THE ACADEMY CORNER

No. 15

Bruce Shawyer

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Memorial University Undergraduate Mathematics Competition

September 25, 1997

Answer as many questions as you can. Complete solutions carry more credit than scattered comments about many problems.

1. Determine whether or not the following system has any real solutions. If so, state how many real solutions exist.
$$x + \frac{1}{x} = y, \qquad y + \frac{1}{y} = z, \qquad z + \frac{1}{z} = x.$$
2. The surface area of a closed cylinder is twice the volume. Determine the radius and height of the cylinder given that the radius and height are both integers.
3. Prove that
$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \; < \; 2.$$
4. Describe the set of points (x, y) in the plane for which
$$\sin(x + y) = \sin x + \sin y.$$
5. In a parallelogram $ABCD$, the bisector of angle ABC intersects AD at P. If $PD = 5$, $BP = 6$ and $CP = 6$, find AB.
6. Show that, where $k + n \leq m$,
$$\sum_{i=0}^{n} \binom{n}{i} \binom{m}{k+i} = \binom{m+n}{n+k}.$$

Send me your nice solutions!

THE OLYMPIAD CORNER

No. 186

R.E. Woodrow

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This number we give the 24 problems proposed to the jury, but not selected for the 37th International Mathematical Olympiad in July 1996 at Mumbai, India. My thanks go to Ravi Vakil, Canadian Team Leader to the 37th IMO for collecting this and other contest material and forwarding it to me.

PROBLEMS PROPOSED TO THE JURY
BUT NOT USED AT THE
37th INTERNATIONAL MATHEMATICAL OLYMPIAD
July 1996 — Mumbai, India

1. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \le 1.$$

When does equality hold?

2. Let $a_1 \ge a_2 \ge \cdots \ge a_n$ be real numbers such that for all integers $k > 0$,

$$a_1^k + a_2^k + \cdots + a_n^k \ \ge\ 0.$$

Let $p = \max\{|a_1|, \dots, |a_n|\}$. Prove that $p = a_1$ and that

$$(x - a_1)(x - a_2)\cdots(x - a_n) \ \le\ x^n - a_1^n$$

for all $x > a_1$.

3. Let $a > 2$ be given, and define recursively:

$$a_0 = 1, \quad a_1 = a, \quad a_{n+1} = \left(\frac{a_n^2}{a_{n-1}^2} - 2\right) a_n.$$

Show that for all integers $k > 0$, we have

$$\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} \ <\ \frac{1}{2}(2 + a - \sqrt{a^2 - 4}).$$

4. Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers, not all zero.

(a) Prove that $x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n = 0$ has precisely one positive real root.

(b) Let $A = \sum_{j=1}^{n} a_j$, and $B = \sum_{j=1}^{n} ja_j$ and let R be the positive real root of the equation in (a). Prove that $A^A \leq R^B$.

5. Let $P(x)$ be the real polynomial, $P(x) = ax^3 + bx^3 + cx + d$. Prove that if $|P(x)| \leq 1$ for all x such that $|x| \leq 1$, then

$$|a| + |b| + |c| + |d| \leq 7.$$

6. Let n be an even positive integer. Prove that there exists a positive integer k such that

$$k = f(x)(x+1)^n + g(x)(x^n + 1)$$

for some polynomials $f(x)$, $g(x)$ having integer coefficients. If k_0 denotes the least such k, determine k_0 as a function of n.

7. Let f be a function from the set of real numbers $\mathbb{R}$ into itself such that for all $x \in \mathbb{R}$, we have $|f(x)| \leq 1$ and

$$f\left(x + \frac{13}{42}\right) + f(x) \;=\; f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right).$$

Prove that f is a periodic function (that is, there exists a non-zero real number c, such that $f(x + c) = f(x)$ for all $x \in \mathbb{R}$).

8. Let the sequence $a(n)$, $n = 1, 2, 3, \ldots$, be generated as follows: $a(1) = 0$, and for $n > 1$,

$$a(n) = a(\lfloor n/2 \rfloor) + (-1)^{n(n+1)/2}.$$

(Here $\lfloor t \rfloor$ is the greatest integer less than or equal to t.)

(a) Determine the maximum and minimum value of $a(n)$ over $n \leq 1996$ and find all $n \leq 1996$ for which these extreme values are attained.

(b) How many terms $a(n)$, $n \leq 1996$, are equal to 0?

9. Let triangle ABC have orthocentre H, and let P be a point on its circumcircle, distinct from A, B, C. Let E be the foot of the altitude BH, let $PAQB$ and $PARC$ be parallelograms, and let AQ meet HR in X. Prove that EX is parallel to AP.

10. Let ABC be an acute-angled triangle with $|BC| > |CA|$, and let O be the circumcentre, H its orthocentre, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$.

11. Let ABC be equilateral, and let P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB in the points A_1, B_1, C_1 respectively. Prove that

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \;\geq\; A_1B \cdot B_1C \cdot C_1A.$$

12. Let the sides of two rectangles be $\{a, b\}$ and $\{c, d\}$ respectively, with $a < c \leq d < b$ and $ab < cd$. Prove that the first rectangle can be placed within the second one if and only if

$$(b^2 - a^2)^2 \;\leq\; (bc - ad)^2 + (bd - ac)^2.$$

13. Let ABC be an acute-angled triangle with circumcentre O and circumradius R. Let AO meet the circle BOC again in A', let BO meet the circle COA again in B' and let CO meet the circle AOB again in C'. Prove that

$$OA' \cdot OB' \cdot OC' \;\geq\; 8R^3.$$

When does equality hold?

14. Let $ABCD$ be a convex quadrilateral, and let R_A, R_B, R_C, R_D denote the circumradii of the triangles DAB, ABC, BCD, CDA respectively. Prove that $R_A + R_C > R_B + R_D$ if and only if $\angle A + \angle C > \angle B + \angle D$.

15. On the plane are given a point O and a polygon $\mathcal{F}$ (not necessarily convex). Let P denote the perimeter of $\mathcal{F}$, D the sum of the distances from O to the vertices of $\mathcal{F}$, and H the sum of the distances from O to the lines containing the sides of $\mathcal{F}$. Prove that $D^2 - H^2 \geq P^2/4$.

16. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and the next number on the circle, moving in a clockwise direction; that is, the numbers
a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$. Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ are primes?

17. A finite sequence of integers $a_0, a_1, \ldots, a_n$ is called *quadratic* if for each i in the set $\{1, 2, \ldots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.

(a) Prove that for any two integers b and c, there exists a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.

(b) Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

18. Find all positive integers a and b for which

$$\left\lfloor \frac{a^2}{b} \right\rfloor + \left\lfloor \frac{b^2}{a} \right\rfloor = \left\lfloor \frac{a^2 + b^2}{ab} \right\rfloor + ab,$$

where, as usual, $\lfloor t \rfloor$ refers to the greatest integer which is less than or equal to t.

19. Let N_0 refer to the set of non-negative integers. Find a bijective function f from N_0 into N_0 such that for all $m, n \in N_0$,

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

20. A square $(n-1) \times (n-1)$ is divided into $(n-1)^2$ unit squares in the usual manner. Each of the n^2 vertices of these squares is to be coloured red or blue. Find the number of different colourings such that each unit square has exactly two red vertices. (Two colouring schemes are regarded as different if at least one vertex is coloured differently in the two schemes.)

21. Let k, m, n be integers such that $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset S of the set $\{1, 2, 3 \ldots, k-1, k\}$ such that no n distinct elements of S add up to m.

22. Determine whether or not there exist two disjoint infinite sets $\mathcal{A}$ and $\mathcal{B}$ of points in the plane satisfying the following conditions:

(a) No three points in $\mathcal{A} \cup \mathcal{B}$ are collinear, and the distance between any two points in $\mathcal{A} \cup \mathcal{B}$ is at least 1.

(b) There is a point of $\mathcal{A}$ in any triangle whose vertices are in $\mathcal{B}$, and there is a point of $\mathcal{B}$ in any triangle whose vertices are in $\mathcal{A}$.

23. A finite number of beans are placed on an infinite row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one bean is chosen. Two beans are taken from this square; one of them is placed on the square immediately to the left while the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one bean on each square. Given some initial configuration, show that any legal sequence of moves will terminate after the same number of steps and with the same final configuration.

24. Let U be a finite set and f, g be bijective functions from U onto itself. Let

$$S = \{w \in U : f(f(w)) = g(g(w))\}$$

and

$$T = \{w \in U : f(g(w)) = g(f(w))\},$$

and suppose that $U = S \cup T$. Prove that, for $m \in U$, $f(w) \in S$ if and only if $g(w) \in S$.

As always we welcome your nice original solutions which differ from the official solutions provided by the proposers and the selection committee.

As an example of an Olympiad which may not be as widely circulated, and for which you may not have already seen solutions, we give the four problems of the 4$^{\text{th}}$ Class for the Croatian National Mathematical Competition of May 13, 1994 and the three problems of the Croatian Mathematical Olympiad of May 14, 1994.

My thanks go to Richard Nowakowski, Canadian Team Leader at the 35$^{\text{th}}$ IMO in Istanbul for collecting these problems.

CROATIAN NATIONAL MATHEMATICAL COMPETITION
Fourth Class
May 13, 1994

1. One member of an infinite arithmetic sequence in the set of natural numbers is a perfect square. Show that there are infinitely many members of this sequence having this property.

2. For a complex number z let $w = f(z) = \dfrac{2}{3-z}$.

(a) Determine the set $\{w : z = 2 + iy,\ \ y \in R\}$ in the complex plane.

(b) Show that the function w can be written in the form

$$\frac{w-1}{w-2} = \lambda\,\frac{z-1}{z-2}.$$

(c) Let $z_0 = \frac{1}{2}$ and the sequence $\{z_n\}$ be defined recursively by

$$z_n = \frac{2}{3 - z_{n-1}}, \quad n \geq 1.$$

Using the property (b) calculate the limit of the sequence $\{z_n\}$.

3. Determine all polynomials $P(x)$ with real coefficients such that for some $n \in \mathbb{N}$ we have $xP(x-n) = (x-1)P(x)$, $\forall\, x \in \mathbb{R}$.

4. In the plane five points P_1, P_2, P_3, P_4, P_5 are chosen having integer coordinates. Show that there is at least one pair (P_i, P_j), for $i \neq j$ such that the line P_iP_j contains a point Q, with integer coordinates, and is strictly between P_i and P_j.

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Additional Competition for the Olympiad
May 14, 1994

1. Find all ordered triples (a, b, c) of real numbers such that for every three integers x, y, z the following identity holds:

$$|ax + by + cz| + |bx + cy + az| + |cx + ay + bz| = |x| + |y| + |z|.$$

2. Construct a triangle ABC if the lengths $|AO|$, $|AU|$ and radius r of the incircle are given, where O is the orthocentre and U the centre of the incircle.

3. Let P be the set of all lines of the plane M. Does there exist a function $f : P \to M$ having the following properties:

(a) the function f is an injection:

(b) $f(p) \in p$, $\forall\, p \in P$?

That should provide some problems for your puzzling pleasure over the next couple of months. Now we return to readers' solutions to problems featured in earlier numbers of the *Corner*.

First, an apology. Somehow, in shifting my files around we misplaced solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain, to two problems that we discussed in the October number of the *Corner*. His name should be added as a solver of problems 6 and 7 of the Telecom 1993 Australian Mathematical Olympiad in the solutions given [1997: 324–325].

Last number we gave solutions by the readers to the first ten problems of the "Baltic Way — 92" contest given in the May 1996 number [1996: 157–159].

MATHEMATICAL TEAM CONTEST "BALTIC WAY — 92" Vilnius, 1992 — November 5–8

11. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Show that there exists one and only one function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the following conditions:

(i) If $0 < q < \frac{1}{2}$ then $f(q) = 1 + f\left(\frac{q}{1-2q}\right)$.
(ii) If $1 < q \leq 2$ then $f(q) = 1 + f(q-1)$.
(iii) $f(q) \cdot f(\frac{1}{q}) = 1$ for all $q \in \mathbb{Q}^+$.

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

By a change of variable $\widetilde{q} = \frac{1}{1-2q}$, we have from (i),

$$f\left(\frac{\widetilde{q}}{1+2\widetilde{q}}\right) = 1 + f(\widetilde{q}), \quad (0 < \widetilde{q} < \infty), \quad \text{or} \quad f\left(\frac{1}{\frac{1}{\widetilde{q}}+2}\right) = 1 + f\left(\frac{1}{\frac{1}{\widetilde{q}}}\right).$$

Calling $t = \frac{1}{\widetilde{q}}$ and using (iii) we have

$$\frac{1}{f(t+2)} = 1 + \frac{1}{f(t)}, \quad 0 < t < \infty, \quad t \in \mathbb{Q}^+. \tag{1}$$

Then

$$\frac{1}{f(t+4)} = \frac{1}{f(t+2+2)} = 1 + \frac{1}{f(t+2)} = 1 + 1 + \frac{1}{f(t)}$$
$$= 2 + \frac{1}{f(t)}.$$

Hence, we can evaluate $f(t+2k)$, $k \geq 0$, k an integer, if we know $f(t)$.

Observe that condition (ii) can be rewritten as $f(1+t) = 1 + f(t)$, $t \in \mathbb{Q}^+$, $0 < t \leq 1$.

We can now evaluate $f(2k+1+q)$ as follows: Since

$$\frac{1}{f(2+q)} = 1 + \frac{1}{f(q)}, \quad \text{we have} \quad \frac{1}{f(2+1+q)} = 1 + \frac{1}{f(1+q)}.$$

If $0 < q \leq 1$, then $\dfrac{1}{f(3+q)} = 1 + \dfrac{1}{1+f(q)}$. Hence $f(3+q)$, $0 < q \leq 1$ can be evaluated if $f(q)$ is known. Once $f(3+q)$ is known, we obtain

$$\frac{1}{f(5+q)} = \frac{1}{f(2+3+q)} = 1 + \frac{1}{f(3+q)},$$

and

$$\frac{1}{f(2k+1+q)} + 1 + \frac{1}{f(2k-1+q)}, \quad 1 \geq q > 0.$$

Therefore, we can now evaluate

$$f(2k+q),\ f(2k+1+q) \qquad 0 < q \leq 1, \tag{2}$$

for all $k \geq 0$, k an integer, if we know $f(q)$.

Furthermore, we can evaluate $f(n)$, $n \geq 1$.

First $f(1) = 1$ since putting $q = 1$ in (iii) gives $(f(1))^2 = 1$. Now $f(2) = 1 + f(1) = 2$ from (ii). We follow recursively, $f(3)$:

$$\frac{1}{f(3)} = 1 + \frac{1}{f(1)} = 2$$

and

$$\frac{1}{f(2k+1)} = 1 + \frac{1}{f(2k-1)}.$$

Similarly

$$\frac{1}{f(2k+2)} = 1 + \frac{1}{f(2k)} \text{ and } f(2) = 2.$$

Thus any such function is uniquely defined on the integers.

Finally, we can evaluate the function at any q from the values on the positive integers. Let $q = \frac{a}{b}$, where $(a, b) = 1$.

Write $a = bq_1 + r_1$ where q_1 is a non-negative integer, and $0 \le r_1 < b$ is an integer. If $r_1 = 0$, $f(q) = f(q_1)$ which is determined.

If $a \le r_1 < b$, we apply $f(\frac{a}{b}) = f(q_1 + \frac{r_1}{b})$. This is determined if the value of $f(\frac{r_1}{b})$ is known using (2). Now $0 < \frac{r_1}{b} < 1$. We now compute $f(\frac{b}{r_1})$. $b = r_1 q_2 + r_2$, $r_2 < r_1$. Continuing, since $0 \le r_{k+1} < r_k$, $r_j = 0$ for some j, and we will have an expression for which f is evaluated at an integer. Hence f exists and is uniquely determined.

12. Let $\mathbb{N}$ denote the set of positive integers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijective function and assume that there exists a finite limit

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = L.$$

What are the possible values of L?

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

We claim L must be 1.

Consider $\max\{\varphi(1), \dots, \varphi(n)\} = j_n$. We note that $j_n \ge n$, since φ is one-to-one. Let $i_n \in \{1, 2, \dots, n\}$ be such that $\varphi(i_n) = j_n$. Then

$$\frac{\varphi(i_n)}{i_n} \ge 1.$$

Since

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = L, \quad \lim_{n\to\infty} \frac{\varphi(i_n)}{i_n} = L \ge 1. \tag{1}$$

Now consider $S_n = \{n \in N : \varphi(n) \le n\}$. S_n must be infinite. First $S_n \ne \emptyset$ for if $S_n = \emptyset$ then $\varphi(k) > k$ for all k and there is no k_0 with $\varphi(k_0) = 1$.

Suppose S_n is finite, with k the largest value in the set. Then $\varphi(n) > n$ for $n \ge k+1$. Consider $\{1, 2, \dots, k+1\}$. Since $\varphi(n) > k+1$ for $n \ge k+1$, the only integers which can be pre-images of $\{1, 2, \dots, k+1\}$ are $\{1, 2, \dots, k\}$. This is not possible, since φ is one-to-one and onto.

Therefore $S_n = \{n \in N : \varphi(n) \le n\}$ is infinite. Choose a sequence, $n_k \in S_n$ with $n_k \to \infty$. We now have $\lim\limits_{k\to\infty} \dfrac{\varphi(n_k)}{n_k} = L$. However

$$\frac{\varphi(n_k)}{n_k} \le 1.$$

Thus

$$L \le 1. \tag{2}$$

From (1) and (2), $L = 1$.

13. Prove that for any positive $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \ \geq \ \frac{4n^2}{\sum_{i=1}^{n}(x_i + y_i)^2}$$

holds.

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Christopher J. Bradley, Clifton College, Bristol, UK; by Michael Selby, University of Windsor, Windsor, Ontario; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We feature Bradley's solution.

Now,

$$\frac{1}{xy} \ \geq \ \frac{4}{(x+y)^2}$$

since $(x+y)^2 \geq 4xy$, as $(x-y)^2 \geq 0$. So

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \ \geq \ \sum_{i=1}^{n} \frac{4}{(x_i + y_i)^2} \qquad (*)$$

Lemma. $(a_1 + a_2 + \cdots + a_n)(a_2 a_3 \cdots a_n + a_1 a_3 a_4 \cdots a_n + a_1 a_2 a_4 \cdots a_n + \cdots + a_1 a_2 \cdots a_{n-1}) \ \geq \ n^2 a_1 a_2 \cdots a_n$.

This follows from separate applications of the AM–GM inequality to the two terms on the left. It follows that

$$a_1 + a_2 + \cdots + a_n \ \geq \ \frac{n^2}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

Now put

$$a_i = \frac{1}{(x_i + y_i)^2}, \qquad i = 1, \ldots, n$$

and then

$$\frac{1}{(x_1+y_1)^2} + \frac{1}{(x_2+y_2)^2} + \cdots + \frac{1}{(x_n+y_n)^2} \ \geq \ \frac{n^2}{\sum_{i=1}^{n}(x_i + y_i)^2}.$$

Combining this with $(*)$ shows

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \ \geq \ \frac{4n^2}{\sum_{i=1}^{n}(x_i + y_i)^2}.$$

14. There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from the other one. Prove that there is a town such that all the remaining towns can be reached from it.

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give the solution by Boase.

We prove the result by induction on the number, n, of towns. If $n \leq 2$ the result is immediate.

Label the towns $A_1, A_2, \ldots, A_k$. We shall prove that if the statement holds for all $n < k$, then it is also true for $n = k$, so by induction it will be true for all n.

We can split up the towns excluding A_1 into two sets M and N, M containing those towns which can be reached from A_1 and N those which cannot be reached from A_1.

Thus, every town in N can reach A_1, and there is no route from a town in M to a town in N.

If N is empty, then A_1 is the desired town.

If this is not the case, then, since for any two towns in N, one of them can be reached from the other, and there is no route from outside N into N, the routes in question must pass through towns in N.

By the induction hypothesis, since $|N| < k$, there is a town in N which can reach all other towns in N. It can also reach A_1, and thus all towns in M. Therefore, this is the town which can reach all the other towns in the country, and the result is proved.

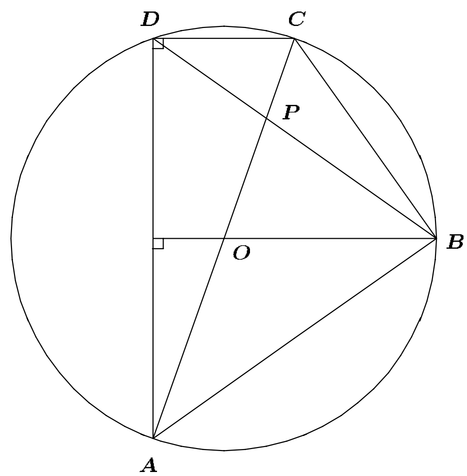
15. Noah has 8 species of animals to fit into 4 cages of the ark. He plans to put species in each cage. It turns out that, for each species, there are at most 3 other species with which it cannot share the accommodation. Prove that there is a way to assign the animals to their cages so that each species shares with compatible species.

Solution by Mansur Boase, student, St. Paul's School, London, England.

Let the animals be vertices of a graph. Join two animals by an edge if they are compatible. Now we have a graph with 8 vertices, and each vertex is joined to at least 4 others. So, by Dirac's theorem on Hamiltonian cycles, there must be a Hamiltonian cycle, and if we take consecutive pairs of animals in this cycle, we can put them in the same cage, and we have the required solution.

17. Quadrangle $ABCD$ is inscribed in a circle with radius 1 in such a way that one diagonal, AC, is a diameter of the circle, while the other diagonal, BD, is as long as AB. The diagonals intersect in P. It is known that the length of PC is $\frac{2}{5}$. How long is the side CD?

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give the solution of Arslanagić.



The triangle ABD is isosceles because $AB = BD$. Let O be the centre of the circumcircle. Then $BO \perp AD$. Because $CD \perp AD$ (AC is a diameter), we get $CD \| BO$; that is, $\triangle PCD \sim \triangle POB$, and it follows that

$$\frac{CD}{OB} = \frac{PC}{PO}; \qquad \text{that is}$$

$$CD = \frac{OB \cdot PC}{PO} = \frac{1 \cdot \frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}.$$

18. Show that in a non-obtuse triangle the perimeter of the triangle is always greater than two times the diameter of the circumcircle.

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Prielipp's solution.

In this solution R, r, s will denote the circumradius, inradius and semiperimeter of a triangle. We shall show that in a non-obtuse triangle the perimeter is always greater than or equal to $2(2R) + 2r$.

