THE ACADEMY CORNER

No. 21

Bruce Shawyer

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THE BERNOULLI TRIALS 1998

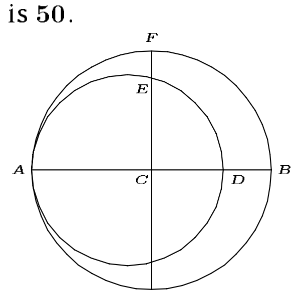
Hints and Answers

Ian Goulden and Christopher Small
University of Waterloo

The questions were printed earlier this year [1998: 257].

1. False. The diameter of the larger circle is 50.

   Let $CD = x$.
   Then $AC = 9 + x$
   and $CE = 4 + x$.
   Consider similar triangles in AED.



2. True. Let X be the number of rounds until his first error, Y the number of subsequent rounds until his second error. Then

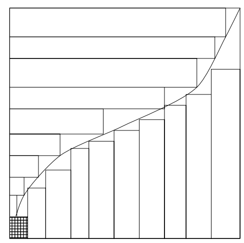
$$E(X+Y) = E(X) + E(Y) = 2\,E(X)$$

   But $E(X) = 2$, which can be found by summing the series.

3. True. Use 1, 2, 4, 8, $\ldots$, 2^{10} and a binary expansion of any number from 8 to 1998.

4. True. Write the points in coordinates and use the Pythagorean formula. Everything will cancel!

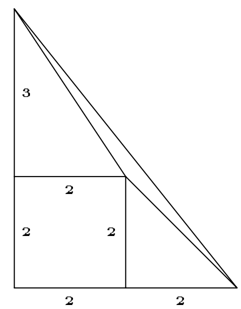
5. True. The volume of the tetrahedron is $\sqrt{2}/12$. To prove this, imbed the tetrahedron into a cube so that the vertices of the tetrahedron are four of the eight vertices of the cube. Show that the tetrahedron has a volume which is one third the volume of the cube. On the other hand, the volume of the sphere is $4\pi r^3/3$ where $r = 1/\pi$.
6. False. Actually $1503^2 = 2259009$ is the smallest.
7. True. Draw a graph of the function in the unit square.



8. True. The choice $k = 9$ works: $\dfrac{9n(n+1)}{2} + 1 = \dfrac{(3n+1)(3n+2)}{2}$.
9. True. The maximum area is achieved with a cyclic quadrilateral. The area of such a cyclic quadrilateral can be determined by Brahmagupta's formula
$$\begin{aligned} A &= \sqrt{(s-a)(s-b)(s-c)(s-d)} \\ &= \sqrt{4!} = 2\sqrt{6}\,. \end{aligned}$$
10. False. Write $\alpha(x) = x/(1+x)$. The equation can be written as
$$f[\alpha(x)] = \alpha[f(x)]\,.$$
This is clearly satisfied by $f(x) = \alpha^n(x)$. In particular, the choice of $f(x) = x/(1+x)$ works.
11. True. $3^{2^3} = 3^8 > 2 \times 2^9 = 2 \times 2^{3^2}$.
So $2^{3^{2^3}} > 2^{2\times 2^{3^2}} = 4^{2^{3^2}} > 3^{2^{3^2}}$.
So $3^{2^{3^{2^3}}} > 3^{3^{2^{3^2}}} > 2 \times 2^{3^{2^{3^2}}}$.
Finally, we have $2^{3^{2^{3^{2^3}}}} > 2^{2\times 2^{3^{2^{3^2}}}} = 4^{2^{3^{2^{3^2}}}} > 3^{2^{3^{2^{3^2}}}}$.
Obviously, this can be continued.
12. True. Write
$$\begin{aligned} \int_0^\infty \frac{x\,dx}{e^x - 1} &= \int_0^\infty \frac{x\,e^{-x}}{1-e^{-x}}\,dx \\ &= \int_0^\infty x(e^{-x} + e^{-2x} + e^{-3x} + \cdots)\,dx \\ &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \end{aligned}$$

13. False.

The answer is one square metre exactly.
To prove this consider the figure to the right:



The thin triangle has the required dimensions and its area is

$$\frac{1}{2} \times 4 \times 5 - \left(2 \times 2 + \frac{1}{2} \times 2 \times 3 + \frac{1}{2} \times 2 \times 2\right) = 1 .$$

14. False. Suppose there were such a function.

The function $e^x - e^{e^x}$ can be seen to be a one-to-one decreasing function. So f must be one-to-one:

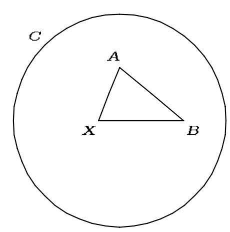
$$f(x) = f(y) \implies f[f(x)] = f[f(y)] \implies x = y .$$

As f is continuous and one-to-one, it must be strictly increasing or strictly decreasing.

But either way, $f[f(x)]$ must then be a strictly increasing function. Contradiction.

15. False.

Consider a third random point X in the circle.



The region R_1 corresponds to the points for X where the angle at X is obtuse. The region R_2 corresponds to the points for X where the angle at B is obtuse.

As the probabilities are the same, the average areas of R_1 and R_2 must be the same.

16. False. The second player wins by forcing bilateral symmetry on remaining petals. For example, if the first player starts by taking petal 1, the second player takes petals 7 and 8 together. If the first player chooses 1 and 2 the second player takes petal 8, etc. This ensures that the first player can never take the last petal.

THE OLYMPIAD CORNER

No. 194

R.E. Woodrow

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We begin this number with the problems from the Bi-National Israel Hungary Competition from 1995. My thanks go to Bill Sands, University of Calgary, who collected these problems for me while assisting at the IMO in Canada in 1995.

BI-NATIONAL ISRAEL-HUNGARY COMPETITION 1995

1. Denote the sum of the first n prime numbers by S_n. Prove that there exists a whole square between S_n and S_{n+1}.

2. Let P, P_1, P_2, P_3, P_4 be five points on a circle. Denote the distance of P from the line P_iP_k by d_{ik}. Prove that $d_{12}d_{34} = d_{13}d_{24}$.

3. Consider the polynomials $f(x) = ax^2 + bx + c$ which satisfy $|f(x)| \leq 1$ for all $x \in [0, 1]$. Find the maximal value of $|a| + |b| + |c|$.

4. Consider a convex polyhedron, whose faces are triangles. Prove that it is possible to colour the edges by red and blue in a way that one can travel from any vertex to any other vertex, passing only through red edges, and also one can travel only through blue edges.

Next we give two rounds of the 31^{st} Spanish Mathematical Olympiad. Both these contests were collected for me by Bill Sands, University of Calgary, while he was assisting at the IMO in Canada in 1995. I also received them from Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain, one of the contest organizers. Many thanks.

31^{st} SPANISH MATHEMATICAL OLYMPIAD

First Round: December 2–3, 1994

(Proposed by the Royal Spanish Mathematical Society)

First Day — Time: 4 hours

1. Let a, b, c be distinct real numbers and $P(x)$ a polynomial with real coefficients. If:

- the remainder on division of $P(x)$ by $x - a$ equals a,

- the remainder on division of $P(x)$ by $x - b$ equals b,
- and the remainder on division of $P(x)$ by $x - c$ equals c;

determine the remainder on division of $P(x)$ by $(x - a)(x - b)(x - c)$.

2. Show that, if $(x + \sqrt{x^2 + 1})(y + \sqrt{y^2 + 1}) = 1$, then $x + y = 0$.

3. The squares of the sides of a triangle ABC are proportional to the numbers 1, 2, 3.

(a) Show that the angles formed by the medians of ABC are equal to the angles of ABC.

(b) Show that the triangle whose sides are the medians of ABC is similar to ABC.

4. Find the smallest natural number m such that, for all natural numbers $n \geq m$, we have $n = 5a + 11b$, with a, b integers ≥ 0.

Second Day — Time: 4 hours

5. A subset $A \subseteq M = \{1, 2, 3, \ldots, 11\}$ is good if it has the following property:

"If $2k \in A$, then $2k - 1 \in A$ and $2k + 1 \in A$".

(The empty set and M are good). How many good subsets has M?

6. Consider the parabolas $y = cx^2 + d$, $x = ay^2 + b$, with $c > 0$, $d < 0$, $a > 0$, $b < 0$. These parabolas have four common points. Show that these four points are concyclic.

7. Show that there exists a polynomial $P(x)$, with integer coefficients, such that $\sin 1°$ is a root of $P(x) = 0$.

8. An aircraft of the airline "Air Disaster" must fly between two cities with $m + n$ stops. At each stop, the aircraft must load or unload 1 ton of goods. In m of the stops, the aircraft loads; in n of the stops, the aircraft unloads. Nobody from the staff has observed that the aircraft cannot handle a load of more than k tons $(n < k < m + n)$, and the stops where the plane loads and unloads are randomly distributed. If the aircraft takes off with n tons of goods, find the probability of the aircraft arriving at its destination.

31st SPANISH MATHEMATICAL OLYMPIAD

Second Round: February 24–25, 1995

First Day — Time: 4 hours

1. Consider sets A of 100 distinct natural numbers, such that the following property holds: "If a, b, c are elements of A (distinct or not), there exists a non-obtuse triangle of sides a, b, c." Let $S(A)$ be the sum of the perimeters of the triangles considered in the definition of A. Find the minimal value of $S(A)$.

2. We have several circles of paper on a plane, such that some of them are overlapped, but no one circle is contained in another. Show that it is impossible to form disjoint circles with the pieces which result from cutting off the non-overlapped parts and re-assembling them.

3. A line through the barycentre G of the triangle ABC intersects the side AB at P and the side AC at Q. Show that

$$\frac{PB}{PA}\cdot\frac{QC}{QA} \le \frac{1}{4}.$$

Second Day — Time: 4 hours

4. Find all the integer solutions of the equation

$$p(x+y) = xy$$

in which p is a prime number.

5. Show that, if the equations

$$\begin{aligned} x^3+mx-n &= 0\,, \\ nx^3-2m^2x^2-5mnx-2m^3-n^2 &= 0\,, \qquad (m \neq 0, n \neq 0) \end{aligned}$$

have a common root, then the first equation would have two equal roots, and determine in this case the roots of both equations in terms of n.

6. AB is a fixed segment and C a variable point, internal to AB. Equilateral triangles ACB' and CBA' are constructed, in the same half-plane defined by AB, and another equilateral triangle ABC' is constructed in the opposite half-plane. Show that:

(a) The lines AA', BB' and CC' are concurrent.

(b) If P is the common point of the lines of (a), find the locus of P when C varies on AB.

(c) The centres A'', B'', C'' of the three equilateral triangles also form an equilateral triangle.

(d) The points A'', B'', C'' and P are concyclic.

As a final problem set for your puzzling pleasure over the Christmas break we include the Final Round problems of the 46^{th} Polish Mathematical Olympiad. Again these problems come to us from more than one source. Bill Sands collected them while assisting at the IMO in Canada. Marcin E. Kuczma, Warszawa, Poland, one of Poland's premiere contest people, also sends us copies of contest materials regularly. Again, many thanks.

46^{th} POLISH MATHEMATICAL OLYMPIAD 1994–5

Problems of the Final Round (March 31-April 1, 1995)

First Day — Time: 5 hours

1. Find the number of those subsets of $\{1, 2, \ldots, 2n\}$ in which the equation $x + y = 2n + 1$ has no solutions.

2. A convex pentagon is partitioned by its diagonals into eleven regions: one pentagon and ten triangles. What is the maximum number of those triangles that can have equal areas?

3. Let $p \geq 3$ be a given prime number. Define a sequence (a_n) by

$$\begin{aligned} a_n &= n & &\text{for } n = 0, 1, 2, \ldots, p-1\,, \\ a_n &= a_{n-1} + a_{n-p} & &\text{for } n \geq p\,. \end{aligned}$$

Determine the remainder left by a_{p^3} on division by p.

Second Day — Time: 5 hours

4. For a fixed integer $n \geq 1$ compute the minimum value of the sum

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n}\,,$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers satisfying the condition

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = n\,.$$

5. Let n and k be positive integers. From an urn containing n tokens numbered 1 through n, the tokens are drawn one by one without replacement, until a number divisible by k appears. For a given n determine all those $k \leq n$ for which the expected number of draws equals exactly k.

6. Let k, l, m be three non-coplanar rays emanating from a common origin P and let A be a given point on k (other than P). Show that there exists exactly one pair of points B, C, with B lying on l and C on m, such that

$$PA + AB = PC + CB \quad \text{and} \quad PB + BC = PA + AC\,.$$

Next an apology. Somehow when preparing the list of solvers of problems for the October issue of ***CRUX with MAYHEM***, I omitted listing Miguel Amengual Covas, Cala Figuera, Mallorca, Spain as a solver of problem 3 of the Swedish Mathematical Olympiad [1998: 327] and for problem 2 of the Dutch Mathematical Olympiad [1998: 330]. Sorry!

Now we turn to solutions by the readers to problems of the Irish Mathematical Olympiad 1994 [1997: 388–389].

1. Let x, y be positive integers with $y > 3$ and

$$x^2 + y^4 = 2[(x-6)^2 + (y+1)^2].$$

Prove that $x^2 + y^4 = 1994$.

Solution by Pavlos Maragoudakis, Pireas, Greece.

Rewriting we get

$$x^2 - 24x - y^4 + 2y^2 + 4y + 74 = 0. \qquad (1)$$

Now (1) has integer solutions only if the discriminant $4(y^4 - 2y^2 - 4y + 70)$ is a perfect square. It is easy to prove that for $y \geq 4$,

$$(y^2-2)^2 < y^4 - 2y^2 - 4y + 70 < (y^2+1)^2. \qquad (*)$$

(Indeed $(*) \iff y^2 - 2y + 33 > 0$ and $4y(y+1) > 69$. The first inequality is true. Since $y \geq 4$, $4y(y+1) \geq 4 \cdot 4 \cdot 5 = 80 > 69$.) The only perfect squares between $(y^2-2)^2$ and $(y^2+1)^2$ are $(y^2-1)^2$ and $(y^2)^2$. Now

$$(y^2-1)^2 = y^4 - 2y^2 - 4y + 70 \iff y = \frac{69}{4} \notin \mathbb{Z},$$

and $y^4 - 2y^2 - 4y + 70 = y^4 \iff y^2 + 2y - 35 = 0 \iff y = 5$ or $y = -7$. Thus, $y = 5$. Now (1) gives $x = 37$ and

$$x^2 + y^4 = 37^2 + 5^4 = 1369 + 625 = 1994.$$

Comment by Jim Totten, The University College of the Cariboo, Kamloops, BC.

The result also works for $y = 1$ and $y = 2$ as well, but fails for $y = 3$ with $x = 1$.

2. Let A, B, C be three collinear points with B between A and C. Equilateral triangles ABD, BCE, CAF are constructed with D, E on one side of the line AC and F on the opposite side. Prove that the centroids of the triangles are the vertices of an equilateral triangle. Prove that the centroid of this triangle lies on the line AC.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Pavlos Maragoudakis, Pireas, Greece. We give the solution of Amengual Covas.

We introduce a rectangular Cartesian system with origin at B and x–axis along AC. Let $\overline{AB} = a$ and $\overline{BC} = b$. The centroids O_1, O_2, O_3 of equilateral triangles ABD, BCE and CAF are

$$G_1 = \left(\frac{-a}{2}, \frac{a}{2\sqrt{3}}\right), \qquad G_2 = \left(\frac{b}{2}, \frac{b}{2\sqrt{3}}\right) \qquad \text{and}$$

$$G_3 = \left(\frac{-a+b}{2}, -\frac{a+b}{2\sqrt{3}}\right).$$

Hence the centroid G of triangle $\triangle G_1G_2G_3$ is

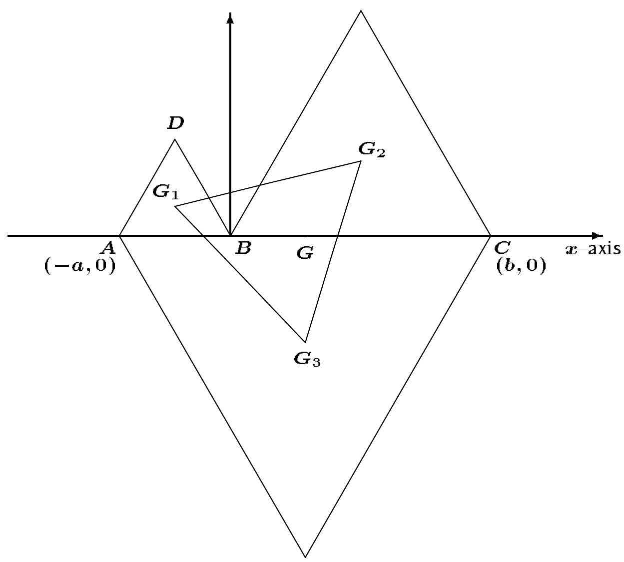
$$G = \left(\frac{-a+b}{3}, 0\right).$$

It is straightforward to verify that

$$\overline{G_1G_2} = \overline{G_2G_3} = \overline{G_3G_1} = \sqrt{\frac{a^2+ab+b^2}{3}}.$$

Therefore $\triangle G_1G_2G_3$ is equilateral.

Finally, since the y–coordinate of G is 0, clearly G is on the x–axis; that is to say, G lies on the line AC.



3. Determine with proof all real polynomials $f(x)$ satisfying the equation

$$f(x^2) = f(x)f(x-1).$$

Solution by Pavlos Maragoudakis, Pireas, Greece.

We will prove that $f(x) = 0$ or $f(x) = (x^2+x+1)^k$, $k = 0, 1, 2, \ldots$. Consider any (possibly complex) root p of $f(x)$. Then

$$f(p^2) = f(p) \cdot f(p-1) = 0 \cdot f(p-1) = 0$$

and

$$f((p+1)^2) = f(p+1) \cdot f(p) = f(p+1) \cdot 0 = 0.$$

So p^2, $(p+1)^2$ are also roots of $f(x)$. Thus p^{2^n} and $(p+1)^{2^n}$ are roots of $f(x)$, $n = 0, 1, 2, \ldots$. If $|p| \neq 1$ or $|p| \neq |p+1|$ then we get an infinite number of roots, so $f(x)$ is a constant polynomial, and having a root p, $f(x) \equiv 0$.

If $|p| \neq 1$ or $|p + 1+ \neq 1$; that is, if $|p| = 1 = |p+1|$ then $p \cdot \overline{p} = 1$ and $p \cdot \overline{p} = (p+1)(\overline{p}+1)$, so $p + \overline{p} = -1$ and $\overline{p} = -(p+1)$, and now $p(-p-1) = 1$. Therefore $p^2+p+1 = 0$. It follows that $f(x) = (x^2+x+1)^k$, for some $k \geq 1$.

On the other hand if $f(x)$ has no roots, then $f(x) = c \neq 0$, is a non-zero constant. Then $f(x^2) = f(x) \cdot f(x-1)$ gives $c = c \cdot c$, and $c \neq 0$ gives $c = 1$. Thus $f(x) = (x^2 + x + 1)^0$. In any case $f(x) = 0$ or $f(x) = (x^2 + x + 1)^k$ for some $k = 0, 1, 2, \ldots$.

6. A sequence $\{x_n\}$ is defined by the rules

$$x_1 = 2$$

and

$$nx_n = 2(2n-1)x_{n-1}\,; \qquad n = 2, 3, \ldots .$$

Prove that x_n is an integer for every positive integer n.

Solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Pavlos Maragoudakis, Pireas, Greece. We give Maragoudakis' solution.

Now

$$\begin{aligned} x_2 &= 2 \cdot \tfrac{3}{2} x_1 \\ x_3 &= 2 \cdot \tfrac{5}{3} x_2 \\ &\cdots \\ x_{n-1} &= 2 \cdot \tfrac{2n-3}{n-1} x_{n-2} \\ x_n &= 2 \cdot \tfrac{2n-1}{n} x_{n-1} \end{aligned}$$

It follows that

$$\begin{aligned} x_n &= 2^{n-1} \frac{(2n-1)(2n-3)\cdots 5 \cdot 3}{n \cdot (n-1) \cdots 3 \cdot 2} \cdot 2 \\ &= \frac{(2n)!}{(n!)^2} = \binom{2n}{n} \in \mathbb{Z} \end{aligned}$$

since $\binom{2n}{n}$ is a binomial coefficient.

7. Let p, q, r be distinct real numbers which satisfy the equations

$$q = p(4-p)\,,$$

$$r = q(4-q)\,,$$

$$p = r(4-r)\,.$$

Find all possible values of $p + q + r$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

From the discriminant of each quadratic equation, it follows that p, q, r are all less than 4. If one of p, q, r is zero, all are zero, so we now assume none are zero. Also, it follows that p, q, r all have the same sign. From the product of the three equations we get $1 = (4-p)(4-q)(4-r)$ so that p, q, r are all positive.

We now let $p = 4\sin^2\theta$, so then successively, $q = 4\sin^2 2\theta$, $r = 4\sin^2 4\theta$, $p = 4\sin^2 8\theta$. Hence, $\sin\theta = \pm\sin 8\theta$ so that we have

$$(\sin 7\theta/2)(\cos 9\theta/2) = 0 \qquad \text{or} \qquad (\sin 9\theta/2)(\cos 7\theta/2) = 0\,.$$

Solving for θ leads to only the following possible non-zero values of $p+q+r$: $4(\sin^2\pi/7 + \sin^2 2\pi/7 + \sin^2 3\pi/7), 4(\sin^2\pi/9 + \sin^2 2\pi/9 + \sin^2 4\pi/9)$ and $3\cdot 4\sin^2\pi/3$; that is, 9.

9. Let w, a, b, c be distinct real numbers with the property that there exist real numbers x, y, z for which the following equations hold:

$$\begin{aligned} x \quad + y \quad + z \quad &= 1 \\ xa^2 + yb^2 + zc^2 &= w^2 \\ xa^3 + yb^3 + zc^3 &= w^3 \\ xa^4 + yb^4 + zc^4 &= w^4. \end{aligned}$$

Express w in terms of a, b, c.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

As known, the following determinant must vanish:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a^2 & b^2 & c^2 & w^2 \\ a^3 & b^3 & c^3 & w^3 \\ a^4 & b^4 & c^4 & w^4 \end{vmatrix}$$

Also, from a known expansion theorem [1] for "alternant" determinants, we have

$$(w-a)(w-b)(w-c)(a-b)(a-c)(b-c)(w(bc+ca+ab)+abc) = 0\,.$$

Hence, $w = a$, or b, or c, or $-abc/[bc+ca+ab]$.

Reference

[1] T. Muir, *A Treatise on the Theory of Determinants*, Dover, N.Y., p. 333, #337.

Now we turn to readers' solutions of problems from the December 1997 number of the *Corner*. We give some solutions to problems proposed to the Jury but not used at the 37th International Olympiad at Mumbai, India [1997; 450–453].

1. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1 .$$

When does equality hold?

Solutions by Pierre Bornsztein, Courdimanche, France; and by Panos E. Tsaoussoglou, Athens, Greece. We give the solution by Tsaoussoglou.

We note first that

$$\frac{a^5 + b^5}{2} \geq \left(\frac{a^3 + b^3}{2}\right)\left(\frac{a^2 + b^2}{2}\right),$$

since $a^5 - a^3b^2 - a^2b^3 + b^5 = (a-b)^2(a+b)(a^2 + ab + b^2) \geq 0$ with equality if and only if $a = b$.

Similarly

$$\frac{a^3 + b^3}{2} \geq \left(\frac{a + b}{2}\right)\left(\frac{a^2 + b^2}{2}\right)$$

because $a^3 - a^2b - ab^2 + b^3 = (a-b)^2(a+b) \geq 0$, with equality if and only if $a = b$. Thus

$$\begin{aligned} \frac{a^5 + b^5}{2} &\geq \left(\frac{a^3 + b^3}{2}\right)\left(\frac{a^2 + b^2}{2}\right) \geq ab\left(\frac{a^3 + b^3}{2}\right) \\ &\geq ab\left(\frac{a + b}{2}\right)\left(\frac{a^2 + b^2}{2}\right) \geq \frac{a^2b^2(a+b)}{2}. \end{aligned}$$

It is enough, therefore, to prove

$$\frac{ab}{ab(a+b)ab + ab} + \frac{bc}{bc(b+c)bc + bc} + \frac{ca}{ca(c+a)ca + ca} \leq 1,$$

or

$$\frac{1}{ab(a+b) + abc} + \frac{1}{bc(b+c) + abc} + \frac{1}{ca(c+a) + abc} \leq 1 .$$

Equivalently,

$$\frac{1}{ab(a+b+c)} + \frac{1}{bc(a+b+c)} + \frac{1}{ca(a+b+c)} \leq 1,$$

or

$$\frac{c}{abc(a+b+c)} + \frac{a}{abc(a+b+c)} + \frac{b}{abc(a+b+c)} \leq 1 .$$

Again, because $abc = 1$ we get

$$\frac{a+b+c}{a+b+c} \leq 1,$$

which is true.

The equality requires $a = b = c = 1$.

2. Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be real numbers such that for all integers $k > 0$,

$$a_1^k + a_2^k + \cdots + a_n^k \;\geq\; 0 .$$

Let $p = \max\{|a_1|, \ldots , |a_n|\}$. Prove that $p = a_1$ and that

$$(x - a_1)(x - a_2)\cdots(x - a_n) \;\leq\; x^n - a_1^n$$

for all $x > a_1$.

Solution by Pierre Bornsztein, Courdimanche, France.

Pour $k \in N^*$, on note $S_k = a_1^k + \cdots + a_n^k$.

On a $S_1 \geq 0$, donc $na_1 \geq s_1 \geq 0$ et alors $a_1 \geq 0$.

Si $a_n \geq 0$, $a_1 \geq \cdots \geq a_n \geq 0$, donc $p = a_1$.

Par contre, si $a_n < 0$, par l'absurde, supposons que $p \neq a_1$.

Alors $p = |a_n|$. Pour $i \in \{1, \ldots , n-1\}$, si $|a_i| < |a_n|$, $\lim_{k\to\infty} \left(\frac{a_i}{a_n}\right)^k = 0$. Cependant, si $|a_i| = |a_n|$, $\lim_{k\to\infty} \left(\frac{a_i}{a_n}\right)^k = 1$.

Par suite pour

$$T_k \;\equiv\; \sum_{i=1}^{n} \left(\frac{a_i}{a_n}\right)^k \;=\; 1 + \sum_{i=1}^{n-1} \left(\frac{a_i}{a_n}\right)^k ,$$

on a $\lim_{k\to\infty} T_k = l$ avec $l \in [1, +\infty)$. En particulier, il existe $k \in N^*$ tel que $T_{2k+1} > 0$ et comme $a_n < 0$ on a $a_n^{2k+1} < 0$, d'où $s_{2k+1} = a_n^{2k+1} \cdot T_{2k+1} < 0$, qui est une contradiction. Par conséquent, $p = a_1$.

Soit $x > a_1$. Alors $x - a_i > 0$ pour $i = 1, 2, \ldots , n$.

D'après AM/GM

$$\prod_{i=2}^{n}(x - a_i) \;\leq\; \left(\frac{\sum_{i=2}^{n}(x - a_i)}{n - 1}\right)^{n-1} \;=\; \left(x - \frac{1}{n-1}\sum_{i=2}^{n} a_i\right)^{n-1} .$$

Or $s_1 \geq 0$, donc $a_1 \geq -\sum_{i=2}^{n} a_i$, et alors

$$\prod_{i=2}^{n}(x - a_i) \;\leq\; \left(x + \frac{a_1}{n-1}\right)^{n-1} \;=\; \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{a_1}{n-1}\right)^i x^{n-1-i} .$$

Mais pour $i \in \{1, 2, \ldots, n-1\}$, $\binom{n-1}{i}\frac{1}{(n-1)^i} \leq \frac{(n-1)(n-2)\ldots(n-i)}{(n-1)^i} \leq 1$. Donc

$$\prod_{i=2}^{n}(x-a_i) \;\leq\; \sum_{i=0}^{n-1} a_1^i x^{n-1-i}$$

et alors

$$\begin{aligned}\prod_{i=1}^{n}(x-a_i) &\leq (x-a_1)\sum_{i=0}^{n-1} a_1^i x^{n-1-i} \\ &= x^n - a_1^n \,.\end{aligned}$$

3. Let $a > 2$ be given, and define recursively:

$$a_0 = 1,\; a_1 = a,\; a_{n+1} \;=\; \left(\frac{a_n^2}{a_{n-1}^2} - 2\right) a_n \,.$$

Show that for all integers $k > 0$, we have

$$\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} \;<\; \frac{1}{2}\left(2 + a - \sqrt{a^2-4}\right) \,.$$

Solution by Pierre Bornsztein, Courdimanche, France.

Comme $a > 2$, il existe $b > 1$ tel que $a = b + \frac{1}{b}$ et on a

$$b^2 - ab + 1 \;=\; 0\,;$$

$$\text{c.à.d.} \qquad b \;=\; \frac{a + \sqrt{a^2-4}}{2} \quad (b > 1)\,,$$

et comme le produit des racines est 1,

$$\frac{1}{b} \;=\; \frac{a - \sqrt{a^2-4}}{2}$$

d'où

$$\frac{1}{2}(2 + a - \sqrt{a^2-4}) \;=\; 1 + \frac{1}{b}\,.$$

On pose $S_n = \sum_{i=0}^{n} \frac{1}{a_i}$ pour $n \geq 1$.

On veut donc prouver que $S_n < 1 + \frac{1}{b}$.

Or $a_1 = b + \frac{1}{b}$, donc $a_2 = (b+\frac{1}{b})(b^2+\frac{1}{b^2})$ et par une récurrence rapide pour $n \geq 1$:

$$a_n \;=\; (b + \frac{1}{b})\left(b^2 + \frac{1}{b^2}\right)\cdots\left(b^{2^{n-1}} + \frac{1}{b^{2^{n-1}}}\right) \,.$$

On en déduit que

$$\begin{aligned} a_n &= \frac{b^2+1}{b}\cdot\frac{b^4+1}{b^2}\cdots\left(\frac{b^{2^n}+1}{b^{2^{n-1}}}\right) \\ &= \frac{(b^2+1)(b^4+1)\cdots(b^{2^n}+1)}{b^{2^n-1}} \end{aligned}$$

et donc pour $n \geq 2$

$$S_n = 1+\frac{b}{b^2+1}+\sum_{i=2}^{n}\frac{b^{2^i-1}}{(b^2+1)\cdots(b^{2^i}+1)}.$$

Or
$$\begin{aligned} \frac{1}{(b^2+1)\cdots(b^{2^{i-1}}+1)} &= \frac{b^{2^i}+1}{(b^2+1)\cdots(b^{2^i}+1)} \\ &= \frac{b^{2^i}}{(b^2+1)\cdots(b^{2^i}+1)}+\frac{1}{(b^2+1)\cdots(b^{2^i}+1)}, \end{aligned}$$

d'où pour $i \geq 2$

$$\begin{aligned} &\frac{b^{2^i-1}}{(b^2+1)\cdots(b^{2^i}+1)} \\ &\quad= \frac{1}{b}\left(\frac{1}{(b^2+1)\cdots(b^{2^{i-1}}+1)}-\frac{1}{(b^2+1)\cdots(b^{2^i}+1)}\right) \end{aligned}$$

et ainsi

$$\begin{aligned} S_n &= 1+\frac{b}{b^2+1}+\frac{1}{b}\left(\frac{1}{b^2+1}-\frac{1}{(b^2+1)\cdots(b^{2^n}+1)}\right) \\ &< 1+\frac{b}{b^2+1}+\frac{1}{b(b^2+1)} \\ &= 1+\frac{b}{b^2+1}+\frac{1}{b}-\frac{b}{b^2+1} \end{aligned}$$

d'où
$$S_n < 1+\frac{1}{b} \qquad \text{pour} \qquad n \geq 2,$$

et comme
$$S_1 = 1+\frac{1}{a} = 1+\frac{b}{b^2+1} < 1+\frac{1}{b},$$

le résultat est vrai pour tout $n \geq 1$.

Remarque : Puisque $b > 1$, $\lim_{n\to+\infty} b^{2^n} = +\infty$, et donc $\lim_{n\to\infty} S_n = 1+\frac{1}{b}$.

4. Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers, not all zero.

(a) Prove that $x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n = 0$ has precisely one positive real root.

(b) Let $A = \sum_{j=1}^{n} a_j$ and $B = \sum_{j=1}^{n} j a_j$, and let R be the positive real root of the equation in (a). Prove that $A^A \le R^B$.

Solution by Pierre Bornsztein, Courdimanche, France.

(a) Pour $x > 0$, $x^n - a_1 x^{n-1} - \ldots - a_n = 0$ est équivalent à $1 - \frac{a_1}{x} - \cdots - \frac{a_n}{x^n} = 0$.

Si l'on pose $f(x) = 1 - \sum_{i=1}^{n} \frac{a_i}{x^i}$, alors :

(i) f est dérivable sur $\mathbb{R}^+$ et pour $x > 0$, $f'(x) = \sum_{i=1}^{n} \frac{i a_i}{x^{i+1}} > 0$ puisque $a_i \ge 0$, non tous nuls. Donc, f est continue, et strictement croissante sur $\mathbb{R}^{+*}$.

(ii) De plus $\lim\limits_{x \to 0^+} f(x) = -\infty$, $\lim\limits_{x \to +\infty} f(x) = 1$. Il existe un unique réel strictement positif, noté R, tel que $f(R) = 0$.

(b) Pour $j \in \{1, \ldots, n\}$, on pose $x_j = \frac{a_j}{A} \ge 0$ et alors $\sum_{j=1}^{n} x_j = 1$.

Comme $\ln$ est une fonction concave sur $\mathbb{R}^{+*}$, on a

$$\begin{aligned} \sum_{j=1}^{n} x_j \ln\left(\frac{A}{R^j}\right) &\le \ln\left(\sum_{j=1}^{n} x_j \frac{A}{R^j}\right) \\ &= \ln\left(\sum_{j=1}^{n} \frac{a_j}{R^j}\right) \\ &= \ln(1) \quad \text{(cf. (a))} \\ &= 0\,. \end{aligned}$$

Donc

$$\sum_{j=1}^{n} [x_j \ln(A) - j x_j \ln(R)] \le 0\,;$$

c.à.d.

$$\ln(A) \cdot A \le \ln(R) \cdot B$$

ou encore

$$\ln(A^A) \le \ln(R^B)$$

d'où le résultat suit.

5. Let $P(x)$ be the real polynomial, $P(x) = ax^3 + bx^2 + cx + d$. Prove that if $|P(x)| \le 1$ for all x such that $|x| \le 1$, then

$$|a| + |b| + |c| + |d| \le 7\,.$$

Solution by Pierre Bornsztein, Courdimanche, France.

On pose $s(P) = |a| + |b| + |c| + |d| \ge 0$.

Quitte à changer P en $(-P)$ et/ou x en $(-x)$ on peut toujours supposer que $a \ge 0$ et $b \ge 0$.

On a $P(0) = d$, $P(1) = a+b+c+d$, $P(-1) = -a+b-c+d$, $8P(\frac{1}{2}) = a+2b+4c+8d$, $8P(-\frac{1}{2}) = -a+2b-4c+8d$, d'où

$$\begin{aligned} a &= \tfrac{2}{3}P(1) - \tfrac{2}{3}P(-1) + \tfrac{4}{3}P(\tfrac{1}{2}) - \tfrac{4}{3}P(-\tfrac{1}{2}) \\ b &= \tfrac{2}{3}P(1) + \tfrac{2}{3}P(-1) - \tfrac{2}{3}P(\tfrac{1}{2}) - \tfrac{2}{3}P(-\tfrac{1}{2}) \\ c &= -\tfrac{1}{6}P(1) + \tfrac{1}{6}P(-1) + \tfrac{4}{3}P(\tfrac{1}{2}) - \tfrac{4}{3}P(-\tfrac{1}{2}) \\ d &= -\tfrac{1}{6}P(1) - \tfrac{1}{6}P(-1) + \tfrac{2}{3}P(\tfrac{1}{2}) + \tfrac{2}{3}P(-\tfrac{1}{2}) \;=\; P(0)\,. \end{aligned}$$

Maintenant si $c \geq 0$ et $d \geq 0$, $S(P) = a+b+c+d = P(1) \leq 1$.

Cependant si $c \geq 0$ et $d \leq 0$, on obtient

$$\begin{aligned} s(P) &= a+b+c-d \\ &= a+b+c+d-2d \\ &= P(1) - 2P(0) \end{aligned}$$

d'où $\quad S(P) \leq |P(1)| + 2|P(0)| = 3$.

On procède au cas $c \leq 0$ et $d \geq 0$, où

$$\begin{aligned} S(P) &= a+b-c+d \\ &= a+b+c+d-2c \end{aligned}$$

et d'après les relations pour a, b, c, d ci-haut,

$$\begin{aligned} S(P) &= \frac{4}{3}P(1) - \frac{1}{3}P(-1) - \frac{8}{3}P\left(\frac{1}{2}\right) + \frac{8}{3}P\left(-\frac{1}{2}\right) \\ &\leq \frac{4}{3} + \frac{1}{3} + \frac{8}{3} + \frac{8}{3}\,; \\ \text{c.à.d.} \quad & S(P) \leq 7\,. \end{aligned}$$

Enfin, si $c \leq 0$ et $d \geq 0$, $S(P) = a+b-c-d$, et suivant la même procédure, on obtient

$$\begin{aligned} S(P) &= \frac{5}{3}P(1) - 4P\left(\frac{1}{2}\right) + \frac{4}{3}P\left(-\frac{1}{2}\right) \\ &\leq \frac{5}{3} + 4 + \frac{4}{3}\,; \\ \text{c.à.d.} \quad & S(P) \leq 7\,. \end{aligned}$$

Finalement, dans tous les cas $S(P) \leq 7$.

Remarque : On peut montrer (voir Olympiades suédoises 1965 – finale) que pour un tel polynôme, on a toujours $|a| \leq 4$ (ce qui est immédiat avec l'expression de a ci-haut).

Ces deux inéqualités sont simultanément vérifiées par $P(x) = 4x^3 - 3x$, qui respecte $|P(x)| \leq 1$ pour $|x| \leq 1$.

7. Let f be a function from the set of real numbers $\mathbb{R}$ into itself such that for all $x \in \mathbb{R}$, we have $|f(x)| \leq 1$ and

$$f\left(x+\frac{13}{42}\right)+f(x) \;=\; f\left(x+\frac{1}{6}\right)+f\left(x+\frac{1}{7}\right).$$

Prove that f is a periodic function (that is, there exists a non-zero real number c such that $f(x+c)=f(x)$ for all $x \in \mathbb{R}$).

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's solution.

We will prove that f is 1-periodic. We have

$$\begin{aligned} & f\left(x+\frac{k}{6}+\frac{l+1}{7}\right)+f\left(x+\frac{k+1}{6}+\frac{l}{7}\right) \\ & \quad = f\left(x+\frac{k}{6}+\frac{l}{7}\right)+f\left(x+\frac{k+1}{6}+\frac{l+1}{7}\right). \end{aligned}$$

If k runs through $1, 2, \ldots, m-1$, where $m \in \mathbb{N}$, and adding these equations we obtain

$$f\left(x+\frac{l+1}{7}\right)+f\left(x+\frac{m}{6}+\frac{l}{7}\right) \;=\; f\left(x+\frac{m}{6}\right)+f\left(x+\frac{m}{6}+\frac{l+1}{7}\right).$$

Similarly when l runs $1, 2, \ldots, n-1$, ($n \in \mathbb{N}$) and adding these equations, we obtain

$$f\left(x+\frac{n}{7}\right)+f\left(x+\frac{m}{6}\right) \;=\; f(x)+f\left(x+\frac{m}{6}+\frac{n}{7}\right).$$

We choose $n=7$ and $m=6$, and find

$$2f(x+1) \;=\; f(x)+f(x+2).$$

This means that the sequence $f(x+n)$ is an arithmetic sequence with common difference $f(x+1)-f(x)$. But, since f is bounded, we must have $f(x+1)-f(x)=0$. Hence 1 is a period of f. Finally, 1 is the "best" period because the function

$$f(x) \;=\; \frac{\{6x\}+\{7x\}}{2}$$

satisfies all the hypotheses of the problem. (Here, $\{x\}$ denotes the fractional part of x.)

8. Let the sequence $a(n)$, $n=1,2,3,\ldots$, be generated as follows: $a(1)=0$, and for $n>1$,

$$a(n) \;=\; a([n/2])+(-1)^{n(n+1)/2}.$$

(Here $[t]$ means the greatest integer less than or equal to t.)

(a) Determine the maximum and minimum value of $a(n)$ over $n \le 1996$ and find all $n \le 1996$ for which these extreme values are attained.

(b) How many terms $a(n)$, $n \le 1996$, are equal to 0?

Solution by Pierre Bornsztein, Courdimanche, France.

Parti (a) On vérifie facilement que $a_1 = 0$, $a_2 = -1$, $a_3 = 1$ et, pour $k \ge 1$

$$\left.\begin{array}{rcl} a_{4k} & = & a_{2k}+1 \\ a_{4k+1} & = & a_{2k}-1 \\ a_{4k+2} & = & a_{2k+1}-1 \\ a_{4k+3} & = & a_{2k+1}+1 \end{array}\right\} \qquad (1)$$

Pour $p \in \mathbb{N}^*$, on pose $E_p = \{2^p, 2^p+1, \dots, 2^{p+1}-1\}$.

Pour $p = 1$, on pose $\varphi_1(2) = (-1)$, $\varphi_1(3) = (1)$, et pour $p = 2$, on pose $\varphi_2(4) = (-1, 1)$, $\varphi_2(5) = (-1, -1)$, $\varphi_2(6) = (1, -1)$, $\varphi_2(7) = (1, 1)$.

On procède à construire $\varphi_p : E_p \to F_p$ pour $p \ge 2$ comme suite où $F_p = \{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p) \mid \forall i \in \{1, \dots, p\},\ \varepsilon_i \in \{-1, 1\}\}$. Alors pour $n \in E_{p+1}$

$$\left[\frac{n}{2}\right] \in E_p, \qquad \text{et on pose} \qquad \varphi_{p+1}(n) = (\varepsilon_1, \dots, \varepsilon_p, \varepsilon_{p+1})$$

où

$$(\varepsilon_1, \dots, \varepsilon_p) = \varphi_p\left(\left[\frac{n}{2}\right]\right) \qquad \text{et} \qquad \varepsilon_{p+1} = (-1)^{(n(n+1))/2}.$$

On construit ainsi une application $\varphi_{p+1} : E_{p+1} \to F_{p+1}$.

Propriété 1. Pour $p \ge 1$, φ_p est une bijection de E_p sur F_p.

Preuve. Par récurrence sur p, φ_1 est clairement une bijection de E_1 sur F_1.

Supposons que φ_p soit une bijection de E_p sur F_p pour $p \ge 1$ fixé. Soient n, k dans E_{p+1} tels que $\varphi_{p+1}(n) = \varphi_{p+1}(k)$. On peut toujours supposer que $n \ge k$. Par l'absurde, si $n > k$, $\varphi_{p+1}(n) = \varphi_{p+1}(k)$ entraine $\varphi_p([\frac{n}{2}]) = \varphi_p([\frac{k}{2}])$ et comme φ_p est bijective, $[\frac{n}{2}] = [\frac{k}{2}]$, d'où $n = k+1$, car n, k sont entiers et si $n \ge k+2$ alors $[\frac{n}{2}] > [\frac{k}{2}]$.

Or si k est impair, $k = 2q+1$ alors $n = 2q+2$ et $[\frac{n}{2}] = q+1 \ne q = [\frac{k}{2}]$, qui est contradictoire.

Si, cependant, $k = 4l$, alors $n = 4l+1$ d'où $\frac{k(k+1)}{2} = 2q(4q+1)$, qui est pair et $\frac{n(n+1)}{2} = (2q+1)(4q+1)$ qui est impair. Donc $\varepsilon_{p+1}(k) \ne \varepsilon_{p+1}(n)$ qui est aussi contradictoire.

Quand $k = rq+2$, alors $n = 4q+3$ d'où $\frac{k(k+1)}{2} = (2q+1)(4q+3)$ qui est impair et $\frac{n(n+1)}{2} = (4q+3)(2q+2)$ qui est pair, et on obtient la même contradiction.

Finalement $n = k$, et φ_{p+1} est injective. Car E_{p+1} et F_{p+1} ont la même cardinalité, 2^p, φ_{p+1} est bijective.

Propriété 2. $\forall p \in \mathbb{N}$, $p \geq 1$, $\forall n \in E_p$, $\varphi_p(n) = (\varepsilon_1, \ldots \varepsilon_p)$. Alors $a_n = \sum_{i=1}^p \varepsilon_i$.

Preuve. Récurrence immédiate sur p en utilisant la construction de φ_p et $a_n = a_{[n/2]} + (-1)^{(n(n+1))/2}$.

Propriété 3. $\forall p \in N^*$, $\forall n \in E_p$, $a_n \equiv p \bmod 2$.

Preuve. Par récurrence sur p. C'est immédiat pour $p = 1$. Supposons le résultat vrai pour $p \geq 1$ fixé. Alors pour $n \in E_{p+1}$ on a $[\frac{n}{2}] \in E_p$ et $a_n = a_{[n/2]} + (-1)^{(n(n+1))/2} \equiv a_{[n/2]} + 1 \equiv p + 1 \bmod 2$ d'où le résultat au rang $p + 1$.

Propriété 4. Pour $n \geq 1$, $a_{2^n - 1} = n - 1$.

Preuve. Par récurrence sur n. C'est immédiat pour $n = 1$ et $n = 2$. Supposons le résultat vrai pour $n \geq 2$ fixé. Alors, d'après (1)

$$\begin{aligned} a_{2^{n+1}-1} &= a_{4(2^{n-1}-1)+3} = a_{2(2^{n-1}-1)+1} + 1 \\ &= a_{2^n - 1} + 1 = n, \end{aligned}$$

d'où la proprieté au rang $n + 1$.

On a $2^{10} = 1024 < 1996 < 2047 = 2^{11} - 1$.

Or si $n < 2^9 - 1$, $n = 1$ ou $n \in E_p$ avec $p \leq 8$, donc $a_n = 0$ ou $a_n = \varphi_p(n) \leq 8$ (cf P_2).

Cependant si $n \in E_9$, $a_n = \varphi_9(n) \leq 9$ (d'après P_2) avec égalité ssi $n = 2^{10} - 1$ (d'après P_4 et car φ_9 est bijective).

Si $n \in E_{10}$, $a_n = \varphi_{10}(n) \leq 10$ et a_n est pair (cf. P_2 et P_3) avec $a_n = 10$ ssi $n = 2^{11} - 1$ (P_4 et φ_{10} bijective).

d'où $a_n \leq 9$ pour $n \leq 1996$.

Finalement la valeur maximale de a_n, pour $n \leq 1996$ est 9, avec $a_n = 9$ ssi $n = 1023$.

On passe des valeurs de a_n pour $n \in E_p$ à celles de a_n pour $n \in E_{p+1}$ en ajoutant ou en retranchant 1. On peut ainsi prévoir que l'on obtiendra les valeurs minimales en utilisant une suite (U_n) d'indices avec $U_1 = 2$ et pour $n \geq 1$

$$U_{n+1} = \begin{cases} 2U_n + 1 & \text{si } U_n \text{ est pair,} \\ 2U_n & \text{si } U_n \text{ est impair.} \end{cases}$$

Propriété 5. Pour $p \geq 1$,

(a) U_p et p sont des entiers de parités contraires.

(b) $U_p = \frac{2^{p+2}-1}{3}$ si p est pair ; $U_p = \frac{s^{p+2}-2}{3}$ si p est impair.

Preuve. Récurrence sur p.

Propriété 6. Pour $n \geq 1$, $a_{U_n} = -n$.

Preuve. Il est clair que pour tout $n \geq 1$, $\left[\frac{U_{n+1}}{2}\right] = U_n$ (par définition de U_{n+1}).

Pour $n \geq 1$, quand n est pair

$$\frac{U_n(U_n+1)}{2} = \frac{2^{n+2}-1}{3} \cdot \frac{2^{n+1}+1}{3} \qquad \text{est impair},$$

et de même quand n est impair

$$\frac{U_n(U_n+1)}{2} = \frac{2^{n+1}-1}{3} \cdot \frac{2^{n+2}+1}{3} \qquad \text{qui est aussi impair}.$$

Donc pour tout $n \geq 1$

$$(-1)^{\frac{U_n(U_n+1)}{2}} = -1.$$

Alors pour tout $n \geq 1$,

$$a_{U_{n+1}} = a_{U_n} - 1$$

et comme $a_{U_1} = a_2 = -1$, une récurrence immédiate sur n permet de conclure que

• si $n \leq 2^{10} - 1$, alors $n = 1$ où $n \in E_p$ avec $p \leq 9$ donc $a_n = 0$ ou $a_n = \varphi_p(n) \geq -9$ (cf. P_2)

• si $n \in E_{10}$, alors $a_n = \varphi_{10}(n) \geq -10$ et a_n est pair (cf. P_2 et P_3). De plus $U_{10} = 1365$ (cf. P_5) donc $U_{10} \in E_{10}$, et $a_{1365} = a_{U_{10}} = -10$ (cf. P_6)

$$a_n = -10 \qquad \text{ssi} \qquad n = 1365.$$

Finalement la valeur minimale de a_n, pour $n \leq 1996$, est -10, avec $a_n = -10$ ssi $n = 1365$.

Parti (b) Remarquons d'abord que $a_1 = 0$.

