THE ACADEMY CORNER

No. 20

Bruce Shawyer

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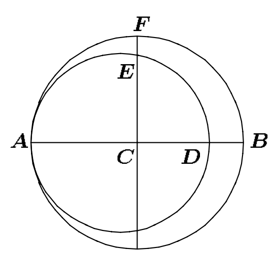
THE BERNOULLI TRIALS 1998

The Bernoulli Trials, an undergraduate mathematics competition, was held Saturday, March 7 at the University of Waterloo. This is the second year for this event, which is a double knockout competition with "true" or "false" as the answers on each round. The participants have 10 minutes for each question, and drop out after their second incorrect answer.

There were 29 student participants in the competition, which lasted 4 hours and 16 rounds. The winner was third year student Frederic Latour. Second place went to first year student Joel Kamnitzer, and third and fourth to Richard Hoshino and Derek Kisman. In keeping with the nature of the answers required, the prizes were awarded in coins: 100 toonies for first place, 100 loonies for second, and quarters for third and fourth.

Ian Goulden and Christopher Small

1. In the figure below, the two circles are tangent at A. The point C is the center of the larger circle, and FC is perpendicular to AB. The line segment DB is of length 9, and the line segment FE is of length 5.



TRUE OR FALSE? The diameter of the larger circle is less than or equal to 49.

2. One of the participants in the Bernoulli Trials answers by flipping a fair coin. He chooses "True" when the coin lands heads and "False" when the coin lands tails.

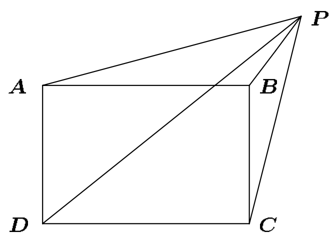
   TRUE OR FALSE? Using this method, on average he will last exactly 4 rounds until he drops out.

3. **TRUE OR FALSE?** There exist 11 distinct positive integers

$$a_1, a_2, \ldots, a_{11}$$

   so that all integers from 8 to 1998 inclusive can be written as sums over subsets of the a_i's.

4. Let $ABCD$ be a rectangle labelled clockwise as shown, and let P be any point in the plane.

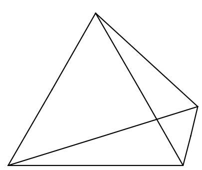
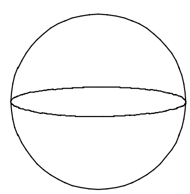


   TRUE OR FALSE?

$$|PA|^2 - |PB|^2 + |PC|^2 - |PD|^2 = 0\,.$$

   (In this expression $|EF|$ denotes the *distance* from point E to point F.)

5. **TRUE OR FALSE?** The volume of a regular tetrahedron of side 1 is less than the volume of a sphere of radius $1/\pi$.

6. **TRUE OR FALSE?** The smallest perfect square ending in 9009 is

$$126503^2 = 16003009009\,.$$

7. Let $f : [0,1] \to [0,1]$ be a continuous strictly increasing function such that $f(0) = 0$ and $f(1) = 1$.

   TRUE OR FALSE? The inequality

$$\sum_{i=1}^{9} f\left(\frac{i}{10}\right) + \sum_{i=1}^{9} f^{-1}\left(\frac{i}{10}\right) \leq \frac{99}{10}$$

   holds true for any such function.

8. Let k be the smallest positive integer such that $kt + 1$ is a triangular number whenever t is a triangular number.

   TRUE OR FALSE? There exists such a k and $k \leq 12$.

   (A number t is triangular if it can be written in the form

$$t = 1 + 2 + 3 + \cdots + n$$

   where $n \geq 1$. The first 4 triangular numbers are 1, 3, 6, and 10.)

9. **TRUE OR FALSE?** It is possible to construct a planar quadrilateral with sides of length 1, 2, 3, and 4, and with area $2\sqrt{6}$.

10. Let $f : [0, \infty) \to [0, \infty)$ be a continuous *strictly* increasing function satisfying the equation

$$f\left(\frac{x}{1+x}\right) = \frac{f(x)}{1+f(x)}$$

   for all $x \geq 0$.

   TRUE OR FALSE? Then $f(x) = x$ for all $x \geq 0$.

11. **TRUE OR FALSE?**

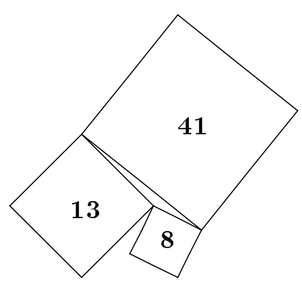
$$2^{3^{2^{3^{2^{3}}}}} > 3^{2^{3^{2^{3^{2}}}}}.$$

12. **TRUE OR FALSE?**

$$\frac{4}{3} < \int_0^\infty \frac{x\,dx}{e^x - 1} < \frac{5}{3}.$$

13. A triangle has squares constructed externally on each of its three sides. Suppose these squares have area 8, 13 and 41 square meters respectively.

   TRUE OR FALSE? The area of the triangle is greater than one square meter.



14. **TRUE OR FALSE?**

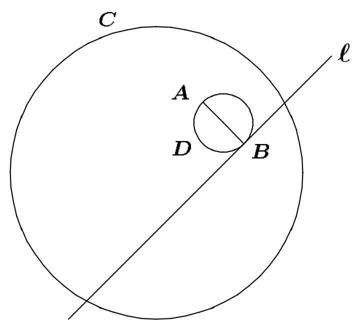
   There exists a continuous function

$$f : \mathbf{R} \to \mathbf{R}$$

   such that

$$f[f(x)] = e^x - e^{e^x}$$

15. Let A and B be two random points in the interior of a circle C. Let D be the circle with diameter AB, and let ℓ be the line tangent to D at B.

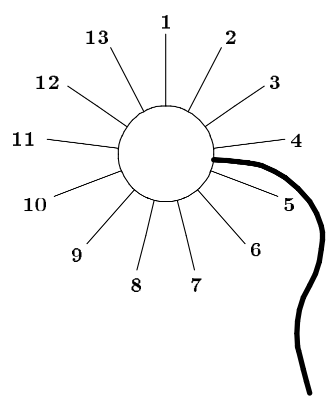


Let R_1 be the area of the region that is interior to both C and D. Let R_2 be the area of the region that is interior to C on the other side of ℓ from D.

TRUE OR FALSE? The average value of R_1 is strictly less than the average value of R_2.

16. The following game is played between two players using the petals of a daisy. Each player takes a turn plucking either one petal or two neighbouring petals from a daisy. The player who takes the last petal wins.

TRUE OR FALSE? With best play by both sides for a daisy with thirteen petals, the first player can force a win.



THE OLYMPIAD CORNER

No. 191

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

Welcome back from the "summer break". I hope all of you have been taking some time to write up your nice solutions to problems given so far. We start this number with two traditional pieces. First we give the problems of the 1998 Canadian Mathematical Olympiad which we reproduce with the permission of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society. Thanks go to Daryl Tingley, University of New Brunswick, and chair of the CMO Committee, for forwarding the questions to me.

1998 CANADIAN MATHEMATICAL OLYMPIAD

Final

November 20

1. Determine the number of real solutions a to the equation

$$\left[\frac{1}{2}\,a\right]+\left[\frac{1}{3}\,a\right]+\left[\frac{1}{5}\,a\right] \;=\; a\,.$$

Here, if x is a real number, then $[\,x\,]$ denotes the greatest integer that is less than or equal to x.

2. Find all real numbers x such that

$$x \;=\; \left(x-\frac{1}{x}\right)^{1/2}+\left(1-\frac{1}{x}\right)^{1/2}.$$

3. Let n be a natural number such that $n \geq 2$. Show that

$$\frac{1}{n+1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right) \;>\; \frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right).$$

4. Let ABC be a triangle with $\angle BAC = 40^\circ$ and $\angle ABC = 60^\circ$. Let D and E be the points lying on the sides AC and AB, respectively, such that $\angle CBD = 40^\circ$ and $\angle BCE = 70^\circ$. Let F be the point of intersection of the lines BD and CE. Show that the line AF is perpendicular to the line BC.

5. Let m be a positive integer. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 0$, $a_1 = m$, and $a_{n+1} = m^2 a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$. Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

The next set of problems is from the twenty-seventh annual United States of America Mathematical Olympiad written April 28, 1998. These problems are copyrighted by the committee on the American Mathematical Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems may be obtained from Professor Walter E. Mientka, AMC Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, USA 68588–0322. As always, we welcome your original, "nice" solutions and generalizations which differ from the published solutions.

27th UNITED STATES OF AMERICA MATHEMATICAL OLYMPIAD
Part I 9 a.m. -12 noon
April 28, 1998

1. Suppose that the set $\{1, 2, \cdots, 1998\}$ has been partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \leq i \leq 999)$ so that for all i, $|a_i - b_i|$ equals 1 or 6. Prove that the sum

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{999} - b_{999}|$$

ends in the digit 9.

2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ $(B \in \mathcal{C}_2)$. Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

3. Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

Part II   1 p.m. - 4 p.m.

4. A computer screen shows a 98×98 chessboard, coloured in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colours in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one colour.

5. Prove that for each $n \geq 2$, there is a set S of n integers such that $(a-b)^2$ divides ab for every distinct $a, b \in S$.

6. Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1A_2\dots A_n$ for which exactly k of the quadrilaterals $A_iA_{i+1}A_{i+2}A_{i+3}$ have an inscribed circle. (Here $A_{n+j} = A_j$.)

As a third Olympiad for this issue we give the 11^{th} Grade and 12^{th} Grade problems of the 1994 Latvian Mathematics Olympiad. My thanks go to Bill Sands, of the University of Calgary, who collected these problems for me when he was assisting with the 1995 International Mathematical Olympiad held in Canada.

45^{th} LATVIAN MATHEMATICAL OLYMPIAD, 1994

11^{th} Grade

1. Prove for each choice of real non-zero numbers $a_1, a_2, \dots, c_3$, the "stars" can be replaced by "<" and ">" so that the system

$$\begin{cases} a_1x + b_1y + c_1 * 0 \\ a_2x + b_2y + c_2 * 0 \\ a_3x + b_3y + c_3 * 0 \end{cases}$$

has no solution.

2. Solve in natural numbers:

$$x(x+1) = y^7$$

3. Given 4 non-coplanar points, how many parallelepipeds having these points as vertices can be constructed?

4. Let $ABCD$ be a convex quadrilateral, $M \in AB$, $N \in BC$, $P \in CD$, $Q \in DA$; $AM = BN = CP = DQ$, and $MNPQ$ is a square. Prove that $ABCD$ is a square, too.

5. A square consists of $n \times n$ cells, $n \geq 2$. A letter is inserted into each cell. It is given that every two rows differ. Prove that there is a column which can be deleted from the square so that all rows are again different after this deletion.

12^{th} Grade

1. Solve the equation $\cos x \cdot \cos 2x \cdot \cos 3x = 1$.

2. All faces of a convex polytope are triangles. What can be the number of the faces?

3. Does there exist a polynomial $P(x, y)$ in two variables such that

(a) $P(x, y) > 0$ for all x, y,

(b) for each $c > 0$ there exist such x and y that $P(x, y) = c$?

4. Let $S(x)$ be the digital sum of natural number x. Prove that $S(2^n) \to \infty$ when $n \to \infty$, n natural.

5. The centres of four equal circles are the vertices of a square. How must A, B, C, D be chosen so that each circle contains at least one of them and the area of $ABCD$ is as big as possible?

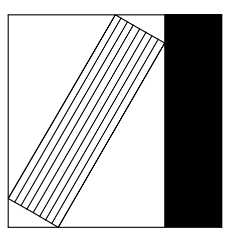
As a final set of problems for you to puzzle over after the hiatus, we give the problems of the Dutch Mathematical Olympiad, Second Round, written 16 September 1994. My thanks again go to Bill Sands for collecting these problems for me while he was helping out at the IMO in Toronto.

DUTCH MATHEMATICAL OLYMPIAD

Second Round

16 September, 1994

1. A unit square is divided in two rectangles in such a way that the smaller rectangle can be put on the greater rectangle with every vertex of the smaller on exactly one of the edges of the greater.



Calculate the dimensions of the smaller rectangle.

2. Given is a sequence of numbers $a_1, a_2, a_3, \ldots$ with the property:

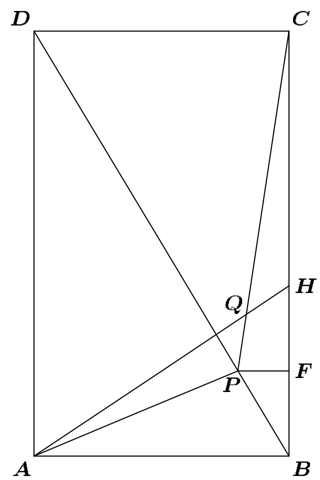
$$a_1 = 2,\ a_2 = 3 \text{ and } \begin{cases} a_{n+1} = 2a_{n-1} & \text{or} \\ a_{n+1} = 3a_n - 2a_{n-1} \end{cases} \quad \text{for all } n \geq 2.$$

Prove that no number between 1600 and 2000 can be an element of the sequence.

3. (a) Prove that every multiple of 6 can be written as the sum of four third powers of integers.

(b) Prove that every integer can be written as the sum of five third powers of integers.

4. Let P be any point on the diagonal BD of a rectangle $ABCD$. F is the projection of P on BC. H lies on BC such that $BF = FH$. PC intersects AH in Q.



Prove: Area $\Delta APQ =$ Area ΔCHQ.

5. Three real numbers a, b and c satisfy the inequality:

$$\left|ax^2 + bx + c\right| \leq 1 \quad \text{for all } x \in [-1, +1].$$

Prove: $\left|cx^2 + bx + a\right| \leq 2$ for all $x \in [-1, +1]$.

We now give the solutions to the Canadian Mathematical Olympiad given earlier this number. (I hope you have already solved them all!) These "official solutions" were selected from the most interesting student solutions by Daryl Tingley, University of New Brunswick and Chair of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society. Of course the original submissions have been somewhat edited.

1998 CANADIAN MATHEMATICAL OLYMPIAD
Solutions of Students

1. *Solution by David Arthur, Upper Canada College, Toronto, Ontario.*

Let $a = 30k + r$, where k is an integer and r is a real number between 0 and 29 inclusive.

Then $\left[\frac{1}{2}\,a\right] = \left[\frac{1}{2}\,(30k + r)\right] = 15k + \left[\frac{r}{2}\right]$. Similarly $\left[\frac{1}{3}\,a\right] = 10k + \left[\frac{r}{3}\right]$ and $\left[\frac{1}{5}\,a\right] = 6k + \left[\frac{r}{5}\right]$.

Now, $\left[\frac{1}{2}\,a\right]+\left[\frac{1}{3}\,a\right]+\left[\frac{1}{5}\,a\right]\;=\;a$, so $(15k+\left[\frac{r}{2}\right])+(10k+\left[\frac{r}{3}\right])+(6k+\left[\frac{r}{5}\right])=30k+r$, and hence $k=r-\left[\frac{r}{2}\right]-\left[\frac{r}{3}\right]-\left[\frac{r}{5}\right]$.

Clearly, r has to be an integer, or $r-\left[\frac{r}{2}\right]-\left[\frac{r}{3}\right]-\left[\frac{r}{5}\right]$ will not be an integer, and therefore, cannot equal k.

On the other hand, if r is an integer, then $r-\left[\frac{r}{2}\right]-\left[\frac{r}{3}\right]-\left[\frac{r}{5}\right]$ will also be an integer, giving exactly one solution for k.

For each r $(0\le r\le 29)$, $a=30k+r$ will have a different remainder $\bmod 30$, so no two different values of r give the same result for a.

Since there are 30 possible values for r $(0,1,2,\ldots,29)$, there are then 30 solutions for a.

2. *Solution by Jimmy Chui, Earl Haig Secondary School, North York, Ontario.*

Since $(x-\frac{1}{x})^{1/2}\ge 0$ and $(1-\frac{1}{x})^{1/2}\ge 0$, then $0\le (x-\frac{1}{x})^{1/2}+(1-\frac{1}{x})^{1/2}=x$.

Note that $x\neq 0$. Else, $\frac{1}{x}$ would not be defined, so $x>0$.

