Congratulations, Andy Liu!

It is with great pleasure that we congratulate Andy Liu on yet another award, confirming the high regard with which he is held in the international mathematical community. ***CRUX with MAYHEM*** is very proud to have had Andy Liu on its Editorial Board for many years.

The following can be seen on the World Wide Web at
`http://www.case.org/awards/canpoy.htm`

The Council for Advancement and Support of Education [USA]
and
The Canadian Council for Advancement and Support of Education

1998–1999 CANADIAN PROFESSORS OF THE YEAR

OUTSTANDING UNIVERSITY PROFESSOR

Andy Liu joined the faculty of the University of Alberta Mathematical Sciences department in 1980 and has been making a profound impact on the institution and its students ever since. His passions include a commitment to the study of mathematics and to developing innovative techniques that allow him to share his knowledge with students of all ages. "Students must not settle into passive learning but must be challenged to participate in the process," explains Liu.

Liu's success stems from his unique knack for presenting difficult concepts in a clear and logical manner. Former student William Willette said, "I have not been inspired to think and achieve by anyone more than Liu. He is not the type of instructor who just gives answers all the time; he inspires students to think." By providing practical examples of theoretical concepts, Liu helps his students understand and learn rather than simply memorize. "Those of us working in the area of education of gifted and talented children have long considered Liu to be a resource par excellence in mathematics education," said a colleague, Carolyn R. Yewchuk.

In the classroom, Liu uses his lively sense of humour to maintain students' interest. His pleasant demeanour creates a comfortable learning environment. "In every class, I know the names of all the students by the time of the midterm test," says Liu. He has even mastered the art of writing upside down so students can follow his written explanations while meeting with him at his desk.

Liu extends himself beyond the campus in a variety of ways. His on- and off-campus lectures as well as the courses he has designed, reflect his extraordinary talent. Liu is a strong supporter of mathematical competitions as a way to motivate and promote interest in mathematics. He has participated in mathematical competitions on the local, provincial, national, and international levels. He has provided training sessions for University of Alberta undergraduate students, prepared training materials for a number of national Mathematical Olympiad teams, and conducted training sessions for the International Mathematical Olympiad (IMO) teams of Australia, Canada, Hong Kong, Taiwan and the United States. In 1995, he received the IMO Certificate of Appreciation and in 1996 received the David Hilbert International Award for the promotion of mathematics worldwide. Liu explains, "I feel that the University is an integral part of the community, and its involvement must extend beyond the confines of the campus. Also, learning is a universal endeavour which transcends political boundaries, and the university is first and foremost an international institution."

THE ACADEMY CORNER

No. 23

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

Here is the first of four articles from the 1998 Canadian Undergraduate Mathematics Conference, held at the University of British Columbia in July 1998.

Abstracts • Résumés

Canadian Undergraduate Mathematics Conference 1998 — Part 1

Tracer Kinetic Modelling in Dynamic Positron Emission Tomography

Mark Andermann
McGill University

Dynamic positron emission tomography (PET) has often been used in the quantitative estimation of physiological parameters in vivo. For example, tracer kinetic modelling of time-series data has been employed on a voxel by voxel basis using various methods of non-linear regression analysis. However, due to the low signal-to-noise ratio and large data sets present in PET studies, such methods are often unstable and computationally intensive. In this lecture, mathematical theory underlying current and past methods which have attempted to increase the precision of tracer kinetic modelling will be reviewed. Subsequently, a novel technique involving principal components analysis, split-and-merge image segmentation using hypothesis testing with both heuristic and statistical likelihood tests, as well as some aspects of spectral analysis will be introduced. Benefits and disadvantages of this method will then be discussed.

You Can *Disprove* That?

Shabnam Beheshti
McGill University

Prove or disprove the following: If two sets A and B of reals are homeomorphic, then they are both dense (respectively meagre). How many times have you been faced with a question like this on a final? If you have experienced the wrath

of a professor who enjoys giving "rigorous multiple choice" examinations, then perhaps the importance of studying and constructing counterexamples has been revealed already. Every student of mathematics is faced with a choice: either conjuring up a creative counterexample or waiting for divine intervention (the latter sometimes takes awhile). Many times, we implicitly assume intuitively sound but analytically false properties of our system, forcing us to draw incorrect conclusions; the power of counterexample is thus discovered. Such an error hindered progress towards the proof of Fermat's Last Theorem. I will attempt to survey the search for a proof to the theorem of Fermat and present a variation of Kummer's counterexample to the purported proof due to Lamé.

Trigonometry, Astronomy, and Computation: The Historical Quest for an Elusive Constant

Abraham Buckingham
The King's University College

Ancient astronomy gave rise to some of the best historical mathematics, especially in the realm of trigonometry. Accurate trigonometric tables were critical to good astronomy, and accurate trigonometric tables relied on an accurate value of the sine of one degree for the majority of values. Unfortunately an exact value for this sine is impossible to find using ruler and compass methods alone. This problem was worked on by Claudius Ptolemy (100-175 AD) and Jamshīd al-Kāshī (?-1429 AD). Ptolemy's geometric method stood for 1200 years, but al-Kāshī was not satisfied and constructed an improved geometric estimation for the sine of one degree. Still not satisfied, al-Kāshī went on to develop a fixed point iteration scheme for the calculation. I will describe and compare these methods within their context.

Invariants and shoelaces

Benoit Charbonneau
Université du Québec à Montréal

In the Montreal phone directory, there are seven entries for Jacques Labelle. If X's name is Jacques Labelle, we need more information to identify him. But we know for sure he is not Nicolas Bourbaki (although this is a subtle question).

One eternal question of mathematics is *Are two objects in fact the same object?*

Up to what we call isomorphisms, some properties of objects don't change. We call them *invariants*. After this talk, you won't look at life the same way!

Invariants et lacets de chaussures

Benoit Charbonneau
Université du Québec à Montréal

Dans le bottin téléphonique de Montréal, il y a sept entrées pour Jacques Labelle. Si le nom de X est Jacques Labelle, on a besoin de plus d'information pour l'identifier. On est cependant sur qu'il n'est pas Nicolas Bourbaki (quoique la question est subtile).

Une question éternelle des mathématiques est *Est-ce que deux objets sont identiques ?*

Certaines propriétés des objets ne changent pas à isomorphisme près, on les appelle des *invariants*. Suite à cet exposé, vous ne verrez plus jamais la vie de la même façon !

Ranking the Participants in a Round-Robin Tournament

Susan Marie Cooper
University of Regina

Various schemes for ranking and comparing the participants in a round-robin tournament have been proposed. However, none of these are considered entirely satisfactory. We will consider the Kendall-Wei method, a method of ranking tournaments using iterated strength vectors, which leads to the Perron vector (i.e. vector of relative strengths) of the corresponding dominance matrix of the tournament. We will investigate such questions as: "how easily can the iterated strength vectors be calculated?" and "can we determine the relative ordering of the strengths of the participants without calculating the Perron vector?"

What is the $(57, 5)$-cage?

Jennifer de Kleine
University of Northern British Columbia

An $(n, 5)$-cage is an n-regular girth-5 graph of smallest order. I will discuss what this means, and the open question of whether there exists a $(57, 5)$-cage with $57^2 + 1$ vertices. Time permitting, I will discuss the connection between estimating the permanent of $(0, 1)$ matrices and a possible approach to using a computer to search for a $57^2 + 1$ vertex $(57, 5)$-cage.

Les nombres de Stirling

Caroline Desjardins
Université du Québec à Montréal

Jacob Stirling, mathématicien anglais du 18^{e} siècle a publié un article intéressant "Methodus differentialis". L'article contient deux triangles de nombres entiers (apparentés au triangle de Pascal) dont les diverses composantes permettent d'obtenir des résultats intéressants en analyse, en combinatoire et en probabilités. Ce sont ces nombres que l'on appelle les nombres de Stirling. On peut obtenir ces nombres en trouvant les matrices de changement de base de deux base de $\mathbb{R}[[x]]$ ou bien, en utilisant les relations de récurrences que ces nombres suivent. Les liens reliant ces nombres aux nombres de partition en k-classes et de fonction surjectives d'un ensemble à n éléments vers un ensemble à k éléments, sont quelquesexemples d'application des nombres de Stirling.

Applications of Group Theory to Chemistry

Norman Dreger
University of British Columbia

The goal of this paper is to present a number of chemical applications of group theory. Some key terms will be defined and the concepts of symmetry elements and symmetry groups introduced. The nomenclature of symmetry elements with regards to chemistry will then be discussed. Finally some applications of symmetry to experimental chemistry will be examined. Only a rudimentary knowledge of group theory is prerequisite.

THE OLYMPIAD CORNER

No. 196

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We lead off this issue with the problems of the 19$^{\text{th}}$ Austrian-Polish Mathematics Competitions, written in Poland, June 26–28, 1996. My thanks go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai as well as to regular supporters Marcin E. Kuczma, Warszawa, Poland and Walther Janous, Ursulinengymnasium, Innsbruck, Austria for supplying copies of the contest material.

19$^{\text{th}}$ AUSTRIAN-POLISH MATHEMATICS COMPETITION 1996

Problems of the Individual Context

June 26–27, 1996 (Time: 4.5 hours)

1. Let $k \geq 1$ be an integer. Show that there are exactly 3^{k-1} positive integers n with the following properties:

(a) The decimal representation of n consists of exactly k digits.

(b) All digits of n are odd.

(c) The number n is divisible by 5.

(d) The number $m = \frac{n}{5}$ has k odd (decimal) digits.

2. A convex hexagon $ABCDEF$ satisfies the following conditions:

(a) The opposite sides are parallel; that is, $AB \| DE$, $BC \| EF$, $CD \| FA$.

(b) The distances between the opposite sides are equal; that is, $d(AB, DE) = d(BC, EF) = d(CD, FA)$, where $d(g, h)$ denotes the distance between lines g and h.

(c) $\angle FAB$ and $\angle CDE$ are right angles.

Show that diagonals BE and CF intersect at an angle of 45°.

3. The polynomials $P_n(x)$ are defined recursively by $P_0(x) = 0$, $P_1(x) = x$ and

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \quad \text{for} \quad n \geq 2\,.$$

For every natural number $n \geq 1$ find all real numbers x satisfying the equation $P_n(x) = 0$.

4. The real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that $-1 \leq xy + yz + zt + tx \leq 0$.

5. A convex polyhedron P and a sphere S are situated in space in such a manner that S intercepts on each edge AB of P a segment XY with $AX = XY = YB = \frac{1}{3}AB$. Prove that there exists a sphere T tangent to all edges of P.

6. Natural numbers k, n are given such that $1 < k < n$. Solve the system of n equations

$$x_i^3 \cdot (x_i^2 + x_{i+1}^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \qquad \text{for} \qquad 1 \leq i \leq n$$

with n real unknowns x_1, x_2, $\ldots$, x_n. Note: $x_0 = x_n$, $x_{n+1} = x_1$, $x_{n+2} = x_2$, and so on.

Problems of the Team Contest (Poland)

June 28, 1996 (Time: 4 hours)

7. Show that there do not exist non-negative integers k and m such that $k! + 48 = 48(k+1)^m$.

8. Show that there is no polynomial $P(x)$ of degree 998 with real coefficients satisfying for all real numbers x the equation

$$P(x)^2 - 1 = P(x^2 + 1)\,.$$

9. We are given a collection of rectangular bricks, no one of which is a cube. The edge lengths are integers. For every triple of positive integers (a, b, c), not all equal, there is a sufficient supply of $a \times b \times c$ bricks. Suppose that the bricks are completely tiling a cubic $10 \times 10 \times 10$ box.

(a) Assume that at least 100 bricks have been used. Prove that there exist at least two bricks situated in parallel, in the sense that if AB is an edge of one of them and $A'B'$ is an edge of one of the other, and if $AB \| A'B'$, then $AB = A'B'$.

(b) Prove the same statement for a number less than 100 (of bricks used). The smaller number, the better the solution.

Next we move to a country whose contest materials have not been very often available in ***CRUX with MAYHEM*** with the problems of the 3rd Turkish Mathematical Olympiad, Second Round, written December 8–9, 1995. My thanks go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai for collecting the problems.

3rd TURKISH MATHEMATICAL OLYMPIAD

Second Round – First Day

December 8, 1995 (Time: 4.5 hours)

1. Let $a_1, a_2, \ldots, a_k$ and $m_1, m_2, \ldots, m_k$ be integers with $2 \le m_1$ and $2m_i \le m_{i+1}$ for $1 \le i \le k-1$. Show that there are infinitely many integers x which do not satisfy any of the congruences

$$x \equiv a_i \pmod{m_1}, \; x \equiv a_2 \pmod{m_2}, \; \ldots, \; x \equiv a_k \pmod{m_k}.$$

2. For an acute triangle ABC, k_1, k_2, k_3 are the circles with diameters $[BC]$, $[CA]$, $[AB]$, respectively. If K is the radical centre of these circles, $[AK] \cap k_1 = \{D\}$, $[BK] \cap k_2 = \{E\}$, $[CK] \cap k_3 = \{F\}$ and Area$(ABC) = u$, Area$(DBC) = x$, Area$(ECA) = y$, and Area$(FAB) = z$, show that $u^2 = x^2 + y^2 + z^2$.

3. Let $\mathbb{N}$ denote the set of positive integers. Let A be a real number and $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 1$ and

$$1 < \frac{a_{n+1}}{a_n} \le A \quad \text{for all} \quad n \in \mathbb{N}.$$

(a) Show that there is a unique non-decreasing surjective function $k : \mathbb{N} \to \mathbb{N}$ such that $1 < \frac{A^{k(n)}}{a_n} \le A$ for all $n \in \mathbb{N}$.

(b) If k takes every value at most m times, show that there exists a real number $C > 1$ such that $C^n \le A a_n$ for all $n \in \mathbb{N}$.

Second Round – Second Day

December 9, 1995 (Time: 4.5 hours)

4. In a triangle ABC with $|AB| \ne |AC|$, the internal and external bisectors of the angle A intersect the line BC at D and E, respectively. If the feet of the perpendiculars from a point F on the circle with diameter $[DE]$ to the lines BC, CA, AB are K, L, M, respectively, show that $|KL| = |KM|$.

5. Let $t(A)$ denote the sum of elements of A for a non-empty subset A of integers, and define $t(\phi) = 0$. Find a subset X of the set of positive integers such that for every integer k there is a unique ordered pair of subsets (A_k, B_k) of X with $A_k \cap B_k = \phi$ and $t(A_k) - t(B_k) = k$.

6. Let $\mathbb{N}$ denote the set of positive integers. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the condition

$$m \mid n \iff f(m) \mid f(n)$$

for all $m, n \in \mathbb{N}$.

Along with the Turkish Olympiad we have the questions of the Turkish Team Selection Examination for the 37$^{\text{th}}$ IMO, written March 23–24, 1996. Thanks again go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai for forwarding these to me.

TURKISH TEAM SELECTION EXAMINATION FOR THE 37$^{\text{th}}$ IMO

First Day — March 23, 1996

Time: 4.5 hours

1. Let $\prod_{n=1}^{1996}(1+nx^{3^n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m}$ where $a_1, a_2, \ldots, a_m$ are non-zero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}.

2. In a parallelogram $ABCD$ with $m(\hat{A}) < 90^\circ$, the circle with diameter $[AC]$ intersects the lines CB and CD at E and F besides C, and the tangent to this circle at A intersects the line BD at P. Show that the points P, F, E are collinear.

3. Given real numbers $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$ with $x_{i+1} - x_i \le h$ for $1 \le i \le 2n$, show that

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) \le \frac{1+h}{2}.$$

Second Day — March 24, 1996

Time: 4.5 hours

4. In a convex quadrilateral $ABCD$, Area(ABC) = Area(ADC) and $[AC] \cap [BD] = \{E\}$, and the parallels from E to the line segments $[AD]$, $[DC]$, $[CB]$, $[BA]$ intersect $[AB]$, $[BC]$, $[CD]$, $[DA]$ at the points K, L, M, N, respectively. Compute the ratio

$$\frac{\text{Area}(KLMN)}{\text{Area}(ABCD)}.$$

5. Find the maximum number of pairwise disjoint sets of the form $S_{a,b} = \{n^2 + an + b : n \in \mathbb{Z}\}$ with $a, b \in \mathbb{Z}$.

6. For which ordered pairs of positive real numbers (a, b) is zero the value of the limit of every sequence $\{x_n\}$ satisfying the condition

$$\lim_{n\to\infty}(ax_{n+1} - bx_n) = 0\,?$$

To round out the contests for your puzzling pleasure we give the two papers of the Australian Mathematical Olympiad 1996. My thanks go to Ravi Vakil, Canadian Team Leader of the IMO at Mumbai, once again, for providing me with the contest materials.

AUSTRALIAN MATHEMATICAL OLYMPIAD 1996

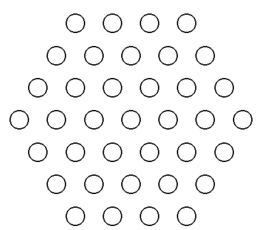
Paper 1

February 6, 1996 (Time: 4 hours)

1. Let $ABCDE$ be a convex pentagon such that $BC = CD = DE$ and each diagonal of the pentagon is parallel to one of its sides. Prove that all the angles in the pentagon are equal, and that all sides are equal.

2. Let $p(x)$ be a cubic polynomial with roots r_1, r_2, r_3. Suppose that $\dfrac{p\left(\frac{1}{2}\right)+p\left(-\frac{1}{2}\right)}{p(0)} = 1000$. Find the value of $\dfrac{1}{r_1r_2}+\dfrac{1}{r_2r_3}+\dfrac{1}{r_3r_1}$.