Si $n \in E_p$, $\varphi_p(n) = (\varepsilon_1, \ldots \varepsilon_p)$ où $\varepsilon_i \in \{-1, 1\}$ et $a_n = \varepsilon_1 + \cdots + \varepsilon_p$ (cf. P_2). Donc $a_n = 0$ ssi il y a autant de $\varepsilon_i = 1$ que de $\varepsilon_j = -1$. Comme φ_p est une bijection de E_p sur F_p, il y a autant de $n \in E_p$ (pour $p \geq$ fixé) tels que $a_n = 0$ que de façons de choisir $\frac{p}{2}$ places pour les "+1" parmi les p places possibles.

Ainsi, d'un part si p est impair $\forall n \in E_p$, $a_n \neq 0$ (on retrouve P_3). D'autre part,

$p = 2$ donne $C_2^1 = 2$ possibilités,

$p = 4$ donne $C_4^2 = 6$ possibilités,

$p = 6$ donne $C_6^3 = 20$ possibilités,

$p = 8$ donne $C_8^4 = 70$ possibilités,

$p = 10$ donne $C_{10}^5 = 252$ possibilités.

Mais $2^{11} - 1 = 2047 > 1996$. Il reste à déterminer combien de $n \in \{1997, \ldots, 2047\}$ vérifient $a_n = 0$.

Or

$$\begin{aligned} a_{62} &= a_{30} + 1 = a_{15} = a_7 + 1 = 3 \quad \text{(on utilise (1)), et} \\ a_{63} &= a_{2^6-1} = 5 \qquad (\text{cf. } P_4). \end{aligned}$$

En utilisant (1) on en déduit

$$a_{124} = 4 = a_{126}, \quad a_{125} = 2, \quad a_{127} = 6\,.$$

Puis

$$\begin{aligned} &a_{249} = a_{251} = a_{253} = 3, \quad && a_{250} = 1, \\ &a_{252} = a_{254} = 5, && a_{255} = 7. \end{aligned}$$

Et encore

$$\begin{aligned} &a_{499} = a_{503} = a_{505} = a_{507} = a_{509} = 4, \\ &a_{504} = a_{508} = a_{510} = 6, \\ &a_{511} = 8, \\ &a_{500} = a_{502} = a_{506} = 2, \\ &a_{501} = 0. \end{aligned}$$

Mais si $n \in \{1997, \ldots, 2047\}$ alors $[\frac{n}{2}] \in \{998, \ldots, 1023\}$;

$$\text{c.à.d.} \qquad \left[\frac{1}{2}\left[\frac{n}{2}\right]\right] \in \{499, \ldots, 511\}\,.$$

Donc $a_n = 0$ ssi

$$a_n = a_{501} + \varepsilon_9 + \varepsilon_{10} \qquad \text{avec} \qquad \varepsilon_9 + \varepsilon_{10} = 0 \quad \text{(deux possibilités)},$$

ou

$$a_n = a_k + \varepsilon_9 + \varepsilon_{10} \quad \text{avec} \quad k \in \{500, 502, 506\} \quad \text{et} \quad \varepsilon_9 = \varepsilon_{10} = -1$$
(trois possibilitiés).

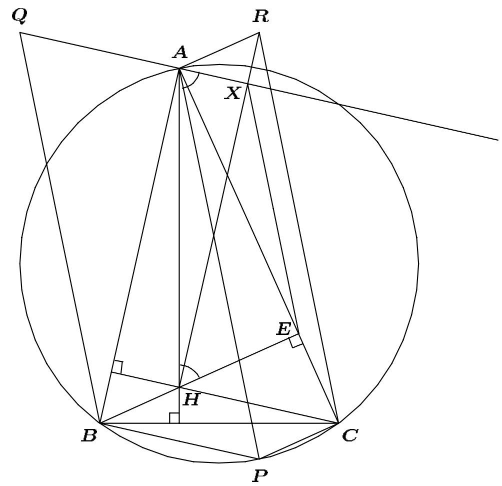
Il ya a donc 5 indices $n \in \{1997, \ldots, 2047\}$ tels que $a_n = 0$ et comme

$$1 + 2 + 6 + 20 + 70 + 252 - 5 = 346\,,$$

le nombre de $n \leq 1996$ tels que $a_n = 0$ est $N = 346$.

9. Let triangle ABC have orthocentre H, and let P be a point on its circumcircle, distinct from A, B, C. Let E be the foot of the altitude BH, let $PAQB$ and $PARC$ be parallelograms, and let AQ meet HR in X. Prove that EX is parallel to AP.

Solution by Toshio Seimiya, Kawasaki, Japan.



Since $AX \| BP$ and A, B, P, C are concyclic we have

$$\angle XAP = \angle APB = \angle ACB .$$

As $AH \perp BC$ and $BH \perp AC$, we get $\angle ACB = \angle AHE$, so that

$$\angle XAP = \angle AHE . \qquad (1)$$

Since $APCR$ is a parallelogram and A, B, P, C are concyclic we have $\angle ARC = \angle APC = \angle ABC$. As $AH \perp BC$ and $CH \perp AB$, we get

$$\angle AHC + \angle ABC = 180^\circ ,$$

so that

$$\angle AHC + \angle ARC = 180^\circ .$$

Hence A, H, C, R are concyclic.

It follows that

$$\angle AHR = \angle ACR = \angle CAP . \qquad (2)$$

From (1) and (2) we have

$$\angle XAE = \angle XAP - \angle CAP = \angle AHE - \angle AHR = \angle XHE .$$

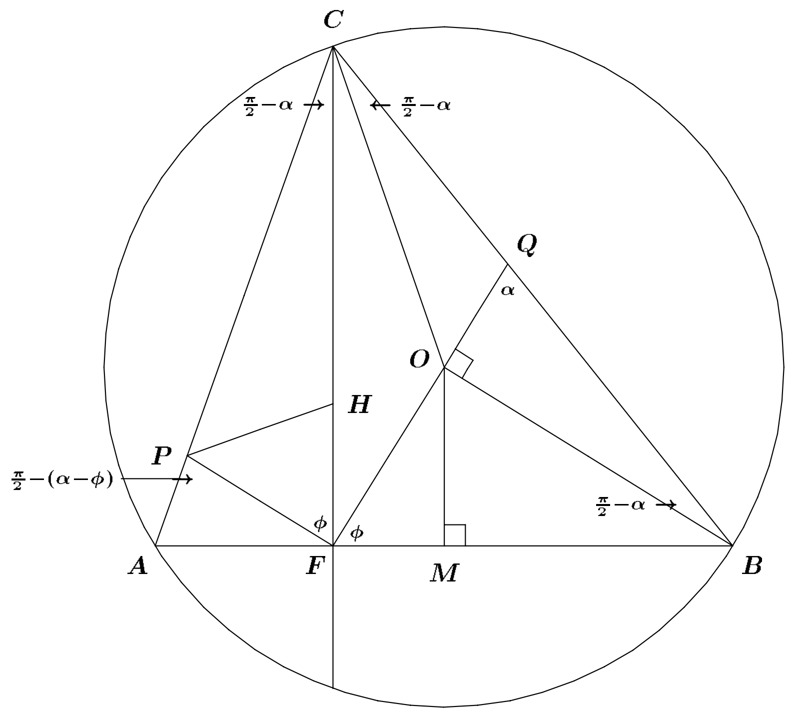
Hence X, A, H, E are concyclic, and

$$\angle AEX = \angle AHX = \angle AHR .$$

Hence we have, from (2), that $\angle AEX = \angle CAP$. Thus $EX \| AP$.

10. Let ABC be an acute-angled triangle with $|BC| > |CA|$, and let O be the circumcentre, H its orthocentre, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$.

Solutions by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.



We denote $\angle CFP = \angle OFB = \phi$. M is the mid-point of AB. Now

$$OM = R\cos\gamma\,, \qquad FM = R\sin(\alpha-\beta)\,,$$

so that

$$\tan\varphi = \frac{\cos\gamma}{\sin(\alpha-\beta)}\,. \tag{1}$$

From the Law of Sines in $\triangle CPF$,

$$\begin{aligned} CF &= 2R\sin\alpha\sin\beta \\ \angle FCP &= \frac{\pi}{2}-\alpha \\ \angle FPC &= \frac{\pi}{2}+\alpha-\varphi \end{aligned} \tag{2}$$

so $CP : CF = \sin\varphi : \cos(\alpha-\varphi)$.

With (2)

$$CP \;=\; \frac{2R\sin\alpha\sin\beta\sin\varphi}{\cos(\alpha-\varphi)} \;=\; \frac{2R\sin\alpha\sin\beta}{\cos\alpha\cot\varphi+\sin(\alpha)}\,. \tag{3}$$

From (1) and (3),

$$CP \;=\; \frac{2R\sin\alpha\cos\gamma}{\sin\beta(\sin^2\alpha-\cos^2\alpha)} \;=\; \frac{-2R\sin\alpha\cos\gamma}{\sin\beta\cos 2\alpha} \tag{4}$$

$$OQ \perp OB, \qquad \angle OBQ \;=\; \frac{\pi}{2}-\alpha \Longrightarrow \angle OQB \;=\; \alpha\,.$$

Now

$$CQ \;=\; a - QB = R\left(2\sin\alpha - \frac{1}{\sin\alpha}\right) \;=\; -R\frac{\cos 2\alpha}{\sin\alpha}\,.$$

Furthermore, $CH = 2R\cos\gamma$, $\quad CO = R$.

It is easy to verify that $CP : CH = CO : CQ$, and

$$-2R\frac{\sin\alpha\cos\gamma}{\sin\beta\cos 2\alpha} : 2R\cos\gamma \;=\; R : \frac{-R\cos 2\alpha}{\sin\alpha}\,. \tag{5}$$

We also have

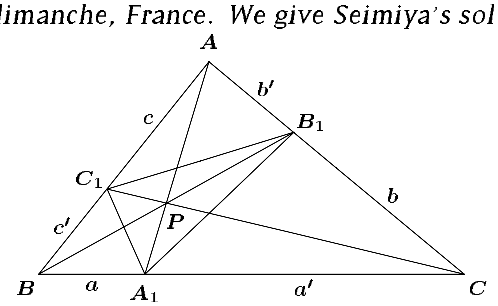
$$\angle PCH \;=\; \angle OCQ \;=\; \frac{\pi}{2}-\alpha\,. \tag{6}$$

From (5) and (6), we have that $\triangle PCH$ and $\triangle OCQ$ are similar. Thus $\angle PHC = \angle OQC = \pi - \alpha$. Thus $\angle FHP = \alpha$.

11. Let ABC be equilateral, and let P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB in the points A_1, B_1, C_1 respectively. Prove that

$$A_1B_1\cdot B_1C_1\cdot C_1A_1 \geq A_1B\cdot B_1C\cdot C_1A\,.$$

Solutions by Toshio Seimiya, Kawasaki, Japan; and by Pierre Bornsztein, Courdimanche, France. We give Seimiya's solution.



We put $A_1B = a$, $A_1C = a'$, $B_1C = b$, $B_1A = b'$, $C_1A = c$ and $C_1B = c'$. Then we have by Ceva's Theorem

$$abc = a'b'c'. \tag{1}$$

Since $\angle B_1AC_1 = 60^\circ$ we have

$$\begin{aligned} B_1C_1^2 &= (b')^2 + c^2 - 2b'c\cos 60^\circ \\ &= (b')^2 + c^2 - b'c \geq b'c. \end{aligned}$$

Similarly we have

$$C_1A_1^2 \geq c'a \quad \text{and} \quad A_1B_1^2 \geq a'b.$$

Multiplying these three inequalities, we get

$$B_1C_1^2 \cdot C_1A_1^2 \cdot A_1B_1^2 \geq b'c \cdot c'a \cdot a'b. \tag{2}$$

From (1) and (2) we have

$$B_1C_1^2 \cdot C_1A_1^2 \cdot A_1B_1^2 \geq a^2b^2c^2.$$

Thus we have

$$B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq abc.$$

That is

$$B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq A_1B \cdot B_1C \cdot C_1A.$$

That completes the *Corner* for this issue. Send me your nice solutions as well as Olympiad contests and materials.

Professor Dan Pedoe

The Editors of ***CRUX with MAYHEM*** are saddened to learn that Dan Pedoe died a few weeks ago, on 27 October 1998. He made major contributions to the success of ***CRUX*** during its first ten years, and always maintained his interest in the journal.

Readers may be interested to read his autobiographical notes in the recent COLLEGE MATH. JOURNAL 29:3 (May 1998) 170-188.

BOOK REVIEWS

Edited by ANDY LIU

This is my last column as editor of the Book Reviews column. Looking back over the past five years, I am happy that many books have been brought to the attention of the readers, obscure titles as well as famed tomes. While some are of marginal quality, and it is not our duty to just sing praises here, the great majority are outstanding publications. They are tremendous resources in our hands. Let us make good use of them.

Grateful acknowledgement is due to the publishers who provided the volumes for review in the first place. The Mathematical Association of America should be singled out for supplying the column with every title it has published in popular mathematics, and it is an ample and steady source. Gratitude is owed to all reviewers, solicited from as wide a spectrum as possible, particularly geographically. My friend and mentor Murray Klamkin regularly graced these pages with his much-esteemed opinion, and deserves a special thank you.

Book Reviews columns as a rule do not generate much fan mail. However, I have been pleasantly surprised how often I hear remarks from satisfied readers, offering corrections, suggesting titles, and providing constructive criticism. Recently, a reader from France called and pointed out that the Mathematical Association of America has *not* acquired Martin Gardner's **Sixth Book of Mathematical Diversions from Scientific American**, as claimed in the Mini-reviews in the February issue, but the right belongs to the University of Chicago Press.

The Last Recreations by Martin Gardner,
published by Springer-Verlag, New York, 1997,
ISBN# 0-387-94929-1, hardcover, 392+ pages, $25.00.
Reviewed by ***Andy Liu***.

This is the fifteenth and last of Martin's Scientific American anthologies, but the first with this publishing giant. This may have something to do with the move from W. H. Freeman to Springer-Verlag of Jerry Lyons, the outstanding editor of mathematics books. The subtitle is **Hydras, Eggs, and Other Mathematical Mystifications**.

The material is from the Scientific American columns from December 1979 to his retirement in December 1981, though that particular column, titled "The Laffer Curve", was in an earlier volume, **Knotted Doughnuts**. Included also are three chapters Martin later wrote as guest columnist, on August and September 1983 as well as June, 1986. The last chapter, titled "Trivalent Graphs, Snarks, and Boojums", is a drastically revised version of his April 1976 column, which he had originally intended to leave out, in view of rapid advancements on the status of the Four Colour Problem. The reviewer is glad to see its inclusion. Thus only the October 1975 column, titled "Extrasensory Perception by Machines", is missing from the fifteen volumes. It is anthologized in a science fiction puzzle collection.

By now, everything that can be said about Martin's writing has been said many times over. Buy this book at once and treasure it! Although Martin continues to contribute to various journals, this is **The Last Recreations** under his own banner!

The Editor-in-Chief, Bruce Shawyer, and his predecessor, Bill Sands, would like to take this opportunity of thanking Andy Liu for his long and sterling work in this column.

"**Thank you very much, Andy. All the best!**"

The 1998 Crossword Puzzle

Peter Hurthig, Columbia College, Burnaby, BC.

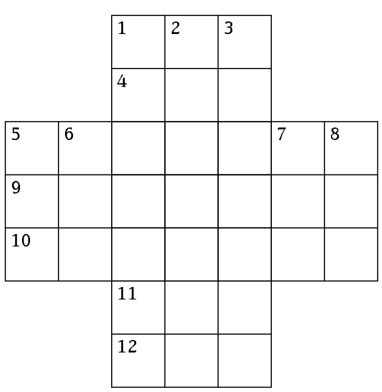
Four Mathematicians, a Detective, a King and a French Lady

Across:

1. A word on a towel.
4. An abbreviated diophantine solution.
5. A Russian who solved number problems.
9. King Zog's country.
10. A graduate student who impressed Gauss!
11. Ruth Rendell's inspector (to his friends).
12. One third of a communist or one half of a fly.

Down:

1. A German whose problems were numbered.
2. Brands with dishonour.
3. British weight.
5. A bony fish.
6. A high priest.
7. She wrote *Delta of Venus*.
8. Professor Pedoe.



THE SKOLIAD CORNER

No. 34

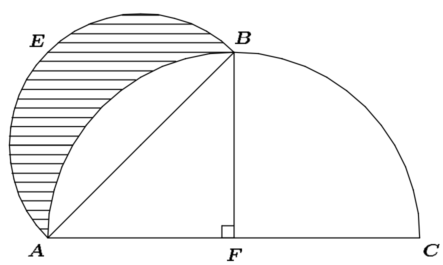
R.E. Woodrow

We begin this issue with the problems of the Old Mutual Mathematical Olympiad 1991 Final Paper 1 and Final Paper 2. The Preliminary Round of this contest is in a multiple choice format and I plan to give an example with the 1992 paper in the new year. My thanks go to John Grant McLoughlin, of the Faculty of Education, Memorial University of Newfoundland, for collecting this contest and forwarding it for use in the *Corner*.

OLD MUTUAL MATHEMATICAL OLYMPIAD 1991
Final Paper 1

Time: 2 hours

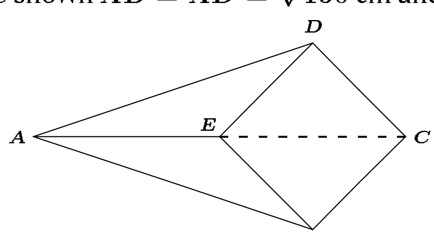
1. In the figure shown ABC and AEB are semi-circles and F is the mid-point of AC and $AF = 1$ cm. Find the area of the shaded region.



2. What is the value of $\sqrt{17 - 12\sqrt{2}} + \sqrt{17 + 12\sqrt{2}}$ in its simplest form?

3. In a certain mathematics examination, the average grade of the students passing was $x\%$, while the average of those failing was $y\%$. The average of all students taking the examination was $z\%$. Find the percentage who failed in terms of x, y and z.

4. In the figure shown $AB = AD = \sqrt{130}$ cm and $BEDC$ is a square.



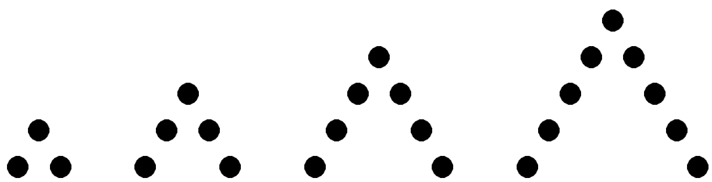
Also the area of $\triangle AEB$ = area of square $BEDC$.

Find the area of $BEDC$.

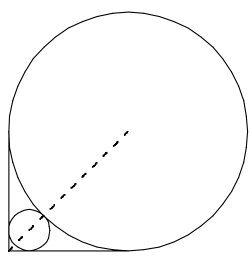
Final Paper 2

Time: 2 hours

1. If the pattern below of dot-figures is continued, how many dots will there be in the 100^{th} figure?



2. It is required to place a small circle in the space left by a large circle as shown. If the radius of the large one is a and that of the small one is b, find the ratio a/b.



3. Find **all** solutions to the simultaneous equations

$$\begin{aligned} x + y &= 2\,, \\ xy - z^2 &= 1\,, \end{aligned}$$

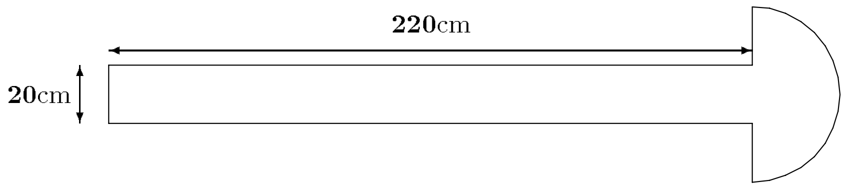
and **prove** that there are no other solutions.

4. If a, b, c and d are numbers such that

$$\begin{aligned} a + b &< c + d\,, \\ b + c &< d + e\,, \\ c + d &< e + a\,, \\ \text{and} \quad d + e &< a + b\,, \end{aligned}$$

prove that the largest number is a and the smallest is b.

5. The diagram below [rotated through 90°] shows a container whose lower part is a hemisphere and whose upper part is a cylinder.



The cylindrical part has internal diameter of 20 cm and is 220 cm long. Water is poured into it and rises to a height of 20 cm in the cylindrical part. The top is then sealed with a flat cover and the container is turned upside down. The water is now 200 cm high in the cylindrical part.

(i) Calculate the volume of the hemisphere in terms of π.

(ii) Find the total height of the container.

[Note: The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.]

Last number we gave the problems of the British Columbia Colleges, Senior High School Mathematics Contest Preliminary Round, 1998. Next we give the "official" solutions, courtesy of Jim Totten, The University College of the Cariboo, and an organizer of the contest.

BRITISH COLUMBIA COLLEGES
Senior High School Mathematics Contest
Preliminary Round 1998

Time: 45 minutes

1. The integer $1998 = (n-1)n^n(10n+c)$ where n and c are positive integers. It follows that c equals: **(d)**.

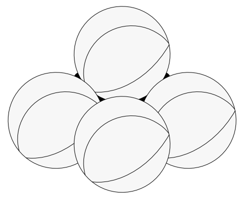
Solution. By comparing the expression $1998 = (n-1)n^n(10n+c)$ with $1998 = 2 \times 3^3 \times 37$, we get $n = 3$, since 3^3 is the only power dividing 1998. Hence, $10n + c = 37$, which gives $c = 7$.

2. The value of the sum $\log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \cdots + \log\frac{9}{10}$ is: **(a)**.

Solution. We have:

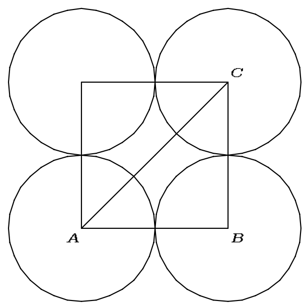
$$\log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \cdots + \log\frac{9}{10} = \log(\frac{1}{2}\frac{2}{3}\frac{3}{4}\cdots\frac{9}{10}) = \log\frac{1}{10} = -1.$$

3. Four basketballs are placed on the gym floor in the form of a square with each basketball touching two others. A fifth basketball is placed on top of the other four so that it touches all four of the other balls, as shown.

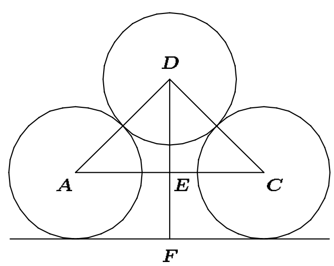


If the diameter of a basketball is 25 cm, the height, in centimetres, of the centre of the fifth basketball above the gym floor is: **(d)**.

Solution. Let us first take a look at the four basketballs lying on the floor:



We find $AC = \sqrt{AB^2 + BC^2} = \sqrt{25^2 + 25^2} = 25\sqrt{2}$. Now, in order to find the height of the centre of the fifth ball, we analyze the cross-section of the pyramid by a vertical plane passing through A and C:



The height $DF = DE + EF$, where $EF = \frac{25}{2}$ and $DE = \sqrt{AD^2 - (\frac{1}{2}AC)^2}$ $= \sqrt{25^2 - (\frac{1}{2}25\sqrt{2})^2} = \frac{25}{2}\sqrt{2}$. Hence, $DF = \frac{25}{2}(1 + \sqrt{2})$.

4. Last summer I planted two trees in my yard. The first tree came in a fairly small pot and the hole that I dug to plant it in filled one wheelbarrow load of dirt. The second tree came in a pot, the same shape as that of the first tree, that was one-and-a-third times as deep as the first pot and one-and-a-half times as big around. Let us make the following assumptions:

(i) The hole for the second tree was the same shape as for the first tree.

(ii) The ratios of the dimensions of the second hole to those of the first hole are the same as the ratios of the dimensions of the pots.

Based on these assumptions, the number of wheelbarrows of dirt that I filled when I dug the hole for the second tree was: **(c)**.

Solution. To solve the problem we need to find the ratio of the volume of the second pot to the volume of the first. We can assume that both pots are generalized cylinders with bases of the same shape. It is convenient to think that the second pot was obtained from the first by stretching it horizontally and vertically. The change of the perimeter is proportional to the change of horizontal dimensions, so that both horizontal dimensions have been stretched one-and-a-half times. Therefore, the area of the base has increased $1\frac{1}{2} \times 1\frac{1}{2} = \frac{9}{4}$ times. Since the height has been stretched $1\frac{1}{3}$ times, the volume increased $\frac{9}{4} \times 1\frac{1}{3} = 3$ times.

5. You have an unlimited supply of 5-gram and 8-gram weights that may be used in a pan balance. If you use only these weights and place them only in one pan, the largest number of grams that you cannot weigh is: **(b)**.