Lemma. If A is an angle of triangle ABC, then $\cos A$ is a root of the equation

$$4R^2t^3 - 4R(R+r)t^2 + (s^2 + r^2 - 4R^2)t + (2R+r)^2 - s^2 = 0 \qquad (*)$$

Proof. Since $a = 2R\sin A$, and $s - a = r\cot(\frac{A}{2})$,

$$\begin{aligned} s &= a + (s-a) = 2R\sin A + r\cot\left(\frac{A}{2}\right) \\ &= 2R\sqrt{(1-\cos A)(1+\cos A)} + r\sqrt{\frac{1+\cos A}{1-\cos A}}\,. \end{aligned}$$

Thus

$$s^2 = 4R^2(1 - \cos A)(1 + \cos A) + 4Rr(1 + \cos A) + r^2\frac{1 + \cos A}{1 - \cos A}$$

so

$$\begin{aligned}4R^2(1 - \cos A)^2(1 + \cos A) + 4Rr(1 + \cos A)(1 - \cos A)\\ +r^2(1 + \cos A) - s^2(1 - \cos A) = 0.\end{aligned}$$

Hence

$$\begin{aligned}rR^2\cos^3 A - 4R(R + r)\cos^2 A + (s^2 + r^2 - 4R^2)\cos A\\ +(2R + r)^2 - s^2 = 0\end{aligned}$$

making $\cos A$ a root of the equation $(*)$.

Corollary 1. If A, B, and C are the angles of triangle ABC, then $\cos A$, $\cos B$, and $\cos C$ are the roots of the equation $(*)$.

Corollary 2. If A, B, and C are the angles of triangle ABC, then

$$\cos A \cos B \cos C = \frac{s^2 - (2R + r)^2}{4R^2}.$$

Corollary 3. If A is the largest angle of triangle ABC, then

$$\begin{aligned}&s > 2R + r \quad \text{if} \quad A < 90^\circ\\ &s = 2R + r \quad \text{if} \quad A = 90^\circ\\ \text{and} \quad &s < 2R + r \quad \text{if} \quad A > 90^\circ.\end{aligned}$$

Corollary 4. In a non-obtuse triangle the perimeter of the triangle is always greater than or equal to $2(2R) + 2r$.

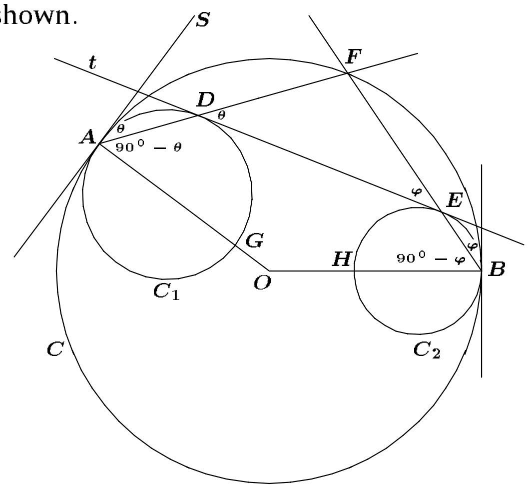
Proof. If the triangle is an acute triangle, then $s > 2R + r$, and $2s > 2(2R) + 2r$. If the triangle is a right triangle, then $s = 2R + r$. Thus $2s = 2(2R) + 2r$.

Corollary 5. In a non-obtuse triangle the perimeter of the triangle is always greater than twice the diameter of the circumcircle.

19. Let C be a circle in the plane. Let C_1 and C_2 be nonintersecting circles touching C internally at points A and B respectively. Let t be a common tangent of C_1 and C_2, touching them at points D and E respectively, such that both C_1 and C_2 are on the same side of t. Let F be the point of intersection of AD and BE. Show that F lies on C.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let SA be the tangent to C_1 and C at A and TB be the tangent to C_2 and C at B, as shown.



Let $\angle SAD = \theta$ and $\angle TBE = \varphi$. Let O be the centre of C. AO meets C_1 again at G and since it is a common radius, AG is a diameter of C_1. BO meets C_2 again at H and BH is likewise a diameter of C_2.

We have $\angle DAG = 90^\circ - \theta$ and $\angle DGA = \theta$. By the alternate segment theorem $\angle GDE = 90^\circ - \theta$ and since $\angle ADG = 90^\circ$ (angle in a semicircle) it follows that $\angle FDE = \theta$. Similarly $\angle FED = \varphi$ and so $\angle DFE = 180^\circ - \theta - \varphi$. Also $\angle EBH = 90^\circ - \varphi$.

Considering the angles of the (re-entrant) quadrilateral $FAOB$ we have *reflex* $\angle AOB = 360^\circ - (90^\circ - \theta) - (90^\circ - \varphi) - (180^\circ - \theta - \varphi) = 2\theta + 2\varphi$. So $\angle AOB = 360^\circ - 2\theta - 2\varphi = 2\angle DFE$. But O is the centre of circle C, and AB is an arc of C, so F lies on C. (Converse of the angle at the centre $=$ twice angle at circumference).

20. Let $a \le b \le c$ be the sides of a right triangle, and let $2p$ be its perimeter. Show that $p(p-c) = (p-a)(p-b) = S$ (the area of the triangle).

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by Mansur Boase, student, St. Paul's School, London, England; by Christopher J. Bradley, Clifton College, Bristol, UK; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; by Michael Selby, University of Windsor, Windsor, Ontario; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Since the triangle is a right triangle we have $c^2 = a^2 + b^2$, $p = \frac{a+b+c}{2}$, and $S = \frac{ab}{2}$.

Then

$$\begin{aligned} p(p-c) &= \frac{a+b+c}{2}\left(\frac{a+b+c}{2}-c\right) = \frac{(a+b)^2-c^2}{4} \\ &= \frac{a^2+b^2+2ab-c^2}{4} = \frac{ab}{2} = S, \end{aligned}$$

and

$$\begin{aligned} (p-a)(p-b) &= \left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right) \\ &= \frac{c+b-a}{2}\frac{c-b+a}{2} \\ &= \frac{c^2-(b-a)^2}{4} = \frac{c^2-(b^2+a^2)+2ab}{4} = \frac{ab}{2} = S, \end{aligned}$$

as required.

We conclude this number of the *Corner* with solutions to some of the problems of the 8th Iberoamerican Mathematical Olympiad, September 14–15, 1993 (Mexico) which we gave last year [1996: 159–160].

8th IBEROAMERICAN MATHEMATICAL OLYMPIAD September 14–15, 1993 (Mexico)

1. (*Argentina*) Let $x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots$ be all the palindromic natural numbers, and for each i, let $y_i = x_{i+1} - x_i$. How many distinct prime numbers belong to the set $\{y_1, y_2, y_3, \dots\}$?

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario. We give Boase's solution.

The first few palindromic numbers are

$$1, 2, 3, \dots, 9, 11, 22, 33, \dots, 99, 101, 111, \dots.$$

Now $11 - 9 = 2$ and $22 - 11 = 11$.

We shall show that these are the only two prime values which a y_i term can take.

If x_i and x_{i+1} have different numbers of digits, then x_i will be of the form $99\dots9$ and x_{i+1} of the form $10\dots01$, so $y_i = x_{k+1} - x_i = 2$.

We can consider, without any loss of generality only y_i where x_i has more than two digits since y_i can only be prime if $y_i = 2$ or 11 for x_i with one or two digits from the above list of the first x_i.

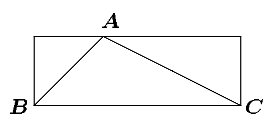
If x_i and x_{i+1} end in the same digit, then 10 divides y_i, so y_i cannot be prime. If x_i and x_{i+1} end in different digits, say r and s, then $s = r + 1$,

x_i is of the form $r999\ldots 9r$, and x_{i+1} is of the form $(r+1)0\ldots 0(r+1)$. Then $y_i = x_{i+1} - x_i = (r+1) - (10-r) = 11$. Thus only two distinct primes belong to the set $\{y_1, y_2, \ldots\}$.

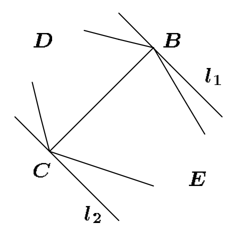
2. (*Mexico*) Show that for any convex polygon of unit area, there exists a parallelogram of area 2 which contains the polygon.

Solution by Mansur Boase, student, St. Paul's School, London, England.

We shall show more generally that there exists a rectangle of area 2 containing the polygon. The result is obviously true for a triangle. To prove this, construct a rectangle on the longest side of the triangle and circumscribing the triangle. If the area of the triangle is 1, then the rectangle will have area 2.

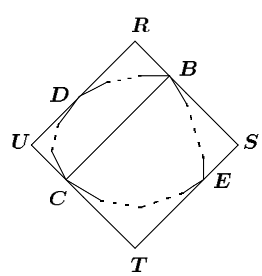


If the polygon has more than three vertices, then choose the two vertices of the polygon which are furthest apart. Call them B and C. Draw perpendiculars to the line BC at B and at C to give lines l_1 and l_2, respectively.



All the vertices of the polygon must lie between these two lines. (Otherwise there would be two vertices further apart than $|BC|$.)

Now consider the smallest rectangle which circumscribes the polygon and with one pair of opposite sides lying on l_1 and l_2. Suppose this polygon touches the polygon again at D and E.



Let the vertices of the rectangle be R, S, T and U with D on UR and E on ST. Then it is easy to see that

$$[RUCB] = 2[BCD]$$

and

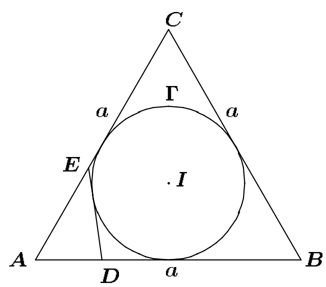
$$[BSTC] = 2[BEC]$$

since $BC \| RU$ and $BC \| ST$. $[RSUT] = 2[CDBE] \leq 2$(area of polygon)$= 2$ since the polygon is convex. We can find an even larger rectangle of area 2 containing the polygon.

4. (*Spain*) Let ABC be an equilateral triangle, and Γ its incircle. If D and E are points of the sides AB and AC, respectively, such that DE is tangent to Γ, show that

$$\frac{AD}{DB} + \frac{AE}{EC} = 1.$$

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and by Mansur Boase, student, St. Paul's School, London, England. We give Arslanagić's solution.



Let $AB = AC = BC = a$ and $BD = p$; that is, $AD = a - p$, and $CE = q$; that is, $AE = a - q$. The circle Γ is inscribed in the quadrilateral and we get

$$ED + BC = BD + CE$$

or

$$ED + a = p + q$$

or

$$ED = p + q - a. \qquad (*)$$

By the law of cosines for the triangle ADE, it follows that

$$ED^2 = AE^2 + AD^2 - 2AE \cdot AD \cos 60^\circ,$$

so, from $(*)$

$$(p + q - a)^2 = (a - q)^2 + (a - p)^2 - (a - q)(a - p)$$

and from this we obtain

$$a = \frac{3pq}{p+q}.$$

Now, we have

$$AD = a - p = \frac{p(2q - p)}{p + q}$$

and

$$AE = a - q = \frac{q(2p-q)}{p+q};$$

that is,

$$\frac{AD}{DB} + \frac{AE}{EC} = \frac{p}{p}\frac{(2q-p)}{p+q} + \frac{q}{q}\frac{(2p-q)}{p+q} = \frac{p+q}{p+q} = 1,$$

as required.

6. (*Argentina*) Two non-negative integer numbers, a and b, are "cuates" (friends in Mexican) if the decimal expression of $a+b$ is formed only by 0's and 1's. Let A and B be two infinite sets of non-negative integers, such that B is the set of all the numbers which are "cuates" of all the elements of A, and A is the set of all the numbers which are "cuates" of all the elements of B. Show that in one of the sets A or B there are infinitely many pairs of numbers x, y such that $x - y = 1$.

Solution by Mansur Boase, student, St. Paul's School, London, England.

Suppose an integer of A ends with the digit 'r'. Then all integers in B must have a last digit the same as for $10 - r$ or $11 - r$ in order that they are all "cuates" of A. If B contains elements with last digits the same as for $10 - r$ and $11 - r$, then every element of A must end in the last digit r to be "cuates" with integers of B of both last digits. Thus either set A or set B has all integers ending in the same digit. Without loss, assume that all elements of A end in 'r'.

Now consider an element of B which is a "cuate" of all the integers of A. Let us say it is of type (i) if it ends with the last digit $10 - r$ and of type (ii) if it ends with the last digit of $11 - r$. If we change the last digit we obtain another number which is a "cuate" of all the elements of A, and hence in B. The difference between these pairs is 1. It follows therefore that there are equal numbers of each type in B, and as B is infinite, there are infinitely many pairs x, y in B such that $x - y = 1$.

That completes the *Corner* for this month. Send me your Olympiad contest materials and your nice solutions to problems in the *Corner*.

BOOK REVIEWS

Edited by ANDY LIU

Learn from the Masters! edited by Frank Swetz, John Fauvel, Otto Bekken, Bengt Johansson, Victor Katz, published by the MAA, Classroom Resource Materials Series, 1995, ISBN# 0-88385-703-0, softcover, 312+ pages, $23.00.

Reviewed by **Maria de Losada**, *Bogotá, Colombia.*

A rich and varied collection of thoughts directed primarily toward the use of the history of mathematics for effective and enriched teaching (and learning), these are papers given at the "Kristiansand (Norway) Conference" of 1988. The areas and topics cover a broad range corresponding to different tastes and personal interests, divided in sections on school and higher mathematics. Frank Swetz's **Using Problems from the History of Mathematics in Classroom Instruction** is a superb example of history transcending anecdotal information and grasping the relationship between problem solving and the gradual construction of meaning (especially clear in the section *Illustrating the growth of mathematical proficiency*), an essential component of each individual student's coming to grips with mathematical concepts. The choice of problems and the orientation involving the specific ways in which they can be used to enrich instruction is excellent. Other papers that focus on mathematical thinking explore algorithms and analogies, modelling and heuristic reasoning, as well as Man-Keung Siu's **Mathematical Thinking and History of Mathematics.**

In the latter section, amidst the very fine selection offered, the article by Israel Kleiner **The Teaching of Abstract Algebra: An Historical Perspective** stands out. Kleiner describes his approach as genetic, but notes that he tries "to show how attempts to solve the problems give rise to the abstract theory. This is, of course, the historical sequence of events." He further addresses the question of how "history provides the opportunity to raise a number of general issues about the nature of mathematics".

It is most unfortunate that the random sprinkling of photographs of notable mathematicians throughout the book should place that of Emmy Noether opposite the title **In Hilbert's Shadow.**

Definitely of interest to those who lean toward using history as a resource for enriching their teaching and the mathematical thinking of their students, as well as to those whose interest in historical subjects is just beginning to be awakened.

Dissecting Rectangular Strips Into Dominoes

Frank Chen
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Kenneth Nearey and Anton Tchernyi
students, Grandview Heights Junior High School, Edmonton

The First Problem

A domino is defined to be a 1×2 or 2×1 rectangle. The first one is said to be horizontal and the second vertical. In our Mathematics Club, we learned to count the number g_n of different ways of dissecting a $3 \times 2n$ strip into dominoes. The sequence $\{g_n\}$ satisfies the recurrence relation

$$g_n = 4g_{n-1} - g_{n-2}, \tag{1}$$

for all $n \geq 2$, with initial conditions $g_0 = 1$ and, as shown in Figure 1, $g_1 = 3$.

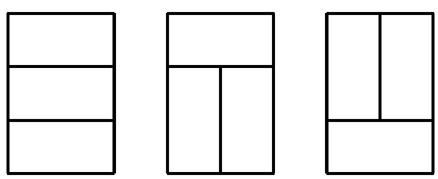

Figure 1

The generating function $G(x)$ for the sequence is defined to be the formal power series

$$g_0 + g_1x + g_2x^2 + \cdots .$$

It is easy to deduce from (1) that

$$G(x) = \frac{1-x}{1-4x+x^2}. \tag{2}$$

In examining the dissections of the $3 \times 2n$ strip, we observed that they fall into two kinds. A dissection of the first kind can be divided by a vertical line into two substrips without splitting any dominoes. Such a line is called a *fault line*. A dissection of the second kind, called a *fault-free dissection*, has no fault lines. Figure 2 shows an example of each, using 3×4 strips.

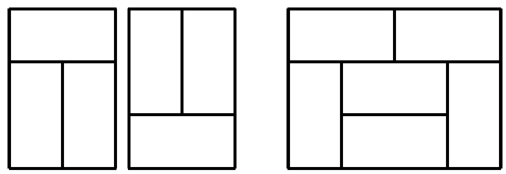

Figure 2

Let f_n be the number of fault-free dissections of the $3 \times 2n$ strip. From Figure 1, $f_1 = 3$. For all $n \geq 2$, a fault-free dissection cannot start with three horizontal dominoes. It must start off as shown in Figure 3, and continue by adding horizontal dominoes except for a final vertical one.

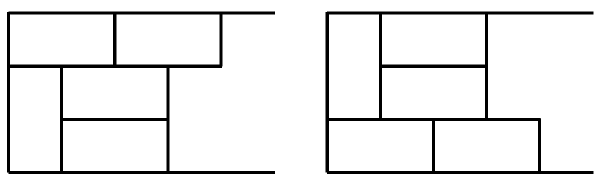

Figure 3

It follows that $f_n = 2$ for all $n \geq 2$, and the sequence satisfies a trivial recurrence relation

$$f_n = f_{n-1}$$

for all $n \geq 3$, with initial conditions $f_0 = 1, f_1 = 3$ and $f_2 = 2$. Let $F(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ be the generating function for the sequence. Then $F(x) = -1 + x + 2(1 + x + x^2 + \cdots) = -1 + x + \frac{2}{1-x}$. This simplifies to

$$F(x) = \frac{1 + 2x - x^2}{1 - x}. \tag{3}$$

Having solved the simpler problem of counting fault-free dissections of the $3 \times 2n$ strip, we make use of our result to find an alternative solution to the general problem of finding all dissections of this strip. They can be classified according to where the first fault line is. This is taken to be the right end of the strip if the dissection is fault-free. Then the strip is divided into a $3 \times 2k$ substrip on the left and a $3 \times 2(n-k)$ substrip on the right, where $1 \leq k \leq n$.

Since the first substrip is dissected without any fault lines, it can be done in f_k ways. The second substrip can be dissected in g_{n-k} ways as we do not care whether there are any more fault lines. Hence

$$g_n = f_1 g_{n-1} + f_2 g_{n-2} + \cdots + f_n g_0. \tag{4}$$

From the values of f_n, $g_n = 3g_{n-1} + 2g_{n-2} + 2g_{n-3} + \cdots + 2g_0$. If we subtract from this expression $g_{n-1} = 3g_{n-2} + 2g_{n-3} + \cdots + 2g_0$, we have $g_n - g_{n-1} = 3g_{n-1} - g_{n-2}$, which is equivalent to (1).

We now derive (2) in another way. It follows from (4) that for all $n \geq 1$,

$$2g_n = f_0g_n + f_1g_{n-1} + f_2g_{n-2} + \cdots + f_ng_0. \tag{5}$$

Multiplying $F(x)$ and $G(x)$ yields

$$\begin{aligned} F(x)G(x) &= (f_0 + f_1x + f_2x^2 + \cdots)(g_0 + g_1x + g_2x^2 + \cdots) \\ &= f_0g_0 + (f_0g_1 + f_1g_0)x + (f_0g_2 + f_1g_1 + f_2g_0)x^2 + \cdots . \end{aligned}$$

In view of (5), this becomes $F(x)G(x) = g_0+2g_1x+2g_2x^2+\cdots = 2G(x)-1$ so that

$$G(x) = \frac{1}{2 - F(x)}. \tag{6}$$

Substituting (3) into (6) yields (2).

The Second Problem

Let g_n be the number of ways of dissecting a $4 \times n$ strip into dominoes. Then $g_0 = 1, g_1 = 1$ and, as shown in Figure 4, $g_2 = 5$. It is not hard to verify that $g_3 = 11$. We wish to determine the infinite sequence $\{g_n\}$ via recurrence relations and generating functions.

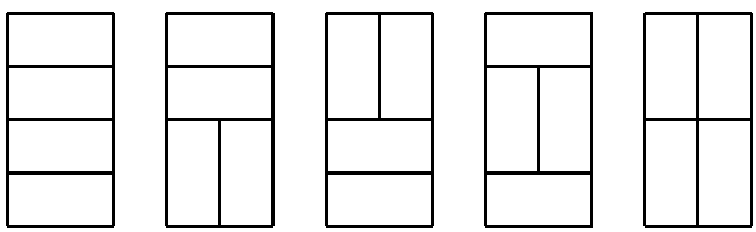

Figure 4

Let f_n be the number of fault-free dissections of the $4 \times n$ strip. We have $f_0 = 0, f_1 = 1$ and from Figure 4, $f_2 = 4$. For odd $n \geq 3$, the only fault-free dissections are the extensions of the second and third ones in Figure 4, with horizontal dominoes except for a final vertical one. Hence $f_n = 2$. For even $n \geq 4$, $f_n = 3$ since we can also include similar extensions of the fourth dissection in Figure 4.

As in the solution of the First Problem, we have

$$g_n = f_1g_{n-1} + f_2g_{n-2} + \cdots + f_ng_0.$$

This leads to

$$g_n = g_{n-1} + 5g_{n-2} + g_{n-3} - g_{n-4} \tag{7}$$

for all $n \geq 4$, with initial conditions $g_0 = 1, g_1 = 1, g_2 = 5$ and $g_3 = 11$. Also,

$$\begin{aligned} F(x) &= 1 + x + 4x^2 + 2x^3 + 3x^4 + 2x^5 + 3x^6 + \cdots \\ &= -2 - x + 2x^2 \\ &\quad +3(1 + x^2 + x^4 + x^6 + \cdots) + 2x(1 + x^2 + x^4 + \cdots) \\ &= -2 - x + x^2 + \frac{3 + 2x}{1 - x^2} \\ &= \frac{1 + x + 3x^2 + x^3 - x^4}{1 - x^2}. \end{aligned} \tag{8}$$

Substituting (8) into (6), which is still valid here, we have

$$G(x) = \frac{1 - x^2}{1 - x - 5x^2 - x^3 + x^4}. \tag{9}$$

We now give an alternative solution to the Second Problem, along the line of the solution to the First Problem we learned at the Mathematics Club. We classify the dissections of the $4 \times n$ strip into five types according to how they start. These correspond to those in Figure 4 if we ignore the vertical dominoes in the second column. Call these Types A, B, C, D and E, and let their numbers be a_n, b_n, c_n, d_n and e_n, respectively.

By symmetry, we have $b_n = c_n$ so that

$$g_n = a_n + 2b_n + d_n + e_n \tag{10}$$

for all $n \geq 1$. In a Type A dissection, we are left with a $4 \times (n - 2)$ strip, which can be dissected in g_{n-2} ways. Hence

$$a_n = g_{n-2} \tag{11}$$

for all $n \geq 3$, with $a_1 = 0$ and $a_2 = 1$.

In a Type B dissection, if we complete the second column with a vertical domino, the remaining $4 \times (n - 2)$ strip can be dissected in g_{n-2} ways. The only alternative is to fill the second column with two horizontal dominoes. The remaining part can be dissected in b_{n-1} ways, so that

$$b_n = g_{n-2} + b_{n-1} \tag{12}$$

for all $n \geq 3$, with $b_1 = 0$ and $b_2 = 1$.

