Letter from the Editors

This edition is somewhat different from usual. A little explanation is necessary.

At the end of July 1998, before all the material for issue 5, 1998, was complete, the Editor-in-Chief, Bruce Shawyer, was admitted to hospital to await surgery. The Associate Editor, Clayton Halfyard, stepped into the breach, and, with the assistance of Bill Sands and Memorial University summer students Paul Marshall and Trevor Rodgers, completed issue 5.

On 25 August 1998, the Editor-in-Chief underwent a quadruple coronary artery bypass operation and is now at home for a two month recuperation period. The Associate Editor has prepared this issue, but since the Academy Corner is the sole responsibility of the Editor-in-Chief, it is missing from this issue.

The Editor-in-Chief hopes to be able to respond to email messages in the near future.

Bruce Shawyer — Clayton Halfyard
Editor-in-Chief — Associate Editor

Lettre de la rédaction

Ce numéro diffère quelque peu de l'ordinaire. Quelques explications s'imposent.

La fin du mois de juillet 1998, avant que tous les textes du numéro 5 soient prêts, le rédacteur en chef, Bruce Shawyer, a été admis à l'hôpital en vue d'une opération. Le rédacteur en chef adjoint, Clayton Halfyard, a donc pris la relève. Avec l'aide de Bill Sands et des étudiants de l'Université Memorial, Paul Marshall et Trevor Rodgers, il a réussi à terminer le numéro 5.

Le 25 août, 1998, le rédacteur en chef a subi un quadruple pontage coronarien. Il est maintenant de retour à la maison, pour une convalescence de deux mois. Le rédacteur en chef adjoint s'est occupé du présent numéro, mais, puisque la chronique Academy Corner relève entièrement du rédacteur en chef, elle ne paraîtra pas dans ce numéro.

Le rédacteur en chef espère être en mesure de répondre à son courriel sous peu.

Bruce Shawyer — Clayton Halfyard
Rédacteur en chef — Rédacteur en chef adjoint

THE OLYMPIAD CORNER

No. 192

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a first Olympiad for this issue we give the problems of the 4^{th} Mathematical Olympiad of the Republic of China (Taiwan) written April 13, 15, 1995. My thanks go to Bill Sands of the University of Calgary, who collected these problems when he was assisting with the 1995 International Mathematical Olympiad held in Canada.

4^{th} MATHEMATICAL OLYMPIAD OF THE REPUBLIC OF CHINA (TAIWAN)

First Day — Taipei

April 13, 1995

1. Let $P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$ be a polynomial with complex coefficients. Suppose the roots of $P(x)$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $|\alpha_1| > 1$, $|\alpha_2| > 1, \ldots, |\alpha_j| > 1$, and $|\alpha_{j+1}| \leq 1, \ldots, |\alpha_n| \leq 1$. Prove:

$$\prod_{i=1}^{j} |\alpha_i| \;\leq\; \sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}\,.$$

2. Given a sequence of integers: $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$, one constructs a second sequence: $|x_2 - x_1|, |x_3 - x_2|, |x_4 - x_3|, |x_5 - x_4|, |x_6 - x_5|, |x_7 - x_6|, |x_8 - x_7|, |x_1 - x_8|$. Such a process is called a single operation. Find all the 8-term integral sequences having the following property: after finitely many applications of the single operation the sequence becomes an integral sequence with all terms equal.

3. Suppose n persons meet in a meeting, and every one among them is familiar with exactly 8 other participants of that meeting. Furthermore suppose that each pair of two participants who are familiar with each other have 4 acquaintances in common at that meeting, and each pair of two participants who are not familiar with each other have only 2 acquaintances in common. What are the possible values of n?

Second Day — Taipei

April 15, 1995

4. Given n distinct integers $m_1, m_2, \ldots, m_n$, prove that there exists a polynomial $f(x)$ of degree n and with integral coefficients which satisfies the following conditions:

(1) $f(m_i) = -1$, for all i, $1 \leq i \leq n$.

(2) $f(x)$ cannot be factorized into a product of two nonconstant polynomials with integral coefficients.

5. Let P be a point on the circumscribed circle of $\triangle A_1A_2A_3$. Let H be the orthocentre of $\triangle A_1A_2A_3$. Let B_1 (B_2, B_3 respectively) be the point of intersection of the perpendicular from P to A_2A_3 (A_3A_1, A_1A_2 respectively). It is known that the three points B_1, B_2, B_3 are collinear. Prove that the line $B_1B_2B_3$ passes through the midpoint of the line segment $\overline{PH}$.

6. Let a, b, c, d be integers such that $ad - bc = k > 0$, $(a, b) = 1$, and $(c, d) = 1$. Prove that there are exactly k ordered pairs of real numbers (x_1, x_2) satisfying $0 \leq x_1, x_2 < 1$ and for which both $ax_1 + bx_2$ and $cx_1 + dx_2$ are integers.

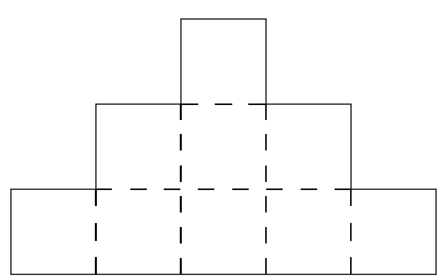
As a second problem set this issue for your puzzling pleasure, we give the XI Italian Mathematical Olympiad written May 5, 1995 at Cesenatico. Thanks go to Bill Sands of the University of Calgary who collected them while at the 1995 IMO in Canada.

XI ITALIAN MATHEMATICAL OLYMPIAD

Cesenatico, May 5, 1995

Time: 4.5 hours

1. Determine for which values of the integer n it is possible to cover up, without overlapping, a square of side n with tiles of the type shown in the picture



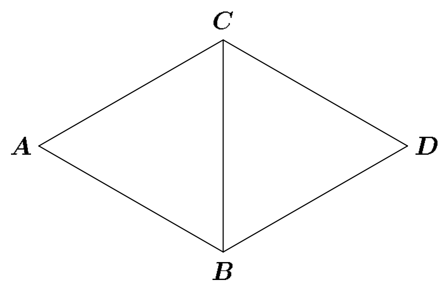
where each small square of the tile has side 1.

2. In a class of 20 students no two of them have the same ordered pair (written and oral examinations) of scores in mathematics. We say that student A is better than B if his two scores are greater than or equal to the corresponding scores of B. The scores are integers between 1 and 10.

(a) Show that there exist three students A, B and C such that A is better than B and B is better than C.

(b) Would the same be true for a class of 19 students?

3. In a town there are 4 pubs, A, B, C and D, connected as shown in the picture.



A drunkard wanders about the pubs starting with A and, after having a drink, goes to any of the pubs directly connected, with equal probability.

(a) What is the probability that the drunkard is at pub C at his fifth drink?

(b) Where is the drunkard more likely to be after n drinks? $(n > 5)$

4. An acute-angled triangle ABC is inscribed in a circle with centre O. Let D be the intersection of the bisector of A with BC, and suppose that the perpendicular to AO through D meets the line AC in a point P interior to the segments AC. Show that $\overline{AB} = \overline{AP}$.

5. Two non-coplanar circles in Euclidean space are tangent at a point and have the same tangents at this point. Show that both circles lie in some spherical surface.

6. Find all pairs of positive integers x, y such that

$$x^2 + 615 = 2^y.$$

As a third set of problems for your attention we give the Third and Fourth Grade and IMO Team selection rounds of the Yugoslav Federal Competition for 1995. Thanks again go to Bill Sands, the University of Calgary, for collecting them for me.

YUGOSLAV FEDERAL COMPETITION 1995
Third and Fourth Grade

1. Let p be a prime number. Prove that the number

$$11\cdots 122\cdots 2\cdots 99\cdots 9 - 123456789$$

is divisible by p, where dots indicate that the corresponding digit appears p times consecutively.

2. A polynomial $P(x)$ with integer coefficients is said to be divisible by a positive integer m if and only if the number $P(k)$ is divisible by m for all $k \in \mathbb{Z}$. If the polynomial

$$P(x) \;=\; a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

is divisible by m, prove that $a_n n!$ is divisible by m.

3. A chord AB and a diameter $CD \perp AB$ of a circle k intersect at a point M. Let P lie on the arc ACB and let $P \not\in \{A, B, C\}$. Line PM intersects the circle k at P and $Q \neq P$, and line PD intersects chord AB at R. Prove that $\overline{RD} > \overline{MQ}$.

4. A tetrahedron $ABCD$ is given. Let P and Q be midpoints of edges AB and CD, and let O and S be the incentre and the circumcentre of the tetrahedron, respectively. If points P, Q and S belong to the same line, prove that the point O also belongs to that line.

Selection of the IMO Team

1. Find all the triples (x, y, z) of positive rational numbers such that $x \le y \le z$ and

$$x + y + z, \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \quad xyz \in \mathbb{Z}.$$

2. Let n be a positive integer having exactly 1995 1's in its binary representation. Prove that 2^{n-1995} divides $n!$.

3. Let $SABCD$ be a pyramid such that all of its edges are of the same length. Let points $M \in BC$ and $N \in AS$ be such that the line MN is perpendicular to line AD as well as to the line BC. Find the ratios BM/MC and SN/NA. [*Editor's note:* $ABCD$ is the base of the pyramid.]

We now turn to readers' solutions to problems of the Swedish Mathematics Contest, 1993 [1997: 196].

SWEDISH MATHEMATICS CONTEST 1993
Final
November 20

1. The integer x is such that the sum of the digits of $3x$ is the same as the sum of the digits of x. Prove that 9 is a factor of x.

Solutions by Jamie Batuwantudawe, student, Sir Winston Churchill High School, Calgary; by Michael Selby, University of Windsor, Windsor, Ontario; and by Enrico Valeriano Cuba, National University of Engineering, Lima, Peru. We give the solution of Valeriano.

Let $S(n)$ be the sum of the digits of n. Working modulo 9

$$3x \equiv S(3x) \bmod 9\,.$$

Also

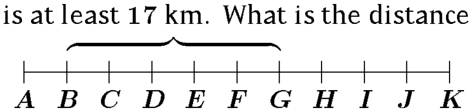
$$x \equiv S(x) \bmod 9\,.$$

Since $S(3x) = S(x)$,

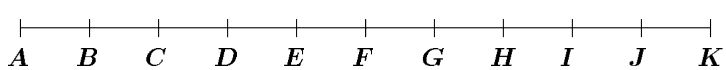
$$2x \equiv 0 \bmod 9\,,$$

and since $(2, 9) = 1$, we have $x \equiv 0 \bmod 9$.

2. A railway line is divided into 10 sections by the stations A, B, C, D, E, F, G, H, I, J and K. The distance between A and K is 56 km. A trip along two successive sections never exceeds 12 km. A trip along three successive sections is at least 17 km. What is the distance between B and G?



Solutions by Jamie Batuwantudawe, student, Sir Winston Churchill High School, Calgary; by Michael Selby, University of Windsor, Windsor, Ontario; and by Enrico Valeriano Cuba, National University of Engineering, Lima, Peru. We give Batuwantudawe's solution.



Now $\overline{AK} = 56$ and $\overline{AK} = \overline{AD} + \overline{DG} + \overline{GJ} + \overline{JK}$. We know that $\overline{AD}, \overline{DG}, \overline{GJ} \geq 17$. Thus $\overline{JK} \leq 5$ to satisfy $\overline{AK} = 56$. We know $\overline{HK} \geq 17$, and since $\overline{JK} \leq 5$, $\overline{HJ} \geq 12$. But, we also know $\overline{HJ} \leq 12$. Thus $\overline{HJ} = 12$. Since $\overline{HK} \geq 17$ and $\overline{HJ} = 12$, $\overline{JK} \geq 5$. The only possibility is that $\overline{JK} = 5$.

Symmetrically we find that $\overline{AB} = 5$ and $\overline{BD} = 12$.

Now,

$$\begin{aligned}\overline{DH} &= \overline{AK} - \overline{AB} - \overline{BD} - \overline{HJ} - \overline{JK} \\ &= 56 - 5 - 12 - 5 - 12 = 22\,.\end{aligned}$$

Now $\overline{GJ} \geq 17$ but $\overline{HJ} = 12$. Hence $\overline{GH} \geq 5$. Since $\overline{DG} \geq 17$ and $\overline{DH} = \overline{DG} + \overline{GH} = 22$, we obtain $\overline{DG} = 17$ and $\overline{GH} = 5$.

Now

$$\begin{aligned} \overline{BG} &= \overline{BD} + \overline{DG} \\ &= 12 + 17 = 29 . \end{aligned}$$

3. Assume that a and b are integers. Prove that the equation $a^2 + b^2 + x^2 = y^2$ has an integer solution x, y if and only if the product ab is even.

Solutions by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; Michael Selby, University of Windsor, Windsor, Ontario; by Sree Sanyal, student, Western Canada High School, Calgary, Alberta; by Enrique Valeriano Cuba, National University of Engineering, Lima, Peru; and by Michael Lebedinsky, student, Henry Wise Wood High School, Calgary, Alberta. We give Selby's solution.

First, we prove that this condition is necessary. Suppose ab is odd. Then a, b are odd and $a^2 \equiv b^2 \equiv 1 \bmod 4$. Now $x^2 \equiv 0$ or $1 \bmod 4$, and $y^2 \equiv 0$ or $1 \bmod 4$. Therefore $a^2 + b^2 + x^2 = y^2$ is not possible, since if we consider this modulo 4, $2 + x^2 \equiv y^2 \bmod 4$, which is impossible since $2 + x^2 \equiv 2$ or $3 \bmod 4$. Therefore ab must be even.

If ab is even, then, without loss of generality, $a = 2k$.

Consider $4k^2 + b^2 + x^2 = y^2$.

If $4k^2 + b^2 = 2t + 1$, t an integer, then set $y - x = 1$ and $y + x = 2t + 1$, $2y = (t + 1)2$, $y = t + 1$ and $x = t$.

Then $2t + 1 + t^2 = (t + 1)^2$. We are done.

If $4k^2 + b^2$ is even, then $b = 2s$ and $4k^2 + b^2 = 4(k^2 + s^2) = 4m$. Again, $y^2 - x^2 = 4m$.

Set $y - x = 2$ and $y + x = 2m$. Then $y = m + 1$ and $x = y - 2 = m - 1$.

Now $4m + (m - 1)^2 = (m + 1)^2$, and again we are done. Hence $a^2 + b^2 + x^2 = y^2$ always has a solution when ab is even.

4. To each pair of real numbers a and b, where $a \neq 0$ and $b \neq 0$, there is a real number $a * b$ such that

$$a * (b * c) = (a * b) \cdot c ,$$

$$a * a = 1 .$$

Solve the equation $x * 36 = 216$.

Solutions by Sree Sanyal and Aliya Walji, students, Western Canada High School, Calgary, Alberta; and by Enrique Valeriano Cuba, National University of Engineering, Lima, Peru.

Now $a * (a * a) = (a * a) \cdot a$, so

$$a * 1 = 1 \cdot a = a\,.$$

Also $a * (b * b) = (a * b) \cdot b$ and $a = a * 1 = (a * b) \cdot b$ so

$$a * b = \frac{a}{b}\,.$$

Finally

$$\frac{x}{36} = 216 \implies x = 7776\,.$$

5. A triangle with perimeter $2p$ has sides a, b and c. If possible, a new triangle with the sides $p - a$, $p - b$ and $p - c$ is formed. The process is then repeated with the new triangle. For which original triangles can the process be repeated indefinitely?

Solutions by Michael Selby, University of Windsor, Windsor, Ontario; by Enrique Valeriano, National University of Engineering, Lima, Peru; and by Sonny Chan, student, Western Canada High School, Calgary, Alberta. We give Valeriano's solution.

Let $a \le b \le c$ and Δ be the difference between the longest and the shortest side.

Original Triangle	New Triangle
Perimeter $= 2p$	Perimeter$^{(1)} = 3p - (a + b + c) = p$
$\Delta = c - a$	$\Delta^{(1)} = (p - a) - (p - c) = c - a$

We can see that the perimeter of the new triangle is half the previous perimeter, but Δ is the same. Then, if $\Delta > 0$, repeating this process we can obtain

$$\text{Perimeter}^{(k)} = \frac{2p}{2^k} < \Delta^{(k)} = c - a\,.$$

If $c^{(k)}$ is the longest side we obtain the absurd relation $c^{(k)} <$ Perimeter$^{(k)} < \Delta^{(k)} < c^{(k)}$. Finally, only with an equilateral triangle as the original triangle ($\Delta = 0$) can we repeat the process indefinitely.

6. Let a and b be real numbers and let $f(x) = (ax + b)^{-1}$. For which a and b are there three distinct real numbers x_1, x_2, x_3 such that $f(x_1) = x_2$, $f(x_2) = x_3$ and $f(x_3) = x_1$?

Solutions by Filip Crnogorac and Sonny Chan, students, Western Canada High School, Calgary, Alberta; and by Michael Selby, University of Windsor, Windsor, Ontario. We give Selby's write-up.

Consider the functions of the form

$$g(x) = \frac{\alpha x + \beta}{\gamma x + \delta}\,.$$

Lemma. $g(x)$ has at least 3 distinct fixed points if and only if $\gamma = \beta = 0$, $\alpha = \delta \neq 0$.

Proof. If $\gamma = \beta = 0$, $\alpha = \delta \neq 0$, $g(x) = x$ and it clearly has at least 3 distinct points x_1, x_2, x_3 such that $g(x_i) = x_i$, $i = 1, 2, 3$. Conversely consider the equation for a fixed point x, $g(x) = x$. This implies $\gamma x^2 + \delta x = \alpha x + \beta$ or $\gamma x^2 + (\delta - \alpha)x - \beta = 0$. Suppose this has three distinct roots. Then the quadratic must be identically 0, or $\gamma = \beta = 0$ and $\alpha = \delta$.