3. A number of tubes are bundled together into a hexagonal form:



A number of tubes in the bundle can be 1, 7, 19, 37 (as shown), 61, $91, \ldots$. If this sequence is continued, it will be noticed that the total number of tubes is often a number ending in 69. What is the 69^{th} number in the sequence which ends in 69?

4. For which positive integers n can we rearrange the sequence $1, 2, \ldots, n$ to $a_1, a_2, \ldots, a_n$ in such a way that $|a_k - k| = |a_1 - 1| \neq 0$ for $k = 2, 3, \ldots, n$?

Paper 2

February 7, 1996 (Time: 4 hours)

5. Let a_1, a_2, $\ldots$, a_n be real numbers and s a non-negative real number such that

(i) $a_1 \le a_2 \le \cdots \le a_n$;

(ii) $a_1 + a_2 + \cdots + a_n = 0$;

(iii) $|a_1| + |a_2| + \cdots + |a_n| = s$.

Prove that

$$a_n - a_1 \ \ge\ \frac{2s}{n}\,.$$

6. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the mid-point of PQ.

7. For each positive integer n, let $\sigma(n)$ denote the sum of all positive integers that divide n. Let k be a positive integer and $n_1 < n_2 < \cdots$ be an infinite sequence of positive integers with the property that $\sigma(n_i) - n_i = k$ for $i = 1, 2, \ldots$. Prove that n_i is a prime for $i = 1, 2, \ldots$.

8. Let f be a function that is defined for all integers and takes only the values 0 and 1. Suppose f has the following properties:

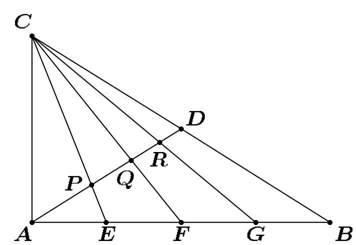
(i) $f(n + 1996) = f(n)$ for all integers n;

(ii) $f(1) + f(2) + \cdots + f(1996) = 45$.

Prove that there exists an integer t such that $f(n + t) = 0$ for all n for which $f(n) = 1$ holds.

Now, an alternate and more general solution to problem 2 of the Dutch Mathematical Olympiad, Second Round, 1993 than the one given in the *Corner* in the October 1998 number [1997: 197], [1998: 330].

2. Given a triangle ABC, $\angle A = 90°$. D is the mid-point of BC, F is the mid-point of AB, E the midpoint of AF and G the mid-point of FB. AD intersects CE, CF and CG respectively in P, Q and R. Determine the ratio $\frac{PQ}{QR}$.



Alternate Solution by Geoffrey A. Kandall, Hamden, Connecticut, USA.

We first establish the following:

Lemma.

$$\frac{PQ}{QR} = \frac{CP}{CE} \cdot \frac{EF}{FG} \cdot \frac{CG}{CR}.$$

Proof.

$$\begin{aligned}\frac{PQ}{QR} = \frac{[CPQ]}{[CQR]} &= \frac{[CPQ]}{[CEF]} \cdot \frac{[CEF]}{[CFG]} \cdot \frac{[CFG]}{[CQR]} \\ &= \frac{CP \cdot CQ}{CE \cdot CF} \cdot \frac{EF}{FG} \cdot \frac{CF \cdot CG}{CQ \cdot CR} = \frac{CP}{CE} \cdot \frac{EF}{FG} \cdot \frac{CG}{CR}.\end{aligned}$$

We now solve the problem, without using the hypothesis that $\angle A = 90^\circ$.

By the lemma

$$\frac{PQ}{QR} = \frac{CP}{CE} \cdot \frac{EF}{FG} \cdot \frac{CG}{RC} = \frac{CP}{CE} \cdot \frac{CG}{RC}\,.$$

By Menelaus' Theorem we have

$$\frac{CD}{DB} \cdot \frac{BA}{AE} \cdot \frac{EP}{PC} = 1\,, \quad \text{hence} \quad \frac{EP}{PC} = \frac{1}{4}\,, \qquad \frac{CP}{CE} = \frac{4}{5}\,; \qquad (1)$$

$$\frac{CD}{DB} \cdot \frac{BA}{AG} \cdot \frac{GR}{CR} = 1\,, \quad \text{hence} \quad \frac{GR}{CR} = \frac{3}{4}\,, \qquad \frac{CG}{CR} = \frac{7}{4}\,. \qquad (2)$$

Consequently $\dfrac{PQ}{QR} = \dfrac{4}{5} \cdot \dfrac{7}{4} = \dfrac{7}{5}$.

This method can be used with different ratios $CD : DB$ and $AE : EF : FG : GB$.

After the February number was finalized we received a package of solutions from Michael Selby, University of Windsor, Windsor, Ontario. This included solutions to problems 1 through 4 of the Croatian National Mathematics Competition (4^{th} Class) May 13, 1994 for which the problems were given [1997: 454] and the solutions [1999: 12]. He also sent a solution to a problem of the *Additional Competition for the Olympiad of the Croatian National Mathematical Competition*, given [1997: 454].

1. Find all ordered triples (a, b, c) of real numbers such that for every three integers x, y, z the following identity holds:

$$|ax + by + cz| + |bx + cy + az| + |cx + ay + bz| = |x| + |y| + |z|\,.$$

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

Set $x = y = z = 1$; we obtain $|a + b + c| = 1$ (1)

Set $x = 1$; $y = z = 0$ we obtain $|a| + |b| + |c| = 1$ (2)

Set $x = 1$; $y = -1$, $z = 0$ we obtain $|a - b| + |b - c| + |c - a| = 2$ (3)

This system is symmetric. Without loss of generality we may assume $a \geq b \geq c$.

Now (3) becomes $2(a - c) = 2$ or $a - c = 1$. Substituting into (1) and (2) gives

$$|1 + b + 2c| = 1 \qquad (4)$$

and

$$|1+c|+|b|+|c| \;=\; 1\,. \qquad (5)$$

Squaring (4) and expanding gives

$$1+(b+2c)^2+2(b+2c) \;=\; 1\,.$$

Thus $b+2c=0$ or $b+2c=-2$.

If $b+2c=0$, then from (5)

$$|1+c|+3|c|=1\,.$$

Since $|c|\le 1$, $1+c\ge 0$, therefore $1+c+3|c|=1$ and $c+3|c|=0$. If $c\ge 0$, we have $4c=0$ and then $c=0$. If $c\le 0$, $-2c=0$ giving $c=0$. Therefore $b=-2c=0$, $a=1+c=1$, in this case.

In case $b+2c=-2$, substitution into (5) yields

$$|1+c|+2|1+c|+|c| \;=\; 1\,.$$

Since $1+c\ge 0$, $3(1+c)+|c|=1$. If $c\ge 0$, $3+4c=1$ and $c=\frac{-1}{2}$. This is impossible.

If $c\le 0$, $3+3c-c=1$ giving $c=-1$. Then $b=0$ and $a=1+c=0$. Therefore we have the solution $a=0$, $b=0$, $c=-1$, and these are the solutions for $a\ge b\ge c$.

Hence there are six solutions

$$(1,0,0)\,,\ (-1,0,0)\,,\ (0,1,0)\,,\ (0,-1,0)\,,\ (0,0,1)\,,\ (0,0,-1)\,.$$

Next we turn to solutions by the readers to problems of the 17^{th} Austrian-Polish Mathematics Competition given in the February 1998 number [1998: 4].

17^{th} AUSTRIAN–POLISH MATHEMATICS COMPETITION Poland, June 29–July 1, 1994

1. The function $f:\mathbb{R}\to\mathbb{R}$ satisfies for all $x\in\mathbb{R}$ the conditions

$$f(x+19)\le f(x)+19 \quad\text{and}\quad f(x+94)\ge f(x)+94\,.$$

Show that $f(x+1)=f(x)+1$ for all $x\in\mathbb{R}$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Pavlos Maragoudakis, Pireas, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Bataille.

Let x be an arbitrary real number. Applying the given conditions to $x - 19$ and $x - 94$ respectively, we obtain

$$f(x-19) \;\geq\; f(x) - 19 \qquad \text{and} \qquad f(x-94) \;\leq\; f(x) - 94\,.$$

Now an easy induction shows that for all $n \in \mathbb{N}$,

$$f(x+19n) \;\leq\; f(x) + 19n\,, \qquad f(x+94n) \;\geq\; f(x) + 94n\,,$$

$$f(x-19n) \;\geq\; f(x) - 19n\,, \qquad \text{and} \qquad f(x-94n) \;\leq\; f(x) - 94n\,.$$

Since $1 = 5 \times 19 - 94$ and $1 = 18 \times 94 - 89 \times 19$, we get:

$$\begin{aligned} f(x+1) \;=\; f(x + 5 \times 19 - 94) &\;\leq\; f(x + 5 \times 19) - 94 \\ &\;\leq\; f(x) + 5 \times 19 - 94 \\ &\;=\; f(x) + 1\,, \end{aligned}$$

and

$$\begin{aligned} f(x+1) \;=\; f(x + 18 \times 94 - 89 \times 19) &\;\geq\; f(x + 18 \times 94) - 89 \times 19 \\ &\;\geq\; f(x) + 18 + 94 - 89 \times 19 \\ &\;=\; f(x) + 1\,, \end{aligned}$$

so that $f(x+1) = f(x) + 1$, as required.

Comment: the same result can be obtained from the more general hypothesis: for all $x \in \mathbb{R}$, $f(x+a) \leq f(x) + a$ and $f(x+b) \geq f(x) + b$ where a and b are positive relatively prime integers. Indeed, the preceding proof adapts easily as we can find positive integers m, n, p, q such that $ma - nb = 1$ and $pb - qa = 1$.

2. The sequence $\{a_n\}$ is defined by the formulae

$$a_0 = \frac{1}{2} \quad \text{and} \quad a_{n+1} = \frac{2a_n}{1 + a_n^2} \quad \text{for} \quad n \geq 0\,,$$

and the sequence $\{c_n\}$ is defined by the formulae

$$c_0 = 4 \quad \text{and} \quad c_{n+1} = c_n^2 - 2c_n + 2 \quad \text{for} \quad n \geq 0\,.$$

Prove that

$$a_n \;=\; \frac{2c_0c_1\ldots c_{n-1}}{c_n} \quad \text{for all} \quad n \geq 1\,.$$

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the solution of Klamkin, which gives an indication of both types of solutions received.

Letting $x_n = c_n - 1$, we have $x_{n+1} = x_n^2$ where $x_0 = 3$. Hence, $x_n = x_0^{2^n}$ and $c_n = 3^{2^n} + 1$. Since $c_1 = 10$ and $a_1 = \frac{4}{5}$ it now suffices to show that $a_n = \frac{2c_0c_1 \ldots c_{n-1}}{c_n}$ satisfies the recurrence $a_{n+1} = \frac{2a_n}{1+a_n^2}$ for $n \geq 0$. Also since $(3^{2^n} + 1)(3^{2^n} - 1) = 3^{2^{n+1}} - 1$, it follows (multiplying by $\frac{3^{2^0}-1}{3^{2^0}-1}$) that

$$\frac{2c_0c_1 \ldots c_{n-1}}{c_n} = \frac{3^{2n} - 1}{3^{2^n} + 1}$$

and by substitution and simplification, this satisfies the recurrence relation for a_n.

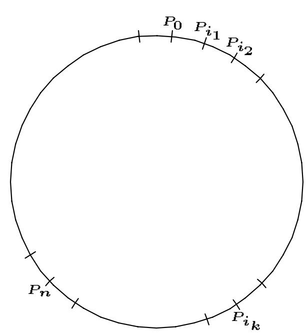
Comment: We can obtain another representation for a_n by letting it equal $\tanh \theta_n$, so that $\tanh \theta_{n+1} = \tanh 2\theta_n$, subject to $\frac{1}{2} = \tanh \theta_0$. It then follows that $a_n = \tanh 2^n\theta_0 = \tanh\left(2^n \arctan \mathrm{h}\frac{1}{2}\right) = \tanh\left(2^{n-1} \ln 3\right)$.

4. Let $n \geq 2$ be a fixed natural number and let P_0 be a fixed vertex of the regular $(n+1)$–gon. The remaining vertices are labelled P_1, P_2, ..., P_n, in any order. To each side of the $(n+1)$–gon assign a natural number as follows: if the endpoints of the side are labelled P_i and P_j, then $|i - j|$ is the number assigned. Let S be the sum of all the $n + 1$ numbers thus assigned. (Obviously, S depends on the order in which the vertices have been labelled.)

(a) What is the least value of S available (for fixed n)?

(b) How many different labellings yield this minimum value of S?

Solution by Pierre Bornsztein, Courdimanche, France.



(a) Soit $\overline{P_0P_n}^{\ominus}$ l'arc réliant P_0 à P_n dans le sens des aiquilles d'une montre, $\overline{P_0P_n}^{\oplus}$ l'arc réliant P_0 à P_n dans le sens contraire.

Notons S^- la somme des nombres assignés sur $\overline{P_0P_n}^{\ominus}$ (idem pour S^+). Par définition,

$$\begin{aligned} S^- &= |0 - i_1| + |i_1 - i_2| + \cdots + |i_{k-1} - i_k| + |i_k - n| \\ &\geq |0 - i_1 + i_1 - i_2 + \cdots + i_{k-1} - i_k + i_k - n| = n \end{aligned}$$

avec egalité ssi $0 \leq i_1 \leq i_2 < \cdots \leq i_k < n$.

De même,

$$S^+ \geq n$$

avec egalité ssi les sommets sont classés dans l'ordre croissant de 1 à n, d'où on en déduit $S = S^- + S^+ \geq 2n$.

(b) Pour P_n fixé il y a i sommets entre P_0 et P_n, le long de $\overline{P_0P_n}^{\ominus}$ où $i \in \{0, \dots, n-1\}$. Il y a donc i nombres à choisir dans $\{1, \dots, n-1\}$, d'où $\binom{n-1}{i}$ choix.

Les nombres, une foix choisis, sont alors disposés dans l'ordre croissant de P_1 à P_n : l'ordre est donc imposé.

De même sur $\overline{P_0P_n}^{\oplus}$ les nombres restants sont imposés ainsi que leur ordre.

Il y a donc $\sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}$ choix pour la disposition.

5. Solve the equation

$$\frac{1}{2}(x+y)(y+z)(z+x) + (x+y+z)^3 = 1 - xyz$$

in integers.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the write-up of Bataille, although all three solvers used the same approach.

Let $s = x + y + z$ and

$$\begin{aligned} P(X) &= (X-x)(X-y)(X-z) \\ &= X^3 - sX^2 + (xy + yz + zx)X - xyz\,. \end{aligned}$$

Then $(x+y)(y+z)(z+x) = P(s) = s(xy+yz+xz) - xyz$ and the given equation may be written

$$s(xy + yz + xz) - xyz + 2s^3 = 2 - 2xyz\,,$$

or $2 + P(-s) = 0$.

As $P(-s) = -(2x+y+z)(2y+z+x)(2z+x+y)$, the equation finally becomes

$$(2x+y+z)(2y+z+x)(2z+x+y) = 2\,.$$

Either one of the three factors of the left-hand side is 2 and the other two are 1, 1 (or -1, -1) or one of the factors is -2 and the other two are 1, -1, (or -1, 1).

The system

$$\begin{cases} 2x + y + z = 2 \\ x + 2y + z = 1 \\ x + y + 2z = 1 \end{cases} \quad \text{is equivalent to} \quad x = 1\,,\ y = 0\,,\ z = 0\,.$$

The system

$$\left\{\begin{array}{rcr} 2x+y+z &=& 2 \\ x+2y+z &=& -1 \\ x+y+2z &=& -1 \end{array}\right. \qquad \text{is equivalent to} \qquad x=2\,,\ y=-1\,,\ z=-1\,.$$

When one of the factors is -2, the two corresponding systems lead to $4(x+y+z)=-2$, which is impossible for integral x, y, z.

Since x, y, z have symmetrical roles, there are six solutions altogether for the triple (x,y,z):

$$(1,0,0)\,,\ (0,1,0)\,,\ (0,0,1)\,,\ (2,-1,-1)\,,\ (-1,2,-1)\,,\ (-1,-1,2)\,.$$

7. Determine all two-digit (in decimal notation) natural numbers $n=(ab)_{10}=10a+b$ $(a\geq 1)$ with the property that for every integer x the difference x^a-x^b is divisible by n.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Clearly, $n\mid x^a-x^b$ for all integers x if $a=b$. We show that besides $11, 22, \ldots, 99$ there are exactly three more such n's. These are: $n=15$, 28, and 48. We assume that $a\neq b$ and start off by eliminating some impossible values of n.

(1) If a is even and b is odd, then setting $x=\pm 2$ leads to $n\mid 2^a-2^b$ and $n\mid 2^a+2^b$. Thus $n\mid 2^{a+1}$, which is clearly impossible since the only possible divisors of 2^{a+1} are powers of two while $n>1$ is odd.

(2) If a is odd and b is even, then setting $x=\pm 2$ again leads to the same conclusion that $n\mid 2^{a+1}$. Hence n must be a power of two. Since a is odd, the only possible values are $n=16$ and 32. However, $16\nmid 2-2^6$ and $32\nmid 2^3-2^2$, showing that there are no solutions in this case either.

(3) If $b=0$, then n is even and $n\mid 2^a-1$, which is clearly impossible.