In a Type D dissection, the situation is similar except that if we fill the second column with two horizontal dominoes, we must then also fill the third column with two more horizontal dominoes. The remaining part can be dissected in d_{n-2} ways, so that

$$d_n = g_{n-2} + d_{n-2} \tag{13}$$

for all $n \geq 3$, with $d_1 = 0$ and $d_2 = 1$.

Finally, in a Type E dissection, after filling the first column with two vertical dominoes, we are left with a $4 \times (n-1)$ strip which can be dissected in g_{n-1} ways. Hence

$$e_n = g_{n-1} \tag{14}$$

for all $n \geq 2$, with $e_1 = 1$. Now (7) follows from (10), (11), (12), (13) and (14) since

$$\begin{aligned}
g_n &= a_n + 2b_n + d_n + e_n \\
&= g_{n-2} + (2g_{n-2} + 2b_{n-1}) + g_{n-2} + d_{n-2} + g_{n-1} \\
&= g_{n-1} + 4g_{n-2} + 2(g_{n-3} + b_{n-2}) + d_{n-2} \\
&= g_{n-1} + 4g_{n-2} + 2g_{n-3} + 2b_{n-2} + d_{n-2} \\
&= g_{n-1} + 4g_{n-2} + 2g_{n-3} + g_{n-2} - a_{n-2} - e_{n-2} \\
&= g_{n-1} + 5g_{n-2} + g_{n-3} - g_{n-4}.
\end{aligned}$$

Using (7), $(1 - x - 5x^2 - x^3 + x^4)G(x)$ simplifies to $1 - x^2$. Hence (9) also follows.

Supplementary Problem

Let f_n denote the number of fault-free dissections of the $4 \times n$ strip. Find a recurrence relation for the sequence $\{f_n\}$ with initial conditions.

Acknowledgement

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THE SKOLIAD CORNER

No. 26

R.E. Woodrow

This number we give the 30 problems of the Kangourou Des Mathématiques, Épreuve Européenne, which was given Friday March 22, 1996 to about 500,000 students in 16 European countries, and in 8 African countries, without counting French schools around the world. My thanks go to Ravi Vakil, Canadian Team leader to the 37^{th} IMO in Mumbai, India, who collected a great deal of contest material and forwarded it to me. My copy is in French, and we give it in that language. The contestants are given 75 minutes, and *no calculators are allowed*.

KANGOUROU DES MATHÉMATIQUES

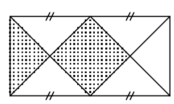
March 22, 1996

Time: 75 minutes

1. Les représentants de 12 pays ont choisi pour vous ces 30 questions. Chaque question a été discutée 10 minutes. Quelle a été la durée totale de la discussion? (3 points)

A. 360 min **B.** 300 min **C.** 120 min **D.** 52 min **E.** 40 min.

2. Dans la figure ci-contre, l'aire de la région laissée en blanc est 6 cm^2. Quelle est l'aire de la région grise? (3 points)



A. 3 cm^2 **B.** 4 cm^2 **C.** 6 cm^2 **D.** 9 cm^2 **E.** 12 cm^2.

3. Quel est le plus grand nombre? (3 points)

A. $1 \times 9 \times 9 \times 6$ **B.** $19 \times 9 \times 6$ **C.** $1 \times 99 \times 6$ **D.** $1 \times 9 \times 96$ **E.** 19×96.

4. En utilisant une et une seule fois chacun des chiffres $1, 2, 3$ et 4, je peux écrire différent nombres. Je peux écrire par exemple 3241. Quelle est le différence entre le plus grand et le plus petit nombre ainsi fabriqués? (3 points)

A. 2203 **B.** 2889 **C.** 3003 **D.** 3087 **E.** 3333.

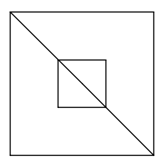
5. Un cercle et un rectangle s'aimaient d'amour tendre, "Mais malheureusement, dit le cercle, que nous croissions ou rétrécissions, nous ne pourrons jamais avoir plus de n points communs!" À votre avis, combien vaut n? (3 points)

A. 2 **B.** 4 **C.** 5 **D.** 6 **E.** 8.

6. Le nombre $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$ est égale à: (3 points)

A. $\frac{3}{1110}$ **B.** $\frac{3}{1000}$ **C.** $\frac{111}{1000}$ **D.** $\frac{111}{1110}$ **E.** $\frac{3}{111}$.

7. La figure ci-contre représente un grand carré d'aire 1 m^2. Une diagonale est partagée en trois segments de même longueur. Le segment médian est une diagonale du petit carré gris. Quelle est l'aire de ce petit carré? (3 points)



A. $\frac{1}{10}m^2$ **B.** $\frac{1}{9}m^2$ **C.** $\frac{1}{6}m^2$ **D.** $\frac{1}{4}m^2$ **E.** $\frac{1}{3}m^2$.

8. La salle d'un théâtre comporte 26 rengées de 24 places chacune. Toutes les places sont numérotées, en commençant par le premier rang. Dans quelle rangée se trouve le siège numéroté 375? (3 points)

A. 12éme **B.** 13éme **C.** 14éme **D.** 15éme **E.** 16éme.

9. Parmi les phrases ci-dessous, quelles sont les phrases vraies?(3 points)

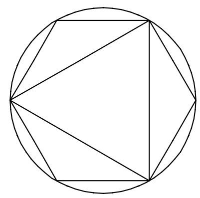
(1)La somme de deux nombres négatifs est toujours négative.
(2)La somme d'une nombre positif et d'un nombre négatif est toujours positive.
(3)La somme d'une nombre négatif et de deux nombres positifs est toujours positive.

A. aucune **B.** la (1) seule **C.** le (1) et la (2)
D. la (2) et la (3) **E.** toutes les trois.

10. a est un nombre de deux chiffres, b est le nombre obtenu en écrivant deux fois, côte à côte, les deux chiffres de a. Quel est le quotient de b par a? (3 points)

A. 10 **B.** 11 **C.** 99 **D.** 100 **E.** 101.

11. Un triangle équilatéral et un hexagone sont inscrits dans un même cercle. Si l'on divise l'aire de l'hexagone par l'aire du triangle, quel est le quotient obtenu? (4 points)



A. 1.5 **B.** 2 **C.** 3 **D.** 4 **E.** π.

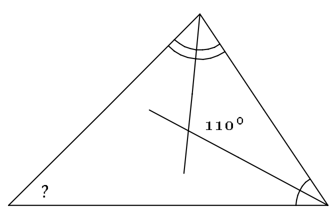
12. Un kangourou a dans sa poche 3 chaussettes blanches, 2 chaussettes noires et 5 chaussettes grises. Sans regarder, il veut en prendre une paire. Quel nombre minimum de chaussettes lui faut-il sortir pour être sûr qu'il en a bien deux de la même couleur? (4 points)

A. 2 **B.** 3 **C.** 4 **D.** 7 **E.** 10.

13. À la fête foraine, une fillette a acheté cinq fléchettes. Chaque fois qu'elle touche la cible, elle a deux fléchettes gratis. Elle a lancé en tout 17 fléchettes. Combien de fois a-t-elle touché la cible. (4 points)

A. 4 **B.** 6 **C.** 7 **D.** 12 **E.** 17.

14. Les bissectrices de deux angles d'un triangle font entre elles un angle de 110°. Combien vaut le troisième angle de ce triangle? (4 points)

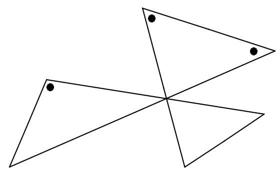


A. 30° **B.** 40° **C.** 45° **D.** 55° **E.** 70°.

15. Une vieille montre retarde de 8 minutes par vingt-quatre heures. De combien de minutes dois-je l'avancer ce soir à 22 heures si j'ai absolument besoin qu'elle me donne l'heure exacte demain matin à 7 heures? (4 points)

A. 1 min 40 s **B.** 2 min 20 s **C.** 3 min **D.** 4 min 30 s **E.** 6 min.

16. La somme, en degrés, des angles marqués sur la figure ci-contre est égale à: (4 points)



A. 120 **B.** 150 **C.** 180 **D.** 270 **E.** 360.

17. Un bidon plein de lait pése 34 kg. Le même bidon quand il est à moitié plein pèse 17, 5 kg. Combien pèse le bidon vide? (4 points)

A. 1 kg **B.** 0, 5 kg **C.** 1, 5 kg **D.** 2 kg

E. il n'y a pas assez de données.

18. Le côtés d'un triangle mesurent 8 cm, 15 cm et 17 cm. Quelle est son aire? (4 points)

A. 40 cm^2 **B.** 60 cm^2 **C.** 68 cm^2 **D.** 80 cm^2

E. on ne peut pas la calculer.

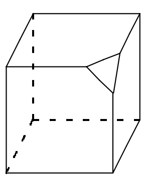
19. Entre 6 heures ce matin et 18 heures ce soir, combien de fois les deux aiguilles de ma montre feront-elles un angle droit? (4 points)

A. 2 **B.** 6 **C.** 12 **D.** 22 **E.** 24.

20. Simone a un gros tas de dalles triangulaires. Toutes ces dalles ont une forme identique: ce sont des triangles équilatéraux de 1 dm de côté. De combien de dalles Simone aura-t-elle besoin pour daller un grand triangle équilatéral de 2 mètres de côte? (4 points)

A. 200 **B.** 300 **C.** 400 **D.** 600 **E.** 800.

21. En découpant un coin d'un cube en bois, on a obtenu le solide ci-contre. Maintenant on découpe de la même façon les sept autres coins du cube. On a alors un solide qui a quatorze faces (les faces triangulaires ne se touchent pas et ne se recoupent pas). Quel est le nombre s de sommets et le nombre a d'arêtes du solide obtenu? (5 points)



A. $s = 24; a = 36$ **B.** $s = 36; a = 24$
C. $s = 10; a = 15$ **D.** $s = 24; a = 32$
E. $s = 36; a = 18$.

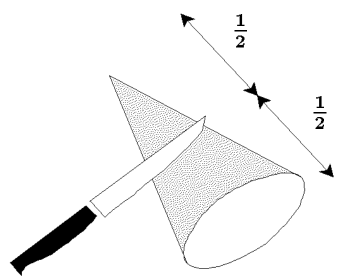
22. On compte le nombre de points d'intersection de quatre droites distinctes. Quel est le nombre qu'l'on est sûr de ne pas trouver? (5 points)

A. 0 **B.** 2 **C.** 3 **D.** 5 **E.** 6.

23. Combien y a-t-il de triangles, dont les côtés ont pour mesures (en centimètres) des nombres entiers, et dont le périmètre est égal à 15 cm? (5 points)

A. 1 **B.** 5 **C.** 7 **D.** 19 **E.** 45.

24. Marine et Claire se partagent un cône glacé en le coupant à mi-hauteur. Marine en a plus que Claire! (5 points)



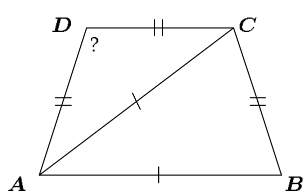
A. 1 fois et demie plus **B.** 2 fois plus **C.** 3 fois plus

D. 7 fois plus **E.** 8 fois plus.

25. Nous sommes sur une ligne de métro circulaire. Vingt-quatre trains s'y déplacent dans la même direction, à intervalles réguliers et roulant tous à la même vitesse. Demain, on doit rajouter des trains afin de diminuer de 20% les intervalles entre deux trains. Combien y aura-t-il de trains supplémentaires demain sur la ligne? (5 points)

A. 2 **B.** 3 **C.** 5 **D.** 6 **E.** 12.

26. Dans la figure ci-contre, (AB) est parallèle à (CD). De plus $AD - DC - CB$ et $AB = AC$. Combien vaut l'angle $\hat{D}$? (5 points)



A. 108° **B.** 120° **C.** 130° **D.** 150° **E.** on ne peut pas la savoir.

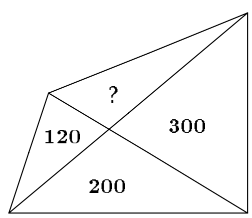
27. Charles a attribué à tous ses livres un code de trois lettres, en utilisant, l'ordre alphabétique: AAA, AAB, AAC, ... , AAZ, ABA, ABB ... Charles a 2203 livres. Quel est le dernier code utilisé par Charles quand il a codé toute sa collection? (5 points)

A. CFS **B.** CHT **C.** DGS **D.** DFT **E.** DGU.

28. Cinq personnes sont assises autour d'une table ronde. Chacune affirme à son tour: "Mes deux voisins, de droite et de gauche, sont des menteurs". On sait que les menteurs mentent toujours et que quelqu'un qui n'est pas un menteur dit toujours la vérité. De plus tout le monde connaît la vérité en ce qui concerne ses deux voisins. Combien y a-t-il de menteurs à cette table? (5 points)

A. 2 **B.** 3 **C.** 4 **D.** 5 **E.** on ne peut pas la savoir.

29. On a coupé quatre drôles de parts suivant les diagonales d'un drôle de gâteau plat à quatre côtés. J'ai mangé une part. Mes amis mécontents ont pesé les trois restantes et ont trouvé 120 g. 200 g. et 300 g. Combien pesait la part que j'ai mangée? (5 points)



A. 120 g **B.** 180 g **C.** 280 g **D.** 330 g **E.** 500 g.

30. Dans la suite de chiffres 122333444455555... , chaque entier est écrit autant de fois que sa valeur. Quel est le 1996$^{\text{ème}}$ chiffre écrit? (5 points)

A. 0 **B.** 3 **C.** 4 **D.** 5 **E.** 6.

Last issue we gave the problems of the Second Round of the Alberta High School Mathematics Competition, of February 11, 1997. The solutions we give were taken from the contest web site

`http://www.math.ualberta.ca/~ahsmc/sample.htm`

where more information about the contest and the solutions may be found. The solutions are selected from contestants' work. My thanks to E. Lewis, University of Alberta, Chair of the contest, for supplying us with materials.

ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION

February 11, 1997

Second Round

1. Find all real numbers x satisfying $|x - 7| > |x + 2| + |x - 2|$.

Remark. Note that $|a|$ is called the absolute value of the real number a. It has the same numerical value as a but is never negative. For example, $|3.5| = 3.5$ while $|-2| = 2$. Of course, $|0| = 0$.

Solution by Laura Harms, Lorne Jenken High School, Barrhead, Alberta.

If x is in $(-\infty, -2)$, the inequality becomes $7 - x > (-2 - x) + (2 - x)$ which simplifies to $x > -7$. Hence all x in $(-7, -2)$ satisfy the inequality.

If x is in $[-2, 2]$, then $|x - 2| + |x + 2| = 4$, while $|x - 7|$ is never less than 5, so all x in $[-2, 2]$ satisfy the inequality.

If x is in $(2, 7)$, then the inequality becomes $7 - x > (x - 2) + (x + 2)$ which simplifies to $x < \frac{7}{3}$. Hence all x in $(2, \frac{7}{3})$ satisfy the inequality.

If x is in $[7, \infty)$, then $x - 7 < x - 2 < (x - 2) + (x + 2)$, and the inequality is not satisfied.

In summary, x satisfies the inequality if and only if $-7 < x < 7/3$.

2. Two lines b and c form a $60°$ angle at the point A, and B_1 is a point on b. From B_1, draw a line perpendicular to the line b meeting the line c at the point C_1. From C_1 draw a line perpendicular to c meeting the line b at B_2. Continue in this way obtaining points C_2, B_3, C_3, and so on. These points are the vertices of right triangles AB_1C_1, AB_2C_2, $AB_3C_3, \ldots$. If area $(AB_1C_1) = 1$, find

$$\text{area }(AB_1C_1) + \text{area }(AB_2C_2) + \text{area }(AB_3C_3) + \cdots + \text{area }(AB_{1997}C_{1997}).$$

Solution by Margaret Tong, James Fowler High School, Calgary, Alberta.

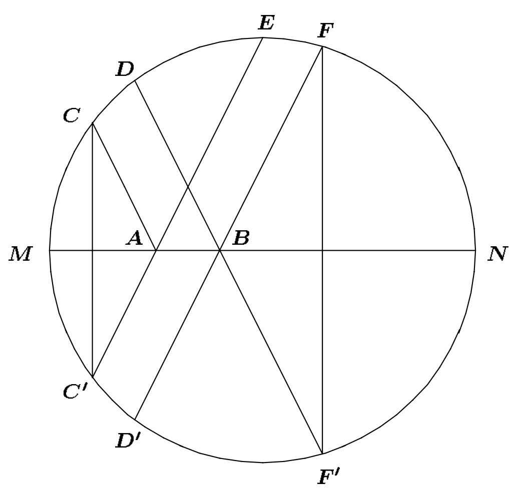
Clearly, triangles AB_nC_n are similar to each other. In a $(30°, 60°, 90°)$ triangle, the hypotenuse is twice as long as the shorter leg. Let $AB_1 = x$. Then $AC_1 = 2x$ and $AB_2 = 4x$. It follows that $\text{area}(AB_nC_n) = 16 \times \text{area}(AB_{n-1}C_{n-1})$, so that the desired total area is given by $T = 1 + 16 + 16^2 + \cdots + 16^{1996}$. Multiplying this by 16, we have $16T =$

$16 + 16^2 + 16^3 + \cdots + 16^{1997}$. Subtraction yields $15T = 16^{1997} - 1$ so that $T = (\frac{1}{15})(16^{1997} - 1)$.

3. A and B are two points on the diameter MN of a semicircle. C, D, E and F are points on the semicircle such that $\angle CAM = \angle EAN = \angle DBM = \angle FBN$. Prove that $CE = DF$.

Solution by Byung-Kyu Chun, Harry Ainlay High School, Edmonton, Alberta.

Complete the circle. Extend EA to cut it at C', and extend DB to cut it at F'. By symmetry, $AC = AC'$ so that $m(AC'C) = m(ACC')$. Similarly, $m(BF'F) = m(BFF')$. Now $m(EC'C) = 180° - m(CAC') = 180° - 2m(CAM) = 180° - 2m(FBN) = 180° - m(FBF') = m(DF'F)$. Since the arcs CE and DF subtend equal angles at the circle, they have equal measure. It follows that the chords CE and DF are equal.



4. (a) Suppose that p is an odd prime number and a and b are positive integers such that p^4 divides $a^2 + b^2$ and p^4 also divides $a(a + b)^2$. Prove that p^4 also divides $a(a + b)$.

(b) Suppose that p is an odd prime number and a and b are positive integers such that p^5 divides $a^2 + b^2$ and p^5 also divides $a(a + b)^2$. Show by an example that p^5 does not necessarily divide $a(a + b)$.

Solution to part (a) by Byung-Kyu Chun, Harry Ainlay High School, Edmonton, Alberta.

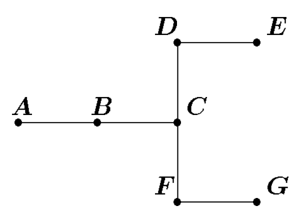
Note that $a(a+b)^2 = a(a^2+b^2)+2a^2b$. Since p^4 divides both $a(a+b)^2$ and a^2+b^2, it must also divide $2a^2b$. Since p is an odd prime, p^4 divides a^2b. Suppose p^2 does not divide a. Then the only powers of p that can possibly divide a^2 are p or p^2. Since p^4 divides a^2b, it follows that p^2 must divide b. Hence p^4 divides b^2. However, this contradicts p^4 dividing $a^2 + b^2$ but not a^2. It follows that we must have p^2 dividing a. Then p^4 divides a^2 so that it also divides b^2. Hence p^2 divides b, and it also divides $a + b$. It follows that

p^4 divides $a(a+b)$.

Solution to part (b) given by Byung-Kyu Chun, Harry Ainlay High School, Edmonton, Alberta; and by Jason Ding, Archbishop MacDonald High School, Edmonton, Alberta.

We look for a, b and p such that p^5 divides a^2+b^2, p^2 divides a and b, but p^3 does not divide $a+b$. Setting $a=p^2x$ and $b=p^2y$, these conditions become p divides x^2+y^2 and p does not divide $x+y$. We can pick $x=2$, $y=1$ and $p=5$. This gives $a=50$ and $b=25$ as an example

5. The picture shows seven houses represented by the dots, connected by six roads represented by the lines. Each road is exactly 1 kilometre long. You live in the house marked B. For each positive integer n, how many ways are there for you to run n kilometres if you start at B and you never run along only part of a road and turn around between houses? You have to use the roads, but you may use any road more than once, and you do not have to finish at B. For example, if $n=4$, then three of the possibilities are: B to C to F to G to F; B to A to B to C to B; and B to C to B to A to B.



Solution by Byung-Kyu Chun, Harry Ainlay High School, Edmonton, Alberta.

The number of ways to travel $n=1$ kilometre is 2, the ways being B to A and B to C. For $n=2$, we have 4 ways, B via A to B, B via C to B, B via C to D and B via C to F. Note that the answer is the same had we started at D or F. We guess that the number of ways for any n is exactly 2^n. We now prove this by induction, in an unusual manner. We consider separately the cases $n=2k$ and $n=2k+1$.

In the even case, the result is true for $k=0$. Suppose that the number of ways for $n=2(k-1)$ is exactly $2^2(k-1)$. At this point, we must be at one of B, D, or F. As pointed out before, there are 4 ways to go another 2 kilometres. Hence, for $n=2k$, the number of ways is $4\cdot 2^{2(k-1)}=2^{2k}$.

For $n=2k+1$, we use the established fact that for $n=2k$, the number of ways is exactly 2^{2k}. Again, after $2k$ kilometres, we must be at one of B, D, or F. In each case, there are two ways to go the extra kilometre, bringing the total to $2\cdot 2^{2(k-1)}=2^{2k+1}$ for $n=2k+1$.

That completes the *Skoliad Corner* for this number. Please send me contest materials for use in the Skoliad as well as any comments or suggestions about what you would like to see featured here.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Richard Hoshino (University of Waterloo), Wai Ling Yee (University of Waterloo), and Adrian Chan (Upper Canada College).

Shreds and Slices

A Note on Convexity

A function f is **convex on** I (I an interval) if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \qquad \forall\, x, y \in I, \quad \alpha \in [0,1],$$

and **J–convex on** I(see [1]) if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \qquad \forall\, x, y \in I.$$

Similarly, f is **concave on** I if

$$f\left(\alpha x + (1-\alpha)y\right) \geq \alpha f(x) + (1-\alpha)f(y) \qquad \forall\, x, y \in I, \ \alpha \in [0,1],$$

and **J–concave** is defined similarly. Several sources claim that J–convexity is sufficient for convexity (see [2]), but we intend to make this more precise. First, f convex clearly implies that f is J–convex. We will prove that:

(i) If f is convex on an open interval I, then f is continuous on I, and

(ii) If f is J–convex and continuous on an interval I, then f is convex on I.

Proof. (i) Let $a \in I$, $\epsilon > 0$. We will show that on some interval around a, $f(x) < f(a) + \epsilon$ and $f(x) > f(a) - \epsilon$.