Now, if $f(x) = \frac{1}{ax+b}$, then

$$f \circ f(x) = \frac{ax + b}{abx + b^2 + a} \quad \text{and} \quad f \circ f \circ f(x) = \frac{abx + a + b^2}{a(a + b^2)x + ab + b(b^2 + a)}.$$

The problem implies $f \circ f \circ f$ has three distinct real fixed points x_1, x_2, x_3. By the above lemma, this is true if and only if

$$a + b^2 = a(a + b^2) = 0 \quad \text{and} \quad ab + b(a + b^2) = ab \neq 0.$$

This is true if and only if $a = -b^2$ and $ab \neq 0$.

To complete this number of the Olympiad Corner we turn to readers' solutions to problems of the Dutch Mathematical Olympiad, second round, September 1993 [1997: 197].

DUTCH MATHEMATICAL OLYMPIAD
Second Round
September, 1993

1. Suppose that $V = \{1, 2, 3, \ldots, 24, 25\}$. Prove that any subset of V with 17 or more elements contains at least two distinct numbers the product of which is the square of an integer.

Solution by Sonny Chan and Filip Crnogorac, students, Western Canada High School, Calgary.

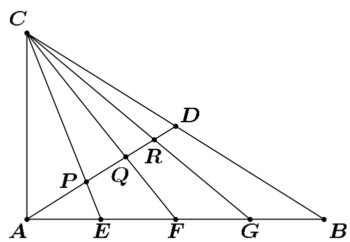
The set of numbers $A = \{1, 2, \ldots, 24, 25\}$ contains a total of five perfect squares $\{1, 4, 9, 16, 25\}$. The product of any two of these will also be a perfect square. There is one triplet, the product of any two of its elements will result in a perfect square: $\{2, 8, 18\}$. The only other pairs of numbers from A whose product is a perfect square are $\{3, 12\}$, $\{5, 20\}$, $\{6, 24\}$. The other eleven elements of the set A are $\{7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23\}$ and they cannot form a perfect square when multiplied with any other element of set A. Group the elements of set A as follows:

$$\{1, 4, 9, 16, 25\},\ \{2, 8, 18\},\ \{3, 12\},\ \{5, 20\},\ \{6, 24\},\ \{7\},$$

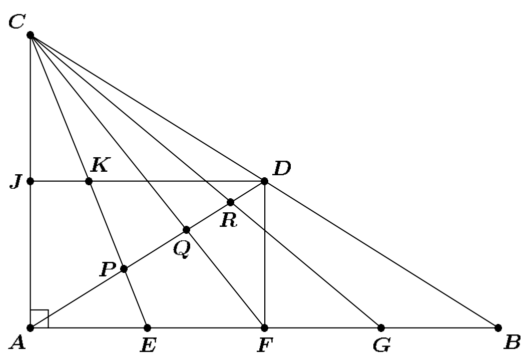
$$\{10\},\ \{11\},\ \{13\},\ \{14\},\ \{15\},\ \{17\},\ \{19\},\ \{21\},\ \{22\},\ \{23\}.$$

If more than one number is chosen from a given group, a perfect square will result. There is a total of 16 groups, so 16 numbers can be chosen without creating a perfect square product. However, if any 17 numbers are chosen, then two must be contained within the same group, and therefore will form a perfect square product.

2. Given is a triangle ABC, $\angle A = 90^\circ$. D is the midpoint of BC, F is the midpoint of AB, E the midpoint of AF and G the midpoint of FB. AD intersects CE, CF and CG respectively in P, Q and R. Determine the ratio $\frac{PQ}{QR}$.



Solution by Filip Crnogorac, student, Western Canada High School, Calgary.



We know that two medians in a triangle divide each other in $2 : 1$ ratio, or in other words the point of intersection is $\frac{2}{3}$ the way from the vertex.

Since CF and AD are both medians in $\triangle ABC$, then $\frac{AQ}{QD} = \frac{2}{1}$, where Q is the point of intersection.

Also, since D is the midpoint of the hypotenuse in the right triangle ABC, then it is the centre of the circumscribed circle with radius $DA = DC = DB$.

Drop a perpendicular from D onto sides AB and CA. The feet of the perpendiculars will be F and J, respectively, where J is the midpoint of AC, since DF and DJ are altitudes in isosceles triangles $\triangle ADB$ and $\triangle ADC$, respectively. Now consider $\triangle CFB$. The segments CG and FD are medians and therefore intersect at H say in the ratio $2 : 1$ so, $\frac{HD}{FD} = \frac{1}{3}$. From here it can be seen that $\triangle ARC$ and $\triangle DRH$ are similar, since their angles are the same. Also, since we know that $\overline{FD} = \overline{JA}$, and $2\overline{JA} = \overline{AC}$ then $\overline{HD} = \frac{1}{6}\overline{CA}$ and $\triangle ARC$ is 6 times bigger than $\triangle DRH$. Now we can see that $\frac{\overline{AR}}{\overline{RD}} = \frac{6}{1}$ and since $\overline{AR} + \overline{RD} = \overline{AD}$, then $\frac{\overline{RD}}{\overline{AD}} = \frac{1}{7}$.

Similarly $\triangle APE \sim \triangle KPD$, where medians DJ and CE meet at K. We know that $\overline{AE} = \frac{1}{4}\overline{AB}$, so then $\overline{JK} = \frac{1}{4}\overline{JD}$, since JD is parallel to AB. It now follows that $\frac{\overline{AE}}{\overline{KD}} = \frac{2}{3}$, and from the similarity of the triangles $\frac{\overline{AP}}{\overline{PD}} = \frac{2}{3}$. Also, since $\overline{AP} + \overline{PD} = \overline{AD}$, then $\frac{\overline{AP}}{\overline{AD}} = \frac{2}{5}$. Combining these results we have $\overline{AP} = \frac{2}{5}\overline{AD}$, $\overline{AQ} = \frac{2}{3}\overline{AD}$, $\overline{QD} = \frac{1}{3}\overline{AD}$ and $\overline{RD} = \frac{1}{7}\overline{AD}$.

Thus
$$\overline{PQ} = \overline{AQ} - \overline{AP} = \frac{2}{3}\overline{AD} - \frac{2}{5}\overline{AD} = \frac{4}{15}\overline{AD}$$
and
$$\overline{QR} = \overline{QD} - \overline{RD} = \frac{1}{3}\overline{AD} - \frac{1}{7}\overline{AD} = \frac{4}{21}\overline{AD}.$$
From these $\frac{\overline{PQ}}{\overline{QR}} = \frac{7}{5}$.

3. A series of numbers is defined as follows: $u_1 = a$, $u_2 = b$, $u_{n+1} = \frac{1}{2}(u_n + u_{n-1})$ for $n \geq 2$. Prove that $\lim_{n\to\infty} u_n$ exists. Express the value of the limit in terms of a and b.

Solution by the Editors.

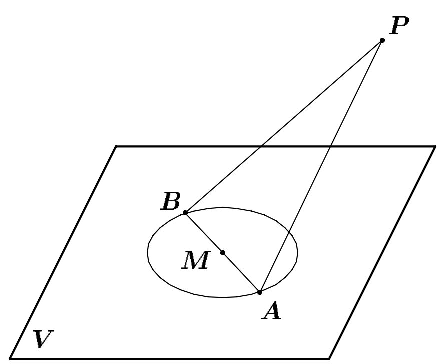
For the recurrence $u_{n+1} = \frac{1}{2}(u_n + u_{n-1})$ we obtain the associated equation $2\lambda^2 - \lambda - 1 = 0$, which has roots $\lambda = -\frac{1}{2}, 1$. Thus we seek a solution of the form $u_n = X\left(-\frac{1}{2}\right)^n + Y$. From $u_1 = a$ and $u_2 = b$ we get

$$-\tfrac{1}{2}X + Y = a$$

$$\tfrac{1}{4}X + Y = b$$

so that $X + \frac{4}{3}(b-a)$ and $Y = \frac{a+2b}{3}$. It is now easy to check by induction that $u_n = \frac{4}{3}(b-a)\left(-\frac{1}{2}\right)^n + \frac{1+2b}{3}$ and as $n \to \infty$, $u_n \to \frac{1}{3}a + \frac{2}{3}b$.

4. In a plane V a circle C is given with centre M. P is a point not on the circle C.



(a) Prove that for a fixed point P, $\overline{AP}^2 + \overline{BP}^2$ is a constant for every diameter AB of the circle C.

(b) Let AB be any diameter of C and P a point on a fixed sphere S not intersecting V. Determine the point(s) P on S such that $\overline{AP}^2 + \overline{BP}^2$ is minimal.

Solution by Jamie Batuwantudawe, student, Sir Winston Churchill High School, Calgary.

(a) With $\triangle PAB$, we can join P and M to create two new triangles, $\triangle PMA$ and $\triangle PMB$. Let $\angle PMA = \theta$. Then $\angle PMB = 180° - \theta$. Because M is the centre of circle C and A and B both lie on circle C, we have $\overline{MA} = \overline{MB} = r$, the radius of the circle.

By the Law of Cosines,

$$\begin{aligned}\overline{BP}^2 &= \overline{MP}^2 + r^2 - 2\overline{MP}r\cos(180° - \theta) \\ &= \overline{MP}^2 + r^2 + 2\overline{MP}r\cos\theta\end{aligned}$$

and

$$\overline{AP}^2 = \overline{MP}^2 + r^2 - 2\overline{MP}r\cos\theta$$

so $\overline{AP}^2 + \overline{BP}^2 = 2\overline{MP}^2 + 2r^2$.

The right hand side is a constant depending only on the radius of the circle and the distance of P from the centre.

(b) From (a), we know that $\overline{AP}^2 + \overline{BP}^2 = 2\overline{MP}^2 + 2r^2$. For any point P on sphere S, the radius of the circle will remain constant. Therefore the only variable affecting the sum $\overline{AP}^2 + \overline{BP}^2$ is $\overline{MP}$, the distance from the point P to the centre of the circle. $\overline{AP}^2 + \overline{BP}^2$ will be a minimum when $\overline{MP}$ is minimum. Therefore we are looking for the point on the sphere closest to M.

Let T be the centre of the sphere S, D be the point on the segment MT that lies on the sphere, and D' be any other point on S.

We know that $\overline{MD} + \overline{DT} < \overline{MD'} + \overline{D'T}$ because the shortest distance between M and T is a straight line. We know that $\overline{DT} = \overline{D'T}$. Thus $\overline{MD} < \overline{MD'}$. Thus D is the point on the sphere which minimizes the sum.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations as well as Olympiad contests.

BOOK REVIEWS

Edited by ANDY LIU

Juegos y acertijos para la enseñanza de las Matemáticas
(**Games and riddles for the teaching of mathematics**)
by Bernardo Recamán Santos,
published by Grupo editorial Norma educativa, 1997, Bogotá, Colombia.
ISBN#958-04-3731-9; softcover, 134 pages.
Reviewed by ***Francisco Bellot Rosado****, Valladolid, Spain.*

This book contains 75 games and riddles, uniformly distributed in five chapters:

I mathematical games;
II arithmetical riddles;
III geometrical riddles;
IV logical riddles;
V algebraic riddles.

A last chapter with solutions (not very detailed), comments and alternatives or variants, and a little Bibliography completes the work.

The difficulty level is indicated by one, two or three stars ($\star$, $\star\star$, $\star\star\star$), meaning that the problem is for students of primary education, of the first years of the secondary, either preuniversity level, respectively; although, as the author said in the preface, this classification is arbitrary.

The origins of the problems are very variable; there are some classical examples (the game of Nim, the snake, a game by Paul Erdős, the Kaprekar algorithm, the Egyptian fractions, the age of the 3 daughters); others can be found in popular books (Martin Gardner, Le petit Archimède) and some are originals from the author. As he says, "los acertijos matemáticos, como los chistes callejeros, no tienen dueño" (*mathematical riddles, like street jokes, are not copyrightable*).

The little book can be used with profit in mathematical clubs, extracurricular activities and also in the classroom. I have used it myself in my classes of "Taller de Matemáticas", a class of 2 hours each week for students of 12 years old in the new educational system of my country (Compulsory Secondary Education).

By the way, the book reveals the identity of one collaborator of ***CRUX***: Ignotus.

THE SKOLIAD CORNER

No. 32

R.E. Woodrow

As a contest this issue we give the Junior High School Mathematics Contest, Preliminary Round 1998 of the British Columbia Colleges which was written March 11, 1998. My thanks go to the contest organizer, Jim Totten, the University College of the Cariboo, for forwarding the 1998 contest materials to me. Students are given 45 minutes to respond to the 15 questions.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Preliminary Round 1998

Time: 45 minutes

1. A number is prime if it is greater than one and divisible only by one and itself. The sum of the prime divisors of 1998 is:

(a) 5 (b) 14 (c) 42 (d) 66 (e) 122

2. Successive discounts of 10% and 20% are equivalent to a single discount of:

(a) 15% (b) 25% (c) 28% (d) 30% (e) 32%

3. Suppose that Ⓐ $= A^2$ and $A \square B = A - 2B$. Then the value of ⑦ $\square$ ③ is:

(a) 1 (b) 16 (c) 961 (d) 43 (e) 31

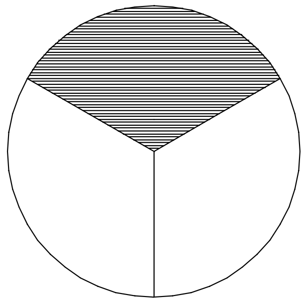
4. The expression that is not equal to the value of the four other expressions listed is:

(a) $1^{\sqrt{9}} + 9 - 8$ (b) $(1 + 9) \div (-\sqrt{9} + 8)$ (c) $-1 \times 9 + \sqrt{9} + 8$ (d) $(1 - \sqrt{9}) \times (9 - 8)$ (e) $19 - 9 - 8$

5. The sum of all of the digits of the number $10^{75} - 75$ is:

(a) 8 (b) 655 (c) 664 (d) 673 (e) 675

6. A circle is divided into three equal parts and one part is shaded as in the accompanying diagram. The ratio of the perimeter of the shaded region, including the two radii, to the circumference of the circle is:



(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{3+\pi}{3\pi}$ (e) $\frac{\pi}{3}$

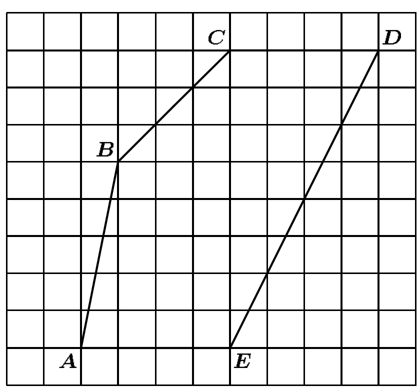
7. The value of

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}$$

is:

(a) $\frac{3}{4}$ (b) $\frac{4}{5}$ (c) $\frac{5}{6}$ (d) $\frac{6}{7}$ (e) $\frac{6}{5}$

8. If each small square in the accompanying grid is one square centimetre, then the area in square centimetres of the polygon $ABCDE$ is:



(a) 38 (b) 39 (c) 42 (d) 44 (e) 46

9. A point P is inside a square $ABCD$ whose side length is 16. P is equidistant from two adjacent vertices, A and B, and the side CD opposite these vertices. The distance PA equals:

(a) 8.5 (b) $6\sqrt{3}$ (c) 12 (d) 8 (e) 10

10. A group of 20 students has an average mass of 86 kg per person. It is known that 9 people from this group have an average mass of 75 kg per person. The average mass in kilograms per person of the remaining 11 people is:

(a) 94 (b) 95 (c) 96 (d) 97 (e) none of these

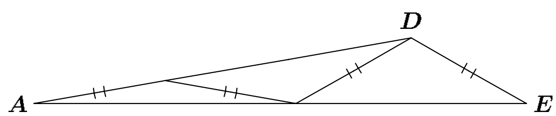
11. In the following display each letter represents a digit:

3	B	C	D	E	8	G	H	I

The sum of any three successive digits is 18. The value of H is:

(a) 3 (b) 4 (c) 5 (d) 7 (e) 8

12. In the accompanying diagram $\angle ADE = 140°$. The sides are congruent as indicated. The measure of $\angle EAD$ is:

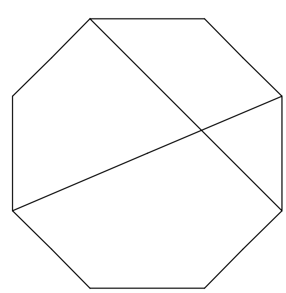


(a) 30° (b) 25° (c) 20° (d) 15° (e) 10°

13. The area (in square units) of the triangle bounded by the x-axis and the lines with equations $y = 2x + 4$ and $y = -\frac{2}{3}x + 4$ is:

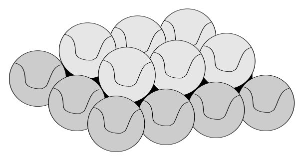
(a) 8 (b) 12 (c) 15 (d) 16 (e) 32

14. Two diagonals of a regular octagon are shown in the accompanying diagram. The total number of diagonals possible in a regular octagon is:



(a) 8 (b) 12 (c) 16 (d) 20 (e) 28

15. A local baseball league is running a contest to raise money to send a team to the provincial championship. To win the contest it is necessary to determine the number of baseballs stacked in the form of a rectangular pyramid. The fifth and sixth levels from the base of the stack of baseballs are shown. If the stack contains a total of seven levels, the number of baseballs in the stack is:



(a) 100 (b) 112 (c) 166 (d) 168 (e) 240

In the May number of the Corner we gave the problems of the 15^{th} W.J. Blundon Contest written by students in Newfoundland and Labrador. Next we give the "official" solutions. My thanks go to Bruce Shawyer for forwarding the contest and solutions to me.