Using (1), (2), and (3) we narrow the possible values of n down to the following set of 32 integers:

$$\{13\,,15\,,17\,,19\,,24\,,26\,,28\,,31\,,35\,,37\,,39\,,42\,,46\,,48\,,51\,,53,\ 57\,,$$
$$59\,,62\,,64\,,68\,,71\,,73\,,75\,,79\,,82\,,84\,,86\,,91\,,93\,,95,\ 97\}\,.$$

Since $n\mid x^a-x^b$ if and only if $n\mid x^b-x^a$ we may assume that $a>b$ when checking whether n satisfies the given property. Note that

$2^3-2=$	6	eliminates	13 and 31 ;
$2^4-2^2=$	12	eliminates	24 and 42 ;
$2^5-2=$	30	eliminates	51 (but **not** 15) ;
$2^5-2^3=$	24	eliminates	35 and 53 ;
$2^6-2^2=$	60	eliminates	26 and 62 ;
$2^6-2^4=$	48	eliminates	46 and 64 ;

$$\begin{array}{llll}
2^7 - 2 \;\; = 126 & \text{eliminates} & 17 \text{ and } 71\,; \\
2^7 - 2^3 = 120 & \text{eliminates} & 37 \text{ and } 73\,; \\
2^7 - 2^5 = \;\; 96 & \text{eliminates} & 57 \text{ and } 75\,; \\
2^8 - 2^2 = 252 & \text{eliminates} & 82 \text{ (but } \textbf{not } 28)\,; \\
2^8 - 2^4 = 240 & \text{eliminates} & 84 \text{ (but } \textbf{not } 48)\,; \\
2^8 - 2^6 = 192 & \text{eliminates} & 68 \text{ and } 86\,; \\
2^9 - 2 \;\; = 510 & \text{eliminates} & 19 \text{ and } 91; \\
2^9 - 2^3 = 504 & \text{eliminates} & 39 \text{ and } 94\,; \\
2^9 - 2^5 = 480 & \text{eliminates} & 59 \text{ and } 95\,; \\
2^9 - 2^7 = 384 & \text{eliminates} & 79 \text{ and } 97\,.
\end{array}$$

Therefore, the only **possible** values of n are: $n = 15$, 28 and 48. We now show that they indeed satisfy the condition that $n \mid x^a - x^b$ for all integers x.

(a) For $n = 15$, we show that $x \equiv x^5 \pmod{15}$. By Fermat's Little Theorem (Fthm), we have $x^3 \equiv x \pmod 3$ and so $x^5 \equiv x^3 \equiv x \pmod 3$. Also, $x^5 \equiv x \pmod 5$. Hence $x^5 \equiv x \pmod{15}$ follows.

(b) For $n = 28$, we show that $x^2 \equiv x^8 \pmod{28}$. Note that $28 = 2^2 \times 7$. By Fthm, we have $x^7 \equiv x \pmod 7$ and so $x^8 \equiv x^2 \pmod 7$. Further, we claim that $x^8 \equiv x^2 \pmod 4$. This is obvious if x is even. On the other hand, if x is odd, then $x^2 \equiv 1 \pmod 4$ implies $x^8 \equiv 1 \pmod 4$ and so $x^8 \equiv x^2 \pmod 4$. Hence $x^8 \equiv x^2 \pmod{28}$ follows.

(c) For $n = 48$, we show that $x^4 \equiv x^8 \pmod{48}$. Note that $48 = 2^4 \times 3$. By Fthm, we have $x^3 \equiv x \pmod 3$ and so $x^4 \equiv x^2 \pmod 3$. Hence $x^8 \equiv x^4 \pmod 3$. It remains to show that $16 \mid x^8 - x^4$. This is clear if x is even. If x is odd, then $x = 2k + 1$ for some integer k and thus

$$\begin{aligned}
x^8 - x^4 &= x^4(x^2 - 1)(x^2 + 1) \\
&= (2k + 1)^4 \left(4k^2 + 4k\right) \left(4k^2 + 4k + 2\right) \\
&= 8k(k + 1) \left(2k^2 + 2k + 1\right) (2k + 1)^4 \ ,
\end{aligned}$$

which is divisible by 16 since $k(k + 1)$ is even.

To summarize, $n = 10a + b$ satisfies $n \mid x^a - x^b$ for all integers x if and only if $n = 11, 22, \dots, 99, 15, 28, 48$.

Comment: This is one of the most intriguing problems that I have seen lately. I will be really surprised if there is a much shorter solution!

8. Consider the functional equation $f(x, y) = a\, f(x, z) + b\, f(y, z)$ with real constants a, b. For every pair of real numbers a, b give the general form of functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying the given equation for all x, y, $z \in \mathbb{R}$.

Solution by Pierre Bornsztein, Courdimanche, France.

Soient a, $b \in \mathbb{R}$ et pour tous x ,y ,$z \in \mathbb{R}$

$$f(x, y) = af(x, z) + bf(y, z)\,. \qquad (*)$$

Alors :

Dans le cas où $x = y = z$, $f(x,x) = (a+b)f(x,x)$ donc $a+b=1$ ou $f(x,x)=0$. Si $a+b \neq 1$, pour tout $x \in \mathbb{R}$, $f(x,x)=0$ et donc pour $z=y$, $(*)$ donne

$$f(x,y) \;=\; af(x,y)+bf(y,y) \;=\; af(x,y)\,.$$

Donc soit $a=1$ ou $f(x,y)=0$.

Dans le cas ou $a=1$

$$f(x,y) \;=\; f(x,z)+bf(y,z)\,,$$

observons qu'avec $x=y$, $f(x,x)=0=f(y,z)(1+b)$, et donc $f\equiv 0$ ou $b=-1$.

Maintenent si $a=1$ et $b=-1$

$$f(x,y) \;=\; f(x,z)-f(y,z)$$

ou encore

$$f(x,z) \;=\; f(x,y)+f(y,z)$$

pour tous x, y, $z \in \mathbb{R}$.

C'est a dire

$$f(x,y) \;=\; f(x,z)+f(z,y)$$

pour tous x, y, $z \in \mathbb{R}$, et donc $f(z,y)=-f(y,z)$. On pose $f(x,0)=g(x)$, alors $f(0,x)=-g(x)$ et

$$\begin{aligned} f(x,y) &= f(x,z)+f(z,y) \\ &= f(x,0)+f(0,y) \\ &= g(x)-g(y)\,. \end{aligned}$$

Reciproquement, $f(x,y)=g(x)-g(y)$ où g est une fonction arbitraire.

Alors $f(x,y)=f(x,z)+f(z,y)$, et f convient.

Dans le cas où $a+b=1$, $b=1-a$, et $(*)$ s'écrit

$$f(x,y)=af(x,z)+(1-a)f(y,z)\,, \qquad (**)$$

et alors $f(x,x)=f(x,z)$ et donc pour tous x, $y \in \mathbb{R}$, $f(x,y)=f(x,x)$. Maintenant $(**)$ donne

$$f(x,x) \;=\; af(x,x)+(1-a)f(y,y)\,,$$

et par conséquence

$$(1-a)f(x,x) \;=\; (1-a)f(y,y)\,.$$

Deux possibilités se présentent. Soit $a=1$ ou $f(x,x)=f(y,y)=f(x,y)$, et f est constante. Si $a=1$, $b=0$, alors $f(x,y) = f(x,z)$ pour tous x,

y, $z \in \mathbb{R}$. Donc $f(x,y) = f(x,x)$ indépendant de y. On a verifié que ces fonctions conviennent.

En conclusion :

• si $(a,b) = (1,-1)$, $f(x,y) = g(x) - g(y)$ où $g : \mathbb{R} \to \mathbb{R}$ est arbitraire ;

• si $a + b \neq 1$ et $(a,b) \neq (1,-1)$, $f \equiv 0$;

• si $a + b = 1$ et $a \neq 1$: f constante ;

• si $(a,b) = (1,0)$: $f(x,y) = g(x)$ pour tous $x, y \in \mathbb{R}$ où $g : \mathbb{R} \to \mathbb{R}$ est arbitraire.

9. On the plane there are given four distinct points A, B, C, D lying (in this order) on a line g, at distances $AB = a$, $BC = b$, $CD = c$.

(a) Construct, whenever possible, a point P, not on g, such that the angles $\angle APB$, $\angle BPC$, $\angle CPD$ are equal.

(b) Prove that a point P with the property as above exists if and only if the following inequality holds: $(a+b)(b+c) < 4ac$.

Solution by Michel Bataille, Rouen, France.

(a) If P is a solution, then the lines PB and PC are interior bisectors in $\triangle APC$ and $\triangle BPD$ respectively. Hence we have: $\dfrac{PA}{PC} = \dfrac{BA}{BC}$ and $\dfrac{PB}{PD} = \dfrac{CB}{CD}$ and P is simultaneously on $E_1 = \left\{M : \dfrac{MA}{MC} = \dfrac{a}{b}\right\}$ and $E_2 = \left\{M : \dfrac{MB}{MD} = \dfrac{b}{c}\right\}$.

In the general case where $a \neq b$, denoting by B' the harmonic conjugate of B with respect to A and C, E_1 is the circle with diameter BB' and, when $a = b$, E_1 is the perpendicular bisector of the segment AC. Similar results hold for E_2.

Conversely, we may construct E_1 and E_2 and, assuming that they are secant, choose for P one of their two distinct points of intersection symmetrical about g. From $\dfrac{PA}{PC} = \dfrac{BA}{BC}$, we deduce that PB is one of the bisectors of $\angle APC$, more precisely the interior bisector in $\triangle APC$ since B is between A and C. Hence $\angle APB = \angle BPC$. Similarly $\angle BPC = \angle CPD$ and finally: $\angle APB = \angle BPC = \angle CPD$.

(b) The above construction provides a point P solution whenever E_1 and E_2 are secant. We first examine the general case where $a \neq b$ and $b \neq c$: E_1 and E_2 are circles with centres I_1, I_2 and radii r_1, r_2 respectively. These circles are secant if and only if:

$$|r_1 - r_2| \;<\; I_1 I_2 \;<\; r_1 + r_2 \qquad (1)$$

Let us denote by k the real number such that $\overline{BI_1} = \dfrac{k}{b}\,\overline{BC}$ (so that $|k| = r_1$).

We may compute: $\overline{I_1A} = -\frac{k+a}{b}\overline{BC}$ and $\overline{I_1C} = \frac{b-k}{b}\overline{BC}$, and from the Newton's relation, $\overline{I_1B}^2 = \overline{I_1A}\cdot\overline{I_1C}$, we obtain easily $k = \frac{ab}{a-b}$, so that $r_1 = \frac{ab}{|a-b|}$. Similarly: $r_2 = \frac{cb}{|c-b|}$.

We also compute: $\overline{I_1I_2} = \frac{b^2-ac}{(b-a)(b-c)}\overline{BC}$ so that $I_1I_2 = \frac{b|b^2-ac|}{|b-a|\,|b-c|}$.

The condition (1) may now be successively written:

$$\big|c|a-b| - a|c-b|\big| \;<\; |b^2-ac| \;<\; a|c-b| + c|a-b|$$

$$a^2(c-b)^2 + c^2(a-b)^2 - 2ac|a-b|\,|c-b| \;<\; (b^2-ac)^2$$
$$<\; a^2(c-b)^2 + c^2(a-b)^2 + 2ac|a-b|\,|c-b|$$

$$\left|(b^2-ac)^2 - a^2(c-b)^2 + c^2(a-b)^2\right| \;<\; 2ac|a-b|\,|c-b|$$

$$|a-b|\,|c-b|\,|b^2+b(a+c)-ac| \;<\; 2ac|a-b|\,|c-b|$$

$$-2ac \;<\; b^2+b(a+c)-ac \;<\; 2ac$$

$$-ac \;<\; b^2+b(a+c) \;<\; 3ac\,.$$

Since $b^2 + b(a+c)$ is positive, the latter condition is equivalent to $b^2 + b(a+c) < 3ac$ or $(a+b)(b+c) < 4ac$.

E_1 and E_2 are both lines when $a = b = c$, but in this case they are strictly parallel so that no point P exists (and the condition $(a+b)(b+c) < 4ac$ is not true either).

Lastly, suppose for instance that E_1 is a line and E_2 is a circle (that is, $a = b$ and $b \neq c$). Since E_1 is perpendicular to g at B, E_1 and E_2 are secant if and only if $I_2B < r_2$. We obtain easily: $I_2B = \frac{b^2}{|c-b|}$ and the condition becomes: $b < c$ (and the inequality $(a+b)(b+c) < 4ac$ reduces to $b < c$ as well). The proof of (b) is now complete.

That completes our file of solutions for problems of the February 1998 number of the *Corner*. The Olympiad Season is nearly upon us. Send me your national and regional Olympiads for use in the *Corner*. We also welcome your nice solutions to problems that appear in the *Corner*.

BOOK REVIEWS

ALAN LAW

A Primer of Real Functions, by **Ralph Boas Jr.**,
published by the Mathematical Association of America, 1996,
ISBN# 0-88385-029-X, softcover, 262+ pages, $32.95.
Reviewed by ***Murray Klamkin****, University of Alberta.*

This is the fourth edition of a popular classic Carus monograph which has been revised, updated and augmented by the author's son Harold P. Boas. The previous editions covered sets, metric spaces, continuous functions and differentiable functions. This edition adds a chapter on measurable sets and functions, the Lebesgue and Stieltjes integrals, and applications. The new material is a rewrite of a draft left over by the author at his death. This book can be likened to a sequence of lectures on a variety of topics selected from the foundations of analysis and is done in a friendly and lively manner.

Mathematically Speaking: A Dictionary of Quotations, selected and arranged by **Carl C. Gaither and Alma E. Cavazos-Gaither**,
published by Institute of Physics Publishing, 1998,
ISBN# 0-7530-0503-7, softcover, 484+xiii pages, $39.00 (US).
Reviewed by ***Bruce Shawyer****, Memorial University of Newfoundland.*

The book contains hundreds of quotations from hundreds of authors, as well as many apocryphal quotations from persons unknown. The quotations are grouped into 199 sets, ordered by topics, running from **ABSTRACTION** to **ZERO**. Most sections are just a few pages long, except for the topics **MATHEMATICIAN** and **MATHEMATICS**, which have 29 and 80 pages respectively.

Also included is a complete bibliography of the source material plus two excellent indices, the **SUBJECT BY AUTHOR INDEX** and the **AUTHOR BY SUBJECT INDEX**.

The quotations vary from the profound to the witty. There are quotations from plays, and quite a few are in poetry, including several mnemonics for π. Unfortunately, this reviewer's favourite is missing!

How I want a drink	3.1415
Alcoholic of course	926
After the heavy lectures	5358
Involving decimal fractions	979

(Engineers can substitute "quantum mechanics" for "decimal fractions".)

The authors quoted come from all different walks of life, from professional mathematicians and scientists to historians, journalists, philosophers, poets, rap artists and writers. Some are famous, their names being almost household words; others are much less known. As might be expected, by far the majority of the quotations come from mathematicians and scientists.

To give a flavour of this book, here is a selection of some of the shorter quotations:

Some quotations from famous mathematicians

- **Richard Guy**

  Mathematics often owes more to those who ask questions than to those who answer them.

- **Paul Halmos**

  The only way to learn mathematics is to do mathematics.

- **Leopold Kronecker**

  Number theorists are like lotus-eaters — having once tasted of this food, they can never give it up.

- **George Pólya**

  Geometry is the art of correct reasoning on incorrect figures.

Some quotations from others

- **Ice-T**

  I write rhymes with addition and algebra, mental geometry.

- **G.K. Chesterton**

  Poets do not go mad; but chess-players do. Mathematicians go mad, and cashiers; but creative artists very seldom.

- **John Updike**

When you look into a mirror	rorrim a otni kool uoy nehW
it is not yourself you see	ees uoy flesruoy ton si ti
but a kind of apish error	rorre hsipa fo dnik a tub
posed in fearful symmetry	yrtemmys lufraef ni desop

- **Mae West**

  A figure with curves always offers a lot of interesting angles.

- **Unknown**

  Trigonometry is a sine of the times!

The book is a wonderful compendium and a great source of useful wisdom for teachers of mathematics. It is very readable, and, once one has started to read it, very difficult to put down.

THE SKOLIAD CORNER

No. 36

R.E. Woodrow

We begin with the problems of the Mini demi-finale 1996 of the Vingt et unième Olympiade Mathématique Belge, organized by the Belgian Mathematics Teachers Society. Twenty-five of the problems are multiple choice. The remaining five require an integer answer between 0 and 999 (inclusive). Students are given 90 minutes. My thanks go to Ravi Vakil, Canadian Team Leader to the International Mathematical Olympiad at Mumbai for collecting the materials.

OLYMPIADE MATHÉMATIQUE BELGE

Mini demi-finale 1996

Mercredi 6 mars 1996

1. Sans réponse préformulée — Quel est le nombre premier le plus proche de 100 ?

2. Laquelle des propositions ci-dessous est la négation de: "Chaque langue européenne est parlée par l'un de nos guides au moins." ?

(a) Chacun de nos guides parle toutes les langues européennes.

(b) Chacun de nos guides parle une langue européenne au moins.

(c) Aucun de nos guides ne parle aucune langue européenne.

(d) L'un de nos guides ne parle aucune langue européenne.

(e) L'une des langues européennes n'est parlée par aucun de nos guides.