Solution. If W is the weight in grams then W can be weighed if $W = 5x + 8y$, where x and y are non-negative integers. Furthermore, y can be written in the form $5z + r$, where z is a non-negative integer and r is either $0, 1, 2, 3$, or 4. Thus $W = 5(x + 8) + 8(0)$, $W = 5(x + 8) + 8(1)$, $W = 5(x+8)+8(2)$, $W = 5(x+8)+8(3)$, or $W = 5(x+8)+8(5)$. These can be simplified to $W = 5k$, $W = 5(k+1)+3$, $W = 5(k+3)+1$, $W = 5(k+4)+4$ and $W = 5(k + 6) + 2$. By putting $k = x + 8z$, we conclude that W can be weighed if $W = 5k + 8r$, with $k \geq 0$ and $r \leq 4$. Thus, according to the value of r, $W = 5k + 8(0)$, $W = 5k$, $W = 5(k + 1) + 3$, $W = 5(k + 3) + 1$, $W = 5(k + 4) + 4$, $W = 5(k + 6) + 2$. On the other hand, W (as well as every integer) can take exactly one of the forms: $5K$, $5K + 1$, $5K + 2$, $5K + 3$, or $5K + 4$. By comparing these forms with the expressions for W that can be weighed, we conclude:

(i) if W is of the form $5K$, then it can be weighed;

(ii) if W is of the form $5K + 1$, then it can be weighed if $K = k + 3$; that is, when $K \geq 3$;

(iii) if W is of the form $5K + 2$, then it can be weighed if $K = k + 6$; that is, when $K \geq 6$;

(iv) if W is of the form $5K + 3$, then it can be weighed if $K = k + 1$; that is, when $K \geq 1$;

(v) if W is of the form $5K + 4$, then it can be weighed if $K = k + 4$; that is, when $K \geq 4$.

Consequently, the largest values of W that cannot be weighed in categories: (ii), (iii), (iv), and (v), are $5(2) + 1 = 11$, $5(5) + 2 = 27$, $5(0) + 3 = 3$, $5(3) + 4 = 19$, respectively. The largest of them is 27.

6. If all the whole numbers from 1 to 1,000,000 are printed, the number of times that the digit 5 appears is: **(c)**.

Solution. **Method I**. The number of times the numeral 5 appears will not change if instead of printing the numbers from 1 to 1,000,000 we will print the numbers from 0 to 999,999. Furthermore, the number of appearances of 5 will not change if we complete the decimal representation of each number to a six-digit sequence, by writing some zeros in front of the number, if necessary. For example, 1998 will be represented by 001,998. The printout of all of the six-digit sequences will have a total of 6×10^6 digits and each digit will appear the same number of times. This gives $\frac{6\times 10^6}{10} = 600{,}000$ appearances of 5.

Method II. (Standard, but more tedious.) Again, as above, we represent the numbers by six-digit sequences. At first, we count the number of sequences that have precisely k copies of 5, for $1 \le k \le 6$. The numeral 5 can appear in $\binom{6}{k}$ different positions in such a sequence, while in each of the remaining $6-k$ positions we can have one of the nine digits other than 5. This gives $\binom{6}{k}9^{6-k}$ different sequences. They provide $k\binom{6}{k}9^{6-k}$ copies of 5. Hence, the number of appearances of 5 in all of the sequences is given by the sum:

$$1\binom{6}{1}9^5 + 2\binom{6}{2}9^4 + 3\binom{6}{3}9^3 + 4\binom{6}{4}9^2 + 5\binom{6}{5}9^1 + 6\binom{6}{6}9^0 \;=\; 600{,}000 .$$

7. The perimeter of a rectangle is x centimetres. If the ratio of two adjacent sides is $a : b$, with $a > b$, then the length of the shorter side, in centimetres, is: **(e)**.

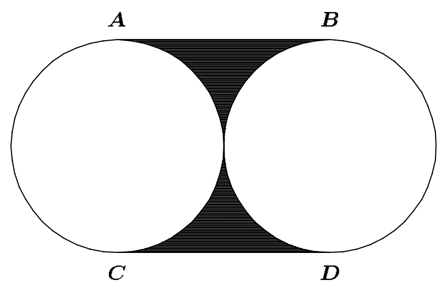
Solution. If a_1 and b_1 are the lengths of the longer and the shorter sides of the rectangle, then $b_1 : x = b : [2(a+b)]$. This gives $b_1 = \frac{bx}{2(a+b)}$.

8. The sum of the positive solutions to the equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$ is: **(e)**

Solution. From $x^{x\sqrt{x}} = (x\sqrt{x})^x$ we get that that $x^{xx^{1/2}} = (xx^{1/2})^x$, which simplifies to $x^{x^{3/2}} = x^{(3/2)x}$. This has the trivial solution $x = 1$. If $x \neq 1$ then the exponents must be equal; that is $x^{3/2} = \frac{3}{2}x$. By squaring both sides, $x^3 = \frac{9}{4}x^2$. Therefore, $x = \frac{9}{4}$, since x is positive. The sum of the solutions is $1 + \frac{9}{4} = 3\frac{1}{4}$.

9. Two circles, each with a radius of one unit, touch as shown. (See diagram on next page.) AB and CD are tangent to each circle. The area, in square units, of the shaded region is: **(d)**.

Solution. The square $ABCD$ has area of $2 \times 2 = 4$ square units. To find the shaded area we need to remove the area of two semicircles from the area of $ACBD$. Thus, the area of the shaded region is $4 - 2(\frac{1}{2})\pi = 4 - \pi$ square units.



10. A parabola with a vertical axis of symmetry has its vertex at $(0, 8)$ and an x–intercept of 2. If the parabola goes through $(1, a)$, then a is: **(c)**

Solution. The parabola has another x–intercept of -2 symmetric to the x–intercept of 2 with respect to the vertical axis of symmetry passing through $(0, 8)$. This implies that the equation of the parabola is of the form $y = A(x - 2)(x + 2)$. The parabola has the vertex at $(0, 8)$: therefore $8 = A(0 - 2)(0 + 2)$. This gives $A = -2$, so that the equation of the parabola is $y = -2(x - 2)(x + 2)$. If a is the value of y corresponding to $x = 1$, then $a = y = -2(1 - 2)(1 + 2) = 6$.

11. A five litre container is filled with pure orange juice. Two litres of juice are removed and the container is filled up with pure water and mixed thoroughly. Then two litres of the mixture are removed and again the container is filled up with pure water. The percentage of the final mixture that is orange juice is: **(d)**.

Solution. Initially, there were 5 litres of juice, and after the removal of the first two litres, 3 litres of juice are left. After mixing this amount with 2 litres of pure water, each litre of the mixture will contain $\frac{3}{5}$ litres of pure juice. Thus, when we remove 2 litres of the mixture, $3 - 2(\frac{3}{5}) = \frac{9}{5}$ litres of pure juice will remain. This gives the concentration of $[(\frac{9}{5}) \div 5] \times 100\% = 36\%$.

12. The lengths of the sides of a triangle are $b + 1$, $7 - b$ and $4b - 2$. The number of values of b for which the triangle is isosceles is: **(b)**

Solution. Potentially, we have three possibilities for the triangle to be isosceles:

(1) $b + 1 = 7 - b$,

(2) $b + 1 = 4b - 2$, and

(3) $7 - b = 4b - 2$.

The first possibility gives $b = 3$ and, consequently, 4, 4, and 10 as possible lengths of the sides of the triangle. The second possibility gives $b = 1$, and 2, 2, and 6 as possible lengths. Finally, the third possibility gives $b = \frac{9}{5}$, and $\frac{14}{5}$, $\frac{26}{5}$, and $\frac{26}{5}$ as the corresponding lengths. However, the sum of lengths of any two sides in a triangle is greater than or equal to the length of the third side. This leaves $b = \frac{9}{5}$ as the only possibility.

13. The number of times in one day when the hands of a clock form a right angle is: **(d)**.

Solution. Suppose that we measure the time from midnight to the next midnight. If ϕ_1 is the angle made by the minute hand and ϕ_2 the angle made by the hour hand turned clockwise from 12 then $\phi_1 = 2\pi t$ and $\phi_2 = \frac{2\pi}{12}t = \frac{1}{6}\pi t$, where t represents time measured in hours. The angle between both hands is 90° when $\phi_1 - \phi_2 = \frac{1}{2}\pi + 2k\pi$ or $\phi_1 = \phi_2 = \frac{3}{2}\pi t + 2k\pi$. By solving these equations for t, we get $t = \frac{6}{11} + \frac{12}{11}k$ or $t = \frac{18}{11} + \frac{12}{11}k$. Since $0 \leq t \leq 24$, each of the equations yields 22 solutions in non-negative integers k. This corresponds to the total of 44 solutions.

14. In my town some of the animals are really strange. Ten percent of the dogs think they are cats and ten percent of the cats think they are dogs. All the other animals are perfectly normal. One day I tested all the cats and dogs in the town and found that 20% of them thought that they were cats. The percentage of the dogs and cats in the town that really are cats is: **(a)**

Solution. Let d and c denote the number of dogs and cats in the town, respectively. Then the number of animals who responded that they are cats is $0.2(d+c)$. This corresponds to $0.1d + 0.9c$, according to the truthfulness of our animals. By equating both expressions we get $0.1d = 0.7c$, or $d = 7c$. Thus, the percentage of cats is $\frac{c}{c+d} \times 100 = \frac{c}{c+7c} \times 100 = \frac{100}{8}\% = 12.5\%$.

15. A short hallway in a junior high school contains a bank of lockers numbered one to ten. On the last day of school the lockers are emptied and the doors are left open. The next day a malicious math student walks down the hallway and closes the door of every locker that has an even number. The following day the same student again walks down the hallway and for every locker whose number is a multiple of three closes the door if it is open and opens it if it is closed. On the next day the student does the same thing with every locker whose number is divisible by four. If the student continues this procedure for a total of nine days, the number of lockers that are closed after the ninth day is: **(d)**.

Solution. A locker number k changes its status from open to closed or vice-versa $n(k)$ times, where $n(k)$ denotes the number of distinct divisors of k that are greater than 1 and less or equal to 10. Thus, $n(1) = 0$, $n(2) = 1$, $n(3) = 1$, $n(4) = 2$, $n(5) = 1$, $n(6) = 3$, $n(7) = 1$, $n(8) = 3$, $n(9) = 2$, $n(10) = 3$. Since all lockers are initially open, the k^{th} locker is closed after the period of nine days if $n(k)$ is odd. Therefore, seven lockers will be closed.

That completes the *Skoliad Corner* for this issue. Please send me your comments and suggests for the evolution of the *Corner*. Also, I need suitable contest material at the pre-Olympiad level for use in future *Corners*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), Richard Hoshino (University of Waterloo), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. We request that solutions from the previous issue be submitted in time for publication in issue 8 of 1999.

High School Solutions

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H229. Here is a simple way to remember how many books there are in the Bible. Remember that there are x books in the Old Testament, where x is a two-digit integer. Then multiply the digits of x to get a new integer y, which is the number of books in the New Testament. Adding x and y, you end up with 66, the number of books in the Bible. What are x and y?

Solution by Katya Permiakova, student, Lisgar Collegiate Institute, Ottawa, Ontario.

Let $x = 10a + b$, where a and b are integers satisfying the conditions $1 \leq a \leq 9$ and $0 \leq b \leq 9$. Then $y = ab$. Now we are given that $10a + b + ab = 66$. Rearranging terms and solving for b, we get $b(a + 1) = 66 - 10a$, so $b = \frac{66-10a}{a+1} = -10 + \frac{76}{a+1}$. Now in order for b to be an integer, $a + 1$ must divide 76. The only positive divisors of 76 are 1, 2, 4, 19, 38, and 76. Since our choice for a is limited to the integers between 1 and 9, the only possibilities for a are 1 and 3 (since that gives us $a + 1 = 2$ and $a + 1 = 4$, respectively).

If $a = 1$, then we have $b = -10 + \frac{76}{2} = 28$, but this does not satisfy $b \leq 9$. However, if $a = 3$, then $b = -10 + \frac{76}{4} = 9$, and this is legitimate.

Hence $x = 39$ and $y = 27$.

Also solved by KEON CHOI, student, A.Y. Jackson Secondary School, North York, Ontario; LINO DEMASI, student, St. Ignatius High School, Thunder Bay, Ontario; and WENDY YU, student, Woburn Collegiate Institute, Scarborough, Ontario.

H230. Dick and Cy stand on opposite corners (on the squares) of a 4×4 chessboard. Dick is telling too many bad jokes, so Cy decides to chase after him. They take turns moving one square at a time, either vertically or horizontally on the board. To catch Dick, Cy must land on the square Dick is on. Prove that:

(i) If Dick moves first, Cy can eventually catch Dick.

(ii) If Cy moves first, Cy can never catch Dick.

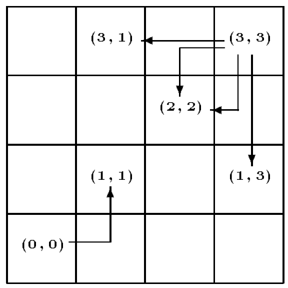
(Can you generalize this to a $2m \times 2n$ chessboard?)

Solution.

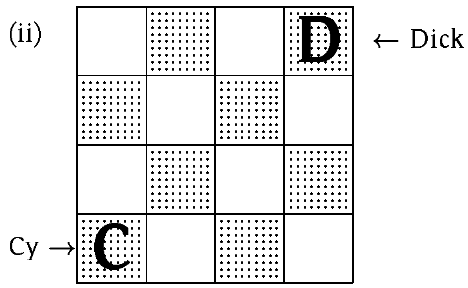
(i) Place coordinates on the board so that Cy is standing on $(0, 0)$ and Dick is standing on $(3, 3)$. We shall show that after a few moves, Cy can catch Dick on a turn. Regardless of what Dick does on his first two turns, Cy can move to $(1, 1)$ after two moves. Now it is Dick's turn. At that time, Dick must be on $(1, 3)$, $(2, 2)$, $(3, 1)$, or $(3, 3)$. So if on his next move, Dick goes to either $(1, 2)$ or $(2, 1)$, Cy is standing one square away and so Cy moves into Dick's square on his next move, and catches Dick. So Dick must move to one of $(0, 3)$, $(2, 3)$, $(3, 2)$, or $(3, 0)$.

If Dick goes to $(0, 3)$ or $(2, 3)$, then Cy can go to $(1, 2)$, and from here it is easy to see that Dick can last at most two moves before he gets caught (since Cy can trap him into a corner). If Dick goes to $(3, 2)$ or $(3, 0)$, then Cy can go to $(2, 1)$, by the same argument, Cy can catch Dick. Thus no matter what, if Dick moves first, then Cy can eventually catch Dick.

(i)



(ii)



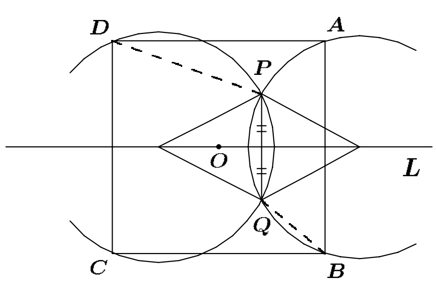
(ii) Colour the 4×4 board in black and white, as in a regular chessboard (so adjacent squares are of different colours). Thus, if $(0, 0)$ is a black square, then $(3, 3)$ must be a black square as well. So for each move, if a person is on a square of a certain colour, then he will move to a square of the other colour.

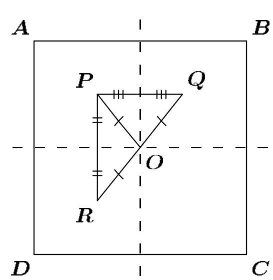
Hence, Dick and Cy both start off on a black square. If Cy moves first, then Cy moves onto a white square (while Dick remains on a black square). Then Dick moves to some white square. Now Dick and Cy are both on white squares, and so on Cy's next move he must move onto a black square (while Dick remains on a white square). Thus whenever Dick is on a square of a certain colour, Cy is moving to a square of the other colour. And so, on any given move, Cy can never move to a square that Dick is currently on, and so Cy will not be able to catch Dick.

Also solved by KEON CHOI, student, A.Y. Jackson Secondary School, North York, Ontario; LINO DEMASI, student, St. Ignatius High School, Thunder Bay, Ontario; and WENDY YU, student, Woburn Collegiate Institute, Scarborough, Ontario.

H231. Let O be the centre of the unit square $ABCD$. Pick any point P inside the square other than O. The circumcircle of PAB meets the circumcircle of PCD at points P and Q. The circumcircle of PAD meets the circumcircle of PBC at points P and R. Show that $QR = 2 \cdot OP$.

Solution by Wendy Yu, student, Woburn Collegiate Institute, Scarborough, Ontario and Keon Choi, student, A.Y. Jackson Secondary School, North York, Ontario.





Construct line L through O parallel to AD and BC. All points on this line are the same distance from A as from B, and the same distance from C as from D. Thus this line contains the centres of the circumcircles of PAB and PCD. Hence, the line L bisects segment PQ. So the point Q must be the reflection of P about the line L, and it follows that $OP = OQ$. Similarly, if we construct line M through O parallel to AB and CD, then R is the reflection of P about the line M. Hence, $OP = OR$. Because PQ and PR are perpendicular (since the lines L and M are perpendicular), PQR is a right-angled triangle. Furthermore, $OP = OQ = OR$, which implies that O is the midpoint of the hypotenuse QR. Hence, we have $QR = OQ + OR = OP + OP = 2OP$, and so $QR = 2OP$, as desired.

Also solved by LINO DEMASI, student, St. Ignatius High School, Thunder Bay, Ontario.

H232. Lucy and Anna play a game where they try to form a ten-digit number. Lucy begins by writing any digit other than zero in the first place, then Anna selects a different digit and writes it down in the second place, and they take turns, adding one digit at a time to the number. In each turn, the digit selected must be different from all previous digits chosen, and the number formed by the first n digits must be divisible by n. For example, 3, 2, 1 can be the first three moves of a game, since 3 is divisible by 1, 32 is divisible by 2 and 321 is divisible by 3. If a player cannot make a legitimate move, she loses. If the game lasts ten moves, a draw is declared.

(i) Show that the game can end up in a draw.

(ii) Show that Lucy has a winning strategy and describe it.

I. *Solution by Lino Demasi, student, St. Ignatius High School, Thunder Bay, Ontario.*

The number 3,816,547,290 has the property that the number formed by the first n digits is divisible by n, for $n = 1, 2, 3, \ldots, 10$. Thus, if the moves are carried out in this order, then the game can end up in a draw.

Here is Lucy's winning strategy. First note that Anna must play an even digit on each of her moves. So Lucy's goal is to play as many even numbers as possible. So Lucy plays a 6 to start. There are three cases to be considered for Anna's second move:

(1) If Anna plays a 4 or a 2, then Lucy plays the other on the third move. Anna must now play an even number because her number now has to be divisible by 4, so if Anna plays an 8, then Lucy plays a 0, and Anna loses because on the sixth move, she would have to play an even number and there are none left. If Anna plays a 0, then Lucy plays a 5, and then Anna also loses because she must now play an even number on the sixth move, the only one of which is an 8, but neither 642,058 or 624,058 is divisible by 6.

(2) If Anna plays a 0, then Lucy plays a 9. Then Anna must play a 2 to make the four-digit number divisible by 4. Lucy then plays a 5. Anna must now play an 8 to make her number divisible by 6. Then Lucy can counter with a 3, since 6,092,583 is divisible by 7. The only even number Anna can now play is a 4, but 60,925,834 is not divisible by 8, so she loses.

(3) If Anna plays an 8, then Lucy plays a 4. Anna's only choice now is a 0. Then Lucy plays a 5. Now Anna's only choice is a 2, but 684052 is not divisible by 6, so she loses.

Note that this covers all the cases because Anna must play an even digit on the second move. Thus, Lucy can always force a win.

II. *Solution by Wendy Yu, student, Woburn Collegiate Institute, Scarborough, Ontario.*

As before, if the game is played in the following order: 3, 8, 1, 6, 5, 4, 7, 2, 9, 0, then the game will end up in a draw.

For Lucy's winning strategy, she can start off with a 4. Then, Anna must counter with an even number. So if she responds with a 2 or an 8, then Lucy's next move is a 0. If Anna's response is a 0 or a 6, then Lucy's next move is a 2. Now, in the case where the number 480 has been written, Anna cannot find a digit to make a four-digit number divisible by 4, so she immediately loses. In the other three cases, there is at least one digit that Anna can pick to remain in the game.

Thus, after four moves, if the game lasts that long, one of the following numbers will be on the board: 4028, 4208, 4620, or 4628. Then Lucy picks a 5, and in each of those four cases, it will be impossible for Anna to then make a move so that the new six-digit number is divisible by 6, since the digits she needs are all taken. Thus, if Lucy follows this strategy, she can always force a win.

Also solved by KEON CHOI, student, A.Y. Jackson Secondary School, North York, Ontario.

Advanced Solutions

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A205. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \leq x$ and $f(x+y) \leq f(x) + f(y)$ for all reals x and y.

Solution.

We have

$$f(x) \leq x, \qquad (1)$$

$$f(x+y) \leq f(x) + f(y). \qquad (2)$$

Let $x = y = 0$ in (2). We have $f(0) \leq 2f(0)$ which implies $0 \leq f(0)$. But for $x = 0$ in (1), we have $f(0) \leq 0$ so $f(0) = 0$.

Now take $y = -x$ in (2) to get $f(0) \leq f(x) + f(-x)$ for all real x. In other words, $-f(x) \leq f(-x)$. But $f(-x) \leq -x$ by (1) so $-f(x) \leq -x$

implies that $f(x) \geq x$ for all real x. Thus combining this with (1) we have $f(x) = x$ for all real x. Now it can be easily checked that this function satisfies the two conditions (1) and (2) in the problem.

A206. Let n be a power of 2. Prove that from any set of $2n-1$ positive integers, one can choose a subset of n integers such that their sum is divisible by n.

Solution.

Let $n = 2^m$, for some integer m. We will prove this result by mathematical induction on m. For $m = 0$, $n = 1$ and the result clearly holds.

Now assume the result is true for some arbitrary $k \geq 0$. In other words, for $n = 2^k$, any set of $2^{k+1} - 1$ positive integers has a subset of n integers whose sum is divisible by n.

Now consider any set, A, of $2^{k+2} - 1$ positive integers. Let A_1 and A_2 be the two subsets of A consisting of the first $2^{k+1} - 1$ integers and the last $2^{k+1} - 1$ integers respectively. By the induction hypothesis, each of these sets has 2^k integers whose sum is divisible by 2^k. Call the subset of 2^k numbers, from A_1, B_1 and the subset of 2^k numbers, from A_2, B_2. Now call the set of integers remaining from A when the 2^{k+1} integers from B_1 and B_2 are removed from A_3. Now A_3 also has $(2^{k+2} - 1) - 2(2^k) = 2^{k+1} - 1$ elements, so again by the induction hypothesis it has a subset, call it B_3, of size 2^k whose sum is divisible by 2^k.

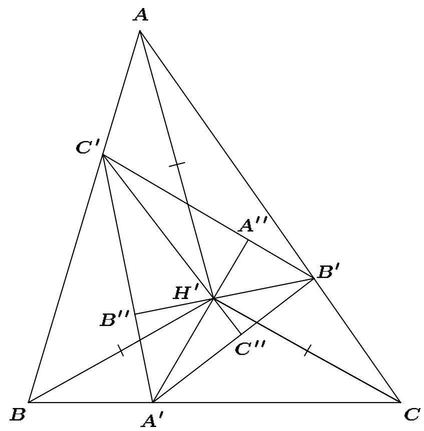
It remains to show that two of the three sets B_1, B_2, and B_3 can be combined to form a set of 2^{k+1} integers whose sum is divisible by 2^{k+1}. Let the sum of the three sets be s_1, s_2, and s_3 respectively. Each is divisible by 2^k, so let $t_i = s_i/2^k$, for $i = 1$, 2, and 3. Now by the Pigeonhole Principle, at least two of these numbers must have the same parity, even or odd. Without loss of generality, let the two sets be A_1 and A_2. The sum of two numbers with the same parity is even, so $t_1 + t_2$ is even. Multiplying by 2^k, we have $s_1 + s_2$ is divisible by 2^{k+1}. Thus there are 2^{k+1} integers from the original set A whose sum is divisible by 2^{k+1}. Our induction on m is complete.

A207. Given triangle ABC, let A', B', and C' be on the sides BC, AC, and AB respectively such that $\triangle A'B'C' \sim \triangle ABC$. Find the locus of the orthocentre of all such triangles $A'B'C'$.

Solution by Alexandre Trichtchenko, student, Brookfield High School, Ottawa, Ontario.

Here we give the solution when the triangle ABC is acute. A similar argument can be given for an obtuse triangle.

Let $\angle BAC = \alpha$, $\angle ABC = \beta$, and $\angle BCA = \gamma$. Let A'', B'', and C'' be the feet of the altitudes from vertices A', B', and C' respectively of triangle $A'B'C'$. Further, let H' be the point of the orthocentre of triangle $A'B'C'$, the point of intersection of its altitudes.



Since triangles $B'H'C''$ and $B'A'B''$ are similar $\angle B'H'C'' = \alpha$. Likewise, $\angle A'H'C'' = \beta$. Thus $\angle B'H'A' = \angle B'H'C'' + \angle A'H'C'' = \alpha + \beta$.