Squaring both sides gives,

$$\begin{aligned} x^2 &= \left(x-\frac{1}{x}\right)+\left(1-\frac{1}{x}\right)+2\sqrt{\left(x-\frac{1}{x}\right)\left(1-\frac{1}{x}\right)}\\ x^2 &= x+1-\frac{2}{x}+2\sqrt{x-1-\frac{1}{x}+\frac{1}{x^2}}. \end{aligned}$$

Multiplying both sides by x and rearranging, we get

$$\begin{aligned} x^3-x^2-x+2 &= 2\sqrt{x^3-x^2-x+1}\\ (x^3-x^2-x+1)-2\sqrt{x^3-x^2-x+1}+1 &= 0\\ (\sqrt{x^3-x^2-x+1}-1)^2 &= 0\\ \sqrt{x^3-x^2-x+1} &= 1\\ x^3-x^2-x+1 &= 1\\ x(x^2-x-1) &= 0\\ x^2-x-1 &= 0 \qquad \text{since } x\neq 0. \end{aligned}$$

Thus $x=\frac{1\pm\sqrt{5}}{2}$. We must check to see if these are indeed solutions.

Let $\alpha=\frac{1+\sqrt{5}}{2}$, $\beta=\frac{1-\sqrt{5}}{2}$. Note that $\alpha+\beta=1$, $\alpha\beta=-1$ and $\alpha>0>\beta$.

Since $\beta < 0$, β is not a solution.

Now, if $x = \alpha$, then

$$\begin{aligned}
&\left(\alpha - \frac{1}{\alpha}\right)^{1/2} + \left(1 - \frac{1}{\alpha}\right)^{1/2} \\
&= (\alpha+\beta)^{1/2} + (1+\beta)^{1/2} \quad (\text{since } \alpha\beta = -1) \\
&= 1^{1/2} + (\beta^2)^{1/2} \quad (\text{since } \alpha+\beta = 1 \text{ and } \beta^2 = \beta + 1) \\
&= 1 - \beta \quad (\text{since } \beta < 0) \\
&= \alpha \quad (\text{since } \alpha + \beta = 1).
\end{aligned}$$

So $x = \alpha$ is the unique solution to the equation.

3. *Solution 1 by Chen He, Columbia International Collegiate, Hamilton, Ontario.*

$$1 + \frac{1}{3} + \ldots + \frac{1}{2n-1} = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \qquad (1)$$

Since

$$\frac{1}{3} > \frac{1}{4}, \ \frac{1}{5} > \frac{1}{6}, \ \ldots, \frac{1}{2n-1} > \frac{1}{2n},$$

(1) gives

$$\begin{aligned}
1 + \frac{1}{3} + \ldots + \frac{1}{2n-1} &> \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} \qquad (2)\\
+ \ldots + \frac{1}{2n} &= \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n}\right).
\end{aligned}$$

Since

$$\frac{1}{2} > \frac{1}{4}, \ \frac{1}{2} > \frac{1}{6}, \ \frac{1}{2} > \frac{1}{8}, \ \ldots, \ \frac{1}{2} > \frac{1}{2n}$$

then

$$\frac{n}{2} = \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}}_{n} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n}$$

so that

$$\frac{1}{2} > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n}\right). \qquad (3)$$

Then (1), (2) and (3) show that

$$\begin{aligned} & 1+\frac{1}{3}+\ldots+\frac{1}{2n-1} \\ & > \frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right) \\ & = \left(1+\frac{1}{n}\right)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right) \\ & = \frac{n+1}{n}\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right). \end{aligned}$$

Therefore $\frac{1}{n+1}\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)$ for all $n \in N$ and $n \geq 2$.

Solution 2 by Yin Lei, Vincent Massey Secondary School, Windsor, Ontario.

Since $n \geq 2$, $n(n+1) \geq 0$. Therefore the given inequality is equivalent to

$$n\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right) \geq (n+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right).$$

We shall use mathematical induction to prove this.

For $n=2$, obviously $\frac{1}{3}\left(1+\frac{1}{3}\right)=\frac{4}{9}>\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}\right)=\frac{3}{8}$.

Suppose that the inequality stands for $n=k$; that is,

$$k\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right) > (k+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right). \qquad (1)$$

Now we have to prove it for $n=k+1$.

We know

$$\begin{aligned} & \left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right) \\ & = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right) \\ & = \frac{1}{1\times 2}+\frac{1}{3\times 4}+\frac{1}{5\times 6}+\ldots+\frac{1}{(2k-1)(2k)}. \end{aligned}$$

Since

$$1\times 2 < 3\times 4 < 5\times 6 < \ldots < (2k-1)(2k) < (2k+1)(2k+2)$$

then

$$\frac{1}{1\times 2}+\frac{1}{3\times 4}+\ldots+\frac{1}{(2k-1)(2k)} > \frac{k}{(2k+1)(2k+2)}$$

Hence

$$1+\frac{1}{3}+\ldots+\frac{1}{2k-1} > \frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}+\frac{k}{(2k+1)(2k+2)}. \qquad (2)$$

Also

$$\frac{k+1}{2k+1}-\frac{k+2}{2k+2}$$
$$= \frac{2k^2+2k+2k+2-2k^2-4k-k-2}{(2k+1)(2k+2)} = -\frac{k}{(2k+1)(2k+2)}.$$

Therefore

$$\frac{k+1}{2k+1} = \frac{k+2}{2k+2}-\frac{k}{(2k+1)(2k+2)}. \qquad (3)$$

Adding 1, 2 and 3 gives

$$\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\frac{k+1}{2k+1}$$
$$> (k+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)$$
$$+\frac{k}{(2k+1)(2k+2)}+\frac{k+2}{2k+2}-\frac{k}{(2k+1)(2k+2)}$$

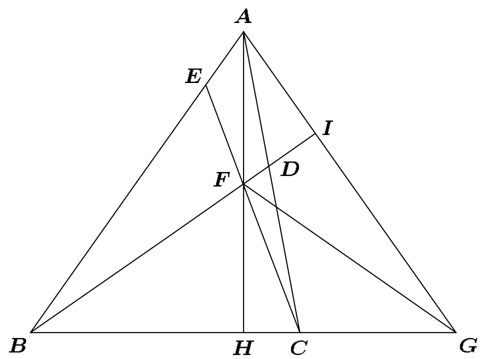
Rearranging both sides, we get

$$(k+1)\left(1+\frac{1}{3}+\ldots+\frac{1}{2k+1}\right) > (k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k+2}\right).$$

This proves the induction.

4. *Solution 1 by Keon Choi, A.Y. Jackson Secondary School, North York, Ontario.*

Suppose H is the foot of the perpendicular line from A to BC; construct equilateral $\triangle ABG$, with C on BG. I will prove that if F is the point where AH meets BD, then $\angle FCB = 70°$. (Because that means AH, and the given lines BD and CE meet at one point, this answers the question.) Suppose BD extended meets AG at I.



Now $BF = GF$ and $\angle FBG = \angle FGB = 40^\circ$, so that $\angle IGF = 20^\circ$. Also $\angle IFG = \angle FBG + \angle FGB = 80^\circ$, so that

$$\begin{aligned}\angle FIG &= 180^\circ - \angle IFG - \angle IGF \\ &= 180^\circ - 80^\circ - 20^\circ \\ &= 80^\circ .\end{aligned}$$

Therefore $\triangle GIF$ is an isosceles triangle, so

$$GI = GF = BF . \qquad (1)$$

But $\triangle BGI$ and $\triangle ABC$ are congruent, since $BG = AB$, $\angle GBI = \angle BAC$, $\angle BGI = \angle ABC$.

Therefore

$$GI = BC . \qquad (2)$$

From (1) and (2) we get

$$BC = BF .$$

So in $\triangle BCF$,

$$\angle BCF = \frac{180^\circ - \angle FBC}{2} = \frac{180^\circ - 40^\circ}{2} = 70^\circ .$$

Thus $\angle FCB = 70^\circ$ and that proves that the given lines CE and BD and the perpendicular line AH meet at one point.

Solution 2 by Adrian Birka, Lakeshore Catholic High School, Port Colborne, Ontario.

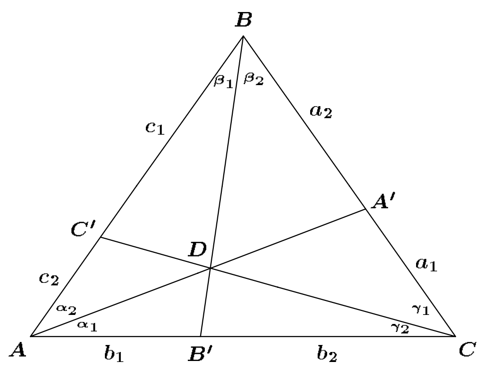
First we prove the following lemma:

In $\triangle ABC$, AA', BB', CC' intersect if and only if

$$\frac{\sin\alpha_1}{\sin\alpha_2}\cdot\frac{\sin\beta_1}{\sin\beta_2}\cdot\frac{\sin\gamma_1}{\sin\gamma_2} = 1,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are as shown in the diagram just below.

[*Editor*: This is a known variant of Ceva's Theorem.]



Proof: Let $\angle BB'C = x$; then $\angle BB'A = 180° - x$. Using the Sine Law in $\triangle BB'C$ yields

$$\frac{b_2}{\sin\beta_2} = \frac{a}{\sin x}. \tag{1}$$

Similarly using the Sine Law in $\triangle BB'A$ yields

$$\frac{b_1}{\sin\beta_1} = \frac{c}{\sin(180° - x)} = \frac{c}{\sin x}. \tag{2}$$

Hence,

$$b_1 : b_2 = \frac{c\sin\beta_1}{a\sin\beta_2} \tag{3}$$

(from (1),(2)). [*Editor*: Do you recognize this when $\beta_1 = \beta_2$?]
Similarly,

$$a_1 : a_2 = \frac{b\sin\alpha_1}{c\sin\alpha_2}, \quad c_1 : c_2 = \frac{a\sin\gamma_1}{b\sin\gamma_2}. \tag{4}$$

By Ceva's Theorem, the necessary and sufficient condition for AA', BB', CC' to intersect is: $(a_1 : a_2)\cdot(b_1 : b_2)\cdot(c_1 : c_2) = 1$. Using (3), (4) on this yields:

$$\frac{b}{c} \cdot \frac{\sin\alpha_1}{\sin\alpha_2} \cdot \frac{a}{b} \cdot \frac{\sin\gamma_1}{\sin\gamma_2} \cdot \frac{c}{a} \cdot \frac{\sin\beta_1}{\sin\beta_2} \;=\; 1\,,$$

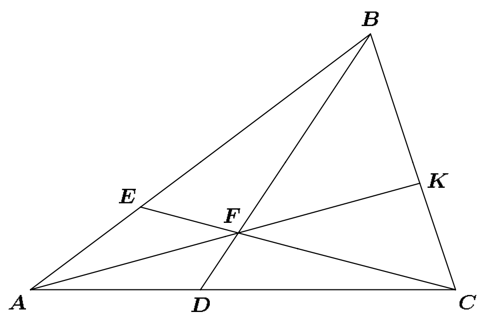
so that

$$\frac{\sin\alpha_1}{\sin\alpha_2} \cdot \frac{\sin\beta_1}{\sin\beta_2} \cdot \frac{\sin\gamma_1}{\sin\gamma_2} \;=\; 1\,. \tag{5}$$

This is just what we needed to show; therefore the lemma is proved.

Now, in our original question, give $\angle BAC = 40^\circ$, $\angle ABC = 60^\circ$. It follows that $\angle ACB = 80^\circ$.

Since $\angle CBD = 40^\circ$, $\angle ABD = \angle ABC - \angle DBC = 20^\circ$. Similarly, $\angle ECA = 20^\circ$.



Now let us show that $\angle FAD = 10^\circ$. Suppose otherwise. Let F' be such that F, F' are in the same side of AC and $\angle DAF' = 10^\circ$. Then $\angle BAF' = \angle BAC - \angle DAF' = 30^\circ$.

Thus

$$\begin{aligned}\frac{\sin\angle ABD}{\sin\angle DBC} \cdot \frac{\sin\angle BCE}{\sin\angle ECA} \cdot \frac{\sin\angle CAF'}{\sin\angle F'AB} &= \frac{\sin 20^\circ}{\sin 40^\circ} \cdot \frac{\sin 70^\circ}{\sin 10^\circ} \cdot \frac{\sin 10^\circ}{\sin 30^\circ} \\ &= \frac{\sin 20^\circ}{2\sin 20^\circ \cos 20^\circ} \cdot \frac{\cos 20^\circ}{\sin 30^\circ} \\ &= \frac{1}{2\sin 30^\circ} \;=\; 1\,.\end{aligned}$$

By the lemma above, AF' passes through $CE \cap BD = F$. Therefore $AF' = AF$, and $\angle FAD = 10^\circ$, contrary to assumption. Thus $\angle FAD$ must be 10°. Now let $AF \cap BC = K$. Since $\angle KAC = 10^\circ$, $\angle KCA = 80^\circ$, it follows that $\angle AKC = 90^\circ$. Therefore $AK \perp BC \Rightarrow AF \perp BC$ as needed.

5. *Solution by Adrian Chan, Upper Canada College, Toronto, Ontario.*

Let us first prove by induction that $\frac{a_n^2+a_{n+1}^2}{a_n \cdot a_{n+1}+1} = m^2$ for all $n \geq 0$.

Proof: Base Case $(n = 0)$: $\frac{a_0^2+a_1^2}{a_0 \cdot a_1+1} = \frac{0+m^2}{0+1} = m^2$.

Now, let us assume that it is true for $n = k$, $k \geq 0$. Then,

$$\begin{aligned}
\frac{a_k^2 + a_{k+1}^2}{a_k \cdot a_{k+1} + 1} &= m^2 \\
a_k^2 + a_{k+1}^2 &= m^2 \cdot a_k \cdot a_{k+1} + m^2
\end{aligned}$$

$$\begin{aligned}
a_{k+1}^2 + m^4 a_{k+1}^2 - 2m^2 \cdot a_k \cdot a_{k+1} + a_k^2 & \\
&= m^2 + m^4 a_{k+1}^2 - m^2 \cdot a_k \cdot a_{k+1} \\
a_{k+1}^2 + (m^2 a_{k+1} - a_k)^2 &= m^2 + m^2 a_{k+1}(m^2 a_{k+1} - a_k) \\
a_{k+1}^2 + a_{k+2}^2 &= m^2 + m^2 \cdot a_{k+1} \cdot a_{k+2}\,.
\end{aligned}$$

Therefore $\frac{a_{k+1}^2 + a_{k+2}^2}{a_{k+1} \cdot a_{k+2} + 1} = m^2$, proving the induction. Hence (a_n, a_{n+1}) is a solution to $\frac{a^2+b^2}{ab+1} = m^2$ for all $n \geq 0$.

Now, consider the equation $\frac{a^2+b^2}{ab+1} = m^2$ and suppose $(a, b) = (x, y)$ is a solution with $0 \leq x \leq y$. Then

$$\frac{x^2 + y^2}{xy + 1} = m^2\,. \tag{1}$$

If $x = 0$ then it is easily seen that $y = m$, so $(x, y) = (a_0, a_1)$. Since we are given $x \geq 0$, suppose now that $x > 0$.

Let us show that $y \leq m^2 x$.

Proof by contradiction: Assume that $y > m^2 x$. Then $y = m^2 x + k$ where $k \geq 1$.

Substituting into (1) we get

$$\begin{aligned}
\frac{x^2 + (m^2 x + k)^2}{(x)(m^2 x + k) + 1} &= m^2 \\
x^2 + m^4 x^2 + 2m^2 xk + k^2 &= m^4 x^2 + m^2 kx + m^2 \\
(x^2 + k^2) + m^2(kx - 1) &= 0.
\end{aligned}$$

Now, $m^2(kx - 1) \geq 0$ since $kx \geq 1$ and $x^2 + k^2 \geq x^2 + 1 \geq 1$ so $(x^2 + k^2) + m^2(kx - 1) \neq 0$.

Thus we have a contradiction, so $y \leq m^2 x$ if $x > 0$.

Now substitute $y = m^2 x - x_1$, where $0 \leq x_1 < m^2 x$, into (1).

We have

$$\begin{aligned}\frac{x^2+(m^2x-x_1)^2}{x(m^2x-x_1)+1} &= m^2\\ x^2+m^4x^2-2m^2x\cdot x_1+x_1^2 &= m^4x^2-m^2x\cdot x_1+m^2\\ x^2+x_1^2 &= m^2(x\cdot x_1+1)\\ \frac{x^2+x_1^2}{x\cdot x_1+1} &= m^2\,. \qquad (2)\end{aligned}$$

If $x_1 = 0$, then $x^2 = m^2$. Hence $x = m$ and $(x_1, x) = (0, m) = (a_0, a_1)$. But $y = m^2x - x_1 = a_2$, so $(x, y) = (a_1, a_2)$. Thus suppose $x_1 > 0$.

