P can be written as a linear combination of the polynomials

$$\binom{j+s+1}{s}, \qquad s = 0, 1, \ldots, q. \qquad \blacksquare$$

[Or just note that these polynomials are linearly independent and thus must form a basis of the vector space $\mathbb{Q}[j]$. –*Ed.*] In particular, there exist constants b_{qs}, $s = 0, 1, \ldots, q$, such that

$$j^q \;=\; \sum_{s=0}^{q} b_{qs}\binom{j+s+1}{s}, \qquad q \geq 0 .$$

Using this in (1),

$$S_n(0,p,q) \;=\; \sum_{r=1}^{p}\sum_{s=0}^{q} a_{pr}b_{qs}\sum_{j=1}^{n}\binom{j+s+1}{s}\binom{j+1}{r+1}\,.$$

The inner sum is

$$\begin{aligned}&\sum_{j=1}^{n}\binom{j+s+1}{r+s+1}\binom{r+s+1}{s}\\ &\quad=\;\binom{r+s+1}{s}\sum_{j=1}^{n}\left[\binom{j+s+2}{r+s+2}-\binom{j+s+1}{r+s+2}\right]\\ &\quad=\;\binom{r+s+1}{s}\binom{n+s+2}{r+s+2}\,,\end{aligned}$$

and so

$$S_n(0,p,q) \;=\; \sum_{r=1}^{p}\sum_{s=0}^{q} a_{pr}b_{qs}\binom{r+s+1}{s}\binom{n+s+2}{r+s+2}\,.$$

The polynomial $\binom{n+s+2}{r+s+2}$ is

$$\frac{(n+s+2)(n+s+1)\ldots(n-r+1)}{(r+s+2)!}\,.$$

Since $s \geq 0$ and $r \geq 1$, $S_n(0,p,q)$ is divisible by $n(n+1)(n+2)$. This finishes the case $\lambda = 0$.

In the general case, if $\lambda < n$, then

$$S_n(\lambda,p,q) \;=\; \sum_{i=1}^{n-\lambda}\sum_{j=i+\lambda}^{n} i^p j^q \;=\; \sum_{i=1}^{n-\lambda}\sum_{k=i}^{n-\lambda} i^p(k+\lambda)^q$$

where $j = k + \lambda$. Thus

$$S_n(\lambda,p,q) \;=\; \sum_{i=1}^{n-\lambda}\sum_{k=i}^{n-\lambda}\sum_{t=0}^{q}\binom{q}{t}\lambda^{q-t}i^pk^t \;=\; \sum_{t=0}^{q}\binom{q}{t}\lambda^{q-t}S_{n-\lambda}(0,p,t)\,.$$

Since $S_{n-\lambda}(0,p,t)$ is divisible by $(n-\lambda)(n-\lambda+1)(n-\lambda+2)$, this completes the proof.

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; and MICHAEL LAMBROU, University of Crete, Crete, Greece. Part (a) *only solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and the proposer.*

For part (a), *here is* $S_n(\lambda,1,1)$, *listed by almost all solvers:*

$$S_n(\lambda,1,1) = \frac{1}{24}(n-\lambda)(n-\lambda+1)(n-\lambda+2)(3n+\lambda+1)\,.$$

2391. [1998: 504] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Consider $d+1$ points, B_1, B_2, ..., B_{d+1} in the unit sphere in $\mathbb{R}^d$, so that the simplex $S_d(B) = B_1 B_2 \ldots B_{d+1}$ includes the origin O. Let $P = \{x \mid B_i \cdot x \leq 1\}$ for all i between 1 and $d+1$.

Prove that there is a point $y \in P$ such that $|y| \geq d$.

Solution by the proposer.

Let π_i be the tangent plane to the sphere (0,1) at the point B_i and we denote $A_i = \cap_{j=1, j\neq i}^{d+1} \pi_j$. Therefore (0,1) is the inscribed sphere in the simplex $S_d(A) = A_1 A_2 \ldots A_{d+1}$ and denoting $\left|\overrightarrow{0A_i}\right| = x_i$ we will have:

$$x_1 + 1 \geq h_1 ,$$

$$x_1 a_1 + a_1 \geq h_1 a_1 ,$$

where h_1, a_1 we denote the altitude from A_1 and the volume of the facet opposite A_1. But $h_1 \cdot a_1 = d \cdot \mathrm{Vol}(S_d(A))$ or

$$x_1 a_1 + a_1 \geq a_1 + a_2 + \ldots + a_{d+1} \quad \text{or}$$

$$x_1 \geq \frac{a_2}{a_1} + \frac{a_3}{a_1} + \ldots + \frac{a_{d+1}}{a_1} . \tag{1}$$

From the remaining vertices, we have d inequalities like (1). Adding, we take

$$\sum_{i=1}^{d+1} x_i \geq \sum_{i<j}^{1,d+1} \left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right) \geq \frac{2d(d+1)}{2} .$$

Therefore $\max x_i = |y| \geq d$.