15^{th} W.J. BLUNDON CONTEST
February 18, 1998

1. (a) Find the exact value of

$$\frac{1}{\log_2 36}+\frac{1}{\log_3 36}\,.$$

Solution. $\frac{1}{\log_2 36}+\frac{1}{\log_3 36}=\log_{36}2+\log_{36}3=\log_{36}6=\frac{1}{2}$.

(b) If $\log_{15}5=a$, find $\log_{15}9$ in terms of a.

Solution. $1=\log_{15}15=\log_{15}(5\cdot 3)=\log_{15}5+\log_{15}3=a+\log_{15}3$. $\log_{15}3=1-a\Longrightarrow\log_{15}9=\log_{15}3^2=2\log_{15}3=2(1-a)$.

2. (a) If the radius of a right circular cylinder is increased by 50% and the height is decreased by 20%, what is the change in the volume?

Solution. $V_2=\pi(1.5r)^2(.8h)=1.8(\pi r^2h)=1.8V_1$. So the volume is increased by 80%.

(b) How many digits are there in the number $2^{1998}\cdot 5^{1988}$?

Solution. $2^{1998}\cdot 5^{1988}=2^{10}\cdot 2^{1988}\cdot 5^{1988}=1024\cdot 10^{1988}$, which has $1988+4=1992$ digits.

3. Solve: $3^{2+x}+3^{2-x}=82$.

Solution.

$$\begin{aligned} 3^{2+x} + 3^{2-x} &= 82 \\ 9 \cdot 3^x + \frac{9}{3^x} &= 82 \\ 9(3^x)^2 - 82(3^x) + 9 &= 0 \\ (9 \cdot 3^x - 1)(3^x - 9) &= 0 \\ 3^x = \frac{1}{9}, &\qquad 3^x = 9 \\ x = -2 &\qquad x = 2 \end{aligned}$$

4. Find all ordered pairs of integers such that $x^6 = y^2 + 53$.

Solution.

$$\begin{aligned} x^6 &= y^2 + 53 \\ x^6 - y^2 &= 53 \\ (x^3 - y)(x^3 + y) &= 53 \end{aligned}$$

$$\begin{array}{cccc} x^3 - y = 53 & x^3 - y = 1 & x^3 - y = -53 & x^3 - y = -1 \\ x^3 + y = 1 & x^3 + y = 53 & x^3 + y = -1 & x^3 + y = -53 \\ \hline x = 3 & x = 3 & x = -3 & x = -3 \\ y = -26 & y = 26 & y = 26 & y = -26 \end{array}$$

The pairs are $(3, -26), (3, 26), (-3, 26), (-3, -26)$.

5. When one-fifth of the adults left a neighbourhood picnic, the ratio of adults to children was $2 : 3$. Later, when 44 children left, the ratio of children to adults was $2 : 5$. How many people remained at the picnic?

Solution. Let A be the number of adults and C be the number of children initially at the picnic. After one-fifth of the adults left, four-fifths remain. So

$$\frac{\frac{4}{5}A}{C} = \frac{2}{3} \Longrightarrow 6A = 5C.$$

After 44 children left

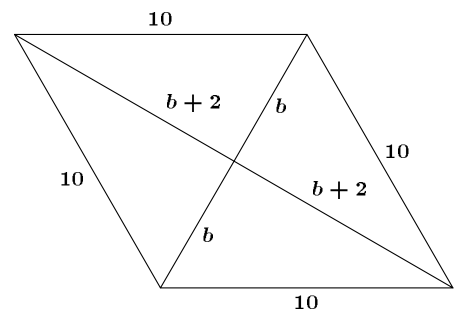
$$\frac{C - 44}{\frac{4}{5}A} = \frac{2}{5} \Longrightarrow 8A = 25C - 1100.$$

Solving the two equations gives $A = 50$, $C = 60$. The number remaining is then

$$\frac{4}{5}(50) + (60 - 44) = 40 + 16 = 56.$$

6. Find the area of a rhombus for which one side has length 10 and the diagonals differ by 4.

Solution.



$$\begin{aligned}(b+2)^2+b^2 &= 100\\ 2b^2+4b-96 &= 0\\ b^2+2b-48 &= 0\\ (b-6)(b+8) &= 0\\ b=6, &\quad b\neq -8\end{aligned}$$

Since the area of a rhombus is one half the product of the diagonals we get

$$A=\frac{1}{2}(2b)(2b+4)=\frac{1}{2}(12)(16)=96.$$

7. In how many ways can 10 dollars be changed into dimes and quarters, with at least one of each coin being used?

Solution. Let q be the number of quarters and d be the number of dimes. Then

$$25q+10d=1000$$

$$d=100-\frac{5}{2}q.$$

Since d must be an integer, q must be even. Also d must be positive. So

$$100-\frac{2}{5}q>0$$

$$q<40.$$

So q must be an even positive integer less than 40, of which there are 19.

8. Solve: $\sqrt{x+10}+\sqrt[4]{x+10}=12$.

Solution. Let $y=\sqrt[4]{x+10}$. Then $y^2=\sqrt{x+10}$, and the equation becomes

$$\begin{aligned} y^2+y &= 12 \\ y^2+y-12 &= 0 \\ (y+4)(y-3) &= 0 \\ y\neq -4,\ y &= 3 \end{aligned} \qquad \text{Then:} \quad \begin{aligned} \sqrt[4]{x+10} &= 3 \\ x+10 &= 81 \\ x &= 71 \end{aligned}$$

9. Find the remainder when the polynomial $x^{135}+x^{125}-x^{115}+x^5+1$ is divided by the polynomial x^3-x.

Solution.

$$\begin{aligned} x^{135}+x^{125}-x^{115}+x^5+1 &= (x^3-x)Q(x)+ax^2+bx+c \\ &= x(x-1)(x+1)Q(x)+ax^2+bx+c \end{aligned}$$

This must be valid for all values of x. Substituting in $x=0$, $x=1$, and $x=-1$ gives:

$$\begin{aligned} x=0: &\quad 1=0+c &&\Longrightarrow\quad c=1 \\ x=1: &\quad 3=0+a+b+c &&\Longrightarrow\quad a+b=2 \\ x=-1: &\quad -1=0+a-b+c &&\Longrightarrow\quad a-b=-2 \end{aligned}$$

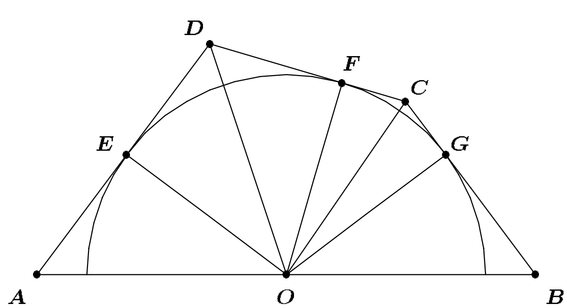
Solving the system

$$\begin{aligned} a+b &= 2 \\ a-b &= -2 \end{aligned}$$

gives $a=0$, $b=2$. So the remainder is $2x+1$.

10. Quadrilateral $ABCD$ below has the following properties: (1) The midpoint O of side AB is the centre of a semicircle; (2) sides AD, DC and CB are tangent to this semicircle. Prove that $AB^2=4AD\times BC$.

Solution. First join the obvious lines in the given figure:



By the properties of tangents, $DE=DF$ and $CF=CG$. Therefore $\angle EDO=\angle FDO=\phi$ and $\angle FCO=\angle GCO=\psi$. Since $OA=OB$, we have $\angle EAO=\angle GBO=\theta$.

Summing the angles of quadrilateral $ABCD$, we get $\theta+2\phi+2\psi+\theta = 360^\circ$. Hence $\theta + \phi + \psi = 180^\circ$; that is, they are the angles of a triangle.

Considering triangles AOD, DOC and COB, we get $\angle AOD = \psi$, $\angle DOC = \theta$ and $\angle COB = \phi$. Thus the three triangles are similar.

Considering the triangles ADO and BOC, we have $\frac{AD}{AO} = \frac{OB}{BC}$, or $AD \times BC = AO \times OB$.

Since $AO = OB = \frac{1}{2}AB$, we get the result.

Last issue we gave the problems of the U.K. Intermediate Mathematical Challenge. Here are the solutions.

1.	C	2.	B	3.	A	4.	A	5.	E
6.	C	7.	C	8.	A	9.	C	10.	B
11.	C	12.	E	13.	D	14.	D	15.	D
16.	D	17.	D	18.	C	19.	A	20.	E
21.	E	22.	D	23.	A	24.	A	25.	B

That completes the *Skoliad Corner* for this number. Send me your comments, suggestions, and most importantly, suitable contest materials for use in future issues.

Advance Announcement

The 1999 Summer Meeting of the Canadian Mathematical Society will take place at Memorial University in St. John's, Newfoundland, from Saturday, 29 May 1999 to Tuesday, 1 June 1999.

The Special Session on Mathematics Education will feature the topic
What Mathematics Competitions do for Mathematics.
The invited speakers are

Ed Barbeau (University of Toronto),
Ron Dunkley (University of Waterloo),
Tony Gardiner (University of Birmingham, UK), and
Rita Janes (Newfoundland and Labrador Senior Mathematics League).

Requests for further information, or to speak in this session, as well as suggestions for further speakers, should be sent to the session organizers:

Bruce Shawyer and Ed Williams
CMS Summer 1999 Meeting, Education Session
Department of Mathematics and Statistics, Memorial University
St. John's, Newfoundland, Canada A1C 5S7

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), Richard Hoshino (University of Waterloo), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions.

High School Solutions

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

First, we present a very elegant solution to a past problem:

H224. Consider square $ABCD$ with side length 1. Select a point M exterior to the square so that $\angle AMB$ is 90°. Let $a = AM$ and $b = BM$. Now, determine the point N exterior to the square so that $CN = a$ and $DN = b$. Find, as a function of a and b, the length of the line segment MN.

Solution by Hoe Teck Wee, Singapore.

Locate point P exterior to the square so that $BP = a$ and $CP = b$. Locate point Q exterior to the square so that $DQ = a$ and $AQ = b$. Then $MPNQ$ is a square of side length $a+b$, and it follows that MN is a diagonal of this square. Thus, the desired length of MN is $\sqrt{2}(a+b)$.

H225. In Cruxmayhemland, stamps can be bought only in two denominations, p and q cents, both of which are at least 31 cents. It is known that if p and q are relatively prime, the largest value that cannot be created by these two stamps is $pq - p - q$. For example, when $p = 5$ and $q = 3$, one can affix any postage that is higher than $15 - 5 - 3$, or 7 cents, but not 7 cents itself. The governor of Cruxmayhemland tells you that 1997 is the largest value that cannot be created by these stamps. Find all possible pairs of positive integers (p, q) with $p > q$.

Solution by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain.

If p and q are not relatively prime, then they have a common divisor d. Then any value that does not divide d cannot be created by these two stamps. Hence there are infinitely many values that cannot be created. Hence, we must have p and q being relatively prime. We are seeking all possible pairs (p, q) with $p > q$ and p and q relatively prime such that $1997 = pq - p - q$. Adding 1 to both sides of this equation, we have $1998 = pq - p - q + 1 = (p-1)(q-1)$. Since $1998 = 2^1 \cdot 3^3 \cdot 37^1$ has $2 \cdot 4 \cdot 2 = 16$ positive divisors, there are exactly eight ways to write down 1998 as the product of two positive integers, namely $1998 = 1 \cdot 1998 = 2 \cdot 999 = 3 \cdot 666 = 6 \cdot 333 = 9 \cdot 222 = 18 \cdot 111 = 27 \cdot 74 = 37 \cdot 54$.

Now p and q are at least 31, so we must have $p - 1$ and $q - 1$ both being at least 30. The only pair for which this is true is $p - 1 = 54$ and $q - 1 = 37$. And we see that 55 and 38 are relatively prime integers. Hence, the only solution is $(p, q) = (55, 38)$.

H226. In right-angled triangle ABC, with BC as hypotenuse, $AB = x$ and $AC = y$, where x and y are positive integers. Squares $APQB$, $BRSC$, and $CTUA$ are drawn externally on sides AB, BC, and CA, respectively. When QR, ST, and UP are joined, a hexagon is formed. Let K be the area of the hexagon $PQRSTU$.

(a) Prove that K cannot equal 1997. (Hint: Try to find a general formula for K.)

(b) Prove that there is only one solution (x, y) with $x > y$ so that $K = 1998$.

Solution by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain.

(a) The hexagon $PQRSTU$ is the union of the right-angled triangle ABC, the three squares, and triangles QRB, STC, and UPA. We will now show that all four triangles have equal area. Consider triangle QRB. We have

$$\begin{aligned} & \angle ABC + \angle QBA + \angle RBQ + \angle CBR = 360^\circ \\ \Longrightarrow \quad & \angle RBQ = 180^\circ - \angle ABC. \end{aligned}$$

Let $[P]$ denote the area of triangle P.

Now, $[QRB] = \frac{1}{2} \cdot QB \cdot RB \cdot \sin \angle RBQ = \frac{1}{2} \cdot AB \cdot CB \cdot \sin \angle ABC = [ABC]$. The argument does not rely on ABC being right-angled, so similarly, we have $[STC] = [UPA] = [ABC] = xy/2$. It follows that $K = 4 \cdot \frac{xy}{2} + x^2 + y^2 + BC^2$, and $BC^2 = x^2 + y^2$ implies $K = 2(x^2 + xy + y^2)$.

Therefore, K is necessarily an even integer, so it cannot be 1997.

(b) Using our formula from the previous part, we seek an integer solution to $999 = x^2 + xy + y^2$. Taking congruences modulo 3, we have $x^2 + xy + y^2 \equiv 0 \pmod 3$.

Suppose $x \equiv 0 \pmod 3$. Then $y^2 \equiv 0 \pmod 3$, which implies that $y \equiv 0 \pmod 3$. Suppose $x \equiv 1 \pmod 3$. Then $y^2 + y + 1 \equiv 0 \pmod 3$, which implies that $y \equiv 1 \pmod 3$. Suppose $x \equiv 2 \pmod 3$. Then $y^2 + 2y + 4 \equiv y^2 + 2y + 1 \equiv (y+1)^2 \equiv 0 \pmod 3$, which implies that $y \equiv 2 \pmod 3$.

Hence, it follows that we must have $x \equiv y \pmod 3$.

Then we make the substitution $x = 3u + a$, $y = 3v + a$ where a is 0, 1 or -1, and this yields $999 = 9u^2 + 9au + 3a^2 + 9v^2 + 9av + 9uv$. All terms are divisible by 9 except possibly $3a^2$. If $3a^2$ does not divide 9, then the left side, 999, is divisible by 9, but the right side is not. Hence, we must have 3 dividing a^2, which implies that $a = 0$, since neither 1^2 nor $(-1)^2$ is divisible by 3. Hence, $a = 0$, and so $x = 3u$ and $y = 3v$, which implies that $u^2 + uv + v^2 = 111$. Multiplying by 4, we get $444 = 4u^2 + 4uv + 4v^2 = (3u^2 + 6uv + 3v^2) + (u^2 - 2uv + v^2) = 3(u+v)^2 + (u-v)^2$. Since $(u-v)^2 < (u+v)^2$, it follows that

$$\begin{aligned} & 3(u+v)^2 < 444 < 4(u+v)^2 \\ \Longrightarrow \quad & 111 < (u+v)^2 < 148 \\ \Longrightarrow \quad & 11 \le u + v \le 12. \end{aligned}$$

Now if $u + v = 12$, then $(u-v)^2 = 444 - 3(u+v)^2 = 12$, and because u and v are integers, this has no solution. However, if $u + v = 11$, then $(u-v)^2 = 81$, and so $u - v = 9$, because $u > v$ (which follows directly from $x > y$). Solving for u and v, we get $u = 10$ and $v = 1$, which implies that $x = 30$ and $y = 3$. Checking, $(x, y) = (30, 3)$ is a solution to $2x^2 + 2xy + 2y^2 = 1998$.

Therefore, there is a unique solution to $K = 1998$, and it is $(x, y) = (30, 3)$.

H227. The numbers $2, 4, 8, 16, \ldots, 2^n$ are written on a chalkboard. A student selects any two numbers a and b, erases them, and replaces them by their average, namely $(a+b)/2$. She performs this operation $n-1$ times until only one number is left. Let S_n and T_n denote the maximum and minimum possible value of this final number, respectively. Determine a formula for S_n and T_n in terms of n.

Solution.

We shall prove by induction that if we perform the operation on the numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, where $a_1 < a_2 < \cdots < a_n$, then

$$T_n = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \cdots + \frac{a_{n-2}}{2^{n-2}} + \frac{a_{n-1}}{2^{n-1}} + \frac{a_n}{2^{n-1}}.$$

For $n = 2$, we have two numbers, a_1 and a_2, and thus T_2 must be $(a_1 + a_2)/2$. Thus the claim holds for $n = 2$.

Assume that for $n = 2, 3, \ldots, k-2, k-1$, the identity holds. Then for the case $n = k$, that is, when we start off with the numbers $a_1, a_2, a_3, \ldots, a_k$, we will perform the operation $k-2$ times and be left with two numbers, x and y. Say x comes from performing the operation on a set of p numbers from the set. Then y comes from performing the operation on the remaining set of $k-p$ numbers.

To illustrate this, say we start off with 2, 4, 8, 16, and 32. Then we can replace 2 and 8 by 5, 16 and 32 by 24, then 4 and 24 by 14. Then we are left with two numbers, 5 and 14. We got 5 from performing the operation on the numbers 2 and 8, and we got 14 from performing the operation on the other three numbers, namely 4, 16, and 32.