Choose any $b \in I$. Assume $b > a$. Then the graph of $f(x)$ on $[a, b]$ lies under the chord joining $(a, f(a))$ and $(b, f(b))$ (see Figure 1), which in turn lies under the line $y = f(a) + \epsilon$ on some interval to the right of a, including a. Applying a similar argument when $b < a$, we find an interval around a on which $f(x) < f(a) + \epsilon$.

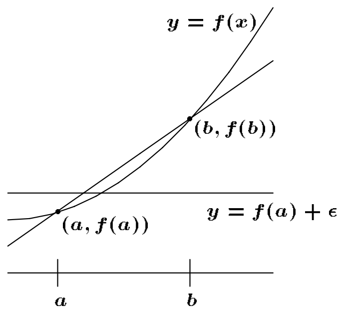


Figure 1.

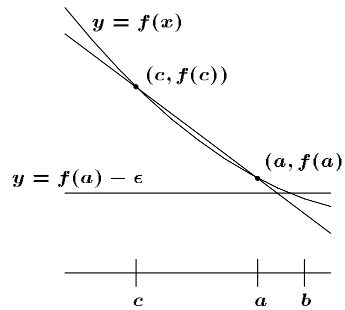


Figure 2.

Now, if $f(x) \geq f(a)$ for all $x \in I$, then we are done, so assume $f(b) < f(a)$ for some $b \in I$. Assume $b > a$. Then $f(x) > f(a)$ for all $x < a$ by convexity, which certainly implies $f(x) > f(a) - \epsilon$. Take some $c < a$, and consider the line joining $(c, f(c))$ and $(a, f(a))$. Then the graph of $f(x)$ to the right of a lies above this line, which in turn lies above the line $y = f(a) - \epsilon$ on some interval to the right of a, including a. The case $b < a$ is similar.

(ii) Since f is J–convex, it is clear that

$$f(\alpha x + (1 - \alpha)y) \ \leq \ \alpha x + (1 - \alpha)f(y)$$

for $\alpha = 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$, and by an induction argument, for any dyadic rational between 0 and 1; that is, a fraction of the form $\frac{m}{2^n}$. Since the dyadic rationals are dense in $[0, 1]$, we can find a sequence which converges to any given real α in $[0, 1]$. By taking a limit along this sequence, the identity is shown to be true for all $\alpha \in [0, 1]$.

Remark. In (i), I must be open, as seen in the example

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 0 \text{ or } x = 1 \end{cases}.$$

Then f is convex, but not continuous.

Hence, (ii) allows for an easy way to check convexity, which is useful for setting up Jensen's inequality, without resorting to the second derivative test if calculus does not appeal to you. Also, the corresponding results hold for f concave.

Example 1. Show that $\sin x$ is concave on $[0, \pi]$.

Proof.

$$\sin x + \sin y \ = \ 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right) \ \leq \ 2 \sin\left(\frac{x + y}{2}\right),$$

so that

$$\sin\left(\frac{x+y}{2}\right) \geq \frac{\sin x + \sin y}{2}.$$

Example 2. Show that $\dfrac{a}{x+b}$ is convex on $(-b, \infty)$, where $a > 0$.

Proof. By two applications of AM-GM,

$$\frac{a}{\frac{x+y}{2}+b} \leq \frac{a}{\sqrt{(x+b)(y+b)}} \leq \frac{1}{2}\left(\frac{a}{x+b} + \frac{a}{y+b}\right).$$

Example 3. Let k be a positive integer. Show that x^k is concave on $[0, \infty)$.

Proof. We show by induction that

$$\left(\frac{x+y}{2}\right)^k \leq \frac{x^k+y^k}{2}$$

for all x, $y \geq 0$. The result is certainly true for $k = 1$. Assume it holds for some k. Assume without loss of generality that $x \geq y$. Then

$$(x^k - y^k)(x-y) = x^{k+1} - x^k y - xy^k + y^{k+1} \geq 0,$$

so that $x^k y + xy^k \leq x^{k+1} + y^{k+1}$. Then,

$$\begin{aligned}
\left(\frac{x+y}{2}\right)^{k+1} &= \left(\frac{x+y}{2}\right)^k \left(\frac{x+y}{2}\right) \\
&\leq \left(\frac{x^k+y^k}{2}\right)\left(\frac{x+y}{2}\right) \\
&= \frac{x^{k+1} + x^k y + xy^k + y^{k+1}}{4} \\
&\leq \frac{2(x^{k+1}+y^{k+1})}{4} = \frac{x^{k+1}+y^{k+1}}{2}.
\end{aligned}$$

References

1. E. Lozansky and C. Rousseau, *Winning Solutions*, Springer-Verlag, New York, 1996.

2. D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

The Equation of the Tangent to the n^{th} Circle

Krishna Srinivasan

Let n circles of radius r be tangent to each other in a row, such that the centre of each lies on the x-axis, and the first circle passes through the origin. Let l be the tangent of the n^{th} circle passing through the origin, as shown in the diagram. What is the equation of this tangent l?

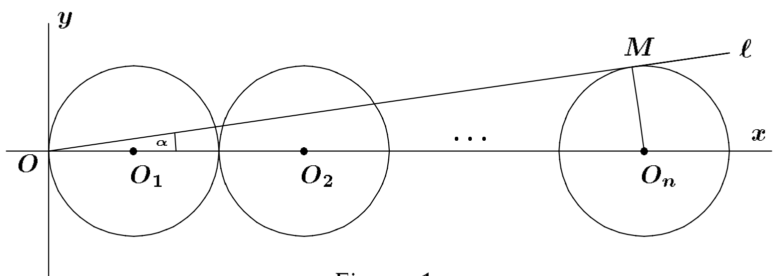


Figure 1.

Let O_n be the centre of the n^{th} circle, M the point of tangency, and α the angle formed between ℓ and the x-axis. Then the slope of ℓ is $\tan\alpha$. Also, $\angle OMO_n = 90°$ (since the radius is perpendicular to the tangent). Therefore, the value of α is $\sin^{-1}(\frac{MO_n}{OO_n})$. Since the radius is r, $MO_n = r$ and $OO_n = 2nr - r$. Consequently, the slope of the tangent is $\tan(\sin^{-1}(\frac{1}{2n-1}))$.

Let us derive $\tan(\sin^{-1} x)$, for $0 \le x < 1$. Let $\theta = \sin^{-1} x$. Then

$$\tan(\sin^{-1} x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{x}{\sqrt{1-x^2}}\,.$$

Substituting $x = \frac{1}{2n-1}$, we find the value of the slope is

$$\frac{1}{\sqrt{(2n-1)^2-1}} = \frac{1}{2\sqrt{n^2-n}}\,.$$

And finally, since the tangent passes through the origin, the equation of ℓ is

$$y = \frac{x}{2\sqrt{n^2-n}}\,. \tag{1}$$

It is interesting to note that the radius r does not appear in (1). This shows that the line $y = x/2\sqrt{2}$ (substituting $n = 2$ into (1)), for example, is always tangent to the second circle, regardless of the radius. [Ed: This makes sense geometrically. Why?]

Coordinatizing the plane, as we have done, can make a problem simpler, as the following problems show:

Problem: Circles P and Q are tangent; each has radius 1. PQ extended meets the circles at A and B, and AC and BTC are tangents to circle P, as shown in the diagram. Compute AC.

(1993 ARML, Team Questions)

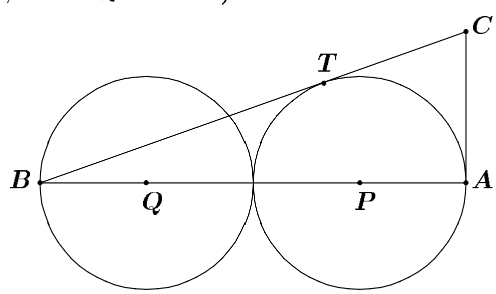


Figure 2.

Solution: The slope of BC is $1/2\sqrt{2}$, by (1). Hence, $AC = AB/2\sqrt{2} = \sqrt{2}$.

Problem: Three equal circles are tangent as shown. The line AD is drawn from point A on the left circle, and tangent to the circle at the right at point D. How long is the chord BC of the circle in the middle?

(1996-1997 Scarborough Mathematics League)

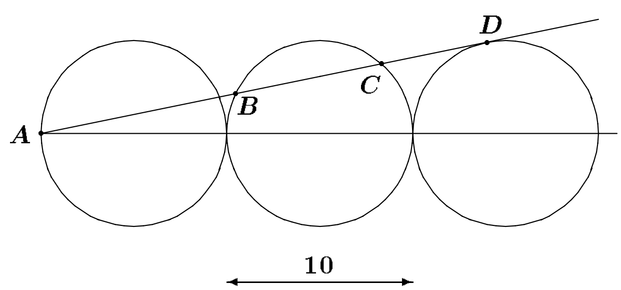


Figure 3.

Solution: Establish a coordinate system, as in Figure 1. As found above, the equation of the tangent is $y = x/2\sqrt{6}$ and the equation of the second circle is $(x-15)^2 + y^2 = 25$. Solving for these two equations gives the points of intersections, which are $B(\frac{72-8\sqrt{6}}{5}, \frac{6\sqrt{6}-4}{5})$ and $C(\frac{72+8\sqrt{6}}{5}, \frac{6\sqrt{6}+4}{5})$. With this information, the length of the chord BC can be calculated to 8 units.

[Ed: Knowing the slope of AD leads to a nice Euclidean solution. By secant theorem, $AB \cdot AC = 10 \cdot 20 = 200$ and by similar triangles, $AB + BC/2 = 6\sqrt{6}$. Now find BC.]

Combinatorial Games

Adrian Chan

Introduction

Definition: One type of *combinatorial game* is a two-person game such that:

(i) Players alternate removing counters from a finite collection according to a set of rules, and

(ii) The last player to remove a counter wins.

The classic combinatorial game of this sort is *Bouton's Nim* or *Nim*. The game consists of some number of piles with some number of counters in each. A player, on his turn, may remove any number of counters from any *one* pile.

Example: Say there are three piles of 1, 3, and 5 counters. The game could proceed as follows:

$$(1,3,5) \xrightarrow{1} (1,3,2) \xrightarrow{2} (1,1,2) \xrightarrow{1} (1,1,0) \xrightarrow{2} (1,0,0) \xrightarrow{1} (0,0,0),$$

and player 1 wins!

A Simple Game: One Pile Nim with Restriction

Let us look at a simple one pile game of Nim, with the restriction that a player may only remove one or two counters at a time. Consider the game where there are 7 counters to play with, denoted $\mathrm{Nim}(7;1,2)$. First, try playing with a friend. You'll probably see a strategy for one of the players. The strategy can be developed by looking at the game in a different way.

Construct a *directed graph*, where the vertices represent the number of counters, and the directed edges represent possible moves. Note that the graph has no cycles since positions cannot repeat. Hence, a game is represented by a path from the initial vertex (7 counters) to the terminal vertex (0 counters).

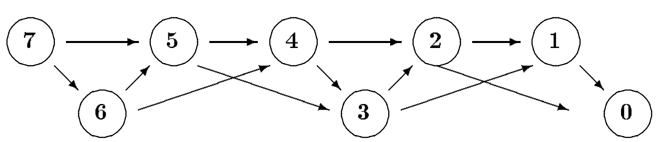


Figure 1.

Safe or Unsafe?

A vertex is *safe* if a player moving to that vertex has a winning strategy, and *unsafe* otherwise. We can label each vertex of our directed graph as safe or unsafe according to the following instructions:

(i) The terminal position is safe,

(ii) If all moves from a vertex lead to an unsafe vertex, then that vertex is safe, and

(iii) If there is a move from a vertex to a safe vertex, then that vertex is unsafe.

We can relabel our directed graph of Nim$(7; 1, 2)$ to get:

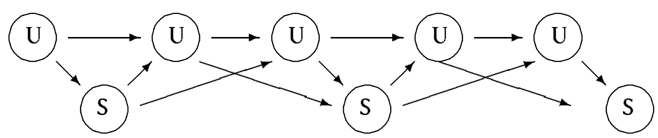


Figure 2.

The Winning Strategy

1. A player starts on an unsafe vertex, and can move to a safe vertex (iii).
2. The other player is forced to move to an unsafe vertex (ii).
3. The original player can again move to a safe vertex (iii). In fact, the original player can always move to a safe vertex, and after some number of moves, he will land on the terminal vertex.

Since the terminal position is a safe vertex, the original player will win!

Exercise: Who should win a similar Nim game with a pile of 11 counters? With 12 counters?

Grundy-Values for Games

For any combinatorial game, we can label each vertex of the directed graph with a non-negative integer instead of the labels safe and unsafe. This number, the *Grundy-Value*, is found by the following process:

(i) The terminal vertex is labelled 0.

(ii) For every other vertex, consider the set of the labels of the vertices it points to. The label of the vertex is then the smallest non-negative integer not appearing in this set.

Example: Our game of Nim$(7; 1, 2)$ with Grundy-Values appears as

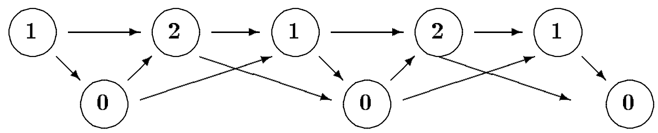


Figure 3.

Is it a coincidence that the safe positions coincide with those vertices assigned a Grundy-Value of 0? Let us investigate. Denote the sets of vertices with safe positions, unsafe positions, Grundy-Values of 0, and positive Grundy-Values as S, U, Z, and N respectively.

Proposition. $S = Z$ and $U = N$.
Proof by Induction. The terminal position is in both S and Z by definition. Assume that there exists some subset of vertices P in which the proposition is true; that is, for any vertex in P, it is in S if and only if it is in Z, and it is in U if and only if it is in N.

Let us look at vertices that are not members of P. There exists a vertex not in P such that all of its emanating edges lead to a vertex in P (to the reader: why?). Let this vertex be A, and consider all vertices that A leads to.

If A leads to a vertex in P that is in S (and so in Z), then A is unsafe so A is in U, and A leads to a vertex with Grundy-Value 0, so A itself must have positive Grundy-Value, and A is in N.

Otherwise, A leads to vertices in P that are only in U (and so in N), so A is safe and A is in S, and every vertex A leads to has a positive Grundy-Value, so A itself has Grundy-Value 0 and A is in Z.

Hence, the statement is true for P and A, a larger set of vertices. We can repeat the argument until we have included all vertices.

The Original Game of Nim

First consider a one-pile game of Nim, with no restrictions on how many counters are taken. Obviously, the first person can take all the counters and win. If we compute the Grundy-Values for such a game, the Grundy-Value of a pile of n counters is n.

Now consider a game of Nim with two piles of counters. Say there are 6 in one and 7 in the other, denoted Nim$(6, 7)$. We can look at the game as a grid instead of a directed graph: You start with a marker at $(6, 7)$ and a move is a translation going left or down, but not both. Whoever lands on $(0, 0)$ wins. (See Figure 4.)

Exercise: Fill the grid with Grundy-Values. Which positions are safe (winning)? Who will win, the first player or the second player?

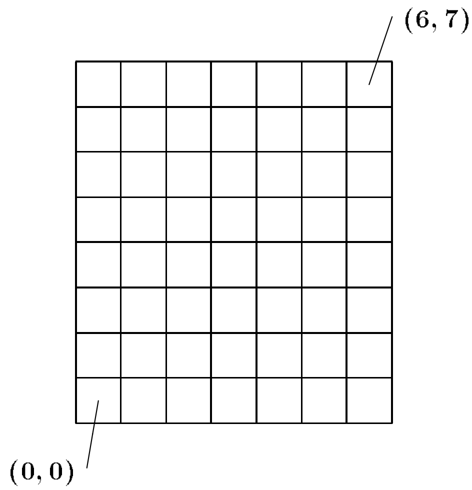


Figure 4.

Finding the Strategy: Nim Sum

Definition: The *Nim Sum* of a combinatorial game is the binary addition of the Grundy-Values of each independent game. However, there is the additional condition that there is no carrying. For example, the nim sum of 1_2 and 1_2 is 0_2, and the nim sum of 1110_2 and 101_2 is 1011_2.

Example: Let us look at the game Nim$(4, 5, 6, 7)$. Notice that the game is the same as the simultaneous games of Nim(4), Nim(5), Nim(6), and Nim(7), where the player can choose which game to play in and make a move in.

	Grundy-Value	Binary Representation
Nim(4)	4	1 0 0
Nim(5)	5	1 0 1
Nim(6)	6	1 1 0
Nim(7)	7	1 1 1
Nim Sum		0 0 0

If the Nim Sum is 0, then the position is *balanced*. Otherwise, it is *unbalanced*.

Problem: Prove that all balanced positions are safe positions. Hint: Look at the definition of safe and unsafe positions.

Exercise: Analyze Nim$(n; 1, 2, 3)$ and Nim$(n; 1, 3, 4)$. What are the Grundy-Values? Does the first or second player win?

With this concept of the Nim Sum, you will be able to find the strategy in any combinatorical game and determine if you will win or lose! The concept of Nim Sum is a very effective tool not only in these games, but also other areas of mathematics.

Mayhem Book Reviews

Donny Cheung

The Art of Problem Solving: A Resource for the Mathematics Teacher, Edited by Alfred S. Posamentier, published by Corwin Press Inc., 2455 Teller Road, Thousand Oaks CA, 91320-2218, 1996, ISBN 0-8039-6362-9, softcover, 465 pages.

The subtitle does not quite do this book justice. This wonderful collection of twenty independent contributions has brought together a diverse spectrum of professionals to cover a wide range of topics. The result is a problem solving resource which is both entertaining and informative.

Having no singular theme other than problem solving in general, this book is ideal as an exposition of the many different aspects of problem solving. Unlike other books on problem solving, this book is not singularly interested in documenting individual problem solving techniques, teaching methods, or historical anecdotes. Rather, a good mixture has been obtained, and this is the strength of the book.

The book begins with a chapter titled "Strategies for Problem Exploration", which individually explores 13 powerful problem solving strategies. But also contained within the book are a chapter promoting the use of problem solving as a teaching paradigm, a chapter investigating the reasons that many students make the errors that they do, a chapter on cooperative learning, a chapter dedicated to the pigeonhole principle, and a chapter exploring mathematically gifted students from a psychological viewpoint.

For the most part, this book is written in a conversational tone, and is very readable. There are problems scattered throughout the book, and there is a handy system of boxed and circled numerals which make it easy to find specific problems from a certain topic of mathematics, or specific problems which highlight a certain problem solving technique.

Perhaps this is not the ideal book for those looking for hundreds of problems to solve, but for anyone who is interested in problem solving in itself, and things related to problem solving, I highly recommend this book.

Students! Get Ready for the Mathematics for SAT I, by Alfred S. Posamentier, published by Corwin Press Inc., 1996, ISBN 0-8039-6415-3, softcover, 206 pages.

Teachers! Prepare Your Students for the Mathematics for SAT I, by Alfred S. Posamentier and Stephen Krulik, published by Corwin Press Inc., 1996, ISBN 0-8039-6416-1, softcover, 116 pages.

Each year, college-bound students around the world write the Scholastic Assessment Test, whose scores are used in college admissions, mostly in the United States. So for these students, preparing for this test is usually considered a very important task both in the classroom and in homes.

To this end, there have been a great number of books which are aimed at helping students prepare for the SAT. The two books reviewed here are designed to help a student prepare for the Mathematics portion of the SAT, and to help a teacher prepare his or her students.

Without an effort on the part of the student, no book can help prepare anyone for the SAT. However, a well-designed book can certainly be helpful.

This set of books is organized and well laid out. The book for students includes a very useful summary of basic mathematical facts, and includes both sample problems with in-depth solutions and timed sample tests for practice. Throughout the sample problems are also good tips for answering different types of questions that are asked on the SAT.

The book for teachers details ten important problem solving strategies, with excellent illustrative problems and in-depth analyses of the problems. It also discusses some various more non-routine types of problems, and some more technical points about the test, such as efficient calculator use. And, it includes, as an appendix, the summary of mathematical facts that appears in the student's book.

On the whole, I think that these books are worth looking into for anyone who is going to write the SAT, or has students who are going to.

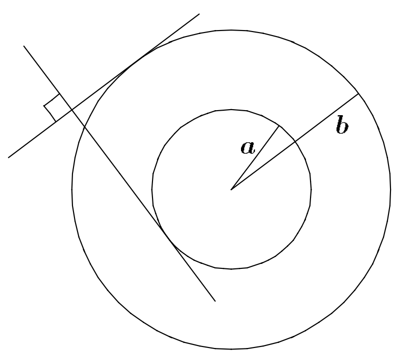
J.I.R. McKnight Problems Contest 1980

1. Sum to k terms the series whose n^{th} term is
$$\frac{n^4+3n^3+2n^2-1}{n^2+n}\,.$$

2. Find all the functions of the form $f(x)=\frac{a+bx}{b+x}$, where a and b are constants, $f(2)=2f(5)$, and $f(0)+3f(-2)=0$.

3. Solve for x and y:
$$\begin{aligned} 2^x\cdot 3^y &= 6 \\ \frac{2^{x+1}}{3^{y-1}} &= 5 \end{aligned}$$

4. Perpendicular tangents are drawn to the circles represented by $x^2+y^2=a^2$ and $x^2+y^2=b^2$. Find the equation of the locus of the point of intersection of these tangents, and name the locus.



5. Prove that
$$\frac{\sin x+\sin y+\sin z-\sin(x+y+z)}{\cos x+\cos y+\cos z+\cos(x+y+z)} = \tan\left(\frac{x+y}{2}\right)\tan\left(\frac{y+z}{2}\right)\tan\left(\frac{z+x}{2}\right)\,.$$

6. A square sheet of tin, a inches on a side, is to be used to make an open-top box by cutting a small square of tin from each corner and bending up the sides. How large a square should be cut from each corner for the box to have as large a volume as possible?

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
Ravi Vakil	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from this issue be submitted by 1 February 1998, for publication in the issue 5 months ahead; that is, issue 4 of 1998. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Solutions

H214. Show that $3(a+b+c) - 8 \leq abc + c$ for all positive integers $a,\ b,\ c \geq 2$.

Solution by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain.

Let $a = x+2$, $b = y+2$, and $c = z+2$, so $x,\ y,\ z \geq 0$. Then

$$3(6+x+y+z) - 8 \leq (2+x)(2+y)(2+z) + 2 + z$$

implies that

$$18 + 3(x+y+z) - 8 \leq 8 + 4(x+y+z) + 2(xy+yz+xz) + xyz + 2 + z\,,$$

which implies that

$$10 + 3(x+y+z) \leq 10 + 4(x+y+z) + 2(xy+yz+xz) + xyz + z\,,$$

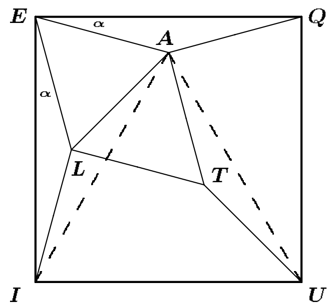
which, in turn, imples that

$$0 \leq \ x + y + 2z + 2(xy+yz+xz) + xyz\,,$$

and the last inequality is trivially true.