There were no other solutions submitted.

2394. [1998: 505] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

The inequality $a^a b^b \geq \left(\frac{a+b}{2}\right)^{a+b}$, where $a, b > 0$, is usually proved using Calculus. Give a proof without the aid of Calculus.

I. *Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA.*

By the weighted Geometric-Harmonic Mean Inequality we have

$$\sqrt[a+b]{a^a b^b} \geq \frac{a+b}{\frac{a}{a} + \frac{b}{b}} = \frac{a+b}{2} .$$

Therefore,

$$a^a b^b \geq \left(\frac{a+b}{2}\right)^{a+b} .$$

II. *Solution by John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta.*

By the Arithmetic-Geometric Mean Inequality we have

$$\left(1+\frac{b}{a}\right)^a\left(1+\frac{a}{b}\right)^b \;\le\; \left[\frac{a}{a+b}\left(1+\frac{b}{a}\right)+\frac{b}{a+b}\left(1+\frac{a}{b}\right)\right]^{a+b} \;=\; 2^{a+b}\,.$$

This implies that

$$\left(\frac{a+b}{2}\right)^{a+b} \;\le\; a^a b^b\,.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; NIELS BEJLEGAARD, Stavanger, Norway; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; EFSTRATIOS RAPPOS, University of Cambridge, Cambridge, UK; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were also two solutions which used Calculus (Taylor series).

The proposer also notes that the generalization is true:

$$\prod_{i=1}^{n} a_i^{a_i} \;\ge\; \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^{a_1+a_2+\cdots+a_n}\,,$$

where $a_i > 0$, $i = 1, 2, \ldots, n$. Indeed, Konečný actually proves this result; his proof is based on a theorem on p. 141 of the Czech book, Metody řešení matematických úloh I, *by* Jiří Herman, Radan Kučera, Jaromír Šimša, *and published by Masaryk University in 1996. (As an application of this theorem they actually prove the above generalization of our result for $n = 3$.)*

2395. [1998: 505] *Proposed by Witold Janicki, Jagiellonian University, Krakow, Poland, Michael Sheard, St. Lawrence University, Canton, NY, USA, Dan Velleman, Amherst College, Amherst, MA, USA, and Stan Wagon, Macalester College, St. Paul, MN, USA.*

Let P be such that

(A) $P(0)$ is true, and

(B) $P(n) \Longrightarrow P(n+1)$.

Find an integer $n > 10^6$ such that $P(n)$ can be proved without using induction, but rather using

(L) the Law of Implication (that is, X and $(X \Longrightarrow Y)$ yield Y)

ten times only.

Solution by E.B. Davies, King's College, London, and Michael Lambrou, University of Crete, Crete, Greece.

We can considerably improve the outcome.

We show that with ten uses of the Law of Implication (L) we can show that $P(n)$ is true for all $n \le 2^{65536} \approx 2 \times 10^{19728}$. This will follow by showing the validity of $S(n)$ for $n = 4$ and $k = m = n = 0$, where $S(n)$, $n \in \mathbb{N}$, is the statement

$$S(n) : \forall k \left[\forall m \left[\forall n \left[P(0) \wedge P(1) \wedge \cdots \wedge P(n) \right.\right.\right.$$
$$\left.\left.\left.\Longrightarrow \quad P(0) \wedge P(1) \wedge \cdots \wedge P\left(2^k \cdot 2^{2^m} \cdot 2^{2^{2^N}} + n\right)\right]\right]\right] . \tag{1}$$

For this purpose define $Q(m)$, $m \in \mathbb{N}$, to be the statement

$$Q(m) : \forall n \, (P(0) \wedge P(1) \wedge \cdots \wedge P(n)$$
$$\Longrightarrow \quad P(0) \wedge P(1) \wedge \cdots \wedge P(2^m + n)) \; . \tag{2}$$