Let $b_1, b_2, b_3, \ldots, b_p$ be the set of numbers that we used to get the number x, where the b_i's are in increasing order, and let $c_1, c_2, c_3, \ldots, c_{k-p}$ be the other numbers from the set that we used to get y, where the c_i's are in increasing order. Then, by our induction hypothesis, we have

$$x \geq \frac{b_1}{2} + \frac{b_2}{4} + \cdots + \frac{b_{p-2}}{2^{p-2}} + \frac{b_{p-1}}{2^{p-1}} + \frac{b_p}{2^{p-1}},$$

and similarly,

$$y \geq \frac{c_1}{2} + \frac{c_2}{4} + \cdots + \frac{c_{k-p-2}}{2^{k-p-2}} + \frac{c_{k-p-1}}{2^{k-p-1}} + \frac{c_{k-p}}{2^{k-p-1}}.$$

Note that the b_i's and the c_i's are just some permutation of the set $a_1, a_2, a_3, \ldots, a_k$. Hence, to minimize the value of $(x + y)/2$, the average of the two numbers will be minimized when the denominators are as large as possible. Hence, we want either p or $k - p$ to be 1, since we will then have a denominator of 2^{k-1} in one of the terms.

Without loss of generality, assume that $k - p = 1$. Thus, we want to now show that

$$\begin{aligned}\frac{x+y}{2} &\geq \frac{c_1}{2} + \frac{b_1}{4} + \frac{b_2}{8} + \cdots + \frac{b_{k-3}}{2^{k-2}} + \frac{b_{k-2}}{2^{k-1}} + \frac{b_{k-1}}{2^{k-1}} \\ &\geq \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \cdots + \frac{a_{k-2}}{2^{k-2}} + \frac{a_{k-1}}{2^{k-1}} + \frac{a_k}{2^{k-1}},\end{aligned}$$

because this will prove the case for $n = k$.

Note that the b_i's are in increasing order, so if $c_1 = a_r$ for some r, then $b_1 = a_1$, $b_2 = a_2$, ... , $b_{r-1} = a_{r-1}$, $b_r = a_{r+1}$, $b_{r+1} = a_{r+2}$, ... , $b_{k-1} = a_k$.

Noting that the a_i's are in increasing order, we have

$$\begin{aligned}\frac{a_r}{2} &= \frac{a_r}{4} + \frac{a_r}{8} + \frac{a_r}{16} + \cdots \\ &\geq \frac{a_r}{4} + \frac{a_r}{8} + \frac{a_r}{16} + \cdots + \frac{a_r}{2^r} \\ &\geq \frac{a_1}{4} + \frac{a_2}{8} + \frac{a_3}{16} + \cdots + \frac{a_{r-1}}{2^r}.\end{aligned}$$

Adding

$$\frac{a_1}{4} + \frac{a_2}{8} + \frac{a_3}{16} + \cdots + \frac{a_{r-1}}{2^r} + \frac{a_{r+1}}{2^{r+1}} + \frac{a_{r+2}}{2^{r+2}} + \cdots + \frac{a_{k-1}}{2^{k-1}} + \frac{a_k}{2^{k-1}}$$

to both sides, we get the desired inequality.

Thus, the induction is proved and hence, for our question, we take $a_i = 2^i$ for $i = 1, 2, \ldots, k$, and we have

$$\begin{aligned}T_n &= \frac{2}{2} + \frac{4}{4} + \frac{8}{8} + \cdots + \frac{2^{n-2}}{2^{n-2}} + \frac{2^{n-1}}{2^{n-1}} + \frac{2^n}{2^{n-1}} \\ &= (n-1) \cdot 1 + 2 = n + 1.\end{aligned}$$

Thus, our formula for T_n is obtained by taking the a_i's in increasing order. In contrast, to get S_n, we want to take the a_i's in decreasing order, namely $a_1 = 2^n$, $a_2 = 2^{n-1}$, ... , $a_{n-1} = 4$, $a_n = 2$. This is so that we can get the maximum possible value at the end. Thus, we have

$$\begin{aligned}S_n &= \frac{2^n}{2} + \frac{2^{n-1}}{4} + \frac{2^{n-2}}{8} + \cdots + \frac{2^3}{2^{n-2}} + \frac{2^2}{2^{n-1}} + \frac{2^1}{2^{n-1}} \\ &= 2^{n-1} + 2^{n-3} + 2^{n-5} + \cdots + 2^{5-n} + 2^{3-n} + 2^{2-n} \\ &= 2^{3-n} \cdot (1 + 2^2 + 2^4 + \cdots + 2^{2n-4}) + 2^{2-n} \\ &= 2^{3-n} \frac{4^{n-1} - 1}{3} + 2^{2-n} \\ &= \frac{8(4^{n-1} - 1)}{3 \cdot 2^n} + \frac{4}{2^n} \\ &= \frac{2^{2n+1} + 4}{3 \cdot 2^n}.\end{aligned}$$

H228. Verify that the following three inequalities hold for positive reals x, y and z:

(i) $x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y) \geq 0$ (this is known as Schur's Inequality),

(ii) $x^4+y^4+z^4+xyz(x+y+z) \geq 2(x^2y^2+y^2z^2+z^2x^2)$,

(iii) $9xyz+1 \geq 4(xy+yz+xz)$, where $x+y+z=1$.

Solution by Vedula N. Murty, Visakhapatnam, India.

We shall show that all three inequalities are equivalent to showing that

$$(x+y-z)(x-y)^2+(y+z-x)(y-z)^2+(z+x-y)(z-x)^2 \geq 0.$$

And we shall then show that this inequality holds for all positive reals x, y, and z. Let

$$p=(x+y-z)(x-y)^2+(y+z-x)(y-z)^2+(z+x-y)(z-x)^2.$$

Expanding and rearranging terms, we find that

$$\begin{aligned}
& 2x(x-y)(x-z)+2y(y-z)(y-x)+2z(z-x)(z-y) \\
= \; & 2x^3+2y^3+2z^3+6xyz-2(x^2y+xy^2+x^2z+xz^2+y^2z+yz^2) \\
= \; & (x^3+y^3+2xyz-x^2y-x^2z-y^2x-y^2z) \\
& \quad +(y^3+z^3+2xyz-y^2x-y^2z-z^2x-z^2y) \\
& \quad +(z^3+x^3+2xyz-z^2x-z^2y-x^2y-x^2z) \\
= \; & (x+y-z)(x-y)^2+(y+z-x)(y-z)^2+(z+x-y)(z-x)^2 \\
= \; & p\,.
\end{aligned}$$

Therefore, we have shown that $x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y)=p/2$. Thus (i) holds if and only if $p \geq 0$.

Note that $x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2=\frac{1}{2}[(x^2-y^2)^2+(y^2-z^2)^2+(z^2-x^2)^2]$. Furthermore, $x^2y^2+y^2z^2+z^2x^2-xyz(x+y+z)=\frac{1}{2}[x^2(y-z)^2+y^2(x-z)^2+z^2(x-y)^2]$. Subtracting the second equation from the first, we have

$$\begin{aligned}
& x^4+y^4+z^4+xyz(x+y+z)-2(x^2y^2+y^2z^2+z^2x^2) \\
= \; & \frac{1}{2}[(x^2-y^2)^2+(y^2-z^2)^2+(z^2-x^2)^2-x^2(y-z)^2-y^2(x-z)^2 \\
& \quad -z^2(x-y)^2] \\
= \; & \frac{1}{2}\{(x-y)^2[(x+y)^2-z^2]+(y-z)^2[(y+z)^2-x^2] \\
& \quad +(z-x)^2[(z+x)^2-y^2]\} \\
= \; & \frac{x+y+z}{2}\cdot[(x-y)^2(x+y-z)+(y-z)^2(y+z-x) \\
& \quad +(z-x)^2(z+x-y)] \\
= \; & \frac{p(x+y+z)}{2}\,.
\end{aligned}$$

Since x, y, and z are positive, $x+y+z$ is also positive, and it follows that (ii) holds if and only if $p \geq 0$.

Suppose that $x+y+z=a$. Then if we replace x by x/a, y by y/a and z by z/a, then the inequality will remain the same. Thus, there is no loss in generality in assuming that $x+y+z=1$, for we can always divide each term by some constant to ensure that the sum of the terms becomes 1.

Thus, letting $x+y+z=1$, we have:

$$\begin{aligned}
\frac{p}{2} &= \frac{1}{2}[(x+y-z)(x-y)^2+(y+z-x)(y-z)^2+(z+x-y)(z-x)^2] \\
&= (x^3+y^3+z^3)+3xyz-(x^2y+x^2z+y^2x+y^2z+z^2x+z^2y) \\
&= (x^3+y^3+z^3)+3xyz-(x^2+y^2+z^2)(x+y+z)+(x^3+y^3+z^3) \\
&= 2(x^3+y^3+z^3)+3xyz-(x^2+y^2+z^2) \\
&= 2(x+y+z)(x^2+y^2+z^2-xy-yz-zx)+6xyz+3xyz \\
&\quad -(x^2+y^2+z^2) \\
&= x^2+y^2+z^2-2(xy+yz+zx)+9xyz \\
&= (x+y+z)^2-4(xy+yz+zx)+9xyz \\
&= 1-4(xy+yz+zx)+9xyz\,.
\end{aligned}$$

Therefore, (iii) holds if and only if $p \geq 0$. Hence, we have shown that all three inequalities are equivalent to $p \geq 0$. Thus, if we can show that this inequality holds, then (i), (ii), and (iii) all hold.

To show that $p \geq 0$, we separate the problem into two cases. If x, y, and z are the sides of a triangle, then $x+y-z$, $x+z-y$, and $y+z-x$ are all positive, and thus it immediately follows that $p \geq 0$. Otherwise, the three sides do not form the sides of a triangle, and so if we assume without loss of generality that $x \leq y \leq z$, then $x+y-z \leq 0$.

And we have

$$\begin{aligned}
p &= (x+y-z)(x-y)^2+(y+z-x)(y-z)^2+(z+x-y)(z-x)^2 \\
&= (x+y-z)[(x-z)+(z-y)]^2+(y+z-x)(y-z)^2 \\
&\quad +(z+x-y)(z-x)^2 \\
&= (x-z)^2(x+y-z)+(z-y)^2(x+y-z) \\
&\quad +2(x-z)(z-y)(x+y-z)+(y+z-x)(y-z)^2 \\
&\quad +(z+x-y)(z-x)^2 \\
&= (x-z)^2(2x)+(y-z)^2(2y)+2(z-x)(z-y)(z-x-y)\,.
\end{aligned}$$

And this last expression is non-negative, since $x \geq 0$, $y \geq 0$, $z-x \geq 0$, $z-y \geq 0$ and $z-x-y \geq 0$. Therefore, $p \geq 0$, and we have proven that all three inequalities hold.

Advanced Solutions

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The following problems first appeared in Volume 23, Issue 5.

A201. Consider an infinite sequence of integers $a_1, a_2, \ldots, a_k, \ldots$ with the property that every m consecutive numbers sum to x and every n consecutive numbers sum to y. If x and y are relatively prime, then show that all numbers are equal.

Erratum. As noted by some of our solvers, the problem as stated is incorrect . The correct problem should read as follows:

Consider an infinite sequence of integers $a_1, a_2, \ldots, a_k, \ldots$ with the property that every m consecutive numbers sum to x and every n consecutive numbers sum to y. If m and n are relatively prime, then show that all numbers are equal.

Solution.

Since every m consecutive numbers sum to the same value, we must have a periodic sequence with $a_k = a_{m+k}$, for all natural numbers k. Likewise, $a_k = a_{n+k}$, for all natural numbers k. What this means is that the sequence is periodic with periods m and n. It may be intuitively clear now why the sequence must be a constant sequence. Here is a rigorous proof. It is enough to show that the n numbers $a_0, a_1, \ldots, a_{n-1}$ are equal since every other number is equal to one of these by periodicity. Now a_0, a_m, $a_{2m}, \ldots$ are equal, so it is enough to show that a_i, for $i = 0, 1, \ldots, n-1$, is equal to a_{km} for some integer k.

We show that each of the n numbers $a_0, a_m, \ldots, a_{(n-1)m}$, leaves a different remainder upon division by n. In other words, the set of numbers $\{0, a_m, \ldots, a_{(n-1)m}\}$ is a complete residue system modulo n. To see this, $im \equiv jm \pmod{n}$ holds if and only if $i \equiv j \pmod{n}$ since m and n are relatively prime. Further, for $0 \leq i, j < n$, we must then have $i = j$. In other words, the n values $a_0, a_m, \ldots, a_{(n-1)m}$ must leave a different remainder upon division by n.

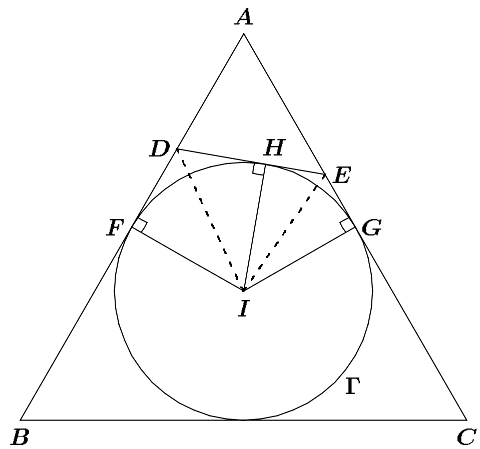
Now, putting it all together, suppose $lm = nq + r$, where $0 \leq r < n$. Then $a_{lm} = a_{nq+r} = a_r$. Now we know that all possible remainders r, from 0 to $n-1$, are achieved for some integer l, so we can conclude that all a_r are equal.

A202. Let ABC be an equilateral triangle and Γ its incircle. If D and E are points on AB and AC, respectively, such that DE is tangent to Γ, show that

$$\frac{AD}{DB} + \frac{AE}{EC} = 1.$$

(8^{th} *Iberoamerican Mathematical Olympiad, Mexico '93*)

Solution I by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain.



Let I be the incentre and r the inradius. Let F, G, and H be the points where Γ is tangent to AB, AC, and DE. Then, $DF = DH$ and $EG = EH$ (this is a well-known property of the two tangents from a point to a circle). This implies that $\angle FID = \angle DIH$ and $\angle GIE = \angle EIH$. Since $\angle FIG = 120°$, $\angle FID + \angle GIE = 60°$. We can write $\angle FID = 30° + \phi$, and $\angle GIE = 30° - \phi$. Now, $FD = r\tan\angle FID$ and $AF = r\tan 60°$, so that

$$\begin{aligned}\frac{AD}{DB} &= \frac{AF - FD}{AF + FD} = \frac{\tan 60° - \tan\angle FID}{\tan 60° + \tan\angle FID} \\ &= \frac{\sqrt{3} - \dfrac{\tan 30° + \tan\phi}{1 - \tan 30° \tan\phi}}{\sqrt{3} + \dfrac{\tan 30° + \tan\phi}{1 - \tan 30° \tan\phi}} \\ &= \frac{\sqrt{3} - \dfrac{1 + \sqrt{3}\tan\phi}{\sqrt{3} - \tan\phi}}{\sqrt{3} + \dfrac{1 + \sqrt{3}\tan\phi}{\sqrt{3} - \tan\phi}} = \frac{1 - \sqrt{3}\tan\phi}{2},\end{aligned}$$

and similarly

$$\frac{AE}{EC} = \frac{1 + \sqrt{3}\tan\phi}{2}.$$

We then have

$$\frac{AD}{DB} + \frac{AE}{EC} = \frac{1 - \sqrt{3}\tan\phi}{2} + \frac{1 + \sqrt{3}\tan\phi}{2} = 1$$

as required.

Solution II. Using the same diagram as above, assume, without loss of generality, that the sides of the equilateral triangle ABC have length 1. Let $x = AD$ and $y = AE$, where $0 \le x, y \le 1/2$. Now using the Cosine Law in triangle ADE, we have

$$(1 - x - y)^2 = x^2 + y^2 - 2xy\cos 60^\circ .$$

This is equivalent to each of the following:

$$\begin{aligned} 1 + x^2 + y^2 - 2x - 2y + 2xy &= x^2 + y^2 - xy\,, \\ 2x + 2y - 1 &= 3xy\,, \\ x - xy + y - xy &= 1 - x - y + xy\,, \end{aligned}$$

and finally,

$$\frac{x}{1-x} + \frac{y}{1-y} = 1\,.$$

The last equation is valid since x and y cannot be equal to 1. This is our desired result.

Also solved by Alexandre Trichtchenko, student, Brookfield High School, Ottawa, Ontario.

A203. Let $S_n = 1 + a + a^a + \cdots + a^{a^{\cdots a}}$, where the last term is a tower of $(n-1)$ a's. Find all positive integers a such that $S_n = na^{S_{n-1}/n}$.

Solution.

Claim: $a = 1$ is the only possibility.

Since a is positive, we have by the AM-GM inequality

$$\frac{1 + a + a^a + \cdots + a^{a^{\cdots a}}}{n} \ge \left(a^{0+1+a+\cdots+a^{a^{\cdots a}}}\right)^{\frac{1}{n}},$$

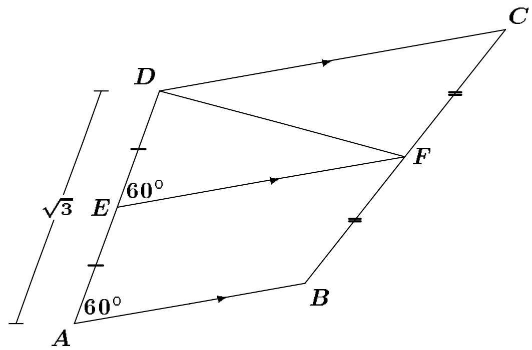
with equality if and only if $1 = a = a^a = \cdots = a^{a^{\cdots a}}$. Using the definition of S_n, we have

$$\frac{S_n}{n} \ge \left(a^{S_{n-1}}\right)^{1/n}.$$

Since equality occurs, we must have $1 = a = a^a = \cdots = a^{a^{\cdots a}}$, $S_n = n$, for all natural numbers n and it follows that $S_n = n(1^{S_{n-1}/n})$.