H215. Consider a square $EQUI$. Let L, A, and T be points on the interior such that EQA and IEL are isoceles triangles with bases EQ and IE respectively. Show that $UT = IL$ if EAL and ALT are equilateral triangles.

Solution.



We have the following facts: $EA = AQ$ and $EL = LI$, since EAQ and ELI are isosceles triangles. Also, $AL = LI$ and $AT = TL = AL$, since EAL and ALT are equilateral triangles.

To show that $UT = IL$, we will show that $\triangle ALI$ is congruent to $\triangle ATU$.

Let $\alpha = \angle AEQ$. Then $\angle LEI = \alpha$, since $\triangle AEQ \cong \triangle LEI$. Hence, $90° = \angle IEQ = 2\alpha + \angle LEA = 2\alpha + 60°$, so $\alpha = 15°$.

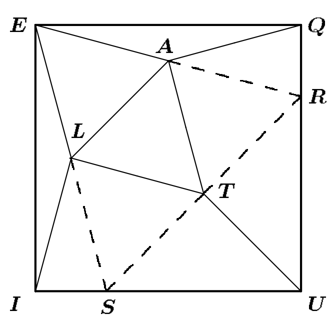
Then $\angle ILA = 360° - \angle ELA - \angle ELI = 360° - 150° - 60° = 150°$. Therefore, $\triangle ALI \cong \triangle ELI$, and $AI = EI = IU$ so $\angle AIU = \angle EIU - \angle EIA = 90° - 30° = 60°$. Therefore, $\triangle AIU$ is equilateral, so $AI = AU$.

Finally, $\triangle ALI \cong \triangle ATU$ since $AL = AT$, $AI = AU$, and $\angle LAI = 60° - \angle IAT = \angle TAU$. Hence, $UT = IL$ as required.

H216. Triangulate a square such that all edges of the triangles on the interior of the square are equal in length and this length is smaller than the length of the side of the square. (Hint: See the previous problem **H215**.)

Solution.

Consider the figure in the solution of problem **H215**. Extend EA through A to meet QU at R and extend EL through L to meet UI at S as shown.



Now ER is the diameter of the circumcircle of $\triangle ERQ$ since $\angle EQR = 90°$. Further, A must be the centre of the circle since $EA = AQ$ and A is on the diameter ER. Thus, $AR = RE$. Likewise, $LS = LE$ in $\triangle ESI$.

Note that $\triangle ERS$ is equilateral and $SR = 2AL$. Therefore, $ST = TR$ and this value is equal to all the other lengths in the interior shown in the solution of problem **H215**.

Rider. Are there other possible triangulations of the square with all the lengths in the interior equal?

Advanced Solutions

A191. Taken over all **ordered** partitions of n, show that

$$\sum_{k_1+k_2+\cdots+k_m=n} k_1k_2\cdots k_m = \binom{m+n-1}{2m-1}.$$

Additional Solution by Waldemar Pompe.

Consider n soccer players. Let $A(n,m)$ be the number of ways they can be divided into m teams (blocks) such that in each team there is a goal-keeper indicated. A partition which has k_1, k_2, ... , k_m players in the blocks gives $k_1k_2\cdots k_m$ ways of choosing goal-keepers. Adding these products over all partitions gives $A(n,m)$.

On the other hand, $A(n,m)$ can be found as follows: Place $m+n-1$ players in a row and choose $2m-1$ of them. Starting with the first of these $2m-1$ players, we make every alternate player a goal-keeper (resulting in m of them), and the remaining $m-1$ are discarded, creating $m-1$ gaps in our row of players. There are then m blocks, with a goal-keeper in each. It can be verified that there is a one to one correspondence between such formations and the formations above. Hence, the equality in question holds.

Challenge Board Solutions

We begin with a couple of corrections. J. Chris Fisher of the University of Regina has kindly pointed out that the statement of problem **C70** (and hence the solution last issue) is wrong. The problem was as follows:

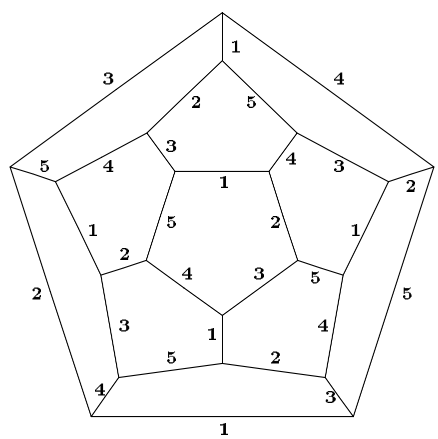
C70. Prove that the group of automorphisms of the dodecahedron is S_5, the symmetric group on five letters, and that the rotation group of the dodecahedron (the subgroup of automorphisms preserving orientation) is A_5.

In fact, the group of automorphisms of the dodecahedron is $A_5 \times C_2$, where C_2 is the two-element group. The published proof showed that the rotation group was A_5, and then incorrectly described the rest of the problem as similar.

In fact, the isomorphism between the automorphism group of the dodecahedron and $A_5 \times C_2$ can be described explicitly: given a dodecahedron with edges numbered as described in the problem, any automorphism induces a permutation of the numbers which can be checked to be even (and hence an element of A_5). Also, the map to C_2 sends an automorphism to the identity if it is a rotation, and to the other element if it is not (that is, if it is a reflection). The reverse map is similar: there is one rotation that permutes the 5 types of edge-labels in any even way, and one reflection.

Chris suggests Coxeter's *Introduction to Geometry* (especially p. 273) as a reference. One of the warning flags which tipped Chris off was that the erroneous proof implied that S_5 is an isometry group of Euclidean three-space. Coxeter lists the possible finite isometry groups, and S_5 does not appear (and indeed S_5 doesn't turn up as an isometry group until Euclidean four-space).

(Also, the figure accompanying the solution contained a typo: the "3" on the central line of symmetry should be a "1".)



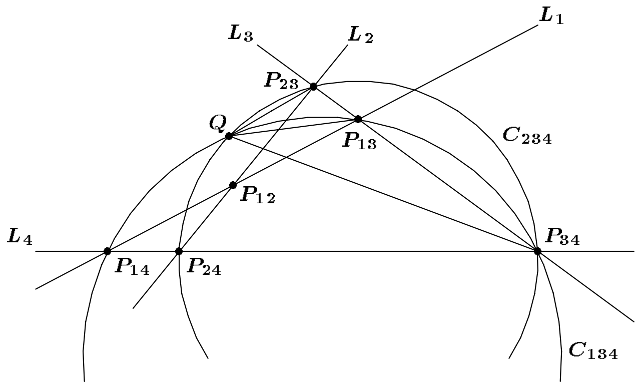
Thank you, Chris, for pointing out the error and explaining how to rectify it! And if any other reader is suspicious of anything written in these pages, please let us know.

C71. Let L_1, L_2, L_3, L_4 be four general lines in the plane. Let p_{ij} be the intersection of lines L_i and L_j. Prove that the circumcircles of the four triangles $p_{12}p_{23}p_{31}$, $p_{23}p_{34}p_{42}$, $p_{34}p_{41}p_{13}$, $p_{41}p_{12}p_{24}$ are concurrent.

Solution.

For convenience, let C_{123}, C_{234}, C_{341}, and C_{412} be the circles. By symmetry, it suffices to show that the first three are concurrent.

This proof is slightly diagram-dependent, but it is short and sweet and geometric. Label the three angles as in the figure. By the exterior angle theorem, $c = a + b$. Let Q be the intersection of C_{234} with C_{341}, other than p_{34}. Then



$$\begin{aligned}
\angle p_{13}Qp_{23} &= \angle p_{23}Qp_{34} - \angle p_{13}Qp_{34} \\
&= c - b \\
&= a \\
&= \angle p_{13}p_{12}p_{23}.
\end{aligned}$$

Thus Q is on the circle C_{123} as well.

Very rough sketch of another solution.

This solution is intended to intrigue and entice the reader, rather than rigorously solve the problem. All statements made here can be made rigorous.

This problem originally came to our attention because of a connection to a sophisticated result:

Lemma. Let C be a degree d curve in the plane. Let A and B be degree e curves that each meet C at de distinct points (say $a_1, \ldots, a_{de}$ and $b_1, \ldots, b_{de}$ respectively). Suppose that $a_i = b_i$ for $i = 1, \ldots, de-1$. Then, if $d \geq 3$, $a_{de} = b_{de}$; that is, the last points are also equal.

Amazingly, this result fails if d is 1 or 2. The proof relies on the fact that there is no rational parametrization of curves of degree greater than 2, while a line such as $x = 0$ has the parametrization $(0, t)$ and a conic such as the circle $x^2 + y^2 = 1$ has the parametrization

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right).$$

(An even more sophisticated result: these deep facts are related to the fact that the diophantine equations $x + y = z$ and $x^2 + y^2 = z^2$ have lots of solutions in integers, while if $n > 2$, $x^n + y^n = z^n$ has only a finite number of solutions even if x, y, and z are allowed to be quite general, for example of the form $r + s\sqrt{2}$ where r and s are integers. This result, stated appropriately, is a consequence of Siegel's Theorem, and relates in an obvious way to Fermat's Last Theorem.)

Even the application of the lemma to this problem has some subtleties. As before, it suffices to show that C_{123}, C_{234}, and C_{341} are concurrent. Let Q be the intersection of C_{234} and C_{341} (other than p_{34}) as before, and let P be the intersection of C_{123} and C_{234} (other than p_{23}). Let C be the cubic curve that is the union of C_{123} and L_4, A the cubic that is the union of C_{234} and L_1, and B the cubic that is the union of C_{341} and L_2. Then C and A intersect at the six points $\{p_{12}, \ldots, p_{34}\}$, and at P. They also intersect at two points at infinity $(\frac{1}{0}, \frac{i}{0})$ and $(\frac{1}{0}, \frac{-i}{0})$ (whatever that means!). The cubics C and B intersect at the same nine points (including the two strange ones!), except the P is replaced by Q. By the lemma, $P = Q$, and we are done.

Of course, a lot of further explanation is required to even make sense of all this, but such explanation is beyond the scope of ***Mayhem***.

Comments.

1. This problem is a first of an infinite sequence of theorems, called the *Clifford Theorems*. Call the intersection of two lines in general position their *Clifford point*. For three lines in general position, there are three Clifford points of pairs of lines; call the circle through the three points the *Clifford circle* of the three lines. Then, according to this problem, given four lines in general position, the four Clifford circles (of the four triples of lines) are concurrent; call this point the *Clifford point* of the four lines. In general, if n is odd, given n general lines, then the n Clifford points of all $(n-1)$-subsets of the n general lines are concyclic, and the resulting circle is called the *Clifford circle* of the n general lines. Similarly, if n is even, given n general lines, then the n Clifford circles of all $(n-1)$-subsets of the n general lines are concurrent, and the resulting point is called the *Clifford point* of the n general lines. The theorems implicit in these definitions are the Clifford theorems.

   This theorem is discussed in Liang-shin Hahn's book *Complex Numbers and Geometry* (published by the Mathematical Association of America). This is a wonderful, beautiful book, and possibly the best place to learn about how complex numbers can be used to make Euclidean plane geometry simple. Hahn proves all the Clifford theorems (in Section 2.3) using a simple lemma, which he proves using complex numbers, but which can also be proved with ordinary Euclidean geometry:

   Lemma. Suppose there are four circles C_1, C_2, C_3, and C_4 in a plane. Let C_1 and C_2 intersect at z_1 and w_1, C_2 and C_3 intersect at z_2 and w_2, C_3 and C_4 intersect at z_3 and w_3, C_4 and C_1 intersect at z_4 and w_4. Then the points z_1, z_2, z_3, z_4 are concyclic if and only if w_1, w_2, w_3, w_4 are concyclic.

   If anyone has a slick proof of the Clifford theorems (or even the next case), we would be interested in seeing it.

2. What needs to be done to make the first (geometric) proof rigorous?

3. Can you make sense of the weird points at infinity $(\frac{1}{0}, \frac{i}{0})$ and $(\frac{1}{0}, \frac{-i}{0})$ described in the sketch of the second solution? In your way of making sense of them, can you see why they lie on every circle?

4. Without understanding the intricacies of the second sketch, can you use the sketch's lemma, with a little hand waving, to prove Hahn's lemma above? Can you use it to prove any other well-known results in Euclidean geometry? (Pappus' theorem seems like a good possibility.) If so, we would love to hear from you!

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 June 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

Where to send your solutions and proposals

There has been an increase in the number of solutions and proposals sent to the Canadian Mathematical Society's Head Office in Ottawa, Ontario. Please note the instructions above and send them directly to the Editor-in-Chief.

Solutions submitted by FAX

There has been an increase in the number of solutions sent in by FAX, either to the Editor-in-Chief's departmental FAX machine in St. John's, Newfoundland, or to the Canadian Mathematical Society's FAX machine in Ottawa, Ontario. While we understand the reasons for solvers wishing to use this method, we have found many problems with it. The major one is that hand-written material is frequently transmitted very badly, and at times is almost impossible to read clearly. We have therefore adopted the policy that we will no longer accept submissions sent by FAX. We will, however, continue to accept submissions sent by email or regular mail. We do encourage email. Thank you for your cooperation.

2287. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Let G denote the point of intersection of the medians, and I denote the point of intersection of the internal angle bisectors of a triangle. Using only an unmarked straightedge, construct H, the point of intersection of the altitudes.

2288. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

In the plane are a circle (without centre) and five points A, B, C, D, E, on it such that $\mathrm{arc}\ AB = \mathrm{arc}\ BC$ and $\mathrm{arc}\ CD = \mathrm{arc}\ DE$. Using only an unmarked straightedge, construct the mid-point of $\mathrm{arc}\ AE$.

2289*. *Proposed by Clark Kimberling, Evansville, IN, USA.*

Use any sequence, $\{c_k\}$, of 0's and 1's to define a *repetition-resistant sequence* $s = \{s_k\}$ inductively as follows:

1. $s_1 = c_1$, $s_2 = 1 - s_1$;
2. for $n \geq 2$, let

$$\begin{aligned} L &= \max\{i \geq 1 \ : \ (s_{m-i+2}, \dots, s_m, s_{m+1}) \\ &\qquad\qquad = (s_{n-i+2}, \dots, s_n, 0) \quad \text{for some } m < n\}, \\ L' &= \max\{i \geq 1 \ : \ (s_{m-i+2}, \dots, s_m, s_{m+1}) \\ &\qquad\qquad = (s_{n-i+2}, \dots, s_n, 1) \quad \text{for some } m < n\}. \end{aligned}$$

(so that L is the maximal length of the tail-sequence of $(s_1, s_2, \dots, s_n, 0)$ that already occurs in $(s_1, s_2, \dots, s_n)$, and similarly for L'), and

$$s_{n+1} = \begin{cases} 0 & \text{if} \quad L < L', \\ 1 & \text{if} \quad L > L', \\ c_n & \text{if} \quad L = L'. \end{cases}$$

$\Big($For example, if $c_i = 0$ for all i, then

$$s = (0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0,$$
$$1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, \dots)\Big)$$

Prove or disprove that s contains every binary word.

2290. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

For $x, y, z \geq 0$, prove that

$$\big((x+y)(y+z)(z+x)\big)^2 \geq xyz(2x+y+z)(2y+z+x)(2z+x+y).$$

2291. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Let a, b, c denote the side lengths of a Pythagorean triangle. Suppose that each side length is the sum of two positive integer squares. Prove that $360|abc$.

2292. *Proposed by K.R.S. Sastry, Dodballapur, India.*

A convex quadrilateral Q has integer values for its angles, measured in degrees, and the size of one angle is equal to the product of the sizes of the other three.

Show that Q is either a parallelogram or an isosceles trapezium.

2293. *Proposed by Claus Mazanti Sorensen, student, Aarhus University, Aarhus, Denmark.*

A sequence, $\{x_n\}$, of positive integers has the properties:

1. for all $n > 1$, we have $x_{n-1} < nx_n$;
2. for arbitrarily large n, we have $x_1x_2 \dots x_{n-1} < nx_n$;
3. there are only finitely many n dividing $x_1x_2 \dots x_{n-1}$.

Prove that $\sum_{k=1}^{\infty} \frac{(-1)^k}{x_k k!}$ is irrational.

2294. *Proposed by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

For the annual Sino-Japanese "Go" tournament, each country sends a team of seven players, C_i's and J_i's, respectively. All players of each country are of different ranks (strengths), so that

$$C_1 < C_2 < \dots < C_7 \qquad \text{and} \qquad J_1 < J_2 < \dots < J_7.$$

Each match is determined by one game only, with no tie. The winner then takes on the next higher ranked player of the opponent country. The tournament continues until all the seven players of one country are eliminated, and the other country is then declared the winner. (For those who are not familiar with the ancient Chinese "Chess" game of "Go", a better and perhaps more descriptive translation would be "the surrounding chess".)

(a) What is the total number of possible sequences of outcomes if each country sends in n players?

(b)* What is the answer to the question in part (a) if there are three countries participating with n players each, and the rule of the tournament is modified as follows:

> The first match is between the weakest players of two countries (determined by lot), and the winner of each match then plays the weakest player of the third country who has not been eliminated (if there are any left). The tournament continues until all the players of two countries are eliminated.

2295. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find three positive integers a, b, c, in arithmetic progression (with positive common difference), such that $a+b$, $b+c$, $c+a$, are all perfect squares.

2296. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

Show that $\sin^2 \dfrac{\pi x}{2} > \dfrac{2x^2}{1+x^2}$ for $0 < x < 1$.

Hence or otherwise, deduce that $\pi < \dfrac{\sin \pi x}{x(1-x)} < 4$ for $0 < x < 1$.

2297. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Given is a circle of radius 1, centred at the origin. Starting from the point $P_0 = (-1,-1)$, draw an infinite polygonal path $P_0P_1P_2P_3\ldots$ going counterclockwise around the circle, where each P_iP_{i+1} is a line segment tangent to the circle at a point Q_i, such that $|P_iQ_i| = 2|Q_iP_{i+1}|$. Does this path intersect the line $y = x$ other than at the point $(-1, 1)$?

2298. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

The "Tickle Me" Feather Company ships its feathers in boxes which cannot contain more than 1 kg of feathers each. The company has on hand a number of assorted feathers, each of which weighs at most one gram, and whose total weight is 1000001/1001 kg.

Show that the company can ship all the feathers using only 1000 boxes.

2299. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, $z > 0$ be real numbers such that $x + y + z = 1$. Show that

$$\prod_{\text{cyclic}} \left(\frac{(1-y)(1-z)}{x}\right)^{(1-y)(1-z)/x} \geq \frac{256}{81}.$$

Determine the cases of equality.

2300. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that ABC is a triangle with circumradius R. The circle passing through A and touching BC at its mid-point has radius R_1. Define R_2 and R_3 similarly. Prove that

$$R_1^2 + R_2^2 + R_3^2 \geq \frac{27}{16}R^2.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2179★. [1996: 318] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For real numbers $a \geq -1$, we consider the sequence

$$F(a) := \left\{ \left(1 + \frac{1}{n}\right)^{\sqrt{n(n+a)}}, \quad n \geq 1 \right\}.$$

Determine the sets D, respectively I, of all a, such that $F(a)$ strictly decreases, respectively increases.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

We show that the sequence $F(a)$ is strictly increasing if and only if

$$-1 \leq a \leq 1$$

and strictly decreasing if and only if

$$a \;>\; \frac{4(\log 1.5)^2 - (\log 2)^2}{(\log 2)^2 - 2(\log 1.5)^2} \;\approx\; 1.168188898,$$

where all logarithms here and below are natural logarithms.

We approach this as follows. For a fixed a, the sequence $F(a)$ is strictly decreasing if and only if for all integers $n \geq 1$ we have

$$\left(1 + \frac{1}{n+1}\right)^{\sqrt{(n+1)(n+1+a)}} \;<\; \left(1 + \frac{1}{n}\right)^{\sqrt{n(n+a)}}.$$

Taking logarithms and squaring, this is equivalent (since all quantities involved are positive) to

$$(n+1)(n+1+a)\left(\log\frac{n+2}{n+1}\right)^2 \;<\; n(n+a)\left(\log\frac{n+1}{n}\right)^2,$$

and so

$$a\left[(n+1)\left(\log\frac{n+2}{n+1}\right)^2 - n\left(\log\frac{n+1}{n}\right)^2\right]$$
$$< \; n^2\left(\log\frac{n+1}{n}\right)^2 - (n+1)^2\left(\log\frac{n+2}{n+1}\right)^2.$$

The quantity in square brackets on the left is negative, because the function

$$f_1(x) \;=\; x\left(\log\frac{x+1}{x}\right)^2 \qquad (1)$$

has derivative

$$\begin{aligned} f_1'(x) &= \left(\log\frac{x+1}{x}\right)\left(\log\frac{x+1}{x}-\frac{2}{x+1}\right) \\ &< \left(\log\frac{x+1}{x}\right)\left(\frac{1}{x}-\frac{2}{x+1}\right) \;<\; 0 \quad \text{for } x\in[1,\infty) \end{aligned}$$

[the first inequality follows because $\log(1+z) < z$ for all $z > 0$] so f_1 decreases for $x \in [1,\infty)$. Thus the sequence $F(a)$ is strictly decreasing if and only if

$$a \;>\; \frac{n^2\left(\log\frac{n+1}{n}\right)^2-(n+1)^2\left(\log\frac{n+2}{n+1}\right)^2}{(n+1)\left(\log\frac{n+2}{n+1}\right)^2-n\left(\log\frac{n+1}{n}\right)^2} \tag{2}$$

for all integers $n \geq 1$.

Next we show that the right-hand side of (2) is a decreasing function of n. It is more convenient to study the right-hand side as a function of a continuous variable x on $[1,\infty)$. Thus we are to show

$$\begin{aligned} &\frac{(x+1)^2\left(\log\frac{x+2}{x+1}\right)^2-(x+2)^2\left(\log\frac{x+3}{x+2}\right)^2}{(x+2)\left(\log\frac{x+3}{x+2}\right)^2-(x+1)\left(\log\frac{x+2}{x+1}\right)^2} \\ &\qquad\qquad < \frac{x^2\left(\log\frac{x+1}{x}\right)^2-(x+1)^2\left(\log\frac{x+2}{x+1}\right)^2}{(x+1)\left(\log\frac{x+2}{x+1}\right)^2-x\left(\log\frac{x+1}{x}\right)^2}\,. \end{aligned}$$

We work on a more general case of an inequality, for appropriate functions $f\colon [1,\infty) \to \mathbb{R}$, of the form

$$\frac{(x+1)f(x+1)-(x+2)f(x+2)}{f(x+2)-f(x+1)} \;<\; \frac{xf(x)-(x+1)f(x+1)}{f(x+1)-f(x)}\,. \tag{3}$$

Here putting f equal to the function f_1 in (1) gives the desired inequality.

We shall need the following easy lemma.

Lemma. If $f : [b,\infty) \to \mathbb{R}$ is a strictly monotonic, twice differentiable function with $f(x) > 0$ for all x in its domain, and further satisfies $f(x)f''(x) < 2\left(f'(x)\right)^2$ there, then inequality (3) holds.