We show $Q(0)$ with one use of (L) by arguing

$$\frac{\begin{array}{l} P(0) \wedge \cdots \wedge P(n) \\ P(n) \Longrightarrow P(n+1) \end{array}}{P(0) \wedge \cdots \wedge P(n) \Longrightarrow P(0) \wedge \cdots \wedge P(n+1)}$$

By a second use of (L) we show that $\forall m$, $(Q(m) \Longrightarrow Q(m+1))$. Indeed, if $Q(m)$ is true, we have for all n:

$$\frac{\begin{array}{ll} P(0) \wedge \cdots \wedge P(n) \Longrightarrow P(0) \wedge \cdots \wedge P(2^m + n) & \text{(this is (2))} \\ P(0) \wedge \cdots \wedge P(2^m + n) \Longrightarrow P(0) \wedge \cdots \wedge P(2^{m+1} + n) & \\ & \text{((2) with } 2^m + n \text{ in place of } n) \end{array}}{P(0) \wedge \cdots \wedge P(n) \Longrightarrow P(0) \wedge \cdots \wedge P(2^{m+1} + n)}$$

as claimed. In particular, we have shown the theorem (call it Theorem 1) that if P satisfies $P(0)$ and $\forall n \; (P(n) \Longrightarrow P(n+1))$, then Q given by (2) satisfies $Q(0)$ and $\forall n \; (Q(n) \Longrightarrow Q(n+1))$.

Define $R(k)$, $k \in \mathbb{N}$ and $S(N)$, $N \in \mathbb{N}$ by

$$R(k) : \quad \forall n \;\; (Q(0) \wedge \cdots \wedge Q(n) \Longrightarrow Q(0) \wedge \cdots \wedge Q\left(2^k + n\right)) \tag{3}$$
$$S(N) : \quad \forall n \;\; (R(0) \wedge \cdots \wedge R(n) \Longrightarrow R(0) \wedge \cdots \wedge R\left(2^N + n\right)) \; . \tag{4}$$

Note that by one use of (L) (see Remarks), Theorem 1 with Q, R in place of P, Q gives $R(0)$ and $\forall k \; (R(k) \Longrightarrow R(k+1))$. Similarly, one more use of Theorem 1 gives $S(0)$ and $\forall N \; (S(N) \Longrightarrow S(N+1))$. Two uses of (L) give us consecutively, $\forall m$,

$$\frac{\begin{array}{l} S(m) \Longrightarrow S(m+1) \\ S(m+1) \Longrightarrow S(m+2) \end{array}}{S(m) \Longrightarrow S(m+2)} \qquad \text{and so} \qquad \frac{\begin{array}{l} S(m) \Longrightarrow S(m+2) \\ S(m+2) \Longrightarrow S(m+n) \end{array}}{S(m) \Longrightarrow S(m+4) \, .}$$

By a seventh use of (L), we have

$$\frac{\begin{array}{l} S(0) \\ S(m) \Longrightarrow S(m+4) \end{array}}{S(4)} \quad \text{(applied to } m = 0\text{)}$$

Going back, we have, by the definition of $S(4)$ and an eighth use of (L), that

$$\frac{\begin{array}{l} R(0) \\ R(0) \Longrightarrow R(0) \wedge \cdots \wedge R(2^4) \end{array}}{R(2^4)}$$

Thus $R(16)$ is true. By the definition of $R(16)$ we have

$$\frac{\begin{array}{l} Q(0) \\ Q(0) \Longrightarrow Q(0) \wedge \cdots \wedge Q(2^{16}) \end{array}}{Q(2^{16})}$$

That is, $Q(65536)$ is true. Finally, a tenth use of (L) gives

$$\frac{\begin{array}{l} P(0) \\ P(0) \Longrightarrow P(0) \wedge \cdots \wedge P(2^{65536}) \end{array}}{P(0) \wedge \cdots \wedge P(2^{65536})}$$

showing that $P(n)$ is true for all $n \leq 2^{65536}$.