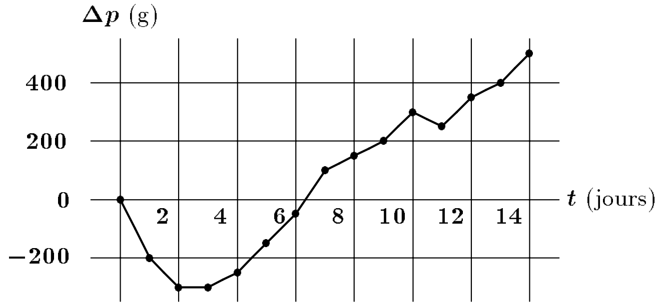
3. Si P désigne le périmètre du triangle ABC et $\|AB\|$, $\|AC\|$, $\|BC\|$ les longueures de ses côtés, laquelle des relations suivantes est correcte ?

(a) $\|AB\| = P/3$ (b) $\|AB\| \leq P/3$ (c) $\|AB\| \leq P/2$
(d) $\|AB\| + \|AC\| = 2P/3$ (e) $\|AB\| + \|AC\| \geq 2P/3$

4. Un parterre rectangulaire de 8 m sur 6 m est entouré extérieurement d'un sentier de $1,5$ m de large. Quelle est l'aire de ce sentier ?

(a) $23,25$ m^2 (b) $37,5$ m^2 (c) 42 m^2 (d) $46,5$ m^2 (e) 51 m^2

5. Le graphique ci-dessous donne, pour les quatorze premiers jours de sa vie, les gains de poids (en grammes, et par rapport au poids à la naissance) d'un bébé dont le poids à la naissance était de $3,250$ kg.



Quel était son poids à une semaine ?

(a) $0,100$ kg **(b)** $3,300$ kg **(c)** $3,350$ kg **(d)** $3,650$ kg **(e)** $4,250$ kg

6. Si X, Y et Z sont les sommets d'un triangle, quel est le nombre de parallélogrammes admettant X, Y et Z pour sommets ?

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 6

7. Que vaut $(3x^2 - 7x)(2x^3 - x^2 + x - 2)$?

(a) $6x^4 - 17x^3 + 10x^2 - 13x + 14$.
(b) $6x^5 - 17x^4 + 10x^3 - 13x^2 + 14x$.
(c) $6x^5 + 17x^4 + 10x^3 + 13x^2 + 14x$.
(d) $6x^6 - 17x^5 + 10x^4 - 13x^3 + 14x^2$.
(e) $6x^6 - 3x^4 - 14x^3 + 10x^2 - 7x + 8$.

8. *Sans réponse préformulée* — Une bille métallique a la propriété de rebondir à une hauteur égale aux huit dixièmes de sa hauteur initiale; si elle est lâchée d'une hauteur de $1,25$ m, quelle sera, en centimètres, la hauteur de son troisième rebond ?

9. De combien augmente l'aire totale d'un cube lorsque la longueur de chacune de ses arêtes augmente de 50% ?

(a) 50% **(b)** 125% **(c)** 225% **(d)** $237,5\%$ **(e)** 2500%

10. Quatre sacs opaques contiennent :

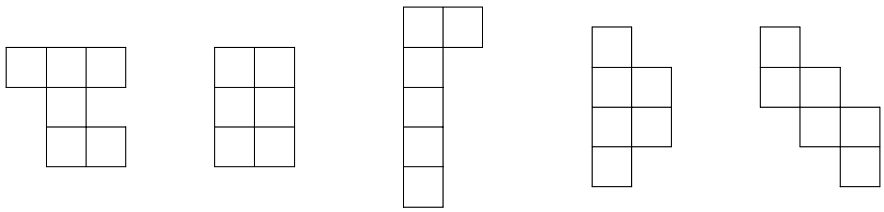
- Le sac A, une bille blanche et une bille rouge ;
- Le sac B, deux billes blanches et deux billes rouges ;
- Le sac C, deux billes blanches, une bille rouge et une bille noire ;
- Le sac D, dix billes blanches et dix billes noires.

De quel sac faut-il tirer une bille au hasard pour avoir le plus de chances que la bille tirée soit blanche ?

(a) Le sac A **(b)** Le sac B **(c)** Le sac C **(d)** Le sac D
(e) Le choix est indifférent

11. Laquelle des figures suivantes est le développement d'un cube ?

(a) **(b)** **(c)** **(d)** **(e)**



12. Si la somme de trois nombres naturels est un nombre de deux chiffres, il est certain que :

(a) chacun des trois nombres est supérieur à 10 ;

(b) deux des nombres, au moins, sont inférieurs à 50 ;

(c) aucun des trois nombres n'est supérieur à 50 ;

(d) les trois nombres sont différents ;

(e) le produit des trois nombres est inférieur à 35 000.

13. Avant son départ en vacances, une personne a acheté 3000 francs français pour 18 270 francs belges. Cette personne, en France, a dû changer à nouveau de l'argent: pour 10 000 francs belges, elle a reçu 1600 francs français. Si elle avait acheté en Belgique, avant son départ, tout l'argent français dont elle a eu besoin,

(a) elle aurait gagné 480 francs belges ;

(b) elle aurait gagné 256 francs belges ;

(c) cela serait revenu au même ;

(d) elle aurait perdu 256 francs belges ;

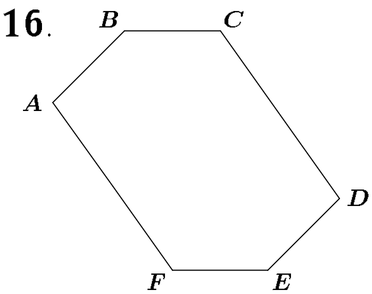
(e) elle aurait perdu 480 francs belges.

14. Une personne a acheté des timbres à 3 F et des timbres à 5 F pour un total de 100 F exactement. Parmi les suivants, quel est le nombre de timbres à 5 F qu'elle *ne* peut *pas* avoir acheté ?

(a) 5 **(b)** 8 **(c)** 9 **(d)** 11 **(e)** 17

15. Un cycliste monte une côte à la vitesse moyenne de 12 km/h, pour la redescendre ensuite à la vitesse moyenne de 48 km/h. Si la montée a duré 22 min 30 s de plus que la descente, quelle est la longueur de cette côte ?

(a) 6 km **(b)** 8 km **(c)** 12 km **(d)** 13, 5 km **(e)** 15 km

16. 

Voici un hexagone $ABCDEF$ dans lequel les côtés opposés sont parallèles et de même longueur. Le triangle ABC est nécessairement appliqué sur le triangle DEF par

(a) une translation **(b)** une symétrie orthogonale
(c) une symétrie central **(d)** une rotation de $90°$
(e) aucune des transformations précédentes

17. Quel est le nombre maximum de points communs à un cercle et au bord d'un losange ?

(a) 2 **(b)** 4 **(c)** 6 **(d)** 8 **(e)** 10

18. Laquelle des affirmations suivantes est vraie ?

(a) Il existe des carrés qui ne sont pas des rectangles.
(b) Un carré n'est jamais un rectangle.
(c) Tout parallélogramme est un losange.
(d) Tout losange est un parallélogramme.
(e) Certains rectangles ne sont pas des parallélogrammes.

19. Dix nombres sont tous inférieurs à 20 et leur moyenne (arithmétique) vaut 18. Laquelle des affirmations suivantes est certainement correcte à propos de ces nombres ?

(a) L'un d'entre eux, au moins, est égal à 18.
(b) Un nombre pair d'entre eux sont égaux à 18.
(c) Un nombre impair d'entre eux sont égaux à 18.
(d) Ils sont tous supérieus à 16.
(e) Ils sont tous positifs.

20. *Sans réponse préformulée* — Les roues avant d'un wagonnet ont 7 cm de circonférence et les roues arrière ont 9 cm de circonférence. Lorsque les roues avant ont fait 10 tours de plus que les roues arrière, de combien de centimètres a avancé le wagonnet ?

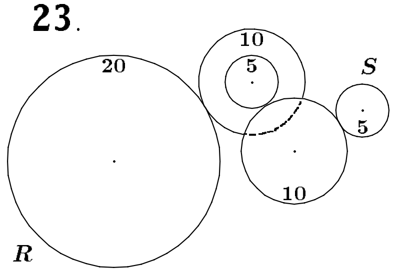
21. La somme de deux nombres premiers est toujours

(a) un nombre pair ; **(b)** un nombre impair ;
(c) un nombre premier ; **(d)** strictement supérieure à 3 ;
(e) inférieure à 1000.

22. Le poids moyen des 30 élèves d'une classe est de 47 kg; si chacun de ces élèves grossit de 3 kg, de combien augmentera le poids moyen ?

(a) $0,1$ kg **(b)** 2 kg **(c)** 3 kg **(d)** 90 kg **(e)** Une autre valeur

23.



Dans le train d'engrenages représenté ci-contre, lorsque la roue R fait un tour dans le sens des aiguilles d'une montre, de combien tourne la roue S ? (Les nombres indiqués donnent le nombre de dents de chaque engrenage. Deux cercles concentriques représentent deux roues solidaires du même axe.)

(a) D'un tour dans le sens des aiguilles d'une montre.

(b) De 2 tours dans le sens des aiguilles d'une montre.

(c) De 4 tours dans le sens des aiguilles d'une montre.

(d) De 2 tours dans le sens opposé à celui des aiguilles d'une montre.

(e) De 4 tours dans le sens opposé à celui des aiguilles d'une montre.

24. *Sans réponse préformulée* — Pour un certain nombre naturel n, $2n + 3$ est un diviseur de $6n + 43$; que vaut n ?

25. Un marchand de jouets a acheté un lot de 100 oursons en peluche qui valaient au total $21\,000$ F, mais le grossiste lui a consenti une remise de 10%. Ce marchand souhaite qu'en accordant une remise de 25% sur le prix de vente affiché, il lui reste encore un bénéfice égal à 30% du prix de vente réel. À combien doit-il afficher l'ourson ?

(a) $341,25$ F **(b)** 360 F **(c)** 400 F **(d)** $34\,125$ F **(e)** Une autre réponse

26. Soit x et y deux demi-droites contenues dans une même droits ; alors,

(a) il existe nécessairement une symétrie centrale appliquant x sur y ;

(b) il existe nécessairement une symétrie orthogonale appliquant x sur y ;

(c) il existe nécessairement une translation appliquant x sur y ;

(d) il existe nécessairement une rotation appliquant x sur y ;

(e) aucune des propositions précédentes n'est vraie.

27. Un avion vole à la vitesse de 400 km/h par rapport à l'air. Pendant son voyage aller, il a de face un vent de 40 km/h; au retour, il a le même vent dans le dos. Quelle est sa vitesse moyenne sur l'ensemble du trajet ?

(a) 390 km/h **(b)** 396 km/h **(c)** 398 km/h **(d)** 400 km/h **(e)** 410 km/h

Last issue we gave the problems of the Old Mutual Mathematical Olympiad 1992. Thanks go to John Grant McLoughlin, Faculty of Education, Memorial University, St. John's, Newfoundland for the problem set and solutions. Here are the answers:

1.	c	**2**.	b	**3**.	d	**4**.	c
5.	a	**6**.	c	**7**.	c	**8**.	d
9.	a	**10**.	b	**11**.	e	**12**.	b
13.	e	**14**.	a	**15**.	c	**16**.	e
17.	a	**18**.	e	**19**.	a	**20**.	e

That completes the *Skoliad Corner* for this issue. Please send me contest materials and suggestions for other features of the *Corner*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Shreds and Slices

Non-mathematical Problem

In *Problem Book for First Year Calculus*, by George W. Bluman, Springer-Verlag, Problem 13.7 asks: Choose a non-mathematician:

(a) John von Neumann.
(b) Mick Jagger.
(c) Georg Cantor.
(d) Pablo Casals.
(e) Stanley I. Grossman.
(f) René Descartes.
(g) Guy Lafleur.

The answer at the back of the book? "Possibly (b), (d), (g)."

Awaiting a Combinatorial Proof

Find a combinatorial proof of the following identity:

$$(n-r)\binom{n+r-1}{r}\binom{n}{r} = n\binom{n+r-1}{2r}\binom{2r}{r}.$$

The individual with the first correct solution that is strictly combinatorial will get a free book prize and the solution published here.

Reference

Tucker, Alan. *Applied Combinatorics*. John Wiley & Sons, Inc. Toronto. 1995 pp. 221.

Discovering the Human Calculator in You

Richard Hoshino
student, University of Waterloo

In his Oscar-winning role in the movie *Rain Man*, Dustin Hoffman plays an idiot savant who can perform complex calculations instantly in his head. Like the "Rain Man", various people have displayed their outstanding capacity for mental arithmetic on TV, and many others have written books teaching these powerful techniques. However, hardly any have ever ventured to justify the validity of these algorithms, as the mathematics involved is surprisingly elementary. In this article, we detail some of the famous tricks that the "human calculators" have used over the years, explain why these methods work, and you will see that, with a little practice, you too can be a human calculator.

Trick 1. Squaring two-digit numbers ending in 5.

To square any two-digit number that ends in 5, add one to the first digit and multiply that sum by the first digit. This will be the first two digits of the answer. The last two digits will always be 25.

For example, $85^2 = 7225$ since $8 \times (8+1) = 8 \times 9 = 72$, and likewise, $25^2 = 625$ since $2 \times 3 = 6$. We can extend this to larger numbers, for example, $195^2 = 38025$, since $19 \times 20 = 380$.

If you are wondering why this method works, a little algebra will quickly convince you:

$$(10A+5)^2 \;=\; 100A^2 + 100A + 25 \;=\; 100A(A+1) + 25\,.$$

Thus, the first two digits will be $A(A+1)$, and the last two digits will be 25.

Let us take this idea one step further. Let us multiply pairs of two-digit numbers whose tens digits are the same, and whose units digits sum to ten. For example, $37 \times 33 = 1221$, $36 \times 34 = 1224$, and $98 \times 92 = 9016$.

Do you see the pattern? Like in the case above, the first two digits of the answer are determined in the same way. But what about the last two digits? Do you see how they are obtained? If so, use a little algebra and convince yourself that it always works.

Copyright © 1999 Canadian Mathematical Society

Trick 2. Squaring any two-digit number.

Take any two-digit number n. Now we know that

$$n^2 = (n^2 - d^2) + d^2 = (n-d)(n+d) + d^2 ,$$

so let us try to find a value of d so that the product $(n-d)(n+d)$ can be easily calculated. Consider the multiple of 10 that is closest to n, and let the difference between the number and this multiple of 10 be d. For example, if we take $n = 87$, then the multiple of 10 that is closest to 87 is 90, and since $90 - 87 = 3$, we have $d = 3$. Similarly, if $n = 94$, we have $d = 4$.

If we perform this calculation for any integer n, then one of $n - d$ or $n+d$ will be a multiple of 10, and the calculation becomes significantly easier. The following examples illustrate this technique:

$$\begin{aligned}
87^2 &= (87+3)(87-3)+3^2 = 90\times 84 + 9 = 7569 ,\\
29^2 &= (29+1)(29-1)+1^2 = 30\times 28 + 1 = 841 ,\\
96^2 &= (96+4)(96-4)+4^2 = 100\times 92 + 16 = 9216 .
\end{aligned}$$

Use this technique to compute the following: 37^2, 52^2, 1999^2.

Trick 3. The Calendar Trick.

One of the more interesting demonstrations performed by "mathemagicians" is the calendar trick. Namely, an audience member calls out her birthday, or some historical date, and the human calculator is able to tell her what day of the week that event took place.

The first thing to do is to memorize the following table:

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1	4	4	0	2	5	0	3	6	1	4	6

It appears challenging to remember this, but there is an interesting pattern here. Reading the row of numbers from left to right in threes, we have 144, 025, 036 and 146. Notice that 144, 025 and 036 are perfect squares, and the last number is just 2 more than the first perfect square, 144. This should make the memorization easier.

Let Y be the last two digits of the year in question. Let D be the day we are searching for, and let M be the integer that corresponds to the month in the above table. Thus, if we are searching for July 25^{th}, 1978, Y would be 78, D would be 25, and M would be 0, since July corresponds to 0.

Compute the value of

$$Y + \left\lfloor \frac{Y}{4} \right\rfloor + D + M ,$$

and divide that sum by 7. Whatever remainder you get corresponds to the day of the week you are seeking, namely 0 is Saturday, 1 is Sunday, 2 is Monday, 3 is Tuesday, 4 is Wednesday, 5 is Thursday, and 6 is Friday.

A small note to remember. If the year is a leap year, and the month is January or February, you must subtract 1 from the total. This is due to the fact that the extra day in a leap year occurs on February 29^{th}, and so if the day you are searching is before that, then the formula is off by one day.

Let us look at a historical date in the 20^{th} century. The famous stock market crash of 1929 occurred on October 29^{th}, so let us use our formula to determine what day of the week "Black Tuesday" occurred.

We have $Y = 29$, $M = 1$, and $D = 29$. Hence $\lfloor \frac{29}{4} \rfloor = 7$, and our sum is $29 + 7 + 29 + 1 = 66$. Dividing this number by 7, we find that the remainder is 3. We conclude that October 29^{th}, 1929 was indeed a Tuesday.

Unfortunately, this formula only works with dates in the 1900's, because in the Gregorian calendar, not all years that are divisible by 4 are leap years. For example, 1800 and 1900 are not leap years, but 2000 is. And thus, we must alter our formula to compensate for this. To calculate dates in the 1800's, use the same formula, but go forward two days in the week. To calculate dates in the 2000's, go one day back. If we use our formula, we find that January 1^{st}, 2000 is a Sunday (remember, 2000 is a leap year!). Go back one day, because it really is a Saturday.