Proof. Indeed, observe that the function $g(x) = 1/f(x)$ satisfies

$$g''(x) \;=\; \frac{2\left(f'(x)\right)^2-f(x)f''(x)}{\left(f(x)\right)^3} \;>\; 0\,,$$

so g is strictly convex. Applying convexity to x, $x+2$ and $\frac{1}{2}(x+(x+2)) = x+1$ we see that

$$\frac{1}{f(x+1)} \;<\; \frac{1}{2}\left(\frac{1}{f(x)}+\frac{1}{f(x+2)}\right);$$

that is,

$$2f(x)f(x+2) \;<\; \big[f(x)+f(x+2)\big]\, f(x+1). \tag{4}$$

If f is strictly monotonic, then the quantity

$$\big(f(x+1)-f(x)\big)\big(f(x+2)-f(x+1)\big)$$

is strictly positive. Multiplying (3) by this quantity and simplifying, we get (4), which is therefore equivalent to (3). □

Returning to the proof, we will show that the function f_1 given by (1), which we have already shown is strictly monotonic, satisfies the rest of the conditions of the lemma. Clearly $f_1(x) > 0$ on $[1,\infty)$, and we need to show $f_1 f_1'' < 2(f_1')^2$. Here

$$f_1'(x) \;=\; \left(\log\frac{x+1}{x}\right)^2 - \frac{2}{x+1}\log\frac{x+1}{x}$$

and

$$f_1''(x) \;=\; \frac{2}{x(x+1)^2} - \frac{2}{x(x+1)^2}\log\frac{x+1}{x}\,.$$

Thus upon cancellation we need to show

$$1-\log\frac{x+1}{x} \;<\; \left[(x+1)\log\frac{x+1}{x} - 2\right]^2 \tag{5}$$

on $[1,\infty)$.

This unfortunately is a little tedious, as the sides of (5) are almost equal for large x, so we will delay its proof until the end. Assuming (5) and returning to inequality (2), if we call its right-hand side a_n, then we have shown that (a_n) is strictly decreasing. Hence the condition $a > a_n$ for all integers $n \geq 1$ is equivalent to

$$a \;>\; a_1 \;=\; \frac{4(\log 1.5)^2-(\log 2)^2}{(\log 2)^2-2(\log 1.5)^2} \;\approx\; 1.168188898.$$

Conclusion: $F(a)$ is strictly decreasing if and only if $a > a_1 = 1.16818889\ldots$.

Similarly, $F(a)$ is strictly increasing if and only if $a < a_n$ for all integers $n \geq 1$. But as (a_n) decreases and is bounded below (by zero), this is equivalent to $a \leq \lim a_n$. We show that $\lim a_n = 1$. This can be done by l'Hôpital's Rule taking $x \to \infty$ on the continuous analogue of a_n; or, arguing asymptotically, we have

$$\begin{aligned}\left(\log \frac{n+1}{n}\right)^2 &= \left(\log\left(1+\frac{1}{n}\right)\right)^2 = \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + O(1/n^4)\right)^2 \\ &= \frac{1}{n^2}\left(1 - \frac{1}{n} + \frac{11}{12n^2} + O(1/n^3)\right)\end{aligned}$$

and hence

$$\begin{aligned} a_n &= \frac{(1 - \frac{1}{n} + \frac{11}{12n^2} + O(1/n^3)) - \left(1 - \frac{1}{n+1} + \frac{11}{12(n+1)^2} + O(1/n^3)\right)}{\left(\frac{1}{n+1} - \frac{1}{(n+1)^2} + O(1/n^3)\right) - (\frac{1}{n} - \frac{1}{n^2} + O(1/n^3))} \\ &= \frac{\frac{-1}{(n+1)^2} + O(1/n^3)}{\frac{-1}{(n+1)^2} + O(1/n^3)} \longrightarrow 1,\end{aligned}$$

as claimed.

To complete the proof we must show (5). Set $w = \frac{1}{x+1}$ for $x \geq 1$ so that $w \leq 1/2$ and $\frac{x}{x+1} = 1 - w$. Now (5) becomes

$$1 + \log(1-w) < \left[\frac{1}{w}\log(1-w) + 2\right]^2,$$

and hence we have to show on $0 < w \leq 1/2$ that

$$[\log(1-w)]^2 + (4w - w^2)\log(1-w) + 3w^2 > 0. \tag{6}$$

Now we need the following estimates of $\log(1-w)$: for $0 < w \leq 1/2$ we have

$$\begin{aligned}&-w - \frac{w^2}{2} - \frac{w^3}{3} - \frac{w^4}{4} - \frac{w^5}{5} - \frac{13w^6}{42} \\ &\qquad < \log(1-w) < -w - \frac{w^2}{2} - \frac{w^3}{3} - \frac{w^4}{4} - \frac{w^5}{5} - \frac{w^6}{6}. \end{aligned} \tag{7}$$

The second inequality is simply a truncation of the Taylor series of $\log(1-w)$. The first simply says

$$\frac{13w^6}{42} > \frac{w^6}{6} + \frac{w^7}{7} + \frac{w^8}{8} + \cdots;$$

this is true enough, for $0 < w \leq 1/2$ implies $0 < \frac{w}{1-w} \leq 1$, and we have

$$\begin{aligned}\frac{w^6}{6} + \frac{w^7}{7} + \frac{w^8}{8} + \cdots &< \frac{w^6}{6} + \frac{w^7}{7}(1 + w + w^2 + \cdots) \\ &= \frac{w^6}{6} + \frac{w^7}{7}\left(\frac{1}{1-w}\right) \leq \frac{w^6}{6} + \frac{w^6}{7} = \frac{13w^6}{42}.\end{aligned}$$

Substituting inequalities (7) into the left-hand side of (6), we get that it is enough to show for $0 < w \leq 1/2$ that

$$\left(w + \frac{w^2}{2} + \frac{w^3}{3} + \frac{w^4}{4} + \frac{w^5}{5} + \frac{w^6}{6}\right)^2$$
$$+ (4w - w^2)\left(-w - \frac{w^2}{2} - \frac{w^3}{3} - \frac{w^4}{4} - \frac{w^5}{5} - \frac{13w^6}{42}\right) + 3w^2 \;>\; 0.$$

The lengthy but routine calculation gives

$$\frac{1}{12}w^4 + \frac{1}{6}w^5 + \frac{19}{90}w^6 - \frac{71}{210}w^7 + \text{(positive terms)} \;>\; 0.$$

But this last inequality is certainly true since, for $0 < w \leq 1/2$, we have

$$\frac{1}{12}w^4 - \frac{71}{210}w^7 \;\geq\; \frac{1}{12}w^4 - \frac{71}{210}w^4\left(\frac{1}{2}\right)^3 \;=\; \frac{69w^4}{1680} > 0.$$

[*Editor's note*: some minor adjustments seemed to be needed to Lambrou's asymptotic proof that $\lim a_n = 1$, and also to one of his coefficients toward the end of the solution. These have been made in the above writeup.]

Also solved by CON AMORE PROBLEM GROUP, The Royal Danish School of Educational Studies, Copenhagen, Denmark; RICHARD I. HESS, Rancho Palos Verdes, California, USA; and VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA. Two other readers sent in incorrect or incomplete solutions.

A couple of readers mentioned that $F(a)$ is eventually *decreasing (for large enough n) whenever $a > 1$, as can be seen from the above solution.*

One reader pointed out the similar problem 442 in the College Mathematics Journal, *solution in Vol.* **23** (1992) *pp.* 71–72, *in which the exponent is $n + a$ instead of $\sqrt{n(n+a)}$.*

2180. [1996: 318] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Prove that if $a > 0, x > y > z > 0, n \geq 0$ (natural), then

1. $a^x(yz)^n(y - z) + a^y(xz)^n(z - x) + a^z(xy)^n(x - y) \geq 0$,
2. $a^x \cosh x(y - z) + a^y \cosh y(z - x) + a^z \cosh z(x - y) \geq 0$.

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA.

A characterization for a function f to be convex is that, for $x > y > z > 0$,

$$(y - z)f(x) + (z - x)f(y) + (x - y)f(z) \;\geq\; 0.$$

[See, for example, D.S. Mitrinović, *Analytic Inequalities*, Springer, Berlin, 1970, p. 16.]

Since result 1 is equivalent to

$$(y-z)\frac{a^x}{x^n}+(z-x)\frac{a^y}{y^n}+(x-y)\frac{a^z}{z^n} \;\geq\; 0,$$

it is sufficient to show that $f(t)=\dfrac{a^t}{t^n}$ is a convex function of t for $n \geq 0$.

The case $n=0$ is easy, so suppose that $n>0$, and without loss of generality, suppose that $a=e$. Then

$$f''(t) \;=\; \frac{e^t\left(t^2-2t+n(n+1)\right)}{t^{n+2}}.$$

The discriminant of the quadratic equation $t^2-2t+n(n+1)$ is $-4(n^2+n-1)<0$. Hence $f''(t)>0$, and so f is convex.

For result 2, we must show that $f(t)=a^t\cosh t$ is convex. Again, assume without loss of generality, that $a=e$. Thus

$$f''(t) \;=\; 2e^t\left(\cosh t+\sinh t\right) \;=\; 2e^{2t} \;>\; 0.$$

Thus f is convex, and the inequality follows.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta (first part only); MICHAEL LAMBROU, University of Crete, Crete; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Klamkin read the second part as $\cosh((x(y-z))$, *etc., which, on reflection, is a reasonable interpretation, and gives a trivial result. We wonder how many other readers did this?*

2181. [1996: 318] *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove that the product of eight consecutive positive integers cannot be the fourth power of any positive integer.

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA. [Slightly modified by the editor.]

Without loss of generality, let

$$P = \;(n-3)(n-2)(n-1)n(n+1)(n+2)(n+3)(n+4).$$

Then

$$P = n^8 + 4n^7 - 14n^6 - 56n^5 + 49n^4 + 196n^3 - 36n^2 - 144n.$$

If P is a 4^{th} power, then it would be of the form

$$F = (n^2 + an + b)^4,$$

where a, b are constants. But then $b = 0$ since P has a zero constant term. Thus $F = n^4(n+a)^4$, and clearly $a \neq 0$.

But this implies that the coefficients of n^3, n^2 and n in F are all zero. However, P has these coefficients non-zero. Hence P cannot be of the form F.

It has been pointed out by many readers that this problem has appeared before. Most readers referred to the American Mathematical Monthly, 1936, p. 310 *for the solution to #3703 (posed by Victor Thébault in 1934, p. 522). Another reference was made to Honsberger's monograph* Mathematical Morsels, *where it appears on p. 156 as "A Perfect 4th Power". Several readers also made reference to the general problem of proving that the product of (two or more) consecutive integers is never a square, which was established in 1975 by Erdős and Selfridge* [1]. *Because the solution by Cautis was quite different from any of these published solutions, we have decided to publish it here. The interested reader is directed to these other sources for a different solution.*

Comments and/or solutions were submitted also by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADAM BROWN, Scarborough, Ontario; GORAN CONAR, student, Varaždin, Croatia; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Greece; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

Seiffert remarks that A. Guibert proved the result in 1862. This is stated by L.E. Dickson in his History of the Theory of Numbers, Vol. II, 1952, pp. 679-680.

Janous notes the following deep theorem by Erdős and Selfridge:

The product of two or more consecutive positive integers is never a power of a positive integer; that is, the Diophantine equation

$$(n+1)(n+2)\dots(n+k) = x^k$$

has no integer solutions with $k, l \geq 2$ and $n \geq 0$.

Reference

1. *P. Erdős and J.L. Selfridge,* The product of consecutive integers is never a power, *Illinois J. Math.* **19** (*1975*), *292–301*.

2182. [1996: 318] *Proposed by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.*

Many ***CRUX*** readers are familiar with the card game "Crazy Eights", of which there are many variations. We define the game of "Solo Crazy Eights" in the following manner:

We are given a standard deck of 52 cards, and are dealt k of these at random, $1 \leq k \leq 52$. We then attempt to arrange these k cards according to three rules:

1. Any card can be chosen as the first card of a sequence;
2. A card can be succeeded by any card of the same suit, or the same number, or by any eight;
3. Anytime in the sequence that an eight appears, any suit can be "called", and the succeeding card must be either of the called suit, or another eight. (This means that, in effect, any card can follow an eight).

The game is won if all dealt cards can be ordered into a sequence according to rules 1–3. If no such sequence is possible, the game is lost.

What is the largest value of k for which it is possible to lose the game?

Solution by Michael Lambrou, University of Crete, Greece.

We show that if 34 or fewer cards are dealt, then it is possible to lose the game, but for any 35 cards or more, the game always ends successfully. Thus $k = 34$.

For example, the 34 or fewer cards may be chosen as follows; (a) ace of hearts, (b) no eights, (c) any subset of the 33 cards consisting of any suit which is not a heart and any rank which is neither an ace nor an eight. In this situation the ace cannot be linked to any other card (as there are no other aces nor hearts nor eights); thus we have a losing hand.

Let us now show that for 35 cards or more, the game can end successfully. Note that if eights were present, we could exchange them with any undealt cards, arrange the cards, and re-exchange the eights (if there are insufficient undealt cards, we could simply place the eights at the end). Thus it is sufficient to show that we can obtain success on 35 or more cards which do not include eights.

The algorithm below is based on two simple observations:

1. Any set of ranks each of which appears 3 times among the dealt cards (that is, is represented by 3 suits) can be arranged in a sequence following the rules of the game. This is so because any two such ranks have at least two suits in common so, running through all suits of a fixed rank, we can link it to the next rank utilising a common suit which we can go through similarly leaving last a suit that links it with a further rank in the set. (Clearly a common suit between the second and the third rank, which is different from the suit that linked the first to the second rank, always exists). This can be repeated until we exhaust the set.

2. Any rank which appears twice among the dealt cards can be linked to a rank which appears three times, since at least one suit is common to both.

Let A_i, $(i = 0, 1, 2, 3, 4)$, denote the set of ranks that appear i times (that is, represented with i suits) among the dealt cards, and let $n_i = |A_i|$, the number of elements in A_i. Since we are excluding eights, we have

$$n_0 + n_1 + n_2 + n_3 + n_4 \;=\; 12$$

and we also have

$$n_1 + 2n_2 + 3n_3 + 4n_4 \;\geq\; 35.$$

We will show that $n_1 + n_2 \leq n_4 + 1$ and that if $n_2 = 0$ this can be improved to $n_1 \leq n_4$. We argue by contradiction. Suppose that $n_4 + 1 < n_1 + n_2$. Then

$$\begin{aligned}
35 &\leq n_1 + 2n_2 + 3n_3 + 4n_4 \\
&= n_1 + 2n_2 + 3(12 - n_0 - n_1 - n_2 - n_4) + 4n_4 \\
&= 36 - 3n_0 - 2n_1 - n_2 + n_4 \\
&\leq 36 - 2n_1 - n_2 + n_4 \\
&< 36 - 2n_1 - n_2 + (n_1 + n_2 - 1) \\
&= 35 - n_1,
\end{aligned}$$

which is impossible since $n_1 \geq 0$. If further we have $n_2 = 0$, and we assume that $n_4 < n_1$ we would get

$$\begin{aligned}
35 &\leq n_1 + 2n_2 + 3n_3 + 4n_4 \\
&= n_1 + 0 + 3(12 - n_0 - n_1 - 0 - n_4) + 4n_4 \\
&= 36 - 3n_0 - 2n_1 + n_4 \\
&\leq 36 - 2n_1 + n_4 \\
&< 36 - 2n_1 + n_1 \\
&= 36 - n_1,
\end{aligned}$$

which forces $n_1 < 1$, but then $0 \le n_4 < n_1$ is impossible.

We are now in position to describe an algorithm arranging all 35 cards according to rules 1–3. Suppose first that $n_2 = 0$. Then $n_1 \le n_4$ and we may write $A_1 = \{b_1, b_2, \ldots, b_{n_1}\}$ and $A_4 = \{c_1, c_2, \ldots, c_{n_4}\}$. We start with a card, say of rank b_1, in A_1. We link progressively this card with each card of A_1 having the same suit. After this suit has been exhausted within A_1, we link with a card, say of rank c_1, of A_4 of the same suit (as the elements of A_4 have all four suits dealt, this is always possible). We then run through all four cards of rank c_1, in some order ending with a suit for which there still exist cards in A_1. (This is always possible unless A_1 is exhausted.) Then exhaust the cards of A_1 with the same suit and jump back to A_4 (this is always possible as $n_1 \le n_4$). Repeat this process until A_1 is exhausted, then go through the rest of A_4 ending with a suit that is the same as some suit for some rank in A_3. Finally jump to A_3 and complete the process, which is possible from observation 1 above.

On the other hand, if $n_2 \ne 0$, and thus $n_1 + n_2 \le n_4 + 1$, we modify the algorithm from above as follows. After we exhaust A_1 and return to A_4, we then go to A_2, select a rank, exhaust the (two) suits of this rank, and jump back to A_4. Continue back and forth between A_4 and A_2, running each rank through all its suits, being sure to order the suits for each rank from A_4 so that it matches a suit for a rank still remaining in A_2. If A_2 is exhausted first we complete the rest of A_4 and jump to A_3 as described for the first case. If A_4 is exhausted first we go from A_2 to A_3 which can be done by observation 2 above, as long as we have taken care to end up for the last rank of A_2 with a suit which matches the suit for at least one rank from A_3. (This may take a little bit of planning for the ordering of the last rank of A_4 as well!)

Also solved by the proposer. There were two incomplete solutions.

Geretschläger also asks about a generalization to replace $52 = 4 \cdot 12 + 4$ *by* $n \cdot k + j$. *He observes that for* $n = 4$ *(that is,* $4k + j$*), an analogous argument to the one presented would yield a maximum number of* $3k - 2$. *He then asks for general conditions such that the resulting maximum number is* $(n-1)(k-1)+1$, *or to find out what other numbers (if any) could turn up.*

2183. [1996: 319] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Suppose that A, B, C are the angles of a triangle and that k, l, $m \ge 1$. Show that:

$$\begin{aligned} 0 &< \sin^k A \sin^l B \sin^m B \\ &\le k^k l^l m^m S^{\frac{S}{2}} \left((Sk^2 + P)^{-\frac{k}{2}}\right) \left((Sl^2 + P)^{-\frac{l}{2}}\right) \left((Sm^2 + P)^{-\frac{m}{2}}\right), \end{aligned}$$

where $S = k + l + m$ and $P = klm$.

Editor's and Proposer's comments.

This problem has already been posed; see **908** [1984: 19] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine the maximum value of

$$P \equiv \sin^{\alpha} A \cdot \sin^{\beta} B \cdot \sin^{\gamma} C,$$

where A, B, C are the angles of a triangle and α, β, γ are given positive numbers.

A solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, is given in [1985: 93]. He sent in a solution this time pointing out that he had done so before!

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

2184. [1996: 319] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let n be a positive integer and let a_n denote the sum

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}.$$

Prove that the sequence $\{a_n : n \geq 0\}$ is periodic.

Composite solution from Michael Lambrou, University of Crete, Crete, Greece and Kee-Wai Lau, Hong Kong.

Since

$$\binom{n-k}{k} = \binom{n-1-k}{k} + \binom{n-1-k}{k-1},$$

we have

$$\begin{aligned} a_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-1-k}{k} + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-1-k}{k-1} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-1-k}{k} - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-2-k}{k} \\ &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} - \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k \binom{n-2-k}{k} \\ &= a_{n-1} - a_{n-2}. \end{aligned}$$

It follows that $a_{n+3} = a_{n+2} - a_{n+1} = (a_{n+1} - a_n) - a_{n+1} = -a_n$, and so $a_{n+6} = -a_{n+3} = a_n$, showing that the sequence $\{a_n : n \geq 0\}$ is periodic with period 6.

In fact, the sequence takes consecutively the values $1, 1, 0, -1, -1, 0$, indefinitely.

Also solved by PAUL BRACKEN, CRM, Université de Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Bradley and Hess commented that the corresponding sum without the factor $(-1)^k$ would yield the sequence of Fibonacci numbers. [Ed.: This can be found in many elementary books on Combinatorics, for example, Exercise 21 on page 87 of Basic Techniques of Combinatorial Theory *by Daniel I.A. Cohen.] Flanigan pointed out that this problem appears, with technical differences in the definition of a_n, in the book* Concrete Mathematics *by Graham, Knuth and Patashuik, (1989), 177–179. Lau pointed out that with the recurrence relation $a_n = a_{n-1} - a_{n-2}$ and the initial values $a_0 = a_1 = 1$, one can prove easily, by induction, that*

$$a_n = \cos\left(\frac{n\pi}{3}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right), \qquad n = 0, 1, 2, \ldots .$$

This fact was also derived by the proposer.

2185. [1996: 319] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Notice that

$$2^2 + 4^2 + 6^2 + 8^2 + 10^2 = 4 \cdot 5 + 5 \cdot 6 + 6 \cdot 7 + 7 \cdot 8 + 8 \cdot 9;$$

that is, the sum of the first n (in this case 5) even positive squares is equal to the sum of some n consecutive products of consecutive pairs of positive integers.

Find another value of n for which this happens.

(NOTE: this problem was suggested by a final exam that I marked recently.)

I. *Solution by Tim Cross, King Edward's School, Birmingham, England.*

We require to find positive integers n, k for which

$$2^2 + 4^2 + \cdots + (2n)^2 = k(k+1) + (k+1)(k+2) + \cdots + (k+n-1)(k+n),$$

which is equivalent to

$$4\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n}(k+r-1)(k+r) = k(k-1)\sum_{r=1}^{n} 1 + (2k-1)\sum_{r=1}^{n} r + \sum_{r=1}^{n} r^2,$$

$$3 \cdot \frac{n}{6}(n+1)(2n+1) = k(k-1)n + (2k-1)\frac{n}{2}(n+1),$$

$$\frac{1}{2}(n+1)(2n+1) + \frac{1}{2}(n+1) = k(k-1) + k(n+1),$$

and finally

$$k^2 + nk - (n+1)^2 = 0. \tag{1}$$

We thus look for positive integer solutions

$$k = \frac{-n + \sqrt{n^2 + 4(n+1)^2}}{2}$$

and we require the discriminant $\Delta = 5n^2 + 8n + 4$ to be a perfect square, say $\Delta = \alpha^2$ for some positive integer α. This condition leads to a Pell equation $(5n+4)^2 - 5\alpha^2 = -4$.

Examining the more general form $x^2 - 5y^2 = -4$, we find solutions

$$(x, y) = (1,1), (4,2), (11,5), (29,13), \ldots \ .$$

The solution $x = 5n + 4 = 29$ gives $n = 5$, the example given.

Pell equations have solution-pairs which satisfy similar second-order recurrence relations. In this case, x_k and y_k both satisfy

$$u_k = 3u_{k-1} - u_{k-2}, \quad k \geq 3, \tag{2}$$

with $(x_1, y_1) = (1,1)$ and $(x_2, y_2) = (4,2)$. (Notice that the sequence $\{y_k\} = 1, 2, 5, 13, \ldots$ is that of alternate Fibonacci numbers.)