Also, $\angle A'H'B' + \angle B'CA' = \alpha + \beta + \gamma = 180°$. Hence, the quadrilateral $CB'H'A'$ is cyclic. Since $\angle B'A'H'$ and $\angle B'CH'$ are inscribed in the circumcircle of $CB'H'A'$ and subtended by the same arc $H'B'$, we have $\angle H'CB' = \angle H'A'B' = 90° - \beta$. Similarly, $\angle H'AB' = 90° - \beta$. So $\angle H'CB' = \angle H'AB'$, and so $AH' = CH'$. By similar reasoning, we can show that $AH' = BH' = CH'$. Thus H' is the circumcentre of triangle ABC and is independent of the choice of triangle $A'B'C'$. Thus the locus of the orthocentre of all triangles $A'B'C'$ is just the single point $H' = O$, the circumcentre of triangle ABC.

Also solved by D.J. SMEENK, Zaltbommel, the Netherlands.

A208. Let p be an odd prime, and let S_k be the sum of the products of the elements $\{1, 2, \dots, p-1\}$ taken k at a time. For example, if $p = 5$, then $S_3 = 1 \times 2 \times 3 + 1 \times 2 \times 4 + 1 \times 3 \times 4 + 2 \times 3 \times 4 = 50$. Show that $p | S_k$ for all $2 \le k \le p-2$.

Solution.

Consider the monic polynomial of degree $p-1$, $x^{p-1} - 1 \equiv 0 \pmod{p}$. There are precisely $p-1$ incongruent solutions modulo p of this polynomial equation, namely, $x = 1, 2, \dots, p-1$. Each of these follows from Fermat's Little Theorem, which states that $a^{p-1} \equiv 1 \pmod{p}$, where p is a prime and a and p are relatively prime. Thus, modulo p, we have

$$\begin{aligned} x^{p-1} - 1 &\equiv (x-1)(x-2)\cdots[x-(p-1)] \\ &\equiv x^{p-1} - S_1 x^{p-2} + S_2 x^{p-3} \\ &\qquad - \cdots + (-1)^{p-2} S_{p-2} + (-1)^{p-1} S_{p-1} \pmod{p}. \end{aligned}$$

Equating coefficients, we have $p | S_k$ for all $2 \le k \le p-2$.

Note: We also get Wilson's Theorem for free. (Why?)

Also solved by ALEXANDRE TRICHTCHENKO, student, Brookfield High School, Ottawa, Ontario.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C77. — Corrected problem (not a solution!)

Let F_i denote the i^{th} Fibonacci number, with $F_0 = 1$ and $F_1 = 1$. (Then $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, etc.)

(a) Prove that each positive integer is uniquely expressible in the form $F_{a_1} + \cdots + F_{a_k}$, where the subscripts form a strictly increasing sequence of positive integers no pair of which are consecutive.

(b) Let $\tau = \frac{1}{2}(1 + \sqrt{5})$, and for any positive integer n, let $f(n)$ equal the integer nearest to $n\tau$. If $n = F_{a_1} + \cdots + F_{a_k}$ is the expression for n from part (a) and if $a_2 \neq 3$, prove that $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$.

(c) Keeping the notation from part (b), if $a_2 = 3$ (so that $a_1 = 1$), it is not always true that the formula $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$ holds. For example, if $n = 4 = F_3 + F_1 = 3 + 1$, then the closest integer to $n\tau = 6.47\ldots$ is 6, not $F_4 + F_2 = 5 + 2 = 7$. Fortunately, in the cases where the formula fails, we can correct the problem by setting $a_1 = 0$ instead of $a_1 = 1$: for example, $4 = F_3 + F_0 = 3 + 1$ as well, and indeed $6 = F_4 + F_1 = 5 + 1$. Determine for which sequences of a_i this correction is necessary.

C78. Let n be a positive integer. An $n \times n$ matrix A is a *magic matrix* of order m if each entry is a non-negative integer and each row and column sum is m. (That is, for all i and j, $\sum_k A_{ik} = \sum_k A_{kj} = m$.) Let A be a magic matrix of order m. Show that A can be expressed as the sum of m magic matrices of order 1.

I. *Solution by Christopher Long, graduate student, Rutgers University.*

Consider the magic matrix A as the adjacency matrix of a weighted bipartite graph G between two sets ("left" and "right") of n vertices: If the $(i,j)^{\text{th}}$ entry of A is greater than 0, place an edge in G between the i^{th} vertex on the left and the j^{th} vertex on the right and give the edge a weight equal to the $(i,j)^{\text{th}}$ entry of A. If the $(i,j)^{\text{th}}$ entry of the A is 0, do not place an edge at all. The condition that the matrix A is a magic matrix implies that total weight of all the edges emanating from any single vertex of G, left or right, is equal to m.

Given a subset S of the left-hand vertices of G, let us compute the size of its neighbourhood (the collection of all vertices on the right which are joined by an edge to a vertex in S). Remove from G all of the edges whose left-hand vertices are not in S. Then the total weight of the remaining edges is exactly $m|S|$, and the neighbourhood of S is exactly the set of right-hand vertices whose weight is still non-zero. (By the weight of a vertex, we mean the weight of all the edges touching that vertex.) But each right-hand vertex has weight at most m, so by the Pigeonhole Principle the number of right-hand vertices with non-zero weight must be at least $m|S|/m = |S|$. That is, every subset on the left has a neighbourhood on the right which is at least as big. Thus, the conditions of the following famous theorem (phrased traditionally, and thus thoroughly objectionable) are satisfied, with the left-hand vertices as boys, the right-hand vertices as girls, and edges as acceptable marriages:

Theorem (Hall's Marriage Theorem). Suppose there are n boys and n girls, and that each boy knows precisely which (possibly more than one) of the girls he is willing to marry. Suppose further that given any set S of boys, the total number of different girls that boys in S are willing to marry is at least S. Then there exists a way of pairing all the boys with the girls in such a way that each boy is willing to marry the girl to whom he is paired.

The proof of the Marriage Theorem is an excellent exercise, and can also be found in almost any graph theory book, so we omit it here. In our case, if the pairing obtained from the Marriage Theorem pairs the vertex i on the left with the vertex σ_i on the right, then we know that the $(i, \sigma_i)^{\text{th}}$ entry of A is positive. Let A' be the matrix whose $(i, \sigma_i)^{\text{th}}$ entry is 1 for all i and whose other entries are all 0. Then A' is a magic matrix of order 1 and $A - A'$ is a magic matrix of order $m - 1$, and the result follows by induction.

II. *Solution.*

As in the previous solution, we prove the result by induction by showing that there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that the (i, σ_i)-entry of A is positive–that is, by constructing a "magic submatrix" of order 1 in A. We do this for all magic matrices A by another induction, this time on $NZ(A)$, the number of non-zero entries of A. If the order of the magic matrix is $m > 0$, then there is a non-zero entry in every row, so the total number of non-zero entries is at least n. However, if the number of non-zero entries is exactly n, then certainly A is m times a magic matrix of order 1, and this completes our base case.

Now assume the result holds for $NZ(A) < k$, where $k > n$. Consider a magic matrix A of order m with exactly k non-zero entries. As $k > n$, by the Pigeonhole Principle, there is a row with at least two non-zero entries, and each is less than m. Let (i_1, j_1) be the position of one of them, and let (i_1, j_2) be the position of the other. As the $(i_1, j_2)^{\text{th}}$ entry of A is less than m, there is a non-zero entry at some position (i_2, j_2) in the same column (and it is also less than m). By the same argument, there is a non-zero entry in the same row (i_2, j_3) as (i_2, j_2). We continue this process to get sequences (i_k), (j_k), such that $i_k \neq i_{k+1}$, $j_k \neq j_{k+1}$, and such that the $(i_k, j_k)^{\text{th}}$ and $(i_k, j_{k+1})^{\text{th}}$ entries of A are all non-zero.

Our goal is to find a loop of an even number of distinct non-zero entries in the matrix, connected by alternating horizontal and vertical moves. Once we have such a loop, find the point (i, j) in the loop whose entry is minimal. Suppose the $(i, j)^{\text{th}}$ entry is q. Decrease the $(i, j)^{\text{th}}$ entry by q to 0, increase the next entry in the loop by q, decrease the next by q, and continue travelling once around the entire loop, alternately adding and subtracting q in this fashion. This will yield a magic matrix B with fewer non-zero entries than A, so by the induction hypothesis B contains a magic submatrix B' of order 1. However, by the construction of B, the non-zero entries of B all correspond to non-zero entries of A, so B' is also a magic submatrix of A. We will thus be done by induction.

Let us proceed with finding this loop. As there are only a finite number of entries of A, at some point in the sequence (i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), $\ldots$ there will be a repeated term. If the first repeated term is of the form (i_l, j_l), and the first appearance of this term is (i_k, j_k) (with $k < l$), then the loop (i_k, j_k), (i_k, j_{k+1}), $\ldots$, (i_{l-1}, j_l), $(i_l, j_l) = (i_k, j_k)$ is exactly the kind of loop we are looking for, and we are done. We are similarly finished if the first repeated term is of the form (i_l, j_{l+1}) and the first occurrence of that term is of the form (i_k, j_{k+1}).

Suppose instead that the first repeated term is of the form (i_l, j_{l+1}) and that the first appearance of the term is (i_k, j_k). Then, replacing j_k by j_l, we obtain the loop $(i_l, j_l) = (i_k, j_l)$, (i_k, j_{k+1}), $\ldots$, (i_{l-1}, j_l), (i_l, j_l), and we are done. In the case when the first repeated term is of the form (i_l, j_l) and the first appearance of the term is (i_k, j_{k+1}), a similar trick works. Thus, we have exhausted all cases, and the proof is complete.

Shreds and Slices

Erratum

There is a mistake in the 1987 Swedish Mathematical Olympiad, as printed in ***CRUX and MAYHEM*** [1998:298]. The expression "$-a + 2b - 3c$" in problem 2 of the Qualifying Round should read "$-a + 2b + 3c$". Thanks to Solomon Golomb for pointing this out.

Goodbye, Richard!

We regret to inform our readership that Richard Hoshino, long time Mayhem staff member and High School Editor, will be leaving us after this issue. We all thank Richard for his strong dedication and numerous contributions to Mayhem over the years, and we hope that he has gained as much in the experience as we have. We wish Richard the best of luck in his new responsibilities, and we will name his successor in the next issue.

1998–1999 Olympiad Correspondence Problems

Please mail your solutions to Professor E.J. Barbeau, Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3.

Set 1

1. ABC is an isosceles triangle with $\angle A = 100°$ and $AB = AC$. The bisector of angle B meets AC in D. Show that $BD + AD = BC$.

2. Let I be the incentre of triangle ABC. Let the lines AI, BI, and CI produced intersect the circumcircle of triangle ABC at D, E, and F respectively. Prove that EF is perpendicular to AD.

3. Let PQR be an arbitrary triangle. Points A, B, and C external to the triangle are determined for which

$$\angle AQR = \angle ARQ = 15°\,, \qquad \angle QPC = \angle RPB = 30°\,,$$
$$\angle PQC = \angle PRB = 45°\,.$$

Prove that: (a) $AC = AB$; (b) $\angle BAC = 90°$.

4. Let a and b be two positive real numbers. Suppose that ABC is a triangle and D a point on side AC for which $\angle BCA = 90°$, $|AD| = a$, and $|DC| = b$. Let $|BC| = x$ and $\angle ABD = \theta$. Determine the values of x and θ for the configuration in which θ assumes its maximum value.

5. Let $\mathfrak{C}$ be a circle with centre O and radius k. For each point $P \neq O$, we define a mapping $P \to P'$ where P' is that point on OP produced for which $|OP| \cdot |OP'| = k^2$.

   In particular, each point on $\mathfrak{C}$ remains fixed, and the mapping at other points has period 2. This mapping is called inversion in the circle $\mathfrak{C}$ with centre O, and takes the union of the sets of circles and lines in the plane to itself. (You might want to see why this is so. Analytic geometry is one route.)

   (a) Suppose that A and B are two points in the plane for which $|AB| = d$, $|OA| = r$, and $|OB| = s$, and let their respective images under the inversion be A' and B'. Prove that $|A'B'| = \dfrac{k^2 d}{rs}$.

   (b) Using (a), or otherwise, show that there exists a sequence $\{X_n\}$ of distinct points in the plane with no three collinear for which all distances between pairs of them are rational.

6. Solve each of the following two systems of equations:

   (a) $$\begin{aligned} x + xy + y &= 2 + 3\sqrt{2}, \\ x^2 + y^2 &= 6. \end{aligned}$$

   (b) $$\begin{aligned} x^2 + y^2 + \frac{2xy}{x+y} &= 1, \\ \sqrt{x+y} &= x^2 - y. \end{aligned}$$

Set 2

7. For a positive integer n, let $r(n)$ denote the sum of the remainders when n is divided by $1, 2, \ldots, n$ respectively.

   (a) Prove that $r(n) = r(n-1)$ for infinitely many positive integers n.

   (b) Prove that $\frac{n^2}{10} < r(n) < \frac{n^2}{4}$ for each integer $n \geq 7$.

8. Counterfeit coins weigh a and genuine coins weigh b ($a \neq b$). You are given two samples of three coins each and you know that each sample has exactly one counterfeit coin. What is the minimum number of weighings required to be certain of isolating the two counterfeit coins by means of an accurate scale (not a balance)?

   (a) Solve the problem assuming a and b are known.

   (b) Solve the problem assuming a and b are not known.

9. Similar isosceles triangles EBA, FCB, GDC, and HAD are erected externally on the four sides of the planar quadrilateral $ABCD$ with the sides of the quadrilateral as their bases. Let M, N, P, and Q be the respective midpoints of the segments EG, HF, AC, and BD. What is the shape of $PQMN$?

10. Given two points A and B in the Euclidean plane, let C be free to move on a circle with A as centre. Find the locus of P, the point of intersection of BC with the internal bisector of angle A of triangle ABC.

11. Let ABC be a triangle; let D be a point on AB and E a point on AC such that DE and BC are parallel and DE is a tangent to the incircle of the triangle ABC. Prove that

$$8DE \leq AB + BC + CA.$$

12. Suppose that n is a positive integer and that $x + y = 1$. Prove that

$$x^{n+1} \sum_{k=0}^{n} \binom{n+k}{k} y^k + y^{n+1} \sum_{k=0}^{n} \binom{n+k}{k} x^k = 1.$$

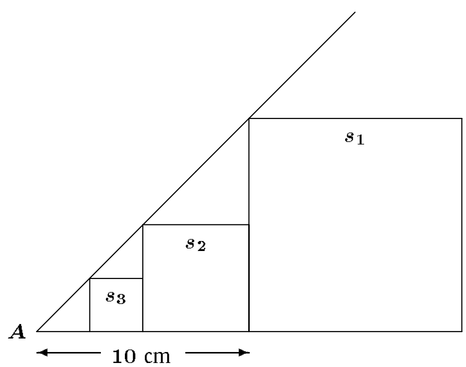
J.I.R. McKnight Problems Contest 1985

1. If $S_n = 3+8x+15x^2+24x^3+\cdots+(n^2+4n+3)x^{n-1}$, determine S_n by first evaluating $(1-x)S_n$. Hence find the limit of S_n as n approaches infinity, given $x = \frac{1}{3}$.

2. (a) P and Q are points $(ap^2, 2ap)$ and $(aq^2, 2aq)$ on the parabola $y^2 = 4ax$. Show that the equation of the chord PQ is $2x - (p+q)y + 2aq = 0$.

   (b) If O is the origin and the chords OP and OQ are perpendicular, prove that the chord PQ cuts the x-axis in the same point for all possible positions of P and Q.

3. In the figure, angle A has a measure of $60°$. At a distance of 10 cm from the vertex, a perpendicular is erected and a square is constructed on it with side s_1. In toward the vertex of the angle a second square of side s_2 is formed. Then similarly a square of side s_3, and so on ad infinitum. Find the sum of the areas of these squares in simplest radical form and then give an approximation to the nearest hundredth of a square centimetre.



4. A wire of length L is to be cut into two pieces, one of which is bent to form a circle and the other to form a square. How should the wire be cut if the sum of the areas enclosed by the two pieces is to be a maximum?

5. (a) Sketch the hyperbola represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a > b,$$

   and draw its asymptotes.

   (b) Draw a tangent to the hyperbola at any point on the hyperbola, and prove that the portion of the tangent between the points where it meets the asymptotes is bisected by the point of contact of the tangent.

(c) Prove that the segment of the tangent in (b) forms with the asymptotes a triangle of constant area.

6. Prove that if $\cos x + \cos y = a$ and $\sin x - \sin y = b$, then

$$\cos(x - y) = \frac{a^2 - b^2}{a^2 + b^2}.$$

Swedish Mathematics Olympiad 1989

1989 Qualifying Round

1. Find the integer t and the hundreds digit a such that

$$(3(320 + t))^2 = 492a04.$$

2. Form all possible six digit numbers, each using the digits 1, 2, 3, 4, 5, 6 exactly once. What is the sum of all these numbers?

3. Let ABC be an acute-angled triangle and let P be a point on the side BC. Let P' be the reflection of P in the side AB, and let P'' be the reflection of P in the side CA. Show that the distance $P'P''$ is least when P is the foot of the perpendicular from A to BC.

4. Show that if x, y, and z are positive real numbers and $x^y = y^z = z^x$ then $x = y = z$.

5. The equations $x^2 + px + q = 0$ and $qx^2 + mqx + 1 = 0$, where m, p, and q are real, and $q > 0$, have roots x_1, x_2, and x_1, $1/x_2$ respectively. Show that $mp \geq 4$.

6. Assume that $a_1 < a_2 < \cdots < a_{995}$ are 995 real numbers. Form all sums $a_i + a_j$, for $1 \leq i \leq j \leq 995$. Show that at least 1989 different numbers are obtained. Show also that exactly 1989 different numbers are obtained if and only if $a_1, a_2, \ldots, a_{995}$ is an arithmetic progression.

The 1989 Final Round has already appeared in a previous issue of ***CRUX***, in Olympiad Corner #125, 1991.

Dividing Points Equally

Cyrus C. Hsia
student, University of Toronto

We start off with a simple problem and follow it through with some related questions that can be reduced to this problem.

Problem 1. Given $2n$ points in the plane, is it possible to draw a straight line so that there are an equal number of points on either side of the line?

Solution. The answer is yes. Since there are a finite number of points, namely $2n$ of them, there are a finite number of lines that pass through a pair of points, at most $\binom{2n}{2}$. Each line passing through a pair of points has a certain direction. So, the $2n$ points have at most $\binom{2n}{2}$ distinct directions. It is not too hard then to pick a direction, call it λ, different from all of these.

Intuitively, it is possible to take a line with direction λ and slide it along (preserving its direction of course) until half the points are on either side. This is possible since the line can pass through only one point at a time. If it passed through more than one point, then the direction between the two points must be the direction λ of this line. This contradicts the choice of λ to be different from any of those pairs from the $2n$ points. Hence, by sliding the line with direction λ along, we pass each point one at a time until we have passed exactly half, and then we are done.

We follow this up with some generalizations and related problems for the reader to solve.

Problem 1A. Prove that given mn points in the plane, we can find $m - 1$ parallel lines that divide the plane into m regions with n points in each region.

Problem 1B. Show that it is possible to divide $2n$ points in the plane by two intersecting lines so that for each line, half the points lie on either side of it. Show that it is possible to divide them by m concurrent lines so that for each line, half the points lie on either side of it.

Problem 1C. Take $4n$ points in the plane and take any three distinct lines, each pair of which divides the set of points into equal quarters. Show that these three lines cannot be concurrent. (Hint: Show that the three lines must form a triangle containing n of the points.)

Problem 1D. Is it possible to divide $2n$ points in the plane by a circle so that half of the points are inside and half are outside? (Note: This does not follow immediately from Problem 1 by inversion.)

Problem 1E. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Can every set of $4n$ points in the plane, no three of which are collinear, be evenly quartered by two mutually perpendicular lines?

Mathematics Magazine, Problem 1513. Vol. 69, No. 5, December 1996, p. 385.

In the next two problems, we consider three dimensions.

Problem 2. There are $2n$ chocolate chips in a roll of frozen cookie dough. Show that it is possible to divide them into two sets of n chips by a plane cut.

Solution. The $2n$ chocolate chips can be thought of as points in space. We want to roll the frozen cookie dough into a position where if the chocolate chips fell straight down, they would not hit each other. Essentially, we want to project the $2n$ points onto a plane so that each point is mapped to a distinct position. Then we are back to Problem 1 of dividing the points in the plane by a line. (Why?) This problem then becomes one of showing that there is such a projection.

In 3 dimensions, we need something that is similar to "slope". This is where the concept of a **direction vector** comes into play. The direction vector between two points from $A = (a_1, a_2, a_3)$ to $B = (b_1, b_2, b_3)$ is given by $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$ (or any multiple of this). To reduce two points having multiple direction vectors, we usually consider the **unit direction vector** which is found simply by taking the given direction vector and dividing each coordinate by the vector's **norm** or length, namely $\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$. (The reader may wish to verify that for any line, any two points on the line will give the same direction vector up to a $\pm$ sign.)

Since there are a finite number of points, there are a finite number of unit direction vectors. We can then pick a unit direction vector different from all of them; call it $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. By projecting each point in this direction onto a plane perpendicular to this vector, each point projects to a different point in the plane.

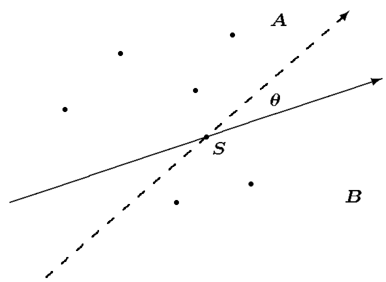
Thus, take the dough so that the direction vector λ points vertically down on the kitchen table. Take a knife and cut along the line that would divide the points in half in the plane.

Problem 3. Suppose the Earth has a population of 6 billion people in the near future. Is it possible to draw an imaginary equator around the world so that each hemisphere contains an equal number of humans?

Here is a lemma that we will need.

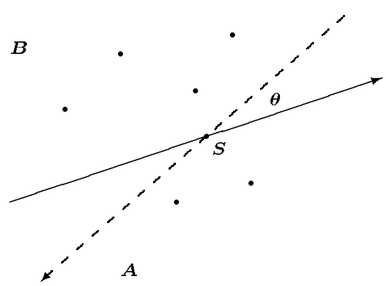
Lemma 3A. $2n$ points and a special point labelled S are given in the plane. Any line passing through S passes through at most one of the $2n$ points. It is then possible to divide the plane by a straight line that passes through S so that n points are on either side of the line.

Solution. Consider the lines passing through S and one other point. There are $2n$ of them. Pick any other line through S which is not one of these. Say there are k points on one side and $2n-k$ points on the other. If $k=n$ then we are finished. Otherwise, rotate this line counter-clockwise about S. This line will go through all of the points one at a time as it rotates. Let $f(\theta)$ be the difference between the number of points on one side labelled A and the number of points on the other side labelled B as shown when the angle is θ from the original line. Points on the line are to be ignored.



A line with sides A and B.

Thus, at the beginning, $f(0°) = (2n-k)-k = 2(n-k)$, and when it has completed a rotation of $180°$, we have $f(180°) = k-(2n-k) = -2(n-k)$. Make sure that sides A and B stay in the same orientation.



A line with sides A and B reversed from the above figure.

Thus these values have different signs and if this were a problem of a continuous function, then we could easily claim by the Intermediate Value Theorem that there is a value $0° < \theta < 180°$ such that $f(\theta) = 0$. Nonetheless, we can still conclude this, since we know that when the value of f changes, it changes by a value of 1.

To see this, note that each time the line passes through a point, the value on one side, say A decreases by 1 and the number of points on side B stays the same. The value of f then decreases by one. Once the line immediately passes by this point, the number of points on side B increases by 1 and the number of points on side A remains the same. Again, the value of f decreases by 1. This will happen for each point that the line crosses.

Note that when the line passes completely over a point the value of f changes by a value of 2. Thus the parity of f changes when the line goes through a point from the state when there are no points on the line and the parity stays the same when the line passes by a point.

Since $2(n - k)$ and its negative are both integers, at some step in the value of f, 0 is reached. Now we must check that this value is not obtained when the line crosses through one of the $2n$ points. Since the values of 0 and $2(n - k)$ have the same parity, when the value of 0 is achieved, it is achieved in the state where the line does not pass through any of the $2n$ points. Hence, for this line, there is an angle θ counter-clockwise away from the original line where the points are equally divided.

Solution to Problem 3.

We must assume, of course, that the people are points occupying a distinct and fixed location on a spherical Earth. Now for each pair of people draw a great circle passing through them. That is, a circle given by the intersection of the sphere with a plane passing through the centre of the sphere; this would be our definition of an equator. There are a finite number of such circles as there are a finite number of pairs. Thus we may always choose another point, call it N, not on any of the great circles. The point diametrically opposite N, call it S, must not be on any of the great circles either. (Why?) If a point was on a great circle then the point diametrically opposite it would be also, by definition.