A204. Given a quadrilateral $ABCD$ as shown, with $AD = \sqrt{3}$, $AB + CD = 2AD$, $\angle A = 60^\circ$ and $\angle D = 120^\circ$, find the length of the line segment from D to the mid-point of BC.

Solution I by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain and Alexandre Trichtchenko, student, Brookfield High School, Ottawa, Ontario.

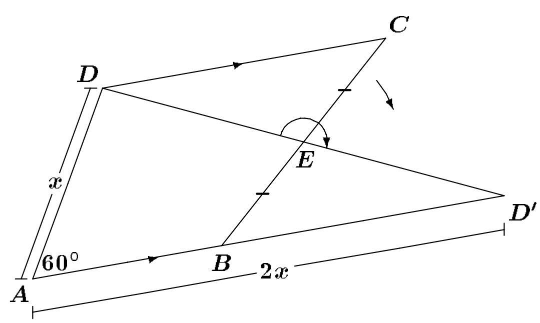


Observe that AB and CD are parallel. Then by Thales' Theorem, a line through the mid-points E and F of AD and BC respectively will also be parallel to AB, and the length of EF will be $(AB+CD)/2 = AD$. Now, applying the Cosine Law to triangle DEF, we have

$$\begin{aligned} DF^2 &= DE^2 + EF^2 - 2 \cdot DE \cdot EF \cos \angle DEF \\ &= \left(\frac{AD}{2}\right)^2 + AD^2 - AD^2 \cos 60^\circ \\ &= \frac{3AD^2}{4}, \end{aligned}$$

so $DF = 3/2$.

Solution II.



Let the mid-point of BC be E. Note that lines AB and CD are parallel. Rotate triangle DEC about point E so that C coincides with B, and D coincides with D' as shown in the figure. This is possible since E was chosen to be the midpoint of line segment BC.

Now A, B, and D' are collinear since lines CD and AB are parallel, so that $\angle D'BE + \angle ABE = \angle DCE + \angle ABE = 180°$.

Let $x = AD$, then $AD' = AB+BD' = AB+CD = 2x$. Thus triangle DAD' is similar to the ubiquitous $1 \colon 2 \colon \sqrt{3}$ triangle. Hence, $DE = DD'/2 = \sqrt{3}/2AD = 3/2$.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C75.

(a) Let n be an integer, and suppose a_1, a_2, a_3, and a_4 are integers such that $a_1a_4 - a_2a_3 \equiv 1 \pmod{n}$. Show that there exist integers A_i, $1 \leq i \leq 4$, such that each $A_i \equiv a_i \pmod{n}$ and $A_1A_4 - A_2A_3 = 1$.

(b) Let $SL(2, \mathbb{Z})$ denote the group of 2×2 matrices with integer entries and determinant 1, and let $\Gamma(n)$ denote the subgroup of $SL(2, \mathbb{Z})$ of matrices which are congruent to the identity matrix modulo n. (Two matrices are congruent modulo n if each pair of corresponding entries is congruent modulo n.) What is the index of $\Gamma(n)$ in $SL(2, \mathbb{Z})$?

Solution.

(a) Without loss of generality, we suppose that the a_i are non-zero. (They may, of course, nevertheless be congruent to $0 \pmod{n}$.) Choose $A_1 = a_1$. Our first order of business is to find $A_2 \equiv a_2 \pmod{n}$ with A_1 and A_2 relatively prime. Let $d = \gcd(a_2, n)$. Then a_2/d and n/d are relatively prime, and so by Dirichlet's Theorem on the infinitude of primes in arithmetic progressions, we can choose a prime p distinct from the prime divisors of A_1 and satisfying

$$p \equiv \frac{a_2}{d} \left(\mathrm{mod}\ \frac{n}{d}\right).$$

Set $A_2 = dp$. It is immediate that $A_2 \equiv a_2$ (mod n). Since a_1 and a_2 must not share any common divisors with n, we see that d and A_1 are relatively prime, and therefore so are A_1 and A_2.

Select any x, y such that $A_1x - A_2y = 1$. We must find k such that $x + A_2k \equiv a_4 \pmod{n}$ and $y + A_1k \equiv a_3 \pmod{n}$. Rewriting these congruences as

$$A_2k \equiv a_4 - x \pmod{n}$$

and

$$A_1k \equiv a_3 - y \pmod{n},$$

if we multiply the first congruence by $-a_3$, the second by a_4, and we add the equations, then we find

$$A_1a_4k - A_2a_3k \equiv a_4(a_3 - y) - a_3(a_4 - x) \pmod{n},$$

and since $A_1a_4 - A_2a_3 \equiv a_1a_4 - a_2a_3 \equiv 1 \pmod{n}$, we obtain

$$k \equiv a_3x - a_4y \pmod{n}.$$

One readily checks that with such a choice of k, setting $A_4 = x + A_2 k$ and $A_3 = y + A_1 k$ gives the A_i the desired properties.

(b) Let $SL(2, \mathbb{Z}/n\mathbb{Z})$ denote the group of 2×2 matrices with entries in $\mathbb{Z}/n\mathbb{Z}$ (the ring of integers modulo n) and determinant 1. The reduction-mod-n homomorphism $\varphi_n : SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/n\mathbb{Z})$, via which the entries of $\varphi_n(A)$ are the congruence classes of the corresponding entries of A, evidently has kernel equal to $\Gamma(n)$. Additionally, part (a) of this problem proves that φ_n is surjective. So, to compute the index of $\Gamma(n)$ in $SL(2, \mathbb{Z})$, we need only compute the order of $SL(2, \mathbb{Z}/n\mathbb{Z})$. Call this order $C(n)$.

Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be the prime factorization of n. Given an element of $SL(2, \mathbb{Z}/n\mathbb{Z})$, we obtain an element in each $SL(2, \mathbb{Z}/p_i^{a_i}\mathbb{Z})$ by reducing modulo $p_i^{a_i}$. Conversely, given one element in each $SL(2, \mathbb{Z}/p_i^{a_i}\mathbb{Z})$, we can use the Chinese Remainder Theorem to obtain a unique matrix whose modulo $p_i^{a_i}$ reductions are our given matrices. It follows, therefore, that $C(n) = C(p_1^{a_1}) \cdots C(p_k^{a_k})$.

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a representative of an element of $SL(2, \mathbb{Z}/p^k\mathbb{Z})$ with p prime and $k > 0$, and suppose

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an element of $SL(2, \mathbb{Z}/p^{k+1}\mathbb{Z})$ whose reduction modulo p^k is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Choose representatives a', b', c', and d' in $\mathbb{Z}/p^{k+1}\mathbb{Z}$ of a, b, c, and d, and write

$$\begin{aligned} A &= a' + a_0 p^k, \\ B &= b' + b_0 p^k, \\ C &= c' + c_0 p^k, \\ D &= d' + d_0 p^k, \end{aligned}$$

with a_0, b_0, c_0, d_0 in $\mathbb{Z}/p\mathbb{Z}$.

Then

$$\det \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = 1 + mp^k$$

in $\mathbb{Z}/p^{k+1}\mathbb{Z}$, and the condition that

$$\det \begin{pmatrix} a' + a_0 p^k & b' + b_0 p^k \\ c' + c_0 p^k & d' + d_0 p^k \end{pmatrix} = 1$$

in $\mathbb{Z}/p^{k+1}\mathbb{Z}$ is transformed into the condition that

$$a_0 d' + a' d_0 - b_0 c' - b' c_0 \equiv -m \pmod{p}.$$

One of a', b', c', d' must be invertible modulo p, say d', without loss of generality. Then for any arbitrary choice of b_0, c_0, and d_0, there exists exactly one a_0 which satisfies this condition. Consequently, we have shown that each element of $SL(2, \mathbb{Z}/p^k\mathbb{Z})$ lifts to precisely p^3 elements in $SL(2, \mathbb{Z}/p^{k+1}\mathbb{Z})$, and so $C(p^{k+1}) = p^3 C(p^k)$. Inductively, $C(p^{k+1}) = p^{3k} C(p)$.

To calculate the number of elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathbb{Z}/p\mathbb{Z}$, note that to have determinant 1, the first row of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

must be non-zero. However, the reader may verify directly that to each of the remaining $p^2 - 1$ possible choices for the pair (a, b), there are exactly p choices for the pair (c, d) which make

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 .$$

Thus,

$$C(p) = (p^2 - 1)p = p^3 \left(1 - \frac{1}{p^2}\right),$$

and putting together all our work, we conclude that

$$C(n) = n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right),$$

where the product is taken over the primes p dividing n.

C76. Let X be any topological space. The n^{th} symmetric power of X, denoted $X^{(n)}$, is defined to be the quotient of the ordinary n-fold product X^n by the action of the symmetric group on n letters—that is, it is the space of unordered n-tuples of points of X. Show that the symmetric power $\mathbb{C}^{(n)}$ is actually homeomorphic to the ordinary product $\mathbb{C}^n$.

Solution.

Every unordered n-tuple $\langle r_1, \ldots, r_n \rangle$ of points in $\mathbb{C}$ is the solution set to a unique monic polynomial, specifically, the polynomial $f(z) = (z - r_1) \cdots (z - r_n)$ whose coefficients are given by the elementary symmetric polynomials in the r_i. Furthermore, by the Fundamental Theorem of Algebra, every polynomial of degree n over $\mathbb{C}$ has exactly n solutions in $\mathbb{C}$, counting multiplicity. Therefore, the map $\varphi : \mathbb{C}^{(n)} \to \mathbb{C}^n$ is a bijection, where φ is defined as

$$\varphi(\langle r_1, \ldots, r_n \rangle) = (s_1, \ldots, s_n)$$

with the s_i given by

$$f(z) = (z - r_1) \cdots (z - r_n) = z^n - s_1 z^{n-1} + \cdots + (-1)^n s_n.$$

It is evident that φ is continuous, and so to prove that φ is a homeomorphism, it remains to show that φ^{-1} is continuous as well. This amounts to proving that the roots of a polynomial vary continuously with the coefficients of the polynomial. The reader may attempt to prove this using elementary methods; instead, we employ a useful theorem from complex analysis:

Theorem. (Rouché) *Let D be a closed disk in $\mathbb{C}$. Suppose that $f(z)$ and $g(z)$ are complex-valued functions which are complex-differentiable in the interior of D, and that at any point z on the boundary of D we have $|f(z) - g(z)| < |f(z)|$. Then $f(z)$ and $g(z)$ have the same number of zeros in D, counting multiplicity.*

Given $\epsilon > 0$, let D_i be a disk centred at r_i and of radius at most ϵ, chosen so that no other roots of f lie on the boundary of D_i. Since $|f(z)|$ achieves its minimum on the boundary of D_i, this minimum is positive. Calling this minimum m_i, we can choose $\delta_i > 0$ to be sufficiently small such that

$$\delta_i(z^{n-1} + \cdots + z + 1) < m_i \leq |f(z)|$$

on the boundary of D_i. Let δ be the minimum of the δ_i, and let $g(z)$ be any monic polynomial whose coefficients each differ from the coefficients of $f(z)$ by at most δ. By construction, the polynomials $f(z)$ and $g(z)$ satisfy the conditions of Rouché's Theorem on each disk D_i, and consequently f and g have the same number of zeros in each D_i. Thus, the roots of $g(z)$ all are distant from corresponding roots of $f(z)$ by no more than ϵ, and so indeed φ^{-1} is continuous and φ is a homeomorphism.

Note for advanced readers: Without too much difficulty, one may use the above result to prove, more generally, that if X is any 2-dimensional manifold, then the symmetric product $X^{(n)}$ is always a manifold again. However, if X is a manifold of any positive dimension k other than 2 and if $n > 1$, then $X^{(n)}$ is never a manifold. One argues this roughly as follows. Let U be a neighbourhood of an unordered n-tuple of points of X with two of the points the same and the rest different. Then U looks like $(\mathbb{R}^k)^{n-2} \times (\mathbb{R}^k \times \mathbb{R}^k)/\{(x,y) \sim (y,x)\}$, where, in the last term of the product, the action of the symmetric group causes (x,y) and (y,x) to be identified. Changing variables in the last term of the product by putting $u = x+y$ and $v = x-y$, we find that U is in fact homeomorphic to $(\mathbb{R}^k)^{n-2} \times \mathbb{R}^k \times V$, where V is the quotient of $\mathbb{R}^k$ under the identification $v = -v$. Then V with the origin removed is homotopic to the projective space $\mathbb{R}P^{k-1}$, whereas $\mathbb{R}^k$ with any point removed is homotopic to the sphere S^{k-1}. Since $\mathbb{R}P^{k-1}$ and S^{k-1} are homotopic only for $k = 2$, we conclude that U is not Euclidean space if $k \neq 2$, and so $X^{(n)}$ can only be a manifold if X is 2-dimensional.

Pushing the Envelope

Naoki Sato
student, Yale University

If C is a differentiable curve in the plane, then we can form a family of lines $\mathcal{F}$ by taking the set of tangents at all points of C. This family $\mathcal{F}$ is called the *envelope* of C (see Figure 1). Obtaining $\mathcal{F}$ from C is easy enough, but consider the converse problem; that is, given a family of lines $\mathcal{F}$, does there exist a curve C which is tangent to every line in $\mathcal{F}$, so making $\mathcal{F}$ the envelope of C.

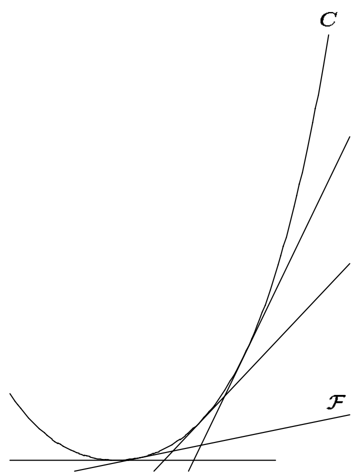


Figure 1

To solve the problem, we assume that $\mathcal{F}$ is a one-parameter family of lines; that is, the lines are parameterized by one parameter, say t, and furthermore, that the lines vary smoothly in t. It should be noted that with respect to the point-line duality in projective geometry, the concept of the envelope is dual to the concept of the locus.

Before tackling the problem, we first give a few examples.

Example 1. The parabola $y = x^2$. The derivative is given by

$$\frac{dy}{dx} = 2x .$$

Hence, the tangent at a particular point $(x_0, y_0) = (t_0, t_0^2)$ is given by

$$y - y_0 = 2t_0(x - x_0) ,$$

or

$$y = 2t_0 x - t_0^2 .$$

Example 2. The general ellipse is given by the following parameterization: $x = a\cos t$, $y = b\sin t$. The derivative is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t .$$

Hence, the tangent at a particular point $(x_0, y_0) = (a\cos t_0, b\sin t_0)$ is given by

$$y - y_0 = -\frac{b}{a}(\cot t_0)(x - x_0) ,$$

or

$$y = -\frac{b}{a}(\cot t_0)x + b\csc t_0 .$$

Now, we tackle the problem. Assume $\mathcal{F}$ is of the form $\{m(t)x + b(t)\}$, so in Example 1, $m(t) = 2t$ and $b(t) = -t^2$. A diagram makes the next step visually clear. If we fix one particular line ℓ in $\mathcal{F}$, and consider other lines in $\mathcal{F}$ that approach ℓ, then the intersections converge to a point on the desired curve C (see Figure 2).

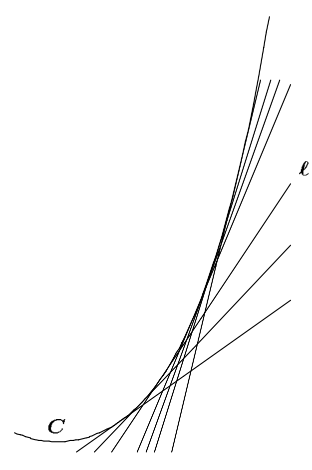


Figure 2

Mathematically, let ℓ be the line corresponding to a fixed t, namely $y = m(t)x + b(t)$. We consider the lines $y = m(t+h)x + b(t+h)$ as h approaches 0. Solving for their intersection, we obtain

$$m(t)x + b(t) = m(t+h)x + b(t+h) ,$$

which implies that

$$x = -\frac{b(t+h)-b(t)}{m(t+h)-m(t)} = -\frac{\frac{b(t+h)-b(t)}{h}}{\frac{m(t+h)-m(t)}{h}}.$$

By definition, this approaches $-\frac{b'(t)}{m'(t)}$ as h approaches 0; hence, this is the x-coordinate of the point on C that is in fact tangent to ℓ. The y-coordinate is

$$-\frac{b'(t)}{m'(t)} \cdot m(t) + b(t) = \frac{m'(t)b(t) - m(t)b'(t)}{m'(t)}.$$

Therefore,

$$C = \left\{ \left(-\frac{b'(t)}{m'(t)}, \frac{m'(t)b(t) - m(t)b'(t)}{m'(t)} \right) \right\}.$$

Checking, for Example 1, we have

$$C = \left\{ \left(\frac{2t}{2}, \frac{(2)(-t^2) - (2t)(-2t)}{2} \right) \right\} = \{(t, t^2)\},$$

and for Example 2,

$$\begin{aligned} C &= \left\{ \left(\frac{a \csc t \cot t}{1 + \cot^2 t}, \frac{b^2(1+\cot^2 t)\csc t - b^2 \cot^2 t \csc t}{b(1+\cot^2 t)} \right) \right\} \\ &= \{(a \cos t, b \sin t)\}. \end{aligned}$$

There is an alternative method for finding curves of envelopes. If the envelope is given in the form $F(x, y, t) = 0$, then the curve is found by solving the equations $F = 0$ and $\frac{\partial F}{\partial t} = 0$. (There are also some regularity conditions which we will ignore here.) In our case, we can take $F(x, y, t) = y - m(t)x - b(t)$, so the curve is found by solving the system of equations

$$\begin{aligned} y - m(t)x - b(t) &= 0, \\ m'(t)x + b'(t) &= 0. \end{aligned}$$

Solving for x and y, we find the same solution as above. Why does this more general method work?