Let us now show that $x_1 < x$.

Proof by contradiction: Assume $x_1 \geq x$.

Then $m^2x - y \geq x$ since $y = m^2x - x_1$, and $\left(\frac{x^2+y^2}{xy+1}\right) x - y \geq x$ since (x, y) is a solution to $\frac{a^2+b^2}{ab+1} = m^2$.

So $x^3 + xy^2 \geq x^2y + xy^2 + x + y$. Hence $x^3 \geq x^2y + x + y$, which is a contradiction since $y \geq x > 0$.

With the same proof that $y \leq m^2x$, we have $x \leq m^2x_1$. So the substitution $x = m^2x_1 - x_2$ with $x_2 \geq 0$ is valid.

Substituting $x = m^2x_1 - x_2$ into (2) gives $\frac{x_1^2+x_2^2}{x_1\cdot x_2+1} = m^2$.

If $x_2 \neq 0$, then we continue with the substitution $x_i = m^2_{x_{i+1}} - x_{i+2}$ (*) until we get $\frac{x_j^2+x_{j+1}^2}{x_j\cdot x_{j+1}+1} = m^2$ and $x_{j+1} = 0$. (The sequence x_i is decreasing, non-negative and integer.)

So, if $x_{j+1} = 0$, then $x_j^2 = m^2$ so $x_j = m$ and $(x_{j+1}, x_j) = (0, m) = (a_0, a_1)$.

Then $(x_j, x_{j-1}) = (a_1, a_2)$ since $x_{j-1} = m^2x_j - x_{j+1}$ (from (*)).

Continuing, we have $(x_1, x) = (a_{n-1}, a_n)$ for some n. Then $(x, y) = (a_n, a_{n+1})$.
Hence $\frac{a^2+b^2}{ab+1} = m^2$ has solutions (a, b) if and only if $(a, b) = (a_n, a_{n+1})$ for some n.

That is all we have room for this issue. Enjoy solving the problems — and send me your nice solutions as well as Olympiad contests for use in the corner.

BOOK REVIEWS

Edited by ANDY LIU

Models that Work, by **Alan Tucker**, published by Mathematical Association of America, 1995, ISBN# 0-88385-096-6, softcover, 88+ pages, US$24.00.
Reviewed by ***Jim Timourian****, University of Alberta.*

At the University of Alberta, like many other institutions, we have large numbers of students enrolling in first year calculus, but very few (outside of engineering students) taking more advanced courses. The number of students choosing a mathematical science as a major is small, given the number of students who attend the university.

This report documents mathematics programs at a variety of institutions that are very successful in attracting mathematics majors and also students for advanced mathematics courses from other programs. In spite of the fact that the effective undergraduate programs studied are at diverse institutions, from two year community colleges to Ph.D.-granting institutions with outstanding reputations for graduate education, the investigators found a lot in common. Most of the report describes that commonality, while the rest discusses specific site visits and answers to a set of questions.

An effective program is considered to be one that succeeds in attracting large numbers of students as majors, or prepares students to pursue advanced study in mathematics, or prepares future teachers or attracts and prepares under-represented groups in mathematics.

The departments are unified by some underlying philosophies, but otherwise they use diverse methods to achieve their goals. Even within departments there is a variety of approaches used by different faculty members.

The report explicitly lists themes that are part of the department culture in each of the cases studied. These are respecting students, caring about their welfare, and enjoying the role of being college instructors. The respect for the students is characterized by the comment "teaching for the students one has, not the students one wished one had." This is mirrored in the curriculum "... geared toward the needs of the students, not the values of the faculty."

The most important part of a successful department's culture is not explicitly mentioned in the report but is obvious throughout. As a whole a department has to adopt the philosophy that it is a good thing for more students to take more courses in the mathematical sciences. In many departments this idea is not shared by enough of the staff members. There are those who think "there are no jobs" so we should not be attracting students. Others think that only the very best students, who have the potential to go on to graduate school in a mathematical science, are worthy and that the rest are "weak" and can be ignored.

A department that wants to improve its programs needs a sense of what it is trying to achieve. Read this report to learn what comparable institutions have done and how they have gone about doing it.

Pythagoras Strikes Again!

K.R.S. Sastry

Is it possible that a given triangle is similar to the triangle formed from its medians? In other words, is it possible that the side lengths of a triangle are proportional in some order to its medians' triangle? For a non-trivial example note the triangle with side lengths $(a, b, c) = (23, 7, 17)$ and the medians' lengths $(m_a, m_b, m_c) = \left(7\frac{\sqrt{3}}{2}, 23\frac{\sqrt{3}}{2}, 17\frac{\sqrt{3}}{2}\right)$ in which $a : m_b = b : m_a = c : m_c$ is possible. You may use the formulas

$$4m_a^2 = 2b^2 + 2c^2 - a^2 \tag{1}$$

etc. to calculate the lengths of the medians in the above example. The formulas (1) themselves are derived by means of the cosine rule. The type of triangle as described exists and shall be called a **self-median** triangle [1].

Is it possible, this time, that a given triangle is similar to the triangle formed from its altitudes? That is, the lengths of the sides of a triangle are proportional in some order to the lengths of its altitudes? Again, trivially it's so in an equilateral triangle but non-trivially we have the triangle with the lengths of the sides $(a, b, c) = (6, 9, 4)$ and the lengths of the altitudes $(h_a, h_b, h_c) = \left(\frac{2\triangle}{6}, \frac{2\triangle}{9}, \frac{2\triangle}{4}\right)$, $\triangle$ being the area of the triangle. Observe that $a : h_a = b : h_c = c : h_b$ holds. This type of triangle too exists and matching the description of self-median triangles we will call the present type **self-altitude** triangles [2].

Interestingly, as we shall see, with the exception of the equilateral triangle, both of these triangle types are simply reincarnations of appropriate right triangles. The theorems that follow illustrate this fact.

Theorems on self-median triangles

Theorem 1 tells us how an appropriate right triangle yields a self-median triangle.

THEOREM 1: Let (a_0, b_0, c_0) be a right triangle in which $a_0^2 + b_0^2 = c_0^2$, $a_0 > b_0$ and $c_0 > 2b_0$ hold. Then $(a, b, c) = (a_0 + b_0, a_0 - b_0, c_0)$ is a self-median triangle in which $a > c > b$ holds.

Proof: Let m_1, m_2, m_3 denote the lengths of the medians drawn to the sides $a_0 + b_0, a_0 - b_0, c_0$ respectively. Then from (1) we can easily deduce that

$$4m_1^2 = 3(a_0 - b_0)^2, 4m_2^2 = 3(a_0 + b_0)^2, 4m_3^2 = 3c_0^2.$$

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Hence
$$\frac{a_0+b_0}{m_2}=\frac{a_0-b_0}{m_1}=\frac{c_0}{m_3}=\frac{2}{\sqrt{3}}.$$
It should be noted that the lengths a_0+b_0, a_0-b_0, c_0 do not form a triangle for every given right triangle (a_0, b_0, c_0). For example $(a_0, b_0, c_0) = (4, 3, 5)$ yields $(a_0+b_0, a_0-b_0, c_0) = (7, 1, 5)$, and there is no triangle with these side lengths. However $(a_0, b_0, c_0) = (12, 5, 13)$ yields the self-median triangle $(a_0+b_0, a_0-b_0, c_0) = (17, 7, 13)$. In order to assure the formation of a self-median triangle, the lengths a_0+b_0, a_0-b_0, c_0 have to satisfy the triangle inequality $(a_0-b_0)+c_0 > a_0+b_0$. This simplifies to $c_0 > 2b_0$. In other words the hypotenuse must exceed twice the shorter leg to give us a self-median triangle. It is trivial to check that the other two triangle inequalities are satisfied. Moreover, in the right triangle (a_0, b_0, c_0), we have $a_0+b_0 > c_0 > a_0-b_0$ which is the same as $a > c > b$.

Theorem 2 tells us how to recover the right triangle from a given self-median triangle.

THEOREM 2: Let (a, b, c) be a self-median triangle in which $a > c > b$ holds. Then $(a_0, b_0, c_0) = (\frac{1}{2}(a+b), \frac{1}{2}(a-b), c)$ is a right triangle in which $a_0^2+b_0^2 = c_0^2$, $a_0 > b_0$ and $c_0 > 2b_0$ hold.

Proof: Since $a > c$, we find that $r(m_a^2 - m_c^2) = 3(c^2 - a^2) < 0$, or, $m_a < m_c$. Thus $a > c > b \iff m_b > m_c > m_a$. From this and the definition of self-median triangle we have $\frac{a}{m_b} = \frac{b}{m_a} = \frac{c}{m_c}$. We square two of these equations, use the formulas (1) and simplify. This yields $(a^2-b^2)(a^2+b^2-2c^2) = 0$. The hypothesis $a > c > b$ shows that this is equivalent to $a^2+b^2 = 2c^2$. We consistently arrive at this same equation (no matter which two equations are squared) which is the same as $\left[\frac{1}{2}(a+b)\right]^2 + \left[\frac{1}{2}(a-b)\right]^2 = c^2$ or $a_0^2+b_0^2 = c_0^2$, with $a_0 > b_0$. Now in triangle (a, b, c), we have $c > a-b$ so that $c_0 > 2b_0$ holds and the proof is complete.

Remark 1: The proof of Theorem 2 says much more than its statement. Since the argument is reversible it has given **two** characterizations of self-median triangles: Suppose the side lengths of triangle (a, b, c) satisfy $a > c > b$. Then the triangle is self-median (i) if and only if $a : m_b = b : m_a = c : m_c$ or (ii) if and only if $a^2+b^2 = 2c^2$. Also, it is easy to see that if triangle (a, b, c) has $a : m_a = b : m_b = c : m_c$ then it **must** be equilateral. This useful observation enables us to give an elegant proof of Theorem 3.

Theorems 1 and 2 considered the determination of self-median triangles in each of which the sides have distinct length measures. Theorem 3 shows that if an isosceles triangle is self-median then it must be equilateral.

THEOREM 3: The equilateral triangle is the only self-median triangle not covered by the Theorems 1 and 2.

Proof: Any self-median triangle (a, b, c) not covered by the Theorems 1 and 2 must have at least two sides equal, say $a = b$. Then $m_a = m_b$. Let

m_1, m_2, m_3 denote the medians m_a, m_b, m_c in some order so that the definition of a self-median triangle enables us to write

$$\frac{a}{m_1} = \frac{b}{m_2} = \frac{c}{m_3}.$$

If $m_1 = m_c$ then $m_2 = m_3 = m_a = m_b$. But then $a = b$ forces $m_a = m_b = m_c$ and hence the triangle is equilateral. If $m_1 = m_2 = m_a = m_b$ then $m_3 = m_c$. This again forces the triangle to be equilateral as mentioned in Remark 1.

Exercise 1: Find the right triangle that generates the self-median triangle $(23, 7, 17)$.

Exercise 2: Suppose (a, b, c) is a right triangle with $a^2 + b^2 = c^2$. Show that the triangle $(a\sqrt{2}, b\sqrt{2}, c)$ is self-median.

Exercise 3: Show that there are exactly two self-median triangles in each of which (i) all the side lengths are integers and (ii) two side lengths are 7 and 17.

Theorems on self-altitude triangles

Theorem 4 tells us how an appropriate right triangle yields a self-altitude triangle.

THEOREM 4: Let (a_0, b_0, c_0) be a right triangle in which $a_0^2 + b_0^2 = c_0^2$ and $a_0 > 2b_0$ hold. Then $(a, b, c) = (a_0, c_0+b_0, c_0-b_0)$ is a self-altitude triangle in which $b > a > c$ holds.

Proof: Let h_a, h_b, h_c denote the altitudes to the sides a, b, c and $\triangle$ denote the area of this triangle. Then $ah_a = bh_b = ch_c = 2\triangle$ or

$$a_0h_a = (c_0 + b_0)h_b = (c_0 - b_0)h_c = 2\triangle\ .$$

Now

$$\frac{a_0}{h_a} = \frac{a_0^2}{a_0h_a} = \frac{a_0^2}{2\triangle} = \frac{c_0^2 - b_0^2}{(c_0 - b_0)h_c} = \frac{c_0^2 - b_0^2}{(c_0 + b_0)h_b}$$

or

$$\frac{a_0}{h_a} = \frac{c_0 + b_0}{h_c} = \frac{c_0 - b_0}{h_b} \Longleftrightarrow \frac{a}{h_a} = \frac{b}{h_c} = \frac{c}{h_b}$$

implying that the triangle (a, b, c) is self-altitude. Furthermore, in triangle (a_0, b_0, c_0), we have that $c_0 + b_0 > a_0 > c_0 - b_0$ holds. This in triangle (a, b, c) is equivalent to $b > a > c$.

Again, not any right triangle (a_0, b_0, c_0) yields a self-altitude triangle. For example, $(a_0, b_0, c_0) = (3, 4, 5)$ yields $(a, b, c) = (3, 9, 1)$ and there is no triangle with these sidelengths. However, $(a_0, b_0, c_0) = (12, 5, 13)$ yields $(a, b, c) = (12, 18, 8)$. We may divide these side lengths by their gcd 2 and take the primitive self-altitude triangle $(6, 9, 4)$. Here the triangle inequality to be satisfied is $a_0 + (c_0 - b_0) > c_0 + b_0$ which simplifies to $a_0 > 2b_0$. It is trivial to check that the other two triangle inequalities are satisfied.

Theorem 5 tells us how to recover the right triangle from a given self-altitude triangle.

THEOREM 5: Let (a, b, c) be a self-altitude triangle in which $b > a > c$ holds. Then

$$(a_0, b_0, c_0) = \left(a, \frac{1}{2}(b-c), \frac{1}{2}(b+c)\right)$$

is a right triangle in which $a_0^2 + b_0^2 = c_0^2$ and $a_0 > 2b_0$ hold.

Proof: From the hypothesis $b > a > c$ we have $h_b < h_a < h_c$. From the definition of self-altitude triangle we have

$$\frac{a}{h_a} = \frac{b}{h_c} = \frac{c}{h_b} \Longleftrightarrow \frac{a^2}{2\triangle} = \frac{bc}{2\triangle} = \frac{bc}{2\triangle},$$

where $\triangle$ is the area of triangle (a, b, c). Thus

$$a^2 = bc = \left[\frac{1}{2}(b+c)\right]^2 - \left[\frac{1}{2}(b-c)\right]^2$$

which is $a_0^2 + b_0^2 = c_0^2$. From $a > b - c$ follows $a_0 > 2b_0$ as required.

Remark 2: The proof of Theorem 5 gives two characterizations of self-altitude triangles: Suppose the side lengths of triangle (a, b, c) satisfy $b > a > c$. Then the triangle is self-altitude

(i) if and only if $a : h_a = b : h_c = c : h_b$, or

(ii) if and only $a^2 = bc$.

It is easy to see that if in triangle (a, b, c) we have $a : h_a = b : h_b = c : h_c$ then it must be equilateral. This observation enables us to give an elegant proof of Theorem 6.

Theorems 4 and 5 considered the determination of self-altitude triangles in each of which the sides have distinct length measures. Theorem 6 shows that if an isosceles triangle is self-altitude then it must be equilateral.

THEOREM 6: The equilateral triangle is the only self-altitude triangle not covered by the Theorems 4 and 5.

Proof: We omit. It is similar to that of Theorem 3.

Exercise 4: Find the right triangle that generates the self-altitude triangle $(35, 49, 25)$.

Exercise 5: Let ABC be a right triangle with right angle at C. CD is drawn perpendicular to AB with the point D on AB. Prove that the triangle whose side lengths are AD, DB, CD is self-altitude. In terms of a, b, c give an answer to the question: when do the lengths AD, DB, CD form a triangle?

Remarks 1 and 2 suggest the following.

OPEN PROBLEM: Suppose AD, BE, CF are three concurrent cevians of triangle ABC. Assume that $\frac{BC}{AD} = \frac{CA}{BE} = \frac{AB}{CF}$ holds. Prove or disprove that the triangle ABC is equilateral.

Acknowledgement: I thank the referee for his painstaking efforts to improve the presentation. He also suggested Theorems 3 and 6.

References

[1] K.R.S. Sastry, Self-Median Triangles, *Mathematical Spectrum*, **22** (1989/90), pp. 58-60.

[2] K.R.S. Sastry, Self-Altitude Triangles, *Mathematical Spectrum*, **22** (1989/90), pp. 88-90.

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Advance Notice

At the summer 1999 meeting of the Canadian Mathematical Society, to be held in St. John's, Newfoundland, there will be a Mathematics Education Session on the topic "What Mathematics Competitions do for Mathematics".