Now place the spherical world on a plane with the point S tangent to it. From point N project each point onto the plane by drawing a line from N through each point to intersect the plane – this projection is called the stereographic map. Now consider lines through S in the plane. Any such line can have at most one other point. If a line contains S and two other points then that would mean that S was on the great circle through the preimage of these points.

By Lemma 3A there is a line through S which cuts the plane into two parts containing an equal number of points. This line projected back onto the sphere is a great circle dividing the world into two equal populations.

Exercises

1. Give solutions to the problems listed above.
2. Is it possible to divide any $4n$ points in the plane by two intersecting lines so that each of the four sectors contains n points?
3. Is it possible to divide the Earth into 4 quarters with an equal number of people in each? (Assume a population of 6 billion.)

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 1999**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2388. *Proposed by Daniel Kupper, Büllingen, Belgium.*

Suppose that for each $k \in \mathbb{N}$, the numbers $a_k, b_k, z_k \in \mathbb{C}$. Suppose that the polynomials

$$A_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \text{and} \qquad B_n(z) = \sum_{k=0}^{n} b_k z^k$$

are related by $A_n(z_j) = B_n(z_j^2) = 0$ for $j \in \{1, 2, \ldots, n\}$.

For each $n \in \mathbb{N}$, find an expression for b_n in terms of $a_0, a_1, \ldots, a_n$.

2389. *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Suppose that f is continuous on $\mathbb{R}^n$ and satisfies

$$f(a_1, a_2, a_3, \ldots, a_n) \le f\left(\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}, a_3, \ldots, a_n\right).$$

Let $m = \dfrac{a_1 + a_2 + \ldots + a_n}{2}$. Prove that

$$f(a_1, a_2, a_3, \ldots, a_n) \le f(m, m, \ldots, m).$$

2390. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $\lambda \geq 0$ and $p, q \geq 1$, let $S_n(\lambda, p, q) := \sum_{i=1}^{n-\lambda} \sum_{j=i+\lambda}^{n} i^p j^q$, where $n > \lambda$.

Given the statement: "$S_n(\lambda, p, q)$, understood as a polynomial in $\mathbb{Q}[n]$, is always divisible by $(n-\lambda)(n-\lambda+1)(n-\lambda+2)$",

(a) give examples for $\lambda = 0, 1, 2, 3, 4$;

(b)* prove the statement in general.

2391. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Consider $d+1$ points, B_1, B_2, ..., B_{d+1} in the unit sphere in $\mathbb{R}^d$, so that the simplex $S_d(B) = B_1B_2\ldots B_{d+1}$ includes the origin O. Let $P = \{x \mid B_i \cdot x \leq 1\}$ for all i between 1 and $d+1$.

Prove that there is a point $y \in P$ such that $|y| \geq d$.

2392. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Suppose that x_i, y_i, $(1 \leq i \leq n)$ are positive real numbers. Let

$$A_n = \sum_{i=1}^{n} \frac{x_i y_i}{x_i + y_i}, \qquad B_n = \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n}(x_i + y_i)},$$

$$C_n = \frac{\left(\sum_{i=1}^{n} x_i\right)^2 + \left(\sum_{i=1}^{n} y_i\right)^2}{\sum_{i=1}^{n}(x_i + y_i)}, \qquad D_n = \sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{x_i + y_i}.$$

Prove that

1. $A_n \leq C_n$,
2. $B_n \leq D_n$,
3. $2A_n \leq 2B_n \leq C_n \leq D_n$.'

2393. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Suppose that a, b, c and d are positive real numbers. Prove that

1. $\Big((a+b)(b+c)(c+d)(d+a)\Big)^{3/2} \geq 4abcd(a+b+c+d)^2$,
2. $\Big((a+b)(b+c)(c+d)(d+a)\Big)^{3} \geq 16(abcd)^2 \prod_{\substack{a,b,c,d \\ \text{cyclic}}} (2a+b+c)$.

2394. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

The inequality $a^a b^b \geq \left(\frac{a+b}{2}\right)^{a+b}$, where $a, b > 0$, is usually proved using Calculus. Give a proof without the aid of Calculus.

2395. *Proposed by Witold Janicki, Jagiellonian University, Krakow, Poland, Michael Sheard, St. Lawrence University, Canton, NY, USA, Dan Velleman, Amherst College, Amherst, MA, USA, and Stan Wagon, Macalester College, St. Paul, Minnesota, USA.*

Let P be such that

(A) $P(0)$ is true, and

(B) $P(n) \Longrightarrow P(n+1)$.

Find an integer $n > 10^6$ such that $P(n)$ can be proved without using induction, but rather using

(L) the Law of Implication (that is, X and $(X \Longrightarrow Y)$ yield Y)

ten times only.

2396. *Proposed by Jose Luis Diaz, Universitat Politecnica de Catalunya, Colum, Terrassa, Spain*

Suppose that $A(z) = \sum_{k=0}^{n} a_k z^k$ is a complex polynomial with $a_n = 1$, and let $r = \max_{0 \leq k \leq n-1} \left\{ |a_k|^{1/(n-k)} \right\}$. Prove that all the zeros of A lie in the disk $\mathcal{C} = \left\{ z \in \mathbb{C} \,:\, |z| \leq \frac{r}{2^{1/n} - 1} \right\}$.

2397. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a right-angled triangle ABC with $\angle BAC = 90^\circ$. Let I be the incentre, and let D and E be the intersections of BI and CI with AC and AB respectively.

Prove that $\dfrac{BI^2 + ID^2}{CI^2 + IE^2} = \dfrac{AB^2}{AC^2}$.

2398. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a square $ABCD$ with points E and F on sides BC and CD respectively, let P and Q be the feet of the perpendiculars from C to AE and AF respectively. Suppose that $\dfrac{CP}{AE} + \dfrac{CQ}{AF} = 1$.

Prove that $\angle EAF = 45^\circ$.

2399★. *Proposed by David Singmaster, South Bank University, London, England.*

In James Dodson's *The Mathematical Repository*, 2nd ed., J. Nourse, London, 1775, pp 19 and 31, are two variations on the classic "Ass and Mule" problem:

> "What fraction is that, to the numerator of which 1 be added, the value will be $1/3$; but if 1 be added to the denominator, its value is $1/4$?"

This is easily done and it is easy to generalize to finding x/y such that $(x+1)/y = a/b$ and $x/(y+1) = c/d$, giving $x = c(a+b)/(ad-bc)$ and $y = b(c+d)/(ad-bc)$. We would normally take $a/b > c/d$, so that $ad - bc > 0$, and we can also assume a/b and c/d are in lowest terms.

> "A butcher being asked, what number of calves and sheep he had bought, replied, 'If I had bought four more of each, I should have four sheep for every three calves; and if I had bought four less of each, I should have had three sheep for every two calves'. How many of each did he buy?"

That is, find x/y such that $(x+4)/(y+4) = 4/3$ and $(x-4)/(y-4) = 3/2$. Again, this is easily done and it is easy to solve the generalisation, $(x+A)/(y+A) = a/b$ and $(x-A)/(y-A) = c/d$, getting $x = A(2ac-bc-ad)/(bc-ad)$ and $y = A(ad+bc-2bd)/(bc-ad)$. We would normally take $a/b < c/d$ so that $bc - ad > 0$, and we can also assume a/b and c/d are in lowest terms.

In either problem, given that a, b, c and d are integers, is there a condition (simpler than computing x and y) to ensure that x and y are integers?

Alternatively, is there a way to generate all the integer quadruples a, b, c, d, which produce integer x and y?

2400. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

(a) Show that $1 + (\pi - 2)x < \dfrac{\cos(\pi x)}{1-2x} < 1 + 2x$ for $0 < x < 1/2$.

[Proposed by Bruce Shawyer, Editor-in-Chief.]

(b)★ Show that $\dfrac{\cos(\pi x)}{1-2x} < \dfrac{\pi}{2} - 2(\pi - 2)\left(x - \dfrac{1}{2}\right)^2$ for $0 < x < 1/2$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2090. [1995: 307, 1997: 433] *Proposed by Peter Ivády, Budapest, Hungary.*

For $0 < x < \pi/2$ prove that

$$\left(\frac{\sin x}{x}\right)^2 < \frac{\pi^2 - x^2}{\pi^2 + x^2}.$$

In the remarks following the solution, Walther Janous gave the extension to: *what is the value of ρ that gives the best inequality of the type*

$$\left(\frac{\sin x}{x}\right)^2 < \frac{\rho^2 - x^2}{\rho^2 + x^2}$$

which is valid for all $x \in (0, \pi/2)$?

Janous has now solved this question himself.

We prove that

$$\left(\frac{\sin x}{x}\right)^2 < \frac{\rho^2 - x^2}{\rho^2 + x^2} \qquad (1)$$

is valid for all $x \in (0, \pi/2)$ if $\rho \geq \sqrt{6}$, and the optimal value is $\rho_{\text{opt}} = \sqrt{6}$.

(**A**) From the Taylor series expansion, we have

$$\begin{aligned} x^2\frac{\rho^2 - x^2}{\rho^2 + x^2} - \sin^2 x &= x^2\frac{\rho^2 - x^2}{\rho^2 + x^2} + \frac{\cos(2x) - 1}{2} \\ &= \left(x^2 + \sum_{k=1}^{\infty}(-1)^k\frac{2}{\rho^{2k}}x^{2k+2}\right) \\ &\qquad + \left(-x^2 + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{2k+1}}{(2k+2)!}x^{2k+2}\right) \\ &= 2\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{\rho^{2k}} - \frac{2^{2k}}{(2k+2)!}\right)x^{2k+2} \\ &= 2x^4\left(\frac{\rho^2 - 6}{4\rho^2} + x^2(\ldots)\right) > 0. \end{aligned}$$

Thus we have (as taking the limit as $x \to 0$ shows) that $\rho^2 - 6 \geq 0$; that is $\rho \geq \sqrt{6}$.

(**B**) Suppose that $x > 0$. Since

$$\begin{aligned} & \frac{\alpha^2 - x^2}{\alpha^2 + x^2} \;<\; \frac{\beta^2 - x^2}{\beta^2 + x^2} \\ \iff\; & \alpha^2\beta^2 + \alpha^2 x^2 - \beta^2 x^2 - x^4 \;<\; \alpha^2\beta^2 - \alpha^2 x^2 + \beta^2 x^2 - x^4 \\ \iff\; & \alpha^2 x^2 \;<\; \beta^2 x^2 \;\iff\; \alpha^2 \;<\; \beta^2 , \end{aligned}$$

it is sufficient to prove (1) for $\rho = \rho_{\text{opt}} = \sqrt{6}$.

(**C**) Next, we prove the auxiliary inequality

$$\sin x \;\leq\; x - \frac{x^3}{6} + \frac{x^5}{120} \tag{2}$$

is valid for all $x > 0$.

Indeed, let $g(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \sin x$. Then $g(0) = 0$. Furthermore, we have

$$\begin{array}{rcll} g'(x) & = & 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cos x\,, & g'(0) = 0\,, \\ g''(x) & = & -x + \frac{x^3}{6} + \sin x\,, & g''(0) = 0\,, \\ g^{(3)}(x) & = & -1 + \frac{x^2}{2} + \cos x\,, & g^{(3)}(0) = 0\,, \\ g^{(4)}(x) & = & x - \sin x\,, & g^{(4)}(0) = 0\,, \quad \text{and} \\ g^{(5)}(x) & = & 1 - \cos x \;\geq\; 0\,. & \end{array}$$

Thus, $g^{(4)}(x)$ increases, so that $g^{(4)}(x) \geq g^{(4)}(0) = 0$.

Further, $g^{(3)}(x)$ increases, so that $g^{(3)}(x) \geq g^{(3)}(0) = 0$. And so on.

Thus, $g(x)$ increases, so that $g(x) \geq g(0) = 0$, as claimed.

(**D**) Because of (**C**), and since $\sin x > 0$ (since $x \in (0, \pi/2)$), inequality (1) with $\rho = \rho_{\text{opt}} = \sqrt{6}$ [that is, $x^2(6 - x^2) > (6 + x^2)\sin^2 x$] will follow from

$$x^2(6 - x^2) - (6 + x^2)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)^2 \;>\; 0\,.$$

This inequality is equivalent to

$$\frac{x^6}{14400}\left(x^6 - 34x^4 + 400x^2 - 960\right) \;:=\; \frac{x^6}{14400} h(x) \;<\; 0\,.$$

But,

$$\begin{aligned} h'(x) &= 6x^5 - 136x^3 + 800x \;=\; 2x\left(x^4 - 68x^2 + 400\right) \\ &> 2x\left(3x^4 - 69x^2 + 399\right) \;=\; 6x\left(x^4 - 23x^2 + 133\right) \\ &= 6x\left[\left(x^2 - \tfrac{23}{2}\right)^2 + \tfrac{3}{4}\right] \;>\; 0\,; \end{aligned}$$

that is, $h(x)$ increases as $x > 0$ increases. Since $\frac{\pi}{2} < 1.6$ and $h(1.6) \approx -142$, the proof is complete.

2259. [1997: 301] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let X, Y, Z, be the projections of the incentre of $\triangle ABC$ onto the sides BC, CA, AB respectively. Let X', Y', Z', be the points on the incircle diametrically opposite to X, Y, Z, respectively. Show that the lines AX', BY', CZ', are concurrent.

I. *Solution by Christopher J. Bradley, Clifton College, Bristol, UK.*

The areal coordinates of the incentre I are $\frac{1}{a+b+c}(a, b, c)$, and those of X are $\frac{1}{2a}(0, a+b-c, a-b+c)$.

Since $\overrightarrow{XI} = \overrightarrow{IX'}$, the areal coordinates of X' are twice those of I minus those of X, and hence are

$$\left(\frac{2a}{a+b+c}, \frac{(c-a+b)(c+a-b)}{2a(a+b+c)}, \frac{(b-a+c)(b+a-c)}{2a(a+b+c)}\right) .$$

The equation of AX' is thus

$$y(b+a-c) \;=\; z(c+a-b) .$$

The equations of BY' and CZ' are obtained by cyclic changes, and all are concurrent at the point P with coordinates

$$\frac{1}{a+b+c}(b+c-a, c+a-b, a+b-c) .$$

II. *Solution by Florian Herzig, student, Cambridge, UK.*

This problem is very similar to problem 2250 [1997:245, 1998:372]. I will use the following known theorem (a proof can be found in [1]):

In an inscriptable quadrilateral the incentre and the midpoints of the diagonals are collinear. (1)

First let A', B', C' be the midpoints of AX, BY and CZ. Because of (1), in the limiting case of a triangle, the lines $A'K$, $B'L$, $C'M$ pass through I. Since $XA' : XA = XI : XX' = 1 : 2$, the lines AX' and KI are parallel. Let G be the centre of gravity of $\triangle ABC$ (and also of $\triangle KLM$). A dilatation with centre G and factor -2 maps $\triangle KLM$ onto $\triangle ABC$. If P is the image of I under that mapping, then AX', BY', CZ' pass through P since the line KI is mapped onto the line AP but also onto the line AX' as $AX' \parallel KI$. Moreover, because of the dilatation, P lies on IG and

$$IG : GP = 1 : 2 .$$

Reference

[1] H. Dörrie, *Triumph der Mathematik*, Physica, Würzburg 1958. (*100 Great Problems in Elementary Mathematics* , Dover, N.Y. 1965.)

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JOHN VLACHAKIS, Athens, Greece; and the proposer.

Bradley commented that if J is the centre of mass of the triangle thought of as a uniform wire framework, then P lies on the extension of IJ and is such that $IJ = JP$. The point P is the Nagel point. Also, G lies on this line, and $IG : GJ = 2 : 1$. Further, G and P are the internal and external centres of similitude of the incircles of triangles ABC and the median triangle (with vertices at the mid-points of the sides of triangle ABC).

Lambrou, who gave an indirect proof, gave a generalization and noted that it is the converse of a result due to Rabinowotz (problem 1353, Mathematics Magazine, **65** 1992, p. 59).

Let X, Y, Z be the points of contact of the incircle of triangle ABC with the sides BC, CA, AB respectively. Let I' be any point within the incircle (not necessarily the incentre). Let XI', YI', ZI' cut the incircle again at X', Y', Z' respectively. Then AX', BY', CZ' are concurrent.

Seimiya also noted that the point of concurrency is the Nagel point of $\triangle ABC$.

2264. [1997: 364] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a right angled triangle with the right angle at A. Points D and E are on sides AB and AC respectively, such that $DE \| BC$. Points F and G are the feet of the perpendiculars from D and E to BC respectively.

Let I, I_1, I_2, I_3 be the incentres of $\triangle ABC$, $\triangle ADE$, $\triangle BDF$, $\triangle CEG$ respectively. Let P be the point such that $I_2P \| I_1I_3$, and $I_3P \| I_1I_2$.

Prove that the segment IP is bisected by the line BC.

Solution by Florian Herzig, student, Cambridge, UK.

Let $\lambda = \frac{AD}{AB} = \frac{AE}{AC}$. Also define x, x_1, x_2, x_3, y to be the lengths of perpendiculars of I, I_1, I_2, I_3, P to BC respectively. It is sufficient to prove that $x = y$. First, (see, for example, Roger A. Johnson, *Modern Geometry* (1929), p. 189),

$$x = \frac{bc/2}{(a+b+c)/2} = \frac{bc}{a+b+c}.$$

Then, by similarity,

$$DF = b \cdot \frac{BD}{a} = (1-\lambda)\frac{bc}{a},$$

and so,

$$x_1 = DF + \lambda x = (1-\lambda)\frac{bc}{a} + \lambda\frac{bc}{a+b+c}.$$

Moreover,

$$x_2 = \frac{BD}{BC}x = (1-\lambda)\frac{bc^2}{a(a+b+c)}, \qquad x_3 = (1-\lambda)\frac{b^2c}{a(a+b+c)}.$$

By definition of P,

$$\begin{aligned} y &= x_1 - x_2 - x_3 \\ &= (1-\lambda)\frac{bc}{a} + \lambda\frac{bc}{a+b+c} - (1-\lambda)\frac{bc(b+c)}{a(a+b+c)} \\ &= x \end{aligned}$$

after an easy calculation, and that is what we wanted to prove.

[Editor's note: To see that $y = x_1 - x_2 - x_3$, let $Q = I_1P \wedge I_2I_3$, $R = I_1P \wedge BC$, and z be the length of the perpendicular from Q to BC. Then $x_2 + x_3 = 2z$ and

$$I_1R = I_1Q + QR = QP + QR = RP + 2QR = I_1R \cdot \frac{y}{x_1} + 2I_1R \cdot \frac{z}{x_1},$$

so $x_1 = y + 2z = y + x_2 + x_3$.]

Solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2267. [1997: 364] *Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA and Peter Yff, Ball State University, Muncie, IN, USA.*

In the plane of $\triangle ABC$, let F be the Fermat point and F' its isogonal conjugate.

Prove that the circles through F' centred at A, B and C meet pairwise in the vertices of an equilateral triangle having centre F.

Solution by Toshio Seimiya, Kawasaki, Japan.

Equilateral triangles BCD, CAE, ABF are erected externally on the sides of $\triangle ABC$. Then AD, BE, CF are concurrent at the Fermat point F, and $\angle AFB = \angle BFC = \angle CFA = 120^\circ$ (see, for example, H.S.M. Coxeter, *Introduction to Geometry* (1961), §1.8).

Let the circles through F' centred at B and C meet at P. Since $BF' = BP$ and $CF' = CP$, P is the reflection of F' across BC. Similarly Q is the reflection of F' across AC, and R' is the reflection of F' across AB, where the circles through F' centred at A and C meet at Q and those centred at A and B meet at R.

Since $\angle CAQ = \angle F'AC = \angle BAF$, we get

$$\angle FAR \;=\; \angle FAC + \angle CAQ \;=\; \angle FAC + \angle BAF \;=\; \angle BAC\,.$$

Similarly we have $\angle RAF = \angle BAC$, so that $\angle FAQ = \angle RAF$.

Since $AQ = AF' = AR$, AF is the perpendicular bisector of QR. Similarly CF is the perpendicular bisector of PQ. Thus we have $FR = FQ = FP$, so that F is the circumcentre of $\triangle PQR$.

Since $AF \perp QR$ and $CF \perp PQ$, and $\angle AFC = 120^\circ$, we have $\angle PQR = 60^\circ$. Similarly, we have $\angle PRQ = 60^\circ$, so that $\triangle PQR$ is equilateral.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain (2 solutions); FLORIAN HERZIG, student, Cambridge, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer.

2269. [1997: 365] *Proposed by Cristóbal Sánchez–Rubio, I.B. Penyagolosa, Castellón, Spain.*

Let $OABC$ be a given parallelogram with $\angle AOB = \alpha \in (0, \pi/2]$.

A. Prove that there is a square inscribable in $OABC$ if and only if

$$\sin\alpha - \cos\alpha \;\leq\; \frac{OA}{OB} \;\leq\; \sin\alpha + \cos\alpha$$

and

$$\sin\alpha - \cos\alpha \;\leq\; \frac{OB}{OA} \;\leq\; \sin\alpha + \cos\alpha\,.$$

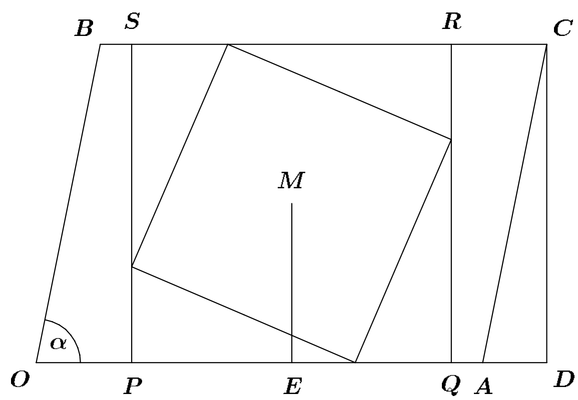
B. Let the area of the inscribed square be S_s and the area of the given parallelogram be S_p. Prove that

$$2S_s \;=\; \tan^2\alpha\left(OA^2 + OB^2 - 2S_p\right).$$

Editor's comment.

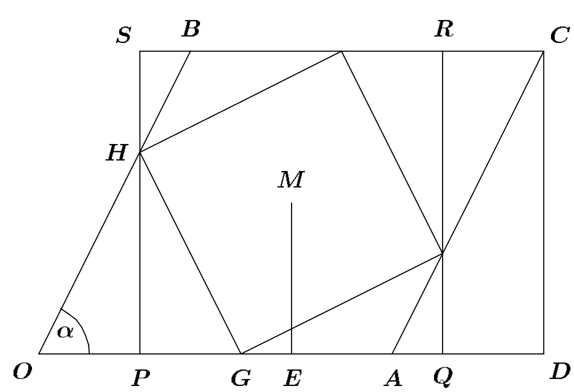
All those who sent in submissions on this problem noted that the proposer had mislabeled the parallelogram $OABC$ instead of $OACB$.

Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany.



Suppose that the diagonals of parallelogram $OACB$ meet at M. The centre of the square must be M. If OB and α are fixed, then $CD = OB\sin\alpha$ is fixed. If the square rotates about M with one vertex on the line OA, then the other vertices lie on the lines QR, RS and SP. Thus, there is an inscribed square if and only if the segment OB meets the segment SP.

Therefore $OA \leq OB\cos\alpha + OB\sin\alpha$ (from $B = S$ and $A = Q$), and $OB\sin\alpha - OB\cos\alpha \leq OA$ (from $O = P$ and $C = R$). The same must hold if we interchange the roles of A and B. So part A follows immediately.



Let $OB = b$, $OP = x$, $HP = y$, $PG = z$. Then $y = x\tan\alpha$ and $z = b\sin\alpha - y$. Therefore $OF = \frac{1}{2}(a + b\cos\alpha)$ and $x = OF - PF = \frac{1}{2}(a + b\cos\alpha - b\sin\alpha)$, giving $4S_s = 4(y^2 + z^2)$ and $S_p = ab\sin\alpha$.

It follows that

$$\begin{aligned} 4S_s &= 4y^2 + 4z^2 \\ &= (a + b\cos\alpha - b\sin\alpha)^2 \tan^2\alpha + 4(b\sin\alpha - x\tan\alpha)^2 \\ &= (a + b\cos\alpha - b\sin\alpha)^2 \tan^2\alpha \\ &\qquad + (2b\sin\alpha - (a + b\cos\alpha - b\sin\alpha)\tan\alpha)^2 \\ &= \tan^2\alpha\,(2(a + b\cos\alpha - b\sin\alpha)^2 + 4b^2\cos^2\alpha \\ &\qquad\qquad -4b(a + b\cos\alpha - b\sin\alpha)\cos\alpha)\ . \end{aligned}$$

Therefore

$$\begin{aligned} 2S_s &= \tan^2\alpha\,((a + b\cos\alpha - b\sin\alpha)(a - b\cos\alpha - b\sin\alpha) \\ &\qquad\qquad +2b^2\cos^2\alpha) \\ &= \tan^2\alpha\,((a - b\sin\alpha)^2 - (b\cos\alpha)^2 + 2b^2\cos^2\alpha) \\ &= \tan^2\alpha\,(a^2 + b^2 - 2ab\sin\alpha) \\ &= \tan^2\alpha\,(OA^2 + OB^2 - 2S_p)\ , \end{aligned}$$

as required.