Orthogonal Curves

A related topic, orthogonal curves, can be interesting in their own right. If $\mathcal{F}$ and $\mathcal{G}$ are each a family of curves in the plane, then they are *orthogonal* if every curve in $\mathcal{F}$ is orthogonal to every curve in $\mathcal{G}$. Two curves are orthogonal if their tangents are perpendicular at any point they intersect. For example, the family of concentric circles centred at the origin and the family of lines passing through the origin are orthogonal (see Figure 3).

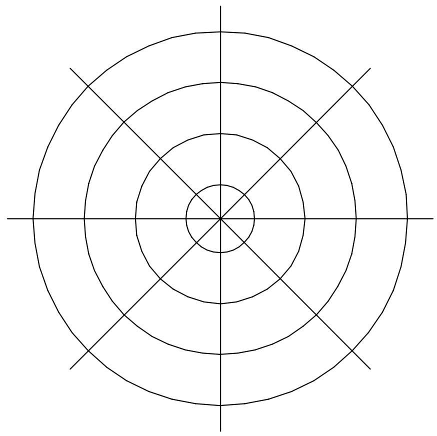


Figure 3

The natural question that arises is: For a family of curves $\mathcal{F}$, does there exist a family $\mathcal{G}$ which is orthogonal to $\mathcal{F}$. Most smooth one-parameter families will have a solution.

Example 3. We take the family of parabolas $\{y = tx^2\}$; in this case, t is again a parameter. For each parabola, differentiating we find

$$\frac{dy}{dx} = 2tx = \frac{2y}{x}.$$

We wish to eliminate t from the expression, because we want to find curves which are orthogonal to every such parabola, and so the expression must be independent of t. As every high school student knows, two lines are perpendicular if and only if the product of their slopes is -1, so we want to solve the differential equation

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{x}{2y} \\
2y\,dy &= -x\,dx \\
\int 2y\,dy &= -\int x\,dx \\
y^2 &= -\frac{x^2}{2} + C \\
\frac{x^2}{2} + y^2 = C \quad \text{or} \quad & \frac{x^2}{2C} + \frac{y^2}{C} = 1\,.
\end{aligned}$$

The orthogonal family is a family of ellipses. In other words, each parabola is orthogonal to each ellipse.

Example 4. We take the family of circles $\{(x - t)^2 + y^2 = t^2 - a^2\}$, where a is a fixed non-negative integer, and $|t| \geq a$. This is a general family of coaxial circles. For each circle, differentiating we find

$$\frac{dy}{dx} = -\frac{x-t}{y},$$

and by re-arranging the equation of the circle, we obtain

$$t = \frac{x^2+y^2+a^2}{2x},$$

so

$$\frac{dy}{dx} = \frac{y^2-x^2+a^2}{2xy}.$$

The orthogonal family must then satisfy

$$\begin{aligned} \frac{dy}{dx} &= \frac{2xy}{x^2-y^2-a^2} \\ (x^2-y^2-a^2)\,dy &= 2xy\,dx \\ 2xy\,dx + (-x^2+y^2+a^2)\,dy &= 0 \end{aligned}$$

By some elementary theory of differential equations, we find y^{-2} is an integrating factor of the above equation, so the equation becomes

$$\begin{aligned} \left(\frac{2x}{y}\right)dx + \left(-\frac{x^2}{y^2}+1+\frac{a^2}{y^2}\right)dy &= d\left(\frac{x^2}{y}+y-\frac{a^2}{y^2}\right) = 0 \\ \frac{x^2}{y}+y-\frac{a^2}{y} &= 2C \end{aligned}$$

We put in a constant of $2C$, because on simplification, the above equation becomes $x^2+(y-C)^2 = a^2+C^2$. This is the family of circles, with centre on the y-axis, passing through the points $(\pm a, 0)$.

In general, families of the same type of curve can have different orthogonal families; it depends on how the curves are "stacked".

Problems

1. For each of the following families of lines, find the curve which it is the envelope of:

   (a) $\{y = -\frac{a}{t^2}x + \frac{2a}{t}\}$. This is the family of lines for which the product of the x-intercept and y-intercept is a constant, $4a$

   (b) $\{y = (\cos t)x - t\cos t + \sin t\}$

   (c) $\{y = -(\tan t)x + a\sin t\}$. This is the family of lines for which the axes are cut off a chord of constant length a.

2. For each of the following families of lines, find the orthogonal family of lines:

   (a) $\{y = tx^3\}$

   (b) $\{y = te^x\}$.

3. Let c be a positive real. Consider the graphs of the equation

$$\frac{x^2}{t^2} + \frac{y^2}{t^2 - c^2} = 1,$$

as t varies. For $|t| < c$, the graph is an ellipse; for $|t| > c$, it is a hyperbola, all with foci $(\pm c, 0)$. Show that the family of such ellipses is orthogonal to the family of such hyperbolae.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 1999**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2364. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

A sequence $\{x_n\}$ is given by the recursion: $x_0 = p$, $x_{n+1} = qx_n + q - 1$ $(n \geq 0)$, where p is a prime and $q \geq 2$ is an integer.

(1) Suppose that p and q are relatively prime. Prove that the sequence $\{x_n\}$ does not consist of only primes.

(2)⋆ Suppose that $p|q$. Prove that the sequence $\{x_n\}$ does not consist of only primes.

2365. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Triangle DAC is equilateral. B is on the line DC so that $\angle BAC = 70^\circ$. E is on the line AB so that $\angle ECA = 55^\circ$. K is the mid-point of ED. Without the use of a computer, calculator or protractor, show that $60^\circ > \angle AKC > 57.5^\circ$.

2366. *Proposed by Catherine Shevlin, Wallsend-upon-Tyne, England.*

Triangle ABC has area p, where $p \in \mathbb{N}$. Let

$$\Sigma \;=\; \min\left(AB^2 + BC^2 + CA^2\right),$$

where the minimum is taken over all possible triangles ABC with area p, and where $\Sigma \in \mathbb{N}$.

Find the least value of p such that $\Sigma = p^2$

2367. *Proposed by K.R.S. Sastry, Dodballapur, India.*

In triangle ABC, the Cevians AD, BE intersect at P. Prove that

$$[ABC] \times [DPE] \;=\; [APB] \times [CDE].$$

(Here, $[ABC]$ denote the area of $\triangle ABC$, etc.)

2368. *Proposed by Iliya Bluskov, Simon Fraser University, Burnaby, BC.*

Let $(a_1, a_2, ..., a_n)$ be a permutation of the integers from 1 to n with the property that $a_k + a_{k+1} + ... + a_{k+s}$ is not divisible by $(n+1)$ for any choice of k and s where $k \geq 1$ and $0 \leq s \leq n-k-1$. Find such a permutation

(a) for $n = 12$;

(b) for $n = 22$.

2369*. *Proposed by Federico Arboleda, student, Bogotá, Colombia* (age 11).

Prove or disprove that for every $n \in \mathbb{N}$, there exists a $2n$-omino such that every n-omino can be placed entirely on top of it.

(An n-omino is defined as a collection of n squares of equal size arranged with coincident sides.)

2370. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the exact values of the roots of the polynomial equation

$$x^5 - 55x^4 + 330x^3 - 462x^2 + 165x - 11 \;=\; 0.$$

2371. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

For n an integer greater than 4, let $f(n)$ be the number of five-element subsets, S, of $\{1, 2, \ldots, n\}$ which have *no isolated points*, that is, such that if $s \in S$, then either $s-1$ or $s+1$ (**not** taken $\mathbf{modulo}\ n$) is in S.

Find a "nice" formula for $f(n)$.

2372. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

For n and k positive integers, let $f(n,k)$ be the number of k–element subsets S of $\{1, 2, \ldots, n\}$ satisfying:

(i) $1 \in S$ and $n \in S$; and

(ii) whenever $s \in S$ with $s < n-1$, then either $s+2 \in S$ or $s+3 \in S$.

Prove that $f(n,k) = f(4k-2-n, k)$ for all n and k; that is, the sequence

$$f(k,k),\ f(k+1,k),\ f(k+2,k),\ \ldots,\ f(3k-2,k)$$

of non-zero values of $\{f(n,k)\}_{n=1}^{\infty}$ is a palindrome for every k.

2373. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $AB > AC$. Let M be the mid-point of BC. Suppose that D is the reflection of M across the bisector of $\angle BAC$, and that A, B, C and D are concyclic.

Determine the value of $\dfrac{AB - AC}{BC}$.

2374. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $\angle BAC < 60^\circ$. Let M be the mid-point of BC. Let P be any point in the plane of $\triangle ABC$.

Prove that $AP + BP + CP \geq 2AM$.

2375. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let D be a point on side AC of triangle ABC. Let E and F be points on the segments BD and BC respectively, such that $\angle BAE = \angle CAF$. Let P and Q be points on BC and BD respectively, such that $EP \| DC$ and $FQ \| CD$.

Prove that $\angle BAP = \angle CAQ$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2247. [1997: 244] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Suppose that $n \geq 3$ is an odd natural number.

Show that the only polynomial $P \in \mathbb{R}[x]$ satisfying the functional equation:

$$(P(x+1))^n = (P(x))^n + \sum_{k=0}^{n-1} \binom{n}{k} x^k, \qquad \text{for all } x \in \mathbb{R},$$

is given by $P(x) = x$.

(b)$\star$ Suppose that $n \geq 1$ is a natural number.

Show that the only polynomial $P \in \mathbb{R}[x]$ satisfying the functional equation:

$$(P(x+1))^n = (P(x))^n + \sum_{k=0}^{n-1} \binom{n}{k} x^k, \qquad \text{for all } x \in \mathbb{R},$$

is given by $P(x) = x$.

(c)$\star$ Suppose that $n \geq 1$ is a natural number.

Show that the only polynomial $P \in \mathbb{R}[x]$ satisfying the functional equation:

$$P\left((x+1)^n\right) = (P(x))^n + \sum_{k=0}^{n-1} \binom{n}{k} x^k, \qquad \text{for all } x \in \mathbb{R},$$

is given by $P(x) = x$.

I. *Solution by David Stone and Vrej Zarikian, Georgia Southern University, Statesboro, Georgia, USA.*

(a) First observe that

$$\sum_{k=0}^{n-1} \binom{n}{k} x^k = (x+1)^n - x^n .$$

Thus the equation satisfied by P can be written as

$$Q(x+1) = Q(x) \qquad \text{for all } x \in \mathbb{R},$$

where $Q(x) = (P(x))^n - x^n$. But $Q \in \mathbb{R}[x]$ and is periodic implies that $Q(x) = c$ for some constant c, so

$$(P(x))^n = x^n + c. \tag{1}$$

By degree considerations, $P(x) = ax + b$ for constants a and b; if P were of degree 2, for instance, the left side of (1) would be of degree $2n$, which exceeds the degree of the right side. Thus $(ax+b)^n = x^n + c$. Since $n \geq 3$ and the right side of this equation has no "middle terms", it must be that $b = 0$. Hence $(ax)^n = x^n + c$, so $c = 0$ and $a^n = 1$. Since n is odd, $a = 1$. In other words, $P(x) = x$.

(b) The proposition is not true as stated. Arguing as in part (a), we reach the same conclusion — $(P(x))^n = x^n + c$ implies $P(x) = ax + b$. If $n = 1$, then $P(x) = x + c$ so $a = 1$ and $b = c$. If $n > 1$, it must be that $b = 0$, so $c = 0$ and $a^n = 1$. If n is odd, $a = 1$. If n is even, $a = 1$ or -1. In summary:

n	$P(x)$
1	$x + c$ (c arbitrary)
> 1, odd	x
> 1, even	x or $-x$

(c) This proposition is not true as stated, either. For example, take $n = 1$ and $P(x) = x + c$ (c arbitrary).

II. *Correction by the editor.*

Unfortunately, part (b) of this problem was incorrectly transcribed into print: the functional equation should have been

$$(P(x+1))^n = P(x^n) + \sum_{k=0}^{n-1} \binom{n}{k} x^k.$$

That is, the first term on the right side should be $P(x^n)$ instead of $(P(x))^n$. With this correction, $P(x) = -x$ is never a solution, although $P(x) = x + c$ is still a counterexample when $n = 1$. We apologize to the proposer for stating his problem incorrectly.

It appears from the proposer's (partial) solution that he had intended $n \geq 2$ to be assumed in part (b) and (c) as well as (a), though his problem statement did not make this clear. In any case, readers are invited to prove or disprove the revised part (b) for $n \geq 2$. For part (c), see below!

III. *Solution to part* (c) *by Michael Lambrou, University of Crete, Crete, Greece.*

[Lambrou first solved (a) and (b), giving the same answer for (b) as in Solution I. — *Ed.*]

In this case we show that for $n > 1$ the only polynomial satisfying the modified identity

$$P((x+1)^n) = (P(x))^n + \sum_{k=0}^{n-1} \binom{n}{k} x^k \,;$$

that is, equivalently

$$P((x+1)^n) = (P(x))^n + (x+1)^n - x^n, \tag{2}$$

is $P(x) = x$. But for $n = 1$ the polynomial could be of the form $P(x) = x+c$ for any constant c.

For a start it is easy to verify identity (2) for the stated polynomials, so let us concentrate on the more interesting converse. The case $n = 1$ is easy: (2) becomes $P(x+1) = P(x) + 1$ which is the same equation as in part (b) for the case $n = 1$. So this case has already been dealt with, giving the stated conclusion for the form of P.

Assume then that $n \geq 2$.

We first observe that (2), valid for the real variable x, can be lifted to an identity valid also for complex numbers z:

$$P((z+1)^n) = (P(z))^n + (z+1)^n - z^n. \tag{3}$$

For a quick proof, set

$$R(z) = P((z+1)^n) - (P(z))^n - (z+1)^n + z^n,$$

which is a polynomial in z. By (2) this polynomial vanishes for an infinity of values of z (all real z!). Hence it is identically zero, proving (3).

We now show that any polynomial P satisfying (3) must be of degree 1.

For this purpose, suppose on the contrary that it is of degree $m \geq 2$. Write

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0 \qquad (a_m \neq 0).$$

Given any real x, set

$$z = e^{2\pi i/n}(1+x) - 1$$

so that

$$(z+1)^n = \left(e^{2\pi i/n}(1+x)\right)^n = (x+1)^n$$

and so $P\left((z+1)^n\right) = P\left((x+1)^n\right)$. Using (3) we find

$$(P(z))^n + (z+1)^n - z^n = (P(x))^n + (x+1)^n - x^n;$$

that is,

$$(P(z))^n - (P(x))^n = z^n - x^n. \tag{4}$$

Hence

$$\prod_{k=0}^{n-1}\left(P(z)-e^{2k\pi i/n}P(x)\right)=z^n-x^n. \tag{5}$$

We have used the fact that the equation $w^n-a^n=0$ has roots $w=e^{2k\pi i/n}a$ $(0\le k\le n-1)$ so w^n-a^n factors as $\prod_{k=0}^{n-1}(w-e^{2k\pi i/n}a)$.

The right hand side of (5) as a polynomial in x is

$$z^n-x^n=\left(e^{2\pi i/n}x+(e^{2\pi i/n}-1)\right)^n-x^n$$

which is of degree n or less. In fact it is of degree $n-1$ or less, as the coefficient of x^n is $e^{2\pi i}-1=0$, leaving powers of x up to $n-1$ at most. We shall derive a contradiction to (5) by showing its left hand side is of degree larger than $n-1$.

For $0\le k\le n-1$ we have

$$\begin{aligned}P(z)-e^{2k\pi i/n}P(x)&=(a_mz^m+\cdots+a_0)-e^{2k\pi i/n}(a_mx^m+\cdots+a_0)\\&=a_m\left(e^{2\pi i/n}x+(e^{2\pi i/n}-1)\right)^m+\cdots+a_0\\&\qquad-e^{2k\pi i/n}(a_mx^m+\cdots+a_0),\end{aligned}$$

and the coefficient of x^m is clearly

$$a_m\left(e^{2\pi mi/n}-e^{2k\pi i/n}\right)=a_me^{2k\pi i/n}\left(e^{2\pi(m-k)i/n}-1\right). \tag{6}$$

Observe that n divides exactly one of the n consecutive numbers $m-0$, $m-1,\ldots,m-(n-1)$. Say $n\mid(m-k_0)$ for some $k_0\in\{0,1,\ldots,n-1\}$, but n does *not* divide any of the remaining $n-1$ terms. Thus the coefficient in (6) is *non-zero* for $k\ne k_0$, $0\le k\le n-1$. In other words for $k\ne k_0$, $0\le k\le n-1$, the polynomial $P(z)-e^{2k\pi i/n}P(x)$ in x is of degree m. Moreover the polynomial $P(z)-e^{2k_0\pi i/n}P(x)$ is not identically zero, since if $P(z)=e^{2k_0\pi i/n}P(x)$ for all real x then $(P(z))^n=(P(x))^n$ for all real x, and so by (4) we would get $z^n=x^n$ for all real x, which is clearly false; otherwise $z=e^{2k'\pi i/n}x$ for some $0\le k'\le n-1$, which is incompatible with $z=e^{2\pi i/n}(1+x)-1$, as the constant term of this last is non-zero.