Invited speakers include Edward Barbeau, Toronto; Tony Gardner, Birmingham, England; Ron Dunkley, Waterloo; and Rita Janes, St. John's. Anyone interested in giving a paper at this session should contact one of the organizers, Bruce Shawyer or Ed Williams, at the Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada.

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THE SKOLIAD CORNER

No. 31

R.E. Woodrow

As a contest this month we give the U.K. Intermediate Mathematical Challenge which was written Thursday, 5^{th} February 1998. This contest is organized from the United Kingdom Mathematics Trust from the School of Mathematics, the University of Leeds. The contest is open to students in School year 11 or below. My thanks go to John Grant McLaughlin of the Faculty of Education, Memorial University of Newfoundland, who sent me a copy of the questions.

U.K. INTERMEDIATE MATHEMATICAL CHALLENGE
February 5, 1998
Time: 1 hour

Five marks are awarded for each correct answer to questions 1–15. Six marks are awarded for each correct answer to questions 16–25. Each incorrect answer to questions 16–20 loses 1 mark. Each incorrect answer to questions 21–25 loses 2 marks.

1. One quarter of a number is 24. What is one third of the original number?

(a) 6 (b) 8 (c) 32 (d) 72 (e) 96

2. 6% of 6 plus 8% of 8 equals

(a) 0.14 (b) 1 (c) 1.4 (d) 1.96 (e) 2

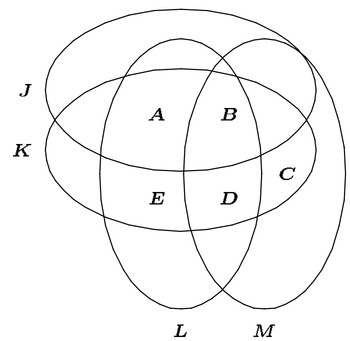
3. Starting at A, a point on a fixed circle with centre, O, I first move anticlockwise one quarter of the way round the circle to a point, W, hop across to X — the opposite end of the diameter through W, then travel one fifth of the way round the circle clockwise to the point Y before hopping across to Z, the point at the opposite end of the diameter through Y. How big is $\angle AOZ$?

(a) $18°$ (b) $22°$ (c) $162°$ (d) $198°$ (e) $270°$

4. Which fraction is the odd one out?

(a) $\frac{1+4}{7+4}$ (b) $\frac{20}{140}$ (c) $\frac{0.2}{1.4}$ (d) $\frac{1\div 11}{7\div 11}$ (e) $\frac{8}{56}$

5. J is the set of High Court judges; K is the set of living things beginning with K; L is the set of all living creatures; M is the set of brilliant mathematicians. Kevin is a very ordinary kangaroo. In which of the five regions A-E of the diagram does Kevin belong?



(a) A **(b)** B **(c)** C **(d)** D **(e)** E

6. $ABCD$ is a square with sides of length 9 cm. How many points (inside or outside the square) are equidistant from B and from C, and are exactly 6 cm from A?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** more than 3

7. Each person's *birthday product* is obtained by multiplying the day of the month in which they were born by the number of the month in which they were born, and then multiplying the answer by the year in which they were born. Here are five English queens and their birthdays. Which of them has the same birthday product as someone born today? [*Ed.* "today" is February 5, 1998.]

(a) Mary I, 18 February 1516 **(b)** Elizabeth I, 7 September 1533
(c) Anne, 6 February 1665 **(d)** Victoria, 24 May 1819
(e) Elizabeth II, 21 April 1926

8. How large will an angle of $2\frac{1}{2}^\circ$ appear to be if you enlarge it by looking through a stack of five magnifying glasses, each one of which magnifies by a factor of 2?

(a) $2\frac{1}{2}^\circ$ **(b)** $12\frac{1}{2}^\circ$ **(c)** 25° **(d)** 40° **(e)** 80°

9. What is the total number of letters in all the incorrect options for this question?

(a) eleven **(b)** twenty two **(c)** thirty three **(d)** forty four **(e)** fifty five

10. On four tests, each marked out of 100, my average was 85. What is the lowest mark I could have scored on any one test?

(a) 0 **(b)** 40 **(c)** 60 **(d)** 81 **(e)** 85

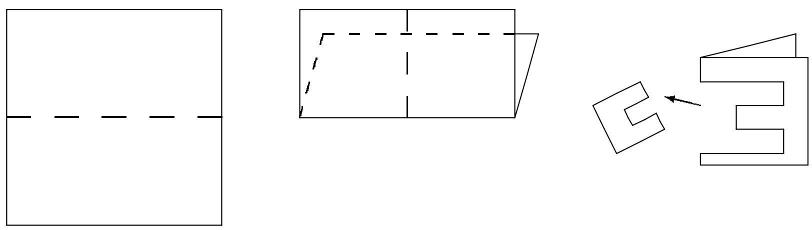
11. The *World Wide Fund For Nature* estimates that 54 acres of Brazilian rainforest are destroyed every minute of every day. Approximately how many acres are lost each week?

(a) 50,000 **(b)** 80,000 **(c)** 200,000 **(d)** 500,000 **(e)** 2,000,000

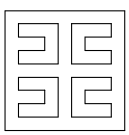
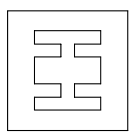
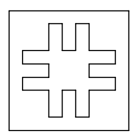
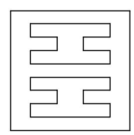
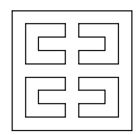
12. If $C°$ Celsius is the same temperature as $F°$ Fahrenheit, then $F = (\frac{9}{5})C + 32$. To avoid working with fractions and awkward numbers, some people use the approximate formula $F' = 2C + 30$. What is the temperature in degrees Celsius when the approximate formula gives an answer which is too large by 1?

(a) 5 (b) 9 (c) 10 (d) 12 (e) 15

13. I fold a piece of paper in half, then in half again before cutting a shape from the folded paper as shown.



When I unfold the paper, what do I see?

A  *B*  *C*  *D*  *E* 

(a) *A* (b) *B* (c) *C* (d) *D* (e) *E*

14. In a sponsored "Animal Streak" the cheetah ran at 90 km/hr while the snail slimed along at 20 hr/km. The cheetah kept going for 18 seconds. Roughly how long would the snail take to cover the same distance?

(a) 9 months (b) 9 weeks (c) 9 days (d) 9 hours (e) 9 minutes

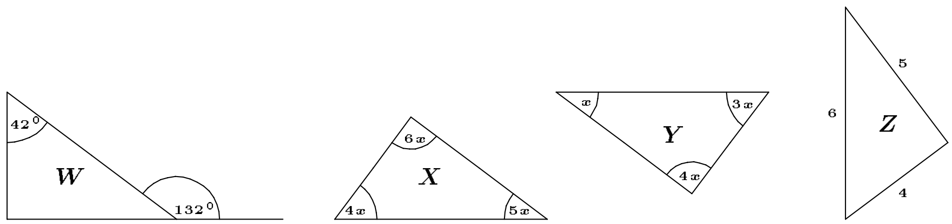
15. Wallace and Gromit are waiting in a queue. There are x people behind Wallace, who is y places in front of Gromit. If there are n people in front of Gromit, how long is the queue?

(a) $n - x + y + 2$ (b) $n + x - y$ (c) $n - x + y - 1$ (d) $n + x - y + 1$ (e) $n - x + y$

16. I made just enough sticky treacle mixture to exactly fill a square tin of side 12 inches. But all I could find were two $8\frac{1}{2}$ inch square tins. How well would the mixture fit?

(a) easily (b) just (with a teeny bit of room to spare) (c) an exact fit (d) nearly (with a small overflow) (e) no way (major overflow)

17. Which of the four triangles W, X, Y, Z are right-angled?



(a) only W (b) W and X (c) X and Z (d) Y and W (e) Y and Z

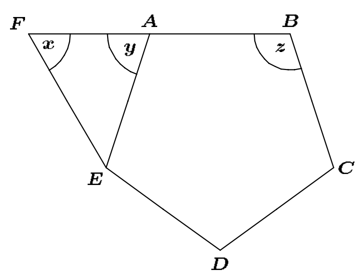
18. The integers from 1 to 20 are listed below in such a way that the sum of each adjacent pair is a prime number. Missing numbers are marked as $*$s.

$20,\ *, 16,\ 15,\ 4,\ *,\ 12,\ *,\ 10,\ 7,\ 6,\ \underline{\ *\ },\ 2,\ 17,\ 14,\ 9,\ 8,\ 5,\ 18,\ *.$

Which number goes in the place which is underlined?

(a) 1 (b) 3 (c) 11 (d) 13 (e) 19

19. $ABCDE$ is a regular pentagon. FAB is a straight line and $FA = AB$. What is the ratio $x : y : z$?



(a) $1 : 2 : 3$ (b) $2 : 2 : 3$ (c) $2 : 3 : 4$ (d) $3 : 4 : 5$ (e) $3 : 4 : 6$

20. The total length of all the edges of a cube is L cm. If the surface area of the cube has the same numerical value L cm^2, what is its volume in cm^3?

(a) 1 (b) L (c) 2 (d) L^3 (e) 8

21. A piece of thin card in the shape of an equilateral triangle with side 3 cm and a circular piece of thin card of radius 1 cm are glued together so that their centres coincide. How long is the outer perimeter of the resulting 2–dimensional shape (in cm)?

(a) 2π (b) $6 + \pi$ (c) 9 (d) 3π (e) $9 + 2\pi$

22. Shape A is made from 6 unit squares; shape B is made from 8, C from 4, D from 8 and E from 3 unit squares. For four of these shapes, four exact copies can be fitted together to make a rectangle. Which is the odd one out?

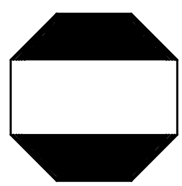
(a) A (b) B (c) C (d) D (e) E

23. In this unusual game of noughts and crosses the first player to form a line of three Os or three Xs loses. It is X's turn. Where should she place her cross to make sure that she does not lose?

A	O	B
C	X	D
E	X	O

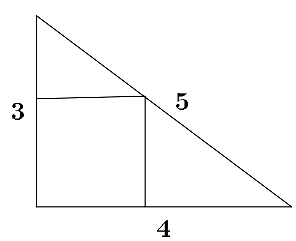
(a) A (b) B (c) C (d) D (e) E

24. Each of the sides of this regular octagon has length 2 cm. What is the difference between the area of the shaded region and the area of the unshaded region (in cm^2)?



(a) 0 (b) 1 (c) 1.5 (d) 2 (e) $2\sqrt{2}$

25. A square is inscribed in a 3–4–5 right-angled triangle as shown. What fraction of the triangle does it occupy?



(a) $\frac{12}{25}$ (b) $\frac{24}{49}$ (c) $\frac{1}{2}$ (d) $\frac{25}{49}$ (e) $\frac{13}{25}$

That completes this number of the corner. Send me contest materials, nice solutions, and any suggestions about items you would like see covered in the *Skoliad Corner*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), Richard Hoshino (University of Waterloo), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Shreds and Slices

Primitive Roots and Quadratic Residues, Part 2

Recall the work in Part 1 [1998: 220]: Let p be an odd prime, and A_p, B_p the quadratic residues and non-residues respectively. Let ζ be a primitive p^{th} root of unity; that is, $\zeta^p = 1$ and $\zeta \neq 1$. These imply $1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1} = 0$, an important fact we will use later. Let

$$x = \sum_{a \in A_p} \zeta^a, \qquad y = \sum_{b \in B_p} \zeta^b .$$

Then $x + y = -1$, and we saw xy always seems to be an integer, but why, and which one?

We introduce some basic quadratic residue theory. Let a be a non-zero integer modulo p. We define the *Legendre symbol* $(\frac{a}{p})$ as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue,} \\ -1 & \text{if } a \text{ is a quadratic non-residue.} \end{cases}$$

Although this notation has some deep consequences, we should think of it as just that: notation. We will now prove a useful relation, known as Euler's Criterion:

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p} .$$

If a is a q.r. (quadratic residue), then $x^2 \equiv a$ for some x, so $a^{(p-1)/2} \equiv x^{p-1} \equiv 1 \pmod{p}$ by Fermat's Little Theorem. Otherwise, if a is a non-q.r.

(quadratic non-residue), then for all x, $1 \leq x \leq p-1$, there is a unique y, $1 \leq y \leq p-1$, $y \neq x$, such that $xy \equiv a$. All integers in $\{1, 2, \ldots, p-1\}$ can be so paired. Their product is $a^{(p-1)/2} \equiv 1 \cdot 2 \cdots (p-1) \equiv (p-1)! \equiv -1 \pmod{p}$, by Wilson's Theorem.

As a corollary, note that

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Consider the case $p \equiv 3 \pmod{4}$, so $p = 4k+3$ for some k, and $(p-1)/2 = 2k+1$. Let $a_1, a_2, \ldots, a_{2k+1}$ be the q.r.'s and $b_1, b_2, \ldots, b_{2k+1}$ the non-q.r.'s. Since $(\frac{-1}{p}) = -1$, a is a q.r. if and only if $-a \equiv p-a$ is a non-q.r.. Hence, for all i, there exists a unique j such that $a_i + b_j = p$. The expression we wish to calculate is

$$\begin{aligned} xy &= (\zeta^{a_1} + \zeta^{a_2} + \cdots + \zeta^{a_{2k+1}})(\zeta^{b_1} + \zeta^{b_2} + \cdots + \zeta^{b_{2k+1}}) \\ &= \zeta^{a_1+b_1} + \zeta^{a_1+b_2} + \cdots + \zeta^{a_1+b_{2k+1}} \\ &\quad + \zeta^{a_2+b_1} + \zeta^{a_2+b_2} + \cdots + \zeta^{a_2+b_{2k+1}} \\ &\quad + \cdots \\ &\quad + \zeta^{a_{2k+1}+b_1} + \zeta^{a_{2k+1}+b_2} + \cdots + \zeta^{a_{2k+1}+b_{2k+1}}. \end{aligned}$$

Note that there are $(2k+1)^2$ terms, and there is a $\zeta^p = 1$ term in each row. This leaves $(2k+1)^2 - (2k+1) = 2k(2k+1)$ terms. We claim that $\zeta, \zeta^2, \ldots, \zeta^{p-1} = \zeta^{4k+2}$ appear the same number of times among these $2k(2k+1)$ terms. Let n be a non-zero integer modulo p. We wish to find the number of ways n can be expressed as the sum of a q.r. u and a non-q.r. v: $n \equiv u + v$.

Assume 1 can be expressed in t distinct such ways: $1 \equiv u_1 + v_1 \equiv u_2 + v_2 \equiv \cdots = u_t + v_t$. Then $n \equiv nu_1 + nv_1 \equiv nu_2 + nv_2 \equiv \cdots \equiv nu_t + nv_t$. If n is a q.r., then so is nu_i, and nv_i remains a non-q.r.. Otherwise, nu_i and nv_i switch roles, and in either case, there are again t distinct ways of expressing n as the sum of a q.r. and a non-q.r..

Therefore, among the $2k(2k+1)$ terms, each power of ζ appears $2k(2k+1)/(p-1) = k$ times. Hence,

$$\begin{aligned} xy &= 2k+1+k(\zeta + \zeta^2 + \cdots + \zeta^{4k+2}) \\ &= 2k+1-k \;=\; k+1 \;=\; \frac{1+p}{4}. \end{aligned}$$

The case $p \equiv 1 \pmod{4}$, say $p = 4k+1$, is similar, and is only briefly sketched here. In this case, $(\frac{-1}{p}) = 1$, so a and $-a \equiv p-a$ both occur as either a_i or b_j. Hence, when we expand xy to obtain $(2k)^2 = 4k^2$ terms, there are no terms of the form $\zeta^p = 1$, and using the same reasoning, the powers of ζ appear evenly among the $4k^2$ terms, so

$$xy \;=\; \frac{4k^2}{p-1}\left(\zeta + \zeta^2 + \cdots + \zeta^{4k}\right) \;=\; -k \;=\; \frac{1-p}{4}\,.$$

In general, we may write

$$xy \;=\; \frac{1 - \left(\frac{-1}{p}\right)p}{4}\,.$$

Hence, x and y are the roots of the quadratic

$$t^2 + t + \frac{1 - \left(\frac{-1}{p}\right)p}{4}$$

which are namely

$$\frac{1}{2}\left(-1 \pm \sqrt{\left(\frac{-1}{p}\right)p}\right)\,.$$

Which is x and which is y? It depends on the choice of ζ. Note that the discriminant of this quadratic is

$$(x-y)^2 \;=\; x^2 - 2xy + y^2 \;=\; (x+y)^2 - 4xy \;=\; \left(\frac{-1}{p}\right)p.$$

Hence, it is possible to express $\left(\frac{-1}{p}\right)p$ in terms of any primitive p^{th} root of unity.