Remarks:

1. It is easy to see that $S(n)$ given in (4) can be re-written as the one given in (1).

2. We deduced $R(0)$ and $\forall k\ (R(k) \Longrightarrow R(k+1))$ by using Theorem 1 and counting one use of (L). By invoking a proof schema called "law of generalization" we could just as legitimately count this as "zero" uses of (L). However, some people might find this unsatisfactory and argue that since Theorem 1 used the Law of Implication twice, each use of Theorem 1 must also count as two uses of (L). If such is the case, here is how we modify our argument: After proving $Q(0)$ and $\forall n\ (Q(n) \Longrightarrow Q(n+1))$, two uses of (L) give $R(0)$ and $\forall k\ (R(k) \Longrightarrow R(k+1))$. Three more uses give $R(k) \Longrightarrow (R(k+2)$, $R(k) \Longrightarrow R(k+4)$, $R(k) \Longrightarrow R(k+8)$ $(k \in \mathbb{N})$. The eighth use is

$$\frac{\begin{array}{l} R(0) \\ R(k) \Longrightarrow R(k+8) \end{array}}{R(8)}$$

from which, as before, we get $Q(2^8) = Q(512)$, and finally $P(0)$, $P(1)$, $\ldots$, $P(2^{512})$ are true.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; and the proposers.

Dergiades showed that $P(n)$ can be proved for $n = 1953125$. The proposers gave shorter and longer versions of the proof of $P(134217728)$.

2396. [1998: 505] *Proposed by Jose Luis Diaz, Universitat Politecnica de Catalunya, Colum, Terrassa, Spain.*

Suppose that $A(z) = \sum_{k=0}^{n} a_k z^k$ is a complex polynomial with $a_n = 1$, and let $r = \max_{0 \le k \le n-1} \left\{ |a_k|^{1/(n-k)} \right\}$. Prove that all the zeros of A lie in the disk $\mathcal{C} = \left\{ z \in \mathbb{C} \; : \; |z| \le \dfrac{r}{2^{1/n} - 1} \right\}$.

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

Let z_0 denote a zero of $A(z)$. Note that for $0 \le k \le n-1$ we have $|a_k| \le r^{n-k}$ so that

$$\begin{aligned} 2|z_0|^n &= |z_0|^n + |z_0|^n \\ &= |-a_0 - a_1 z_0 - a_2 z_0^2 \cdots - a_{n-1} z_0^{n-1}| + |z_0|^n \\ &\le |a_0| + |a_1|\,|z_0| + |a_2|\,|z_0|^2 + \cdots + |a_{n-1}|\,|z_0|^{n-1} + |z_0|^n \\ &\le r^n + r^{n-1}|z_0| + r^{n-2}|z_0|^2 + \cdots + r|z_0|^{n-1} + |z_0|^n \\ &\le r^n + \tbinom{n}{1} r^{n-1}|z_0| + \tbinom{n}{2} r^{n-2}|z_0|^2 + \cdots + \tbinom{n}{n-1} r|z_0|^{n-1} + |z_0|^n \\ &= \left(r + |z_0|\right)^n . \end{aligned}$$

Extracting n^{th} roots, we have $2^{1/n}|z_0| \le r + |z_0|$, from which the required result follows.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (slightly modified by the editor.)*

We shall prove the stronger result that, in fact, all the zeros of $A(z)$ lie in the disk $D = \{z \in \mathbb{C} : \; |z| \le \alpha_n\, r\}$ where α_n denotes the unique positive solution of $x^{n+1} - 2x^n + 1 = 0$ which lies in the interval $(1, \infty)$ when $n \ge 2$ and $\alpha_1 = 1$. The case $n = 1$ is trivial since $A(z) = a_0 + z$, the only root of which is $z = -a_0$, and $r = |a_0|$. If $r = 0$, then $A(z) = z^n$, which has $z = 0$ as the only root (with multiplicity n) and the conclusion clearly holds. Hence we assume that $n \ge 2$ and $r > 0$.

We first show that the function $f(x) = x^{n+1} - 2x^n + 1$ has a unique zero α_n in $(1, \infty)$. Since $f'(x) = x^{n-1}((n+1)x - 2n)$, $f'(x) = 0$ if and only if $x = x_0 = \dfrac{2n}{n+1} = 2 - \dfrac{2}{n+1} \in (1, 2)$. Since f is decreasing on $(1, x_0)$ and increasing on (x_0, ∞) we see that f has a relative and absolute minimum over $(1, \infty)$ at $x = x_0$. Since $f(1) = 0$ and $f(2) = 1$ we conclude that there is a unique α_n in $(1, \infty)$ such that $f(\alpha_n) = 0$.

Now let z_0 be a zero of $A(z)$. If $|z_0| \le r$, then z_0 is clearly inside D. Thus we assume that $|z_0| > r$ and show that $|z_0| \le \alpha_n\, r$.