Also solved by FLORIAN HERZIG, student, Cambridge, UK; and the proposer.

Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Michael Lambrou, University of Crete, Crete, Greece commented on the error in notation. Janous gave a condition for the problem to be true.

2276. [1997: 430] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Quadrilateral $ABCD$ is cyclic with circumcircle $\Gamma(0, R)$.

Show that the nine-point (Feuerbach) circles of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$ and $\triangle ABC$ have a point in common, and characterize that point.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, and María Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain and a summary of their comments.

This problem is a classical result about cyclic quadrangles. We will present a solution, then discuss some references. Let H_A, B_B, H_C, H_D be the respective orthocentres of the triangles BCD, CDA, DAB, and ABC, and let N_A, N_B, N_C, N_D be their nine-point centres.

Theorem 1. For any cyclic quadrangle, the centres of the nine-point circles of the four triangles formed, taking the vertices of the quadrangle three at a time, form a homothetic quadrangle.

Proof. The nine-point centres N_A, N_B, N_C, N_D are the midpoints of the segments OH_A, OH_B, OH_C, OH_D; therefore the quadrangles $N_AN_BN_CN_D$ and $H_AH_BH_CH_D$ are homothetic (with ratio $\frac{1}{2}$). But

$H_AH_BH_CH_D$ is also homothetic (and oppositely congruent) to $ABCD$. Hence $N_AN_BN_CN_D$ and $ABCD$ are homothetic (with ratio $-\frac{1}{2}$).

As a consequence, the circumcentre M of $N_AN_BN_CN_D$ is the midpoint of the segment joining O to the circumcentre of $H_AH_BH_CH_D$, while the circumradius is $\frac{R}{2}$. In other words, M is the centre of symmetry of $ABCD$ and $H_AH_BH_CH_D$ (the common midpoint of AH_A, etc). Since our four nine-point circles all have radius $\frac{R}{2}$, they must all pass through M, which is the desired result.

Comments. It seems that this result is from Lemoine, 1869, *Nouvells Annals de Mathématiques* (pp. 174 and 317). M is sometimes called the *Mathot point*, after Jules Mathot, *Mathesis*, 1901, p. 25. In [1] M is called the *anticentre* of the cyclic quadrangle $ABCD$. Here is a summary of some of the properties of this point:

Theorem 2. If $ABCD$ is cyclic, the nine-point circles of the triangles BCD, CDA, DAB, ABC, $H_BH_CH_D$, $H_CH_DH_A$, $H_DH_AH_B$, $H_AH_BH_C$, and the Simson lines of these triangles (with respect to the fourth vertex of each quadrangle) all contain the Mathot point of $ABCD$, as do the lines perpendicular to a side of $ABCD$ that pass through the midpoint of the opposite side.

Convenient references that present a solution to our problem include

1 Nathan Altshiller Court, *College Geometry*. Barnes and Noble, 1965 (theorem 263, p. 132.

2 Roger A. Johnson, *Advanced Euclidean Geometry*. Dover, 1960 (pp. 209 and 243).

Part of Theorem 1 was used in the 1984 Balkan Math. Olympiad. See [1985: 243] for related references.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary (2 solutions); TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

2279. [1997: 431] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

With the usual notation for a triangle, prove that

$$\sum_{\text{cyclic}} \sin^3 A \cos B \cos C \;=\; \frac{sr}{4R^4}\left(2R^2 - s^2 + (2R+r)^2\right).$$

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

On [1997: 447–448] it was shown by Herzig that

$$\sum \sin^3 A \cos B \cos C \;=\; \left(\prod \sin A\right)\left(\sum \cos^2 A\right) .$$

Using (see, for example, [1996: 130])

$$\prod \sin A \;=\; \frac{sr}{2R^2}$$

and

$$\begin{aligned}\sum \cos^2 A &= 3 - \sum \sin^2 A \;=\; 3 - \frac{s^2 - 4Rr - r^2}{2R^2} \\ &= \frac{6R^2 - s^2 + 4Rr + r^2}{2R^2},\end{aligned}$$

the desired identity easily follows.

Also solved by HAYO AHLBURG, Benidorm, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; SAI C. KWOK, San Diego, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; VEDULA N. MURTY, Maharanipeta, India; ISAO NAOI, Seki-Shi, Gifu, Japan; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

Ahlburg and Prielipp gave the same solution as Seiffert. Several other solvers also used the identity from [1997: 447–448].

2280. [1997: 431] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with incentre I. Let D be the second intersection of AI with the circumcircle of $\triangle ABC$. Let X, Y be the feet of the perpendiculars from I to BD, CD respectively.

Suppose that $IX + IY = \frac{1}{2}AD$. Find $\angle BAC$.

Combination of solutions by Michael Lambrou, University of Crete, Crete, Greece and by Florian Herzig, student, Perchtoldsdorf, Austria.

We show that

$$IX + IY \;=\; (\sin A)AD .$$

The given condition then implies that $\sin A = \dfrac{1}{2}$, so that $A = \dfrac{\pi}{6}$ or $A = \dfrac{5\pi}{6}$.

Note first that $\triangle IBD$ is isosceles with $ID = BD$:

$$\begin{aligned}\angle IBD &= \angle IBC + \angle CBD = \frac{B}{2} + \angle CAD \\ &= \frac{B}{2} + \frac{A}{2} = \angle ABI + \angle BAI = \angle BID .\end{aligned}$$

Thus $IY = ID \sin\angle CDI = BD \sin\angle CDA$. Let the diameter of the circumcircle of $\triangle ABC$ be 1 (so that a chord equals the sine of either angle it subtends on the circumference), in which case,

$$IY = BD \times AC .$$

Similarly,

$$IX = CD \times AB .$$

Thus (by Ptolemy's Theorem)

$$IX + IY = BD \times AC + CD \times AB = BC \times AD = (\sin A)AD .$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; NIKOLAOS DERGIADES, Thessaloniki, Greece; GERRY LEVERSHA, St. Paul's School, London, England;VEDULA N. MURTY, Visakhapatnam, India; VICTOR OXMAN, University of Haifa, Haifa, Israel; ISTVÁN REIMAN, Budapest, Hungary; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2281. [1997: 431] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle, and D is a point on the side BC produced beyond C, such that $AC = CD$. Let P be the second intersection of the circumcircle of $\triangle ACD$ with the circle on diameter BC. Let E be the intersection of BP with AC, and let F be the intersection of CP with AB.

Prove that D, E, F, are collinear.

Solution by Nikoloas Dergiades, Thessaloniki, Greece (with his notation modified to make use of directed line segments).

Let G be the point where AP intersects BD. The (convex) quadrilateral $APCD$ is cyclic and $\triangle ACD$ is isosceles so

$$\begin{aligned}\angle GPC &= \angle CDA && \text{(cyclic)} \\ &= \angle CAD && \text{(isosceles)} \\ &= \angle CPD && \text{(cyclic).}\end{aligned}$$

Hence PC is the bisector of $\angle GPD$; since $PB \perp PC$, PB is the exterior bisector of the same angle. It follows that

$$\frac{GC}{CD} = \frac{BG}{BD}, \qquad \text{or} \qquad \frac{BG}{GC} = -\frac{BD}{DC} .$$

Ceva's Theorem applied to $\triangle ABC$ gives

$$\frac{BG}{GC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Thus

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1,$$

which, by Menelaus's Theorem, means that D, E, F are collinear.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; D.J. SMEENK, Zaltbommel, the Netherlands (2 solutions); and the proposer.

2282. [1997: 431] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

A line, ℓ, intersects the sides BC, CA, AB, of $\triangle ABC$ at D, E, F respectively such that D is the mid-point of EF.

Determine the minimum value of $|EF|$ and express its length as elements of $\triangle ABC$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let M be the midpoint of BC. Drop perpendiculars from E, F onto BC having lengths h_1, h_2 respectively. Observe that $h_1 = h_2$ by similarity, as $ED = DF$. Hence the triangles BMF, CME are equal in area. This implies that $[ABC] = [AFME]$, using the common notation for areas. Therefore

$$[ABC] = [AFME] = \tfrac{1}{2}AM \cdot EF \cdot \sin\angle(AM, EF) \leq \tfrac{1}{2}AM \cdot EF$$

and so

$$EF \geq \frac{2[ABC]}{AM}$$

with equality if and only if $EF \perp AM$. Equality is always possible by a continuity argument: Consider all lines ℓ perpendicular to AM; the lines passing through B and C yield the extreme ratios $FD : DE = 0$ or infinity, whence there is a desired line in between.

[Ed: Dou gave three constructions for the points E', F' that provide the minimum value of $|EF|$. Here is one of them: Define B' as the point where the perpendicular from B to AM meets the side AC; and D' to be where the line joining A to the midpoint of BB' meets BC; the perpendicular from D' to AM meets AB and AC in the desired points F' and E'.]

Remark: The lines EF envelope a parabola (see [1991: 97]) that touches BC in M, AC in Y and AB in Z where $AC = CY$, $AB = BZ$. Indeed this problem shows that in this case the shortest segment of a tangent intercepted by AB and AC comes from the tangent perpendicular to the axis of the parabola. I will sketch a proof: Let the directrix of the parabola be d and the focus be P. Drop perpendiculars from Y, Z on d which have feet K, L. Then, as AY, AZ are tangents to the parabola, AY, AZ are perpendicular bisectors of PK, PL using the tangent property (for parabolae). Hence A is the circumcentre of $\triangle PLK$ and so the perpendicular from A to d is actually the bisector of the lines KY, LZ; that is, it passes through M. Thus AM is perpendicular to the directrix; that is, it is parallel to the axis of the parabola.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; JORDI DOU, Barcelona, Spain; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, Missouri, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; GERRY LEVERSHA, St. Paul's School, London, England: TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Dou also proved the parabola property of the minimum EF. Several readers interpreted the question as calling for the minimum value of EF in terms of the side lengths of $\triangle ABC$:

$$\min|EF| = \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{\sqrt{2b^2+2c^2-a^2}}.$$

2283. [1997: 432] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

You are given triangle ABC with $\angle C = 60°$. Suppose that E is an interior point of line segment AC such that $CE < BC$. Suppose that D is an interior point of line segment BC such that

$$\frac{AE}{BD} = \frac{BC}{CE} - 1.$$

Suppose that AD and BE intersect in P, and the circumcircles of AEP and BDP intersect in P and Q. Prove that $QE \parallel BC$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

Define F to be the point on AC such that $\angle CBF = 60°$, and R to be the point on BF such that ER is parallel to BC.

We shall show that the circumcircles of AEP, BDP intersect at R. In other words R and Q coincide and the required conclusion $QE||BC$ will follow.

If the projections of R and E on BC are R_1 and E_1 respectively, then $BR_1 = CE_1 = \frac{1}{2}CE$ (as $\angle C = 60°$), so that $RE = R_1E_1 = BC - BR_1 - CE_1 = BC - CE$. Hence from the stated condition we have

$$\begin{aligned} \frac{AE}{BD} &= \frac{BC}{CE} - 1, \\ &= \frac{RE}{CE} \end{aligned}$$

so that

$$AE \cdot CE = BD \cdot RE. \tag{1}$$

Suppose that the circumcircle of RAC cuts the extension of RE at G. As RG, AC are intersecting chords we have $AE \cdot CE = EG \cdot RE$. Comparing with (1) we see $EG = BD$. Using this we see that triangles BRD, ECG are congruent: they have $BD = EG, BR = EC$ (clear) and $\angle B = 60° = \angle ECB = \angle CEG$ (alternate on parallels). Thus $\angle EGC = \angle RDB$ and thus quadrilateral $RGCD$ is cyclic. In other words C, G, A, R and D are all on one circle. Hence

$$\begin{aligned} \angle RAD &= \angle RCD && \text{(same arc)} \\ &= \angle CBE && \text{(symmetry)} \\ &= \angle BER && \text{(parallel lines);} \end{aligned}$$

that is, $\angle RAP = \angle REP$, showing that quadrilateral $RAEP$ is cyclic, as required.

It remains to verify that quadrilateral $BDPR$ is cyclic:

$$\begin{aligned} \angle BPD &= \angle APE && \\ &= \angle ARE && \text{(as triangle } RAE \text{ is cyclic)} \\ &= \angle ACG && \text{(as quadrilateral } RACG \text{ is cyclic)} \\ &= \angle BRD && \text{(as } \triangle BDR \cong ECG\text{).} \end{aligned}$$

This completes the proof.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Pompe shows more generally that $QE||BC$ still holds if the given condition were

$$\frac{AE}{BD} = \frac{BC}{CE} - 2\cos C.$$

Note that our featured solution accommodates this generalization: define F to be the point on AC for which $\angle CBF = \angle C$.

2284. [1997: 432] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a rhombus with $\angle A = 60^\circ$. Suppose that E, F, are points on the sides AB, AD, respectively, and that CE, CF, meet BD [Ed: not BC as was originally printed in error] at P, Q respectively. Suppose that $BE^2 + DF^2 = EF^2$.

Prove that $BP^2 + DQ^2 = PQ^2$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria. Apparently, there is a misprint in the published condition: "... meet BC at ..." should read "... meet BD at ...". We solve the corrected version. Without loss of generality, let $AB = 1$, and put $EB = x$, and $FD = y$. Then $DB = 1$, $AE = 1 - x$, and $AF = 1 - y$.

By applying the Cosine Law to $\triangle AEF$ ($\angle EAF = 60^\circ$) we transform the condition $x^2 + y^2 = EF^2$ to

$$x^2 + y^2 = (1-x)^2 + (1-y)^2 - 2 \cdot \frac{1}{2} \cdot (1-x)(1-y),$$

which simplifies to

$$1 - x - y - xy = 0,$$

and so,

$$y = \frac{1-x}{1+x}. \tag{1}$$

Furthermore, BP is the angle-bisector of $\angle EBC$ in $\triangle EBC$, so, by a well-known formula,

$$BP = \frac{2EB \cdot BC \cdot \cos 60^\circ}{EB + BC} = \frac{x}{x+1}.$$

Similarly, $DQ = \frac{y}{y+1}$. Using (1), $DQ = \frac{1-x}{2}$. Now,

$$PQ = DB - BP - DQ = 1 - \frac{x}{x+1} - \frac{1-x}{2} = \frac{1+x^2}{2(1+x)}.$$

Finally,

$$\begin{aligned} BP^2 + DQ^2 &= \left(\frac{x}{x+1}\right)^2 + \left(\frac{1-x}{2}\right)^2 \\ &= \frac{1 + 2x^2 + x^4}{4(1+x)^2} = \frac{(1+x^2)^2}{4(1+x)^2} = PQ^2, \end{aligned}$$

which completes the proof.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NICOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; ANGEL JOVAL ROQUET, I.E.S. Joan Brudieu, La Seu d'Urgell, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Most of the submitted solutions are similar to the above. All solvers have noticed the misprint and solved the corrected problem.

2285. [1997: 432] *Proposed by Richard I. Hess, Rancho Palos Verdes, California, USA.*

An isosceles right triangle can be 100% covered by two congruent tiles.

Design a connected tile so that two of them maximally cover a non-isosceles right triangle. (The two tiles must be identical in size and shape and may be turned over so that one is the mirror image of the other. They must not overlap each other or the border of the triangle.)

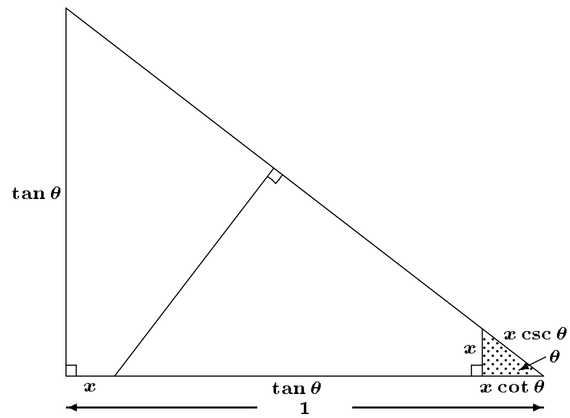
What coverage is achieved for a 30–60–90 right triangle?

Editorial comments by Bill Sands.

Two readers sent in solutions to this problem, but both seem to have assumed that the tile must be a triangle, so the amount of the right triangle their solutions cover is quite a bit less than optimal. The proposer's solution is the best of the three received, although it has not been proved optimal either, and this in fact may be quite difficult to do! The proposer's solution also had an error at one point, which slightly affected the end results. So below is a corrected version of his solution. (The proposer has independently verified the numerical results given below.)

Let the given right triangle have angle $\theta \leq 45°$, with the side adjacent to this angle having length 1, so the other side has length $\tan\theta$ and the hypotenuse has length $\sec\theta$. The area of the triangle is $(\tan\theta)/2$. The proposer gives two methods for partially tiling this triangle with two congruent tiles, each method better than the other for certain values of θ.

Method A.



From the figure, $x + \tan\theta + x\cot\theta = 1$, giving

$$x = \frac{1 - \tan\theta}{1 + \cot\theta}\,.$$

The uncovered area is

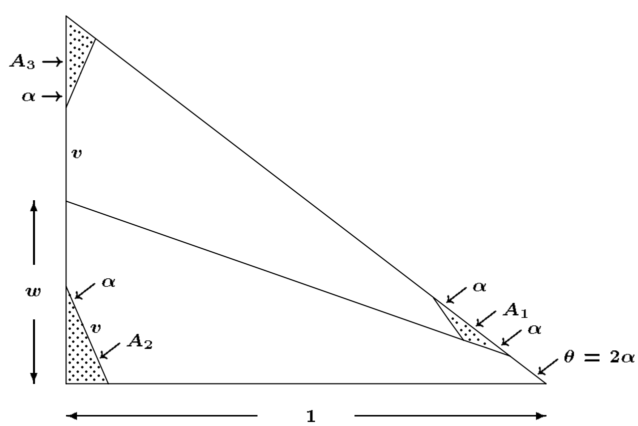
$$\frac{x^2 \cot\theta}{2} = \frac{(1 - \tan\theta)^2 \cot\theta}{2(1 + \cot\theta)^2}\,,$$

so the fraction of uncovered area is

$$F_a = \left(\frac{1 - \tan\theta}{1 + \tan\theta}\right)^2.$$

Note that this method is only valid for $\theta \le 45^\circ$, and for $\theta = 45^\circ$ gives $F_a = 0$, the perfect tiling of the isosceles right triangle.

Method B.



Let $\theta = 2\alpha$. From the figure one can derive, for arbitrary v between 0 and $\tan\theta/(1+\cos\alpha)$,

$$w = \frac{v\sin^2\alpha + \sin\theta}{\cos\alpha + \cos\theta}$$

[this formula was given incorrectly by the proposer],

$$A_1 = \frac{\sin\theta(w\cot\alpha - 1)^2}{2}\,, \qquad A_2 = \frac{v^2\sin\theta}{4}\,,$$

and

$$A_3 = \frac{\sin\alpha(w - v\cos\alpha)(\tan\theta - v - w)}{2}\,,$$

where A_1, A_2, A_3 are the uncovered areas of the triangle as shown. For a given θ, and hence α, vary v so that $A = A_1 + A_2 + A_3$ is minimized. After much calculation, one obtains: the minimum value of A occurs at

$$v = \frac{\sin\alpha\cos\alpha}{\cos^3\alpha + 2\cos^2\alpha - 1}\,,$$

whence

$$w = \frac{\sin\alpha\cos\alpha}{\cos^2\alpha + \cos\alpha - 1}\,,$$

and the minimum value of A is

$$A = \frac{\sin^3\alpha\cos\alpha(3\cos^2\alpha - \cos^4\alpha - 1)}{(2\cos^2\alpha - 1)(\cos^3\alpha + 2\cos^2\alpha - 1)^2}\,.$$

Thus the uncovered fraction is

$$F_b = \frac{\sin^2\alpha(3\cos^2\alpha - \cos^4\alpha - 1)}{(\cos^3\alpha + 2\cos^2\alpha - 1)^2}\,.$$

When $\theta = 30^\circ$ ($\alpha = 15^\circ$), we get

$$F_b = \frac{15\sqrt{3} - 22}{20\sqrt{6} + 15\sqrt{3} + 36\sqrt{2} + 74} \approx .019915536,$$

that is, just under 2% of the 30–60–90 right triangle remains uncovered. Can anyone do better?

One can also calculate that Method A and Method B give the same uncovered area when α satisfies the remarkably simple equation

$$4\sin\alpha\cos^4\alpha = 1,$$

namely for $\alpha \approx 17.648^\circ$, or $\theta \approx 35.3^\circ$. Method B gives the better result when $\theta < 35.3^\circ$, and Method A when $35.3^\circ < \theta \leq 45^\circ$. In particular, Method B is the better way for the 30-60-90 triangle.

If anyone can find a reasonably *short* derivation of these relations, let us know! And of course the entire problem is still open, in the sense that no proof of minimality has been given, including for the 30–60–90 triangle. Maybe some reader can find a third method that beats both of the proposer's methods for some values of θ.

2287. [1997: 501] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Let G denote the point of intersection of the medians, and I denote the point of intersection of the internal angle bisectors of a triangle. Using only an unmarked straightedge, construct H, the point of intersection of the altitudes.

Solution - see problem 2234 [1997:168, 1998: 247]

New solutions were sent in by FLORIAN HERZIG, student, Cambridge, UK; and MICHAEL LAMBROU, University of Crete, Crete, Greece.

2288. [1997: 501] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

In the plane are a circle (without centre) and five points A, B, C, D, E, on it such that $\text{arc } AB = \text{arc } BC$ and $\text{arc } CD = \text{arc } DE$. Using only an unmarked straightedge, construct the mid-point of $\text{arc } AE$.

Solution - see problem 2251 [1997: 300, 1998: 373]

New solutions were sent in by NIKOLAOS DERGIADES, Thessaloniki, Greece; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; and GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria.

2289*. [1997: 501] *Proposed by Clark Kimberling, Evansville, IN, USA.*

Use any sequence, $\{c_k\}$, of 0's and 1's to define a *repetition-resistant sequence* $s = \{s_k\}$ inductively as follows:

1. $s_1 = c_1$, $s_2 = 1 - s_1$;
2. for $n \geq 2$, let

$$\begin{aligned} L &= \max\{i \geq 1 \ : \ (s_{m-i+2}, \ldots, s_m, s_{m+1}) \\ &\qquad = (s_{n-i+2}, \ldots, s_n, 0) \quad \text{for some } m < n\}, \\ L' &= \max\{i \geq 1 \ : \ (s_{m-i+2}, \ldots, s_m, s_{m+1}) \\ &\qquad = (s_{n-i+2}, \ldots, s_n, 1) \quad \text{for some } m < n\}. \end{aligned}$$

(so that L is the maximal length of the tail-sequence of $(s_1, s_2, \ldots, s_n, 0)$ that already occurs in $(s_1, s_2, \ldots, s_n)$, and similarly for L'), and

$$s_{n+1} = \begin{cases} 0 & \text{if} \quad L < L', \\ 1 & \text{if} \quad L > L', \\ c_n & \text{if} \quad L = L'. \end{cases}$$

(For example, if $c_i = 0$ for all i, then

$$s = (0,1,0,0,0,1,1,0,1,0,1,1,1,0,0,1,0,0,$$
$$1,1,1,1,0,1,1,0,0,0,0,0,1,0,1,0,\ldots)\Big)$$

Prove or disprove that s contains every binary word.

No solutions have been received — the problem remains open.

2290. [1997: 501] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

For $x, y, z \geq 0$, prove that

$$\left((x+y)(y+z)(z+x)\right)^2 \;\geq\; xyz(2x+y+z)(2y+z+x)(2z+x+y)\,.$$

Solution by Florian Herzig, student, Cambridge, UK.

First note that we can assume that x, y, z are greater than zero since otherwise the inequality becomes trivial. Now let $a = \frac{1}{x}$, $b = \frac{1}{y}$ and $c = \frac{1}{z}$. The inequality is therefore equivalent to

$$(a+b)^2(b+c)^2(c+a)^2 \;\geq\; (bc+m)(ca+m)(ab+m)\,,$$

where $m = bc + ca + ab$. Since $(a+b)(a+c) = a^2 + m$, etc., this is equivalent to

$$(a^2+m)(b^2+m)(c^2+m) \;\geq\; (bc+m)(ca+m)(ab+m)\,,$$

which is a consequence of Cauchy's Inequality:

$$(a^2+m)(b^2+m) \;\geq\; (ab+m)^2$$

(the three such expressions are then multiplied together). Equality therefore holds if and only if $a = b = c$; that is, $x = y = z$ or two of the x, y, z are zero.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MICHAEL LAMBROU, University of Crete, Crete, Greece (3 solutions); KEE-WAI LAU, Hong

Kong; VEDULA N. MURTY, Dover, PA, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; GEORGE TSAPAKIDIS, Agrinio, Greece; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer.

2291. [1997: 501] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Let a, b, c denote the side lengths of a Pythagorean triangle. Suppose that each side length is the sum of two positive integer squares. Prove that $360|abc$.