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions.

High School Problems

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H241. Find the integer n that satisfies the equation

$$1 \cdot 1998 + 2 \cdot 1997 + 3 \cdot 1996 + \cdots + 1997 \cdot 2 + 1998 \cdot 1 \;=\; \binom{n}{3}.$$

H242. Let a and b be real numbers that satisfy $a^2 + b^2 = 1$. Prove the inequality

$$|a^2b + ab^2| \;\leq\; \frac{\sqrt{2}}{2}.$$

Determine the values of a and b for which equality occurs. (See how many ways you can solve this!)

H243. For a positive integer n, let $f(n)$ denote the remainder of $n^2 + 2$ when divided by 4. For example, $f(3) = 3$ and $f(4) = 2$. Prove that the equation

$$x^2 + (-1)^y f(z) \;=\; 10y$$

has no integer solutions in x, y, and z.

H244. For a positive integer n, let $P(n)$ denote the sum of the digits of n. For example, $P(123) = 1 + 2 + 3 = 6$. Find all positive integers n satisfying the equation $P(n) = \frac{n}{74}$.

Advanced Problems

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada. M1G 1C3 <hsia@math.toronto.edu>

A217. *Proposed by Alexandre Trichtchenko, OAC student, Brookfield High School, Ottawa.*

Show that for any odd prime p, there exists a positive integer n such that n, n^n, n^{n^n}, ... all leave the same remainder upon division by p where n does not leave a remainder of 0 or 1 upon division by p.

A218. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

(a) Suppose $f(x) = x^n + qx^{n-1} + t$, where q and t are integers, and suppose there is some prime p such that p divides t but p^2 does not divide t. Show, by imitating the proof of Eisenstein's Theorem, that either f is irreducible or f can be reduced into two factors, one of which is linear and the other irreducible.

(b) Deduce that if both q and t are odd then f is irreducible.

(Generalization of Question 1, IMO 1993)

A219. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Solve the following system:

$$3\left(x+\frac{1}{x}\right) = 4\left(y+\frac{1}{y}\right) = 5\left(z+\frac{1}{z}\right),$$
$$xy+yz+zx = 1.$$

A220. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

P is an interior point of triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Let Q be the interior point of triangle ABC such that

$$\angle ACP = \angle BCQ \quad \text{and} \quad \angle BAQ = \angle CAP.$$

Prove that $\angle DEF = 90^\circ$ if and only if Q is the orthocentre of triangle BDF.

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C79. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function, and assume that there are two constants $p > 0$ and $T > 0$ such that

$$\int_t^\infty f(s)^{p+1}\,ds \;\le\; Tf(0)^p f(t)$$

for all $t \in \mathbb{R}^+$. Prove that

$$f(t) \;\le\; f(0)\left(\frac{T+pt}{T+pT}\right)^{-\frac{1}{p}}$$

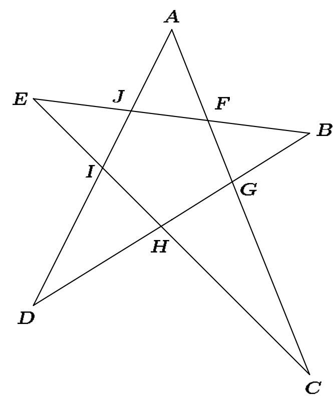
for all $t \ge T$.

C80. Suppose $a_1, a_2, \ldots, a_m$ are transpositions in S_n (the symmetric group on n elements) such that $a_1a_2\cdots a_m = 1$. Show that if the a_i generate S_n, then $m \ge 2n-2$.

The Pentagram Theorem

Hiroshi Kotera
teacher, Todaijigakuen Junior High School, Japan

A pentagram $ABCDE$ is a five-pointed star-shaped figure made by extending the sides of a convex pentagon $FGHIJ$ until they meet, and is sometimes used as a mystic symbol.

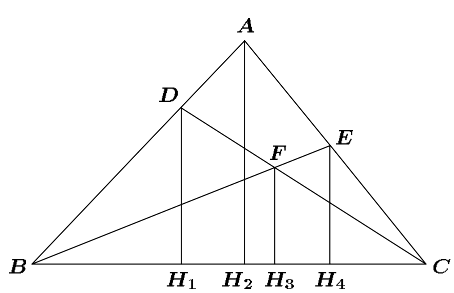


In my math class, Eiji Konishi, a Todaijigakuen Junior High School student in Japan, found a beautiful theorem concerning the pentagram. I would like to show the theorem he proved by himself. We need two lemmas.

Lemma 1. In triangle ABC, let D and E be points on sides AB and AC respectively, and let F be the intersection of BE and CD. Then

$$\frac{AC}{AB}\cdot\frac{DB}{DC}\cdot\frac{EB}{EC}\cdot\frac{FC}{FB} = 1\,.$$

Proof. Let H_i $(i = 1, 2, 3, 4)$ be points on side BC such that DH_1, AH_2, FH_3, and EH_4 are all perpendicular to BC.



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Since

$$FH_3 = DH_1 \cdot \frac{FC}{DC} \qquad \text{and} \qquad DH_1 = AH_2 \cdot \frac{DB}{AB},$$

we have

$$FH_3 = AH_2 \cdot \frac{DB}{AB} \cdot \frac{FC}{DC}. \tag{1}$$

Similarly, since

$$FH_3 = EH_4 \cdot \frac{FB}{EB} \qquad \text{and} \qquad EH_4 = AH_2 \cdot \frac{EC}{AC},$$

we have

$$FH_3 = AH_2 \cdot \frac{EC}{AC} \cdot \frac{FB}{EB}. \tag{2}$$

Equations (1) and (2) imply

$$\frac{DB}{AB} \cdot \frac{FC}{DC} = \frac{EC}{AC} \cdot \frac{FB}{EB}.$$

It follows that

$$\frac{AC}{AB} \cdot \frac{DB}{DC} \cdot \frac{EB}{EC} \cdot \frac{FC}{FB} = 1.$$

Lemma 2. Let D and E be two points on side AB of triangle ABC, and let F be a point on side BC. Let G be a point on side BC, extended. Let H be the intersection of DG and AC, let I be the intersection of DG and AF, and let J be the intersection of GE and AC. Then

$$\frac{DE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CJ}{JH} \cdot \frac{HI}{ID} = 1.$$

Proof. Let L be the point on AC such that $BL \| DG$, and let N be the point on DG such that $NB \| AC$. Let M be the intersection of AF and BL, and let P be the intersection of GE and BN.

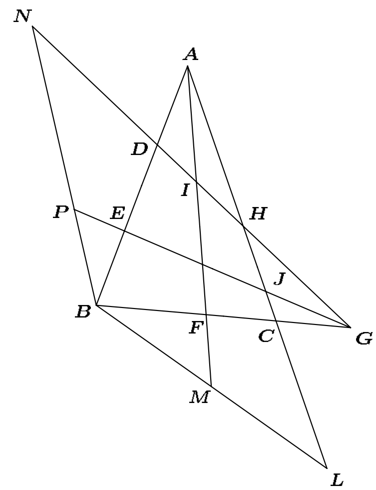
By Menelaus's Theorem in triangles BCL and BDN, we have

$$\frac{BF}{FC} \cdot \frac{CA}{AL} \cdot \frac{LM}{MB} = -1, \tag{3}$$

$$\frac{BP}{PN} \cdot \frac{NG}{GD} \cdot \frac{DE}{EB} = -1. \tag{4}$$

By similar triangles,

$$\frac{LM}{MB} = \frac{HI}{ID},$$



and

$$\frac{AH}{AD} = \frac{AL}{AB},$$

so by (3),

$$\frac{BF}{FC} \cdot \frac{LM}{MB} = \frac{BF}{FC} \cdot \frac{HI}{ID} = -\frac{AL}{CA} = -\frac{AH}{AD} \cdot \frac{AB}{AC}. \tag{5}$$

Similarly,

$$\frac{BP}{PN} = \frac{CJ}{JH},$$

and

$$\frac{GC}{GH} = \frac{GB}{GN},$$

so by (4),

$$\frac{BP}{PN} \cdot \frac{DE}{EB} = \frac{CJ}{JH} \cdot \frac{DE}{EB} = -\frac{GD}{NG} = -\frac{GD}{GH} \cdot \frac{GC}{GB}. \tag{6}$$

Equations (5) and (6), and Lemma 1 imply

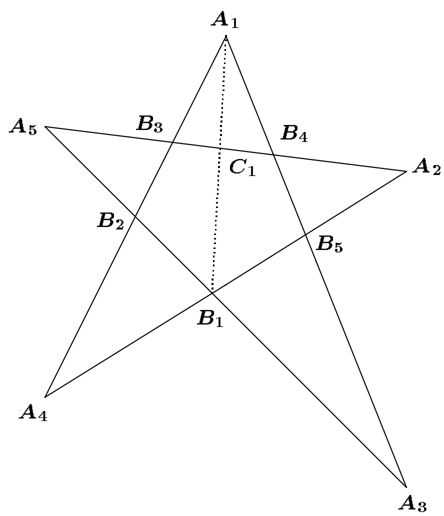
$$\frac{BF}{FC} \cdot \frac{HI}{ID} \cdot \frac{CJ}{JH} \cdot \frac{DE}{EB} = \frac{AH}{AD} \cdot \frac{AB}{AC} \cdot \frac{GD}{GH} \cdot \frac{GC}{GB} = 1.$$

It follows that

$$\frac{DE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CJ}{JH} \cdot \frac{HI}{ID} = 1.$$

Theorem. Let $A_1A_2A_3A_4A_5$ be a pentagram with pentagon $B_1B_2B_3B_4B_5$ as shown in the following figure. Let C_k be the intersection of the line A_kB_k with the side $B_{k+2}B_{k+3}$, $k = 1, 2, 3, 4, 5$. Then

$$\frac{B_3C_1}{C_1B_4} \cdot \frac{B_4C_2}{C_2B_5} \cdot \frac{B_5C_3}{C_3B_1} \cdot \frac{B_1C_4}{C_4B_2} \cdot \frac{B_2C_5}{C_5B_3} = 1 .$$



Proof. Lemma 2 yields

$$\frac{B_3C_1}{C_1B_4} \cdot \frac{B_4C_2}{C_2B_5} = \frac{B_1A_4}{B_5B_1} \cdot \frac{B_2B_3}{A_4B_2},$$

and

$$\frac{B_5C_3}{C_3B_1} \cdot \frac{B_1C_4}{C_4B_2} = \frac{B_3A_1}{B_2B_3} \cdot \frac{B_4B_5}{A_1B_4}.$$

By Menelaus's Theorem in triangle $A_4B_5C_5$, we have

$$\frac{A_4B_1}{B_1B_5} \cdot \frac{B_5A_5}{A_5C_5} \cdot \frac{C_5B_2}{B_2A_4} = -1,$$

and in triangle $A_1B_5C_5$, we have

$$\frac{A_1B_4}{B_4B_5} \cdot \frac{B_5A_5}{A_5C_5} \cdot \frac{C_5B_3}{B_3A_1} = -1.$$

It follows that

$$\begin{aligned} & \frac{B_3C_1}{C_1B_4}\cdot\frac{B_4C_2}{C_2B_5}\cdot\frac{B_5C_3}{C_3B_1}\cdot\frac{B_1C_4}{C_4B_2}\cdot\frac{B_2C_5}{C_5B_3} \\ = & \frac{B_1A_4}{B_5B_1}\cdot\frac{B_2B_3}{A_4B_2}\cdot\frac{B_3A_1}{B_2B_3}\cdot\frac{B_4B_5}{A_1B_4}\cdot\frac{B_2C_5}{C_5B_3} \\ = & \left(\frac{B_1A_4}{B_5B_1}\cdot\frac{B_2C_5}{A_4B_2}\right)\cdot\left(\frac{B_3A_1}{A_1B_4}\cdot\frac{B_4B_5}{C_5B_3}\right) \\ = & \frac{A_5C_5}{B_5A_5}\cdot\frac{B_5A_5}{A_5C_5} \\ = & 1\,. \end{aligned}$$

This result may be something already well known, but I presume the fact that its discovery has been made by a junior high school student is significant.

Have you heard about ATOM?

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Publiés par la Société mathématique du Canada (SMC), les livrets de la collection Aime-t-on les mathématiques (ATOM) sont destinés au perfectionnement des étudiants du cycle secondaire qui manifestent un intérêt et des aptitudes pour les mathématiques. Certains livrets de la collection ATOM servent également de matériel de préparation aux concours de mathématiques sur l'échiquier national et international.

This volume contains the problems and solutions from the 1995–1996 Mathematical Olympiads' Correspondence Program. This program has several purposes. It provides students with practice at solving and writing up solutions to Olympiad-level problems, it helps to prepare student for the Canadian Mathematical Olympiad and it is a partial criterion for the selection of the Canadian IMO team.

For more information, contact the Canadian Mathematical Society.

A Combinatorial Triad

Cyrus Hsia
student, University of Toronto

One of the beautiful aspects of mathematics is that solving a problem can be done in so many ways. In combinatorics, the study of counting and ways of counting, this is more than evident in its many interesting problems. See any standard textbook on combinatorics [1] to convince yourself. Consider the following three standard combinatorial problems and see if you can find the link between them.

Problem 1. What is the number of ways to distribute 10 identical apples into 3 baskets labelled A, B, and C?

Problem 2. How many ways are there to select 10 ice-cream scoops from 3 different flavours on an ice-cream cone?

Problem 3. Find the number of non-negative integer solutions to the diophantine equation $x_1 + x_2 + x_3 = 10$.

If you said the answers are all the same, then you are correct. The three problems above are different ways of counting something equivalent. Each way of thinking may be more helpful in certain situations than others. The following are solutions to approaching the apparently different, but similar, problems.

Solution to Problem 1. The standard approach to solving this type of problem rests on the following key trick. Let the apples be represented by 1's since they are all identical. Now use two 0's to place between the 1's to split them into 3 piles. Put the first pile into basket A, the second into basket B and the remaining pile into basket C. The problem then reduces to finding the number of ways of placing the 0's among the 1's.

There are $10 + 1 = 11$ positions to place the first 0, namely 9 spots between the 10 1's and the front and back of the list of 1's. Now there are 11 numbers and $11 + 1 = 12$ positions to place the second 0. However, we are counting everything twice! If we place 0 in front and then a 0 after the fifth 1, this is the same as if we place the first 0 after the fifth 1 and the second 0 in the first spot. Hence, there are

$$\frac{11 \times 12}{2}$$

ways in total.

Solution to Problem 2. This is exactly the same as Problem 1 if we consider the ice-cream scoops to be 1's and the type of flavour it is by its

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position in one of the three piles of numbers. We use two 0's again to be dividing points for the three piles. The number of ways of selecting scoops from three flavours becomes a problem of arranging the 10 1's and two 0's. There are

$$\binom{12}{2} = \frac{11 \times 12}{2}$$

ways of doing this.

Solution to Problem 3. Here, the problem and solution have become completely algebraic. The standard way of solving this would be to use generating functions. Consider the following function:

$$(1 + x + x^2 + x^3 + \cdots)^3.$$

Now consider the coefficient of x^{10}. How do you get this term? Take x^a in the first factor, x^b in the second, and x^c in the third so that $a+b+c = 10$. Thus, we want all possible ways of doing this, so that a, b, and c are solutions to the equation in the problem: $x_1 + x_2 + x_3 = 10$. The coefficient is the number of nonnegative integer solutions to the equation!

Of course, one must check that the coefficient of x^{10} is indeed $\binom{12}{2}$.

Here are some problems to drive this idea home.

Standard Exercises

1. Reformulate the following problems to the other two types and solve them.
   (a) How many ways are there to select 98 DNA codons if there are 4 different codons to choose from?
   (b) Solve the diophantine equation $x_1 + x_2 + \cdots + x_{19} = 97$, with $x_i \geq 3$ for all $i = 1, 2, \ldots, 19$.
   (c) How many ways are there to put 98 pigeons into 97 pigeonholes? (Aside: Must one pigeonhole have more than one pigeon?)
2. How many ways are there to distribute 9 apples, 8 bananas and 7 oranges to 6 hungry school children? How would you reformulate this problem to the other two?

Other Exercises

1. How many ways are there to select 3 numbers from the integers 1 to 20 so that no two are consecutive?
2. How many ways are there to distribute 10 distinct balls into three tubes labelled A, B, and C taking the order of the balls in each tube into account? (Careful! This is harder than meets the eye.)

