THE ACADEMY CORNER

No. 12

Bruce Shawyer

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In the last issue, we printed the Bernoulli Trials. Here are the hints and answers.

Bernoulli Trials 1997

HINTS — 1

1. The curve is a circle centred at $(1, -1)$ with radius 5.
2. Write the two series in terms of the $\log$ function.
3. For any integer m, one of the numbers m, $m + 1$ and $m + 2$ must be divisible by three.
4. Find the highest power of 5 which divides $1000!$
5. A real polynomial of odd degree has a positive and finite number of roots.
6. Show that, in fact, the probability equals $1/2$.
7. Show that $\lim\limits_{x\to 0+} x^{(2p)} = 1$ and $\lim\limits_{x\to 0+} x^{(2p+1)} = 0$.
8. Expand $(x - a_1)(x - a_2)\ldots(x - a_n)$.
9. Show that equality can be obtained.
10. By an elementary application of the Pigeonhole Principle, there will eventually be two pairs of adjacent Fibonacci numbers, say f_m, f_{m+1} and f_n, f_{n+1} whose last four digits respectively are the same. That is, f_m and f_n are congruent modulo 10^4, and similarly, so are f_{m+1} and f_{n+1}.
11. Factor the denominator as $(k^2 + k + 1)(k^2 - k + 1)$ and use partial fractions.

12. The number 1997 is prime and congruent to 1 modulo 4.
13. There is more than one way to skin a cat: Multiply top and bottom by $1 - \sin x$.
14. Make the substitution $x = y^p$ to turn this into a polynomial.
15. Show that f has period $2a$.

HINTS — 2

1. Show that the Euclidean distance from $(1, -1)$ to the given point is greater than 5.
2. Show that $-\log(1-p) = \log(1+q)$.
3. Note that 2^n is never divisible by three.
4. Use Legendre's formula for the prime factorization of $n!$
5. Consider the set $\{(t, t^3, t^5) : -\infty < t < +\infty\}$.
6. Write the number of heads that Johann gets in 1997 tosses as x and the number of heads that Jakob gets in 1996 tosses as y. Compare the distribution of coins for the two Bernoulli sequences with the distribution with heads and tails reversed.

   Note that $x > y$ if and only if $1997 - x \leq 1996 - y$.
7. Show that $x^p \log x \to 0$ as $x \to 0$ for all $p > 0$. Use induction.
8. Expanding out, we see that the elementary symmetric functions appear as the coefficients of the polynomial. Suppose the a_i's are distinct. Show that if any a_i is negative, then $x > 0$ can be chosen so that the polynomial is negative.
9. Symmetry suggests trying the substitution $a = c$ to find a counterexample. [This does not work!] To investigate this question further, prove the inequality is true when strict inequality is relaxed to $\geq$. The inequality is equivalent to $[h(a, b, c)]^2 \geq 0$ for an appropriate function h.
10. Having found two pairs of adjacent Fibonacci numbers with the same last four digits, now work backwards. Compare f_{m-1} and f_{n-1} etc. Keep going.
11. Note that

$$\frac{2k}{k^4 + k^2 + 1} = \frac{1}{k^2 - k + 1} - \frac{1}{k^2 + k + 1}$$

   Using partial fractions, show that the sum can be written as a telescoping sum.

12. Apply Fermat's Great Theorem.

13. There is more than one way to skin a cat: use integration by parts and the transformation $t = \tan(x/2)$.

14. Descartes Rule of Signs tells us that if z is the number of positive zeros of a polynomial, and c is the number of changes of sign of the sequence of coefficients, then $c - z$ is a nonnegative even number.

15. Show that

$$[f(x+a)]^2 - f(x+a) = -[f(x) - 1/2]^2.$$

Then write out a formula for $f(x+2a)$ using the original equation given in the question.

ANSWERS

1.	True	2.	True	3.	False	4.	True	5.	False
6.	False	7.	False	8.	True	9.	False	10.	False
11.	True	12.	True	13.	False	14.	True	15.	True

Comment

For the information of readers, to the knowledge of the editor, other versions of the Bernoulli Trials have been held at least in his home institution and at the Canadian IMO Residential Training Program. In the latter instance, local high school students were added to the list of competitors. On both occasions, all the competitors reported that it was an enjoyable experience, and a pleasant social event as well.