If we take the sequence $\{x_k\} = 1, 4, 11, 29, \ldots$ we see that terms are alternately $\equiv 1 \bmod 5$ and $\equiv 4 \bmod 5$. Since we need $x \equiv 4 \bmod 5$, we put $v_k = x_{2k}$ and can then derive from (2) the sequence

$$v_k = 7v_{k-1} - v_{k-2}, \quad k \geq 3, \qquad \text{with } v_1 = 4 \quad \text{and } v_2 = 29.$$

[*Editor's note*. For example, from (2) we get

$$\begin{aligned} x_{2k} &= 3x_{2k-1} - x_{2k-2} = 3(3x_{2k-2} - x_{2k-3}) - x_{2k-2} \\ &= 7x_{2k-2} - (3x_{2k-3} - x_{2k-2}) = 7x_{2k-2} - x_{2k-4} \end{aligned}$$

and the recurrence for the v's follows.] Then, since $n_k = (v_k - 4)/5$, we can deduce the sequence $\{n_k\}$ defined by

$$n_k = 7n_{k-1} - n_{k-2} + 4, \quad k \geq 3, \qquad \text{with } n_1 = 0 \quad \text{and } n_2 = 5.$$

This gives the sequence

$$\{n_k\} = 0 \text{ (trivially)},\ 5,\ 39,\ 272,\ 1869,\ 12815,\ \ldots$$

of suitable values of n.

II. *Solution by John Oman and Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA.*

[*Editor's note*: Oman and Prielipp first derived equation (1), which however they wrote in the form

$$n^2 - (k-2)n - (k^2-1) = 0. \qquad (3)$$

Here and below, their notation has been changed to agree with Solution I.]

Considering equation (3) as a quadratic in n, a necessary condition for it to have integer solutions is for the discriminant $k(5k-4)$ to be a perfect square. Thus $5k-4 = x^2$ and $k = y^2$ for some positive integers x and y. [*Editor's note*. Since $\gcd(k, 5k-4) = 1, 2$ or 4, the only alternative is $5k-4 = 2x^2$ and $k = 2y^2$, which implies $5y^2 - 2 = x^2$ and thus $x^2 \equiv 3 \bmod 5$, impossible. Note that we thus get the same equation $x^2 = 5y^2 - 4$ as in Solution I.]

The following sequences for k_m and n_m solve (3) and provide additional solutions to the problem:

$$k_m = y_m^2 \qquad \text{and} \qquad n_m = \frac{k_m - 2 + \sqrt{k_m(5k_m-4)}}{2} = \frac{y_m^2 - 2 + x_m y_m}{2},$$

where

$$x_m = (2+\sqrt{5})\left(\frac{3+\sqrt{5}}{2}\right)^m + (2-\sqrt{5})\left(\frac{3-\sqrt{5}}{2}\right)^m$$

and

$$y_m = \left(\frac{5+2\sqrt{5}}{5}\right)\left(\frac{3+\sqrt{5}}{2}\right)^m + \left(\frac{5-2\sqrt{5}}{5}\right)\left(\frac{3-\sqrt{5}}{2}\right)^m.$$

The first few values generated by these formulas are

m	x	y	k	n
0	4	2	4	5
1	11	5	25	39
2	29	13	169	272
3	76	34	1156	1869

[*Editor's note*. Oman and Prielipp then noted that

$$x_m = L_{2m+3} \qquad \text{and} \qquad y_m = F_{2m+3},$$

the $(2m+3)$rd Lucas and Fibonacci numbers, respectively (where, as usual, $F_1 = F_2 = 1$ and $L_1 = 1$, $L_2 = 3$, both sequences then generated by the familiar Fibonacci recurrence). This follows because

$$\frac{3 \pm \sqrt{5}}{2} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^2 \qquad \text{and} \qquad 2 \pm \sqrt{5} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^3,$$

and so

$$x_m = \left(\frac{1+\sqrt{5}}{2}\right)^{2m+3} + \left(\frac{1-\sqrt{5}}{2}\right)^{2m+3} = L_{2m+3}$$

and

$$y_m = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2m+3} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2m+3} = F_{2m+3}.$$

Thus $k_m = F_{2m+3}^2$ and (after some manipulations) $n_m = F_{2m+3}F_{2m+4} - 1$.]

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADAM BROWN, Scarborough, Ontario; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, Washington, USA; JEFFREY K. FLOYD, Newnan, Georgia, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. One incorrect solution was sent in.

Besides Oman and Prielipp, and (to a smaller extent) Cross, no other readers seem to have noticed the presence of the Fibonacci numbers in the solution to this problem. One more occurrence, which Oman and Prielipp don't mention, is that the largest number $k+n$ on the right-hand side of the given equation is F_{2m+4}^2, which fits nicely with the fact that the smallest number on the right-hand side is $k = F_{2m+3}^2$. In fact, only Brown commented on the fact that these numbers are squares! So for example, the equation arising from the next-smallest solution $n = 39$ is

$$2^2 + 4^2 + \cdots + 78^2 = 25 \cdot 26 + 26 \cdot 27 + \cdots + 63 \cdot 64,$$

where $25 = F_5^2$ *and* $64 = F_6^2$ (*and* $78 = 2F_5F_6 - 2$, *as follows from Solution* II). *In general, the required equation reads*

$$2^2 + 4^2 + \cdots + (2F_{2t-1}F_{2t} - 2)^2$$

$$= F_{2t-1}^2(F_{2t-1}^2 + 1) + (F_{2t-1}^2 + 1)(F_{2t-1}^2 + 2) + \cdots + (F_{2t}^2 - 1)F_{2t}^2$$

for any integer $t \geq 2$.

2186. [1996: 319] *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

Let a, b, c respectively denote the lengths of the sides BC, CA, AB of triangle ABC. Let G denote the centroid, let I denote the incentre, let R denote the circumradius, r denote the inradius, and let s denote the semi-perimeter.

Prove that

$$GI^2 = \frac{1}{9(a+b+c)}\Big((a-b)(a-c)(b+c-a)$$

$$+ (b-c)(b-a)(c+a-b) + (c-a)(c-b)(a+b-c)\Big).$$

Deduce the (known) result

$$GI^2 = \frac{1}{9}\left(s^2 + 5r^2 - 16Rr\right).$$

Solution by Kee-Wai Lau, Hong Kong.

Let A be the origin, $\mathbf{AB} = \mathbf{u}$ and $\mathbf{AC} = \mathbf{v}$. It is well known that

$$\mathbf{AG} = \frac{1}{3}(\mathbf{u} + \mathbf{v}) \text{ and } \mathbf{AI} = \frac{b}{a+b+c}\mathbf{u} + \frac{c}{a+b+c}\mathbf{v}.$$

Hence

$$\mathbf{GI} = \left(\frac{b}{a+b+c} - \frac{1}{3}\right)\mathbf{u} + \left(\frac{c}{a+b+c} - \frac{1}{3}\right)\mathbf{v}.$$

Since $\mathbf{u}\cdot\mathbf{u}=c^2,\ \mathbf{v}\cdot\mathbf{v}=b^2$ and $\mathbf{u}\cdot\mathbf{v}=bc\cos A=\frac{1}{2}(b^2+c^2-a^2)$, so

$$\begin{aligned}
GI^2 &= \left(\frac{b}{a+b+c}-\frac{1}{3}\right)^2 c^2+\left(\frac{c}{a+b+c}-\frac{1}{3}\right)^2 b^2\\
&\quad+\left(\frac{b}{a+b+c}-\frac{1}{3}\right)\left(\frac{c}{a+b+c}-\frac{1}{3}\right)(b^2+c^2-a^2)\\
&= \frac{1}{9(a+b+c)^2}\Big((a+c-2b)^2c^2+(a+b-2c)^2b^2\\
&\quad+(a+c-2b)(a+b-2c)(b^2+c^2-a^2)\Big)\\
&= \frac{1}{9(a+b+c)}\Big(-a^3-b^3-c^3+2a^2b+2ab^2\\
&\quad+2b^2c+2bc^2+2c^2a+2ca^2-9abc\Big)\\
&= \frac{1}{9(a+b+c)}\Big((a-b)(a-c)(b+c-a)\\
&\quad+(b-c)(b-a)(c+a-b)+(c-a)(c-b)(a+b+c)\Big)
\end{aligned}$$

as required. Since

$$s=\frac{a+b+c}{2},\quad r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\quad\text{and}\quad R=\frac{abc}{4rs},$$

we have

$$\begin{aligned}
&\frac{1}{9}\left(s^2+5r^2-16Rr\right)\\
&= \frac{1}{9}\left(\frac{(a+b+c)^2}{4}+\frac{5(b+c-a)(a+c-b)(a+b-c)}{4(a+b+c)}-\frac{8abc}{a+b+c}\right)\\
&= \frac{1}{9(a+b+c)}\left(\frac{(a+b+c)^3+5(b+c-a)(a+c-b)(a+b-c)-32abc}{4}\right)\\
&= \frac{1}{9(a+b+c)}\Big(-a^3-b^3-c^3+2a^2b+2ab^2\\
&\quad+2b^2c+2bc^2+2c^2a+2ca^2-9abc\Big)\\
&= GI^2,
\end{aligned}$$

as required.

Bellot Rosado notes that a variation of this problem was proposed by Cezar Coşniţă, solved by T.C. Esty, with the second form of GI given by D.L. MacKay [Problem E415, American Mathematical Monthly (1940), solution p. 712].

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, *Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.*

2187. [1996: 320] *Proposed by Syd Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is easy to show that the maximum number of bishops that can be placed on an 8×8 chessboard, so that no two of them attack each other, is 14.

(a) Prove or disprove that in any configuration of 14 non-attacking bishops, all the bishops must be on the boundary of the board.

(b) Describe all of the configurations with 14 non-attacking bishops.

Solution by Kee-Wai Lau, Hong Kong.

(a) We prove the result.

Clearly we need consider only the black bishops. The diagram below shows the black squares of a chessboard rearranged so that the bishops will now move vertically or horizontally. [The diagram is labelled the same as the standard labelling of the squares of a chessboard, with the rows (ranks) numbered 1 to 8 and the columns (files) labelled a to h. — *Ed.*]

$$\begin{array}{ccccccc}
 & & & h8 & & & \\
 & & f8 & g7 & h6 & & \\
 & d8 & e7 & f6 & g5 & h4 & \\
b8 & c7 & d6 & e5 & f4 & g3 & h2 \\
a7 & b6 & c5 & d4 & e3 & f2 & g1 \\
 & a5 & b4 & c3 & d2 & e1 & \\
 & & a3 & b2 & c1 & & \\
 & & & a1 & & &
\end{array}$$

In any column and any row there is at most one bishop. [Thus there are at most 7 black bishops and similarly at most 7 white bishops, for the given maximum of 14. Moreover, to attain this maximum, there must be a black bishop in each column in the above diagram. — *Ed.*]

Denote the bishop in the kth column by B_k. If B_1 is on $a7$ then B_7 must be on $h2$, and if B_1 is on $b8$ then B_7 must be on $g1$. Next, B_2 must be on $a5$ or $d8$ and correspondingly B_6 must be on $h4$ or $e1$. Next B_3 must be on $a3$ or $f8$ and correspondingly B_5 must be on $h6$ or $c1$. Finally B_4 must be on $a1$ or $h8$. This shows that the non-attacking bishops must be on the boundary of the board.

(b) From part (a) we see that there are $2^4 = 16$ ways to locate the black bishops. Similarly there are 16 ways to locate the white bishops. Thus there are 256 configurations with 14 non-attacking bishops.

Also solved (usually the same way) by SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposers. Part (a) *was also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria.*

2188. [1996: 320] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Suppose that a, b, c are the sides of a triangle with semi-perimeter s and area $\triangle$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{s}{\triangle}.$$

I. *Solution Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let C be the angle opposite to side c of the triangle. Since $\triangle=\frac{1}{2}ab\sin C\leq\frac{1}{2}ab$, we have $\frac{1}{a}\leq\frac{b}{2\triangle}$. Here, equality holds if and only if the angle opposite to c is a right angle.

Similarly, $\frac{1}{b}\leq\frac{c}{2\triangle}$, and $\frac{1}{c}\leq\frac{a}{2\triangle}$. Since equality cannot hold simultaneously, we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{b+c+a}{2\triangle}=\frac{s}{\triangle}.$$

II. *Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

I will prove the stronger inequality:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{\sqrt{3}}{2}\cdot\frac{s}{\triangle}.$$

Let $a=x+y,\ b=y+z,\ c=z+x$, where $x,y,z>0$. Then

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &= \left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)^2\\ &\leq \left(\frac{1}{2\sqrt{xy}}+\frac{1}{2\sqrt{yz}}+\frac{1}{2\sqrt{zx}}\right)^2\\ &= \frac{1}{4}\left(\frac{1}{\sqrt{xy}}+\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}}\right)^2\\ &\leq \frac{3}{4}\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right) \text{ (by Cauchy-Schwarz)}\end{aligned}$$

Now $\frac{s}{\triangle}=\frac{x+y+z}{\sqrt{(x+y+z)xyz}}=\sqrt{\frac{x+y+z}{xyz}}=\sqrt{\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}}$.

Thus $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{\sqrt{3}}{2}\cdot\frac{s}{\triangle}$. The equality holds only when $a=b=c$.

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADAM BROWN, Scarborough, Ontario; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; STEFAN and ALEXANDER LAMBROU, students, Crete, Greece; MICHAEL LAMBROU, University of Crete, Crete, Greece; CAN AN+H MINH, University of California, Berkeley, California; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria (2 solutions); BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; ADRIÁN UBIS MATIÍNEZ, student, I.B. Sagasta, Logroño, Spain; MELETIS VASILIOU, Elefsis, Greece; and the proposer.

2189. [1996: 361] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The incircle of a triangle ABC touches BC at D. Let P and Q be variable points on sides AB and AC respectively such that PQ is tangent to the incircle. Prove that the area of triangle DPQ is a constant multiple of $BP \cdot CQ$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let E, F, T be the points of contact of the incircle with CA, AB and PQ; define $x := AP$, $y := AQ$ as well as $z := PQ$. Then

$$\begin{aligned} x+y+z &= x+y+PT+QT \\ &= (x+PF)+(y+QE) \\ &= AF+AE \;=\; 2(s-a) \end{aligned} \qquad (1)$$

where s is the semi-perimeter of $\triangle ABC$. Applying the cosine rule to $\triangle APQ$ yields

$$z^2 \;=\; x^2+y^2-2xy\cos A \;=\; x^2+y^2-2xy\cdot\frac{b^2+c^2-a^2}{2bc}.$$

From equation (1) we get

$$4(s-a)^2 - 4(s-a)(x+y) + x^2 + 2xy + y^2 \;=\; x^2 + y^2 - xy \cdot \frac{b^2+c^2-a^2}{bc}$$

which is equivalent to

$$\begin{aligned}
xy \cdot \left(\frac{b^2+2bc+c^2-a^2}{bc}\right) + 4(s-a)^2 &= 4(s-a)(x+y) \\
xy\left((b+c)^2 - a^2\right) + 4bc(s-a)^2 &= 4bc(s-a)(x+y) \\
4xys(s-a) + 4bc(s-a)^2 &= 4bc(s-a)(x+y) \\
xys + bc(s-a) &= bc(x+y). \qquad (2)
\end{aligned}$$

Now let R be the circumradius of $\triangle ABC$ and use the sine law [in the second equality] and (2) [in the third] to get

$$\begin{aligned}
[DPQ] &= [ABC] - \frac{AP \cdot AQ \cdot \sin A}{2} - \frac{BP \cdot BD \cdot \sin B}{2} \\
&\quad - \frac{CD \cdot CQ \cdot \sin C}{2} \\
&= \frac{1}{4R}\Big(abc - xya - (c-x)(s-b)b - (b-y)(s-c)c\Big) \\
&= \frac{1}{4R}\Big[abc - axy - bc(s-b) - bc(s-c) + bx(s-b) \\
&\qquad + \; cy(s-c) + \Big(xys + bc(s-a) - bc(x+y)\Big)\Big] \\
&= \frac{1}{4R}\Big(abc - axy - abc + bx(s-b-c) + cy(s-b-c) \\
&\qquad + xys + bc(s-a)\Big) \\
&= \frac{1}{4R}\Big(xy(s-a) - bx(s-a) - cy(s-a) + bc(s-a)\Big) \\
&= \frac{(s-a)}{4R}(c-x)(b-y) \;=\; \frac{(s-a)}{4R} \cdot BP \cdot CQ.
\end{aligned}$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. Several solvers mentioned that this problem generalizes the proposer's earlier 1862 [1993: 203], [1994: 172-173].

2190. [1996: 361] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the range of

$$\frac{\sin^2 A}{A} + \frac{\sin^2 B}{B} + \frac{\sin^2 C}{C}$$

where A, B, C are the angles of a triangle.

Solution by Kee-Wai Lau, Hong Kong.

Denote the function of the problem by $f(A, B, C)$. We show that

$$0 \ < \ f(A, B, C) \ \le \ \frac{27}{4\pi}.$$

The first inequality follows from the definitions. By considering the degenerate triangle $A = \pi$, $B = 0$, $C = 0$, we see that it is also sharp. We now prove the second inequality.

For $0 < x \le \frac{\pi}{2}$, let $g(x) = \frac{\sin^2 x}{x}$. We have

$$\begin{aligned} \frac{dg}{dx} &= \frac{\sin x(2x\cos x - \sin x)}{x^2} \quad \text{and} \\ \frac{d^2 g}{dx^2} &= \frac{h(x)}{x^3}, \quad \text{where} \quad h(x) = 1 - \cos(2x) + 2x^2\cos(2x) - 2x\sin(2x). \end{aligned}$$

Since $\frac{dh}{dx} = -4x^2\sin(2x) \le 0$ and $h(0) = 0$, we have $h(x) \le 0$. It follows that $\frac{d^2g}{dx^2} \le 0$, so that g is concave. Hence, if $\triangle ABC$ is acute angled, then

$$f(A, B, C) \ \le \ 3g\left(\frac{A+B+C}{3}\right) \ = \ 3g\left(\frac{\pi}{3}\right) \ = \ \frac{27}{4\pi}.$$

Equality holds if and only if $A = B = C = \frac{\pi}{3}$.

Now, suppose that one of the angles of $\triangle ABC$ is obtuse. We may assume that $\angle A > \frac{\pi}{2}$, $\angle B + \angle C < \frac{\pi}{2}$.

Hence $f(A, B, C) \ < \ \frac{2}{\pi} + \frac{\sin^2 B}{B} + \frac{\sin^2 C}{C} \ \le \ \frac{2}{\pi} + 2\frac{\sin^2\left(\frac{B+C}{2}\right)}{\frac{B+C}{2}}$

[by the concavity of $g(x)$ together with $\frac{\sin^2 A}{A} < \frac{1}{A} < \frac{2}{\pi}$ for $A > \frac{2}{\pi}$].

It is easy to show that $\tan x \le 2x$ for $0 \le x \le \frac{\pi}{4}$. Hence $\frac{dg(x)}{dx} \ge 0$ for $0 \le x \le \frac{\pi}{4}$. It follows that

$$f(A, B, C) \ < \ \frac{2}{\pi} + 2\frac{\sin^2\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} \ = \ \frac{6}{\pi} \ < \ \frac{27}{4\pi}.$$

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; and the proposer.

Three unsatisfactory solutions were received. Two of these depended on an inequality that was passed off as "known". Such claims must be accompanied by a reference. Otherwise, how are the editors supposed to know that the claim is true? More importantly, the purpose of the solution should be to explain **why** *the result is true; consequently any reference should be accessible to* ***CRUX with MAYHEM*** *readers.*

A key step in the third rejected submission was claimed to be "obvious". The only thing obvious to this editor was that such claims do not belong in mathematical arguments.

2191. [1996: 361] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all positive integers n, that satisfy the inequality

$$\frac{1}{3} < \sin\left(\frac{\pi}{n}\right) < \frac{1}{2\sqrt{2}}.$$

Editor's composite solution based on the ones submitted by the solvers whose names appear below.

Since $\sin\pi = 0$, $n = 1$ is clearly not a solution.

Since $\sin\left(\frac{\pi}{n}\right)$ is decreasing for $n \geq 2$ and since

$$\sin\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}} > \frac{1}{2\sqrt{2}}$$

we have $\sin\left(\frac{\pi}{n}\right) > \frac{1}{2\sqrt{2}}$ for $2 \leq n \leq 8$.

On the other hand, since $\sin x < x$ for $x > 0$, we have, for all $n \geq 10$, $\sin\left(\frac{\pi}{n}\right) < \frac{\pi}{n} \leq \frac{\pi}{10} < \frac{1}{3}$.

We now show that

$$\frac{1}{3} < \sin\left(\frac{\pi}{9}\right) < \frac{1}{2\sqrt{2}} \tag{1}$$

and so $n = 9$ is the only solution. [Ed: Though (1) can be verified numerically by using a calculator, as many solvers did, the proposer's original intention was an "analytic" proof like the one presented below.]

Let $\theta = \frac{\pi}{9}$ and let $r = \sin\theta$. Then from

$$\frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right) = \sin(3\theta) = 3\sin\theta - 4\sin^3\theta = 3r - 4r^3,$$

we see that r is a positive root of the polynomial $f(x) = 4x^3 - 3x + \frac{\sqrt{3}}{2}$.

Note that

$$f(-1) = -1 + \frac{\sqrt{3}}{2} < 0, \qquad f(0) = \frac{\sqrt{3}}{2} > 0,$$

$$f\left(\frac{1}{3}\right) = -\frac{23}{27} + \frac{\sqrt{3}}{2} > 0, \qquad f\left(\frac{1}{2\sqrt{2}}\right) = \frac{-5}{4\sqrt{2}} + \frac{\sqrt{3}}{2} < 0,$$

and

$$f(1) = 1 + \frac{\sqrt{3}}{2} > 0.$$

Since f is a continuous function, we conclude that it has three real roots, one in each of the three intervals: $(-1,0)$, $\left(\frac{1}{3}, \frac{1}{2\sqrt{2}}\right)$ and $\left(\frac{1}{2\sqrt{2}}, 1\right)$. But $\sin\left(\frac{\pi}{9}\right) < \frac{\pi}{9} \approx 0.349 < \frac{1}{2\sqrt{2}}$, and so (1) follows.

Solved by GERALD ALLEN, CHARLES DIMINNIE, TREY SMITH and ROGER ZARNOWSKI (jointly) Angelo State University, San Angelo, TX, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DIEGO SOTÉS and JAVIER GUTIÉRREZ, students, University of Rioja, Logroño, Spain; and DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA.