Reference

[1] Tucker, Alan. *Applied Combinatorics*. John Wiley & Sons, Inc., Toronto. 1995 pp. 205.

Swedish Mathematics Olympiad

1987 Qualifying Round

1. Kalle has a litre flask A full of orange juice and an empty litre flask B. He pours part of the juice from the full flask A to the empty flask B. He then adds water to B until it is full, and shakes it so that the mixture is thoroughly blended. Lastly, he adds the blended mixture from flask B to flask A until flask A is full. Show that flask A now contains at least 75% of the original juice.

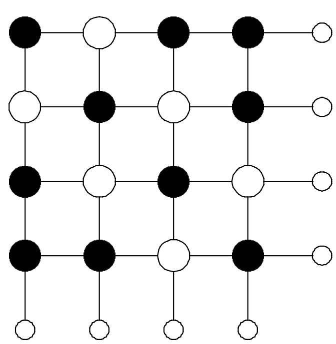
2. Show that the number abc, which denotes a 3-digit number written in the usual way, is divisible by 7 if and only if the number $-a + 2b - 3c$ is divisible by 7.

3. Solve the system of equations

$$\begin{aligned} (x+y)(x^2-y^2) &= 1176 \\ (x-y)(x^2+y^2) &= 696. \end{aligned}$$

4. In the triangle ABC, arbitrarily choose a point P on the side BC. Through the midpoint M of the same side, draw DM parallel to AP, where D is a point on one of the other two sides. Show that the line segment DP divides the triangle into two regions of equal area.

5. Sixteen light bulbs are connected in a square network as shown in the figure. Each row and column has a switch which works as follows: one half-turn of the switch extinguishes all the lighted bulbs in the row (or column), and turns on all the bulbs that are off in that row (or column).



In the initial state, six bulbs are on, as shown in the figure. Prove that it is impossible to turn on all 16 bulbs, using the switches. What is the maximum number of bulbs that can be turned on?

6. A small school with 50 pupils has put the pupils into 8 groups. There are 4 more pupils in the largest group than there are in the smallest group. These groups are combined two by two into larger groups as follows. The two smallest of the original groups are combined, numbers 3 and 4 form the next combined group, and so on. The largest of the four groups so formed now has 5 more pupils than the smallest. The two smallest of these groups form class **A** and the two largest form class **B**. Class **B** has 6 more pupils than class **A**.

   Find the numbers of pupils in the original groups.

1987 Final Round

1. Sixteen real numbers are arranged in a "magic square" of side 4 in such a way that the sum of the numbers in each row, in each column and in each of the two main diagonals is always k. Prove that the sum of the four numbers in the corners of the square is also k.

2. A circle of radius R is divided into two equal parts by the arc of another circle. Show that the arc of this circle (which lies inside the given circle) is longer than $2R$.

3. Assume that 10 closed intervals all of length 1 are placed in the interval $[0, 4]$. Show that there exists some point in the larger interval that belongs to at least 4 of the smaller intervals.

4. f is a differentiable function defined on the interval $0 \leq x \leq 1$ and $f(0) = f(1) = 1$. Show that there exists at least one point y in the interval such that

$$|f'(y)| = 4 \int_0^1 |f(x)|\, dx.$$

5. The numbers a, b, c, and d are positive and $abcd = 1$. Show that there exists a positive number t such that for all such a, b, c, and d,

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} + \frac{1}{1+d} > t.$$

   Find the largest t with this property.

6. A baker with access to a number of different spices bakes 10 cakes. He uses more than half of the different kinds of spices in each cake. No two of the combinations of spices are exactly alike. Show that there exist three spices a, b, c, such that every cake contains at least one of these.

J.I.R. McKnight Problems Contest 1983

1. Prove that in any triangle ABC, where AD is a median and $|AD| = m$, $4m^2 = b^2 + c^2 + 2bc\cos A$.

2. Find the equations of the three normals to the parabola with the equation $y^2 = 4x$, from $(21, 30)$.

   Note: A normal ℓ to a curve C at a point P is a line ℓ which intersects C at P and is perpendicular to the tangent of C at P.

3. Find the equations of the four common tangents of the circles having equations $x^2 + y^2 = 16$ and $(x - 25)^2 + y^2 = 121$.

4. A tank is filled using two taps. It takes taps one and two t_1 and t_2 minutes respectively to fill the tank each on its own. From the following information, determine what the values t_1 and t_2 are:

   (i) If the first tap is open for one-third of the time t_2 and the second tap is open for one-third of the time t_1, then the fraction of the volume of the tank that is filled is $11/18$.

   (ii) If both taps are used simultaneously, it would take 3 hours and 36 minutes to fill the tank.

5. Two bodies are moving with constant acceleration in a straight line going in the same direction. The first body starts 20 km ahead of the other. It travels a distance of 25 and $50/3$ km in 1 and 2 hours respectively. The second body travels a distance of 30 and $59/2$ km in 1 and 2 hours respectively. How long does it take for the second body to catch up to the first?

Editor's comment. Problem 4 of the above contest, as stated, has no solution. The equation arising from condition (i) is not solvable for positive t_1 and t_2. However, the problem is solvable with nice values if the fraction $\frac{11}{18}$ is replaced by $\frac{13}{18}$.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 March 1999**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2326* Correction. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove that if A, B and C are the angles of a triangle, then

$$\frac{2}{\pi} < \sum_{\text{cyclic}} \frac{\left(1-\sin\frac{A}{2}\right)\left(1+2\sin\frac{A}{2}\right)}{\pi - A} \leq \frac{9}{2\pi}.$$

2329* Correction. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that p and $t > 0$ are real numbers. Define

$$\lambda_p(t) := t^p + t^{-p} + 2^p \qquad \text{and} \qquad \kappa_p(t) := \left(t + t^{-1}\right)^p + 2.$$

(a) Show that $\lambda_p(t) \leq \kappa_p(t)$ for $p \geq 2$.

(b) Determine the sets of p: A, B and C, such that

1. $\lambda_p(t) \le \kappa_p(t)$,
2. $\lambda_p(t) = \kappa_p(t)$,
3. $\lambda_p(t) \ge \kappa_p(t)$.

2351. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

A triangle with integer sides is called ***Heronian*** if its area is an integer.

Does there exist a Heronian triangle whose sides are the arithmetic, geometric and harmonic means of two positive integers?

2352. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Determine the shape of $\triangle ABC$ if

$$\cos A\cos B\cos(A-B)+\cos B\cos C\cos(B-C)$$
$$+\cos C\cos A\cos(C-A)+2\cos A\cos B\cos C \;=\; 1\,.$$

2353. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Determine the shape of $\triangle ABC$ if

$$\sin A\sin B\sin(A-B)+\sin B\sin C\sin(B-C)$$
$$+\sin C\sin A\sin(C-A) \;=\; 0\,.$$

2354. *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In triangle $P_1P_2P_3$, the line joining $P_{i-1}P_{i+1}$ meets a line σ_j at the point $S_{i,j}$ ($i,j=1,2,3$, all indices taken modulo 3), such that all the points $S_{i,j}$, P_k are distinct, and different from the vertices of the triangle.

1. Prove that if all the points $S_{i,j}$ are non-collinear, then any two of the following conditions imply the third condition:

   (a) $\dfrac{P_1S_{3,1}}{S_{3,2}P_2}\cdot\dfrac{P_2S_{1,2}}{S_{1,3}P_3}\cdot\dfrac{P_3S_{2,3}}{S_{2,1}P_1} \;=\; -1;$

   (b) $\dfrac{S_{1,2}S_{1,1}}{S_{1,1}S_{1,3}}\cdot\dfrac{S_{2,3}S_{2,2}}{S_{2,2}S_{2,1}}\cdot\dfrac{S_{3,1}S_{3,3}}{S_{3,3}S_{3,2}} \;=\; 1;$

   (c) σ_1, σ_2, σ_3 are either concurrent or parallel.

2. Prove further that (a) and (b) are equivalent if the $S_{i,i}$ are collinear.

Here, AB denotes the signed length of the directed line segment $[AB]$.

2355. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

For $j = 1, 2, \ldots, m$, let A_j be non-collinear points with $A_j \neq A_{j+1}$. Translate every even-numbered point by an equal amount to get new points A'_2, A'_4, $\ldots$, and consider the sequence B_j, where $B_{2i} = A'_{2i}$ and $B_{2i-1} = A_{2i-1}$. The last member of the new sequence is either A_{m+1} or A'_{m+1} according as m is even or odd.

Find a necessary and sufficient condition for the length of the path $B_1B_2B_3 \ldots B_m$ to be greater than the length of the path $A_1A_2A_3 \ldots A_m$ for all such non-zero translations.

CRUX 1985 [1994: 250; 1995: 280] provides an example of such a configuration. There, $m = 2n$, the A_i are the vertices of a regular $2n$–gon and $A_{2n+1} = A_1$.

2356. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Five points, A, B, C, K, L, with whole number coordinates, are given. The points A, B, C do not lie on a line.

Prove that it is possible to find two points, M, N, with whole number coordinates, such that M lies on the line KL and $\triangle KMN$ is similar to $\triangle ABC$.

2357. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

An unsteady man leaves a place to commence a one-dimensional random walk. At each step he is equally likely to stagger one step to the east or one step to the west. Let his expected **absolute** distance from the starting point after $2n$ steps be a. Now consider $2n$ unsteady men each engaging in independent random walks of this type. Let the expected number of men at the starting point after $2n$ steps be b. Show that $a = b$.

2358. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

In triangle ABC, let the mid-points of BC, CA, AB be L, M, N, respectively, and let the feet of the altitudes from A, B, C be D, E, F, respectively. Let X be the intersection of LE and MD, let Y be the intersection of MF and NE, and let Z be the intersection of ND and LF. Show that X, Y, Z are collinear.

2359. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Let $PQRS$ be a parallelogram. Let Z divide PQ internally in the ratio $k : l$. The line through Z parallel to PS meets the diagonal SQ at X. The line ZR meets SQ at Y.

Find the ratio $XY : SQ$.

2360. *Proposed by K.R.S. Sastry, Dodballapur, India.*

In triangle ABC, let BE and CF be internal angle bisectors, and let BQ and CR be altitudes, where F and R lie on AB, and Q and E lie on AC. Assume that E, Q, F and R lie on a circle that is tangent to BC.

Prove that triangle ABC is equilateral.

2361. *Proposed by K.R.S. Sastry, Dodballapur, India.*

The lengths of the sides of triangle ABC are given by relatively prime natural numbers. Let F be the point of tangency of the incircle with side AB. Suppose that $\angle ABC = 60^\circ$ and $AC = CF$. Determine the lengths of the sides of triangle ABC.

2362. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Suppose that a, b, $c > 0$. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \;\geq\; \frac{3}{1+abc}.$$

2363. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For natural numbers $a, b, c > 0$, let

$$q(a,b,c) \;:=\; a + \cfrac{a + \cfrac{a+\cfrac{a+\frac{a+\cdots}{b+\cdots}}{b+\frac{c+\cdots}{a+\cdots}}}{b+\cfrac{c+\frac{a+\cdots}{b+\cdots}}{a+\frac{c+\cdots}{a+\cdots}}}}{b + \cfrac{c+\cfrac{a+\frac{a+\cdots}{b+\cdots}}{b+\frac{c+\cdots}{a+\cdots}}}{a+\cfrac{c+\frac{a+\cdots}{b+\cdots}}{a+\frac{c+\cdots}{a+\cdots}}}}$$

(in the n^{th} "column" above, from the third one onwards, we have, from top to bottom, the sequence a, b, c, a repeated 2^{n-3} times), where it is assumed that the right side (understood as an infinite process) yields a well-defined positive real number.

The original, a *Talent Search Problem*, asked to determine $q(1,3,5)$. The value is $\sqrt[3]{2}$ (see *Mathematics and Informatics Quarterly*, 7 (1997), No. 1, p. 53).

Determine whether or not there exist infinitely many triples (a,b,c) such that $q(a,b,c)$ is the cube root of a natural number.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

R.P. Sealy was accidentally omitted from the list of solvers for problem 2338.

2015. [1995: 53, 129] *Proposed by Shi-Chang Shi and Ji Chen, Ningbo University, China.*

Prove that

$$(\sin A + \sin B + \sin C)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \geq \frac{27\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle.

Editor's comment. The first solution printed was incorrect [1996: 47, solution I]. Its "correction" [1996: 125] was also incorrect. Finally, the second solution printed, [1996: 48, solution II], was also incorrect. It is time to correct these wrongs!!

In solution I, solver Grant claims that both terms

$$\sin A + \sin B + \sin C \qquad \text{and} \qquad 1/A + 1/B + 1/C$$

are minimized when $A = B = C (= 60^\circ)$. This is true for the second term, but NOT for the first! It is obvious that for the degenerate triangle $A = 180^\circ$, $B = C = 0^\circ$, the sine sum is zero while the sum is positive for $A = B = C$. And of course this means that for 'real' triangles sufficiently close to the degenerate one, the sine sum will be less than the equilateral sine sum too. Thus the proof falls apart.

In Solution II, the proposers first show that the function $y(x) = x^{-1/3}\cos x$ is convex on $(0, \pi/2]$. However, in the second inequality of their second displayed equation, they appear to apply Jensen's inequality to the function $f(x) = \log y(x)$, which is NOT convex everywhere in this interval! That is, they want to prove that $6y(A/2)y(B/2)y(C/2) \geq 6(y(\pi/6))^3$, which by taking logarithms is equivalent to $f(A/2) + f(B/2) + f(C/2) \geq 3f(\pi/6)$; that is,

$$\frac{1}{3}\left[f(A/2) + f(B/2) + f(C/2)\right] \;\geq\; f\left[(1/3)(A/2 + B/2 + C/2)\right].$$

For this we need that f is convex; it is not. In fact, the last inequality on [1996: 48] is incorrect, as can be seen again for the degenerate triangle $A = 180$, $B = C = 0$. However, this time we must use limits, so put

$B/2 = C/2 = x$ and $A/2 = \pi/2 - 2x$. Then you can find that the limit of

$$\left(\frac{\cos(\pi/2 - 2x)\cos^2(x)}{(\pi/2 - 2x)x^2}\right)^{1/3}$$

as $x \to 0$ is ZERO, so the inequality fails.

Thanks to Waldemar Pompe and Bill Sands for pointing these out. So, we must belatedly present a correct solution.

III *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Since $\sum \sin A = \frac{s}{R}$, the given inequality is equivalent to

$$\sum \frac{1}{A} \;\geq\; \frac{27\sqrt{3}}{2\pi}\,\frac{R}{s}. \tag{1}$$

Now, from item 6.60 on p. 188 in D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989, the following inequality (due to V. Mascioni) is known:

$$\sum \frac{1}{A} \;\geq\; \frac{9}{\pi}\sqrt{\frac{R}{2r}}. \tag{2}$$

We now show that (2) is stronger than (1). Indeed, we have:

$$\frac{9}{\pi}\sqrt{\frac{R}{2r}} \;\geq\; \frac{27\sqrt{3}}{2\pi}\,\frac{R}{s}, \tag{3}$$

since this is equivalent to $s\sqrt{2} \geq 3\sqrt{3}\sqrt{Rr}$; that is,

$$2s^2 \;\geq\; 27Rr\,. \tag{4}$$

Finally, (4) is simply item 5.12 on p. 52 in O. Bottema et al., *Geometric Inequalities*, Groningen, 1969.

A very similar solution was submitted by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA.

Remarks by Janous

(i) We can also show the inequality

$$\sqrt[3]{\prod \sin A}\sum \frac{1}{A} \;\geq\; \frac{9\sqrt{3}}{2\pi}\,. \tag{5}$$

(Because of the AM–GM Inequality, (5) is stronger than (1).)

Since $\prod \sin A \;=\; \frac{sr}{2R^2}$, we find that (5) can be re-written as

$$\sum \frac{1}{A} \;\geq\; \frac{9\sqrt{3}}{2\pi}\sqrt[3]{\frac{2R^2}{sr}}. \tag{6}$$

We shall also show that

$$\frac{9}{\pi}\sqrt{\frac{R}{2r}} \;\geq\; \frac{9\sqrt{3}}{2\pi}\sqrt[3]{\frac{2R^2}{sr}}. \tag{7}$$

A simple algebraic manipulation shows that this is the same as (4).