In the past, it was believed that a year had precisely 365.25 days, and so we compensated for the extra quarter day by adding February 29^{th} to our calendar once every four years. Unfortunately, a year has 365.2422 days, so we cannot add an extra day exactly once every four years. It would be nice however if we could, for then this formula would always hold.

Using this method, determine what day of the week you were born on.

Trick 4. Extracting Cube Roots.

We now detail the method for determining the cube roots of all perfect cubes under one billion.

First, you must first learn the cubes of the integers 0 through 9.

n	0	1	2	3	4	5	6	7	8	9
n^3	0	1	8	27	64	125	216	343	512	729

Let us first find the cube root of numbers that are below one million. Hence, the cube root will be at most 99. Say we want to find the cube root of 314, 432. We separate the number into two parts, separated by the comma. Thus, 314 is the first part, and 432 is the second part. The desired cube root has two digits. We will use the first part to get the first digit, and we will use the second part to get the second digit.

Take the first part and determine where it lies in the table of cubes. In our example, 314 lies between 216 and 343. Thus, $216,000 < 314,432 < 343,000$, which implies that the desired cube root lies between 60 and 70, since $60^3 = 216,000$ and $70^3 = 343,000$. Hence, it follows that the first digit of our cube root must be 6.

Now we determine the second digit. If we look at the table of cubes, notice that each cube ends in a different digit. So if a certain cube ends in 2, we know that its cube root must end in 8, because 8 is the only digit whose cube ends in 2. Since 432 ends in 2, the second digit of the cube root must be 8. Thus, the desired cube root is 68.

Before, you go further, determine the cube roots of the following numbers in your head: $157,464$; $185,193$; $778,688$; $12,167$.

With a little practice, you will find that it is faster to do this exercise in your head rather than punching it in your calculator!

So that we do not confuse the digits in the following example, let us introduce some variables. Let $n = (100p + 10q + r)^3$; that is n is the perfect cube, and $100p + 10q + r$ is the cube root of n, where p, q and r are its digits.

To do this trick, once again, you need to memorize a small table:

A	1	2	3	4	5	6	7	8	9	10
B	1	7	9	5	3	8	6	2	4	10

Just keep repeating "one-seven-nine-five-three, eight-six-two-four-ten", to remember the right order in the B row. It might help to notice that the first five entries are odd, and the last five entries are even.

Now, let us move on to the cubes of three-digit numbers. Let us say we wanted to find the cube root of $n = 101,847,563$. First, let us separate this large number into three smaller ones, separated at the commas.

As we did before, we look at the first part, 101, to determine what the first digit of the cube root is. Since 101 lies between $64 = 4^3$ and $125 = 5^3$, we conclude that the $p = 4$. The last part is 563, which ends in a 3, and since $7^3 = 343$, we see that $r = 7$. Hence, we determine the first and last digits the same way as we did before with the smaller numbers.

Now take n, and add and subtract the digits of this number in alternating fashion starting from the right. Thus, we compute

$$3 - 6 + 5 - 7 + 4 - 8 + 1 - 0 + 1 = -7.$$

Now repeatedly add or subtract 11 to this number until we get a number between 0 and 10 inclusive (more formally, we say we take this number modulo 11). Hence, -7 becomes 4. Let this number be A. If you have ever seen the test for divisibility by 11, you will see that $n \equiv A \pmod{11}$.

Now take the number A, and look in the above table to determine the corresponding number B. We see that $A = 4$ corresponds to $B = 5$; thus we have $B = 5$. Finally, our second digit q is the value of $p + r - B \pmod{11}$; that is we add or subtract 11 until we get an integer between 0 and 10 inclusive, and this will be our digit q. Thus, in our case, $p + r - B = 4 + 7 - 5 = 6$, and so we have shown that $q = 6$.

Therefore, $p = 4$, $q = 6$ and $r = 7$; thus the cube root of $n = 101,847,563$ is 467. Checking, we see that this is correct.

Let us summarize the algorithm.

(i) Use the first part of n to determine the first digit of the cube root. Call this digit p.

(ii) Use the last digit of n to determine the last digit of the cube root. Call this digit r.

(iii) Take the alternating sum of n to determine A, where A is between 0 and 10. That is, find the A for which $n \equiv A \pmod{11}$.

(iv) Use the table to find the number B that corresponds to A.

(v) Determine the sum $p + r - B$, and reduce it modulo 11. This is q.

(vi) The number with first digit p, second digit q and third digit r is your desired cube root.

Now try to find the cube roots of the following numbers. Note, you can use a pencil and paper, but you are not allowed to use a calculator!

(a) $17,173,512$, (b) $1,860,867$, (c) $758,550,528$,
(d) $84,604,519$, (e) $170,953,875$.

Let us now justify why this algorithm works. First of all, take each value B in the table, and compute $B^3 \pmod{11}$. You will find that we will get the corresponding value of A in each case. That is how the numbers are determined. Note that there is a bijection between the elements of the two rows in the table. If this were not the case, then this algorithm would not work. We have

$$\begin{aligned} n &= (100p + 10q + r)^3 \\ &\equiv (p - q + r)^3 \\ &\equiv A \pmod{11}. \end{aligned}$$

And from the table, this implies that $(p - q + r) \equiv B \pmod{11}$; that is $q \equiv (p + r - B) \pmod{11}$.

Therefore, q is uniquely determined, as are p and r from before.

With a little practice, you will become comfortable with all of the methods described in this article. Maybe you will even be able to come up with better ways to perform the tricks described. Nevertheless, through practice and perseverance, you too can be a human calculator.

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from the last issue be submitted in time for issue 2 of 2000.

High School Solutions

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <a11238@ipoline.com>

H223. In each of the following alphametics, each letter in the addition represents a unique digit:

$$\begin{array}{ccccc} & 1 & 9 & 9 & 7 \\ + & & O & L & D \\ \hline & Y & E & A & R \end{array} \quad \text{and} \quad \begin{array}{ccccc} & 1 & 9 & 9 & 8 \\ + & & O & L & D \\ \hline & Y & E & A & R \end{array}.$$

For each alphametic, find a solution, or prove that a solution does not exist.

Solution. First, we show that the second alphametic has no solution. If either L or D is greater than 1, then there will be a carry involved in each step of the addition. Most importantly, from the third column, we obtain that $1 + 9 + O \equiv E \pmod{10}$, which implies that $O \equiv E \pmod{10}$.

Thus, O and E represent the same digit, which we cannot have. Hence, a solution must satisfy $LD = 01$; that is $L = 0$ and $D = 1$. But this leads to

$$\begin{array}{ccccc} & 1 & 9 & 9 & 8 \\ + & & O & 0 & 1 \\ \hline & Y & E & 9 & 9 \end{array},$$

so $A = R = 9$, which we cannot have either. Therefore, there is no solution to this alphametic.

A solution to the first alphametic must satisfy $L = 0$ and $D = 1$, or $L = 0$ and $D = 2$ by the same reasoning as above. Setting $L = 0$ and

$D = 1$ leads to $A = R = 9$, but setting $L = 0$ and $D = 1$ leads to several possibilities, for example

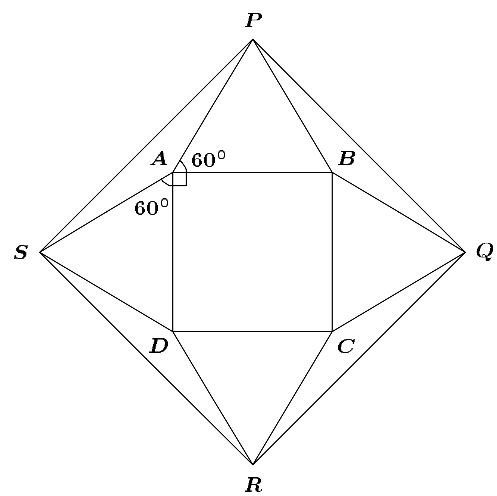
$$\begin{array}{rcccc} & 1 & 9 & 9 & 7 \\ + & & 4 & 0 & 1 \\ \hline & 2 & 3 & 9 & 8 \end{array}.$$

H224. Let $ABCD$ be a square. Construct equilateral triangles APB, BQC, CRD, and DSA, where P, Q, R, and S are points outside of the square.

(a) Prove that $PQRS$ is a square.

(b) Determine the ratio PQ/AB. (See how many ways you can solve this!)

Solution.



(a) In the figure, $\angle PAS = \angle PBQ = \angle QCR = \angle RDS = 150°$, and $AP = AS = BP = BQ = CQ = CR = DR = DS$, so triangles PAS, PBQ, QCR, and RDS are congruent. Hence, $PQ = QR = RS = SP$. Also, $\angle PAS = 150°$, so $\angle SPA = \angle PSA = 15°$, and $\angle SPQ = \angle APB + \angle BPQ + \angle BPQ = 15° + 60° + 15° = 90°$, and by symmetry, $\angle PQR = \angle QRS = \angle RSP = 90°$ as well, so $PQRS$ is a square.

(b) By the Cosine Law,

$$\begin{aligned} PQ^2 &= PB^2 + BQ^2 - 2PB \cdot PQ \cos 150° \\ &= 2AB^2 - 2AB^2 \cdot \cos 150° \;=\; 2AB^2(1 + \cos 30°) \\ \Longrightarrow \frac{PQ^2}{AB^2} &= 2\left(1 + \frac{\sqrt{3}}{2}\right) \;=\; 2 + \sqrt{3} \;=\; \frac{4 + 2\sqrt{3}}{2} \\ \Longrightarrow \frac{PQ}{AB} &= \sqrt{\frac{4 + 2\sqrt{3}}{2}} \;=\; \frac{1 + \sqrt{3}}{\sqrt{2}} \;=\; \frac{\sqrt{2} + \sqrt{6}}{2}. \end{aligned}$$

H225. Consider a row of five chairs, numbered 1, 2, 3, 4, and 5. You are originally sitting on 1. On each move, you must stand up and sit down on an adjacent chair. Make 19 moves, then take away chairs 1 and 5. Then make another 97 moves, with the three remaining chairs. No matter how the moves are made, you will always end up on chair 3. Why is this the case?

Solution. Say that chairs 1, 3, and 5 are *odd chairs* and that chairs 2 and 4 are *even chairs*. After 1 move, we are on an even chair. After 2 moves, we are on an odd chair. By a simple parity argument, after 19 moves we must be on an even chair. Thus, we take away chairs 1 and 5, and are left with two even chairs and one odd chair.

By parity again, after 97 moves, we will be on an odd chair, namely chair 3. Hence, we will always end up on chair 3.

H226. The smallest multiple of 1998 that consists of only the digits 0 and 9 is 9990.

(a) What is the smallest multiple of 1998 that consists of only the digits 0 and 3?

(b) What is the smallest multiple of 1998 that consists of only the digits 0 and 1?

Solution. (a) Let N be the smallest multiple of 1998 that only consists of the digits 0 and 3. Since $1998 = 6 \times 333$, we have that 333 divides $N/6$.

By definition, N is of the form

$$N = 3 \cdot 10^{a_1} + 3 \cdot 10^{a_2} + \cdots + 3 \cdot 10^{a_n}$$

for some integers $a_1 > a_2 > \cdots > a_n > 0$. Then,

$$\frac{N}{6} = 5 \cdot 10^{a_1 - 1} + 5 \cdot 10^{a_2 - 1} + \cdots + 5 \cdot 10^{a_n - 1},$$

which implies that $N/6$ is an integer that consists of only the digits 0 and 5. Since 9 divides 333, 9 must divide $N/6$. By a well known divisibility test, the number of 5's in $N/6$ must be a multiple of 9. Hence, $N/6 \geq 555\,555\,555$, or $N \geq 3\,333\,333\,330$.

Checking, we find that 1998 does indeed divide 3 333 333 330, so this is the multiple of 1998 we seek.

(b) Let N be the smallest multiple of 1998 that consists of only the digits 0 and 1. Then N is of the form

$$N = 10^{a_1} + 10^{a_2} + \cdots + 10^{a_n}$$

for some integers $a_1 > a_2 > \cdots > a_n > 0$. The last digit of N must be a 0. Note that 1998 divides N if and only if 999 divides $N/2$, which in turn occurs if and only if 999 divides $N/10$, so henceforth we consider $N/10 = 10^{a_1-1} + 10^{a_2-1} + \cdots + 10^{a_n-1}$. Let $b_i = a_i - 1$ for all i.

Let a, b, and c be the number of exponents b_i which are congruent to 0, 1, and 2 modulo 3 respectively. Then

$$\begin{aligned} \frac{N}{10} &= 10^{a_1-1} + 10^{a_2-1} + \cdots + 10^{a_n-1} \\ &= 10^{b_1} + 10^{b_2} + \cdots + 10^{b_n} \\ &\equiv a + 10b + 100c \\ &\equiv 0 \pmod{999}, \end{aligned}$$

since $10^3 \equiv 1 \pmod{999}$. Note that $a + 10b + 100c \equiv 0 \pmod{999}$ implies that $10a + 100b + c \equiv 0$ and $100a + b + 10c \equiv 0$, which we obtain by multiplying by 10 and 100 respectively. We consider possible values of a, b, and c.

Suppose that $c \geq 10$. Since $a + 10b + 100c \equiv a + 10b + 100(c - 10) + 1000 \equiv (a + 1) + 10b + 100(c - 10) \pmod{999}$, the triple $(a + 1, b, c - 10)$ is also a solution. Similarly, if $b \geq 10$ or $a \geq 10$, then we can obtain another solution by using the two congruences derived above. Hence, by using this reduction, for any given solution, we can obtain a solution where a, b, and c are all at most 9. There is no danger of being caught in a cycle, since each application of the reduction decreases the sum $a + b + c$ by 9.

Now consider such a reduced solution. Each of a, b, and c is at most 9, but if any is less than 9, then it is apparent that $a + 10b + 100c < 999$, so the congruence $a + 10b + 100c \equiv 0 \pmod{999}$ cannot hold. Therefore, the only solution with all of a, b, and c at most 9 is $a = b = c = 9$. This leads to the number

$$\frac{N}{10} = \underbrace{111...1}_{27\ 1\text{'s}}.$$

We have ruled out the case $a + b + c < 27$, and all values such that $a + b + c \geq 27$ must lead to a number with at least 27 digits. Therefore, the desired minimal multiple is

$$N = \underbrace{111...1}_{27\ 1\text{'s}}0.$$

Advanced Solutions

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A209. Is there an infinite number of squares among the triangular numbers?

Solution.

The answer is yes. The problem reduces to a special case of Pell's equation. Pell's equation is the name given to diophantine equations of the form $x^2 - dy^2 = 1$, where d is a positive, non-square integer.

We want to show that there is an infinite number of integers n such that the n^{th} triangular number $n(n-1)/2$ is a perfect square. Assume that $n(n-1)/2 = a^2$, or equivalently $n(n-1) = 2a^2$. By completing the square, we obtain $(2n-1)^2 - 8a^2 = 1$, or $(2n-1)^2 - 2(2a)^2 = 1$.

In other words, the problem reduces to showing that there is an infinite number of pairs of integer solutions to the equation $x^2 - 2y^2 = 1$. Here $x = 2n-1$ and $y = 2a$. This is a particular case of Pell's equation, which has infinitely many solutions. First, note that $(x_0, y_0) = (3, 2)$ is a specific solution. Then note that for any solution (x, y), $(3x+4y, 2x+3y)$ is also a solution:

$$\begin{aligned}(3x+4y)^2 - 2(2x+3y)^2 &= 9x^2 + 24xy + 16y^2 - 8x^2 - 24xy - 18y^2 \\ &= x^2 - 2y^2 = 1\,.\end{aligned}$$

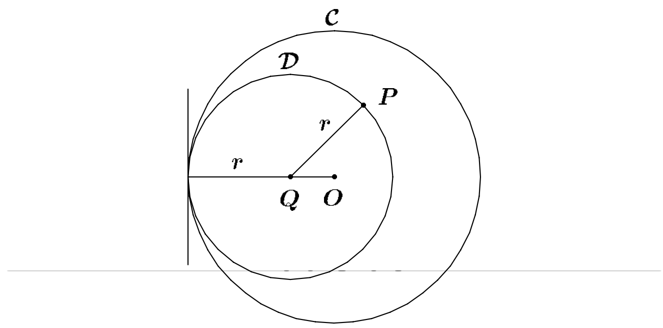
Furthermore, if x is odd and y is even, then $3x+4y$ is odd and $2x+3y$ is even. In this way, we can generate an infinite number of solutions to $x^2 - 2y^2 = 1$ starting from $(3, 2)$. Thus, setting $n = (x+1)/2$, we find that there is an infinite number of integers n such that the n^{th} triangular number is a perfect square.

A210. *Proposed by Naoki Sato.*

Let P be a point inside circle $\mathcal{C}$. Find the locus of the centres of all circles ω which pass through P and are tangent to $\mathcal{C}$.

Solution.

Let the centre of circle $\mathcal{C}$ be O and its radius R. We claim that the locus is an ellipse with foci at P and O, with semi-major axis length $R/2$. To see this, let $\mathcal{D}$ be a circle passing through P and tangent to the circle $\mathcal{C}$ as shown. Let Q be the centre of this circle.



To show that Q lies on the ellipse with foci P and O, it is sufficient to show that $PQ + QO$ is a constant. Let the radius of circle $\mathcal{D}$ be r. Since the two circles $\mathcal{C}$ and $\mathcal{D}$ are tangent, the common tangent point is collinear with the centres O and Q as shown. Then $OQ = R - r$, and PQ is the radius of the circle. Therefore, $PQ + QO = r + R - r = R$. Thus, the locus is an ellipse with foci at P and O. Further, the constant is R, so the length of the ellipse's semi-major axis is $R/2$.

