THE ACADEMY CORNER

No. 16

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

This month, we present some more of the solutions to a university entrance scholarship examination paper from the 1940's, which appeared in the April 1997 issue of ***CRUX with MAYHEM*** [1997: 129].

4. (a) Suppose that $a \neq 0$ and $c \neq 0$, and that $ax^3 + bx + c$ has a factor of the form $x^2 + px + 1$. Show that $a^2 - c^2 = ab$.

   (b) In this case, prove that $ax^3 + bx + c$ and $cx^3 + bx^2 + a$ have a common quadratic factor.

Solution.

(a) The given conditions on the coefficients show that $x = 0$ is not a solution. Also, the other factor must be linear, and, on examining coefficients, must be of the form $ax + c$. Therefore

$$ax^3+bx+c = (x^2+px+1)(ax+c) = ax^3+(c+ap)x^2+(a+cp)x+c.$$

Comparing coefficients gives

$$\begin{aligned} c + ap &= 0, \\ a + cp &= b. \end{aligned}$$

Thus, $p = -\frac{c}{a}$, so that $a - \frac{c^2}{a} = b$, giving the required result.

(b) Since $x \neq 0$, we write $y = \frac{1}{x}$, and examine the corresponding equations in y to get this part.

5. Prove that all the circles in the family defined by the equation

$$x^2 + y^2 - a(t^2 + 2)x - 2aty - 3a^2 = 0$$

(a fixed, t variable) touch a fixed straight line.

Solution.

We re-write the equation as

$$\left(x - a\left(1 + \frac{t^2}{2}\right)\right)^2 + (y - at)^2 = \left(a\left(2 + \frac{t^2}{2}\right)\right)^2.$$

This shows that the line $x = -a$ is a tangent line. (The y–coordinate is $2at$.)

6. Find the equation of the locus of a point P which moves so that the tangents from P to the circle $x^2 + y^2 = r^2$ cut off a line segment of length $2r$ on the line $x = r$.

Solution.

From the circle, $x^2 + y^2 = a^2$, we find the points of intersection with the line $(y - q) = m(x - p)$.

Substituting $y = mx - mp + q$, we get $x^2 + (mx - mp + q)^2 = a^2$, leading to the two solutions:

$$x = \frac{m(mp - q) \pm \sqrt{a^2(m^2 + 1) - m^2p^2 + q(2mp - q)}}{m^2 + 1}.$$

For the line to be a tangent, the discriminant, $\Delta = a^2(m^2 + 1) - m^2p^2 + q(2mp - q)$, must be zero. So we solve $\Delta = 0$, to get

$$m = \frac{pq}{p^2 - a^2} \pm \frac{a\sqrt{p^2 + q^2 - a^2}}{p^2 - a^2}.$$

So, we have that the equations of the tangent lines from (p, q) to the circle $x^2 + y^2 = a^2$ are:

$$\begin{aligned} y &= \left(-\frac{pq}{a^2 - p^2} \pm \frac{a\sqrt{a^2 - p^2 - q^2}}{p^2 - a^2}\right) x \\ &\quad - \left(-pq/(a^2 - p^2) + \pm \frac{\sqrt{a^2 - p^2 - q^2}}{p^2 - a^2}\right) p + q. \end{aligned}$$

Setting $x = a$ in these, and subtracting the two values leads to

$$(a + p)^2 = p^2 + q^2 - a^2.$$

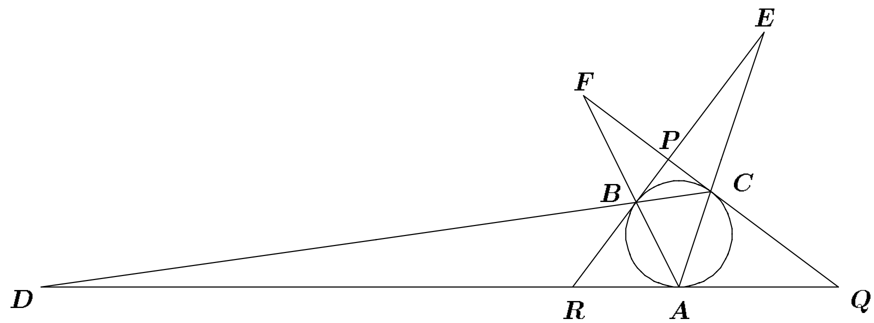
Thus the locus of P is given by

$$y^2 = 2a(a + x).$$

This is a parabola opening to the right with central axis, the x–axis and nose at $x = -a$. There are, of course, three points that should be excluded: $(-a, 0)$, $(a, 2a)$, and $(a, -2a)$.

7. If the tangents at A, B, C, to the circumcircle of triangle $\triangle ABC$ meet the opposite sides at D, E, F, respectively, prove that D, E, F, are collinear.

Solution.



For notational ease, we write $PB = PC = \alpha$, $QC = QA = \beta$ and $RB = RA = \gamma$.

Apply Menelaus' Theorem for triangle $\triangle PQR$ with transversals BDC, EAC and BAF

$$\begin{aligned} -1 &= \frac{PB}{BR}\frac{RD}{DQ}\frac{QC}{CP} = \frac{QC}{BR}\frac{RD}{DQ}, \\ -1 &= \frac{PE}{ER}\frac{RA}{AQ}\frac{QC}{CP} = \frac{PE}{ER}\frac{RA}{CP}, \\ -1 &= \frac{PB}{BR}\frac{RA}{AQ}\frac{QF}{FP} = \frac{PB}{AQ}\frac{QF}{FP}. \end{aligned}$$

These lead to

$$\frac{\gamma+\beta+QD}{QD} = -\frac{BR}{QC} = -\frac{\gamma}{\beta}, \tag{1}$$

$$\frac{\alpha+\gamma+ER}{ER} = -\frac{CP}{RA} = -\frac{\alpha}{\gamma}, \tag{2}$$

$$\frac{QF}{FQ+\alpha+\beta} = -\frac{AQ}{PB} = -\frac{\beta}{\alpha}. \tag{3}$$

Now consider the product of ratios to prove D, E and F collinear by the converse of Menelaus:

$$\frac{PE}{ER}\frac{RD}{DQ}\frac{QF}{FP}.$$

This is equal to

$$\frac{\alpha+\beta+ER}{ER}\frac{\gamma+\beta+QD}{QD}\frac{QF}{FQ+\alpha+\beta}.$$

Using (1), (2) and (3) above, we get a value of -1, and the result is proved.

THE OLYMPIAD CORNER

No. 187

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

Another year has passed — How the time seems to fly. I would particularly like to thank Joanne Longworth for her excellent help in pulling together the information for the column and her work in producing a good LaTeX copy for the Editor-in-Chief – usually under enormous time pressure because I am behind schedule.

The Corner could not exist without its readers who supply me with good Olympiad materials and send in their interesting solutions and comments to problems in the Corner. I hope we have missed no one in the following list of contributors.

Miguel Amengual Covas	Joel Kamnitzer	Bob Prielipp
Šefket Arslanagić	Deepee Khosla	Toshio Seimiya
Mansur Boase	Derek Kisman	Michael Selby
Christopher Bradley	Murray Klamkin	Zun Shan
Sabin Cautis	Marcin Kuczma	D.J. Smeenk
Adrian Chan	Andy Liu	Daryl Tingley
Byung-Kuy Chun	Beatriz Margolis	Panos Tsaoussoglou
Mihaela Enachescu	Vedula Murty	Ravi Vakil
George Evagelopoulos	Richard Nowakowski	Dan Velleman
Shawn Godin	Colin Percival	Stan Wagon
Joanne Juszuńska		Edward Wang

Thank you all, and all the best for 1998!

The first Olympiad we give for the new year is the 17$^{\text{th}}$ Austrian-Polish Mathematics Competition, written in Poland, June 29–July 1, 1994. My thanks go to Richard Nowakowski, Canadian Team Leader to the 35$^{\text{th}}$ IMO in Hong Kong and to Walther Janous, Ursulinengymnasium, Innsbruck, Austria for sending a copy to me.

17$^{\text{th}}$ AUSTRIAN–POLISH MATHEMATICS COMPETITION
Poland, June 29–July 1, 1994

1. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies for all $x \in \mathbb{R}$ the conditions

$$f(x+19) \leq f(x) + 19 \quad \text{and} \quad f(x+94) \geq f(x) + 94.$$

Show that $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$.

2. The sequence (a_n) is defined by the formulas

$$a_0 = \frac{1}{2} \quad \text{and} \quad a_{n+1} = \frac{2a_n}{1+a_n^2} \quad \text{for} \quad n \geq 0,$$

and the sequence (c_n) is defined by the formulas

$$c_0 = 4 \quad \text{and} \quad c_{n+1} = c_n^2 - 2c_n + 2 \quad \text{for} \quad n \geq 0.$$

Prove that

$$a_n = \frac{2c_0c_1 \ldots c_{n-1}}{c_n} \quad \text{for all} \quad n \geq 1.$$

3. A rectangular building consists of two rows of 15 square rooms (situated like the cells in two neighbouring rows of a chessboard). Each room has three doors which lead to one, two or all the three adjacent rooms. (Doors leading outside the building are ignored.) The doors are distributed in such a way that one can pass from any room to any other one without leaving the building. How many distributions of the doors (in the walls between the 30 rooms) can be found so as to satisfy the given conditions?

4. Let $n \geq 2$ be a fixed natural number and let P_0 be a fixed vertex of the regular $(n+1)$-gon. The remaining vertices are labelled $P_1, P_2, \ldots, P_n$, in any order. To each side of the $(n+1)$-gon assign a natural number as follows: if the endpoints of the side are labelled P_i and P_j, then $i - j$ is the number assigned. Let S be the sum of all the $n+1$ numbers thus assigned. (Obviously, S depends on the order in which the vertices have been labelled.)

(a) What is the least value of S available (for fixed n)?
(b) How many different labellings yield this minimum value of S?

5. Solve the equation

$$\frac{1}{2}(x+y)(y+z)(z+x) + (x+y+z)^3 = 1 - xyz$$

in integers.

6. Let $n > 1$ be an odd positive integer. Assume that the integers $x_1, x_2, \ldots, x_n \geq 0$ satisfy the system of equations

$$\begin{aligned} (x_2 - x_1)^2 + 2(x_2 + x_1) + 1 &= n^2 \\ (x_3 - x_2)^2 + 2(x_3 + x_2) + 1 &= n^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (x_1 - x_n)^2 + 2(x_1 + x_n) + 1 &= n^2. \end{aligned}$$

Show that either $x_1 = x_n$ or there exists j with $1 \leq j \leq n-1$ such that $x_j = x_{j+1}$.

7. Determine all two-digit (in decimal notation) natural numbers $n = (ab)_{10} = 10a + b$ $(a \geq 1)$ with the property that for every integer x the difference $x^a - x^b$ is divisible by n.

8. Consider the functional equation $f(x, y) = a\, f(x, z) + b\, f(y, z)$ with real constants a, b. For every pair of real numbers a, b give the general form of functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying the given equation for all $x, y, z \in \mathbb{R}$.

9. On the plane there are given four distinct points A, B, C, D lying (in this order) on a line g, at distances $AB = a$, $BC = b$, $CD = c$.

(a) Construct, whenever possible, a point P, not on g, such that the angles $\angle APB$, $\angle BPC$, $\angle CPD$ are equal.

(b) Prove that a point P with the property as above exists if and only if the following inequality holds: $(a + b)(b + c) < 4ac$.

As a second source of problem pleasure for winter evenings (that is if you are having winter this January!), we give the Second Round of the Iranian National Mathematical Olympiad. My thanks go to Richard Nowakowski, Canadian Team Leader to the 35^{th} IMO in Hong Kong for collecting the problems and forwarding them to me.

IRANIAN NATIONAL MATHEMATICAL OLYMPIAD
February 6, 1994
Second Round

1. Suppose that p is a prime number and is greater than 3. Prove that $7^p - 6^p - 1$ is divisible by 43.

2. Let ABC be an acute angled triangle with sides and area equal to a, b, c and S respectively. Show that the necessary and sufficient condition for existence of a point P inside the triangle ABC such that the distance between P and the vertices of ABC be equal to x, y and z respectively is that there be a triangle with sides a, y, z and area S_1, a triangle with sides b, z, x and area S_2 and a triangle with sides c, x, y and area S_3 where $S_1 + S_2 + S_3 = S$.

3. Let n and r be natural numbers. Find the smallest natural number m satisfying this condition: For each partition of the set $\{1, 2, \ldots, m\}$ into r subsets $A_1, A_2, \ldots, A_r$ there exist two numbers a and b in some A_i $(1 \leq i \leq r)$ such that $1 < \frac{a}{b} \leq 1 + \frac{1}{n}$.

4. G is a graph with n vertices $A_1, A_2, \ldots, A_n$ such that for each pair of nonadjacent vertices A_i and A_j there exists another vertex A_k that is adjacent to both A_i and A_j.

(a) Find the minimum number of edges of such a graph.

(b) If $n = 6$ and $A_1, A_2, A_3, A_4, A_5, A_6$ form a cycle of length 6, find the number of edges that must be added to this cycle such that the above condition holds.

5. Show that if D_1 and D_2 are two skew lines, then there are infinitely many straight lines such that their points have equal distance from D_1 and D_2.

6. $f(x)$ and $g(x)$ are polynomials with real coefficients such that for infinitely many rational values x, $\frac{f(x)}{g(x)}$ is rational. Prove that $\frac{f(x)}{g(x)}$ can be written as the ratio of two polynomials with rational coefficients.

Now we turn to readers' solutions and comments for problems posed in the Corner. First solutions to a problem of the 25^{th} United States of America Mathematical Olympiad [1996: 203–204].

1. Prove that the average of the numbers

$$n \sin n^\circ, \quad n = 2, 4, 6, \dots, 180$$

is $\cot 1^\circ$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.
Let

$$\begin{aligned} x &= 2\sin 2^\circ + 4\sin 4^\circ + \cdots + 90\sin 90^\circ + \cdots + 178\sin 178^\circ \\ &= (2+178)\sin 2^\circ + (4+176)\sin 4^\circ + \cdots \\ &= 180(\sin 2^\circ + \sin 4^\circ + \cdots + \sin 88^\circ) + 90\sin 90^\circ. \end{aligned}$$

Then

$$\bar{x} = \frac{x}{90} = 2\sin 2^\circ + 2\sin 4^\circ + \cdots + 2\sin 88^\circ + 1$$

$$\bar{x}\sin 1^\circ = 2\sin 2^\circ \sin 1^\circ + 2\sin 4^\circ \sin 1^\circ + \cdots + 2\sin 88^\circ \sin 1^\circ + \sin 1^\circ.$$

Now

$$\begin{aligned} 2\sin 2^\circ \sin 1^\circ &= \cos 1^\circ - \cos 3^\circ \\ 2\sin 4^\circ \sin 1^\circ &= \cos 3^\circ - \cos 5^\circ \\ &\cdots \\ 2\sin 88^\circ \sin 1^\circ &= \cos 87^\circ - \cos 89^\circ. \end{aligned}$$

Hence

$$\begin{aligned} \bar{x}\sin 1^\circ &= \cos 1^\circ - \cos 89^\circ + \sin 1^\circ \\ &= \cos 1^\circ. \end{aligned}$$

Thus $\bar{x} = \cot 1^\circ$, as required.

Next we give a comment on a problem, and one solution to another from the 1994 Italian Mathematical Olympiad.

2. [1996:204] *Italian Mathematical Olympiad* 1994.

Find all integer solutions of the equation

$$y^2 = x^3 + 16.$$

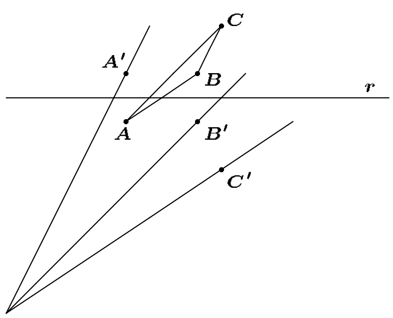
Comment by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

The equation has no solutions in integers different from $x = 0$, $y = \pm 4$. For a proof see Theorem 20, page 102 of W. Sierpinski's *Elementary Theory of Numbers*.

4. [1996: 204] *Italian Mathematical Olympiad* 1994

Let r be a line in the plane and let ABC be a triangle contained in one of the half-planes determined by r. Let A', B', C' be the points symmetric to A, B, C with respect to r; draw the line through A' parallel to BC, the line through B' parallel to AC and the line through C' parallel to AB. Show that these three lines have a common point.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.



There is no need that $\triangle ABC$ be contained in one of the half planes determined by r.

If the coordinates of A are (x_1, y_1), the coordinates of B are (x_2, y_2) and those of C are (x_3, y_3) in a Cartesian coordinate system with r as x-axis, then A' has coordinates $(x_1, -y_1)$, B' has coordinates $(x_2, -y_2)$ and C' has coordinates $(x_3, -y_3)$.