(ii) With the usual notation for the power mean

$$M_t(x,y,z) \;=\; \left(\frac{x^t+y^t+z^t}{3}\right)^{\frac{1}{t}}, \quad t \neq 0 \quad \text{and} \quad M_0(x,y,z) \;=\; (xyz)^{\frac{1}{3}},$$

we find that (5) is

$$M_t(\sin A, \sin B, \sin C) \;\geq\; \frac{3\sqrt{3}}{2\pi}M_{-1}(A,B,C) \quad \text{with} \quad t = 0\,. \tag{8}$$

This raises the questions:

1. Is $t = 0$ the minimum value of t such that (8) holds?
2. What is the maximum value of t such that (8) holds with "$\leq$" instead of "$\geq$"?
3. What about analogous questions if we replace the right side of (8) by $\frac{3\sqrt{3}}{2\pi}M_p(A,B,C)$?

(iii) Note that $\prod \sin(A/2) \;=\; \frac{r}{4R}$. So, inequality (3) can be re-written as

$$\frac{1}{3}\sum \sin A \;\geq\; \sqrt{6}\sqrt{\prod \sin\left(\frac{A}{2}\right)}.$$

This suggests a further question:

Let $0 < \lambda \leq 1$ be a given real number. Determine the optimum constant $C = C(\lambda)$ such that

$$\frac{1}{3}\sum \sin A \;\geq\; C\left(\prod \sin(\lambda A)\right)^{\lambda}$$

for all triangles. I conjecture that [*Ed. using* $A = B = C = 60^\circ$]

$$C(\lambda) \;=\; \frac{\sqrt{3}}{2}\left(\sin\left(\frac{\lambda\pi}{3}\right)\right)^{-3\lambda}.$$

Similarly, inequality (7) is equivalent to

$$\sqrt[3]{\prod \sin A} \;\geq\; \sqrt{6}\sqrt{\prod \sin\left(\frac{A}{2}\right)},$$

leading to the analogous question of finding an optimal constant $D = D(\lambda)$ such that

$$\sqrt[3]{\prod \sin A} \;\geq\; S\left(\prod \sin(\lambda A)\right)^{\lambda},$$

where $0 < \lambda \leq 1$. I conjecture that $D(\lambda) = C(\lambda)$.

220A***.** [1996: 363] *Proposed by Ji Chen, Ningbo University, China.*

Let P be a point in the interior of the triangle ABC, and let $\alpha_1 = \angle PAB$, $\beta_1 = \angle PBC$, $\gamma_1 = \angle PCA$.

Prove or disprove that $\sqrt[3]{\alpha_1\beta_1\gamma_1} \le \pi/6$.

Solution by Kee-Wai Lau, Hong Kong (slightly modified by the editor).

We prove the inequality. We write $\alpha = \alpha_1$, $\beta = \beta_1$, $\gamma = \gamma_1$. Applying the Sine Rule to the triangles PAB, PBC, PCA we obtain

$$\sin\alpha\sin\beta\sin\gamma = \sin(A-\alpha)\sin(B-\beta)\sin(C-\gamma)\,.$$

The function $\log\sin x$ is concave for $0 < x < \pi$; hence

$$\log\sin(A-\alpha) + \log\sin(B-\beta) + \log\sin(C-\gamma) \le 3\log\sin\left(\frac{A+B+C-\alpha-\beta-\gamma}{3}\right)\,.$$

Therefore, we get

$$\log\sin\alpha + \log\sin\beta + \log\sin\gamma \le 3\log\sin\left(\frac{\pi-\alpha-\beta-\gamma}{3}\right)\,. \qquad (1)$$

For $\alpha, \beta, \gamma > 0$ and $\alpha+\beta+\gamma < \pi$, set

$$f(\alpha,\beta,\gamma) = \log\sin\alpha + \log\sin\beta + \log\sin\gamma - 3\log\sin\left(\frac{\pi-\alpha-\beta-\gamma}{3}\right),$$

and define

$$D = \left\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \mid \alpha,\beta,\gamma > 0,\ \alpha+\beta+\gamma < \pi,\ \alpha\beta\gamma > \tfrac{\pi^3}{6^3}\right\}.$$

In view of (1), it is enough to prove the following

Proposition: If $(\alpha,\beta,\gamma) \in D$, then $f(\alpha,\beta,\gamma) > 0$.

Proof: Fix $\varepsilon > 0$, and let

$$D_\varepsilon = \left\{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \mid \alpha,\beta,\gamma > 0,\ \alpha+\beta+\gamma < \pi-\varepsilon,\ \alpha\beta\gamma > \tfrac{\pi^3}{6^3}\right\}.$$

It is easily seen that $D_{\varepsilon'} \subset D_\varepsilon$ if $\varepsilon < \varepsilon'$ and $D = \bigcup_{\varepsilon>0} D_\varepsilon$. Thus, it is enough to prove that there exists $\delta > 0$ such that, if $\varepsilon < \delta$, then the inequality $f(\alpha,\beta,\gamma) > 0$ holds for all $(\alpha,\beta,\gamma) \in D_\varepsilon$.

[*Editorial note:* These ε-δ complications are caused by the fact that f is not defined if $\alpha+\beta+\gamma = \pi$; that is, on the closure of D in $\mathbb{R}^3$. One may avoid these complications by putting $f(\alpha,\beta,\gamma) = +\infty$ if $\alpha+\beta+\gamma = \pi$, and showing that such an f is "continuous" on the closure of D. Nothing new, however, arises from this approach, nor does the proof become simpler!]

Since f is continuous on $\overline{D_\varepsilon}$ (the closure of D_ε in $\mathbb{R}^3$), and since $\overline{D_\varepsilon}$ is a compact subset of $\mathbb{R}^3$, the minimum value of f is attained on $\overline{D_\varepsilon}$. But if

$(\alpha, \beta, \gamma) \in D_\varepsilon$, then one of the numbers α, β, γ (say γ) is less than $\pi/2$, so that

$$\frac{\partial f}{\partial \gamma}(\alpha, \beta, \gamma) = \cot \gamma + \cot \left(\frac{\pi - \alpha - \beta - \gamma}{3} \right) \neq 0 \,. \tag{2}$$

This means that f attains its minimum value on ∂D_ε — the boundary of D_ε in $\mathbb{R}^3$. We have

$$\begin{aligned} \partial D_\varepsilon \quad &= \quad \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \alpha, \beta, \gamma > 0,\ \alpha + \beta + \gamma = \pi - \varepsilon,\ \alpha\beta\gamma \geq \tfrac{\pi^3}{6^3} \right\} \cup \\ &\qquad \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \alpha, \beta, \gamma > 0,\ \alpha + \beta + \gamma < \pi - \varepsilon,\ \alpha\beta\gamma = \tfrac{\pi^3}{6^3} \right\} \\ &=: \quad \partial D'_\varepsilon \cup \partial D''_\varepsilon . \end{aligned}$$

Assume that $(\alpha, \beta, \gamma) \in \partial D'_\varepsilon$. Then $\alpha, \beta, \gamma < \pi$, so that $\alpha \pi^2 > \alpha\beta\gamma \geq \frac{\pi^3}{6^3}$, which implies that $\alpha > \frac{\pi}{216}$. Analogously: $\beta, \gamma > \frac{\pi}{216}$. Also:

$$\alpha < \pi - \beta - \gamma < \pi - \tfrac{2\pi}{216} = \tfrac{214\pi}{216} \,.$$

Analogously: $\beta, \gamma < \frac{214\pi}{216}$. Since there exists a real number M such that

$$\log \sin x > M \qquad \text{for } \ x \in [\tfrac{\pi}{216}, \tfrac{214\pi}{216}],$$

we obtain that

$$f(\alpha, \beta, \gamma) > 3M - 3 \log \sin \tfrac{\varepsilon}{3} \qquad \text{for } \ (\alpha, \beta, \gamma) \in \partial D'_\varepsilon \,.$$

Since $\log \sin \frac{\varepsilon}{3} \to -\infty$ as $\varepsilon \to 0$, there exists $\delta > 0$ such that $M > \log \sin \frac{\varepsilon}{3}$ for all $0 < \varepsilon < \delta$; that is, if $0 < \varepsilon < \delta$, then

$$f(\alpha, \beta, \gamma) > 0 \qquad \text{for } \ (\alpha, \beta, \gamma) \in \partial D'_\varepsilon \,. \tag{3}$$

In order to complete the proof of the proposition, it is sufficient to show that, if $0 < \varepsilon < \delta$, then $f(\alpha, \beta, \gamma) \geq 0$ on $\partial D''_\varepsilon$ (because then (2) implies that $f(\alpha, \beta, \gamma) > 0$ for $(\alpha, \beta, \gamma) \in D_\varepsilon$). Set $k = \pi^3/6^3$. Define

$$E_\varepsilon = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \mid \alpha, \beta > 0,\ \alpha + \beta + \tfrac{k}{\alpha\beta} < \pi - \varepsilon \right\}$$

and

$$F(\alpha, \beta) = \log \sin \alpha + \log \sin \beta + \log \sin \frac{k}{\alpha\beta} - 3 \log \sin \left(\frac{\pi - \alpha - \beta - \frac{k}{\alpha\beta}}{3} \right) .$$

If $(\alpha, \beta, \gamma) \in \partial D''_\varepsilon$, then $(\alpha, \beta) \in E_\varepsilon$ and $F(\alpha, \beta) = f(\alpha, \beta, \gamma)$. Therefore it is sufficient to prove that

$$F(\alpha, \beta) \geq 0 \qquad \text{for } \ (\alpha, \beta) \in E_\varepsilon .$$

Similarly, as before, if $(\alpha, \beta) \in \partial E_\varepsilon$, where

$$\partial E_\varepsilon = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \mid \alpha, \beta > 0,\ \alpha + \beta + \tfrac{k}{\alpha\beta} = \pi - \varepsilon \right\},$$

then $(\alpha, \beta, \frac{k}{\alpha\beta}) \in \partial D'_\varepsilon$, and, by (3),

$$F(\alpha,\beta) = f(\alpha,\beta,\tfrac{k}{\alpha\beta}) > 0 \qquad \text{for } (\alpha,\beta) \in \partial E_\varepsilon .$$

Therefore it suffices to prove that $F(\alpha,\beta) \geq 0$ for all stationary points $(\alpha,\beta) \in E_\varepsilon$. So, we assume that $(\alpha,\beta) \in E_\varepsilon$. We have

$$\frac{\partial F}{\partial \alpha}(\alpha,\beta) = \cot\alpha + \left(1 - \frac{k}{\alpha^2\beta}\right)\cot\left(\frac{\pi-\alpha-\beta-\frac{k}{\alpha\beta}}{3}\right) - \frac{k\cot\frac{k}{\alpha\beta}}{\alpha^2\beta}$$

and

$$\frac{\partial F}{\partial \beta}(\alpha,\beta) = \cot\beta + \left(1 - \frac{k}{\alpha\beta^2}\right)\cot\left(\frac{\pi-\alpha-\beta-\frac{k}{\alpha\beta}}{3}\right) - \frac{k\cot\frac{k}{\alpha\beta}}{\alpha\beta^2}.$$

From the equations

$$\frac{\partial F}{\partial \alpha}(\alpha,\beta) = \frac{\partial F}{\partial \beta}(\alpha,\beta) = 0, \tag{4}$$

we obtain that

$$\alpha\cot\alpha - \beta\cot\beta + (\alpha-\beta)\cot\left(\frac{\pi-\alpha-\beta-\frac{k}{\alpha\beta}}{3}\right) = 0. \tag{5}$$

We show that the above equality implies that $\alpha = \beta$. Suppose that $\alpha \neq \beta$ and without loss of generality assume that $\beta > \alpha$. Set

$$G(\alpha,\beta) = \frac{\alpha\cot\alpha - \beta\cot\beta}{\beta-\alpha} - \cot\left(\frac{\pi-\alpha-\beta-\frac{k}{\alpha\beta}}{3}\right).$$

We have

$$2\sqrt{\tfrac{k}{\pi}} + \beta < 2\sqrt{\tfrac{k}{\beta}} + \beta \leq \alpha + \tfrac{k}{\alpha\beta} + \beta < \pi, \tag{6}$$

which gives $0 < \beta < \pi - 2\sqrt{\frac{k}{\pi}}$. The function $g_1(\beta) = \beta\csc^2\beta - \cot\beta$ is non-decreasing in $\left(0, \pi - 2\sqrt{\frac{k}{\pi}}\right)$. This, together with (6), implies that

$$\begin{aligned} G(\alpha,\beta) &= \frac{1}{\beta-\alpha}\int_\alpha^\beta (x\csc^2 x - \cot x)\,dx - \cot\left(\frac{\pi-\alpha-\beta-\frac{k}{\alpha\beta}}{3}\right) \\ &< (\beta\csc^2\beta - \cot\beta) - \cot\left(\frac{\pi-2\sqrt{\frac{k}{\pi}}-\beta}{3}\right) < -0.75. \end{aligned}$$

[Ed: The last inequality can be verified by computer algebra using, for example, **DERIVE**.] Hence from (5) we deduce that $\alpha = \beta$.

Now (4) becomes

$$H(\alpha) := \cot\alpha + \left(1 - \frac{k}{\alpha^3}\right)\cot\left(\frac{\pi - 2\alpha - \frac{k}{\alpha^2}}{3}\right) - \frac{k}{\alpha^3}\cot\frac{k}{\alpha^2} = 0\,. \tag{7}$$

We know that $\pi - 2\alpha - \frac{k}{\alpha^2} > 0$. The equation $\pi - 2\alpha - \frac{k}{\alpha^2} = 0$ has two positive roots α_0, α_1 with $0.231 < \alpha_0 < 0.232$ and $1.540 < \alpha_1 < 1.541$, so $\alpha \in (\alpha_0, \alpha_1)$. We have

$$H'(\alpha) = h_1(\alpha) - h_2(\alpha) + h_3(\alpha) + h_4(\alpha) - h_5(\alpha)\,,$$

where

$$h_1(\alpha) = 3\cot\left(\frac{\pi - 2\alpha - \frac{k}{\alpha^2}}{3}\right), \quad h_2(\alpha) = \alpha\csc^2\alpha, \quad h_3(\alpha) = 3\cot\alpha\,,$$

$$h_4(\alpha) = \frac{2}{3\alpha^5}(\alpha^3 - k)^2\csc^2\left(\frac{\pi - 2\alpha - \frac{k}{\alpha^2}}{3}\right), \quad h_5(\alpha) = \frac{2k^2}{\alpha^5}\csc^2\left(\frac{k}{\alpha^2}\right).$$

We next verify [using again **DERIVE**] that

$$h_1(\alpha) - h_2(\alpha) > 2.97 \qquad \text{and} \qquad h_3(\alpha) + h_4(\alpha) - h_5(\alpha) > 0.995$$

for $\alpha \in (\alpha_0, \alpha_1)$. Therefore, the equation $H(\alpha) = 0$ has at most one solution. Thus, (7) implies that $\alpha = \frac{\pi}{6}$. Hence $(\alpha, \beta) = (\frac{\pi}{6}, \frac{\pi}{6})$ is the only stationary point of F in E_ε and we have $F(\frac{\pi}{6}, \frac{\pi}{6}) = 0$.

The proof is now complete.

2206. [1997: 46; 1998: 61, 62] *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let a and b denote distinct positive real numbers.

(a) Show that if $0 < p < 1$, $p \neq \frac{1}{2}$, then

$$\frac{1}{2}\left(a^p b^{1-p} + a^{1-p} b^p\right) < 4p(1-p)\sqrt{ab} + (1 - 4p(1-p))\frac{a+b}{2}.$$

(b) Use (a) to deduce Pólya's Inequality:

$$\frac{a-b}{\log a - \log b} < \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right).$$

Note: "log" is, of course, the natural logarithm.

III. *Solution to part* (a) *by the proposer, slightly adapted by the editor.*

With $r = \sqrt{ab}$ and $x = \log(a/r)$, we have $x \neq 0$, $a = re^x$, and $b = re^{-x}$. With $q = 2p - 1$ (so that $|q| < 1$), the desired inequality (after dividing by r) becomes $\cosh(qx) < (1 - q^2) + q^2 \cosh(x)$. Note that

$$(1 - q^2) + q^2 \cosh(x) - \cosh(qx) \;=\; \sum_{k=1}^{\infty} \frac{x^{2k+1}}{2k+1!} q^2 \left(1 - q^{2k}\right) \;>\; 0$$

since $(1 - q^{2k}) > 0$ for $k \geq 1$. Thus the desired inequality is proved.

2240. [1997: 243] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that $\dfrac{BD}{DC} \leq \dfrac{BF}{FA} \leq 1$ and $\dfrac{AE}{EC} \leq \dfrac{AF}{FB}$.

Prove that $[DEF] \leq \dfrac{[ABC]}{4}$ with equality if and only if two of the three points D, E, F, (at least) are mid-points of the corresponding sides.