Solution by Florian Herzig, student, Cambridge, UK.

More generally we show that this is even true for all right triangles with each side being the sum of any two integer squares. We use the following well-known criterion ($\star$) throughout the solution: *a positive integer n is the sum of two integer squares if and only if all prime factors p with $p \equiv 3$ (mod 4) are contained in n to an even power*. We may assume that a, b, c have no common factor since if $d > 1$ is the highest common factor of them and $a = a_1 d$, etc., then by ($\star$) a_1, etc. are again sums of two squares (d has to contain each prime of the form $4k + 3$ to an even power). Hence we can write

$$\begin{aligned} a &= m^2 - n^2 \\ b &= 2mn \\ c &= m^2 + n^2 \end{aligned}$$

where $m > n$ are relatively prime and $m \not\equiv n$ (mod 2).

For the power of 3 in abc we know that $a^2 + b^2 = c^2$ and since $x^2 \equiv 1 \pmod 3$ for x not divisible by 3, it follows that at least one of a, b, c is divisible by 3. By ($\star$) this implies that this one is even divisible by 9, whence $9|abc$.

For the power of 2 in abc notice that m even, n odd is impossible since it would then follow that $a \equiv 3 \pmod 4$. Thus m is odd and n is even. As $b/2$ is the sum of two integer squares and m, n have no common factor, it follows that m, n are each the sum of two integer squares. The same follows for $m - n$ and $m + n$ as $a = (m - n)(m + n)$ is the sum of two integer squares. Hence m and $m - n$ are of the form $4k + 1$ as both are odd. But this implies that n is divisible by 4 and so $8|abc$.

Finally as $x^2 \equiv 1$ or $4 \pmod 5$ for x not divisible by 5, we conclude (since $a^2 + b^2 = c^2$) that at least one of a, b, c is divisible by 5 and hence also abc. This shows that $360|abc$.

Moreover this is the best possible, even for side lengths being the sum of two *positive* integer squares: just consider $(a, b, c) = (225, 272, 353)$, $(153, 104, 185)$, and $(65, 72, 97)$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL

LAMBROU, University of Crete, Crete, Greece; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

There were three incorrect and two incomplete solutions submitted.

2292. [1997: 502] *Proposed by K.R.S. Sastry, Dodballapur, India.*

A convex quadrilateral Q has integer values for its angles, measured in degrees, and the size of one angle is equal to the product of the sizes of the other three.

Show that Q is either a parallelogram or an isosceles trapezium.

Solution by Florian Herzig, student, Cambridge, UK (slightly edited).

Let the angles be x, y, z, and xyz in degrees, which are positive integers. Then $x + y + z + xyz = 360$, and since the quadrilateral is convex, $xyz < 180$. Without loss of generality, $x \leq y \leq z$. Hence

$$3z \ \geq \ x+y+z \ = \ 360 - xyz \ > \ 180\,,$$

and so $z > 60$. But then $180 > xyz > 60xy$, which gives $xy \leq 2$, and leaves us two cases to check.

If $x = 1$ and $y = 2$, then $z = 119$, which contradicts $xyz < 180$.

If $x = y = 1$ we obtain $z = 179$ and the last angle is also 179 degrees. Depending on whether the angles of one degree are opposite or adjacent, Q is a parallelogram or isosceles trapezium.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; VICTOR OXMAN, University of Haifa, Haifa, Israel; D.J. SMEENK, Zaltbommel, the Netherlands; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

2293. [1997: 502] *Proposed by Claus Mazanti Sorensen, student, Aarhus University, Aarhus, Denmark.*

A sequence, $\{x_n\}$, of positive integers has the properties:

1. for all $n > 1$, we have $x_{n-1} < nx_n$;
2. for arbitrarily large n, we have $x_1x_2\ldots x_{n-1} < nx_n$;
3. there are only finitely many n dividing $x_1x_2\ldots x_{n-1}$.

Prove that $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{x_k k!}$ is irrational.

Solution by Michael Lambrou, University of Crete, Crete, Greece.
We shall show the required irrationality without using conditions 1 and 3. We argue by contradiction.

Suppose condition 2 is valid for all $n \geq n_0$ for some fixed $n_0 \in \mathbb{N}$, and suppose on the contrary that

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{x_k k!} = \frac{p}{q}$$

with $p, q \in \mathbb{N}$. By multiplying both terms of the fraction p/q by a positive integer, if necessary, we may assume that $q \geq n_0$. In particular we have by condition 2 that

$$x_1x_2\ldots x_q < (q+1)x_{q+1} \qquad (1)$$

and, for all $k \geq q+1$, (since $x_i \geq 1$ and $0 < x_{k-1} \leq x_1x_2\ldots x_{k-1} < kx_k$) that $0 < x_{k-1} < kx_k$. So for $k \geq q+1$ we have

$$\frac{1}{x_{k-1}(k-1)!} > \frac{1}{x_k k!}.$$

Thus the terms of

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{x_k k!}$$

are decreasing in absolute value. So by a well-known estimate of alternating series (see, for example, G.H. Hardy, *A Course of Pure Mathematics*, Cambridge University Press, Tenth Edition, 1952, p. 377) we have

$$\begin{aligned} 0 &< \frac{1}{x_{q+1}(q+1)!} - \frac{1}{x_{q+2}(q+2)!} \\ &< (-1)^{q+1}\sum_{k=q+1}^{\infty}\frac{(-1)^k}{x_k k!} < \frac{1}{x_{q+1}(q+1)!}, \end{aligned}$$

and so

$$0 \ < \ (-1)^{q+1}\left(\frac{p}{q} - \sum_{k=1}^{q}\frac{(-1)^k}{x_k k!}\right) \ < \ \frac{1}{x_{q+1}(q+1)!}\,.$$

Multiplying across by $q!(x_1x_2\ldots x_q)$ we obtain

$$\begin{aligned} 0 \ &< \ (-1)^{q+1}\left[q!(x_1x_2\ldots x_q)\frac{p}{q} - \sum_{k=1}^{q}\frac{(-1)^k q!(x_1x_2\ldots x_q)}{x_k k!}\right] \\ &< \ \frac{x_1x_2\ldots x_q}{x_{q+1}(q+1)}\,. \end{aligned}$$

The middle term is an integer (as a sum of integers) but by inequality (1) it is strictly between 0 and 1. This is absurd, and the required irrationality follows.

Also solved by FLORIAN HERZIG, student, Cambridge, UK; and the proposer. Herzig also avoided use of condition (3).

2294. [1997: 502] *Proposed by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

For the annual Sino-Japanese "Go" tournament, each country sends a team of seven players, C_i's and J_i's, respectively. All players of each country are of different ranks (strengths), so that

$$C_1 < C_2 < \ldots < C_7 \qquad \text{and} \qquad J_1 < J_2 < \ldots < J_7.$$

Each match is determined by one game only, with no tie. The winner then takes on the next higher ranked player of the opponent country. The tournament continues until all the seven players of one country are eliminated, and the other country is then declared the winner. (For those who are not familiar with the ancient Chinese "Chess" game of "Go", a better and perhaps more descriptive translation would be "the surrounding chess".)

(a) What is the total number of possible sequences of outcomes if each country sends in n players?

(b)* What is the answer to the question in part (a) if there are three countries participating with n players each, and the rule of the tournament is modified as follows:

> The first match is between the weakest players of two countries (determined by lot), and the winner of each match then plays the weakest player of the third country who has not been eliminated (if there are any left). The tournament continues until all the players of two countries are eliminated.

Solution to (a) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and the proposers.*

Let T_n denote the number sought. Suppose, for some $k = 1, 2, \ldots, n$, the winning country has lost $k - 1$ of its players before the tournament is over; that is, the k^{th} player of the winning country defeats the strongest player of the losing country. Since each match eliminates exactly one player, the total number of matches before the last game is $n-1+k-1$, or $n+k-2$, and among these $n+k-2$ matches, $k-1$ of them resulted in the elimination of a player from the winning country. Hence, for each k, the total number of such sequences of outcomes is $W_k = \binom{n+k-2}{k-1}$, and so

$$\begin{aligned} T_n &= 2\sum_{k=1}^{n} W_k = 2\sum_{k=1}^{n}\binom{n+k-2}{k-1} \\ &= 2\left\{\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \ldots + \binom{2n-2}{n-1}\right\} \\ &= 2\binom{2n-1}{n-1} = \frac{2n}{n}\binom{2n-1}{n-1} = \binom{2n}{n}. \end{aligned}$$

[Ed: The summation formula used above is a special case of the well-known identity

$$\sum_{k=0}^{r}\binom{n+k}{k} = \binom{n+r+1}{r}.$$

This identity can readily be proved by induction. See, for example, p. 207 of *Applied Combinatorics* (2nd Edition), John Wiley and Sons, Inc., 1984, by Alan Tucker.]

Alternate solution to (a) *by the proposers.*

We represent the $2n$ players by a row of $2n$ white marbles. Choose any n of the marbles and colour them black. Let the white (black) marbles represent the players from China (Japan), and so they are listed (from left to right) in order of their elimination, so that the colour of the last marble indicates the winning country. Clearly, there is a one-to-one correspondence between all such colourings and all the possible sequences of outcomes; for example, the colouring depicted below corresponds to the following sequence of outcomes:

C_1 defeats J_1 and J_2, but loses to J_3;
C_2 defeats J_3 and J_4, but loses to J_5;
C_3 defeats J_5, but loses to J_6;
C_4 loses to J_6;
C_5 defeats J_6, but loses to J_7;
finally, C_6 defeats J_7.

Therefore $T_n = \binom{2n}{n}$ as in solution I.

No other solutions were received to either (a) *or* (b)*. *Hence,* (b)* *remains open.*

If one lets $f(n)$ denote the number sought, then using brute-force computations on a tree diagram, the proposers found that $f(1) = 12$ and $f(2) = 84$. They feel that, for general n, this is a difficult problem.

2295. [1997: 503] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find three positive integers a, b, c, in arithmetic progression (with positive common difference), such that $a+b$, $b+c$, $c+a$, are all perfect squares.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

We take the arithmetic progression as $x - y$, x, and $x + y$. Then we must have

$$2x - y = u^2, \quad 2x = v^2, \quad 2x + y = w^2$$

so that $x = (w^2 + u^2)/4$, $y = (w^2 - u^2)/2$, and $u^2 + w^2 = 2v^2$.

The problem of finding three squares in arithmetic progression is well-known and is obtained easily from the identity

$$(1 + i)(p + iq)^2 = (p^2 - q^2 - 2pq) + i(p^2 - q^2 + 2pq).$$

On taking the square of the absolute values of each side, we have

$$2(p^2 + q^2)^2 = (2pq + p^2 - q^2)^2 + (2pq + p^2 - q^2)^2.$$

Hence we can take

$$u = 2pq + q^2 - p^2, \quad w = 2pq + p^2 - q^2, \quad \text{and} \quad v = p^2 + q^2 \quad (\text{with } p > q).$$

In order to have $x > y$, we must have $3u^2 > w^2$ or $(p^2+q^2)^2 > 8pq(p^2-q^2)$. This can be satisfied for an infinite number of pairs (p, q) in which p is close to q and not too small. For example, if we choose $p = 10$ and $q = 9$, we get $x = (199^2 + 161^2)/4$ and $y = (199^2 - 161^2)/2$. Since the Diophantine equation is homogeneous, we can multiply x and y by 4, to give $x = 65{,}522$ and $y = 27{,}360$, and then $2x - y = 322^2$, $2x = 362^2$, and $2x + y = 398^2$.

Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; SAM BAETHGE, Nordheim, Texas, USA; NIELS BEJLEGAARD, Stavanger, Norway; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete,

Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There was one incorrect solution submitted.

Hess and Wilke both determine constraints on the ratio p/q which will guarantee that $x > y$ above:

$$\frac{p}{q} < 2-\sqrt{3}+2\sqrt{2-\sqrt{3}} \approx 1.303225373$$

or

$$\frac{p}{q} > 2+\sqrt{3}+2\sqrt{2+\sqrt{3}} \approx 7.595754113\,.$$

Many solvers also pointed out that to get integers for x and y we need to have p and q having the same parity, which explains why Klamkin needed to multiply his values by 4 to get integers.

2296. [1997: 503] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Show that $\sin^2\frac{\pi x}{2} > \frac{2x^2}{1+x^2}$ for $0 < x < 1$.

Hence or otherwise, deduce that $\pi < \frac{\sin \pi x}{x(1-x)} < 4$ for $0 < x < 1$.

Solution by Florian Herzig, student, Cambridge, UK.

From the infinite product formula for $\cos x$ [Ed.: See, for example, 1.431, #3 on p. 45 of *Tables of Integrals, Series and Products* (5^{th} edition), by I.S. Gradshteyn and I.M. Ryzhik, Academic Press, 1994], we get

$$\cos\left(\frac{\pi x}{2}\right) = \prod_{k=0}^{\infty}\left(1-\frac{x^2}{(2k+1)^2}\right) \le 1-x^2 \qquad \text{for } 0 \le x \le 1\,. \qquad (1)$$

Since $0 < (1-x^2)(1+x^2) < 1$ for $0 < x < 1$, we have

$$\cos^2\left(\frac{\pi x}{2}\right) \le (1-x^2)^2 < \frac{1-x^2}{1+x^2}\,,$$

or, equivalently,

$$\sin^2\left(\frac{\pi x}{2}\right) > \frac{2x^2}{1+x^2}\,.$$

From (1), we deduce that $\cos(\pi y) \le 1-4y^2$ for $0 \le y \le \frac{1}{2}$. Hence, for $0 < x \le \frac{1}{2}$ and $y = \frac{1}{2}-x$, we get

$$\sin \pi x \le 1-(1-2x)^2 = 4x(1-x)\,.$$

Dividing by $x(1-x)$ and noting that $f(x) = \frac{\sin(\pi x)}{x(1-x)} = f(1-x)$, we have proved the right side of the double inequalities. Note that equality holds when $x = \frac{1}{2}$. [Ed.: Hence, the right inequality, as given, was incorrect.]

To prove the left inequality, note first that $y = \cos(\pi t)$ is concave on $[0, \frac{1}{2}]$. Since the chord joining $(0,1)$ to $(\frac{1}{2}, 0)$ has equation $y = 1 - 2t$, we have, by integrating from 0 to x $(0 < x \leq \frac{1}{2})$ that $\frac{1}{\pi}\sin(\pi x) > x(1-x)$, or $\frac{\sin(\pi x)}{x(1-x)} > \pi$. Again, since $f(x) = f(-x)$, the inequality holds for all $x \in (0,1)$.

Also solved by PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There were also one incomplete and one incorrect solution.

The infinite product formula was also used by Konečný, Lambrou and Seiffert. Besides Herzig, Bradley, Lambrou and Seiffert also pointed out the error mentioned in the solution above.

Herzig remarked that the same problem appeared on p. 277 of G.H. Hardy's A Course of Pure Mathematics, *Cambridge University Press, and that it was also used in the Mathematical Tripos 1993 for part IA (Cambridge exams after the first year), paper 3, problem 7D. Janous asked the question:*

> Determine the set of all real numbers α such that $\sin^{\alpha}\left(\frac{\pi x}{2}\right) \geq \frac{2x^{\alpha}}{1+x^{\alpha}}$ holds for all $x \in [0,1]$.

Lambrou strengthened the first inequality to

$$\sin^2\left(\frac{\pi x}{2}\right) > \frac{20x^2}{9(1+x^2)} \qquad \textit{if } 0 < x \leq \tfrac{1}{2};$$

$$\sin^2\left(\frac{\pi x}{2}\right) \geq x > \frac{2x^2}{1+x^2} \qquad \textit{if } \tfrac{1}{2} \leq x < 1.$$

Both Konečný and Seiffert pointed out that the left hand side of the double inequalitiers is weaker than the following known inequality:

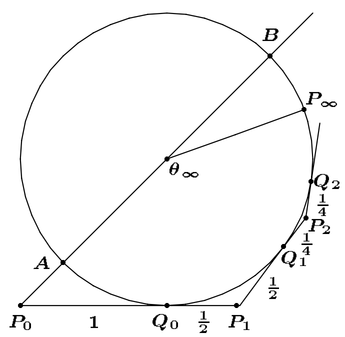
$$\frac{1-x^2}{1+x^2} \leq \frac{\sin(\pi x)}{\pi x} \qquad \textit{for all } x \neq 0.$$

See ***CRUX*** [1993: 433]. *According to Klamkin and Manes, this inequality is "not as easy to prove".*

2297. [1997: 503] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Given is a circle of radius 1, centred at the origin. Starting from the point $P_0 = (-1,-1)$, draw an infinite polygonal path $P_0P_1P_2P_3\ldots$ going counterclockwise around the circle, where each P_iP_{i+1} is a line segment tangent to the circle at a point Q_i, such that $|P_iQ_i| = 2|Q_iP_{i+1}|$. Does this path intersect the line $y = x$ other than at the point $(-1,1)$?

Solution by Richard I. Hess, Rancho Palos Verdes, California, USA.



The total length of the infinite polygon $P_0P_1P_2P_3\ldots$ is

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\cdots = 3.$$

The total length of the arc from A to B is $\pi > 3$. Therefore $P_0P_1P_2\ldots$ will never reach B.

By calculation the limit point P_∞ has angle $\theta_\infty \approx 154.76$ degrees.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; VICTOR OXMAN, University of Haifa, Haifa, Israel; and the proposer.

Janous and Oxman also gave the simple solution above.

2298. [1997: 503] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

The "Tickle Me" Feather Company ships its feathers in boxes which cannot contain more than 1 kg of feathers each. The company has on hand a number of assorted feathers, each of which weighs at most one gram, and whose total weight is 1000001/1001 kg.

Show that the company can ship all the feathers using only 1000 boxes.

Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, New York, USA.

Before packing the feathers in the boxes, first sort them by weight, from heaviest to lightest. Now put 1000 feathers in each box, beginning with the heaviest. Thus, the 1000000 heaviest feathers are now in the boxes with no box containing more than 1 kg of feathers. For each remaining feather (beginning still with the heaviest), put it in the first box that it will "fit" in; that is, the first which it will not put over weight capacity. Continue this process until all feathers are packed or until you reach a feather that will not fit anywhere.

Suppose you find a feather that does not fit. Then in each box, adding that feather would push the weight over 1 kg. Since there are at least 1000 feathers already in each box, which are each the same weight as, or heavier than, the new feather that will not fit, adding this feather would increase the weight by a factor of at most $1001/1000$. Thus the weight of each box without this feather must be strictly greater than $1000/1001$ kg. If we add the weight of this feather to that of the feathers in any one box, it exceeds 1 kg. The total weight of the feathers in the other 999 boxes is more than $999 \cdot 1000/1001$ kg. Putting these weights together, we get a value strictly greater than the $1000001/1001$ kg which is supposed to be the total weight of all the feathers. Therefore this situation cannot happen, so it must be possible to pack all the feathers.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the proposer.

Janous and the proposer note the more general result:

let $a \geq 1$ and let n be a positive integer, and let $\{w_1, w_2, \ldots\}$ be positive real numbers satisfying

$$\max w_i \leq \frac{1}{a} \qquad \text{and} \qquad \sum w_i \leq \frac{na+1}{a+1}\,.$$

Then the w_i's can be partitioned into at most n sets so that the sum of each set is at most 1.

Readers may like to prove this. The given problem is the case $a = n = 1000$.

The proposer also notes that the result is best possible in two ways:

- *if there are 1000001 feathers each of weight $\frac{1}{1001} + \epsilon < \frac{1}{1000}$ kg, for ϵ sufficiently small, then the total weight of the feathers is just slightly greater than $1000001/1001$ kg, but if only 1000 boxes are used, some box must contain at least 1001 feathers which would weigh altogether more than 1 kg;*
- *if there are 999001 feathers each of weight $\frac{1}{1000} + \epsilon$ kg, for ϵ sufficiently small, then each feather weighs just slightly more than 1 gm, and the total weight of the feathers is still at most $1000001/1001$ kg, but if only 1000 boxes are used, some box must contain at least 1000 feathers which would weigh altogether more than 1 kg.*

YEAR END FINALE

Again, a year has flown by! This year was flowing smoothly until I had a three month sick-leave break. Our Associate Editor, CLAYTON HALFYARD, completed issue 5, and did issues 6 and 7. He did a splendid job. In particular, Memorial University student PAUL MARSHALL did almost all the diagrams for issues 6 and 7. He too did a splendid job. I have resumed responsibilities with this issue. Thanks to all for their patience and assistance.

The online version of ***CRUX with MAYHEM*** continues to attract attention. There is a "new look" now. We recommend it highly to you. Thanks are due to LOKI JORGENSON, NATHALIE SINCLAIR, FREDERIC TESSIER, and the rest of the team at SFU who are responsible for this.

There are many people that I wish to thank most sincerely for particular contributions. Again, first and foremost is BILL SANDS. Bill is of such value to me and to the continuance of ***CRUX with MAYHEM***. As well, I thank most sincerely, CATHY BAKER, ILIYA BLUSKOV, ROLAND EDDY, CHRIS FISHER, BILL SANDS, JIM TOTTEN, and EDWARD WANG, for their regular yeoman service in assessing the solutions; DENIS HANSON, DIETER RUOFF, CHRIS FISHER, RICHARD MCINTOSH, DAIHACHIRO SATO, DOUG FARENICK, HARLEY WESTON, for ensuring that we have quality articles; TED LEWIS, ANDY LIU, MARÍA FALK de LOSADA, BILL SANDS, JIM TIMOURIAN, for ensuring that we have quality book reviews, ROBERT WOODROW, who carries the heavy load of two corners, and RICHARD GUY for sage advice whenever necessary.

The editors of the ***MAYHEM*** section, NAOKI SATO, CYRUS HSIA, ADRIAN CHAN, RICHARD HOSHINO, DAVID SAVITT and WAI LING YEE, all do a sterling job.

I also thank two of our regulars who assist the editorial board with proof reading; THEODORE CHRONIS and WALDEMAR POMPE. The quality of these people is a vital part of what makes ***CRUX with MAYHEM*** what it is. Thank you one and all.

As well, I would like to give special thanks to our Associate Editor, CLAYTON HALFYARD, for guiding issues 5, 6 and 7 to fruition, and for keeping me from printing too many typographical and mathematical errors; and to my colleagues, PETER BOOTH, RICHARD CHARRON, ROLAND EDDY, EDGAR GOODAIRE, ERIC JESPERS, MIKE PARMENTER, DONALD RIDEOUT, NABIL SHALABY, in the Department of Mathematics and Statistics at Memorial University, and to JOHN GRANT MCLOUGHLIN, Faculty of Education, Memorial University, for their occasional sage advice. I have also been helped by some Memorial University students, MIKE GILLARD, DON HENDER, PAUL MARSHALL, JON MAUGER, SHANNON SULLIVAN, TREVOR RODGERS, as well as a WISE Summer student, SHAWNA LEE.

The staff of the Department of Mathematics and Statistics at Memorial University deserve special mention for their excellent work and support: ROS ENGLISH, MENIE KAVANAGH, WANDA HEATH, and LEONCE MORRISSEY; as well as the computer and networking expertise of RANDY BOUZANE, DAVID VINCENT and PAUL DROVER.

Also the LaTeX expertise of JOANNE LONGWORTH at the University of Calgary, ELLEN WILSON at Mount Allison University, and the ***MAYHEM*** staff, is much appreciated.

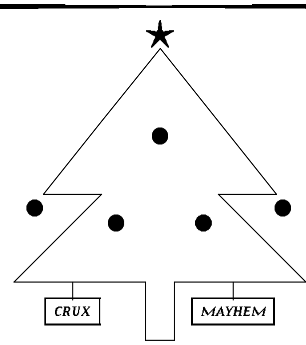
GRAHAM WRIGHT, the Managing Editor, continues to be a tower of strength and support. Graham has kept so much on the right track. He is a pleasure to work with. The CMS's TeX Editor, MICHAEL DOOB has been very helpful in ensuring that the printed master copies are up to the standard required for the U of T Press who continue to print a fine product.

Finally, I would like to express real and heartfelt thanks to the Head of my Department, HERBERT GASKILL, and to the Acting Dean of Science of Memorial University, WILLIE DAVIDSON. Without their support and understanding, I would not be able to do the job of Editor-in-Chief.

Last but not least, I send my thanks to you, the readers of ***CRUX with MAYHEM***. Without you, ***CRUX with MAYHEM*** would not be what it is. Keep those contributions and letters coming in. We need your ARTICLES, PROPOSALS and SOLUTIONS to keep ***CRUX with MAYHEM*** alive and well. I do enjoy knowing you all.

In a letter from Ron Weedon, Reading, UK, concerning the reference to F. G.-M., the author of the valuable 1912 book entitled **Exercises de Géométrie**, we learned that the author's full name is F. Gabriel-Marie (which members of the Editorial Board had not known).

The letter made us wonder if there is a copy of the F. G.-M. book anywhere in Canada. Let us know if you have a copy. Apparently, the University of Toronto does not have a copy, and the copy in the British Library is damaged.



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