We have:

The equation of the line through A' parallel to BC is

$$(y_2 - y_3)x - (x_2 - x_3)y - x_1(y_2 - y_3) - y_1(x_2 - x_3) = 0.$$

The equation of the line through B' parallel to CA is

$$(y_3 - y_1)x - (x_3 - x_1)y - x_2(y_3 - y_1) - y_2(x_2 - x_1) = 0.$$

The equation of the line through C' parallel to AB is

$$(y_1 - y_2)x - (x_1 - x_2)y - x_3(y_1 - y_2) - y_3(x_1 - x_2) = 0.$$

Since the three equations when added together vanish identically, the lines represented by them meet in a point.

Its coordinates are found, by solving between any two, to be

$$\left(\frac{(x_1^2+x_2x_3)(y_2-y_3)+(x_2^2+x_3x_1)(y_3-y_1)+(x_3^2+x_1x_2)(y_1-y_2)}{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)},\right.$$

$$\left.\frac{(y_1^2+y_2y_3)(x_2-x_3)+(y_2^2+y_3y_1)(x_3-x_1)+(y_3^2+y_1y_2)(x_1-x_2)}{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)}\right).$$

Now we jump to the October 1996 number of the Corner, and two solutions to problems of the 10^{th} IBEROAMERICAN Mathematical Olympiad [1996: 251–252].

1. (*Brazil*). Determine all the possible values of the sum of the digits of the perfect squares.

Solution by Mansur Boase, student, St. Paul's School, London, England.

The squares can only be 0, 1, 4 or $7 \bmod 9$.

Thus the sum of the digits of a perfect square cannot be 2, 3, 5, 6 or $8 \bmod 9$, since the number itself would then be 2, 3, 5, 6 or $8 \bmod 9$.

We shall show that the sum of the digits of a perfect square can take every value of the form 0, 1, 4 or $7 \bmod 9$.

$$(10^m-1)^2 = 10^{2m} - 2\cdot 10^m + 1 = \underbrace{9\,9\cdots 9}_{m-1}\;8\;\underbrace{0\cdots 0}_{m-1}\;1, \quad m \geq 1.$$

The sum of the digits is $9m$, giving all the values greater than or equal to 9 congruent to $0 \bmod 9$

$$(10^m-2)^2 = 10^{2m} - 4\cdot 10^m + 4 = \underbrace{9\,9\cdots 9}_{m-1}\;6\;\underbrace{0\cdots 0}_{m-1}\;4, \quad m \geq 1.$$

The sum of the digits is $9m+1$, which gives all values greater than or equal to 10 congruent to $1 \bmod 9$.

$$(10^m-3)^2 = 10^{2m} - 6\cdot 10^m + 9 = \underbrace{9\,9\cdots 9}_{m-1}\;4\;\underbrace{0\cdots 0}_{m-1}\;9, \quad m \geq 1.$$

The sum of the digits is $9m+4$, which takes every value greater than or equal to 13 which is congruent to $4 \bmod 9$

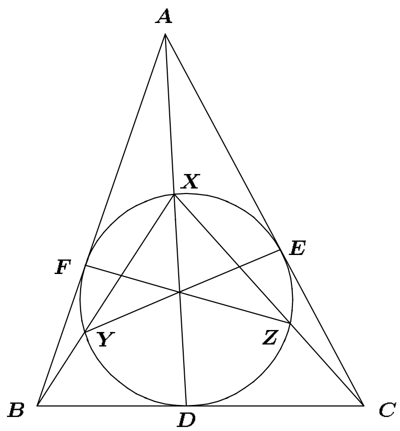
$$(10^m-5)^2 = 10^{2m} - 10^{m+1} + 25 = \underbrace{9\cdots 9}_{m-1}\underbrace{0\,0\cdots 0}_{m-1}\;2\,5.$$

The sum of the digits is $9(m-1)+7 = 9m-2$, from which we get every value greater than or equal to 7 congruent to $7 \bmod 9$.

We have taken care of all the integers apart from 0, 1, 4, which are the sums of the digits of 0^2, 1^2, and 2^2 respectively.

5. *(Spain)*. The inscribed circumference in the triangle ABC is tangent to BC, CA and AB at D, E and F, respectively. Suppose that this circumference meets AD again at its mid-point X; that is, $AX = XD$. The lines XB and XC meet the inscribed circumference again at Y and Z, respectively. Show that $EY = FZ$.

Solution by Toshio Seimiya, Kawasaki, Japan.



Since $\angle BFY = \angle BXF$ and $\angle FBY = \angle XBF$ we have $\triangle BFY$ and $\triangle BXF$ are similar, so that

$$FY : FX = BF : BX. \qquad (1)$$

Similarly we get

$$DY : DX = BD : BX. \qquad (2)$$

As $BF = BD$, we have from (1) and (2) that

$$FY : FX = DY : DX.$$

Since $AX = DX$ we get

$$FY : FX = DY : AX. \qquad (3)$$

Since X, F, Y, D are concyclic we have

$$\angle FYD = \angle AXF. \qquad (4)$$

Thus we get from (3) and (4) that $\triangle FYD$ is similar to $\triangle FXA$.

Hence $\angle YFD = \angle XFA = \angle XDF$ so that $FY \| XD$. Similarly we have $EZ \| XD$. Thus $FY \| EZ$.

Therefore $FYZE$ is an isosceles trapezoid and then $EY = FZ$.

We now turn to two solutions to the Maxi Finale 1994 of the Olympiade Mathématique Belge [1996: 253–254].

1. Un pentagone plan convexe a deux angles droits non adjacents. Les deux côtés adjacents au premier angle droit out des longueurs égales. Les deux côtés adjacents au second angle droit ont des longueurs égales. En remplaçant par leur point milieu les deux sommets du pentagone situés sur un seul côté de ces angles droits, nous formons un quadrilatère. Ce quadrilatère admet-il nécessairement un angle droit?

Solution by Mansur Boase, student, St. Paul's School, London, England.

Let the pentagon be $ABCDE$ and let $\angle ABC = \angle AED = 90°$. Let the midpoint of CD be M.

Then the problem is equivalent to proving that given any triangle ACD with isosceles right triangles constructed on AC and AD, giving points B and E, $\angle BM = 90°$ and so quadrilateral $BMEA$ must have a right angle.

This result is well known. One very nice proof is that if L and N are the midpoints of AC and AD respectively then $\triangle BLM \equiv \triangle MNE$ ($BL = LC = MN$, $LM = ND = EN$ and

$$\begin{aligned}\angle BLM = 90° + \angle CLM &= 90° + \angle CAD \\ &= 90° + \angle MND = \angle MNE).\end{aligned}$$

And since $ML \| ND$ and $ND \perp EN$, $ML \perp EN$ and similarly $BL \perp NM$. Therefore $BM \perp EM$ since the other two pairs of sides are perpendicular.

4. Le plan contient-il 1994 points (distincts) non tous alignés tels que la distance entre deux quelconques d'entre eux soit un nombre entier?

Solution by Mansur Boase, student, St. Paul's School, London, England.

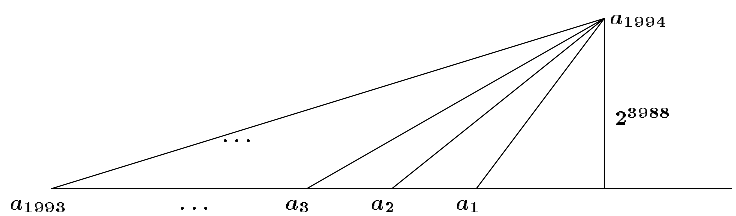
There are 1994 pairs of solutions (m, n) with $m > n$ satisfying

$$2mn = 2^{3988}$$

$$[(2^0, 2^{3987}), \ldots, (2^{1993}, 2^{1994})].$$

So there are 1994 Pythagorean triples of the form $(m^2 - n^2, 2mn, m^2 + n^2)$ with one side of length 2^{3988}.

Hence we can form a line of 1993 collinear points and place a 1994^{th} point a distance 2^{3998} above this line so that, with the correct arrangement of the points, if it is joined to any other point, the resulting line is the hypotenuse corresponding to a Pythagorean triple and is therefore integral. Also the distance between any two points on the line will be integral.



That completes the Corner for this issue. Olympiad season is fast upon us. Send me your contest materials and your nice solutions for use in the Corner.

BOOK REVIEWS

Edited by ANDY LIU

Mini-Reviews Update

Andy Liu

This is an update of earlier Mini-Reviews (see [1] to [9]), and is an abbreviated form of what appeared in the journals *Delta K* and **AGATE**. Written permission for reprint has been granted by the Alberta Teachers' Association which publishes the above two journals.

A. Mir Publishers' Little Mathematics Library Series (see [1])

This excellent series has become an unfortunate casualty of the demise of the former Soviet Union. Lost also are *Mathematics Can Be Fun* and *Fun with Maths and Physics* featured in Section I.

B. New Mathematical Library of the Mathematical Association of America (see [2])

In additional to the new titles listed below, there is also a revised edition of an earlier volume, *Graphs and Their Uses*. It was #10 in the series, but is now #34. Two further volumes have also been published recently.

USA Mathematical Olympiads: 1972-1986, by Murray Klamkin, 1988. **ISBN#** 0-88385-634-4.

This book collects the problems of the first fifteen USA Mathematical Olympiads. While they are presented chronologically, the solutions are grouped according to subject matters, which facilitates using this book for training sessions. There is a very useful 10-page glossary of mathematical terms and results, and a most extensive bibliography.

Exploring Mathematics with your Computer, by Arthur Engel, 1993. (see [16])

Game Theory and Strategy, by Philip Straffin, 1993. (see [17])

C. Martin Gardner's Scientific American Series (see [3])

Two more volumes have appeared, and there will be a fifteenth and final volume, about to be released by Springer-Verlag. Several earlier volumes have also changed publishers. The Mathematical Association of America has acquired *Martin Gardner's New Mathematical Diversions from Scientific American, Martin Gardner's 6th Book of Mathematical Diversions from Scientific American, Mathematical Carnival, Mathematical Magic Show* and *Mathematical Circus*. The University of Chicago Press has acquired *The Scientific American Book of Mathematical Puzzles and Diversions, The 2nd Scientific American Book of Mathematical Puzzles and Diversions* and *The Unexpected Hanging and Other Mathematical Diversions*.

Penrose Tiles to Trapdoor Ciphers, 1989, W. H. Freeman, 1997, Mathematical Association of America. **ISBN#** 0-88385-521-6.

Topics covered are Penrose tilings, Mandebrot's fractals, Conway's surreal numbers, mathematical wordplay, Wythoff's version of the game "Nim", mathematical induction, negative numbers, dissection puzzles, trapdoor ciphers, hyperbolas, the new version of the game "Eleusis", Ramsey theory, the mathematics of Berrocal's sculptures, curiosities in probability, Raymond Smullyan's logic puzzles, as well as two collections of short problems. The book contains a surprise ending, the resurrection of Dr. Matrix! There is also an update chapter. Unfortunately, Figures 3 to 6 are inadvertently left out. They are reproduced below.

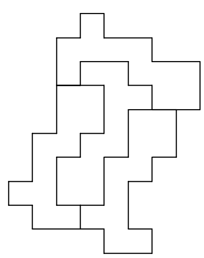


Figure 3.

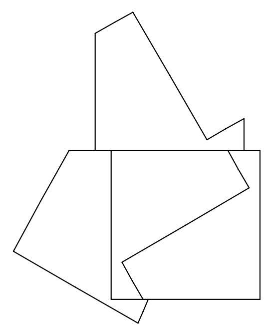


Figure 4.

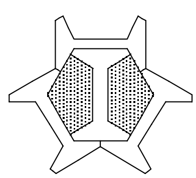


Figure 5.

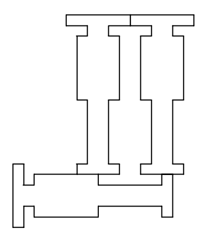


Figure 6.

Fractal Music, Hypercards and More, 1992, W.H. Freeman.
ISBN# 0-7167-2189-9.

Topics covered are fractal music, the Bell numbers, mathematical zoo, Charles Sanders Peirce, twisted prismatic rings, coloured cubes, Egyptian fractions, minimal sculptures, tangent circles, time, generalized tick-tack-toe, psychic wonders and probability, mathematical chess problems, Hofstadter's *Gödel, Escher, Bach*, imaginary numbers, some accidental patterns, packing squares, Chaitin's irrational number Ω, as well as one collection of short problems.

D. Books from W.H. Freeman & Company, Publishers (see [4])

Note that the most recent book in Martin Gardner's Scientific American Series is a Freeman publication. The first of the books listed below has actually gone out of print, but fortunately Dover Publications Inc. has decided to pick it up.

The Puzzling Adventures of Dr. Ecco, by Dennis Shasha, 1988.

The title character calls himself an omniheurist, solver of all problems (mathematical). The narrative is by a Watsonesque companion, Prof. Scarlet. Ecco's clients range from government officials, industrialists, eccentric millionaires to no less than the President of a Latin American country. They brought him important, instructive and interesting problems in discrete mathematics, all of which Ecco solves to their satisfaction. The book concludes with the mysterious disappearance of Dr. Ecco.

Codes, Puzzles, and Conspiracy, by Dennis Shasha, 1992. (see [11])

New Book of Puzzles, by Jerry Slocum and Jack Botermans, 1992. (see [12])

Another Fine Math You've Got Me Into . . . , by Ian Stewart, 1992. (see [13])

E. Oxford University Press Series on Recreations in Mathematics (see [5])

The Mathematics of Games, by John D. Beasley, 1989.
ISBN# 0-19-853206-7.

This book analyses mathematically some card and dice games, nim-type games, a version of John Conway's "Hackenbush", as well as providing a mathematical model for the study of some sports games. The principal techniques are counting, probability and game theory. Some mathematical puzzles are also considered.

The Puzzling World of Polyhedral Dissections, by Stewart T. Coffin, 1990.
ISBN# 0-19-853207-5.

This is a labour of love from an expert craftsman. Starting with two chapters of two-dimensional geometric puzzles, the author eases the readers gently into the third dimension and soon launches into his specialty, the burrs, which are assemblies of interlocking notched sticks. The book is profusely illustrated with black-and-white line drawings and photographs. It concludes with a chapter on woodworking techniques.

More Mathematical Byways, by Hugh ApSimon, 1990.
ISBN# 0-19-217777-X.

This book contains fourteen chapters. The first three form a sequence but the others are independent of each other. Unlike the earlier volume by the same author, the problems are of uneven level of difficulty, ranging from the relatively simple Alphametics to others which require a considerable amount of what the author calls "slog". One of the chapters, titled *Potential Pay*, is not really a problem but a commentary on a classic paradox.

F. Raymond Smullyan's Logic Series (see [6])

Satan, Cantor, and Infinity, 1992, Alfred A. Knopf.
ISBN# 0-679-40688-3.

In this book, a remarkable character known as the Sorcerer makes his debut. He escorts the readers on a wonderful guided tour, visiting familiar grounds such as the domains of the knights and the knaves, and those bordering on the land of Gödel. There are also ventures into new territories, including an island where intelligent robots create others which can continue this process *ad infinitum*. This eventually leads to the pioneering discoveries on infinity of the great mathematician, Georg Cantor. The readers may be amused to discover how Satan got into the picture.

G. Dolciani Mathematical Expositions Series of the Mathematical Association of America (see [7])

In addition to the new titles listed below, three further volumes have been published recently.

More Mathematical Morsels, by Ross Honsberger, 1991. (see [10])
ISBN# 0-88385-313-2.

This is a collection of 57 problems, almost all of which are taken from the Canadian Mathematical Society's journal *Crux Mathematicorum*, plus further "gleanings" from its famed *Olympiad Corner*.

Old and New Unsolved Problems in Plane Geometry and Number Theory, by Victor Klee and Stan Wagon, 1991. **ISBN#** 0-88385-315-9.

This book is divided into two halves, as suggested by the title, though the second half also covers problems about some interesting real numbers. Each half consists of two parts. In the first, twelve problems are presented, giving the statement, known results and background information. In the second, the same twelve problems are reexamined for further results and extensions. Each half concludes with a comprehensive bibliography. Although the problems are unsolved, and therefore hard, it is not impossible for them to yield to an inspired attack. Even if this does not happen, gifted students who are willing to attempt them will find their mathematical talent enhanced.

Problems for Mathematicians Young and Old, by Paul Halmos, 1992.
ISBN# 0-88385-320-5.

The fourteen chapters of this book are titled Combinatorics, Calculus, Puzzles, Numbers, Geometry, Tilings, Probability, Analysis, Matrices, Algebra, Sets, Spaces, Mappings and Measures. The author, a ranking mathematician and master expositor, wrote this book for fun, and hoped that it will be read the same way.

Excursions in Calculus, by Robert Young, 1992. **ISBN#** 0-88385-317-5.