THE OLYMPIAD CORNER

No. 183

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin with the 1997 Canadian Mathematical Olympiad which we reproduce with the permission of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society. Thanks go to Daryl Tingley, University of New Brunswick, and Chair of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society, for forwarding the questions to me.

PROBLEMS FROM THE 1997 CANADIAN MATHEMATICAL OLYMPIAD

1. How many pairs of positive integers x, y are there, with $x \leq y$, and such that $\gcd(x, y) = 5!$ and $\operatorname{lcm}(x, y) = 50!$?

Note:

$\gcd(x, y)$ denotes the greatest common divisor of x and y,
$\operatorname{lcm}(x, y)$ denotes the least common multiple of x and y,
and $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

2. The closed interval $A = [0, 50]$ is the union of a finite number of closed intervals, each of length 1. Prove that some of the intervals can be removed so that those remaining are mutually disjoint and have total length ≥ 25.

Note: For $a \leq b$, the closed interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ has length $b - a$; disjoint intervals have *empty* intersection.

3. Prove that

$$\frac{1}{1999} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{1997}{1998} < \frac{1}{44}.$$

4. The point O is situated inside the parallelogram $ABCD$ so that

$$\angle AOB + \angle COD = 180^\circ.$$

Prove that $\angle OBC = \angle ODC$.

5. Write the sum

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k^3 + 9k^2 + 26k + 24}$$

in the form $\frac{p(n)}{q(n)}$, where p and q are polynomials with integer coefficients.

The next set of problems are from the twenty-sixth annual United States of America Mathematical Olympiad written May 1, 1997. These problems are copyrighted by the Committee on the American Mathematical Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems may be obtained from Professor Walter E. Mientka, AMC Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, USA 68588–0322. As always, we welcome your original, "nice" solutions and generalizations which differ from the published solutions.

26th UNITED STATES OF AMERICA MATHEMATICAL OLYMPIAD

Part I 9 a.m. - 12 noon

May 1, 1997

1. Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k, define

$$x_k = \begin{cases} 0 & if\ x_{k-1} = 0, \\ \left\{\dfrac{p_k}{x_{k-1}}\right\} & if\ x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. (The fractional part of x is given by $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

2. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines $\overleftrightarrow{EF}, \overleftrightarrow{FD}, \overleftrightarrow{DE}$, respectively, are concurrent.

3. Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \ldots, 9\}$ such that $Q(-2) = Q(-5) = n$.

Part II 1 p.m. - 4 p.m.

4. To *clip* a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than $1/3$, for all $n \geq 6$.

5. Prove that, for all positive real numbers a, b, c,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (c^3 + a^3 + abc)^{-1} \leq (abc)^{-1}.$$

6. Suppose the sequence of nonnegative integers $a_1, a_2, \ldots, a_{1997}$ satisfies

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$$

for all $i, j \geq 1$ with $i + j \leq 1997$. Show that there exists a real number x such that $a_n = \lfloor nx \rfloor$ (the greatest integer $\leq nx$) for all $1 \leq n \leq 1997$.

As a third Olympiad for this issue we give selected problems of the 3rd Ukrainian Mathematical Olympiad, written March 26 and 27, 1994. My thanks go to Richard Nowakowski, Canadian Team leader to the IMO in Hong Kong for collecting the set.

3rd UKRAINIAN MATHEMATICAL OLYMPIAD
Selected Problems
March 26–27, 1994

1. (class 9) A convex quadrangle $ABCD$ is given. Bisectors of external angles of $ABCD$ form a new quadrangle $PQRS$. Prove that sum of diagonals of $PQRS$ is more than the perimeter of $ABCD$.

2. (9–10) A convex polygon and point O inside it are given. Prove that for any $n > 1$ there exist points $A_1, A_2, \ldots, A_n$ on the sides of the polygon such that $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \ldots + \overrightarrow{OA_n} = \overrightarrow{O}$.

3. (10) A sequence of natural numbers a_k, $k \geq 1$, such that for each k, $a_k < a_{k+1} < a_k + 1993$ is given. Let all prime divisors of a_k be written for every k. Prove that we receive an infinite number of different prime numbers.

4. (10) Inside the triangle ABC the point D is given such that angles $\angle ABC$ and $\angle DBC$ are equal. Let K and L be the projections of D on lines (AC) and (BD) respectively. Prove that the midpoints of AB, CD and KL are collinear.