Using the triangle inequality, we have

$$\begin{aligned} 0 &= |A(z_0)| = \left| z_0^n + \sum_{k=0}^{n-1} a_k\, z_0^k \right| \\ &\ge |z_0|^n - \sum_{k=0}^{n-1} |a_k|\, |z_0|^k \ge |z_0|^n - \sum_{k=0}^{n-1} r^{n-k} |z_0|^k \\ &= |z_0|^n - r \left(\frac{|z_0|^n - r^n}{|z_0| - r} \right) . \end{aligned}$$

Multiplying both sides by $|z_0| - r$, we get $|z_0|^{n+1} - 2r|z_0|^n + r^{n+1} \le 0$.

Thus $\left(\dfrac{|z_0|}{r}\right)^{n+1} - 2\left(\dfrac{|z_0|}{r}\right)^n + 1 \le 0$; that is, $f\left(\dfrac{|z_0|}{r}\right) \le 0$.

Hence $\dfrac{|z_0|}{r} \le \alpha_n$ from which $|z_0| \le \alpha_n r$ follows.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; KEITH EKBLAW, Walla Walla, WA, USA; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany and the proposer.

Using similar arguments, Lambrou also obtained the stronger result given by Janous. He pointed out that as a consequence, all the zeros of $A(z)$ lie in the disk $D' = \{z \in \mathbb{C} : |z| \le 2r\}$ and commented that the upper bound $2r$ is considerably better than $\dfrac{r}{2^{1/n} - 1}$ for large values of n since $\dfrac{1}{2^{1/n} - 1} \to \infty$ as $n \to \infty$. He also gave an example to show that the bound $\alpha_n r$ is the best possible. A similar comment about the "crudeness" of the bound $\dfrac{r}{2^{1/n} - 1}$ was also given by Leversha.

2397. [1998: 505] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a right-angled triangle ABC with $\angle BAC = 90^\circ$. Let I be the incentre, and let D and E be the intersections of BI and CI with AC and AB respectively.

Prove that $\dfrac{BI^2 + ID^2}{CI^2 + IE^2} = \dfrac{AB^2}{AC^2}$.

Nearly identical solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Gottfried Perz, Pestalozzi-gymnasium, Graz, Austria; Iftimie Simion, Stuyvesant HS, New York, NY, USA; and by D.J. Smeenk, Zaltbommel, the Netherlands.

Let r be the inradius of $\triangle ABC$. By the relevant definitions,

$$\sin\frac{B}{2} = \frac{r}{BI}, \quad \cos\frac{B}{2} = \frac{r}{ID}, \quad \sin\frac{C}{2} = \frac{r}{CI}, \quad \text{and} \quad \cos\frac{C}{2} = \frac{r}{IE}.$$

Hence,

$$\frac{BI^2+ID^2}{CI^2+IE^2} = \frac{\left(\frac{r}{\sin(\frac{B}{2})}\right)^2+\left(\frac{r}{\cos(\frac{B}{2})}\right)^2}{\left(\frac{r}{\sin(\frac{C}{2})}\right)^2+\left(\frac{r}{\cos(\frac{C}{2})}\right)^2} = \frac{\sin^2\frac{C}{2}\cdot\cos^2\frac{C}{2}}{\sin^2\frac{B}{2}\cdot\cos^2\frac{B}{2}}$$

$$= \frac{\sin^2 C}{\sin^2 B} = \frac{AB^2}{AC^2}\,,$$

as desired.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; ANDREI SIMION, student, Brooklyn Tech. HS, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ECKARD SPECHT, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Miranda, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2398. [1998: 505] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a square $ABCD$ with points E and F on sides BC and CD respectively, let P and Q be the feet of the perpendiculars from C to AE and AF respectively. Suppose that $\dfrac{CP}{AE}+\dfrac{CQ}{AF}=1$.

Prove that $\angle EAF = 45^\circ$.

The solution is a combination of the solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

Suppose that the square has unit side length. Let $\angle CAP = \theta$ and $\angle CAQ = \phi$. Then $0 \le \theta$, $\phi \le 45^\circ$ and

$$\frac{CP}{AE} = \sqrt{2}\sin\theta\cos(45^\circ-\theta) = \frac{1}{2}+\frac{1}{\sqrt{2}}\sin(2\theta-45^\circ)\,,$$

since $\sin\lambda\cos\mu = \frac{1}{2}(\sin(\lambda+\mu)+\sin(\lambda-\mu))$. Similarly,

$$\frac{CQ}{AF} = \frac{1}{2}+\frac{1}{\sqrt{2}}\sin(2\phi-45^\circ)\,.$$

It follows that $\frac{CP}{AE}+\frac{CQ}{AF}=1$ if and only if $\sin(2\theta-45^\circ)+\sin(2\phi-45^\circ)=0$. This is possible only when $2\theta-45^\circ = 45^\circ-2\phi$; that is, $\theta+\phi=45^\circ$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, *USA; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; ÁNGEL JOVAL ROQUET, Instituto Español de Andorra, Andorra; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Miranda, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.*