The subtitle of this book is *An Interplay of the Continuous and the Discrete*. Using calculus as a unifying theme, the author branches into number theory, algebra, combinatorics and probability. The book contains a large collection of exercises and problems.

The Wohascum County Problem Book, by George Gilbert, Mark Krusemeyer and Loren Larson, 1993. (see [14])

Lion Hunting & Other Mathematical Pursuits, edited by Gerald Alexanderson and Dale Muggler, 1995. (see [18])

The Linear Algebra Problem Book, by Paul Halmos, 1995.
ISBN# 0-88385-322-1.

The whole book is a sequence of structured problems. Like the preceding volume, most of this book is beyond high school level. However, the introductory problems are certainly not intimidating, and inquisitive students may be lured into a most rewarding exploration, laying a good foundation for their undergraduate studies.

H. Books from Dover Publications, Inc. (see [8])

Excursions in Number Theory, by Stanley Ogilvy and John Anderson, 1988. **ISBN#** 0-486-25778-9.

This book covers the basics of classical number theory. Topics include prime numbers, congruences, Diophantine equations and Fibonacci numbers. The narrative style is very soothing. It concludes with 20 pages of elaborations and commentary on some finer points raised in the text.

Excursions in Geometry, by Stanley Ogilvy, 1990.
ISBN# 0-486-26530-7.

The first half of this book is on inversive geometry, and the second half on projective geometry. These two topics are linked by the concept of cross-ratio and the study of the conic sections. It is in the same style as the preceding volume.

Excursions in Mathematics, by Stanley Ogilvy, 1994.
ISBN# 0-486-28283-X.

The original title of this volume was *Through the Mathescope*. The opening chapter is titled *What Do Mathematicians Do?* It is followed by lively tours of number theory, algebra, geometry and analysis. The last chapter is titled *Topology and Apology*.

I. Books from Various Publishers (see [9])

Selected Problems and Theorems in Elementary Mathematics has been acquired by Dover and renamed *The USSR Olympiad Problem Book*. Dover has also picked up *The Moscow Puzzles*. The Mathematical Association of America has published *Five Hundred Mathematical Challenges*, comprising the first five booklets of the project, *1001 Problems in High School Mathematics*. (see [19]) The Canadian Mathematical Society has published *The Canadian Mathematical Olympiad, 1969–1993*, edited by Michael Doob and Claude Laflamme. This book combines two earlier volumes, *The First Ten Canadian Mathematics Olympiads, 1969–1978* and *The Canadian Mathematics Olympiads, 1979-1985*, and adds the contests from 1986 to 1993. (see [15])

Cross References to other entries in *Crux Mathematicorum*:

1. A. Liu, Mini-reviews, MIR Publishers' Little Mathematics Library Series, 1989, 142-147.
2. A. Liu, Mini-reviews, MAA's New Mathematical Library Series, 1989, 171-176.
3. A. Liu, Mini-reviews, Martin Gardner's Mathematical Games Series, 1989, 202-206
4. A. Liu, Mini-reviews, Books on popular mathematics from W. H. Freeman, 295-297.
5. A. Liu, Mini-reviews, Oxford University Press' Recreation in Mathematics Series, 1990, 42-43.
6. A. Liu, Mini-reviews, Raymond Smullyan's books on logic puzzles, 1990, 106-108.
7. A. Liu, Mini-reviews, MAA's Dolciani Mathematical Exposition Series, 1990, 237-238.
8. A. Liu, Mini-reviews, Books on popular mathematics from Dover, 1991, 11-13.
9. A. Liu, Mini-reviews, Books on popular mathematics from various publishers, 1991, 74-77.
10. A. Liu, Review of *More Mathematical Morsels*, 1991, 235-236.
11. A. Liu, Review of *Codes, Puzzles and Conspiracy*, 1992, 204-205.
12. A. Liu, Review of *New Books of Puzzles*, 1993, 13-14.
13. R.K. Guy, Review of *Another Fine Math You've Got Me Into*, 1993, 46-47.
14. A. Liu, Review of *The Wohascum County Problem Book*, 1993, 202.
15. R. Geretschläger and G. Perz, Review of *The Canadian Mathematical Olympiad Book, 1969-1993*, 1994, 15-16.
16. G.C. Denham, Review of *Exploring Mathematics with Your Computer*, 1994, 134-135.
17. M. Larsen, Review of *Game Theory and Strategy*, 1995, 88.
18. A. Liu, Review of *Lion Hunting and Other Mathematical Pursuits*, 1995, 304-305.
19. M. Kuczma, Review of *Five Hundred Mathematical Challenges*, 1996, 313-316.

CANADIAN MATHEMATICAL SOCIETY AWARD FOR CONTRIBUTIONS TO MATHEMATICAL EDUCATION

The Adrien Pouliot Award

Eric Muller, Chair, CMS Education Committee

In 1995, in celebration of its 50th Anniversary, the Canadian Mathematical Society (CMS) instituted an Award for Contributions to Mathematics Education. This award is to recognize individuals or teams of individuals who have made significant and sustained contributions to mathematics education within Canada. Such contributions are to be interpreted in the broadest possible sense and might include: community outreach programmes, the development of a new programme in either an academic or industrial setting, publicizing mathematics so as to make it accessible to the general public, developing mathematics displays, establishing and supporting mathematics conferences and competitions for students, etc..

The prize is named after Adrien Pouliot. But who was Adrien Pouliot? Pouliot is certainly one of the main scientific figures of the twentieth century in the Province of Quebec. Trained as a civil engineer at the Ecole Polytechnique de Montreal he started teaching mathematics at the Universite Laval in the early 1920s. He completed a 'licence' in sciences (mathematics) at the Sorbonne in Paris (1928). In the words of Professor Hodgson of Laval [1] "Pouliot was the main person behind the growth of science and mathematics in the Province of Quebec. To understand this one has to recall that the first part of the century, the education system in Quebec was preparing more for traditional jobs (priests, physicians) than for science. Pouliot led (with success) a public movement to have science and math become a major part of the education system. (Because of that he was accused to be anti-humanities. His response was to learn Greek; when he died in 1980 he was working on a Greek-French dictionary)". Pouliot's influence extended to other parts of Canada and beyond through his work as a governor and officer of Radio-Canada and through his participation on many missions to other countries. Pouliot was awarded a number of honorary doctorates in Canada, France and Italy.

Recipients of the first three awards are known to many Crux readers.

- The first award was made in 1995, and was awarded to Professor **Edward Barbeau** of the University of Toronto, Ed as he is known to students, mathematics teachers, mathematicians and university colleagues, the press, local school board trustees, and all who come into contact with him. Ed was recognized for his numerous contributions to mathematics education, sustained over so many years and in so many different forums. He has always been willing to spend the time to communicate with interested mathematics teachers, students and the general public. He continues to be involved with the International Mathematical Olympiad, an involvement that started some 16 years ago. Many have benefited from his innovative and challenging mathematics publications accessible to secondary and post secondary students.

- The 1996 recipient was Professor **Bruce Shawyer** of Memorial University of Newfoundland. Bruce is of course the present Editor-in-Chief of Crux. He was recognized for his substantial contributions to many different mathematical competitions and challenges. He proposed and was the founding member of the Newfoundland and Labrador Mathematical Association and was involved in the development of the Newfoundland Teachers' Association Senior Math League and the Junior High Math Challenge. Bruce was most visible as the Chief Operating Officer of the 36th International Mathematical Olympiad that was held in Canada in 1995.

- The third award, in 1997, went to a team from the University of Waterloo, a team of university and school mathematics educators, **Ed Anderson**, **Don Attridge**, **Ron Dunkley** and **Ron Scoins**. They were instrumental in starting the Canadian Mathematics Competition that grew out of their local competition activities in 1962. They have continued to nurture and sustain these competitions that have grown and, in the past year, involved over 200,000 students. These contests not only involve students but also support a network of mathematics teachers throughout Canada. They provide an environment that stimulates professional development and provide opportunities for mathematics teachers from schools and universities to meet and exchange ideas.

Reference

[1] Hodgson, Bernard, CMS Notes, Jan-Feb 1996

Cyclic ratio sums and products

Branko Grünbaum

The well known classical theorems of Menelaus and Ceva deal with certain properties of triangles by relating them to the products of three ratios of directed lengths of collinear segments. Less well known is a theorem of Euler [2] which states, in the notation of Figure 1, that

$$\sum_{j=1}^{3} ||QB_j/A_jB_j|| \;=\; 1$$

for every triangle $T = [A_1A_2A_3]$.

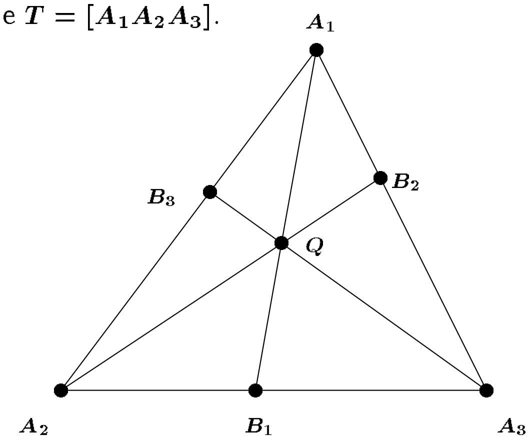


Figure 1.
A theorem of Euler states that if B_j is the intersection of the line A_jQ with the side of the triangle $A_1A_2A_3$ opposite to A_j, then $\sum_{j=1}^{3} ||QB_j/A_jB_j|| = 1$. Here, and throughout this note, $||MN/RS||$ means the ratio of signed lengths of the collinear segments MN and RS.

However, while the theorems of Menelaus and Ceva have been generalized to arbitrary polygons, and in many other ways — see, for example, [4] [5] [6] — until very recently there have been no analogous generalizations of Euler's result. One explanation for this situation may be that attempts at straightforward generalizations lead to invalid statements. An example of such a failed "theorem" is given by the question whether, in the notation of Figure 2,

$$\sum_{j=1}^{n} ||QB_j/A_jB_j||$$

equals 1 or some other constant independent of Q and the polygon.

Research supported in part by NFS grant DMS-9300657.

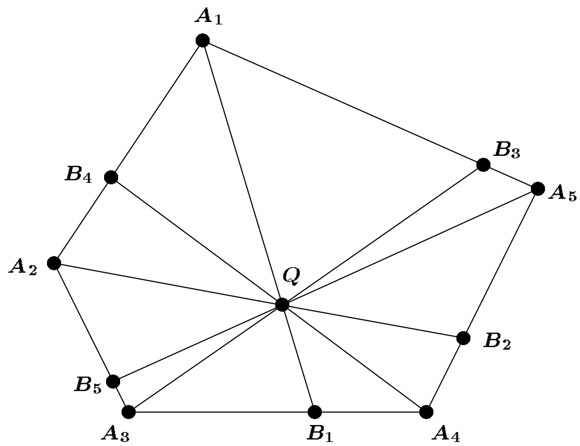


Figure 2.
Attempts to generalize Euler's theorem in the form $\sum_{j=1}^{n} ||QB_j/A_jB_j|| = \text{const}$ necessarily fail for $n > 3$ (here $n = 5$). However, as shown by Shephard [9], it is possible to find weights w_j which depend on the polygon but not on the position of Q, such that $\sum_{j=1}^{n} w_j||QB_j/A_jB_j|| = 1$.

Recently, Shephard [9] had the idea, apparently not considered previously, of attaching to the ratios $r_j = ||(QBj)/(AjBj)||$ certain weights w_j, which depend on the polygon $\mathbf{P} = [A_1, A_2, \dots, A_n]$ but not on the point Q, such that $\sum_{j=1}^{n} w_j r_j = 1$. (In fact, Shephard established a much more general result in this spirit; its complete formulation would lead us too far from the present aims.)

By sheer chance, the same day I received from Shephard a preprint of [9], I happened to read [7], in which two different sums of ratios appear, one in Bradley's solution, the other in Konečný's comments. This coincidence led me to consider whether these results could be generalized along Shephard's idea. As it turns out, the answer is affirmative, and leads to a number of other results.

Let $\mathbf{P} = [A_1, A_2, \dots, A_n]$ be an arbitrary n–gon, and Q an arbitrary point, subject only to the condition that all the points B_j mentioned below are well determined. On each side A_jA_{j+1} of $\mathbf{P}$ (understood as the unbounded line) the point B_j is the intersection with the line through Q parallel to $A_{j+1}A_{j+2}$. (Here, and throughout the present note, subscripts are understood $\bmod n$). This is illustrated in Figure 3 by an example with $n = 5$. We are interested in the ratios

$$r_j = ||B_jA_{j+1}/A_jA_{j+1}||.$$

We denote by $\triangle(UVW)$ the signed area of the triangle $\triangle UVW$ with respect to an arbitrary orientation of the plane and, more generally, by $\triangle(\mathbf{P})$ the signed area of any polygon $\mathbf{P}$, calculated with appropriate multiplicities for the different parts if $\mathbf{P}$ has self-intersections.

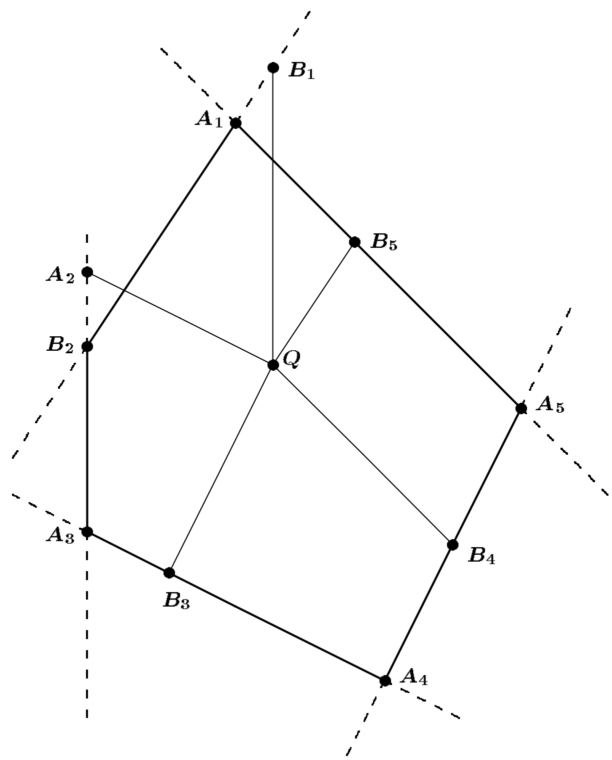


Figure 3.
The point B_j is the intersection of the line A_jA_{j+1} with the parallel through Q to the line $A_{j+1}A_{j+2}$. The ratios $r_j = ||B_jA_{j+1}/A_jA_{j+1}||$ of directed segments are considered in Theorem 1.

Theorem 1. For each polygon $\mathbf{P}$ we have $\sum_{j=1}^{n} w_j r_j = \triangle(\mathbf{P})$ for all Q, where $w_j = \triangle(A_jA_{j+1}A_{j+2})$ are weights that depend on the polygon $\mathbf{P}$ but are independent of the point Q.

For a proof it is sufficient to note that

(i) by straightforward calculations or by easy geometric arguments it can be shown that $r_j = \triangle(QA_{j+1}A_{j+2})/\triangle(A_jA_{j+1}A_{j+2})$; and

(ii) therefore the sum $\sum_{j=1}^{n} w_j r_j$ is equal to $\sum_{j=1}^{n} \triangle(QA_{j+1}A_{j+2}) = \triangle(\mathbf{P})$, since the triangles with vertex Q triangulate the polygon $\mathbf{P}$.

As a corollary we deduce at once that $\sum_{j=1}^{n} w_j s_j = \left(\sum_{j=1}^{n} w_j\right) - \triangle(\mathbf{P})$, where $s_j = ||A_jB_j/A_jA_{j+1}|| = 1 - r_j$.

In the special case that $\mathbf{P}$ is a *regular* (n/d)–gon, all the weights w_j are equal to the value $w = 4\sin^3(d\pi/n)\cos(d\pi/n)$. (The regular (n/d)–gon has n vertices and surrounds its centre d times. Successive vertices are obtained by rotation through $2\pi d/n$, see [1]. It is usually assumed that n and d are coprime, but this is a restriction that is unnecessary here and in most other contexts, and downright harmful in some cases — see, for example, [3]). Hence, in this case one can divide throughout by w, and the result becomes

$$\sum_{j=1}^{n} r_j \;=\; \frac{\triangle(\mathbf{P})}{w} \;=\; \frac{n}{4\sin^2(d\pi/n)}. \tag{1}$$

Since the ratios r_j involve only collinear lengths, the sum is invariant under affinities, and so the result (1) remains valid for all *affine*-regular (n/d)–gons $\mathbf{P}$. (An (n/d)–gon is affine-regular if it is the image of a regular (n/d)–gon under a nonsingular affinity.) Thus in this special case we actually achieve the analogue of the generally invalid statement mentioned above. Since all triangles are affine-regular, this establishes the condition for concurrency found by Konečný, mentioned in [7]. (We note that Shephard obtains in [9] the analogous generalization of Euler's result to affine-regular n–gons.) In the affine case, the above corollary can be simplified in the same way. For $n = 3$ this yields the condition for concurrency obtained by Bradley in [7].