Note: $[XYZ]$ denotes the area of triangle $\triangle XYZ$.

[*Editor's note*: Most solvers noted that the condition for equality should actually be:

F plus at least one of D or E be the midpoints of the corresponding sides.]

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

There exist numbers $u \in [0, 1)$, $v \in (0, 1]$, $w \in (0, 1)$ such that

$$\begin{aligned} D &= & (1-u)B + \quad uC \quad, \\ E &= \quad vA & + (1-v)C, \\ F &= (1-w)A + \quad wB & . \end{aligned}$$

Then $BD = uBC$, $DC = (1 - u)BC$, $AE = (1 - v)CA$, $EC = vCA$, $AF = wAB$ and $BF = (1 - w)AB$, so that the conditions

$$\frac{BD}{DC} \leq \frac{BF}{FA} \qquad \text{and} \qquad \frac{AE}{EC} \leq \frac{AF}{FB}$$

give $u/(1 - u) \leq (1 - w)/w$ and $(1 - v)/v \leq w(1 - w)$ or

$$u + w \leq 1 \leq v + w. \tag{1}$$

As is well-known, we have

$$\frac{[DEF]}{[ABC]} = \begin{vmatrix} 0 & 1-u & u \\ v & 0 & 1-v \\ 1-w & w & 0 \end{vmatrix}.$$

[*Editor's note*: A triangle, $X = (x_1, x_2),\ Y = (y_1, y_2),\ Z = (z_1, z_2)$, has area

$$[XYZ] = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix};$$

see, for example, 13.45, H.S.M. Coxeter, *Introduction to Geometry*, John Wiley and Sons, Inc., London (1961). Then

$$[DEF] = \frac{1}{2}\begin{vmatrix} d_1 & d_2 & 1 \\ e_1 & e_2 & 1 \\ f_1 & f_2 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & 1-u & u \\ v & 0 & 1-v \\ 1-w & w & 0 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.]$$

So

$$\frac{[DEF]}{[ABC]} = w(1-w) - (1-u-w)(v+w-1). \qquad (2)$$

Since $w(1-w) \leq \frac{1}{4},\ 0 < w < 1$, with equality if and only if $w = \frac{1}{2}$, from (1) and (2) we obtain the desired inequality. There is equality if and only if $w = \frac{1}{2}$ and $u = \frac{1}{2}$ or $v = \frac{1}{2}$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. There was one incomplete solution.

2241. [1997: 243] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Triangle ABC ($AB \neq AC$) has incentre I and circumcentre O. The incircle touches BC at D. Suppose that $IO \perp AD$.

Prove that AD is a symmedian of triangle ABC. (The symmedian is the reflection of the median in the internal angle bisector.)

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Lemma. *In a non-equilateral triangle ABC with side lengths a, b, c, incentre I and circumcentre O, let P be the point on the half-line starting at B in the direction of C for which $BP = c$, and Q be on the half-line from A to C with $AQ = c$; then $PQ \perp IO$.*

Proof. Let O' and O'' be the projections of O on BC and AC respectively, and let I' and I'' be the projections of I on those sides. Let S be the point where OO' intersects II''. Consider triangles CPQ and SOI. Since $CO' = \frac{1}{2}a$ and $CI' = \frac{1}{2}(a+b-c)$, $O'I' = \frac{1}{2}(b-c)$; similarly, $O''I'' = \frac{1}{2}(a-c)$.

Furthermore $\angle OSI = \angle SOO'' = \gamma$ (because the sides of the first two angles are perpendicular to the sides of the third). It follows that

$$OS = \frac{O''I''}{\sin\gamma} = \frac{a-c}{2\sin\gamma}, \quad \text{and} \quad IS = \frac{O'I'}{\sin\gamma} = \frac{b-c}{2\sin\gamma}.$$

This means that $\triangle CPQ \sim \triangle SOI$ (by side-angle-side). Since $SO \perp CP$ and $SI \perp CQ$, it follows that $OI \perp PQ$.

For the solution to our problem we must show that $CD : BD = b^2 : c^2$ (which, according to standard references, is a property that holds if and only if AD is a symmedian). Because we take $AD \perp IO$, the lemma implies that $AD \parallel PQ$, so that $\triangle ACD \sim \triangle QPC$. Thus $CP : CQ = CD : AC$, or $\frac{c-a}{c-b} = \frac{a+b-c}{2b}$, so that

$$a = \frac{b^2+c^2}{b+c}. \tag{1}$$

From $CD = s - c$ and $BD = s - b$ we conclude

$$\frac{CD}{BD} = \frac{a+b-c}{a-b+c}. \tag{2}$$

Plugging (1) into (2) gives the desired conclusion that $CD : BD = b^2 : c^2$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer.

It is noteworthy that this result and that of **2246** *(which appears later in this issue) follow immediately from the same lemma, even though they seem to have little else in common except for the same four solvers. Perhaps a peculiar kind of sunspot activity would account for mathematicians as far apart as the Netherlands and Japan conceiving such related problems at the same time. Smeenk believes that his lemma is known, but he found it easier to prove than to find; in fact, he supplied two proofs.*

2242. [1997: 243] *Proposed by K.R.S. Sastry, Dodballapur, India.*

$ABCD$ is a parallelogram. A point P lies in the plane such that

1. the line through P parallel to DA meets DC at K and AB at L,
2. the line through P parallel to AB meets AD at M and BC at N, and
3. the angle between KM and LN is equal to the non-obtuse angle of the parallelogram.

Find the locus of P.

Editor's comment. Even though the number of solvers was relatively small, no two solutions were alike. This seemed to result, in part, from various interpretations of the proposal. For example, if one considers a parallelogram as the convex hull of its vertices (M is on the line segment AD, for example) and that the angle at vertex A is acute, then the locus of P is the empty set. Nonempty loci included various conic arcs according to the aforementioned interpretations. It would not seem instructive to try to report all of these; rather, we attempt to give a representative summary.

The common approach was to use a standard analytic argument to derive the equation of a conic which passes through some, or all, of the vertices of the parallelogram, again, depending on interpretation. However, the actual derivation of the conic equations required the omission of the vertices so that they are not properly in the locus of P. Smeenk was the only solver to point this out specifically.

Most of the solvers assumed that $\angle BAD < \pi/2$ which causes P to be outside the parallelogram. Smeenk, assuming further that $\angle LSM = \angle BAD$, where $S = MK \cap LN$, derives the locus equation

$$x(x-a) + y(y-b) - 2\cos\angle BAD(x-a)(y-b) = 0,$$

which is that of an ellipse through the vertices $B(a,0)$, $C(a,b)$, $D(0,b)$.

Con Amore pointed out that for P to be in the convex hull of the vertices $\angle BAD > \pi/2$. Also, assuming that $MK \cap LN$ lies in the half-plane from BD containing A, they derive the equation of the composite quartic

$$[(a-x)x - (b-y)y]\,[(a-x)x + (b-y)y - 2(a-x)(b-y)\cos v] \;=\; 0\,,$$

where $v = \angle ABC < \pi/2$. The first component gives the locus as the intersection of a hyperbola with the interior of the parallelogram while the second (similar to Smeenk's equation) gives the locus as an elliptic arc also in the interior of the parallelogram. If $MK \cap LN$ lies in the half-plane from BD not containing A, then the resulting elliptic arc is symmetric to the derived one with respect to the centre of the parallelogram.

The proposer points out that, since the locus is the circumcircle of a rectangle if the angle at vertex A is a right angle, this could be the springboard for a more elegant solution, presumably, via some affine transformation. Konečný remarks that the parallelogram can, in fact, be obtained by right-angled projection of the rectangle onto the plane which has in common the diagonal, say AC of the parallelogram.

Bradley derives the result that the lines KM, LN and AC are concurrent.

Solutions were received from CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2243. [1997: 243] *Proposed by F.J. Flanigan, San Jose State University, San Jose, California, USA.*

Given $f(x) = (x - r_1)(x - r_2)\dots(x - r_n)$ and $f'(x) = n(x - s_1)(x - s_2)\dots(x - s_{n-1})$, $(n \geq 2)$, consider the harmonic mean h of the $n(n-1)$ differences $r_i - s_j$.

If $f(x)$ has a multiple root, then h is undefined, because at least one of the differences is zero.

Calculate h when $f(x)$ has no multiple roots.

All solutions submitted were essentially the same.

From $f(x) = \prod_{k=1}^{n}(x - r_k)$, we get, by logarithmic differentiation, that

$$\frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{1}{x - r_k},$$

giving

$$\frac{f'(s_j)}{f(s_j)} = \sum_{k=1}^{n} \frac{1}{s_j - r_k} = 0 \tag{1}$$

for $k = 1, 2, \dots, n-1$.

Now, the harmonic mean h is defined by

$$\frac{1}{h} = \frac{1}{n(n-1)} \sum_{j=1}^{n-1}\sum_{k=1}^{n} \frac{1}{r_k - s_j}.$$

In view of (1), we may say, according to temperament, that h is infinite or undefined.

Solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; DAVID STONE and VREJ ZARIKIAN, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

The proposer said that the problem was inspired by A.G. Clark's solution to problem 3034 in the American Mathematical Monthly ***1930****, p. 317. See also the American Mathematical Monthly* ***1923****, p. 276 and* ***1930****, p. 94.*

2244. [1997: 243] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle and D is a point on AB produced beyond B such that $BD = AC$, and E is a point on AC produced beyond C such that $CE = AB$. The perpendicular bisector of BC meets DE at P.

Prove that $\angle BPC = \angle BAC$.

Solution by Stergiou Haralampos, Chalkis, Greece.

Let $|AC| = b$ and $|AB| = c$. We draw $DF||AC$ with $|DF| = b$ and F on the opposite side of DE from A. Then $ACFD$ is a parallelogram; so $|CF| = b+c$ and $\angle CAD = \angle CFD$. Let $H = CF \wedge DE$. Since $CH||AD$ and $\triangle ADE$ is isosceles, $\angle CHE = \angle ADE = \angle CEH$. It follows that $|CH| = c$, so $|HF| = b$. If we draw BH, then $ABHC$ is a parallelogram. Hence $\angle BHC = \angle BAC = \angle CFD$ (above). It is easy to see that $BHFD$ is a rhombus, so $|PF| = |PB| = |PC|$ and $\angle PBH = \angle PFH = \angle PCH$. But if $\angle PBH = \angle PCH$, it follows that $BCHP$ is a cyclic quadrilateral, so $\angle BPC = \angle BHC = \angle BAC$.

[Note that this proof works whether P is between D and E or not.]

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; CHARLES DIMINNIE and TREY SMITH, Angelo State University, San Angelo, Texas; JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VICTOR OXMAN, University of Haifa, Haifa, Israel; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; K.R.S. SASTRY and JEEVAN SANDHYA, Bangalore, India; D.J. SMEENK, Zaltbommel, the Netherlands; KAREN YEATS, student, St. Patrick's High School, Halifax, Nova Scotia; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

2245. [1997: 244] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Prove that $\dfrac{3^n + (-1)^{\binom{n}{2}}}{2} - 2^n$ is divisible by 5 for $n \geq 2$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

I will prove that the assertion is true for all non-negative integers n. [Ed: with the usual convention that $\binom{0}{2} = \binom{1}{2} = 0$.]

Since $3^4 \equiv 1 \pmod 5$, since $2^4 \equiv 1 \pmod 5$ and since

$$\binom{n+4}{2} = \frac{(n+4)(n+3)}{2} \equiv \frac{n(n-1)}{2} \pmod 2 = \binom{n}{2} \pmod 2,$$

we deduce that $f(n+4) \equiv f(n) \pmod 5$, where $f(n) = \dfrac{3^n + (-1)^{\binom{n}{2}}}{2} - 2^n$.

Since $f(0) = f(1) = f(2) = 0$ and $f(3) = 5$, it follows that $f(n)$ is divisible by 5 for all non-negative integers n.

Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; SAM BAETHGE, Nordheim, Texas, USA; MANSUR BOASE, student, St. Paul's School, London, England; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; ALAN LING, student, University of Toronto, Toronto, Ontario; VEDULA N. MURTY, Visakhapatnam, India; VICTOR OXMAN, University of Haifa, Haifa, Israel; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; CHRISTOS SARAGIOTIS, student, Aristotle University, Thessaloniki, Macedonia, Greece; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; REZA SHAHIDI, student, University of Waterloo, Waterloo, Ontario; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

2246. [1997: 244] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that G, I and O are the centroid, the incentre and the circumcentre of a non-equilateral triangle ABC.

The line through B, perpendicular to OI intersects the bisector of $\angle BAC$ at P. The line through P, parallel to AC intersects BC at M.

Show that I, G and M are collinear.

Solution by the proposer.

Lemma. *In a non-equilateral triangle ABC with side lengths a, b, c, incentre I and circumcentre O, let D be the point on the half-line starting at C in the direction of A for which $CD = a$, and E be on the half-line from B to A with $BE = a$; then $DE \perp IO$.*

This lemma was proved as part of the foregoing solution to **2241** (in the notation of that problem). We assume that all points are well defined. [See the remarks below, after the list of solvers.] From the lemma,

$AD = a - b, AE = a - c$, and $ED \perp IO$ so that (because we are given $BP \perp IO$) $BP \parallel ED$. Let Q be the point where BP intersects AC. Then $\triangle ADE \sim AQB$. It follows that $AB : AQ = AE : AD = (a-c) : (a-b)$. Since AP bisects $\angle BAQ$, $AB : AQ = BP : PQ$, while $PM \parallel AC$ implies $BP : PQ = BM : MC$. Thus, $BM : MC = (a-c) : (a-b)$.

We introduce trilinear coordinates with respect to $\triangle ABC$. The coordinates of I are $(1,1,1)$, of G are (bc, ca, ab), and of M are

$$(0, MC\sin\gamma, MB\sin\beta) = (0, c(a-b), b(a-c)).$$

Then I, G, and M are collinear if and only if

$$\det\begin{bmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ 0 & c(a-b) & b(a-c) \end{bmatrix} = 0,$$

which is easily confirmed.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; and TOSHIO SEIMIYA, Kawasaki, Japan.

All solvers used trilinear coordinates. Orthogonality is generally awkward with these coordinates, and each of the submitted solutions displayed clever insight to overcome the difficulty that arises. The lemma in our featured solution simplifies matters considerably.

Note that one must assume that $\triangle ABC$ is not equilateral — otherwise $O = I$; furthermore, as Bradley points out, IG must not be parallel to BC — otherwise M is not defined. Smeenk's problem implies that the latter condition is equivalent to forbidding $AI \perp IO$. Bradley determined that the required assumption should be that $2a \neq b + c$. Putting these remarks together we deduce the unexpected consequence,

For a triangle ABC with side lengths a, b, c, and with distinct centroid G, incentre I, and circumcentre O, $IG \parallel BC$ if and only if $AI \perp IO$, if and only if $2a = b + c$.

This observation is closely related to problem 1506 in Mathematics Magazine **70** : 4 *(October, 1997) 302-303:* Prove that $\angle AIO \leq 90^\circ$ if and only if $2a \leq b + c$, with equality holding only simultaneously.

2248. [1997: 245] *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Find the value of the sum $\sum_{k=1}^{\infty} \frac{d(k)}{k^2}$, where $d(k)$ is the number of positive integer divisors of k.

Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.

The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges absolutely to $\pi^2/6$. Hence the terms of the series $\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^2$ may be rearranged in any order without changing the value of the sum $\pi^4/36$. Thus

$$\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^2 = \left(\sum_{i=1}^{\infty} \frac{1}{i^2}\right)\left(\sum_{j=1}^{\infty} \frac{1}{j^2}\right) = \sum_{k=1}^{\infty} \sum_{ij=k} \frac{1}{i^2 j^2} = \sum_{k=1}^{\infty} \frac{d(k)}{k^2},$$

since there are exactly $d(k)$ pairs (i, j) such that $ij = k$. Thus

$$\sum_{k=1}^{\infty} \frac{d(k)}{k^2} = \frac{\pi^4}{36}.$$

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; MANSUR BOASE, student, St. Paul's School, London, England; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; ALAN LING, University of Toronto, Toronto, Ontario; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

Several solvers gave references to places where this problem or generalizations were treated. Indeed, DOSTER gives a reference to the best generalization with his comment:

The series is a Dirichlet series. Hardy and Wright (*An Introduction to the Theory of Numbers*, 5th ed., pp. 248-250) work out the general theory of this type of sum and prove, more generally, that $\zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^k}$, where $s > 1$, and $s > k+1$, where $\sigma_k(n) = \sum_{d|n} d^k$ and $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Let $s = 2$ and $k = 0$ to get the required result.

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