2399★. [1998: 506] *Proposed by David Singmaster, South Bank University, London, England.*

In James Dodson's *The Mathematical Repository*, 2nd ed., J. Nourse, London, 1775, pp 19 and 31, are two variations on the classic "Ass and Mule" problem:

> "What fraction is that, to the numerator of which 1 be added, the value will be $1/3$; but if 1 be added to the denominator, its value is $1/4$?"

This is easily done and it is easy to generalize to finding x/y such that $(x+1)/y = a/b$ and $x/(y+1) = c/d$, giving $x = c(a+b)/(ad-bc)$ and $y = b(c+d)/(ad-bc)$. We would normally take $a/b > c/d$, so that $ad - bc > 0$, and we can also assume a/b and c/d are in lowest terms.

> "A butcher being asked, what number of calves and sheep he had bought, replied, 'If I had bought four more of each, I should have four sheep for every three calves; and if I had bought four less of each, I should have had three sheep for every two calves'. How many of each did he buy?"

That is, find x/y such that $(x+4)/(y+4) = 4/3$ and $(x-4)/(y-4) = 3/2$. Again, this is easily done and it is easy to solve the generalisation, $(x+A)/(y+A) = a/b$ and $(x-A)/(y-A) = c/d$, getting $x = A(2ac - bc - ad)/(bc - ad)$ and $y = A(ad + bc - 2bd)/(bc - ad)$. We would normally take $a/b < c/d$ so that $bc - ad > 0$, and we can also assume a/b and c/d are in lowest terms.

In either problem, given that a, b, c and d are integers, is there a condition (simpler than computing x and y) to ensure that x and y are integers?

Alternatively, is there a way to generate all the integer quadruples a, b, c, d, which produce integers x and y?

To date, no solutions have been received.

2400. [1998: 506] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

(a) Show that $1+(\pi-2)x \;<\; \dfrac{\cos(\pi x)}{1-2x} \;<\; 1+2x$ for $0<x<1/2$.

[Proposed by Bruce Shawyer, Editor-in-Chief.]

(b)* Show that $\dfrac{\cos(\pi x)}{1-2x} \;<\; \dfrac{\pi}{2}-2(\pi-2)\left(x-\dfrac{1}{2}\right)^2$ for $0<x<1/2$.

Solution by Kee-Wai Lau, Hong Kong (modified and expanded by the editor).

(a) [Ed: We present only the proof of the left inequality since, as pointed out by Seiffert, the right inequality was already established by Herzig in his solution to #2296 [1998: 533].]

It suffices to prove that for $0<x<\frac{1}{2}$,

$$1+(\pi-4)x-2(\pi-2)x^2 \;<\; \cos(\pi x) \tag{1}$$

Let $f(x)=\cos(\pi x)+2(\pi-2)x^2-(\pi-4)x-1$, for $0\le x\le\frac{1}{2}$. Then $f'(x)=-\pi\sin(\pi x)+4(\pi-2)x+4-\pi$ and $f''(x)=-\pi^2\cos(\pi x)+4(\pi-2)$. Setting $f''(x)=0$, we find that $x=x_0=\dfrac{1}{\pi}\cos^{-1}\left(\dfrac{4(\pi-2)}{\pi^2}\right)=0.346\cdots$. Since $f''(x)<0$ for $0<x<x_0$, the graph of f is concave down on $[0,x_0]$. Since $f(0)=0$ and $f(x_0)=0.035\cdots>0$, we conclude that $f(x)>0$ on $(0,x_0]$. On the other hand, since $f''(x)>0$ for $x_0<x<\frac{1}{2}$, $f'(x)$ is increasing on $[x_0,\frac{1}{2}]$. This, together with the fact that $f'(\frac{1}{2})=0$, imply that $f'(x)<0$ for $x_0<x<\frac{1}{2}$. Hence $f(x)$ is decreasing on $[x_0,\frac{1}{2}]$. Since $f(\frac{1}{2})=0$ we conclude that $f(x)>0$ on $(x_0,\frac{1}{2})$ as well and (1) follows.