From the above it follows that in the case of affine-regular polygons (but not for general polygons) we have

$$\sum_{j=1}^{n} ||B_jC_j/A_jA_{j+1}|| \;=\; -\,\frac{n\cos(2d\pi/n)}{2\sin^2(d\pi/n)}, \tag{2}$$

where the C_j is the intersection of the line A_jA_{j+1} with the parallel through Q to the line $A_{j-1}A_j$ (see Figure 4). For $n = 3$, the right-hand side of (2) equals 1, and the result coincides with Problem 16 in [8].

It may be observed that for $n = 3$ and $d = 1$, the right-hand side of condition (1) equals 1, and the equality to 1 of the ratio sum is necessary and sufficient for the three parallels to the sides of the triangle to be concurrent, just as the equality to 1 of the product in Ceva's theorem for triangles is necessary and sufficient for the concurrence of the Cevians. However, for $n > 3$ it is not obvious that the weights given above are the only ones which yield the right-hand constants for all Q, although one may conjecture that this is the case. Naturally, for particular choices of $\mathbf{P}$ and Q other weights may be used.

The expression for r_j obtained in (i), together with the analogous formula for the ratio $t_j = ||A_jC_j/A_jA_{j+1}||$ (in the notation of Figure 4) leads at once to the following:

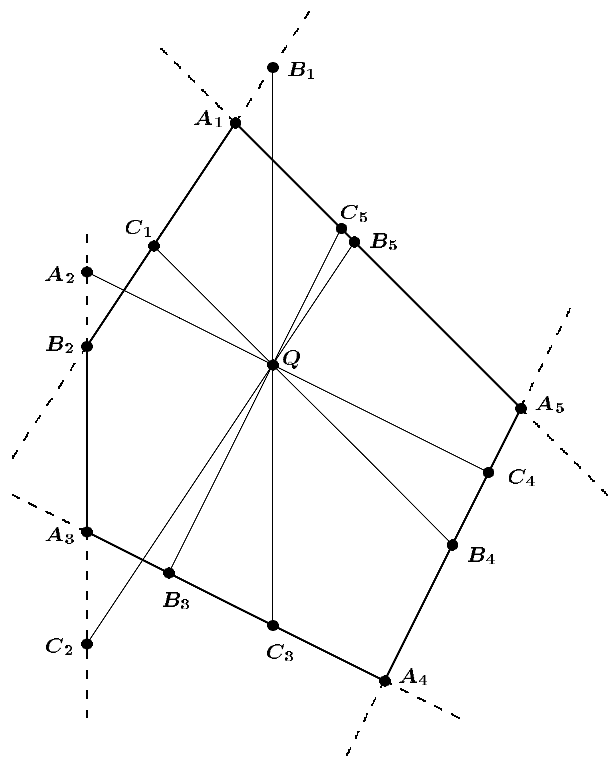


Figure 4.
The point B_j is obtained as in Figure 3, while the point C_j is the intersection of the line $A_j A_{j+1}$ with the parallel through Q to the line $A_{j-1} A_j$. The ratios $r_j = ||B_j A_{j+1} / A_j C_j||$ of directed segments are considered in Theorem 2.

Theorem 2. For each polygon $\mathbf{P}$ with we have

$$\prod_{j=1}^{n} \frac{r_j}{t_j} = \prod_{j=1}^{n} ||B_j A_{j+1} / A_j C_j|| = 1$$

for all Q.

Finally, since $QB_j A_{j+1} C_{j+1}$ is a parallelogram for every j, we also have $\prod_{j=1}^{n} ||B_j Q / Q C_{j+2}|| = 1$.

This last is a Ceva-type result which seems not to have been noticed previously.

A referee's suggestions for improved presentation are acknowledged with thanks.

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University of Washington, Box 354350,
Seattle, WA 98195-4350
e-mail: grunbaum@math.washington.edu

THE SKOLIAD CORNER

No. 27

R.E. Woodrow

To start the new year we give the problems of the Preliminary round of the British Columbia Colleges Junior High School Mathematics Contest for 1997. This round of the contest is written in the schools. The top students are invited to a mathematics day at which the final round is written, and they get to participate in talks, tours, and a presentation ceremony. My thanks go to John Grant McLoughlin, now of the Faculty of Education, Memorial University of Newfoundland, who participated in the writing process while he was at the Okanagan University College.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Preliminary Round 1997

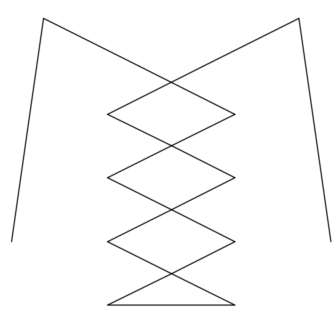
1. A number is prime if it is greater than one and is divisible only by itself and one. The number 1997 is prime but the sum of the digits of 1997 is not. The sum of the prime divisors of the *sum* of the digits of 1997 is:

(a) 6 (b) 8 (c) 10 (d) 13 (e) 15

2. If $x = 3$, which expression has a value different from the other four?

(a) $2x^2$ (b) $x^2 + 9x$ (c) $12x$ (d) $x^2(x-1)^2$ (e) $2x^2(x-1)$

3. Suppose that you lace your shoes using the pattern shown below and that the horizontal spacing between the eyelets is 4 centimetres and the vertical spacing is 3 centimetres. If there is a total of ten eyelets and there are 10 centimetres of lace left at each of the upper eyelets, then the total length, in centimetres, of the lace used for one shoe is:



(a) 44 (b) 54 (c) 63 (d) 64 (e) 73

4. Consider two unequal numbers. If we subtract half the smaller number from both numbers, the result with the larger number is three times as large as the result with the smaller number. How many times is the larger number greater than the smaller number?

(a) 2 (b) 3 (c) 6 (d) 8 (e) 9

5. Suppose your hockey card collection grew by 20% last year and last year it was 30% larger than the year before. By what percentage has your collection grown during the last two years?

(a) 25 (b) 50 (c) 56 (d) 60 (e) 66

6. Antonino, the chocolate freak, eats x bars of chocolate every y days. In a week he eats how many bars of chocolate?

(a) $\frac{7x}{y}$ (b) $\frac{7y}{x}$ (c) $7xy$ (d) $\frac{1}{7xy}$ (e) $\frac{x}{7y}$

7. Six mattresses, each of which was originally 20 centimetres thick, are piled in a stack in a warehouse. Each mattress is compressed by one tenth each time an additional mattress is added to the stack. The height h of the stack, in centimetres, satisfies:

(a) $h < 70$ (b) $70 \leq h < 86$ (c) $86 \leq h < 92$ (d) $92 \leq h < 110$ (e) $h \geq 110$

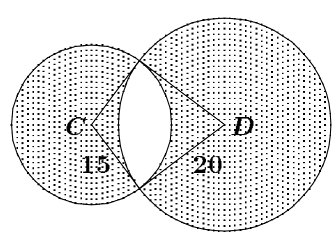
8. If a man walks to work and rides back home it takes him an hour and a half. When he rides both ways, it takes 30 minutes. How long would it take him to make the round trip by walking?

(a) $2\frac{1}{2}$ hrs (b) $1\frac{1}{4}$ hrs (c) $1\frac{1}{2}$ hrs (d) $3\frac{1}{2}$ hrs (e) $2\frac{3}{4}$ hrs

9. Tanya was asked to add 14 to a certain number and then divide the result by 4. Instead she first added 4 and then divided the answer by 14. Her result was 5. If Tanya had followed the instructions correctly, her result would have been:

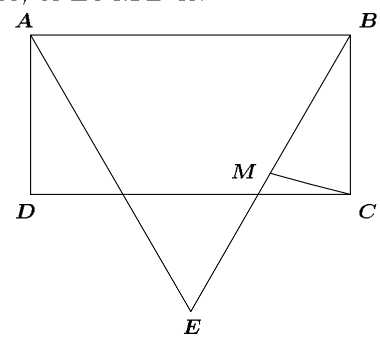
(a) 5 (b) 20 (c) 25 (d) 66 (e) 70

10. In the diagram, a circle with centre at point C and radius 15 overlaps a circle with centre at point D and radius 20. How many square units larger is the shaded region on the right than that on the left?



(a) 400π (b) $400\pi - 400$ (c) $175\pi - 400$ (d) 175π (d) $225\pi - 400$

11. $ABCD$ is a rectangle in which AB is twice as long as BC. E is a point such that ABE is an equilateral triangle. M is the midpoint of BE. The measure, in degrees, of $\angle CMB$ is:



(a) 30 **(b)** 60 **(c)** 75 **(d)** 90 **(e)** 150

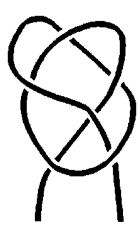
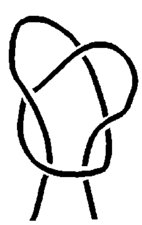
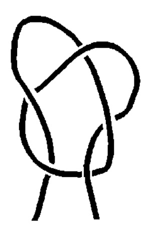
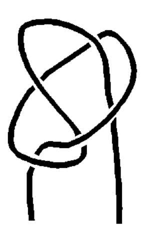
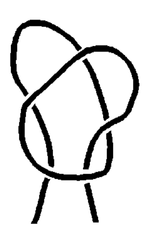
12. The numbers 2^{800}, 3^{600}, 5^{400} and 6^{200} listed in increasing order are:

(a) $2^{800}, 3^{600}, 5^{400}, 6^{200}$ **(b)** $6^{200}, 2^{800}, 3^{600}, 5^{400}$ **(c)** $6^{200}, 2^{800}, 5^{400}, 3^{600}$ **(d)** $2^{800}, 5^{400}, 3^{600}, 6^{200}$ **(e)** $3^{600}, 6^{200}, 2^{800}, 5^{400}$

13. Think about a stopped clock. It shows the correct time twice a day. Now consider a working clock that gains five seconds per day. If it is never adjusted, it will first show the correct time after:

(a) 48 days **(b)** 360 days **(c)** 720 days **(d)** 8640 days **(e)** 17280 days

14. The diagrams below show five pieces of rope. Imagine grasping the two loose ends of each piece firmly, then imagine pulling them until you have a straight piece of rope — either with a knot or without one. Which of the five will give you a knot?

(a)

(b)

(c)

(d)

(e)


15. In how many ways can 75 be expressed as the sum of at least two positive integers, all of which are consecutive?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Last issue we gave the thirty problems of the 1996 Kangourou des Mathématiques [1997: 473–478]. Here are solutions.

1.	B	**2**.	B	**3**.	E	**4**.	D	**5**.	E
6.	C	**7**.	B	**8**.	E	**9**.	B	**10**.	E
11.	B	**12**.	C	**13**.	B	**14**.	B	**15**.	C
16.	E	**17**.	A	**18**.	B	**19**.	D	**20**.	C
21.	A	**22**.	D	**23**.	B	**24**.	D	**25**.	D
26.	A	**27**.	D	**28**.	B	**29**.	B	**30**.	C

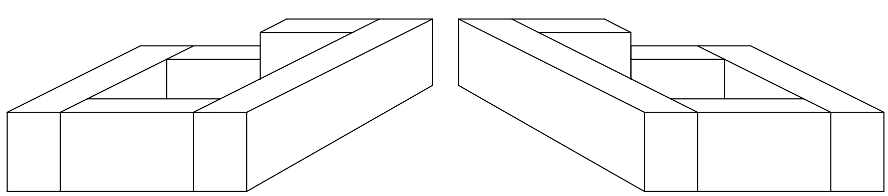
Note: A number of typos were missed in proof-reading the problems. In the Figure for Number 2, apparently four, not three, of the triangles should have been shaded — but didn't look to be the case on the photocopy I had!

In the Figure for Number 16 the six angles not sharing a common vertex should be marked.

Number 18 should start "Les côtés".

More seriously in Number 26: $AD = DC = CB$.

That completes the Skoliad Corner for this number. I need good contest materials at this level from around the world. Please send me materials for use in the Corner as well as suggestions for future directions for the column.



MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), Richard Hoshino (University of Waterloo), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Shreds and Slices

The K Method

In this brief article, we will present a useful geometrical tool, which we coin "The K Method". As is convention, let $[P]$ denote the area of polygon P. We will further stipulate that if P is labelled counter-clockwise, then $[P]$ is positive, and negative otherwise. This sign convention will matter.

Let ABC be a triangle, labelled counter-clockwise, and let P be a point in the plane. Let $K = [ABC]$, $K_A = [PBC]$, $K_B = [PCA]$, and $K_C = [PAB]$ (see Figure 1). Because of the signed areas, we have that for all points P, inside or outside the triangle ABC,

$$K = K_A + K_B + K_C .$$

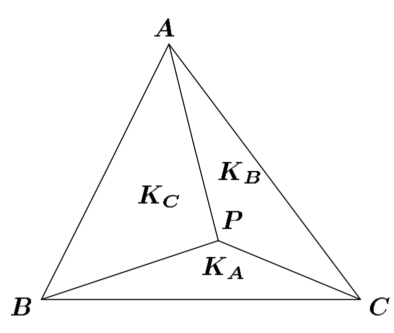


Figure 1

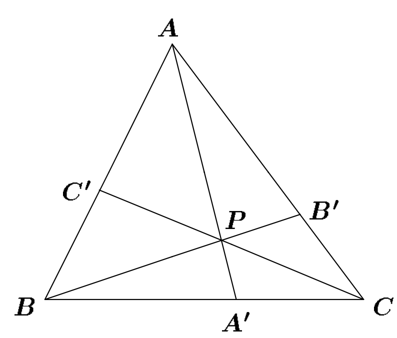


Figure 2

Now, extend AP to A' on BC, and define B' and C' similarly (see Figure 2). Recall that triangles with the same height have areas in proportion to their bases. Then we have

$$\frac{AP}{PA'} = \frac{[PAB]}{[PBA']} = \frac{[PCA]}{[PA'C]} = \frac{[PAB]+[PCA]}{[PBA']+[PA'C]} = \frac{K_B + K_C}{K_A}, \quad \text{and}$$

$$\frac{BA'}{A'C} = \frac{[ABA']}{[AA'C]} = \frac{[PBA']}{[PA'C]} = \frac{[ABA']-[PBA']}{[AA'C]-[PA'C]} = \frac{K_C}{K_B}.$$

We can similarly derive that

$$\frac{BP}{PB'} = \frac{K_A + K_C}{K_B}, \quad \frac{CP}{PC'} = \frac{K_A + K_B}{K_C}, \quad \frac{CB'}{B'A} = \frac{K_A}{K_C}, \text{ and } \frac{AC'}{C'B} = \frac{K_B}{K_A}.$$

Again, these hold regardless of whether P is inside or outside triangle ABC, but only with directed line segments (that is, if PQ and PR are in different directions, then their ratio will be negative).

These expressions can be very useful in problems which involve these ratios. In fact, one half of Ceva's theorem is now trivial (and with some consideration, so is the other half).

Problem 1. Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \quad \frac{P_2P}{PQ_2}, \quad \frac{P_3P}{PQ_3},$$

at least one is less than or equal to 2 and at least one is greater than or equal to 2. (1961 IMO, Problem #4)

Solution. Let us use the same understood notation as above. Without loss of generality, we can assume that $K_A \le K_B \le K_C$ (re-label the triangle if necessary). Then

$$\frac{P_3P}{PQ_3} = \frac{K_A + K_B}{K_C} \le \frac{K_C + K_C}{K_C} = 2, \quad \text{and}$$

$$\frac{P_1P}{PQ_1} = \frac{K_B + K_C}{K_A} \ge \frac{K_A + K_A}{K_A} = 2.$$

Problem 2. In triangle ABC, A', B', and C' are on sides BC, AC, and AB, respectively. Given that AA', BB', and CC' are concurrent at the point O, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find the value of

$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}.$$

(1992 AIME)

Solution. The given implies

$$\frac{K_A + K_B}{K_C} + \frac{K_A + K_C}{K_B} + \frac{K_B + K_C}{K_A} = 92,$$

and further that

$$K_A^2 K_B + K_A K_B^2 + K_A^2 K_C + K_A K_C^2 + K_B^2 K_C + K_B K_C^2 = 92 K_A K_B K_C.$$

Then

$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} = \left(\frac{K_B + K_C}{K_A}\right)\left(\frac{K_A + K_C}{K_B}\right)\left(\frac{K_A + K_B}{K_C}\right)$$

$$= \frac{K_A^2 K_B + K_A K_B^2 + K_A^2 K_C + K_A K_C^2 + K_B^2 K_C + K_B K_C^2 + 2K_A K_B K_C}{K_A K_B K_C}$$

$$= \frac{92 K_A K_B K_C + 2K_A K_B K_C}{K_A K_B K_C} = 94.$$

Problems

1. Prove Ceva's Theorem, which states that AA', BB', and CC' (as in Figure 2) are collinear if and only if

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

2. Let P be a point inside the triangle ABC. Let AP meet BC at D, BP meet CA at E, and CP meet AB at F. Prove that

$$\frac{PA}{PD} \cdot \frac{PB}{PE} + \frac{PB}{PE} \cdot \frac{PC}{PF} + \frac{PC}{PF} \cdot \frac{PA}{PD} \geq 12.$$

(*The Red Book of Mathematical Problems*, Williams and Hardy)

Powers of Two

Wai Ling Yee
student, University of Waterloo

We will solve several problems involving the number 2 and, in the process, survey various interesting mathematical results.