To summarize, we have shown:

- the degree of $P(z)-e^{2k\pi i/n}P(x)$ for $k\ne k_0$, $0\le k\le n-1$, is m;
- $P(z)-e^{2k_0\pi i/n}P(x)$ is not identically zero.

Hence the left hand side of (5) is of degree at least $m(n-1)>n-1$. This is a contradiction. Hence we conclude that P is a first degree polynomial, $P(x)=ax+b$ for some a,b. Recall that we are seeking a polynomial $P\in\mathbb{R}[x]$, so a,b are restricted to be real.

By (2) we have

$$a(x+1)^n + b = (ax+b)^n + (x+1)^n - x^n$$

for all real x. Comparing coefficients of x^n and x^{n-1} and the constant term (recall $n \geq 2$) we find in turn $a = a^n$, $na = na^{n-1}b + n$ (that is, $a = a^{n-1}b + 1$), and $a + b = b^n + 1$. The second equation gives $a \neq 0$ so our equations become

$$a^{n-1} = 1, \tag{7}$$

$$a = 1 \cdot b + 1, \tag{8}$$

$$a + b = b^n + 1. \tag{9}$$

By (7), we have $a = \pm 1$ (in fact for even n it only gives $a = 1$). But if $a = -1$ then by (8), we would get $b = -2$ and in (9), we would have $-1 - 2 = (-2)^n + 1$. This last is clearly impossible as $(-2)^n$ is never equal to -4. Hence $a = 1$ and (8) gives $b = 0$. To summarize, we have shown $P(x) = 1 \cdot x + 0$, showing that $P(x) = x$ is the only (real) polynomial satisfying our condition, and the proof is complete.

Remarks. (1) For even $n = 2t$ we do not have to go through the consideration of complex numbers. By setting $z = -(1+x) - 1 = -x - 2$ and making the obvious adaptations and simplifications of the above we can arrive at the proof. For example the analysis into products in (5) can be replaced by

$$(P(z))^{2t} - (P(x))^{2t} = \left[(P(z))^t - (P(x))^t\right]\left[(P(z))^t + (P(x))^t\right]$$

and then estimating the degree of the right hand side as at least mt.

The difficulty of the above approach for odd n arises from the fact that $(1+x)^n$ is, for the real variable x, a 1-1 function, so cannot be put in the form $(1+z)^n$ for a different (real) z. Thus we had to go through complex numbers.

(2) If one could somehow show $P(0) = 0$ (as expected), then a quick way to complete the proof would be as follows. Set $x_0 = 0$ and $x_{k+1} = (1+x_k)^n$ for $k = 0, 1, \ldots$. It is easy to see that each x_k is a fixed point of P; that is, $P(x_k) = x_k$, inductively based on

$$P(x_{k+1}) = P((1+x_k)^n) = (P(x_k))^n + (1+x_k)^n - x_k^n = (1+x_k)^n = x_{k+1}.$$

But the values of x_k are all distinct (the x_k's are strictly increasing), so $P(x) = x$ identically as required. Unfortunately, however, I do not have an easy proof of $P(0) = 0$, other than a proof resembling the proof given above. Can the readers find such an easy proof of $P(0) = 0$ (or, just as well, $P(1) = 1$)?

Parts (a) *and the incorrectly stated* (b) *were also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; and DIGBY SMITH, Mount Royal College, Calgary, Alberta.*

Hess and Smith found all functions satisfying the given equation, as did our featured solvers Stone and Zarikian (Solution I*) and Lambrou; Bradley and Herzig just showed that* $P(x) = x$ *is not the only such function. Part* (a) *only was solved by the proposer. There was also one incorrect solution sent in.*

Stone and Zarikian also consider the similar functional equation

$$P((x+1)^n) = P(x^n) + \sum_{k=0}^{n-1} \binom{n}{k} x^k,$$

which, in view of the correction to part (b)*, is the obvious fourth variation on the proposer's original three equations. They prove easily that the only polynomials satisfying this equation for all real* x *are of the form* $P(x) = x + c$ *for* c *constant. Readers may like to check this out for themselves, and then try finishing off part* (b)*!*

2249. [1997: 245] *Proposed by K.R.S. Sastry, Dodballapur, India.*

How many distinct acute angles α are there such that

$$\cos\alpha\,\cos 2\alpha\,\cos 4\alpha \;=\; \tfrac{1}{8}\,?$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

There are exactly three such values: $\alpha = \frac{\pi}{9}$, $\frac{2\pi}{7}$, and $\frac{\pi}{3}$.

Let $P = \cos\alpha\,\cos 2\alpha\,\cos 4\alpha$. Then, for all α such that $\sin\alpha \neq 0$, we have $P = \frac{1}{8}$ if and only if

$$\sin\alpha \;=\; 8P\sin\alpha \;=\; \sin(8\alpha)\,.$$

This is equivalent to having $8\alpha = 2k\pi + \alpha$ or $8\alpha = (2k+1)\pi - \alpha$ for some integer k. That is, either

(i) $\alpha = \frac{2k\pi}{7}$, or
(ii) $\alpha = \frac{(2k+1)\pi}{9}$.

Note that $\alpha = 0$ is not a solution. Since $\frac{\pi}{2} < \frac{5\pi}{9} < \frac{4\pi}{7}$, we see that the only possible values for k, to ensure that $0 < \alpha \leq \frac{\pi}{2}$, are $k = 1$ in (i) and $k = 0, 1$ in (ii). These yield $\alpha = \frac{2\pi}{7}$, $\frac{\pi}{9}$, and $\frac{\pi}{3}$, respectively. Since $\sin\alpha \neq 0$ for these values, the result $P = \frac{1}{8}$ follows.

Also solved by (NOTE: a dagger † before a name indicates that the solver's solution is very similar to the one above,) HAYO AHLBURG, Benidorm, Spain; GERALD ALLEN, CHARLES DIMINNIE, TREY SMITH and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, *Bosnia and Herzegovina; SAM BAETHGE, Nordheim, Texas, USA; † MICHAEL BATAILLE, Rouen, France; † FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ADRIAN BIRKA, student, Lakeshore Catholic High School, Port Colbourne, Ontario; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; † THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; † GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; † KEE-WAI LAU, Hong Kong; † ALAN LING, student, University of Toronto, Toronto, Ontario; VEDULA N. MURTY, Visakhapatnam, India; VICTOR OXMAN, University of Haifa, Haifa, Israel; † BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; † CHRISTOPHER SO, student, Francis Libermann Catholic High School, Scarborough, Ontario; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; DAVID VELLA, Skidmore College, Saratoga Springs, NY, USA; and the proposer. There were also two incorrect and four incomplete submissions, two of which gave the correct answer without proof.*

2250. [1997: 245] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a scalene triangle with incentre I. Let D, E, F be the points where BC, CA, AB are tangent to the incircle respectively, and let L, M, N be the mid-points of BC, CA, AB respectively.

Let l, m, n be the lines through D, E, F parallel to IL, IM, IN respectively.

Prove that l, m, n are concurrent.

Solution by Michael Lambrou, University of Crete, Greece.

Introduce position vectors, $\vec{c} = \overrightarrow{BC}$, $\vec{a} = \overrightarrow{BA}$, $\vec{d} = \overrightarrow{BD}$, $\vec{e} = \overrightarrow{BE}$, $\vec{f} = \overrightarrow{BF}$, all originating from B. If we denote, as usual, the lengths $|BC| = a$, $|CA| = b$, and $|AB| = c$, then as $|BD| = |BF| = \frac{a+c-b}{2}$, we have $\vec{d} = \left(\frac{a+c-b}{2a}\right)\ \vec{c}$ and $\vec{f} = \left(\frac{a+c-b}{2c}\right)\ \vec{a}$. Also $\vec{e} = \left(\frac{(a+b-c)\vec{a}+(b+c-a)\vec{c}}{2b}\right)$.

If the bisector, AI, cuts BC at P, then $|BP| = \frac{ac}{b+c}$, so $\overrightarrow{BP} = \left(\frac{c}{b+c}\right)\ \vec{c}$. So, as $\frac{|AI|}{|IP|} = \frac{|AB|}{|BP|} = \frac{b+c}{a}$, we have,

$$\overrightarrow{BI} = \overrightarrow{BP} + \overrightarrow{PI} = \left(\frac{a}{a+b+c}\right)\ \vec{a} + \left(\frac{c}{a+b+c}\right)\ \vec{c}.$$

Consider X, where $\overrightarrow{BX} = \vec{x} = \left(\frac{3a-b-c}{a+b+c}\right)\ \vec{a} + \left(\frac{3c-b-a}{a+b+c}\right)\ \vec{c}$. We show that X is on lines l, m and n, so these lines are concurrent, as is required. All we need to do is to verify that: $\vec{x} - \vec{d}$, $\vec{x} - \vec{e}$ and $\vec{x} - \vec{f}$ are parallel to $\overrightarrow{IL}$, $\overrightarrow{IM}$ and $\overrightarrow{IN}$, respectively.

Then

$$\vec{x} - \vec{d} = \left(\frac{3a-b-c}{a+b+c}\right)\vec{a} + \left(\frac{3c-b-a}{a+b+c} - \frac{a+c-b}{a}\right)\vec{c}$$

and

$$\begin{aligned}\overrightarrow{IL} \;=\; \overrightarrow{BL} - \overrightarrow{BI} &= \frac{\vec{c}}{2} - \left(\frac{a}{a+b+c}\,\vec{a} + \frac{c}{a+b+c}\,\vec{c}\right)\\ &= \left(\frac{a+b-c}{2(a+b+c)}\right)\vec{c} - \left(\frac{a}{a+b+c}\right)\vec{a}\\ &= \left(\frac{-a}{3a-b-c}\right)(\vec{x}-\vec{d}).\end{aligned}$$

That $\vec{x} - \vec{f}$ is parallel to $\overrightarrow{IN}$ follows immediately from the above by interchanging the roles of c and a. Similar routine calculation shows $\vec{x} - \vec{e}$ is parallel to $\overrightarrow{IM}$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. There was one incomplete solution.

2251. [1997: 300] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

In the plane, you are given a circle (but not its centre), and points A, K, B, D, C on it, so that $\text{arc } AK = \text{arc } KB$ and $\text{arc } BD = \text{arc } DC$.

Construct, using only an unmarked straightedge, the mid-point of $\text{arc } AC$.

Preliminary comment. There are two arcs AC; the midpoint of one of them will always be easily constructed; the other is more challenging. Most solvers (including the proposer) constructed only the easier one, which does not require the lemma. Note that it is essentially the same as the lemma in Oxman's solutions to his earlier problem: **2234** [1998: 247].

Solution by Toshio Seimiya, Kawasaki, Japan.

Lemma. *If we are given points A, M, B, and C on a line with $AM = MB$, then using only an unmarked ruler we can construct D on the line such that $BC = CD$.*

Construction. Let P be a point not on the line AB, and let Q be a point on the segment AP.

Let $R = BQ \cap MP$, $\quad S = AR \cap BP$,

$N = PM \cap QS$, $\quad T = NC \cap SB$,

$D = QT \cap AB$. $\quad$ Then $BC = CD$.

Proof. By Ceva's Theorem we have $\frac{PQ}{QA} \cdot \frac{AM}{MB} \cdot \frac{BS}{SP} = 1$. Since $AM = MB$ we get $\frac{PQ}{QA} = \frac{PS}{SB}$, so that $QS \parallel AB$. It follows that since QD, NC, SB are concurrent at T,

$$\frac{BC}{CD} = \frac{SN}{NQ},$$

and since SA, NM, QB are concurrent at R,

$$\frac{SN}{NQ} = \frac{AM}{MB}.$$

Using these two equalities together with $AM = MB$, we conclude that

$$\frac{BC}{CD} = \frac{AM}{MB} = 1,$$

so that $BC = CD$.

We now turn to the main construction.

Construction. We denote the given circle by Γ. Draw AD meeting CK at I. Let M be the second intersection of BI with Γ; then M is the midpoint of the arc AC not containing B. Let N be the intersection of KD with BI; then $BN = NI$. By the lemma we can construct the point I_B such that $IM = MI_B$. Let I_C be the intersection of AI_B with KC, and let L be the second intersection of I_CB with Γ. Then L is the midpoint of the arc ABC.

Proof. Since arc AK = arc KB, we have that CK bisects $\angle ACB$. Similarly AD bisects $\angle BAC$, so that I is the incentre of $\triangle ABC$.
[*Editor's comment.* The statement of the problem might be construed as allowing C and K to lie on the same side of the line AB, in which case I would be an excentre. Seimiya's argument is based on the interpretation that the points lie on the circle in the prescribed order, namely $AKBDC$; it easily can be modified to accommodate the alternative interpretation.]
Hence BM bisects $\angle ABC$, so that M is the midpoint of the arc AC not containing B.

Since I is the incentre and K is the midpoint of arc AB, we have $KI = KB$. Similarly we have $DI = DB$. Hence KD is the perpendicular bisector of BI, and N is the midpoint of BI. By the lemma we can therefore construct point I_B such that $IM = MI_B$ using only an unmarked ruler. Since M is the midpoint of arc AC while I is the incentre and $IM = MI_B$, I_B is an excentre. Thus I_C is an excentre, I_CB is the exterior bisector of $\angle ABC$, and L is the midpoint of arc ABC.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, *Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.*

2252. [1997: 300] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Prove that the nine-point circle of a triangle trisects a median if and only if the side lengths of the triangle are proportional to its median lengths in some order.

The first part of the solution is by Christopher J. Bradley, Clifton College, Bristol, UK. The rest is a compilation of ideas from other solutions liberally sprinkled with comments by editor, Cathy Baker.

If l, m and n are the lengths of the medians from A, B and C respectively, then Apollonius' Theorem gives the formulae

$$4l^2 = 2b^2 + 2c^2 - a^2$$

$$4m^2 = 2c^2 + 2a^2 - b^2$$

$$4n^2 = 2a^2 + 2b^2 - c^2.$$

Suppose $a \geq b \geq c$. Then clearly from these formulae, $n \geq m \geq l$. If the medians are proportional to the sides, these orderings must be preserved and so a constant k exists such that

$$2a^2 + 2b^2 - c^2 = ka^2$$

$$2c^2 + 2a^2 - b^2 = kb^2$$

$$2b^2 + 2c^2 - a^2 = kc^2.$$

Adding, one finds $k = 3$ and each of the three equations reduces to $a^2 + c^2 = 2b^2$.

Note that if the triangle has two equal sides, then it must be equilateral. If, for example, $a = b \geq c$, then $2b^2 = a^2 + c^2$ and $2a^2 = b^2 + c^2$, so adding gives $2c^2 = a^2 + b^2$, and subtraction, $a = b = c$.

Conversely, if $a^2 + c^2 = 2b^2$, then substitution in the formulae gives $4l^2 = 3c^2, 4m^2 = 3b^2$ and $4n^2 = 3a^2$, so $\frac{l}{c} = \frac{m}{b} = \frac{n}{a} = \frac{\sqrt{3}}{2}$; that is, the medians are proportional to the sides. Note that in this case, b must lie between a and c. If, for example, $b < c \leq a$ [$c \leq a < b$], then $2b^2 = a^2 + c^2 \geq 2c^2$ [$2b^2 = a^2 + c^2 \leq 2a^2$]; a contradiction.

We have shown that a triangle with sides of length a, b, c is self median if and only if $2b^2 = a^2 + c^2$, where b lies between a and c.

Let D and E be the midpoints of AC and BC, respectively; F the foot of the altitude from A; P the other point where the nine point circle meets

BD. Then $BD = kBP$, for some $k > 0$. Then using the power of B with respect to the nine-point circle, we get:

$$BP \cdot BD = BF \cdot BE.$$

But $BF = c\cos B$ and, by the Cosine Law, $\cos B = \frac{a^2+c^2-b^2}{2ac}$. Since $BD^2 = m^2 = \frac{2c^2+2a^2-b^2}{4}$ and $BE = \frac{a}{2}$, we have

$$\frac{2c^2+2a^2-b^2}{4k} = \frac{ac(a^2+c^2-b^2)}{4ac}$$

$$(k-1)b^2 = (k-2)(a^2+c^2).$$

If the nine-point circle trisects BD, then $k = 3$, so $2b^2 = a^2 + c^2$, b lies between a and c, and the medians are proportional to the sides.

Conversely, if $2b^2 = a^2 + c^2$, then $(k-3)b^2 = 0$ and $k = 3$. Hence if $2b^2 = a^2 + c^2$, then the nine-point circle trisects the median from B.

Solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

2253. [1997: 300] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle and I_b, I_c are the excentres of $\triangle ABC$ relative to sides CA, AB respectively.

Suppose that

$I_bA^2 + I_bC^2 = BA^2 + BC^2$ and that $I_cA^2 + I_cB^2 = CA^2 + CB^2$.

Prove that $\triangle ABC$ is equilateral.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let us turn the problem around and start from $\triangle ABC$ (with sides a, b, c) and the triangle formed by the feet of the altitudes D, E, F. Then this $\triangle ABC$ plays the role of the given $\triangle I_aI_bI_c$, while $\triangle DEF$ corresponds to the original $\triangle ABC$ (since I_aA, etc. are the altitudes in $\triangle I_aI_bI_c$). In this notation our given condition is that $BD^2 + BF^2 = ED^2 + EF^2$ and $CD^2 + CE^2 = FD^2 + FE^2$, and we are to show that $\triangle DEF$ is equilateral. Since $AE : AF = c\cos A : b\cos A = AB : AC$, we have $\triangle AEF \sim \triangle ABC$ and, hence, $EF = a\cos A$. Similarly $FD = b\cos B$ and $DE = c\cos C$. Hence we get

$$(a^2+c^2)\cos^2 B = a^2\cos^2 A + c^2\cos^2 C, \qquad (1)$$
$$(a^2+b^2)\cos^2 C = a^2\cos^2 A + b^2\cos^2 B. \qquad (2)$$

Adding $b^2\cos^2 B$ to the first and $c^2\cos^2 C$ to the second equation yields

$$(a^2+b^2+c^2)\cos^2 B + (a^2+b^2+c^2)\cos^2 C,$$

and, hence, $B = C$(since $B + C \neq \pi$). Thus (2) becomes

$$(a^2+b^2)\cos^2 B = a^2\cos^2 A + b^2\cos^2 B,$$

whence $A = B$. Therefore $\triangle ABC$ and also $\triangle DEF$ are equilateral.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; VEDULA N. MURTY, Visakhapatnam, India; ISTVÁN REIMAN, Budapest, Hungary; MARAGOUDAKIS PAVLOC, Pireas Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was one incorrect solution.