A211. Do there exist a convex polyhedron and a plane, not passing through any of its vertices, and intersecting *more* than $2/3$ of all of the edges of the polyhedron?

(Polish Mathematical Olympiad, first round)

Solution.
We first prove a lemma.

Lemma. For any polyhedron with E edges and F faces, we have the inequality $2E \geq 3F$.

Proof. Each face has at least three edges, so there is a total of at least $3F$ edges, counting each edge twice. Each edge is counted once for the two faces it attaches. The total number of edges is given by E, so $2E \geq 3F$.

Suppose on the contrary that there are a convex polyhedron and a plane, not passing through any of its vertices, and intersecting more than $2/3$ of all the edges. Let E, F, and V denote the number of edges, faces, and vertices of the polyhedron respectively. The plane cuts the polyhedron into a polygonal region. Suppose the polygon formed in this way has n vertices, and hence n sides. Each vertex is the intersection of an edge of the polyhedron with the plane. By assumption, $n > 2E/3$. Now each edge of the polygon is the intersection of a face of the polyhedron with the plane. Thus, the number of faces of the polyhedron is at least n. That is, $F \geq n$.

Together with the lemma, we obtain that $2E/3 \geq F \geq n > 2E/3$, which is a contradiction.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

This issue, we will be delaying the challenge board solutions until a later issue. The observant reader will have noticed that the schedule for the high school and advanced sections have been asynchronous with that of the challenge board section; this will fix this problem.

Note: Although the editor listed, as the proposer for **C83** (***CRUX with MAYHEM***, Vol. 25, Issue 1), the person from whom he heard the problem, the editor is aware that **C83** is an old question. In fact, a version of the problem has appeared in a competition as recently as 1996, when it was used in a Romanian IMO team selection contest. Thanks to Mohammed Aassila for bringing this to our attention.

Problem of the Month

Jimmy Chui, student, Earl Haig S.S.

Problem. Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \;\le\; \sqrt{a}+\sqrt{b}+\sqrt{c}\,,$$

and determine when equality occurs.

(1996 APMO, Problem 5)

Solution. Let $s = (a+b+c)/2$, the semi-perimeter, and let $x = s-a$, $y = s-b$, and $z = s-c$. Then $a = y+z$, $b = z+x$, and $c = x+y$. Note that x, y, and z are all positive, since $x = (b+c-a)/2$, etc., and a, b, and c are the lengths of the sides of a triangle. (This is known as the infamous Ravi Substitution [*Ed*. at least in Canadian IMO circles].) Hence, the inequality is equivalent to

$$\sqrt{2x}+\sqrt{2y}+\sqrt{2z} \;\le\; \sqrt{z+x}+\sqrt{x+y}+\sqrt{y+z}\,.$$

Recall that the Arithmetic Mean–Quadratic Mean (AM–QM) inequality states that

$$\frac{a+b}{2} \;\le\; \sqrt{\frac{a^2+b^2}{2}}$$

for all $a, b \geq 0$. Then, we have that

$$\begin{aligned}
\sqrt{2x}+\sqrt{2y}+\sqrt{2z} &= \frac{\sqrt{2x}+\sqrt{2y}}{2}+\frac{\sqrt{2y}+\sqrt{2z}}{2}+\frac{\sqrt{2z}+\sqrt{2x}}{2} \\
&\leq \sqrt{\frac{2x+2y}{2}}+\sqrt{\frac{2y+2z}{2}}+\sqrt{\frac{2z+2x}{2}} \quad \text{(AM–QM)} \\
&= \sqrt{z+x}+\sqrt{x+y}+\sqrt{y+z}\,.
\end{aligned}$$

Equality holds if and only if $x = y = z$; that is, if and only if $a = b = c$.

When dealing with an expression involving the lengths of the sides of a triangle, implementing the Ravi Substitution will often alter the expression to a more manageable form.

J.I.R. McKnight Problems Contest 1987

1. (a) Three people, Aretha, Bob, and Chai, throw dice upon the condition that the one who has the lowest result shall give each of the others the sum of money each of the two winners has already. Aretha loses first, Bob loses second, and Chai loses the third game. They discovered that each finished with the same amount of money. Express the amount of money that each one had at the beginning in terms of the amount that each had at the end of the third game.

   (b) Find all integer solutions for x, y, and z:

$$\begin{aligned}
x(y+z) &= 32\,, \\
y(x+z) &= 65\,, \\
z(x+y) &= 77\,.
\end{aligned}$$

2. (a) Find the sum of the first 22 terms of the geometric series having first term i and ratio $1+i$, where $i = \sqrt{-1}$. Give your answer in the form $a+bi$, where $a, b \in \mathbb{R}$.

   (b) A three-dimensional figure is defined by the equation

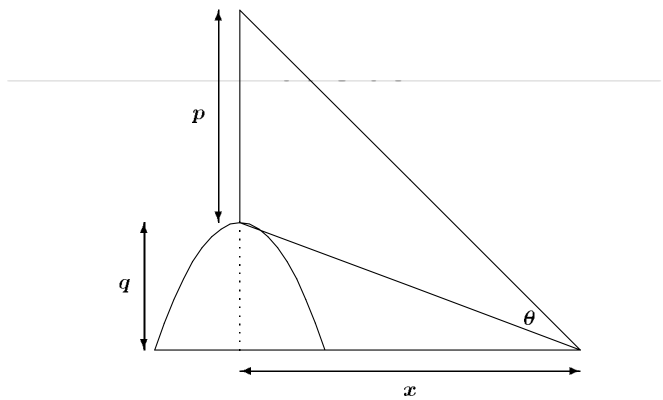
$$4|x|+3|y|+6|z| \;=\; 12\,.$$

   Identify this figure and determine its volume.

3. A sequence of integers is defined by the following recursion: $x_1 = 2$, $x_2 = 5$, and $x_k = x_{k-1} + 2x_{k-2}$ for $k > 2$. Prove that

$$x_n \;=\; \frac{7\cdot 2^{n-1}+(-1)^n}{3}\,.$$

4. The screen of a drive-in theatre is p units tall and is situated on a hill q units high. A car is situated a distance x units from the screen such that the angle θ subtended by the screen is a maximum. Show that the maximum value of θ occurs when $x = \sqrt{q(p+q)}$.



5. The lengths of the sides of a triangle are 8, 8, and 11. Find the length of one of the angle trisectors drawn to the longest side.

6. Consider the set of odd numbers $\{1, 3, 5, \ldots, 101\}$.

   (a) How many combinations of two distinct numbers can be formed from this set?

   (b) Determine the sum of the products of the pairs in (a).

7. For any convex quadrilateral $ABCD$, the diagonals AC and BD intersect at E. The centroids of triangles ABE, BCE, CDE, and DAE are P, Q, R, and S respectively.

   (a) Prove that $PQRS$ is a parallelogram.

   (b) Find the ratio of the area of the parallelogram $PQRS$ to the original quadrilateral.

Do question 8 or 9:

8. Seventeen dots are arranged so that no three are collinear. Each pair of dots is connected by a line segment which may be drawn using one of three colours. Prove that there are at least three points connected to each other with the same colour.

9. Prove the following theorem: If the bisectors of a pair of opposite exterior angles of a cyclic quadrilateral are parallel, then the angles at the other two vertices are right angles.

Swedish Mathematics Olympiad 1985

Final Round

4. The polynomial $p(x)$ of degree n has real coefficients, and $p(x) \geq 0$ for all x. Show that

$$p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x) \;\geq\; 0\,.$$

Solution by Hadi Salmasian, Sharif University of Technology, Tehran, Iran.

Put $\phi(x) = p(x) + p'(x) + \cdots + p^{(n)}(x)$. Since $p^{(n+1)}(x) = 0$, we have

$$\phi(x) = p(x) + \phi'(x)\,. \tag{1}$$

Now, define $\psi(x) = \phi(x)e^{-x}$. Then

$$e^x\psi(x) = p(x) + e^x\psi(x) + e^x\psi'(x)\,,$$

which is a result of (1). This gives us the following equation:

$$\psi'(x) \;=\; -p(x)e^{-x}\,,$$

and obviously, for all $x \in \mathbb{R}$, $\psi'(x) \leq 0$, so $\psi(x)$ is a decreasing function. On the other hand, the degrees of p', p'', ... , are less than n, and the leading coefficient in both polynomials $p(x)$ and $\phi(x)$ are equal. The following lemma is easy to prove:

Lemma. For a non-zero polynomial $p(x)$, the following two are equivalent:

(i) $\lim_{x\to+\infty} p(x) = +\infty$.

(ii) The leading coefficient of $p(x)$ is positive.

Using the lemma, we find that $\phi(x) \geq 0$ for sufficiently large values of x, and the same is true for $\psi(x)$. But $\psi(x)$ is a decreasing function, so $\psi(x) \geq 0$ for all x. The same is also true for $\phi(x)$.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 October 1999**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2414. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, Guang Dong Province, China, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

For $1 < x \leq e \leq y$ or $e \leq x < y$, prove that $x^x y^{x^y} > x^{y^x} y^x$.

2415. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Given a point Z on a line segment AB, find a Euclidean construction of a right-angled triangle ABC whose incircle touches hypotenuse AB at Z.

2416. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Given $\triangle ABC$, where C is an obtuse angle, suppose that M is the mid-point of BC and that the circle with centre A and radius AM meets BC again at D. Assume also that $MD = AB$. The circle, Γ, with centre M and radius MB meets AB at E. Let H be the foot of the perpendicular from A to BC (extended). Suppose that AC and EH intersect at I.

Find the angles $\angle IAH$ and $\angle AHI$ as function of $\angle ABC$.

[This proposal was inspired by problem **2316**.]

2417. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

In $\triangle ABC$, with $AB \neq AC$, the internal and external bisectors of $\angle BAC$ meet the circumcircle of $\triangle ABC$ again in L and M respectively. The points L' and M' lie on the extensions of AL and AM respectively, and satisfy $AL = LL'$ and $AM = MM'$. The circles ALM' and $AL'M$ meet again at P.

Prove that $AP \parallel BC$.

2418. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

In $\triangle ABC$, the lengths of the sides BC, CA, AB are 1998, 2000, 2002 respectively.

Prove that there exists exactly one point P (distinct from A and B) on the minor arc AB of the circumcircle of $\triangle ABC$ such that PA, PB, PC are all of integral length.

2419. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Find all solutions to the alphametic:

	M	I	X	·	E
+	D	B	A	·	S
	E	S	U	·	M

1. The letters before the decimal points represent base ten digits, and addition is done in that base.
2. The letters after the decimal points represent base six digits, and addition is done in that base.
3. The same letter stands for the same digit, distinct letters stand for distinct digits, and initial digits are non-zero.

Readers familiar with cricket will realize that this is a real world problem! [*Ed.* Readers not familiar with cricket may be interested to learn that an ‘over’ consists of six ‘deliveries’!!]

2420. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that x, y and z are integers. Solve the equation:

$$x^2 + y^2 = 2420z^2 .$$

2421. *Proposed by Ice B. Risteski, Skopje, Macedonia.*

What is the probability that the k numbers in the Las Vegas lottery on a given payout day do not include two consecutive integers? (The winning numbers are an unordered random choice of k distinct integers from 1 to n, where $n > k$.)

2422*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, C be the angles of an arbitrary triangle. Prove or disprove that

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C} \;\geq\; \frac{9\sqrt{3}}{2\pi\left(\sin A\sin B\sin C\right)^{1/3}}\,.$$

2423. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x_1, x_2, \ldots, x_n > 0$ be real numbers such that $x_1+x_2+\ldots+x_n = 1$, where $n > 2$ is a natural number. Prove that

$$\prod_{k=1}^{n}\left(1+\frac{1}{x_k}\right) \;\geq\; \prod_{k=1}^{n}\left(\frac{n-x_k}{1-x_k}\right)\,.$$

Determine the cases of equality.

2424. *Proposed by K.R.S. Sastry, Dodballapur, India.*

In $\triangle ABC$, suppose that I is the incentre and BE is the bisector of $\angle ABC$, with E on AC. Suppose that P is on AB and Q on AC such that PIQ is parallel to BC. Prove that $BE = PQ$ if and only if $\angle ABC = 2\angle ACB$.

2425. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Suppose that D is the foot of the altitude from vertex A of an acute-angled Heronian triangle ABC (that is, one having integer sides and area). Suppose that the greatest common divisor of the side lengths is 1. Find the smallest possible value of the side length BC, given that $BD - DC = 6$.

NOTE OF THANKS

In the December 1998 issue of ***CRUX with MAYHEM*** [1998: 538] reference was made to whether a copy of the book *Exercises de Géométrie* by F. Gabriel-Marie was available anywhere in Canada.

Dr. Kenneth Williams, Department of Mathematics and Statistics, Carleton University, has donated a copy to the Canadian Mathematical Society for the use of the Editors of ***CRUX with MAYHEM***. We are very grateful to Dr. Williams for his generous gift. Having ready access to this book will be of considerable assistance to the members of the Editorial Board of ***CRUX with MAYHEM***.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2255. [1997: 300; 1998: 378] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that $|\angle PAB - \angle PAC| \geq |\angle PBC - \angle PCB|$.

The published solution [1998: 378-379] is incorrect. More precisely, that solution deals with an inequality that is much simpler than what is given; among other things, our given inequality does not extend to arbitrary isosceles triangles. We thank Toshio Seimiya for these remarks. Here is his solution.

II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let M be the mid-point of BC. Then AM is the perpendicular bisector of BC, and if P lies on it, the given relation holds with both sides zero. We may therefore assume, without loss of generality, that P is an interior point of $\triangle ABM$. We then have $\angle PAB < \angle PAC$ and $\angle PBC > \angle PCB$, so that $|\angle PAB - \angle PAC| = \angle PAC - \angle PAB$ and $|\angle PBC - \angle PCB| = \angle PBC - \angle PCB$. Thus the relation we wish to prove reduces to

$$\angle PAC - \angle PAB \;>\; \angle PBC - \angle PCB\,. \tag{1}$$

Let Q be the reflection of P in the line AM; then $\angle PAB = \angle QAC$ and $\angle PCB = \angle QBC$. Thus

$$\angle PAC - \angle PAB \;=\; \angle PAC - \angle QAC \;=\; \angle PAQ\,,$$

$$\text{and} \qquad \angle PBC - \angle PCB \;=\; \angle PBC - \angle QBC \;=\; \angle PBQ\,,$$

so that (1) becomes

$$\angle PAQ \;>\; \angle PBQ\,. \tag{2}$$

Since $PQ \perp AM$ and $AM \perp BC$ we get $PQ \| BC$. Let PQ meet AB and AC at R and S respectively, and let T be the reflection of B in RS. Then

$$\angle PTQ \;=\; \angle PBQ\,. \tag{3}$$

Since AM is the perpendicular bisector of both PQ and RS, the circumcentres of $\triangle APQ$ and $\triangle ARS$ lie on AM, so that the circumcircle of $\angle ABC$ is tangent at A to the circumcircles of both $\triangle ARS$ and $\triangle APQ$, which we denote by Γ and Γ' respectively. Note that Γ' is contained in Γ. Because $\angle TRA = \angle TRQ - \angle ARQ = \angle BRQ - \angle ARQ = 120° - 60° = 60°$, while $\angle ASR = 60°$, it follows that $\angle TRA = \angle ASR$. Hence RT is tangent to Γ so that T is a point outside Γ and, consequently, T is a point outside Γ'. Since A and T are on the same side of PQ, we have

$$\angle PAQ > \angle PTQ . \tag{4}$$

The desired relation (2) is a consequence of (3) and (4).

Summary of Seimiya's further comments.

The above argument extends to triangles for which $\angle B = \angle C \leq 60°$; that is $|\angle PAB - \angle PAC| \geq |\angle PBC - \angle PCB|$ also for isosceles triangles when $\angle A \geq 60°$. On the other hand, when $\angle B = \angle C > 60°$ one can find positions for P where the given inequality fails to hold.

2309. [1998: 47] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that ABC is a triangle and that P is a point of the circumcircle, distinct from A, B and C. Denote by S_A the circle with centre A and radius AP. Define S_B and S_C similarly. Suppose that S_A and S_B intersect at P and P_C. Define P_B and P_A similarly.

Prove that P_A, P_B and P_C are collinear.

Solution by 12 of the 17 solvers; the notation of our featured solution is by Florian Herzig, student, Cambridge, UK.

Since S_A and S_B are symmetric with respect to their line of centres AB, their intersections P and P_C are also symmetric with respect to this line. Let Q_C be the mid-point of PP_C, etc. By the above conclusion, Q_C is the foot of the perpendicular from P onto AB. The points Q_A, Q_B and Q_C are collinear since they determine the Simson [or Wallace] line (of P with respect to $\triangle ABC$). A dilatation with centre P and factor 2 takes Q_A to P_A, etc. Hence P_A, P_B and P_C are collinear as well [in a line parallel to the Simson line].

Also solved by MICHEL BATAILLE, Rouen, France; NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANSUR BOASE, student, Cambridge, England; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; JOEL SCHLOSBERG, student, Bayside, NY, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PARAYIOU THEKLITOS, Limassol, Cyprus; G. TSINTSIFAS, Thessaloniki, Greece; JOHN VLACHAKIS, Athens, Greece; and the proposer.