Problem 1. Prove that for all positive integers n, $p_n(x) = x^n - 2$ is irreducible in $\mathbb{Q}[x]$.

Remark. $\mathbb{Q}[x]$ is the set (or more precisely, ring) of polynomials in x with rational coefficients. Hence, a polynomial is irreducible in $\mathbb{Q}[x]$ if and only if it cannot be factored non-trivially as a product of polynomials, also with rational coefficients.

Solution. This follows directly from Eisenstein's Criterion, but we will take a more basic approach. By DeMoivre's Theorem, the n roots of $p_n(x)$ (in $\mathbb{C}$) are $2^{\frac{1}{n}}\operatorname{cis}(\frac{2k\pi}{n})$, $k = 0, 1, 2, \ldots, n-1$, so that

$$p_n(x) = \left(x - 2^{\frac{1}{n}}\operatorname{cis} 0\right)\left(x - 2^{\frac{1}{n}}\operatorname{cis}\left(\frac{2\pi}{n}\right)\right)\cdots\left(x - 2^{\frac{1}{n}}\operatorname{cis}\left(\frac{2(n-1)\pi}{n}\right)\right).$$

Suppose $p_n(x) = f(x)g(x)$ for some $f(x), g(x) \in \mathbb{Q}[x]$, with $\deg f(x) = m$ and $n > m > 0$. (Note: we only need to consider $n \geq 2$.) Let the roots of $f(x)$, respectively $g(x)$, be ω_1, ω_2, $\ldots$, ω_m, respectively ω_{m+1}, ω_{m+2}, $\ldots$, ω_n, so ω_1, ω_2, $\ldots$, ω_n is a permutation of the roots of $p_n(x)$. Let $\omega_i = 2^{1/n}\operatorname{cis}\theta_i$, and let $\theta = \sum_{k=1}^{m}\theta_i$. Note $(-1)^m\prod_{k=1}^{m}\omega_k$ is the constant term of $f(x)$, so

$$(-1)^m\prod_{k=1}^{m}\omega_k = (-1)^m 2^{\frac{m}{n}}\prod_{k=1}^{m}\operatorname{cis}\theta_k = (-1)^m 2^{\frac{m}{n}}\operatorname{cis}\theta \in \mathbb{Q}.$$

But $\operatorname{cis}\theta = \cos\theta + i\sin\theta \in \mathbb{Q}$ implies that $\sin\theta = 0$ and further, that $\cos\theta = \pm 1$. Therefore, the constant term of $f(x)$ is $\pm 2^{\frac{m}{n}}$. It is easy to show that $2^{\frac{m}{n}} \notin \mathbb{Q}$, a contradiction. Hence, $x^n - 2$ is irreducible in $\mathbb{Q}[x]$.

Problem 2. If the sum of the proper divisors of n (that is, the divisors of n that are less than n) is n, then n is called a *perfect number*. Equivalently, the sum of the divisors of n, $\sigma(n)$, is $2n$. For example, $2 \cdot 6 = \sigma(6) = 1 + 2 + 3 + 6$. Classify all even perfect numbers.

Solution. By the Fundamental Theorem of Arithmetic, we can express n uniquely, up to permutations, in the form $p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$, where the p_i are distinct primes and the a_i are positive integers. Then, all divisors of n are of the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $0 \le \alpha_i \le a_i$. Hence,

$$\begin{aligned} \sigma(n) &= \prod_{j=1}^{k} \left(1 + p_j + p_j^2 + \cdots + p_j^{a_j}\right) \\ &= \left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right) \left(\frac{p_2^{a_2+1} - 1}{p_2 - 1}\right) \cdots \left(\frac{p_k^{a_k+1} - 1}{p_k - 1}\right). \end{aligned}$$

It follows that if $\gcd(m, n) = 1$, then $\sigma(mn) = \sigma(m)\sigma(n)$.

We claim that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is a perfect number. In such a case, we have

$$\begin{aligned} \sigma(2^{n-1}(2^n - 1)) &= \frac{2^n - 1}{2 - 1} \cdot \frac{(2^n - 1)^2 - 1}{(2^n - 1) - 1} \\ &= (2^n - 1)(2^n - 1 + 1) = 2 \cdot 2^{n-1}(2^n - 1). \end{aligned}$$

Now consider an arbitrary even perfect number n. Then $n = 2^k m$, where $k \ge 1$ and $\gcd(2^k, m) = 1$. Also, $2^{k+1}m = \sigma(2^k m) = \sigma(2^k)\sigma(m) = (2^{k+1} - 1)\sigma(m)$. Now, $2^{k+1} - 1$ is odd, so $(2^{k+1} - 1) \mid m$. Let $m = d(2^{k+1} - 1)$, so d is a divisor of m. By the above,

$$\sigma(m) = \frac{2^{k+1}m}{2^{k+1} - 1} = m + \frac{m}{2^{k+1} - 1} = m + d,$$

but $\sigma(m)$ is defined as the sum of *all* of the divisors of m. Therefore, m only has the two divisors m and d, so m is prime, d is 1, and so $2^{k+1} - 1$ is prime.

We have shown that all even perfect numbers are of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime. Primes of this form are called *Mersenne primes*.

Problem 3. Perform the following transformation T on the vector $(x_1, x_2, x_3, \ldots, x_{2^n})$, where the x_i are non-negative integers:

$$T(x_1, x_2, x_3, \ldots, x_{2^n}) = (|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_{2^n} - x_1|).$$

Prove that a finite number of applications of T will transform any such vector to the zero vector.

Solution 1. Notice that $\bmod 2$ has the following special property: $1 \equiv -1 \pmod 2 \Rightarrow a \equiv -a \pmod 2$ for all integers a. This implies $|a| \equiv a \pmod 2$ for all integers a, and that subtraction is the same as addition in $\bmod 2$. Therefore, the values $|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_{2^n} - x_1| \bmod 2$, are congruent to $x_1 + x_2, x_2 + x_3, \ldots, x_{2^n} + x_1 \bmod 2$, which will be easier to work with.

Define $S(x_1, x_2, \dots, x_{2^n}) = (x_1 + x_2, x_2 + x_3, \dots, x_{2^n} + x_1)$. By the previous remarks, S and T give the same results $\bmod 2$. Notice that

$$\begin{aligned} &S^2(x_1, x_2, x_3, \dots, x_{2^n}) \\ &\quad = \quad (x_1 + 2x_2 + x_3, x_2 + 2x_3 + x_4, \dots, x_{2^n} + 2x_1 + x_2), \end{aligned}$$

and

$$\begin{aligned} &S^3(x_1, x_2, \dots, x_{2^n}) \\ &\quad = \quad (x_1 + 3x_2 + 3x_3 + x_4, x_2 + 3x_3 + 3x_4 + x_5, \\ &\qquad\qquad \dots, x_{2^n} + 3x_1 + 3x_2 + x_3). \end{aligned}$$

The coefficients seem to be entries in Pascal's triangle. Indeed, we can prove by induction that the i^{th} entry of $S^k(x_1, x_2, x_3, \dots, x_{2^n})$ is $\displaystyle\sum_{j=0}^{k} \binom{k}{j} x_{j+i}$, where $x_r = x_s$ if $r \equiv s \pmod{2^n}$.

What does $S^{2^n}(x_1, x_2, x_3, \dots, x_{2^n})$ look like? First, we claim that if $1 \le j \le 2^n - 1$, then $\binom{2^n}{j}$ is even. We prove the more general result which states that if p is prime and $1 \le j \le p^k - 1$, then p divides $\binom{p^k}{j}$.

Step 1: For p prime, we determine the highest power of p dividing $m!$.

There are $\lfloor \frac{m}{p} \rfloor$ natural numbers less than or equal to m that are divisible by p, $\lfloor \frac{m}{p^2} \rfloor$ natural numbers less than or equal to m that are divisible by p^2, and so on. For each number divisible by at most p^a, we have counted it exactly a times in the sums.

Therefore, the highest power of p dividing $m!$ is

$$\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \left\lfloor \frac{m}{p^3} \right\rfloor + \cdots .$$

Step 2: We calculate the highest power of p dividing $\binom{p^k}{j}$, where $1 \le j \le p^k - 1$ and p is prime, which is simply the highest power of p dividing the numerator subtracted by the highest power of p dividing the denominator.

Applying the result from the previous step, the highest power of p dividing $\binom{p^k}{j} = \frac{p^k!}{j!(p^k-j)!}$ is

$$\sum_{i=1}^{\infty} \left\lfloor \frac{p^k}{p^i} \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{j}{p^i} \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{p^k - j}{p^i} \right\rfloor = \sum_{i=1}^{k} \left(\left\lfloor \frac{p^k}{p^i} \right\rfloor - \left\lfloor \frac{j}{p^i} \right\rfloor - \left\lfloor \frac{p^k - j}{p^i} \right\rfloor \right).$$

Each term in the sum is non-negative, since $\lfloor x + y \rfloor \ge \lfloor x \rfloor + \lfloor y \rfloor$ for all reals x, y, and when $i = k$,

$$\left\lfloor \frac{p^k}{p^k} \right\rfloor - \left\lfloor \frac{j}{p^k} \right\rfloor - \left\lfloor \frac{p^k - j}{p^k} \right\rfloor = 1,$$

since $1 \le j \le p^k - 1$. Hence, there is at least one factor of p in $\binom{p^k}{j}$.

Now we relate this result to the problem. The parity of the i^{th} entry of $T^{2^n}(x_1, x_2, x_3, \dots, x_{2^n})$ is the same as the parity of the i^{th} entry of $S^{2^n}(x_1, x_2, x_3, \dots, x_{2^n})$, which is

$$\sum_{j=0}^{2^n} \binom{2^n}{j} x_{j+i} \;\equiv\; \binom{2^n}{0} x_i + \binom{2^n}{2^n} x_{i+2^n} \;\equiv\; 2x_i \;\equiv\; 0 \pmod 2.$$

Therefore, the i^{th} entry of $T^{2^n}(x_1, x_2, x_3, \dots, x_{2^n})$ is even. We can then pull a factor of 2 out of each of the resulting elements and apply T 2^n times to that vector (since T is linear), and again, obtain a vector with all entries divisible by 2, and so on. Hence, after $t \cdot 2^n$ applications of T, each entry in the vector must be divisible by 2^t.

If y_k is used to denote the largest of the 2^n entries of

$$T^k(x_1, x_2, x_3, \dots, x_{2^n}),$$

then $\{y_k\}$ is a non-increasing sequence. Furthermore, there exists t such that $y_1 < 2^t$. Then after $t \cdot 2^n$ applications of T, the resulting vector has entries all divisible by 2^t. Since the entries are non-negative and less than 2^t, they must all be zero. Therefore, we have transformed the original vector to the zero vector with a finite number of applications of T.

Solution 2. We shall prove by induction that for all natural n, there exists k such that every entry of $T^k(x_1, x_2, x_3, \dots, x_{2^n})$ and, equivalently, every entry of $S^k(x_1, x_2, x_3, \dots, x_{2^n})$ is even for all non-negative integers $x_1, x_2, \dots, x_{2^n}$. When $n = 0$, $T(x_1) = x_1 - x_1 = 0$ and our hypothesis holds. Assume the result holds for some $n = r$, with k_r iterations of T always producing a result whose entries are all divisible by 2. Let $n = r + 1$. Then

$$\begin{aligned}
&S(x_1, x_2, x_3, \dots, x_{2^{r+1}}) \\
&\quad = \; (x_1 + x_2, x_2 + x_3, x_3 + x_4, \dots, x_{2^{r+1}} + x_1), \\
&S^2(x_1, x_2, x_3, \dots, x_{2^{r+1}}) \\
&\quad = \; (x_1 + 2x_2 + x_3, x_2 + 2x_3 + x_4, x_3 + 2x_4 + x_5, \dots) \\
&\quad \equiv \; (x_1 + x_3, x_2 + x_4, x_3 + x_5, \dots, x_{2^{r+1}} + x_2) \pmod 2 .
\end{aligned}$$

If we take every other element of the last vector, starting with the first, we obtain $(x_1 + x_3, x_3 + x_5, \dots, x_{2^{r+1}-1} + x_1)$, which is also what we obtain when we apply T to $(x_1, x_3, x_5, \dots, x_{2^{r+1}-1})$, and similarly with the even indexed elements. Therefore, by the induction hypothesis, after $2k_r$ applications of T, we have a vector whose entries are all divisible by 2. By mathematical induction, for all natural n, a finite number of applications of T will transform any vector into a vector divisible by 2. As we concluded in the previous solution, a finite number of applications of T will transform any vector into the zero vector.

The Cantor Set and Cantor Function

Naoki Sato

We begin by considering a problem that was given out at IMO training this past summer.

Problem. Let $f : [0,1] \to [0,1]$ be a function satisfying the following three properties:

(i) f is non-decreasing (that is; $x < y \Rightarrow f(x) \le f(y)$),

(ii) $f(x) = 1 - f(1-x)$ for all $x \in [0,1]$, and

(iii) $f(3x) = 2f(x)$ for all $x \in [0, \frac{1}{3}]$.

Evaluate $f(1/7)$ and $f(1/13)$.

The reader at this point should stop and try to work out the values the problem asks for; this is a good way to get a feel for the dynamics of this function, which turns out to be a very special function, and well-known in analysis and chaos theory.

What is the first of many remarkable facts is that the three properties given above are enough to determine the value of $f(x)$ for any point $x \in [0,1]$, even though they do not seem to be – what if x is irrational?

We will come back to these questions later. We will first describe a seemingly unrelated concept, the Cantor set. There are several possible definitions, and this is perhaps the most straightforward.

Let A_0 be the interval $[0,1]$. From this, we remove the "middle-third" $\left(\frac{1}{3}, \frac{2}{3}\right)$, and we obtain $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, which is the union of two intervals. We remove the middle-third from these two intervals, and obtain $A_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so forth (see Figure 1). We call C, the set obtained in this limiting process, the Cantor set. Rigorously speaking,

$$A_n = [0,1] \backslash \bigcup_{k=1}^{n} \bigcup_{j=1}^{3^{k-1}} \left(\frac{3j-2}{3^k}, \frac{3j-1}{3^k} \right),$$

$$C = [0,1] \backslash \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{3^{k-1}} \left(\frac{3j-2}{3^k}, \frac{3j-1}{3^k} \right).$$

Note that A_n is the union of closed intervals, and that the sum of the lengths of these intervals is $\left(\frac{2}{3}\right)^n$ (since we remove one third at each step), which goes to 0 as n approaches infinity. In this sense, we have removed "most" of the interval [0,1] in obtaining C.

We mentioned that C has alternate definitions. Each $x \in [0,1]$ has a base 3 expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} ,$$

where $a_i \in \{0,1,2\}$. Then C is the set of all $x \in [0,1]$ such that $a_i \neq 1$ for all i. In other words, each digit is a 0 or 2.

Caution. There are some cases where this is not exactly true — some numbers have two possible base 3 expansions, for example $\frac{1}{3} = 0.1_3 = 0.0222\ldots_3$. Which one should we use?

The set of numbers which have $a_1 = 1$ is precisely the interval $(\frac{1}{3}, \frac{2}{3})$, which is what we threw away from A_0 to get A_1. Then, the set of numbers which have $a_1 \neq 1$ and $a_2 = 1$ is $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, which is what we threw away from A_1 to get A_2, and so on. Hence, the two constructions are the same.

The Cantor set has many uses. For example, although most of $[0,1]$ has been removed to obtain C, it can be shown that there is an onto map from C to $[0,1]$; that is, every element in $[0,1]$ is equal to the image of x for some $x \in C$. Not even $\mathbb{Q}$, the rationals, can make this claim. In the language of set theory, $\mathbb{Q}$ is a countable set, and C and $[0,1]$ are uncountable sets, which are "bigger" (at this level, the size, or cardinality of sets are measured by the existence of 1–1, onto, or bijective maps between them).