(b) Let $y=\frac{\pi}{2}(1-2x)$. Then $0<y<\frac{\pi}{2}$ and the given inequality becomes

$$\frac{\pi\cos(\frac{\pi}{2}-y)}{2y} \;<\; \frac{\pi}{2}-\frac{2(\pi-2)}{4}\left(\frac{2y}{\pi}\right)^2$$

or

$$\sin y \;<\; y-\frac{4(\pi-2)}{\pi^3}y^3\,. \tag{2}$$

Let $g(y)=\sin y-y+\dfrac{4(\pi-2)}{\pi^3}y^3$, for $0\le y\le\frac{\pi}{2}$. We show that $g(y)<0$ for $0<y<\frac{\pi}{2}$. Note first that

$$\begin{aligned} g(y) &< \left(y-\frac{y^3}{6}+\frac{y^5}{120}\right)-y+\frac{4(\pi-2)}{\pi^3}y^3 \\ &= \frac{y^3}{120\pi^3}(\pi^3y^3+480\pi-20\pi^3-960). \end{aligned}$$

Straightforward computations show that $\pi^3y^3 + 480\pi - 20\pi^3 - 960 < 0$ when $y = 1.52$. Hence $g(y) < 0$ for $0 < y \le 1.52$. Next,

$$\begin{aligned} g'(y) &= \cos y - 1 + \frac{12(\pi-2)}{\pi^3}y^2 \\ &= 2y^2\left(\frac{6(\pi-2)}{\pi^3} - \frac{\sin^2(\frac{y}{2})}{y^2}\right). \end{aligned}$$

Note that $\dfrac{\sin y}{y}$ is decreasing for $0 < y < \frac{\pi}{2}$.

[Ed: $\dfrac{d}{dy}\left(\dfrac{\sin y}{y}\right) = y^{-2}(y\cos y - \sin y)$ and $y\cos y < \sin y$ as it is well known that $\cos y < \dfrac{\sin y}{y}$].

If $1.52 \le y < \frac{\pi}{2}$, then $0.76 \le \frac{y}{2} < \frac{\pi}{4}$ and so

$$\begin{aligned} \frac{\sin^2(\frac{y}{2})}{y^2} &= \frac{1}{4}\left(\frac{\sin(\frac{y}{2})}{\frac{y}{2}}\right)^2 < \frac{1}{4}\left(\frac{\sin(0.76)}{0.76}\right)^2 \\ &< \frac{(0.907)^2}{4} = 0.2057 < \frac{6(\pi-2)}{\pi^3}. \end{aligned}$$

Hence $g'(y) > 0$ for $1.52 \le y < \frac{\pi}{2}$ which implies that $g(y)$ is increasing on $[1.52, \frac{\pi}{2}]$. Since $g(\frac{\pi}{2}) = 0$ we conclude that $g(y) < 0$ for $1.52 \le y < \frac{\pi}{2}$ as well. Therefore, $g(y) < 0$ on $(0, \frac{\pi}{2})$ and (2) follows.

Also solved by OSCAR CIAURRI, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; PHIL MCCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer who apparently found a proof of (b)* *after the problem had appeared. There was also one incorrect and one incomplete solution.*

YEAR END FINALE

Again, a year has flown by! It is difficult to realize that I have now done this job for four years. My term has one year to go, but I have agreed to a two year extension until the end of 2002, when you will have a new Editor-in-Chief.

There are many people that I wish to thank most sincerely for particular contributions. Again, first and foremost is BILL SANDS. Bill is of such value to me and to the continuance of ***CRUX with MAYHEM***. As well, I thank most sincerely, CATHY BAKER, ILIYA BLUSKOV, ROLAND EDDY, CHRIS FISHER, BILL SANDS, JIM TOTTEN, and EDWARD WANG, for their regular yeoman service in assessing the solutions; DENIS HANSON, DOUG FARENICK, CHRIS FISHER, CHRIS LEIER, TOM MacDONALD, JUDI McDONALD, RICHARD McINTOSH, DIETER RUOFF, NABIL SHALABY, JASON STEIN, MICHAEL TSATSOMEROS, BRUCE WATSON, HARLEY WESTON, for ensuring that we have quality articles; ALAN LAW, RICHARD CHARRON, MURRAY S. KLAMKIN, T.W. LEUNG, ANDY LIU, JACK W. MACKI, CHRISTOPHER SMALL, IAN VANDERBURGH, for ensuring that we have quality book reviews, ROBERT WOODROW, who carries the heavy load of two corners, and RICHARD GUY for sage advice whenever necessary.