Schlosberg found the problem in a slightly different form on page 43 of Ross Honsberger's Episodes in 19th and 20th Century Euclidean Geometry. *Bellot informs us that the line $P_AP_BP_C$ is called the* Steiner line *of P with respect to $\triangle ABC$; its properties are discussed in Y. and R. Sortais,* La géométrie du triangle, Exercises résolus, *Hermann, Paris, 1987. Both he and Seimiya report that the line passes through the orthocentre of $\triangle ABC$.*

2312. [1998: 47] *Proposed by K.R.S. Sastry, Dodballapur, India.*

The r^{th} n–gonal number is given by $P(n,r) = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2}$, where $n \geq 3$, $r = 1, 2, \ldots$.

Prove that, in the interval $[P(n,r), P(n,r+1)]$, there is an $(n-1)$–gonal number.

Solution by Florian Herzig, student, Cambridge, UK.

Assume that for some $n \geq 3$ and r, $s \geq 1$,

$$P(n-1,s) < P(n,r) < P(n,r+1) < P(n-1,s+1)\,. \qquad (1)$$

Therefore $P(n-1,s+1) - P(n-1,s) > P(n,r+1) - P(n,r)$.

Now

$$\begin{aligned} P(n,r+1) - P(n,r) &= (n-2)\left(\frac{(r+1)^2 - r^2}{2}\right) - (n-4)\frac{1}{2} \\ &= r(n-2) + 1\,. \end{aligned}$$

Similarly, $P(n-1,s+1) - P(n-1,s) = s(n-3) + 1$.

Hence, from the last inequality above, $s(n-3)+1 > r(n-2)+1$, and thus $n \geq 4$ and $s > \dfrac{r(n-2)}{n-3}$.

Using the fact that $P(n-1,s)$, as a real function in s, is strictly increasing for all $s \geq 1$ (verified by differentiation), we have

$$\begin{aligned} P(n-1,s) &> P\left(n-1, \frac{r(n-2)}{n-3}\right) = \frac{(n-2)^2}{(n-3)}\frac{r^2}{2} - \frac{(n-2)(n-5)}{(n-3)}\frac{r}{2} \\ &\geq P(n,r), \end{aligned}$$

since $\dfrac{(n-2)^2}{n-3} \geq n-2$ and $\dfrac{(n-2)(n-5)}{n-3} \leq n-4$ .This contradicts (1) and hence it follows that in $[P(n,r), P(n,r+1)]$ there is an $(n-1)$–gonal number.

Also solved by MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer.

It is worth noting that all seven solutions involved different methods. Boase showed that $P\left(n-1, \left\lceil\sqrt{\frac{n-2}{n-3}}\,\right\rceil\right)$ always lies in the interval $[P(n,r), P(n,r+1)]$. Janous and others noted that the involvement of $(n-1)$–gonal numbers implies that $n-1 \geq 3$; that is, $n \geq 4$. Referring to the geometry involved, the proposer noted that the problem can be reformulated: Prove that there are just one or just two $(n-1)$–gonal numbers in $[P(n,r), P(n,r+1)]$.

2313. [1998: 47] *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let N be a non-negative integer and let a and b be complex numbers with a, $b \notin \{0, -1, -2, \ldots, -(n-1)\}$. Find a closed form expression for

$$\sum_{k=0}^{n} \frac{(-1)^k}{(a)_k\,(b)_{n-k}}\,,$$

where $(a)_k$ denotes the Pochhammer symbol, defined by $(a)_0 = 1$, $(a)_k = a(a+1)\ldots(a+k-1)$, $k \in \mathbb{N}$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

If $S(a,b,n)$ denotes the sought sum, we show by induction on n that

$$S(a,b,n) \;=\; \frac{(a-1)_{n+1} + (-1)^n(b-1)_{n+1}}{(a+b+n-2)(a)_n(b)_n}\,.$$

Indeed, for $n = 1$ it is true since

$$S(a,b,1) \;=\; \frac{1}{b} - \frac{1}{a} \;=\; \frac{(a-b)(a+b-1)}{(a+b-1)ab} \;=\; \frac{(a-1)a-(b-1)b}{(a+b-1)ab}\,.$$

The induction step uses the facts

$$c(c+1)_t \;=\; (c)_{t+1} \qquad \text{and} \qquad (c+t)(c)_t \;=\; (c)_{t+1}$$

for all c and t, and the observation

$$\begin{aligned}
&S(a,b,n+1)\\
&\quad= \sum_{k=0}^{n+1} \frac{(-1)^k}{(a)_k(b)_{n+1-k}} \;=\; \frac{1}{b}\sum_{k=0}^{n}\frac{(-1)^k}{(a)_k(b+1)_{n-k}} + \frac{(-1)^{n+1}}{(a)_{n+1}}\\
&\quad= \frac{1}{b}S(a,b+1,n) + \frac{(-1)^{n+1}}{(a)_{n+1}}\\
&\quad= \frac{1}{b}\left(\frac{(a-1)_{n+1}+(-1)^n(b)_{n+1}}{(a+b+n-1)(a)_n(b+1)_n}\right) + \frac{(-1)^{n+1}}{(a)_{n+1}}\\
&\quad= \frac{(a-1)_{n+2}+(-1)^n(b)_{n+1}(a+n)+(-1)^{n+1}(a+b+n-1)(b)_{n+1}}{(a+b+n-1)(a)_{n+1}(b)_{n+1}}\\
&\quad= \frac{(a-1)_{n+2}+(-1)^{n+1}(b-1)_{n+2}}{(a+b+n-1)(a)_{n+1}(b)_{n+1}}\,,
\end{aligned}$$

as required.

Also solved by KEE-WAI LAU, Hong Kong; and the proposer. One other reader submitted a solution which was not in closed form.

Lau notes that $a+b+n-2 \neq 0$ is required for this problem, and that if $a+b+n-2 = 0$ then the given expression likely does not have a closed form.

The proposer gives a similar expression for $\sum_{k=0}^{n}(-1)^k(a)_k(b)_{n-k}$, *and lists some combinatorial identities as special cases of these two results. For instance, from the formula for* $S(1/2, 1/2, 2n)$ *and the observation that* $(1/2)_k = 4^{-k}(2k)!/k!$ *he obtains*

$$\sum_{k=0}^{2n} \frac{(-1)^k\binom{4n}{2k}}{\binom{2n}{k}} = \frac{1}{1-2n};$$

and as a special case of his formula for $\sum_{k=0}^{n}(-1)^k(a)_k(b)_{n-k}$ *he gets*

$$\sum_{k=0}^{2n} \frac{(-1)^k}{\binom{2n}{k}} = \frac{2n+1}{n+1}.$$

2314. [1998: 107] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $AB < AC$. The bisectors of angles B and C meet AC and AB at D and E respectively, and DE intersects BC at F.

Suppose that $\angle DFC = \frac{1}{2}(\angle DBC - \angle ECB)$. Determine angle A.

Solution by Florian Herzig, student, Cambridge, UK.

First we show that $DI = IE$ where I is the incentre. We have

$$\begin{aligned} \frac{1}{2}\angle DIC &= \frac{1}{2}(\angle DBC + \angle ECB) = \angle DFC + \angle ECB \\ &= \angle ECF + \angle EFC = \angle DEC. \end{aligned}$$

Since $\angle DIC = \angle DEC + \angle EDB$, it follows that $DI = EI$. Now by the Sine Law:

$$\frac{AI}{\sin\angle ADI} = \frac{ID}{\sin\frac{A}{2}} = \frac{IE}{\sin\frac{A}{2}} = \frac{AI}{\sin\angle AEI}$$

and hence either $\angle ADI = \angle AEI$ or $\angle ADI + \angle AEI = 180^\circ$. From the first of these it easily follows that the angles B and C are equal, a contradiction. From the second it follows that the quadrilateral $AEID$ is cyclic. Hence

$$180^\circ = A + \angle EID = A + \left(90^\circ + \frac{A}{2}\right),$$

and hence $A = 60^\circ$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; D.J. SMEENK, Zaltbommel, the Netherlands; PARAYIOU THEOKLITOS, Limassol, Cyprus; and the proposer. There were two incorrect solutions.

2315. [1998: 107] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Prove or disprove that $F(n) = \sqrt{n\left(1-\frac{1}{n}\right)^{n-1}}$, where $F(n)$ is the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sin x_1 \cos x_2 \ldots \cos x_n + \cos x_1 \sin x_2 \ldots \cos x_n$$

$$+ \ldots + \cos x_1 \cos x_2 \ldots \sin x_n ,$$

$x_k \in [0, \pi/2]$, $k = 1, 2, \ldots, n$, and $n > 1$ is a natural number.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

Let $n \geq 2$ and let

$$C(n) = \frac{n-1}{\sqrt[n-1]{n^{n-2}}}.$$

Then [1998: 119] if $a_1, \ldots, a_n \geq 0$,

$$\sum_{k=1}^{n} \sqrt{a_k} \leq \prod_{k=1}^{n} \sqrt{a_k + C(n)}.$$

Replacing a_k by $C(n)b_k$ and noting that $C(n)^{(n-1)/2} = F(n)$ give

$$\sum_{k=1}^{n} \sqrt{b_k} \leq F(n) \prod_{k=1}^{n} \sqrt{b_k + 1}, \qquad (1)$$

valid for all $b_1, \ldots, b_n \geq 0$.

If $x_1, \ldots, x_n \in [0, \pi/2)$, let $b_k = \tan^2 x_k$. Since $\sqrt{b_k + 1} = \sec x_k$, we obtain, from (1), after multiplying by $\prod_{j=1}^{n} \cos x_j$,

$$\sum_{k=1}^{n} \sin x_k \prod_{j=1, j\neq k}^{n} \cos x_j \leq F(n).$$

Clearly, the latter inequality remains valid if $x_k = \pi/2$ for some k. Since there is equality if $x_k = \arcsin(1/\sqrt{n})$ for all k, it follows that $F(n)$, in fact, is the maximum value of the considered expression.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; and PANOS E. TSAOUSSOGLOU, Athens, Greece.

The problem was inspired by problem 2214. *Janous's solution also made use of* 2214.

2316. [1998: 108] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with angles B and C satisfying $C = 90^\circ + \frac{1}{2}B$. Suppose that M is the mid-point of BC, and that the circle with centre A and radius AM meets BC again at D. Prove that $MD = AB$.

Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.

Let E be the other point where the circle centred at A with radius AC meets line BC. Then $\angle AEC = \angle ACE$, so that $\angle AED = \angle ACM$. Also, $AD = AM$ and $\angle ADE = \angle AMC$. Therefore, $\triangle ADE \cong \triangle AMC$. Hence, $DE = CM$. But $CM = MB$; therefore, $DE = MB$, so that $MD = BE$.

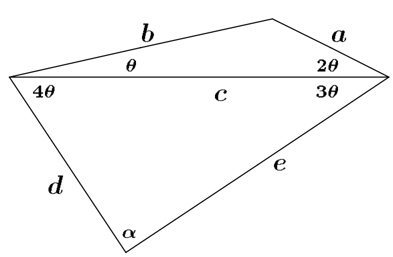
Note that $\angle AEC = \angle ACE = 90^\circ - \frac{1}{2}\angle B$. Hence,

$$\angle BAE \;=\; 180^\circ - \left(\angle B + 90^\circ - \frac{1}{2}\angle B\right) \;=\; 90^\circ - \frac{1}{2}\angle B\,.$$

Thus, $AB = BE$. Therefore, $MD = AB$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; RON SHEPLER, Ferris State University, Big Rapids, Michigan, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PARAYIOU THEOKLITOS, Limassol, Cyprus; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

2317. [1998: 108] *Proposed by Richard I. Hess, Rancho Palos Verdes, California, USA.*



The quadrilateral shown at the left has integer elements a through e. The angles as shown are integer multiples of the smallest.

(a) What is the smallest possible value of c?

(b) What is the smallest possible value of c if α must be obtuse?

Solution by Michael Lambrou, University of Crete, Crete, Greece.

We show that the answer to (a) is $c = 5 \cdot 17 \cdot 29 \cdot 1049 = 2585785$ and to (b) it is $c = 19 \cdot 41 \cdot 59 \cdot 139 \cdot 50539 = 322872394081$.

(a) From the Law of Sines on the upper triangle we have

$$\frac{a}{\sin\theta} = \frac{b}{\sin 2\theta} = \frac{c}{\sin 3\theta}$$

and as $\sin 2\theta = 2\sin\theta\cos\theta$ and $\sin 3\theta = (4\cos^2\theta - 1)\sin\theta$, this reduces to

$$a = \frac{b}{2\cos\theta} = \frac{c}{4\cos^2\theta - 1}.$$

Eliminating $\cos\theta$ we find $\cos\theta = b/2a$ and so

$$a^2 + ac - b^2 = 0. \tag{1}$$

As a quadratic in a this last must have discriminant a perfect square, say x^2. That is,

$$c^2 + (2b)^2 = x^2. \tag{2}$$

Hence there exist s, t, $m \in \mathbb{N}$ with $(s,t) = 1$ and

$$c = (s^2 - t^2)m, \qquad 2b = 2stm, \qquad x = (s^2 + t^2)m. \tag{3}$$

(Note that if c is odd this is certainly so, but when c is even we may have the dual relations:

$$2b = (s^2 - t^2)m, \qquad c = 2stm, \qquad x = (s^2 + t^2)m.$$

However, this amounts to the same thing because then, by (2), x is even; so either m is even or $s^2 + t^2$ is even. In the first case set $m = 2M$, $s_1 = s + t$, $t_1 = s - t$ and we then have $c = (s_1^2 - t_1^2)M$, $b = s_1 t_1 M$, $x = (s_1^2 + t_1^2)M$ which is the same as (3). If, instead, m is odd, so $s^2 + t^2$ is even, we see that s, t are of the same parity; set $s_1 = (s+t)/2$, $t_1 = (s-t)/2$, $M = 2m$, and we recapture (3), since s_1, t_1, $M \in \mathbb{N}$).

From (1) (keeping the positive sign only) we have

$$a = \frac{-c + \sqrt{x^2}}{2} = t^2 m, \quad \text{so} \quad \cos\theta = \frac{b}{2a} = \frac{s}{2t}.$$

From the lower triangle we have $0 < 3\theta + 4\theta < 180°$; that is

$$1 > \cos\theta > \cos\tfrac{\pi}{7} \approx 0.90096887 > 0.9.$$

Using $\cos\theta = s/2t$ we have

$$1.8t < 2t\cos\tfrac{\pi}{7} < s < 2t. \tag{4}$$

In particular t cannot be 1, 2, 3, 4, or 5, as there is no integer s in the open interval $\left(2t\cos\frac{\pi}{7}, 2t\right)$. The least allowable t is $t = 6$ (whence $s = 11$). For the record the next few allowable pairs are $(t,s) = (7,13)$, $(8,15)$, $(9,17)$.

From the Law of Sines on the lower triangle we have

$$\frac{d}{\sin 3\theta} = \frac{e}{\sin 4\theta} = \frac{c}{\sin 7\theta}.$$

Hence $d \sin 4\theta = e \sin 3\theta$ or

$$d(8\cos^3\theta - 4\cos\theta)\sin\theta = e(4\cos^2\theta - 1)\sin\theta.$$

Using $\cos\theta = s/2t$ we get

$$ds(s^2 - 2t^2) = et(s^2 - t^2). \tag{5}$$

Recall that $(s, t) = 1$. We show that we also have

$$(s, s^2 - t^2) = (t, s^2 - 2t^2) = (s^2 - 2t^2, s^2 - t^2) = 1.$$

Indeed, if p is a prime and $p|s$, $p|s^2 - t^2$, then $p|t^2$, so $p|t$. Thus $p|(s, t) = 1$, showing that $(s, s^2 - t^2) = 1$. Similarly $(t, s^2 - 2t^2) = 1$, and if $p|s^2 - 2t^2$, $p|s^2 - t^2$, then $p|s^2 - t^2 - (s^2 - 2t^2) = t^2$, so $p|t$, etc., as before.

These conditions of relative primality applied to (5) show that there exists a constant $\lambda \in \mathbb{N}$ such that

$$d = \lambda t(s^2 - t^2) \quad \text{and} \quad e = \lambda s(s^2 - 2t^2). \tag{6}$$

Now

$$c = \frac{d \sin 7\theta}{\sin 3\theta} = \frac{d(64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1)\sin\theta}{(4\cos^2\theta - 1)\sin\theta}$$
$$= \frac{\lambda(s^6 - 5s^4t^2 + 6s^2t^4 - t^6)}{t^3} \quad \text{by (6)}.$$

From (3) we have

$$mt^3(s^2 - t^2) = \lambda(s^6 - 5s^4t^2 + 6s^2t^4 - t^6). \tag{7}$$

Observe that

$$(t, s^6 - 5s^4t^2 + 6s^2t^4 - t^6) = 1 = (s^2 - t^2, s^6 - 5s^4t^2 + 6s^2t^4 - t^6).$$

Indeed the first, as above, is clear. For the second, note that

$$s^6 - 5s^4t^2 + 6s^2t^4 - t^6 = (s^2 - t^2)(s^4 - 4s^2t^2 + 2t^4) + t^6,$$

so any prime dividing both terms must divide $s^2 - t^2$ and t^6 (and so t), which is impossible, as above. We conclude from (7) that there is a $\tau \in \mathbb{N}$ such that

$$m = (s^6 - 5s^4t^2 + 6s^2t^4 - t^6)\tau, \qquad \lambda = t^3(s^2 - t^2)\tau,$$

and hence

$$\begin{aligned} c &= (s^2-t^2)m = (s^2-t^2)(s^6-5s^4t^2+6s^2t^4-t^6)\tau \\ &= (s^2-t^2)(s^3+s^2t-2st^2-t^3)(s^3-s^2t-2st^2+t^3)\tau\,. \end{aligned}$$

For the least c, we clearly need $\tau = 1$ (any solution for (s, t, τ) is larger than for $(s, t, 1)$). We shall show that the least c comes from $t = 6$, $s = 11$ (and $\tau = 1$).