Also, as the reader might have suspected, the Cantor set is also one of the most basic examples of a fractal, an object, roughly speaking, with self-similarity. In fact, the Cantor set is one of the most important fractals, despite its almost bland simplicity and lack of interesting detail. For example, certain Julia sets are modified Cantor sets. But how does all this relate to our original problem?

We give a formulation for $f(x)$. If $x \in C$, then $a_i \in \{0,2\}$ for all i in the base 3 expansion of x, and

$$f(0.a_1 a_2 a_3 \ldots_3) = 0.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\ldots_2 .$$

Note we have base 3 on the left and base 2 on the right. If $x \notin C$, then $a_i = 1$ for some i. Choose the smallest such i. Then

$$f\left(0.a_1 a_2 a_3 \ldots a_i \ldots_3\right) = 0.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\ldots\frac{a_{i-1}}{2}a_i\,_2 .$$

For example,

$$f\left(\frac{4}{13}\right) = f\left(0.022022\ldots_3\right) = 0.011011\ldots_2 = \frac{3}{7} ,$$

$$f\left(\frac{38}{243}\right) = f\left(0.02102_3\right) = 0.011_2 = \frac{3}{8} .$$

We leave it to the reader to verify that these are the correct expressions, but we will show that the properties above are sufficient to determine f.

First, assume that $x \in C$, so $a_i \in \{0, 2\}$ for all i. We proceed by induction on the number of digits of x. Let $0.b_1 b_2 b_3 \ldots_2$ be the binary expansion of $f(x)$. Using the properties, we see that $f(0) = 0$, $f(1) = 1$, and $f\left(\frac{1}{3}\right) = f\left(\frac{2}{3}\right) = \frac{1}{2}$, or $f\left(0.0222\ldots_3\right) = 0.0111\ldots_2$, and $f\left(0.2_3\right) = 0.1_2$. Hence, by expressing $\frac{1}{2}$ in these two forms, and since f is non-decreasing, we see that if $a_1 = 0$, then $b_1 = 0$, and if $a_1 = 2$, then $b_1 = 1$, so $b_1 = a_1/2$.

Now assume $b_i = a_i/2$ for all i from 1 up to some n. We must show that $b_{n+1} = a_{n+1}/2$. First, consider the case $a_1 = 0$. Then $b_1 = 0$, as shown above, and $f(3x) = 2f(x)$, or

$$f(0.a_2 a_3 \ldots a_{n+1} \ldots_3) = 0.b_2 b_3 \ldots b_{n+1} \ldots_2 .$$

By the induction hypothesis, $b_{n+1} = a_{n+1}/2$. The case where $a_1 = 2$ is left to the reader [Hint: This is where we must use $f(x) = 1 - f(1 - x)$]. Therefore, by induction, $b_i = a_i/2$ for all i.

Also, it can be seen that if x and y are the end-points of one of the intervals removed to obtain C, then $f(x) = f(y)$. Thus, again since f is non-decreasing, f on each missing interval is the common value at the end-points.

We also get another remarkable property as a bonus. Note that f is onto. Pick $y \in [0, 1]$, express y in base 2, and it should be obvious which $x \in [0, 1]$ satisfies $f(x) = y$. We can even choose $x \in C$, so that f, when restricted to C, is an example (in fact, the standard example) of a map from C to $[0, 1]$ that is onto. Moreover, f is non-decreasing, so the only possible discontinuities of f are jump discontinuities. But f is onto, so it cannot have any jump discontinuities, so f is continuous.

Figure 1.

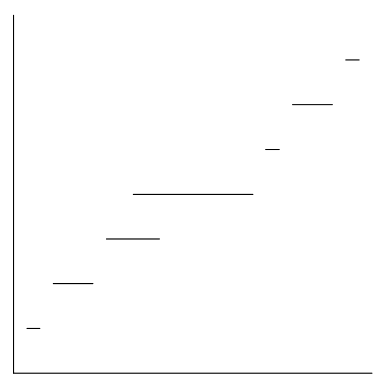


Figure 2.

When f is graphed, the features alluded to above become immediately apparent (see Figure 2). On the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, f is indeed the constant $\frac{1}{2}$, on $(\frac{1}{9}, \frac{2}{9})$, the constant $\frac{1}{4}$, and on $(\frac{7}{9}, \frac{8}{9})$, the constant $\frac{3}{4}$, and so on. As mentioned, this function f, sometimes called the Devil's Staircase, has some exceptional properties. By the observation just made, $f'(x) = 0$ almost everywhere ("almost everywhere" does have a technical meaning, and it is related to the fact that most of $[0, 1]$ is missing); in fact, the derivative is not defined precisely at points in the Cantor set. But $f(0) = 0$ and $f(1) = 1$, so f must somehow climb up abruptly at points in the Cantor set, since it is flat almost everywhere.

So, a fairly innocuous problem on a functional equation turns out to have some large ramifications. We end with a few more miscellaneous facts.

Let $T_1(x) = x/3$ and $T_2(x) = (x + 2)/3$. Pick a random point $x \in [0, 1]$, and recursively apply T_i to x (where i is randomly chosen at each step). Plot each x; you can do this on your computer. What do you get?

Define $T(x)$ on $[0, 1]$ as follows:

$$T(x) = \begin{cases} 3x & \text{if } 0 \le x \le 1/2 \\ 3 - 3x & \text{if } 1/2 \le x \le 1 \end{cases}.$$

This function is sometimes called the tent function, for obvious reasons. Some points in the range of T are outside of $[0, 1]$; which ones? These are bad points we wish to remove from the domain of T, because we wish to apply T an arbitrarily number of times to values in $[0, 1]$. For which x is it true that $T^2(x) \in [0, 1]$? What about T^3?

In fact, what is the set of all x such that $T^k(x) \in [0, 1]$ for all k?

You guessed it, it is the Cantor set. In terms of the base 3 expansion, how does $T(x)$ relate to x?

Problems

1. Show that for all $z \in [0, 2]$, there exist x, $y \in C$ such that $x + y = z$. Symbolically, $C + C = [0, 2]$.
2. Show that C is totally disconnected; that is, show that for all x, $z \in C$, there is a $y \notin C$ such that $x < y < z$.

J.I.R. McKnight Problems Contest 1981

1. If x, y, and z are all between 0 and $\frac{\pi}{2}$ inclusive, solve for x, y and z if

$$\begin{aligned}\log_2(\sin x \sin y \sin z) &= -\frac{3}{2}\\ \log_2\left(\frac{\sin^2 x \sin^3 y}{\sin z}\right) &= -\frac{5}{2}\\ \log_2\left(\frac{\sin x \sin^4 y}{\sin^2 z}\right) &= -3\end{aligned}$$

2. A variable chord of an ellipse subtends a right angle at the centre. Show that the chord always touches a fixed circle.

3. (a) Obtain the equation of the tangent with slope m to the parabola whose equation is $y^2 = 4px$. Assume $p > 0$.
   (b) Obtain the equation of the tangent perpendicular to the tangent in (a).
   (c) Find the equation of the locus of the points of an intersection of pairs of perpendicular tangents to the parabola in (a).

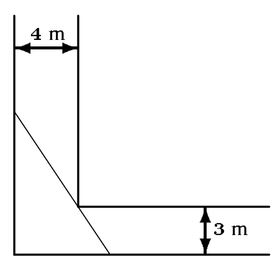
4. Find the first four terms in ascending powers of x in the expansion as an infinite series of

$$\frac{1-2x}{1+x-2x^2}$$

and state the restrictions on x.

5. A tent has the shape of a cone and has a capacity of 1000 m^3. Find the radius of the base if the amount of canvas used is to be a minimum. (No canvas is used on the floor.)

6. Find the length of the longest ladder that can be carried horizontally around the corner of the corridor shown in the diagram.



Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from this issue be submitted by 1 April 1998, for publication in the issue 5 months ahead; that is, issue 6 of 1998. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others.

High School Problems

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H223. In each of the following alphametics, each letter in the addition represents a unique digit:

```
  1997     and      1998
+ OLD             + OLD
------            ------
  YEAR              YEAR
```

For each alphametic, find a solution, or prove that a solution does not exist.

H224. Let $ABCD$ be a square. Construct equilateral triangles APB, BQC, CRD, and DSA, where P, Q, R, and S are points outside of the square.

(a) Prove that $PQRS$ is a square.

(b) Determine the ratio $\frac{PQ}{AB}$. (See how many ways you can solve this!)

H225. Consider a row of five chairs, numbered 1, 2, 3, 4, and 5. You are originally sitting on 1. On each move, you must stand up and sit down on an adjacent chair. Make 19 moves, then take away chairs 1 and 5. Then make another 97 moves, with the three remaining chairs. No matter how the moves are made, you will always end up on chair 3. Why is this the case?

H226. The smallest multiple of 1998 that only consists of the digits 0 and 9 is 9990.

(a) What is the smallest multiple of 1998 that only consists of the digits 0 and 3?

(b) What is the smallest multiple of 1998 that only consists of the digits 0 and 1?

Advanced Problems

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada. M1G 1C3 <hsia@math.toronto.edu>

A209. Are there an infinite number of squares among the triangular numbers? Triangular numbers are numbers of the form $T_n = n(n+1)/2$.

A210. Let P be a point inside circle C. Find the locus of the centres of all circles ω which pass through P and are tangent to C.

A211. Does there exist a convex polyhedron and a plane, not passing through any of its vertices, and intersecting more than $\frac{2}{3}$ of all of the edges of the polyhedron?

(Polish Mathematical Olympiad, first round)

A212. Let A and B be real $n \times n$ matrices such that $A^2 + B^2 = AB$. Prove that if $AB - BA$ is an invertible matrix, then n is divisible by 3.

(International Competition for University Students in Mathematics)

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

In this issue, we introduce David Savitt as the new Challenge Board editor. David is currently a graduate student at Harvard, and comes to us with much problem solving experience under his belt, including high ranking on the Putnam competition. It looks like he may be asking some very tough problems!

Ravi Vakil has recently graduated from Harvard University, and will be going on to Princeton and the IAS for post-doctoral work, and we wish him the best of luck.

C75. (a) Let n be an integer, and suppose a_1, a_2, a_3, and a_4 are integers such that $a_1a_4 - a_2a_3 \equiv 1 \pmod{n}$. Show that there exist integers A_i, $1 \le i \le 4$, such that each $A_i \equiv a_i \pmod{n}$ and $A_1A_4 - A_2A_3 = 1$.

(b) Let $SL(2, \mathbb{Z})$ denote the group of 2×2 matrices with integer entries and determinant 1, and let $\Gamma(n)$ denote the subgroup of $SL(2, \mathbb{Z})$ of matrices which are congruent to the identity matrix modulo n. (By this we mean that all pairs of corresponding entries are congruent modulo n.)

What is the index of $\Gamma(n)$ in $SL(2, \mathbb{Z})$?

C76. Let X be any topological space. The n^{th} symmetric power of X, denoted $X^{(n)}$, is defined to be the quotient of the ordinary n-fold product X^n by the action of the symmetric group on n letters – that is, it is the space of unordered n-tuples of points of X. Show that the symmetric power $\mathbb{C}^{(n)}$ is actually homeomorphic to the ordinary product $\mathbb{C}^n$.

From the files

For the benefit of those readers who are new to ***MATHEMATICAL MAYHEM***, we present a few problems from back issues:

J14. [MAYHEM 1988: Vol 1, #1, 19]

In trapezoid $ABCD$, we have $AB \| CD$ and $|AB| = 2|CD|$. Suppose that AC meets BD at X.

Find the ratio $BX : XD$.

S12. [MAYHEM 1988: Vol 1, #1, 20]

Find all functions with domain $[0, \infty)$ such that

$$f(x) = \int_0^x f(t)\,dt\,.$$

U10. [MAYHEM 1988: Vol 1, #1, 21]

Prove that

$$\sum_{r=0}^{n} \binom{2n}{r} \binom{2n-2r}{n-r} = 4^n\,.$$

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 September 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2301. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that ABC is a triangle with sides a, b, c, that P is a point in the interior of $\triangle ABC$, and that AP meets the circle BPC again at A'. Define B' and C' similarly.

Prove that the perimeter $\mathcal{P}$ of the hexagon $AB'CA'BC'$ satisfies

$$\mathcal{P} \;\geq\; 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right).$$

2302. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that the bisector of angle A of triangle ABC intersects BC at D. Suppose that $AB + AD = CD$ and $AC + AD = BC$.

Determine the angles B and C.

2303. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that ABC is a triangle with angles B and C satisfying $C = 90^\circ + \frac{1}{2}B$, that the exterior bisector of angle A intersects BC at D, and that the side AB touches the incircle of $\triangle ABC$ at E.

Prove that $CD = 2AE$.

2304. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

An acute angled triangle ABC is given, and equilateral triangles ABD and ACE are drawn outwardly on the sides AB and AC. Suppose that CD and BE meet AB and AC at F and G respectively, and that CD and BE intersect at P.

Suppose that the area of the quadrilateral $AFPG$ is equal to the area of the triangle PBC. Determine angle BAC.

2305. *Proposed by Richard I. Hess, Rancho Palos Verdes, California, USA.*

An integer sided triangle has angles $p\theta$ and $q\theta$, where p and q are relatively prime integers. Prove that $\cos\theta$ is rational.

2306. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

(a) Give an elementary proof of the inequality:

$$\left(\frac{\sin\pi x}{2}\right)^2 > \frac{2x^2}{1+x^2}; \qquad (0 < x < 1). \tag{1}$$

(b) Hence (or otherwise) show that

$$\tan\pi x \begin{cases} < \frac{\pi x(1-x)}{1-2x}; & \left(0 < x < \frac{1}{2}\right), \\ > \frac{\pi x(1-x)}{1-2x}; & \left(\frac{1}{2} < x < 1\right). \end{cases} \tag{2}$$

(c) Find the maximum value of $f(x) = \dfrac{\sin\pi x}{x(1-x)}$ on the interval $(0,1)$.

2307. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

It is known that every regular $2n$–gon can be dissected into $\binom{n}{2}$ rhombuses with the same side length.

(a) How many different classes of rhombuses are there?

(b) How many rhombuses are there in each class?

2308. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

A sequence $\{v_n\}$ has initial value $v_0 = 1$ and, for $n \geq 0$, satisfies the recurrence relation

$$v_{n+1} = 2^{n+1} - \sum_{k=0}^{n} v_k\, v_{n-k}; \ .$$

Find a formula for v_n in terms of n.

2309. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that ABC is a triangle and that P is a point of the circumcircle, distinct from A, B and C. Denote by S_A the circle with centre A and radius AP. Define S_B and S_C similarly. Suppose that S_A and S_B intersect at P and P_C. Define P_B and P_A similarly.

Prove that P_A, P_B and P_C are collinear.

2310. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Let $n \in \mathbb{N}$. I call a positive integral divisor of n, say d, a *unitary* divisor if $\gcd(d, n/d) = 1$.

Let $\Upsilon(n)$ denote the sum of the unitary divisors of n.

Find a characterization of n so that $\Upsilon(n) \equiv 2 \pmod 4$.

2311. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Let $\Upsilon_e(n)$ denote the sum of the even unitary divisors, and $\Upsilon_o(n)$, the sum of the odd unitary divisors, of n. Assume that $\Upsilon_e(n) - \Upsilon_o(n) = n$.

(a) If n is composed of powers of exactly two distinct primes, show that n must be the product of two consecutive integers, one of which is a Mersenne prime.

(b) Give an example of a natural number n that is composed of powers of more than two distinct primes.

2312. *Proposed by K.R.S. Sastry, Dodballapur, India.*

The r^{th} n-gonal number is given by $P(n, r) = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2}$, where $n \geq 3$, $r = 1, 2, \ldots$.

Prove that, in the interval $[P(n, r), P(n, r+1)]$, there is an $(n-1)$-gonal number.

2313. *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let N be a non-negative integer and let a and b be complex numbers with $a, b \notin \{0, -1, -2, \ldots, -(n-1)\}$. Find a closed form expression for

$$\sum_{k=0}^{n} \frac{(-1)^k}{(a)_k\,(b)_{n-k}},$$

where $(a)_k$ denotes the Pochhammer symbol, defined by $(a)_0 = 1$, $(a)_k = a(a+1)\ldots(a+k-1)$, $k \in \mathbb{N}$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Erratum

On page 510 of issue 8, 1997, for "Cautis", read "Howard". However, the solution printed for 2181 was incorrect — see below.

The name of RICHARD I. HESS, Rancho Palos Verdes, California, USA was inadvertantly omitted from the list of solvers of problem 2153. He also sumbitted a late solution to problem 2136.

2145. [1996: 170, 1997: 302] *Proposed by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.*

Prove that $\prod_{k=1}^{n}(ak+b^{k-1}) \leq \prod_{k=1}^{n}(ak+b^{n-k})$ for all a, $b > 1$.