Chronis notes that the result can be generalized by replacing the exponent 2 by any positive real: $\triangle ABC$ is equilateral if for some $t > 0$, we have $I_bA^t + I_bC^t = BA^t + BC^t$ and $I_cA^t + I_cB^t = CA^t + CB^t$. Indeed, our featured solution extends to the proof of Chronis's more general statement.

2254. [1997: 300] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle with $AB = AC$. Let D be the point on side AC such that $CD = 2AD$. Let P be the point on the segment BD such that $\angle APC = 90^\circ$.

Prove that $\angle ABP = \angle PCB$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let Q be the midpoint of BC and complete the rectangle $AQCS$. It follows that B, D and S are collinear, as $AS||BC$ and $AS : BC = AD : DC$. Now observe that A, P, Q, C and S are concyclic by Thales' Theorem [$\angle APC = \angle AQC = \angle ASC = 90^\circ$, so P, Q and S are all on the circle with diameter AC] and $AB||SQ$. Hence

$$\angle ABP = \angle PSQ = \angle PCQ = \angle PCB.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; RUSSELL EULER and JAWAD SADEK, NW

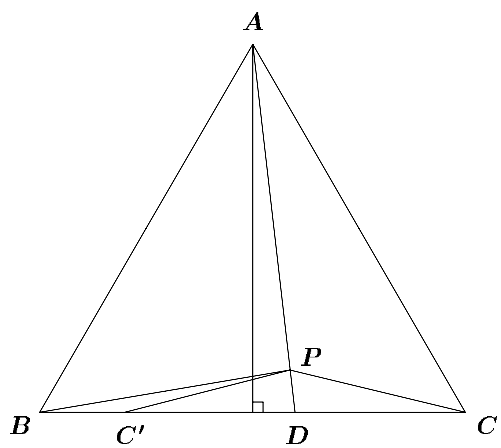
Missouri State University, Maryville, Missouri, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; DAG JONSSON, Uppsala, Sweden; GEOFFREY A. KANDALL, Hamden, Connecticut, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (2 solutions); KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; VICTOR OXMAN, University of Haifa, Haifa, Israel; PAVLOS MARAGOUDAKIS, Hatzikeriakio, Pireas, Greece; LUIS A. PONCE, Santos, Brazil; D.J. SMEENK, Zaltbommel, the Netherlands (2 solutions); PANOS E. TSAOUSSOGLOU, Athens, Greece; ENRIQUE VALERIANO, National University of Engineering, Lima, Peru; JOHN VLACHAKIS, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

2255. [1997: 300] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that $|\angle PAB - \angle PAC| \geq |\angle PBC - \angle PCB|$.

Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.

Without loss of generality we suppose that $\angle PAB \geq \angle PAC$.



In the figure $PC = PC'$; we deduce that $\angle BPD \geq \angle CPD$ [since $\angle BPD \geq \angle C'PD \geq \angle CPD$]. Therefore

$$\angle PAB + \angle ABP \geq \angle PAC + \angle ACP \quad \text{[opposite interior angles]},$$

which implies

$$\angle PAB + (60^\circ - \angle PBC) \geq \angle PAC + (60^\circ - \angle PCB),$$

which in turn implies

$$\angle PAB - \angle PAC \geq \angle PBC - \angle PCB.$$

Editor's comment. It is clear from Chronis's argument that the result continues to hold for *any isosceles triangle* ABC *with* $\angle B = \angle C$. Just replace the 60° angle by the base angle.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; JORDI DOU, Barcelona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VICTOR OXMAN, University of Haifa, Haifa, Israel; PANOS E. TSAOUSSOGLOU, Athens, Greece; ENRIQUE VALERIANO, Lima, Peru; and the proposer.

2256. [1997: 300] *Proposed by Russell Euler and Jawad Sadek, Department of Mathematics and Statistics, Northwest Missouri State University, Maryville, Missouri, USA.*

If $0 < y < x \leq 1$, prove that $\dfrac{\ln(x) - \ln(y)}{x - y} > \ln\left(\dfrac{1}{y}\right)$.

I. *Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*

First we show that $\ln u > 1 - \frac{1}{u}$ for $u > 1$. Let $g(u) = \ln u - 1 + \frac{1}{u}$. Then $g'(u) = \frac{u-1}{u^2} > 0$, which implies that g is strictly increasing on $[1, \infty)$. Hence $g(u) > g(1) = 0$ for $u > 1$.

Now, hold y fixed at $y = a$, where $0 < a < x \leq 1$, and let $f(x) = \dfrac{\ln x - \ln a}{x - a} + \ln a$. Then $f'(x) = \dfrac{1 - \dfrac{a}{x} - \ln\left(\dfrac{x}{a}\right)}{(x-a)^2}$.

Since $\frac{x}{a} > 1$, we have $f'(x) < 0$ by the inequality above. Hence f is strictly decreasing on $(a, 1]$ and so $f(x) > f(1) = \frac{-a\ln a}{1-a} > 0$.

II. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

For a fixed $y \in (0, 1)$, consider the function $f(x) = x\ln y + \ln x$, where $y < x \leq 1$.

Since $0 < y < 1$, we have $y\ln y + \ln y < \ln y$; that is, $f(y) < f(1)$.

Since $f''(x) = -\frac{1}{x^2} < 0$, we have that f is strictly concave on $[y, 1]$, and since $y < x \leq 1$, we get $f(x) > f(y)$, which is readily seen to be equivalent to the proposed inequality.

Also solved by PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; LUZ M. DeALBA, Drake University, Des Moines, IA, USA; HANS ENGELHAUPT, Franz-Lud-

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Most of the submitted solutions are similar to either I *or* II *given above. Herzig proved the stronger result that* $\dfrac{\ln x - \ln y}{x - y} > \ln\left(1 + \dfrac{1}{y}\right)$ *for all distinct* x, y *with* $0 < x, y \leq 1$ *by using the following inequality known as Bernoulli's Inequality:* $(1+x)^{\alpha} > 1 + \alpha x$, *where* $x > -1$, $x \neq 0$ *and* $\alpha > 1$ *or* $\alpha < 0$.

[*Ed.* See, for example, page 34 of D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.]

2258. [1997: 301] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

In a right-angled triangle ABC (with $\angle C = 90°$), D lies on the segment BC so that $BD = AC\sqrt{3}$. E lies on the segment AC and satisfies $AE = CD\sqrt{3}$. Find the angle between AD and BE.

I. *Solution by D. Kipp Johnson, Valley Catholic High School, Beaverton, Oregon.*

Without loss of generality we put A at $(0,1)$, C at $(0,0)$, and D at $(x,0)$. Then $BD = AC\sqrt{3} = \sqrt{3}$, so B is at $(x+\sqrt{3},0)$. Since $AE = CD\sqrt{3} = x\sqrt{3}$, E is at $(0, 1 - x\sqrt{3})$. The slope of the line AD is $-1/x$ and the slope of the line BE is $(x\sqrt{3}-1)/(x+\sqrt{3})$. If θ is the angle between AD and BE, then:

$$\tan\theta = \frac{\dfrac{x\sqrt{3}-1}{x+\sqrt{3}} + \dfrac{1}{x}}{1 + \left(-\dfrac{1}{x}\right)\left(\dfrac{x\sqrt{3}-1}{x+\sqrt{3}}\right)} = \frac{\sqrt{3}(x^2+1)}{x^2+1} = \sqrt{3},$$

implying $\theta = 60°$.

II. *Solution by Niels Bejlegaard, Stavanger, Norway.*

Let $\overrightarrow{CA} = a\vec{\jmath}$ and $\overrightarrow{EA} = y\vec{\jmath}$. Then we have that $\overrightarrow{CD} = \dfrac{y}{\sqrt{3}}\vec{\imath}$ and $\overrightarrow{CB} = \left(a\sqrt{3} + \dfrac{y}{\sqrt{3}}\right)\vec{\imath}$. Thus

$$\begin{aligned}\overrightarrow{EB} &= \overrightarrow{EC} + \overrightarrow{CB} = \overrightarrow{EA} + \overrightarrow{AC} + \overrightarrow{CB} \\ &= \overrightarrow{EA} - \overrightarrow{CA} + \overrightarrow{CB} = (y-a)\vec{\jmath} + \left(a\sqrt{3} + \frac{y}{\sqrt{3}}\right)\vec{\imath}, \\ \overrightarrow{AD} &= \overrightarrow{AC} + \overrightarrow{CD} = -a\vec{\jmath} + \frac{y}{\sqrt{3}}\vec{\imath}.\end{aligned}$$

Then the angle θ between the vectors $\overrightarrow{EB}$ and $\overrightarrow{AD}$ satisfies:

$$\begin{aligned}\cos\theta &= \frac{\overrightarrow{EB}\cdot\overrightarrow{AD}}{|\overrightarrow{EB}|\cdot|\overrightarrow{AD}|} = \frac{(y-a)(-a) + \left(a\sqrt{3} + \frac{y}{\sqrt{3}}\right)\cdot\frac{y}{\sqrt{3}}}{\sqrt{4a^2 + \frac{4}{3}y^2}\cdot\sqrt{a^2 + \frac{1}{3}y^2}} \\ &= \frac{a^2 + \frac{1}{3}y^2}{2\left[\sqrt{a^2 + \frac{1}{3}y^2}\right]^2} = \frac{1}{2}.\end{aligned}$$

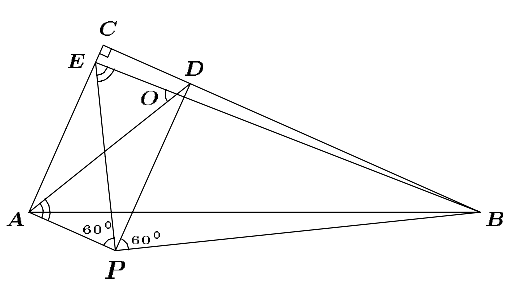
Therefore, $\theta = 60^\circ$.

III. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let P be a point such that $AP \parallel CB$ and $DP \parallel CA$. Since $CAPD$ is a rectangle, we get $PD = AC$, $AP = CD$, and $\angle CDP = 90^\circ = \angle CAP$. As $BD = AC\sqrt{3}$ we get $BD = PD\sqrt{3}$, so that $\angle DPB = 60^\circ$. Since $AE = CD\sqrt{3}$ we have $AE = AP\sqrt{3}$, so that $\angle APE = 60^\circ$. Since $\triangle PAE \sim \triangle PDB$, we have $PA : PD = PE : PB$, and $\angle APD = 60^\circ + \angle EPD = \angle EPB$. Therefore $\triangle PAD \sim \triangle PEB$. Thus we have

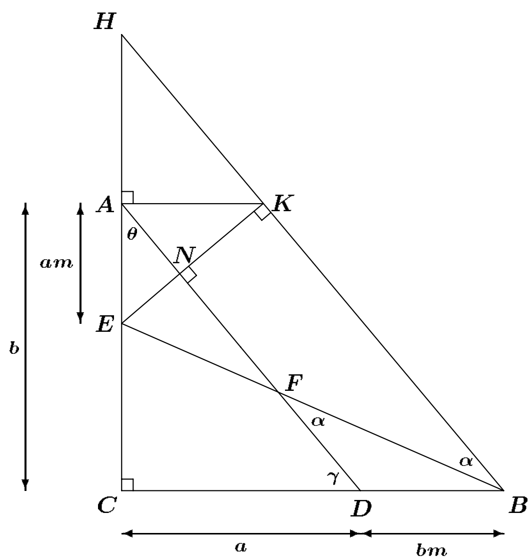
$$\angle PAD = \angle PEB. \qquad (1)$$

Let O be the intersection of AD and BE. Then we know from (1) that $\angle EOA = \angle EPA = 60^\circ$. Therefore the angle between AD and BE is 60°.



IV. *Solution by Luiz A. Ponce, Santos, Brazil.*

This is a generalization to the case where $BD = AC \cdot m$ and $AE = CD \cdot m$ for some $m > 0$. Let $AC = b$, $CD = a$, and $AD = c$. Define F to be the point of intersection of AD and BE. Let $\alpha = \angle DFB$, $\gamma = \angle CDA$, and $\theta = \angle CAD$.



Since $AE = CD \cdot m$ and $BD = AC \cdot m$, we have $AE = am$ and $BD = bm$. Since ACD is a right-angled triangle, we have

$$\gamma + \theta = 90^\circ. \tag{1}$$

Through the point B construct a line parallel to AD intersecting the line AC (extended) at H, and through the point A construct a line parallel to CB intersecting the line BH at K. It is now clear that $\angle EAK = 90^\circ$, $\angle EBK = \alpha$, $AK = BD = bm$, and

$$BK = AD = c\,. \tag{2}$$

Note that

$$\frac{AE}{CD} = \frac{am}{a} = m = \frac{bm}{b} = \frac{AK}{AC}\,.$$

Since we also have $\angle DCA = \angle EAK = 90^\circ$, it follows that $\triangle AEK \sim \triangle CDA$ (SAS). According to similarity we can write

$$EK = AD \cdot m = cm\,, \tag{3}$$

and also $\angle KEA = \angle ADC = \gamma = \angle NEA$, where N is the point of intersection of AD with EK. Moreover, in triangle ANE we have

$$\angle N + \angle NEA + \angle EAN = 180^\circ;$$

that is,

$$\angle N + \gamma + \theta = 180^\circ,$$

or

$$\angle N = 90^\circ$$

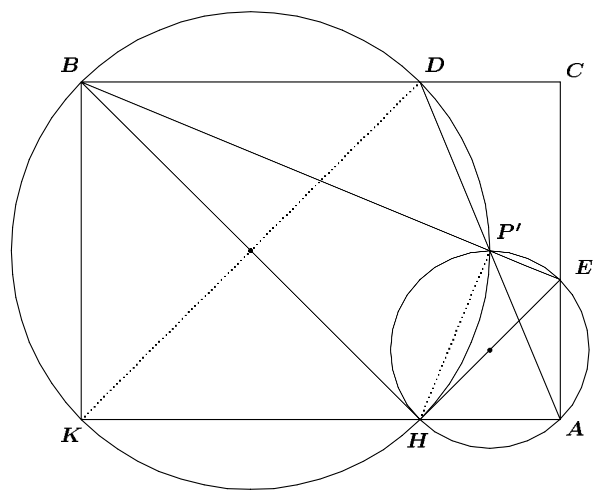
by considering (1). Thus $AD \perp EK$, which implies that $EK \perp BK$ since $AD \parallel BK$. Consequently we conclude that EKB is a right-angled triangle. This together with (2) and (3) yield

$$\tan\alpha = \tan(\angle EBK) = \frac{EK}{BK} = \frac{cm}{c} = m\,.$$

Therefore, the acute angle between AD and BE is α, such that $\tan\alpha = m$, where $m > 0$ is the given ratio.

V. *Solution by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let $r = \frac{BC}{AC}$. It is possible to locate the points D and E if and only if $\sqrt{3} < r < \frac{4}{3}\sqrt{3}$. Then the angle between AD and BE is $60°$. This is a special case of the following situation. Let ACB be a right triangle with $r = \frac{BC}{AC} > 1$. For any positive $t < r$ satisfying $t^2 - rt + 1 > 0$, let D and E be points on the segments BC and AC respectively, so that $BD = t \cdot AC$, and $AE = t \cdot CD$. Then, the angle between AD and BE is $\arctan t$.



Complete the right triangle into a rectangle $ACBK$. For $t < r$, we choose points D on BC and H on KA such that $DBKH$ is a rectangle with $DB = t \cdot KB$. Since $t^2 - rt + 1 > 0$, $AC > t(AK - t \cdot BK) = t \cdot AH$, there is a point E on AC such that $AE = t \cdot AH$. Note that HE and HB are perpendicular to each other, since the right triangles HAE and BKH are similar. Consider the circumcircles of the rectangle $DBKH$ and the right triangle AEH. The centres of these circles lie on HB and HE respectively. Let P' be the intersection of these circles other than H. Note that $HP'E$ and $HP'B$ are both right angles. This means that B, P', and E are collinear. Note also that

$$\angle BP'D = \angle BKD = \angle AHE = \angle AP'E.$$

From this we conclude that A, P', and D are collinear. This means that P' is the intersection of the segments AD and BE, and the angle between them is the same as $\angle BKD$, which is clearly $\arctan t$.

Remark. If $r \leq 2$, the condition $t^2 - rt + 1 > 0$ is always satisfied for $t < r$. For $r > 2$, this is satisfied by values of t in either of the ranges $\left(0, \frac{r-\sqrt{r^2-4}}{2}\right)$ and $\left(\frac{r+\sqrt{r^2-4}}{2}, r\right)$. For example, for $r = \frac{5}{2}$, t must satisfy either $0 < t < \frac{1}{2}$ or $2 < t < \frac{5}{2}$.

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Both Hess and Lambrou consider the case when the points D and E are external to the segments BC and AC. Neither of them consider all the possible combinations, however. Perhaps the interested reader would like to pursue this investigation.

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