5. (11) 1994^2 squares 1×1 of k different colours are given. Find all natural numbers k for which it is always possible to construct a 1994×1994 square coloured symmetrically with respect to the diagonal from any collection of 1994^2 such squares.

6. (11) A mathematical olympiad consists of two days. The Jury intends to place the participants in rooms so that any two students will be in rooms with different numbers of participants in one of the days. Is it possible for

(a) 9 participants?

(b) 14 participants?

(The Jury has the necessary number of rooms.)

7. (11) Three points P, Q and S in space are given. Two rays from P and two rays from Q are taken such that each ray from P intersects two rays from Q. Let A, B, C, D be the intersection points of these four rays. It is known that pyramid $SABCD$ has a section $KLMN$ of rectangular form ($K \in SA$, $L \in SB$, $M \in SC$, $N \in SD$). The area of $ABCD$ equals 1. Prove that the volume of $SABCD$ is not larger than $PQ/6$.

8. (11) Two circles with radii R and r ($R > r$) in the plane are tangent at point M with one circle inside the other. Chord AB of the bigger circle is tangent to the smaller circle. Find the largest possible value of the perimeter of ABM.

Eagle-eyed readers Dan Velleman, Stan Wagon and Andy Liu spotted trouble with problem 4 of the Swedish Mathematics Contest 1993 as we gave it in the last number of the *Corner*. Indeed, as they point out, setting $a*b = 1$ for all a, b satisfies the conditions of the problem, but does not give a unique solution! I must not have been seeing stars when I proof-read that one. Here is the correct version to try.

4. *Swedish Mathematics Contest 1993* [1997: 196]

To each pair of real numbers a and b, where $a \neq 0$ and $b \neq 0$, there is a real number $a * b$ such that

$$a * (b * c) = (a * b)c$$

$$a * a = 1.$$

Solve the equation $x * 36 = 216$.

In somewhat of a departure from past practice we now give the solutions given to the Canadian Mathematical Olympiad given earlier this issue! (I hope you have solved them by now.) The "official solutions" provided for me by Daryl Tingley, University of New Brunswick, and Chair of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society, feature solutions by contestants in the CMO itself. Each solution is credited to one of the contestants.

PROBLEM SOLUTIONS FOR THE 1997 CANADIAN MATHEMATICAL OLYMPIAD

1. How many pairs of positive integers x, y are there, with $x \leq y$, and such that gcd $(x, y) = 5!$ and lcm $(x, y) = 50!$?

Note:

gcd (x, y) denotes the greatest common divisor of x and y,
lcm (x, y) denotes the least common multiple of x and y,
and $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

Solution by Deepee Khosla, Lisgar Collegiate Institute, Ottawa, ON

Let $p_1, \ldots, p_{12}$ denote, in increasing order, the primes from 7 to 47. Then

$$5! = 2^3 \cdot 3^1 \cdot 5^1 \cdot p_1^0 \cdot p_2^0 \cdots \cdot p_{12}^0$$

and

$$50! = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot p_1^{b_1} \cdot p_2^{b_2} \cdots \cdot p_{12}^{b_{12}}.$$

Note that $2^4, 3^2, 5^2, p_1, \ldots, p_{12}$ all divide $50!$, so all its prime powers differ from those of 5!

Since $x, y | 50!$, they are of the form

$$\begin{aligned} x &= 2^{n_1} \cdot 3^{n_2} \cdot \cdots \cdot p_{12}^{n_{15}} \\ y &= 2^{m_1} \cdot 3^{m_2} \cdot \cdots \cdot p_{12}^{m_{15}}. \end{aligned}$$

Then $\max(n_i, m_i)$ is the ith prime power in $50!$ and $\min(n_i, m_i)$ is the ith prime power in $5!$

Since, by the above note, the prime powers for p_{12} and under differ in $5!$ and $50!$, there are 2^{15} choices for x, only half of which will be less than y. (Since for each choice of x, we have that y is forced and either $x < y$ or $y < x$.) So the number of pairs is $2^{15}/2 = 2^{14}$.

2. The closed interval $A = [0, 50]$ is the union of a finite number of closed intervals, each of length 1. Prove that some of the intervals can be removed so that those remaining are mutually disjoint and have total length greater than or equal to 25.