There were also nine incomplete or (partially) incorrect solutions. Though these solutions all give the correct answer $n = 9$, they contain various errors. Most of these errors pertain to the analysis of the three roots of $f(x)$ which is actually more subtle than it appears to be. Some solvers erroneously claimed that one of the roots is greater than one. Using **MAPLE**, *we find easily that the three roots are: -0.9848077, 0.3420201, and 0.6427876. Only one (joint) solver stated that the three roots are, in fact, $\cos\left(\frac{19\pi}{18}\right)$, $\sin\left(\frac{\pi}{9}\right)$, and $\sin\left(\frac{7\pi}{9}\right)$. However, they made the mysterious and wrong statement that $\sin\left(\frac{7\pi}{9}\right) < \frac{1}{3}$.*

Some other solvers, after checking that $f\left(\frac{1}{3}\right)$ and $f\left(\frac{1}{2\sqrt{2}}\right)$ have opposite signs, jump to the conclusion that $\sin\left(\frac{\pi}{9}\right)$ must be the root in the interval $\left(\frac{1}{3}, \frac{1}{2\sqrt{2}}\right)$. Clearly, to make this argument valid, one has to show that the other positive root is not in this interval.

2192. [1996: 362] *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Let $\{a_n\}$ be a sequence defined as follows:

$$a_{n+1} + a_{n-1} = \left(\frac{a_2}{a_1}\right) a_n, \qquad n \geq 1.$$

Show that if $\left|\frac{a_2}{a_1}\right| \geq 2$, then $\left|\frac{a_n}{a_1}\right| \geq n$.

Solution by Can Anh Minh, University of California, Berkeley.

By the triangle inequality we have

$$|a_{n+1}| + |a_{n-1}| \geq |a_{n+1} + a_{n-1}| = \left|\frac{a_2}{a_1}\, a_n\right| = \left|\frac{a_2}{a_1}\right| |a_n| \geq 2|a_n|$$

Thus we have $|a_{n+1}| - |a_n| \geq |a_n| - |a_{n-1}|$, and therefore

$$|a_n| - |a_{n-1}| \geq |a_{n-1}| - |a_{n-2}| \geq \cdots \geq |a_2| - |a_1| \geq |a_1|$$

Thus we have

$$|a_n| - |a_1| = \sum_{k=2}^{n} (|a_k| - |a_{k-1}|) \geq (n-1)|a_1|$$

It follows that $|a_n| \geq n|a_1|$, or equivalently

$$\left|\frac{a_n}{a_1}\right| \geq n.$$

Also solved by GERALD ALLEN, CHARLES DIMINNIE, TREY SMITH, and ROGER ZARNOWSKI (jointly), Angelo State University, San Angelo, Texas; CHETAN T. BALWE, Pune, India; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the proposer.

2193. [1996: 362] *Proposed by Luis V. Dieulefait, IMPA, Rio de Janeiro, Brazil.*

(a) Prove that every positive integer is the difference of two relatively prime composite positive integers.

(b) Prove that there exists a positive integer n_0 such that every positive integer greater than n_0 is the sum of two relatively prime composite positive integers.

Solution to (a) *by Michael Lambrou, University of Crete, Crete, Greece, modified slightly by the editor.*

Note that for all natural numbers k, we have:

$$2k+1 = (k+1)^2 - k^2 \tag{1}$$

and

$$2k = (2k+1)(8k+1) - (4k+1)^2 \tag{2}$$

Since $\gcd(k, k+1) = 1$, (1) gives a required representation for all odd integers greater than 3. But $1 = 9 - 8$ and $3 = 25 - 22$, and so a required representation exists for all odd natural numbers. On the other hand, straightforward computations show that

$$-(8k+4)(2k+1)(8k+1) + (8k+5)(4k+1)^2 = 1$$

and so

$$\gcd\left((2k+1)(8k+1), (4k+1)^2\right) = 1.$$

Thus (2) gives a required representation for all even natural numbers.

Solution to (b) *by the proposer, modified by the editor.*

Let $\phi(n)$ denote Euler's totient function and let $\pi(n)$ denote the prime-counting function. [Ed: that is, $\pi(n)$ is the number of primes p such that $p \leq n$.] It is known (see Hardy and Wright, *An Introduction to the Theory of Numbers*) that

$$\liminf_{n\to\infty} \frac{\phi(n)}{\dfrac{n}{\log\log n}} = e^{-\gamma} > \frac{1}{2},$$

where γ is Euler's constant. Hence, for sufficiently large n, we have

$$\phi(n) > \frac{2}{2\log\log n}. \tag{1}$$

On the other hand, by Tschebycheff's theorem, we have

$$\pi(n) < \frac{6}{5}\frac{n}{\log n} \tag{2}$$

if n is large enough. From (1) and (2), we obtain

$$\lim_{n\to\infty} \frac{\phi(n)}{\pi(n)} > \lim_{n\to\infty} \frac{5\log n}{12\log\log n} = \infty,$$

which implies, for large enough n, that

$$\phi(n) \geq 2\left(\pi(n)+1\right). \tag{3}$$

Note that, if $n = a + b$, then $(a, n) = 1$ is equivalent to $(b, n) = 1$, and to $(a, b) = 1$.

Consider the $\phi(n)$ ordered decompositions of $n : n = k + (n - k)$, where $1 \leq k \leq n$, such that $\gcd(k, n) = 1$. If we strike out those pairs in which the first or second summand is a prime or 1, then we are deleting at most $2(\pi(n) + 1)$ pairs, and so from (3), we conclude that there is at least one decomposition $n = a + b$ where a and b are relatively prime composite natural numbers, and our proof is complete.

Solved (both parts) by RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer. Part (a) *only was solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.*

Regarding (a), *Janous actually obtained a stronger result by showing that there are, in fact, infinitely many required representations. Regarding* (b), *Hess remarked that a computer search seems to indicate that* $n_0 \leq 210$. *Using arguments similar to those given in the proof above, Lambrou proved a stronger result, namely:*

For any given integer $k \geq 1$, there exists a positive integer m_k such that every integer greater than m_k can be written as the sum of two relatively prime composite natural numbers in at least m_k different ways.

It is not difficult to see that this result also follows easily from the proposer's proof presented above.

2194. [1996: 362] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Prove or disprove that it is possible to find a triangle ABC and a transversal NML with N lying between A and B, M lying between A and C, and L lying on BC produced, such that BC, CA, AB, NB, MC, NM, ML, and CL are all of integer length, and $NMCB$ is a cyclic inscriptable quadrilateral.

Editor's comment. There is a terminology problem here. It appears (from his solution) that the proposer had intended the word *inscriptable* to mean that a circle can be inscribed in the quadrangle $NMCB$. Although several of the solvers assumed this to be the case, all reference books that I consulted disagree. The *Oxford English Dictionary* and Nathan Altshiller Court (in his 1925 *College Geometry*) both say that an *inscriptible quadrangle* (note the spelling) is one that can be inscribed in a circle; today one more commonly calls such quadrangles (that can be inscribed in circles) *cyclic, concyclic*, or *inscribable* — take your choice. On the other hand, one must keep in mind that English (the language in which, for example, *inflammable* means *flammable* and *unravel* means *ravel*) is so unpredictable that the proposer might well have references to back up his terminology. For a quadrangle that contains an inscribed circle, one would say *circumscribing* or avoid the issue and just say "a quadrangle with an inscribed circle". The matter really becomes muddy here: Court calls the latter quadrangle *circumscriptible*, while the *OED* agrees in one place (under *inscriptible*), but contradicts itself in another, where a quadrangle is said to be *circumscribable* if it "may be circumscribed by a circle". The moral of this story: be circumspect (or maybe, circumspectable). As it turns out, our featured solutions all produce cyclic quadrangles having inscribed circles!

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece* (somewhat edited).

We show that *any* triangle ABC with rational sides and $\angle A \neq \angle C$ has a transversal LMN such that $NMCB$ satisfies both circle conditions, and has all relevant lengths rational. Hence, multiplying by an appropriate number, we can obtain a similar triangle with the required properties.

Let a triangle ABC with rational sides be given. By renaming, we may assume that B, which may be acute, right or obtuse, is larger than C. Note that B cannot equal C, as the transversal — in order to produce a cyclic quadrangle — would then be parallel to BC.

Consider as transversal LMN such that

$$\angle AMN = B \text{ (making } NMCB \text{ cyclic), and} \qquad (4)$$
$$LM \text{ is tangent to the incircle of triangle } ABC \qquad (5)$$

(forcing $NMCB$ to have an inscribed circle). We must show that the lengths NB, MC, NM, ML and CL are all rational.

We have that the triangles ANM and ACB are similar. Let $\lambda = AM/AB$ be the similarity ratio. We use the fact that the sides of triangle ABC are rational to show that $\lambda \in \mathbb{Q}$.

$$\begin{aligned}
&\lambda(BC + AB + AC) \\
&\quad = \ MN + AM + AN \\
&\quad = \ MN + (AC - MC) + (AB - NB) \\
&\quad = \ [(MN + BC) - (NB + MC)] + AB + AC - BC \\
&\quad = \ 0 + AB + AC - BC \ \in \mathbb{Q}\,.
\end{aligned}$$

But $BC + AB + AC \in \mathbb{Q}$, so that $\lambda \in \mathbb{Q}$ as well.

Next, by Menelaus's Theorem applied to triangle ABC with transversal LMN, it follows that

$$\frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = 1 \in \mathbb{Q},$$

so that $BL/CL \in \mathbb{Q}$. Since $BC/CL = (BL - CL)/CL$, we have $LC \in \mathbb{Q}$ and $BL \in \mathbb{Q}$.

Similarly, by Menelaus's Theorem applied to triangle MLC with transversal AB, we see that $LN \in \mathbb{Q}$, and the desired result follows.

II. *Families of solutions by Michael Lambrou, University of Crete, Crete, Greece and (independently) by Richard I. Hess, Rancho Palos Verdes, California, USA.*

Let AMN and LBN be congruent right triangles with right angles at M and B. Denote the legs by p ($= AM = LB$), q ($= MN = BN$), and the hypotenuse by r ($= NL = NA = \sqrt{p^2+q^2}$). The solution is achieved by choosing p, q, r so that p divides $r+q$. Hess provided the examples in column 2, and Lambrou, column 3.

	Hess	Lambrou
$p = AM = LB$	$2k+1$	λ^2,
$q = MN = BN$	$2k(k+1)$	$\lambda\mu$,
$r = NL = NA$	$(k+1)^2 + k^2$	$\lambda\nu$, where $\lambda^2 + \mu^2 = \nu^2$,
$r + q = AB = LM$	$(2k+1)^2$	$(\mu+\nu)\lambda$,
$q(q+r)/p = MC = BC$	$2k(k+1)(2k+1)$	$(\mu+\nu)\mu$,
$r(q+r)/p = AC = LC$	$(2k+1)\times((k+1)^2+k^2)$	$(\mu+\nu)\nu = \lambda^2 + \mu^2 + \mu\nu$.

In each case, $NMCB$ has a circumscribed circle because of the right angles at B and M, while it has an inscribed circle because of its symmetry (so that $NM + CB = NB + CM$).

Lambrou also sent in a family of asymmetric solutions for positive integers $m > n$:

$$\begin{aligned} AB &= nm^2(m-n)(n^2+1), \\ AC &= n^2m(m-n)(m^2+1), \\ BC &= mn(m-n)(m+n)(mn-1), \quad \text{and} \\ AN &= \frac{1}{mn}AC, \quad AM = \frac{1}{mn}AB. \end{aligned}$$

We leave it to the reader to verify the details. (Remember to check the various triangle inequalities.)

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2195. [1996: 362] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

A barrel contains $2n$ balls, numbered 1 to $2n$. Choose three balls at random, one after the other, and with the balls replaced after each draw.

What is the probability that the three-element sequence obtained has the properties that the smallest element is odd and that only the smallest element, if any, is repeated?

For example, the sequences 453 and 383 are acceptable, while the sequences 327 and 388 are not.

(NOTE: this problem was suggested by a final exam that I marked recently.)

I. *Solution by David Hankin, Hunter College Campus Schools, New York, NY, USA.*

Let u_k be the number of acceptable sequences chosen from balls numbered 1 to $2k$, and let a_k be the number of these acceptable sequences that contain 1. We first show that, for $k \geq 1$,

$$u_k - a_k = u_{k-1},$$

where $u_0 = 0$. Note that $u_k - a_k$ is the number of acceptable sequences that do not contain 1. The number of these sequences is the same as the number of acceptable sequences that can be chosen from balls numbered 3 to $2k$. Clearly, this is equal to u_{k-1}.

To find a_k, note that sequences that contain 1 must have one, two or three 1's. There is one sequence with three 1's. For sequences with two 1's, there are $2k-1$ ways to choose the third element and 3 arrangements of the three elements; so there are $3(2k-1)$ such sequences. Similarly, there are $6\binom{2k-1}{2} = 6(2k-1)(k-1)$ sequences with one 1. Thus

$$a_k = 1+3(2k-1)+6(2k-1)(k-1) = 12k^2-12k+4 = 4[k^3-(k-1)^3].$$

Now (since $u_0 = 0$)

$$u_n = \sum_{k=1}^{n}(u_k - u_{k-1}) = \sum_{k=1}^{n} a_k = 4\sum_{k=1}^{n}[k^3-(k-1)^3] = 4n^3.$$

Thus the requested probability is

$$\frac{4n^3}{(2n)^3} = \frac{1}{2}\,.$$

II. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

We show that there is a one-to-one onto pairing that pairs each favourable triplet abc (which for convenience we will denote (a,b,c)) with an unfavourable one. Once this is done, the independence of events shows that the sought probability is $1/2$, as the favourable events exactly match the unfavourable ones.

We denote the smallest element among (a, b, c) by $2s - 1$, where $1 \leq s \leq n$. The pairing will be described for the case when $2s - 1$ occurs in the first position (and is perhaps repeated in one or both of the other positions). The other two possibilities are dealt with in a similar fashion, by cyclic change of order. Note that some care must be taken not to double count the triplets that belong to more than one situation.

Here is how we do our pairing.

- For fixed s and p with $2s - 1 < p \leq 2n - 1$, map, in any one-to-one onto fashion, the $2n - 2s$ triplets of the form

  $$(2s - 1, p, q) \qquad \text{where} \quad 2s - 1 < q \leq 2n \quad \text{and} \quad q \neq p$$

  to the $2n - 2s$ triplets of the form

  $$(2s, p + 1, v) \qquad \text{where} \quad 2s < v \leq 2n;$$

- for $2s - 1 < q \leq 2n - 1$, map $(2s - 1, 2s - 1, q)$ to $(2s, 2s, q + 1)$;
- for $2s - 1 < q \leq 2n - 1$, map $(2s - 1, 2n, q)$ to $(2s - 1, q, q)$;
- map $(2s - 1, 2s - 1, 2n)$ to $(2n, 2n, 2s - 1)$; and finally
- map $(2s - 1, 2s - 1, 2s - 1)$ to $(2s, 2s, 2s)$.

It is easy to see that this map is not ambiguous. It is also easy to check that all favourable and all unfavourable triplets have been taken into account once and once only. This completes the argument.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; W. MOSER, McGill University, Montréal, Québec; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; D. N. SHETH, Sir Parashuram College, Pune, India; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the proposer. Two incorrect solutions were sent in.

Lambrou was the only solver to find a "combinatorial" proof that the probability is exactly $1/2$. He also sent in two other solutions, one of which is similar to solution I. *Bradley's solution is also similar to solution* I.

2196. [1996: 362] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Find all solutions of the diophantine equation

$$2(x+y)+xy=x^2+y^2,$$

with $x>0$, $y>0$.

Solution by Sam Baethge, Nordheim, Texas, USA.

Let $y=rx$ with r rational. The given equation becomes a quadratic in r:

$$r^2x^2-r(x^2+2x)+(x^2-2x)=0$$

Then $r=\left(x+2\pm\sqrt{-3x^2+12x+4}\right)/2x$. The discriminant can be written as $16-3(x-2)^2$ and must be the square of an integer. Since x is a positive integer the only solutions are $(x,r)=(4,1)$, $(4,\frac{1}{2})$, or $(2,2)$. This produces $(x,y)=(4,4)$, $(4,2)$, or $(2,4)$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; GORAN CONAR, student, Varaždin, Croatia; PAUL-OLIVIER DEHAYE, Brussels, Belgium; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; C. FESTRAETS-HAMOIR, Brussels, Belgium; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; IGNOTUS, Villeta, Colombia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; CAN ANH MINH, University of California, Berkeley; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; ADRIÁN UBIS MATIÍNEZ, Logroño, Spain; DAVID C. VELLA, Skidmore College, Saratoga Springs, New York; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There were seven incorrect solutions and one incomplete solution.

Vella (in one of his two submitted solutions) used a change of variables to turn the given equation into the equation of an ellipse with centre on the

x–axis and major axis aligned with the x–axis, and then used a geometric approach to solve the problem. The proposer also suggests a generalization:

Solve the diophantine equation:

$$z(x+y)+xy = x^2+y^2$$

2197. [1996: 363] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let n be a positive integer. Evaluate the sum:

$$\sum_{k=n}^{\infty} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}}.$$

Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.

The value of the sum is $\dfrac{1}{4^n}\dbinom{2n}{n}$. This follows directly from

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}} = 1 \tag{1}$$

and

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}} = 1 - \frac{1}{4^n}\binom{2n}{n}. \tag{2}$$

To prove (1), note that it is easy to show that

$$\frac{\binom{2k}{k}}{(k+1)2^{2k+1}} = -\binom{\frac{1}{2}}{k+1}(-1)^{k+1}$$

$\left(\text{they both simplify to } \dfrac{1\cdot 3\cdot 5\cdots(2k-1)}{2^{k+1}(k+1)!}\right)$. Therefore

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}} &= -\sum_{k=0}^{\infty}(-1)^{k+1}\binom{\frac{1}{2}}{k+1} = -\sum_{k=1}^{\infty}(-1)^{k}\binom{\frac{1}{2}}{k} \\ &= -\left(-1+\sum_{k=0}^{\infty}(-1)^{k}\binom{\frac{1}{2}}{k}\right) = 1-\sqrt{1+(-1)} = 1.\end{aligned}$$

To prove (2), we use induction. After checking at $n = 1$, we use the inductive assumption to get

$$\begin{aligned}\sum_{k=0}^{n} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}} &= 1 - \frac{1}{4^n}\binom{2n}{n} + \frac{\binom{2n}{n}}{(n+1)2^{2n+1}} \\ &= 1 - \frac{1}{4^n}\binom{2n}{n}\left(1 - \frac{1}{2(n+1)}\right) = 1 - \frac{1}{4^n}\binom{2n}{n}\frac{2n+1}{2(n+1)} \\ &= 1 - \frac{1}{4^{n+1}}\binom{2n}{n}\frac{(2n+1)(2n+2)}{(n+1)^2} = 1 - \frac{1}{4^{n+1}}\binom{2n+2}{n+1},\end{aligned}$$

which proves the identity.

Also solved by PAUL BRACKEN, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; STAN WAGON, Macalester College, St. Paul, Minnesota, USA; and the proposer.

Janous finds the sum by considering the generating function of the sequence $\{C_k\} = \left\{\frac{1}{k+1}\binom{2k}{k}\right\}$ *of Catalan numbers and in doing so, obtains, as a by-product, the recurrence relation*

$$C_n = \frac{1}{n+1}\left(4^n - 2\sum_{k=0}^{n-1} C_k \cdot 4^{n-1-k}\right).$$

He believes that this is a new identity, and wonders whether there is a purely combinatorial proof and/or interpretation of it.

Wagon obtained the sum by using **MATHEMATICA**, *and remarked that more sophisticated symbolic algebra software (work of Petkovsek, Wilf and Zeilberger) can provide certificates that the closed-form formula obtained is actually valid. He also commented that:* "Thus sums such as this are really best left to computers, . . . ". *This editor is interested in knowing how many of our readers would agree with him.*

YEAR END FINALE

Again, a year has flown by! This year saw the merger of ***CRUX*** and ***MAYHEM*** come to fruition. The logistical problems were greater than I expected, and we were slow at getting the first few issues to press. But now, it seems to flow smoothly. Thanks to all for their patience and assistance.

The online version of ***CRUX with MAYHEM*** continues to attract attention. Thanks are due to LOKI JORGENSON, NATHALIE SINCLAIR, and the rest of the team at SFU who are responsible for this.

There are many people that I wish to thank most sincerely for particular contributions. First and foremost is BILL SANDS. Bill is of such value to me and to the continuance of ***CRUX with MAYHEM***. As well, I thank most sincerely, CATHY BAKER, ROLAND EDDY, CHRIS FISHER, BILL SANDS, JIM TOTTEN, and EDWARD WANG, for their regular yeoman service in assessing which solutions should be highlighted; DENIS HANSON, D. FARENICK, C. FISHER, A. LIU, R. MCINTOSH, J. MACLAREN, D. RUOFF, M. TSATSOMEROS, H. WESTON, for ensuring that we have quality articles; ANDY LIU, ROBERT GERETSCHLÄGER, MURRAY S. KLAMKIN, MARÍA FALK de LOSADA, JÓSZEF PELIKÁN, GOTTFRIED PERZ, JIM TOTTEN, for ensuring that we have quality book reviews, ROBERT WOODROW (and JOANNE LONGWORTH), who carries the heavy load of two corners, one somewhat new and the other of long standing, and RICHARD GUY for sage advice whenever necessary. The editors of the ***MAYHEM*** section, NAOKI SATO, CYRUS HSIA, ADRIAN CHAN, RICHARD HOSHINO, RAVI VAKIL and WAI LING YEE, all do a sterling job. I also thank two of our regulars who assist the editorial board with proof reading; THEODORE CHRONIS and WALDEMAR POMPE. The quality of these people are vital parts of what makes ***CRUX with MAYHEM*** what it is. Thank you one and all.

As well, I would like to give special thanks to our retiring Associate Editor, CLAYTON HALFYARD, for keeping me from printing too many typographical errors; and my colleagues, PETER BOOTH, RICHARD CHARRON, EDGAR GOODAIRE, ERIC JESPERS, MIKE PARMENTER, DONALD RIDEOUT, NABIL SHALABY, in the Department of Mathematics and Statistics at Memorial University for their occasional sage advice. I have also been helped by some Memorial University students, DON HENDER, COLIN HILLIER, PAUL MARSHALL, SHANNON SULLIVAN, as well as a WISE Summer student, ALYSON FORD. The staff of the Department of Mathematics and Statistics at Memorial University deserve special mention for their excellent work and support: ROS ENGLISH, MENIE FRENCH, WANDA HEATH, and LEONCE MORRISSEY; as well as the computer and networking expertise of RANDY BOUZANE. Also the assistance of ELLEN WILSON at Mount Allison University is much appreciated. Not to mention GRAHAM WRIGHT, Managing Editor, would be a travesty. Graham has kept so much on the right track. He is a pleasure to work with. The CMS's TEX Editor, MICHAEL DOOB has been very helpful in ensuring that the printed master copies are up to the standard required for the U of T Press who continue to print a fine product. Finally, I would like to express real and heartfelt thanks to the Heads of my Department, BRUCE WATSON and HERBERT GASKILL, and to ALAN LAW, Dean of Science of Memorial University, and WILLIE DAVIDSON, Acting Dean of Science of Memorial University, without whose support and understanding, I would not be able to do the job of Editor-in-Chief.

Last but not least, I send my thanks to you, the readers of ***CRUX***. Without you, ***CRUX*** would not be what it is. Keep those contributions and letters coming in. I do enjoy knowing you all.

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