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. The given inequality is a special case of the following more general re-arrangement inequality due to Oppenheim [1]:

If $0 < a_1 \leq a_2 \leq \ldots \leq a_n$, $0 < b_1 \leq b_2 \leq \ldots \leq b_n$, and the sequence $\{c_n\}$ is any permutation of the sequence $\{b_n\}$, then

$$\prod_{k=1}^{n}(a_k+b_k) \leq \prod_{k=1}^{n}(a_k+c_k) \leq \prod_{k=1}^{n}(a_k+b_{n-k+1}) \, .$$

Reference

1. A. Oppenheim, Inequalities connected with definite Hermition forms II, *Amer. Math. Monthly* **61** (1954), 263–266.

2181. [1996: 318, 1997: 509] *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove that the product of eight consecutive positive integers cannot be the fourth power of any positive integer.

Corrected solution — several readers of the on-line version pointed out that the printed solution in the previous issue was incorrect — thank you!

It has been pointed out by many readers that this problem has appeared before. Most readers referred to the American Mathematical Monthly, 1936, p.310 *for the solution to #3703 (posed by Victor Thébault in 1934,* *p.522). Another reference was made to Honsberger's monograph* Mathematical Morsels, *where it appears on p.156 as "A Perfect 4th Power". Several readers also made reference to the general problem of proving that the product of (two or more) consecutive integers is never a square, which was established in 1975 by Erdős and Selfridges (Illinois Journal of Math 19 (1975), 292-301). Because the solution appears elsewhere, we will simply refer the interested reader to these other sources for a solution.*

Comments and/or references and/or solutions were submitted by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADAM BROWN, Scarborough, Ontario; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer. There were two incorrect solutions submitted.

Seiffert remarks that A. Guibert proved the result in 1862 according to L.E. Dickson in History of the Theory of Numbers, Vol. II, 1952, pp. 679-680.

2198. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Prove that, if a, b, c are the lengths of the sides of a triangle,

$$(b-c)^2\left(\frac{2}{bc}-\frac{1}{a^2}\right)+(c-a)^2\left(\frac{2}{ca}-\frac{1}{b^2}\right)+(a-b)^2\left(\frac{2}{ab}-\frac{1}{c^2}\right)\ \geq\ 0,$$

with equality if and only if $a=b=c$.

Editor's comment: Several solvers (indicated by a † beside their name in the list of solvers below) pointed out that a proof of this inequality had already appeared in the comment (given by the proposer) on problem 1940 [1994: 321].

Solved by RICHARD I. HESS, Rancho Palos Verdes, California, USA; †WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece(two proofs); †HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and †PANOS E. TSAOUSSOGLOU, Athens, Greece.

2199. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*

Find the maximum value of c for which $(x + y + z)^2 > cxz$ for all $0 \leq x < y < z$.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

With $x = 1$, $y = 1 + u$, and $z = 2$, where $0 < u < 1$, the inequality gives $(4 + u)^2 > 2c$. Letting u approach zero, we find $c \leq 8$. If $0 \leq x < y < z$, then

$$(x + y + z)^2 > (2x + z)^2 = (2x - z)^2 + 8xz \geq 8xz.$$

Hence, 8 is the maximum value of c.

Also solved by MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PAUL-OLIVIER DEHAYE, Bruxelles, Belgium; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There were two incomplete solutions.

Klamkin used a similar proof to verify the more general result:

The maximum value of c, for which

$$(x_1 + x_2 + \ldots + x_n)^2 > cx_1 \ldots x_n$$

for all $0 \leq x_1 < x_2 < \ldots < x_n$, is $c = 4(n-1)$.

Janous provided a more general result still (note the inequalities here are not strict):

Let $n \geq 2, 1 \leq k < l \leq n$ be natural numbers. Then the maximum constant $c_{k,l}$ such that

$$(x_1 + x_2 + \ldots + x_n)^2 \geq c_{k,l} x_k x_l$$

is valid for all $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n$ *is given by:*

$$c_{k,l} = \begin{cases} 4(l-k)(n-l+1) & \text{if} \quad 2l-k \geq n+1 \\ (n-k+1)^2 & \text{if} \quad 2l-k \leq n. \end{cases}$$

[*In our case, this allows us to conclude* $c \leq c_{1,3} = 8$.]

2200. *Proposed by Jeremy T. Bradley, Bristol, UK and Christopher J. Bradley, Clifton College, Bristol, UK.*

Find distinct positive integers a, b, c, d, w, x, y, z, such that

$$z^2 - y^2 \;=\; x^2 - c^2 \;=\; w^2 - b^2 \;=\; d^2 - a^2$$

and

$$c^2 - a^2 \;=\; y^2 - w^2.$$

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California, USA.*

Let d_1, d_2, d_3, d_4 and $n/d_1, n/d_2, n/d_3, n/d_4$ all be distinct divisors of some positive integer n. Then

$$z = \frac{1}{2}\left(d_1 + \frac{n}{d_1}\right), \quad y = \frac{1}{2}\left(d_1 - \frac{n}{d_1}\right), \quad \text{etc.}$$

satisfy

$$n = z^2 - y^2 = x^2 - c^2 = w^2 - b^2 = d^2 - a^2. \qquad (1)$$

The other condition in the problem becomes

$$\left(\frac{d_2}{2} - \frac{n}{2d_2}\right)^2 - \left(\frac{d_4}{2} - \frac{n}{2d_4}\right)^2 = \left(\frac{d_1}{2} - \frac{n}{2d_1}\right)^2 - \left(\frac{d_3}{2} + \frac{n}{2d_3}\right)^2,$$

or

$$n^2\left(\frac{1}{d_2^2} + \frac{1}{d_3^2} - \frac{1}{d_1^2} - \frac{1}{d_4^2}\right) + 4n + (d_2^2 + d_3^2 - d_1^2 - d_4^2) = 0, \qquad (2)$$

or $An^2 + 4n + C = 0$ where

$$A = \left(\frac{1}{d_2^2} + \frac{1}{d_3^2} - \frac{1}{d_1^2} - \frac{1}{d_4^2}\right), \qquad C = d_2^2 + d_3^2 - d_1^2 - d_4^2.$$

A computer search exposed the following solutions.

[*Editorial note*. Hess searched for integer solutions of (2), which yield *rational* values for z, y etc. He then multiplied all of z, y, x, c, w, b, d, a by an appropriate positive integer λ, and n by λ^2, to clear all denominators. He gave several solutions, some very large. We give just two of the smaller ones.]

(i) If $A = 0$, then a solution to (2) is

$$d_1 = 5,\ d_2 = 6,\ d_3 = 9,\ d_4 = 90,\ n = 2002,$$

which with $\lambda = 90$ gives the solution

$$z = 18243,\ y = 17793,\ x = 15285,\ c = 14745,$$

$$w = 10415,\ b = 9605,\ d = 5051,\ a = 3049.$$

(ii) A solution to (2) with $A \neq 0$ is

$$d_1 = 24,\ d_2 = 20,\ d_3 = 15,\ d_4 = 12,\ n = 24$$

which leads ($\lambda = 10$) to the solution

$$z = 125,\ y = 115,\ x = 106,\ c = 94,\ w = 83,\ b = 67,\ d = 70,\ a = 50.$$

II. *More solutions, and editorial comments.* There were only two other contributions to this problem. The proposers give two solutions:

$$z = 109,\ y = 89,\ x = 81,\ c = 51,\ w = 73,\ b = 37,\ d = 63,\ a = 3,$$

and

$$z = 97,\ y = 83,\ x = 79,\ c = 61,\ w = 57,\ b = 27,\ d = 51,\ a = 9,$$

found by computer. They wonder if there are infinitely many solutions. (Of course, any solution can be multiplied through by a positive integer to create another solution, but we are only interested in *primitive* solutions, that is, ones in which the eight numbers $z, y, \ldots$ have no common factor.) Hess claims to generate infinitely many primitive solutions, but this editor was unable to untangle his argument. Does anyone have a simple proof?

VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA, uses an idea similar to Hess's, namely he finds parametric expressions for the eight variables of the problem which automatically satisfy the first given condition, and then searches for values of the parameters so that the second condition is satisfied too. In particular he notes that

$$z = fgh + e,\ y = fgh - e,\ x = egh + f,\ c = egh - f,$$

$$w = efh + g,\ b = efh - g,\ d = efg + h,\ a = efg - h,$$

satisfy (1) whenever e, f, g, h are positive integers yielding positive distinct values for these expressions. The second condition then becomes

$$(egh - f)^2 - (efg - h)^2 = (fgh - e)^2 - (efh + g)^2,$$

and Konečný gives the two solutions

$$e = 4,\ f = 5,\ g = 6,\ h = 11 \quad \text{and} \quad e = 4,\ f = 6,\ g = 5,\ h = 11,$$

which result in the respective solutions

$$z = 334,\ y = 326,\ x = 269,\ c = 259,\ w = 226,\ b = 214,\ d = 131,\ a = 109,$$

and

$$z = 334,\ y = 326,\ x = 226,\ c = 214,\ w = 269,\ b = 259,\ d = 131,\ a = 109.$$

Note that these solutions differ only in that b, c have been switched and w, x have been switched. In fact this pair of switches is always possible. Condition (1) obviously remains true whenever these switches are made; moreover, in Hess's solution, the other condition (2) is symmetric in d_2 and d_3, which shows that these switches do not affect the truth of (2) either. And Hess's solution is *general*: given any solution of (1), put $z + y = d_1$ (and $z - y = n/d_1$), and similarly for the other variables, and Hess's expressions for z, y etc. follow.

As a consequence, we need only look for solutions to the problem which satisfy $b < c$ and $w < x$, which hold for all solutions above except for Konečný's second. Similarly (but more easily), given any solution to the problem, switching z with d and y with a will always result in another solution, which means we can also assume that $d < z$ and $a < y$, which hold for all the above solutions.

At the moment this problem is still in an unsatisfactory state of disarray, and needs someone to bring some order to the chaos! Readers are encouraged to try.

2201. [1997: 45] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral, and O is the intersection of its diagonals. Let L, M, N be the midpoints of DB, BC, CA respectively. Suppose that AL, OM, DN are concurrent. Show that

$$\text{either} \quad AD \parallel BC \qquad \text{or} \qquad [ABCD] = 2[OBC],$$

where $[\mathcal{F}]$ denotes the area of figure $\mathcal{F}$.

I. *Solution by C. Festraets-Hamoir, Brussels, Belgium.*

Let O be the origin of a coordinate system where A, B, C, D are represented by $(a,0)$, $(0,b)$, $(c,0)$, $(0,d)$ with a, b positive and c, d negative. Thus L is the point $\left(0, \frac{(b+d)}{2}\right)$, M is $\left(\frac{c}{2}, \frac{b}{2}\right)$, N is $\left(\frac{(a+c)}{2}, 0\right)$ and

$$AL : (b+d)x + 2ay - a(b+d) = 0$$

$$OM : bx - cy = 0$$

$$DN : 2dx + (a+c)y - d(a+c) = 0.$$

These lines are concurrent if and only if

$$\begin{vmatrix} b & -c & 0 \\ b+d & 2a & -a(b+d) \\ 2d & a+c & -d(a+c) \end{vmatrix} = 0.$$

This equation reduces (after some manipulation) to

$$(ab - cd)[(a - c)(b - d) + 2bc] = 0.$$

Consequently, either

(a) $ab = cd$, in which case $AD||BC$, or

(b) $\frac{1}{2}(a - c)(b - d)\sin\alpha = 2\left(-\frac{1}{2}bc\sin\alpha\right)$ (where $\alpha = \angle AOB$), in which case $[ABCD] = 2[OBC]$.

II. *Comment based on a solution by Michael Lambrou, University of Crete, Crete, Greece.*

Using barycentric coordinates one can obtain a geometric characterization of those quadrilaterals $ABCD$ for which AL, OM, and DN are concurrent at J: Either $AB||BC$ (in which case J always exists), or $JCOB$ is a parallelogram whose area equals $ABCD$. Thus to draw an accurate figure of the latter case, begin with a parallelogram $JCOB$ and let L be any point on BO. (For $ABCD$ to be convex, L should lie between O and the midpoint of BO.)

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece, (3 solutions); GERRY LEVERSHA, St Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2202. [1997: 45] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose than $n \geq 3$. Let $A_1, \ldots, A_n$ be a convex n-gon (as usual with interior angles $A_1, \ldots, A_n$).

Determine the greatest constant C_n such that

$$\sum_{k=1}^{n} \frac{1}{A_k} \geq C_n \sum_{k=1}^{n} \frac{1}{\pi - A_k}.$$

Determine when equality occurs.

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We assume that the intention of the problem is to determine the largest constant C_n so that the given inequality holds *for all* convex n-gons.

If $n \geq 4$ then clearly $C_n = 0$, since we could let one of the interior angles tend to π without forcing any other interior angle to tend to zero. Then

$$\sum_{k=1}^{n} \frac{1}{\pi - A_k} \to +\infty \qquad \text{while} \qquad \sum_{k=1}^{n} \frac{1}{A_k} \quad \text{is bounded.}$$

Thus the inequality cannot hold unless $C_n \leq 0$. Hence $C_n = 0$ and equality can never occur.

If $n = 3$, then by the arithmetic–harmonic–mean inequality,

$$(A_1 + A_2)\left(\frac{1}{A_1} + \frac{1}{A_2}\right) \geq 4$$

and thus

$$\frac{1}{A_1} + \frac{1}{A_2} \geq \frac{4}{A_1 + A_2} = \frac{4}{\pi - A_3}\,.$$

Similarly,

$$\frac{1}{A_2} + \frac{1}{A_3} \geq \frac{4}{\pi - A_1} \qquad \text{and} \qquad \frac{1}{A_3} + \frac{1}{A_1} \geq \frac{4}{\pi - A_2}\,.$$

Adding, we get

$$\sum_{k=1}^{3} \frac{1}{A_k} \geq 2 \sum_{k=1}^{3} \frac{1}{\pi - A_k}\,.$$

It is easily seen that equality holds if and only if all the A_k's are equal. Hence $C_3 = 2$ and equality holds if and only if $\Delta A_1A_2A_3$ is equilateral.

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece (three solutions); GERRY LEVERSHA, St Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. The case $n = 3$ only was solved by GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia. Two incorrect solutions were also sent in.

Seiffert uses the known inequality

$$\sum_{k=1}^{n} \frac{1}{1 - a_k} \sum_{k=1}^{n} (1 - a_k) \leq \sum_{k=1}^{n} \frac{1}{a_k} \sum_{k=1}^{n} a_k\,,$$

where $0 < a_k \leq 1/2$ (see pages 25–27 of Mitrinović, Pečarić and Fink, Classical and New Inequalities in Analysis, *Kluwer, 1993), to prove the following* *related inequality: if $n \geq 4$ and $A_1A_2\ldots A_n$ is a convex n-gon such that $A_k \geq \pi/2$ for all k, then*

$$\sum_{k=1}^{n} \frac{1}{A_k} \leq \frac{2}{n-2} \sum_{k=1}^{n} \frac{1}{\pi - A_k}$$

(put $a_k = 1 - A_k/\pi$). Furthermore, equality holds if and only if the n-gon is regular, since equality holds in the earlier inequality if and only if the a_k's are equal.

One reader pointed out that the proposer had also published this problem, in German, in the journal Wissenschaftliche Nachrichten *in January 1996, and that a solution was published on page 36 of the January 1997 issue.*

Readers are reminded that a problem submitted to *CRUX with MAYHEM* should not be submitted for publication elsewhere, unless and until the problem has either been rejected by *CRUX with MAYHEM* or withdrawn by the proposer — Editor-in-Chief.

2203. [1997: 46] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $ABCD$ be a quadrilateral with incircle $\mathcal{I}$. Denote by P, Q, R and S, the points of tangency of sides AB, BC, CD and DA, respectively with $\mathcal{I}$.

Determine all possible values of $\angle(PR, QS)$ such that $ABCD$ is cyclic.

Comments. A quadrilateral that is simultaneously inscribed in one circle and circumscribed about another is call bicentric. Konečný and Reiman both report that in [2, section 39] you can find the solution to our problem and lots more, including its converse and the fact (Brianchon's Theorem) that AC, BD, PR, and ST all pass through the same point (see also [1, p. 79]). Bellot Rosado reminds us that our problem was part of a problem proposed by India (but not used) at the 1989 IMO. We present two of the many possible solutions.

I. *Solution by C. Festraets-Hamoir, Brussels, Belgium.*

We denote by $\widehat{XY}$ the angle subtended at the centre of $\mathcal{I}$ by the arc XY. [Note that since $\mathcal{I}$ is an incircle, $ABCD$ is convex and the diagonals PR and QS intersect in the interior.]

$\angle A = \frac{1}{2}(\widehat{PQ} + \widehat{QR} + \widehat{RS} - \widehat{SP})$, and

$\angle C = \frac{1}{2}(\widehat{RS} + \widehat{SP} + \widehat{PQ} - \widehat{QR})$,

so that

$\angle A + \angle C = \widehat{PQ} + \widehat{RS}$.