Note: For $a \leq b$, the closed interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ has length $b - a$; disjoint intervals have *empty* intersection.

Solution by Byung-Kuy Chun, Harry Ainlay Composite High School, Edmonton, AB.

Look at the first point of each given unit interval. This point uniquely defines the given unit interval.

Lemma. In any interval $[x, x+1)$ there must be at least one of these first points $(0 \le x \le 49)$.

Proof. Suppose the opposite. The last first point before x must be $x - \varepsilon$ for some $\varepsilon > 0$. The corresponding unit interval ends at $x - \varepsilon + 1 < x + 1$. However, the next given unit interval cannot begin until at least $x + 1$.

This implies that points $(x - \varepsilon + 1,\ x + 1)$ are not in set A, a contradiction.

Therefore there must be a first point in $[x,\ x+1)$. ∎

Note that for two first points in intervals $[x, x+1)$ and $[x+2, x+3)$ respectively, the corresponding unit intervals are disjoint, since the intervals are in the range $[x, x+2)$ and $[x+2, x+4)$ respectively.

Thus, we can choose a given unit interval that begins in each of

$$[0,1)\ [2,3)\ \ldots\ [2k, 2k+1)\ \ldots\ [48,49).$$

Since there are 25 of these intervals, we can find 25 points which correspond to 25 disjoint unit intervals.

Solution by Colin Percival, Burnaby Central Secondary School, Burnaby, BC.

I prove the more general result, that if $[0, 2n] = \bigcup_i A_i, |A_i| = 1, A_i$ are intervals then $\exists\ a_1 \ldots a_n$, such that $A_{a_i} \cap A_{a_j} = \emptyset$.

Let $0 < \varepsilon \le \frac{2}{n-1}$ and let $b_i = (i-1)(2+\varepsilon), i = 1 \ldots n$. Then

$$\begin{aligned} \min\{b_i\} = 0, \max\{b_i\} &= (n-1)(2+\varepsilon) \le (n-1)\left(2 + \frac{2}{n-1}\right) \\ &= (n-1)\left(\frac{2n}{n-1}\right) = 2n. \end{aligned}$$

So all the b_i are in $[0, 2n]$.

Let a_i be such that $b_i \in A_{a_i}$. Since $\bigcup A_i = [0, 2n]$, this is possible.

Then since $(b_i - b_j) = (i-j)(2+\varepsilon) \ge 2+\varepsilon > 2\ (i > j)$, and the A_i are intervals of length 1, $\min A_{a_i} - \max A_{a_j} > 2 - 1 - 1 = 0$, so $A_{a_i} \cap A_{a_j} = \emptyset$.

Substituting $n = 25$, we get the required result.

3. Prove that

$$\frac{1}{1999} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{1997}{1998} < \frac{1}{44}.$$

Solution by Mihaela Enachescu, Dawson College, Montréal, PQ.

Let $P = \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{1997}{1998}$. Then $\frac{1}{2} > \frac{1}{3}$ [because $2 < 3$], $\frac{3}{4} > \frac{3}{5}$ [because $4 < 5$], ..., ... $\frac{1997}{1998} > \frac{1997}{1999}$ [because $1998 < 1999$].

So

$$P > \frac{1}{3} \cdot \frac{3}{5} \cdot \ldots \cdot \frac{1997}{1999} = \frac{1}{1999}. \qquad (1)$$

Also $\frac{1}{2} < \frac{2}{3}$ [because $1 \cdot 3 < 2 \cdot 2$], $\frac{3}{4} < \frac{4}{5}$ [because $3 \cdot 5 < 4 \cdot 4$], ..., $\frac{1997}{1998} < \frac{1998}{1999}$ [because $1997 \cdot 1999 = 1998^2 - 1 < 1998^2$].

So $P < \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{1998}{1999} = \underbrace{\left(\frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \ldots \cdot \frac{1998}{1997}\right)}_{1/P} \frac{1}{1999}$.