The editors of the ***MAYHEM*** section, NAOKI SATO, CYRUS HSIA, ADRIAN CHAN, DONNY CHEUNG, JIMMY CHUI, DAVID SAVITT and WAI LING YEE, all do a sterling job.

I also thank those who assist with proofreading. The quality of all these people is a vital part of what makes ***CRUX with MAYHEM*** what it is. Thank you one and all.

As well, I would like to give special thanks to our Associate Editor, CLAYTON HALFYARD, for continuous sage advice, and for keeping me from printing too many typographical and mathematical errors; and to my colleagues, YURI BAHTURIN, RICHARD CHARRON, ROLAND EDDY, EDGAR GOODAIRE, MIKE PARMENTER, DONALD RIDEOUT, NABIL SHALABY, in the Department of Mathematics and Statistics at Memorial University, and to JOHN GRANT McLOUGHLIN, Faculty of Education, Memorial University, for their occasional sage advice. I have also been helped by some Memorial University students, KARELYN DAVIS, PAUL MARSHALL, SHANNON SULLIVAN, TREVOR RODGERS, as well as WISE Summer students, JANINE RYDER, DENISE VATCHER and REBECCA WHITE.

The staff of the Department of Mathematics and Statistics at Memorial University deserve special mention for their excellent work and support: ROS ENGLISH, MENIE KAVANAGH, WANDA HEATH, and LEONCE MORRISSEY; as well as the computer and networking expertise of STEVE POPE and CRAIG SQUIRES.

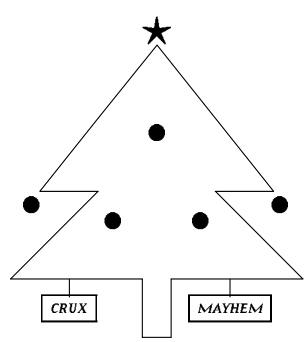
Also the LaTeX expertise of JOANNE LONGWORTH at the University of Calgary, ELLEN WILSON at Mount Allison University, the ***MAYHEM*** staff, and all others who produce material, is much appreciated.

GRAHAM WRIGHT, the Managing Editor for the first four issues, was a tower of strength and support. Graham kept so much on the right track. He has been a pleasure to work with. We welcome BOB QUACKENBUSH as the new Managing Editor. The CMS's TeX Editor, MICHAEL DOOB has been very helpful in ensuring that the printed master copies are up to the standard required for the U of T Press, who continue to print a fine product.

The online version of ***CRUX with MAYHEM*** continues to attract attention. We recommend it highly to you. Thanks are due to LOKI JORGENSON, JEN CHANG, CONG LY, FREDERIC TESSIER, PAUL WOLSTENHOLME, and the rest of the team at SFU who are responsible for this.

Finally, I would like to express real and heartfelt thanks to the Head of my Department, HERBERT GASKILL, to the former Acting Dean of Science of Memorial University, WILLIE DAVIDSON, and to the new Dean of Science, BOB LUCAS. Without their support and understanding, I would not be able to do the job of Editor-in-Chief.

Last but not least, I send my thanks to you, the readers of ***CRUX with MAYHEM***. Without you, ***CRUX with MAYHEM*** would not be what it is. Keep those contributions and letters coming in. We need your ARTICLES, PROPOSALS and SOLUTIONS to keep ***CRUX with MAYHEM*** alive and well. I do enjoy knowing you all.



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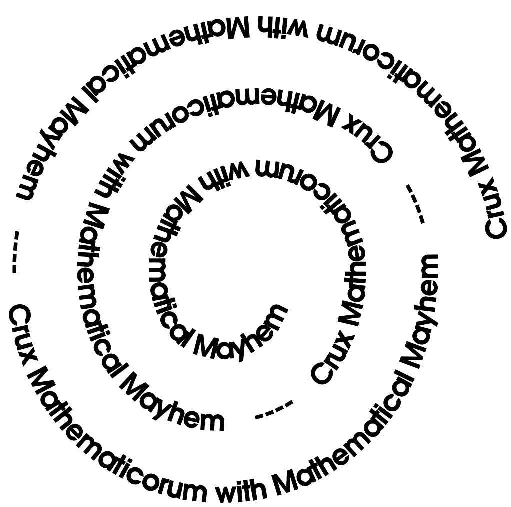

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