Denote by $c(t, s)$ the value of c at the pair (t, s) (and $\tau = 1$). For $6 \le t \le 12$ the relatively prime pairs (t, s) satisfying (4) are $(6, 11)$, $(7, 13)$, $(8, 15)$, $(9, 17)$, $(10, 19)$, $(11, 20)$, $(11, 21)$, and $(12, 23)$. They give $c(6, 11) = 2585785$ (the least), $c(7, 13) \approx 1.7 \times 10^7$, $c(8, 15) \approx 7.4 \times 10^7$, $c(9, 17) \approx 2.4 \times 10^8$, $c(10, 19) \approx 6.6 \times 10^8$, $c(11, 20) \approx 1.6 \times 10^8$, $c(11, 21) \approx 1.6 \times 10^9$, and $c(12, 23) \approx 3.6 \times 10^9$. We now show that for $t \ge 13$ we still get larger values than $c(6, 11)$ by eliminating c as follows. By (4) we have for the factors of c:

$$\begin{aligned} s^2-t^2 &\ge (1.8t)^2-t^2 = 2.24t^2\,; \\ s^3+s^2t-2st^2-t^3 &\ge (1.8t)^3+(1.8t)^2-2(2t)t^2-t^3 \\ &= 4.072t^3\,. \end{aligned}$$

The other factor must be positive (since c, and thus the product of the remaining two factors, is positive). As this factor is an integer, it must be at least 1. Hence

$$c \ge (2.24t^2)(4.072t^3)1 > 9t^5\,.$$

Hence if $t \ge 13$ we have $c > 9 \cdot 13^5 \approx 3.3 \times 10^6 > c(6, 11)$.

To conclude part (a) the least value of c is $c(6, 11) = 2585785$.

(b) If we insist that α is obtuse, we have equivalently $3\theta + 4\theta < 90°$; that is, $\theta < \frac{\pi}{14}$. So instead of (4) we have the sharper requirements:

$$\frac{s}{2t} = \cos\theta > \cos\tfrac{\pi}{14} \approx 0.9749279\,;$$

that is,

$$(1.949)t < 2t\cos\tfrac{\pi}{14} < s < 2t\,. \tag{8}$$

It follows that $t \ge 20$, as for $1 \le t \le 19$ there is no $s \in \mathbb{N}$ in the open interval $(1.949t, 2t)$. The least t is $t = 20$ with corresponding $s = 39$ (and the next few pairs are $(21, 41)$, $(22, 43)$, $(23, 45)$, …). We show that the least value of c is $c(20, 39) = 19 \cdot 41 \cdot 59 \cdot 139 \cdot 50539 \approx 3.2 \times 10^{11}$. Certainly, $c(20, 39) < c(21, 41)$ as the expression for c gives $c(21, 41) = 1240 \cdot 58799 \cdot 6719 \approx 4.89 \times 10^{11}$. Moreover, for larger t we use estimates arising from (8), namely

$$\begin{aligned} s^2 - t^2 &> (1.949t)^2 - t^2 > 2.798t^2\,; \\ s^3 + s^2t - 2st^2 - t^3 &> (1.949t)^3 + (1.949t)^2t - 2(2t)t^2 - t^3 \\ &> 6.202t^3\,; \\ s^3 - s^2t - 2st^2 + t^3 &> (1.949t)^3 - (2t)^2t - 2(2t)t^2 + t^3 \\ &> 0.4034t^3\,. \end{aligned}$$

Multiplying together we find $c > 7t^8$. Thus for $t \geq 22$ we have $c > 7 \cdot 22^8 \approx 3.8 \times 10^{11} > c(20, 39)$. This concludes the proof.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

2318. [1998: 108] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Suppose that ABC is a triangle with circumcentre O and circumradius R.

Consider the bisector (ℓ) of any side (say AC), and let P (the "pedal point") be any point on ℓ inside the circumcircle.

Let K, L, M denote the feet of the perpendiculars from P to the lines AB, BC, CA respectively.

Show that $[KLM]$ (the area of the pedal triangle KLM) is a decreasing function of $\rho = \overline{OP}$, $\rho \in (0, R)$.

Combination of the solutions by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; Michael Lambrou, University of Crete, Crete, Greece; Gerry Leversha, St. Paul's School, London, England; and Toshio Seimiya, Kawasaki, Japan.

Either by referring to problem 2236, or R.A. Johnson's *Advanced Euclidean Geometry*, Dover, 1960, theorem 198, page 139, it is known that

$$[KLM] = \left(\frac{R^2 - OP^2}{4R^2}\right) [ABC]\,.$$

Thus, it follows that $[KLM]$ is a decreasing function of ρ.

Also solved (in full) by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; PARAYIOU THEOKLITOS, Limassol, Cyprus; and the proposer.

Bellot Rosado and Leversha note that there is no need for P to lie on a perpendicular bisector of a side.

Dergiades and Janous observe that if $d = R$, then $[KLM] = 0$, and so K, L, M lie on the Simson line.

Dergiades recalls the result as being known as **Steggals' Theorem**. *He observes that if* $[ABC] = E$ *and if* E_0, E_1, E_2 *and* E_3 *are the areas of the triangle* $[KLM]$ *when* P *is the incentre, and the three excentres respectively, then we can deduce, from Euler's Theorem, that*

$$d^2 = R(R-2r) \qquad d_1^2 = R(R-2r_1) \qquad d_2^2 = R(R-2r_2) \qquad d_3^2 = R(R-2r_3)\,.$$

Steggal's Theorem now gives

$$\frac{E_0}{E} = \frac{r}{2R}\,, \qquad \frac{E_1}{E} = \frac{r_1}{2R}\,, \qquad \frac{E_2}{E} = \frac{r_2}{2R}\,, \qquad \frac{E_3}{E} = \frac{r_3}{2R}\,,$$

and since $r_1 + r_2 + r_3 = r + 4R$ *(Bobillier's Theorem), we have*

$$E_1 + E_2 + E_3 = E_0 + 2E\,.$$

2319. [1998: 108] *Proposed by Florian Herzig, student, Cambridge, UK.*

Suppose that UV is a diameter of a semicircle, and that P, Q are two points on the semicircle with $UP < UQ$. The tangents to the semicircle at P and Q meet at R. Suppose that S is the point of intersection of UP and VQ.

Prove that RS is perpendicular to UV.

Editorial note. Some solvers sent in more than one correct solution. They are indicated by a † after their names.

I. *Solution by Diane and Roy Dowling, University of Manitoba, Winnipeg, Manitoba.*

The problem may be generalized slightly as follows: UV is a diameter of a circle; P and Q are distinct points on the circle; PQ is not a diameter; $P \neq U$ and $Q \neq V$; the tangents to the circle at P and Q meet at R; the lines UP and VQ meet at S; prove that RS is perpendicular to UV. (The given conditions ensure that the points of intersection R and S exist, are unique and distinct from each other.)

Choose coordinate axes and scale so that the origin O is the centre of the circle, $V = (1, 0)$ and consequently $U = (-1, 0)$. Let $P = (a, b)$; then $a^2 + b^2 = 1$. Let $Q = (c, d)$; then $c^2 + d^2 = 1$. Let $R = (x_1, y_1)$ and $S = (x_2, y_2)$.

The equation of PR is $ax + by = 1$ and the equation of QR is $cx + dy = 1$. Since PR and QR intersect at exactly one point, the determinant of the coefficients of this system is non-zero. Solving the system for x we get the x–coordinate of R:

$$x_1 = \frac{d-b}{ad-bc}\,.$$

The equation of UP is $bx - (a+1)y = -b$ and the equation of VQ is $dx - (c-1)y = d$. Since UP and VQ intersect at exactly one point, namely S, the determinant of the coefficients of this system is also non-zero. Solving the system for x we get the x–coordinate of S:

$$x_2 = \frac{ad+bc+d-b}{ad-bc+d+b}\,.$$

Therefore

$$\begin{aligned} x_1 - x_2 &= \frac{d-b}{ad-bc} - \frac{ad+bc+d-b}{ad-bc+d+b} = \frac{d^2(1-a^2)-b^2(1-c^2)}{(ad-bc)(ad-bc+d+b)} \\ &= \frac{d^2b^2-b^2d^2}{(ad-bc)(ad-bc+d+b)} = 0\,. \end{aligned}$$

It follows that RS is perpendicular to UV.

II. *Solution by Toshio Seimiya †, Kawasaki, Japan.*

Let T be the intersection of UQ and VP, and let M be the mid-point of ST. Since $\angle UPV = \angle UQV = 90^\circ$, T is the orthocentre of triangle SUV, so that $ST \perp UV$. Hence we have $\angle PTS = \angle PUV$. Since $\angle SPT = 90^\circ$ and M is the mid-point of ST we get $MP = MT$. Thus

$$\angle MPV = \angle MPT = \angle MTP = \angle PTS = \angle PUV\,.$$

Hence PM is tangent to the semicircle. Similarly QM is tangent to the semicircle. Therefore M coincides with R. As $ST \perp UV$, we have $RS \perp UV$.

III. *Solution by Keivan Mallahi †, student, Sharif University of Technology, Tehran, Iran.*

Let T be the intersection of UQ and PV. Consider the hexagon $QQUPPV$. Note that all of its vertices lie on the semicircle. Applying Pascal's Hexagon Theorem we see that the following points are collinear:

$$QQ \cap PP = R\,, \qquad QU \cap PV = T\,, \qquad UP \cap VQ = S$$

(where QQ and PP are intended to be the tangents at Q and P, respectively). Thus it is sufficient to show that the line ST is perpendicular to UV. To this end note that $\angle UQV = \angle VPU = 90^\circ$, so the lines UQ and PV are the altitudes of the triangle SUV. Since the altitudes of a triangle are concurrent, ST must be the third altitude, which completes the proof.

IV. *Solution by Michael Lambrou †, University of Crete, Crete, Greece.*

Consider the nine-point (Feuerbach) circle of triangle UVS. This passes through P and Q (as P and Q are the feet of the perpendiculars from V and U, respectively, because $\angle UPV = \angle UQV = 90^\circ$, being angles on the semicircle). If PV, QU meet at T, the orthocentre, then ST is also an altitude meeting UV at D, say. Note that the nine-point circle also passes through D, through the mid-point M of ST, and through the mid-point O of UV (so O is the centre of the semicircle on UV). Moreover, MO is a diameter of the nine-point circle (as $\angle MDO = 90^\circ$). Thus we also have $PM \perp PO$. But OP is a radius of the semicircle on UV, so PM, being perpendicular to PO at its endpoint P, is a tangent to the semicircle. Similarly QM is a tangent. In other words the point M is where these two tangents meet. Thus M and R coincide and clearly SR (being the same as SM) is an altitude of SUV. This conclude the proof that $SR \perp UV$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; STERGIOU HARAFAPOS †, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MASOUD KAMGARPOUR, Carson Graham Secondary School, North Vancouver, British Columbia; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; PARAYIOU THEOKLITOS, Limassol, Cyprus; GEORGE TSAPAKIDIS, Agrinio, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer †. There was one incorrect solution submitted.

Although four methods of solving are given above, this was not exhaustive. Bataille used inversion and Theoklitos used radical axes in their proofs.

2320. [1998: 108] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Two circles on the same side of the line ℓ are tangent to it at D. The tangents to the smaller circle from a variable point A on the larger circle intersect ℓ at B and C. If b and c are the radii of the incircles of triangles ABD and ACD, prove that $b + c$ is independent of the choice of A.

Solution by Florian Herzig, student, Cambridge, UK.

Let m be the parallel to ℓ which is also tangent to the smaller circle. We will show that

> $b + c$ is constant if A is a point outside the region between ℓ and m [so that D is between B and C], and $b : c$ is constant if A is between these lines.

Denote the larger circle by $C_1(M; R)$ and the smaller circle by $C_2(N; r)$. Let $S \in AB$ and $T \in AC$ be the points of contact with C_2. Also let I_B and I_C be the incentres of triangles ABD and ACD. U and V are the feet of the perpendiculars from I_B and I_C to ℓ. We will now calculate the ratio $AS : AD$ which is constant in both cases: let P be the second intersection of C_2 and AD. Then $AP \cdot AD = AS^2$. Also by similarity $PD : AD = r : R$. Hence

$$AS^2 \;=\; AP \cdot AD \;=\; AD^2 \left(1 - \frac{r}{R}\right) ,$$

and thus the ratio $AS : AD$ is constant for all positions of A.

If A is not between ℓ and m, then

$$DU \;=\; \frac{BD + AD - AB}{2} \;=\; \frac{AD - AS}{2} \;=\; \frac{CD + AD - AC}{2} \;=\; DV$$

(because $BD = BS$, $CD = CT$ and $AS = AT$). Therefore the two incircles touch line AD at the same point, call it Y. It also follows that

$I_B I_C \perp AD$. Moreover DN is the middle parallel between $I_B U$ and $I_C V$, whence $2DX = b + c$, where $X = DN \bigcap I_B I_C$. If L is the mid-point of AD, then $\triangle DLM \sim \triangle DYX$ so that $DX : DM = DY : DL$.

Using $2DX = b + c$, $DM = R$, and $2DY = 2DU = AD - AS$ it follows that

$$b + c = \frac{2R(AD - AS)}{AD} = 2R\left(1 - \sqrt{1 - \frac{r}{R}}\right) = 2(R - \sqrt{R^2 - Rr})$$

is constant.

Otherwise, if A is between lines ℓ and m assume, without loss of generality, that $DB > DC$. Then $2DV = AD - AS$ as before, but

$$DU = \frac{AD + BD - AB}{2} = \frac{AD + AS}{2}.$$

Therefore $b : c = DU : DV = \dfrac{AD + AS}{AD - AS} = \dfrac{R + \sqrt{R^2 - Rr}}{R - \sqrt{R^2 - Rr}}$ is constant.

Finally, note that all the calculations of the first case remain valid (mutatis mutandis) if instead of the incircle in the second case we take the excircle opposite B [so that b is the radius of the excircle of $\triangle ABD$ that touches the side AD].

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; PARAYIOU THEOKLITOS, Limassol, Cyprus; HOE TECK WEE, Singapore; and the proposer.

Bellot found the problem in the Japanese Encyclopedia of Geometry, *volume 4, page 261, which provided the reference* Journal de Mathématiques Élémentaires **34** (*1909*), *problem 7000. Several readers noted that the statement of our problem was not entirely correct, but Seimiya was the only solver besides Herzig to provide a correct alternative using an excircle.*

2323. [1998: 109] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Determine a positive constant c so that the Diophantine equation

$$uv^2 - v^2 - uv - u = c$$

has exactly four solutions in positive integers u and v.

Solution by Digby Smith, Mount Royal College, Calgary, Alberta.

Since $u = -1 - c < 0$ when $v = 1$, we have $v \geq 2$. Then

$$v^2 - v - 1 = (v-2)^2 + 3(v-2) + 1 > 0$$

and

$$u = \frac{v^2 + c}{v^2 - v - 1} > 0 \qquad (1)$$

When $c = 61$, one verifies easily that (1) admits no positive integer solutions for u if $v = 5, 6, 7, 8$, while $v = 2, 3, 4, 9$ would yield four solutions in u and v: $(u, v) = (65, 2), (14, 3), (7, 4)$ and $(2, 9)$. Finally, if $v > 9$, then $(v - 9)(v + 7) > 0$ would imply that $2(v^2 - v - 1) > v^2 + 61$, making $u = \frac{v^2+61}{v^2-v-1} < 2$. However, if $u = 1$, then $v = -62 < 0$. Hence (1) admits no positive integer solutions for u if $v > 9$. Therefore (1) has exactly four solutions as listed above.

Also solved by SAM BAETHGE, Nordheim, Texas, USA; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was also one incorrect solution.

While most solutions gave only one value of c, Manes gave two. Baethge and Chronis each gave three values of c, while Hess gave the following twelve values and claimed that they are the twelve "lowest" values of c: 51, 61, 156, 321, 336, 402, 431, 486, 526, 611, 761 and 771. In terms of the "frequency" of the value of c given, the top three are $c = 61$, 336 (each given six times) and $c = 51$ (given five times). The largest value of c given was $c = 26461$, obtained by Janous. Both Hess and Herzig considered similar problems in which one seeks the value of c for which the given equation has exactly n solutions for $n = 4, 5, 6, \ldots$. Herzig used a computer to find that, for $4 \le n \le 10$, the corresponding minimum values of c are 51, 1381, 3966, 33776, 51816 and 14686766, respectively, while Hess list the ten lowest values of c for each n such that $4 \le n \le 8$. His lowest values agree with Herzig's.

For those curious, the editor has obtained (by computer) the next few lowest values of c for $n = 4$: 776, 816, 1066, 1071, 1146, 1153, 1172, 1201, 1271 and 1360 as those less than 1381, the lowest value with five solutions.

Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell

Editors emeriti / Rédacteur-emeriti: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer

Mathematical Mayhem

Founding Editors / Rédacteurs-fondateurs: Patrick Surry & Ravi Vakil

Editors emeriti / Rédacteurs-emeriti: Philip Jong, Jeff Higham, J.P. Grossman, Andre Chang, Naoki Sato, Cyrus Hsia