$ABCD$ is cyclic if and only if $\angle A + \angle C = \pi$, and so, if and only if $\angle(PR, QS) = \frac{\pi}{2}$.

II. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

Consider inversion in $\mathcal{I}$. The images A', B', C', D' of A, B, C, D are the respective midpoints of SP, PQ, QR, and RS (as in [1, Figure 5.3A]), so that $A'B'C'D'$ is a parallelogram. Since circles are preserved by inversion, $A'B'C'D'$ is cyclic if and only if $ABCD$ is, in which case the parallelogram $A'B'C'D'$ would be a rectangle. Because each side of a midpoint quadrangle is parallel to a diagonal (PR or QS) we conclude that $PR \perp QS$ if and only if $ABCD$ is cyclic.

References.

[1] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*. MAA, 1967.

[2] Heinrich Dörrie, *Triumph der Mathematik*, Würzburg, 1958. English translation: *100 Great Problems of Elementary Mathematics*, Dover, 1965.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JORDI DOU, Barcelona, Spain; YEO KENG HEE, Hua Chong Junior College, Singapore; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlandsand the proposer.

2204. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

For triangle ABC such that $R(a+b) = c\sqrt{ab}$, prove that

$$r < \frac{3}{10}a.$$

Here, a, b, c, R, and r are the three sides, the circumradius and the inradius of $\triangle ABC$.

Solution by Kee-Wai Lau, Hong Kong.

Since $c = 2R\sin C$,

$$R(a+b) = 2R\sin C\sqrt{ab}.$$

Using the AM–GM inequality,

$$\sin C = \frac{a+b}{2\sqrt{ab}} \geq 1.$$

Hence, $\sin C = 1$, $C = 90^0$, $a = b$ and $c = \sqrt{2}a$. Thus

$$r = \frac{\text{Area of } \triangle ABC}{\text{Semiperimeter of } \triangle ABC} = \frac{a^2}{2a + \sqrt{2}a} = \frac{a}{2 + \sqrt{2}} < \frac{3}{10}a,$$

as required.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; C. FESTRAETS-HAMOIR, Brussels, Belgium; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; PANOS E. TSAOUSSOGLOU, Athens, Greece(two solutions); and the proposer.

Most of the submitted solutions are similar to the one given above.

2205. [1997: 46] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Find the least positive integer n such that the expression

$$\sin^{n+2} A \ \sin^{n+1} B \ \sin^n C$$

has a maximum which is a rational number (where A, B, C are the angles of a variable triangle).

Many of the solvers cited the relationship between this problem and [1984: 19] proposed by M.S. Klamkin which asks for the maximum value of $P = \sin^\alpha A \sin^\beta B \sin^\gamma C$, where A, B, C are the angles of a triangle and α, β, γ are given. The published solution by Walther Janous [1985: 908] gives this maximum as:

$$P_{\max} = \left(\frac{\alpha(\alpha+\beta+\gamma)}{(\alpha+\beta)(\alpha+\gamma)}\right)^{\alpha/2} \left(\frac{\beta(\alpha+\beta+\gamma)}{(\beta+\gamma)(\beta+\alpha)}\right)^{\beta/2} \left(\frac{\gamma(\alpha+\beta+\gamma)}{(\gamma+\alpha)(\gamma+\beta)}\right)^{\gamma/2}.$$

Substitution and simplification yields

$$P_{\max} = \frac{(n+1)^{n+1}(n+2)3^{n+1}}{2^{n+1}(2n+3)^{n+1}(2n+1)^n} \cdot \frac{n^{n/2}(n+2)^{n/2} \cdot 3^{(n+1)/2}}{(2n+1)^{1/2}(2n+3)^{1/2}}.$$

The first term is rational and the smallest integer for which the second part is rational is $n = 12$. (See also problem 2183 [1996: 319; 1997: 514], 2116 [1996: 75; 1997: 116].

Solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

However, some of our solvers were not content to leave it here. The first word goes to Walther Janous who provided the solution in [1985: 908].

Generalization, Part I, by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We now show that there are infinitely many integers n such that $P_{\max}(n)$ is rational.

Let n be even. Then $n/2$ is an integer and

$$P_{\max}(n) \in \mathbb{Q} \qquad \text{is equivalent to} \qquad \sqrt{\frac{3}{(2n+1)(2n+3)}} \in \mathbb{Q}.$$

Because gcd $(2n+1, 2n+3) = 1$, one of the following two possibilities must occur:

(i) $2n+1 = x^2$ and $2n+3 = 3y^2$, or

(ii) $2n+1 = 3x^2$ and $2n+3 = y^2$,

where x, y are integers. We show that (i) leads to the claimed infinitely many n's. Indeed, (i) implies $x^2 + 2 = 3y^2$; that is,

$$x^2 - 3y^2 = -2, \qquad (*)$$

which has $(x, y) = (1, 1)$ as one solution. The associated "pure" Pellian equation

$$X^2 - 3Y^2 = 1$$

has $(X, Y) = (2, 1)$ as its fundamental solution. Hence, we get infinitely many solutions (x_m, y_m) of $(*)$:

$$x_m + y_m\sqrt{3} = (1+\sqrt{3})(2+\sqrt{3})^m, \text{ for } m = 0, 1, 2, \ldots$$

Simple congruence considerations, mod 4, show that all solutions (x, y) of $(*)$ consist of odd numbers. Then

$$n = \frac{x_m^2 - 1}{2}, \quad m = 1, 2, 3, \ldots$$

as claimed.

If $m = 1$, $x_1 = 5$ and $n = 12$. In general, via conjugation, $x_m - y_m\sqrt{3} = (1-\sqrt{3})(2-\sqrt{3})^m$. Thus,

$$x_m = \frac{1}{2}((1+\sqrt{3})(2+\sqrt{3})^m + (1-\sqrt{3})(2-\sqrt{3})^m)$$

and

$$n = \frac{x_{m^2}-1}{2} = \frac{1}{4}((2+\sqrt{3})^{2m+1} + (2-\sqrt{3})^{2m+1} - 4)$$

(so $n = 12$; 180; 2520; 35112; 489060; 6811740; 94875312; etc.)

Generalization, Part II, by Michael Lambrou, University of Crete, Crete, Greece.

We show that there are no solutions with n odd.

Let n be odd and write $n = 2N-1$, $N \geq 1$. Then the second part of $P_{\max}$ is

$$\frac{(2N-1)^{\frac{2N-1}{2}}(2N+1)^{\frac{2N-1}{2}}3^N}{(4N-1)^{1/2}(4N+1)^{1/2}} = \frac{\text{(a rational number)}}{\sqrt{(2N-1)(2N+1)(4N-1)(4N+1)}}$$

We now show that the integer inside the square root sign is never a perfect square. We have

$$(2N-1)(2N+1)(4N-1)(4N+1) = 64N^4 - 20N^2 + 1$$

which is easily seen to be strictly between the perfect squares

$$(8N^2-2)^2 = 64N^4-32N^2+4 \quad \text{and} \quad (8N^2-1)^2 = 64N^4-16N^2+1,$$

so is not itself a perfect square.

2206. [1997: 46] *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let a and b denote distinct positive real numbers.

(a) Show that if $0 < p < 1$, $p \neq \frac{1}{2}$, then

$$\frac{1}{2}\left(a^p b^{1-p} + a^{1-p}b^p\right) < 4p(1-p)\sqrt{ab} + (1-4p(1-p))\left(\frac{a+b}{2}\right).$$

(b) Use (a) to deduce Pólya's Inequality:

$$\frac{a-b}{\log a - \log b} < \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right).$$

Note: "log" is, of course, the natural logarithm.

I. *Solution to part (a) by Christopher J. Bradley, Clifton College, Bristol, UK.*

The symmetry of the inequality to be proved, between a and b, and about $p = \frac{1}{2}$, allows us, without loss of generality, to suppose that $b > a$ and $0 < p < \frac{1}{2}$. To save writing, set $k = 4p(1-p)$ and note that $0 < k < 1$.

We are then required to prove:

$$\frac{1}{2}\left(a^p b^{1-p} + a^{1-p} b^p\right) \ < \ k\sqrt{ab} + (1-k)\left(\frac{a+b}{2}\right) .$$

This is equivalent to each of the following three inequalities:

$$\begin{aligned}
\frac{1}{2}\left(a^p b^{1-p} + a^{1-p} b^p\right) \ &< \ \frac{(1-k)}{2}\left(a+b-2\sqrt{ab}\right) + \sqrt{ab}, \\
\left(a^p b^{1-p} + a^{1-p} b^p - 2\sqrt{ab}\right) \ &< \ (1-k)\left(a+b-2\sqrt{ab}\right) , \\
a^p b^p\left(b^{1-2p} - 2a^{\frac{1}{2}-p} b^{\frac{1}{2}-p} + a^{1-2p}\right) \ &< \ (1-2p)^2\left(a - 2a^{\frac{1}{2}} b^{\frac{1}{2}} + b\right) .
\end{aligned}$$

It is sufficient to prove the inequality obtained from taking the positive square root of this, namely:

$$b^{\frac{1}{2}-\frac{p}{2}} a^{\frac{p}{2}} - a^{\frac{1}{2}-\frac{p}{2}} b^{\frac{p}{2}} \ < \ (1-2p)\left(b^{\frac{1}{2}} - a^{\frac{1}{2}}\right) .$$

Setting $a = c^2$ and $b = d^2$ (with $d > c > 0$), this is the same as

$$2p(d-c) \ < \ d - c + c^{1-p} d^p - d^{1-p} c^p$$

or

$$2p(d-c) \ < \ \left(d^p - c^p\right)\left(d^{1-p} - c^{1-p}\right) .$$

We now prove this inequality when p is rational. Let $p = \frac{m}{n}$, with m and n co-prime integers and $2m < n$ (since $p < \frac{1}{2}$). Also, we set $c = x^n$, $d = y^n$ with $y > x > 0$. With these restrictions and substitutions, it is now sufficient to prove that

$$2m\left(y^n - x^n\right) \ < \ n\left(y^m - x^m\right)\left(y^{n-m} + x^{n-m}\right)$$

or

$$(n-2m)\left(y^n - x^n\right) \ > \ n y^m x^m\left(y^{n-2m} - x^{n-2m}\right) .$$

On division by $(y-x)$, which is valid since it is a positive quantity, it is sufficient to prove that

$$\begin{aligned}
&(n-2m)\left(y^{n-1} + xy^{n-2} + \ldots + x^{n-1}\right) \\
&\quad > \ n x^m y^m\left(y^{n-2m-1} + xy^{n-2m-2} + \ldots + x^{n-2m-1}\right) .
\end{aligned}$$

This is true by repeated application of the Power Means Inequality. This result is then extended to irrational values of p by the familiar continuity arguments.

II. *Solution by Kee-Wai Lau, Hong Kong.*

(a) For $t > 0$, let

$$f(t) \;=\; \tfrac{1}{2}\left(t^p + y^{1-p}\right) - 4p(1-p)\sqrt{t} - \Big(1 - 4p(1-p)\Big)\left(\frac{t+1}{2}\right).$$

Differentiating, we have:

$$f'(t) \;=\; \frac{pt^{p-1} + (1-p)t^{-p}}{2} - 2p(1-p)t^{-\frac{1}{2}} - \frac{1 - 4p(1-p)}{2}, \qquad (1)$$

and

$$f''(t) \;=\; \tfrac{1}{2}p(1-p)t^{-2}\left(2t^{\frac{1}{2}} - t^p - t^{1-p}\right). \qquad (2)$$

It is easy to check that $f'(1) = f''(1) = 0$. Since $p \neq \frac{1}{2}$, we have, for $t \neq 1$,

$$2t^{\frac{1}{2}} - t^p - t^{1-p} \;<\; 2t^{\frac{1}{2}} - 2\sqrt{t^{1-p}t^p} \;=\; 0\,,$$

and hence that $f''(t) < 0$. Thus $f(t) \le f(1) = 0$ and $f(t) = 0$ if and only if $t = 1$. Now putting $t = \frac{a}{b}$, we easily obtain inequality (a).

(b) From part (a), we see that, for $t \neq 1$ and $p \neq \frac{1}{2}$, we have $f(t) < 0$, or

$$\tfrac{1}{2}\left(t^p + t^{1-p}\right) \;<\; 4p(1-p)t^{\frac{1}{2}} + (2p-1)^2\left(\frac{t+1}{2}\right).$$

By integrating this inequality with respect to p from $p = \frac{1}{2}$ to $p = 1$, we obtain

$$\frac{1}{2}\left(\frac{t-1}{\log t}\right) \;<\; \frac{1}{3}\sqrt{t} + \frac{t+1}{12}.$$

Pólya's Inequality follows by substituting $t = \frac{a}{b}$.

Part (a) was also solved by PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (two solutions); VEDULA N. MURTY, Visakhapatnam, India; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

Part(b) was also solved by PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (two solutions); VEDULA N. MURTY, Visakhapatnam, India; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

All solvers, except Bradley, used Calculus in some way or other.

2207. [1997: 46] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Let p be a prime. Find all solutions in positive integers of the equation:

$$\frac{2}{a}+\frac{3}{b}=\frac{5}{p}.$$

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Equivalently, we have $p(3a+2b)=5ab$; hence there are three cases to consider.

First case: $p=5$. Then we get

$$3a+2b=ab \quad \text{or} \quad (a-2)(b-3)=6=1\cdot 6=2\cdot 3.$$

Hence all pairs of positive integers (a,b) are $(3,9)$, $(4,6)$, $(5,5)$ and $(8,4)$.

Second case: p divides a. Let $a=a_1p$. Hence

$$3a_1p=b(5a_1-2).$$

Then p divides either b or $5a_1-2$. If $b=b_1p$, then $3a_1=b_1(5a_1-2)$ or

$$(5a_1-2)(5b_1-3)=6$$

which has only one solution: $(a_1,b_1)=(1,1)$, whence $(a,b)=(p,p)$. Otherwise, $5a_1-2=a_2p$. Therefore,

$$3\frac{a_2p+2}{5}=ba_2 \quad \text{or} \quad 3a_2p+6=5ba_2.$$

Hence a_2 divides 6.

If $a_2=1$, then $(a,b)=\left(\frac{p(p+2)}{5},\frac{3(p+2)}{5}\right)$, but only if $p\equiv 3 \pmod 5$.

If $a_2=2$, then $(a,b)=\left(\frac{2p(p+1)}{5},\frac{3(p+1)}{5}\right)$, but only if $p\equiv 4 \pmod 5$.

If $a_2=3$, then $(a,b)=\left(\frac{p(3p+2)}{5},\frac{3p+2}{5}\right)$, but only if $p\equiv 1 \pmod 5$.

If $a_2=6$, then $(a,b)=\left(\frac{2p(3p+1)}{5},\frac{3p+1}{5}\right)$, but only if $p\equiv 3 \pmod 5$.

Third case: p divides b. Let $b=b_1p$. Hence

$$2b_1p=a(5b_1-3).$$

Then p divides either a or $5b_1-3$. If p divides a, then we have the same case (p divides a and b) as already considered above. Again (p,p) is the (only) solution. Otherwise, $5b_1-3=b_2p$. Therefore,

$$2\frac{b_2p+3}{5}=ab_2 \quad \text{or} \quad 2b_2p+6=5ab_2.$$

Hence b_2 divides 6.

If $b_2 = 1$, then $(a, b) = \left(\frac{2(p+3)}{5}, \frac{p(p+3)}{5}\right)$, but only if $p \equiv 2 \pmod 5$.

If $b_2 = 2$, then $(a, b) = \left(\frac{2p+3}{5}, \frac{p(2p+3)}{5}\right)$, but only if $p \equiv 1 \pmod 5$.

If $b_2 = 3$, then $(a, b) = \left(\frac{2(p+1)}{5}, \frac{3p(p+1)}{5}\right)$, but only if $p \equiv 4 \pmod 5$.

If $b_2 = 6$, then $(a, b) = \left(\frac{2p+1}{5}, \frac{3p(2p+1)}{5}\right)$, but only if $p \equiv 2 \pmod 5$.

[Note that if $p = 2$ or 3, this generates only two distinct solutions: $(2, 2)$, $(1, 6)$ for $p = 2$ and $(3, 3)$, $(12, 2)$ for $p = 3$. If $p > 5$, then the three solutions are all distinct.]

Also solved by GERALD ALLEN, CHARLES DIMINNIE, TREY SMITH and ROGER ZARNOWSKI (jointly), Angelo State University, San Angelo, TX, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; KEITH EKBLAW, Walla Walla, Washington, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer. There were six incorrect and two incomplete solutions.

2153. [1996:217; 1997:313]
The conjectured inequality should read $|x^n p(1/x)| \leq 2^{n-1}$.

2167. [1996:274; 1997:381]
The equation in the last line should read $2^{(n+4)/2}n + n = 2$.