Hence $P^2 < \frac{1}{1999} < \frac{1}{1936} = \frac{1}{44^2}$ and $P < \frac{1}{44}$. $\qquad (2)$

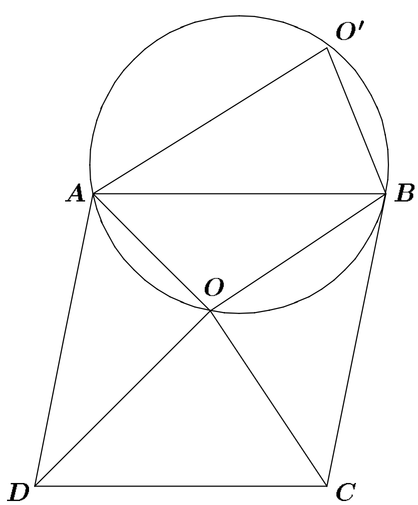
Then (1) and (2) give $\frac{1}{1999} < P < \frac{1}{44}$.

4. The point O is situated inside the parallelogram $ABCD$ so that

$$\angle AOB + \angle COD = 180^\circ .$$

Prove that $\angle OBC = \angle ODC$.

Solution by Joel Kamnitzer, Earl Haig Secondary School, North York, ON.



Consider a translation which maps D to A. It will map $O \to O'$ with $\overrightarrow{OO'} = \overrightarrow{DA}$, and C will be mapped to B because $\overrightarrow{CB} = \overrightarrow{DA}$.

This translation keeps angles invariant, so $\angle AO'B = \angle DOC = 180^\circ - \angle AOB$.

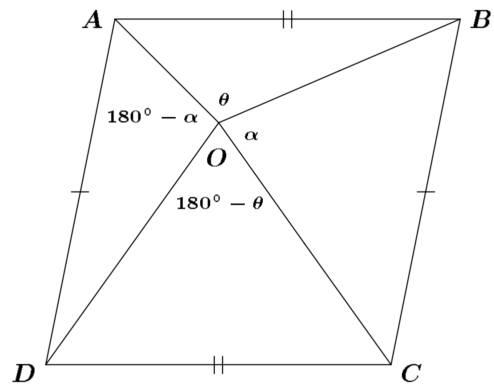
Therefore $AOBO'$ is a cyclic quadrilateral. And further, $\angle ODC = \angle O'AB = \angle O'OB$. But, since $O'O$ is parallel to BC,

$$\angle O'OB = \angle OBC.$$

Therefore

$$\angle ODC = \angle OBC.$$

Solution by Adrian Chan, Upper Canada College, Toronto, ON.



Let $\angle AOB = \theta$ and $\angle BOC = \alpha$. Then $\angle COD = 180^\circ - \theta$ and $\angle AOD = 180^\circ - \alpha$.

Since $AB = CD$ (parallelogram) and $\sin\theta = \sin(180^\circ - \theta)$, the sine law on $\triangle OCD$ and $\triangle OAB$ gives

$$\frac{\sin\angle CDO}{OC} = \frac{\sin(180^\circ - \theta)}{CD} = \frac{\sin\theta}{AB} = \frac{\sin\angle ABO}{OA}$$

so

$$\frac{OA}{OC} = \frac{\sin\angle ABO}{\sin\angle CDO}. \tag{1}$$

Similarly, the Sine Law on $\triangle OBC$ and $\triangle OAD$ gives

$$\frac{\sin\angle CBO}{OC} = \frac{\sin\alpha}{BC} = \frac{\sin(180^\circ - \alpha)}{AD} = \frac{\sin\angle ADO}{OA}$$

so

$$\frac{OA}{OC} = \frac{\sin\angle ADO}{\sin\angle CBO}. \tag{2}$$

Equations (1) and (2) show that

$$\sin\angle ABO \cdot \sin\angle CBO = \sin\angle ADO \cdot \sin\angle CDO.$$

Hence

$$\begin{aligned}
&\frac{1}{2}[\cos(\angle ABO + \angle CBO) - \cos(\angle ABO - \angle CBO)] \\
&\quad = \frac{1}{2}[\cos(\angle ADO + \angle CDO) - \cos(\angle ADO - \angle CDO)].
\end{aligned}$$

Since $\angle ADC = \angle ABC$(parallelogram) and $\angle ADO + \angle CDO = \angle ADC$ and $\angle ABO + \angle CBO = \angle ABC$ it follows that

$$\cos(\angle ABO - \angle CBO) = \cos(\angle ADO - \angle CDO).$$

There are two cases to consider.

Case (i): $\angle ABO - \angle CBO = \angle ADO - \angle CDO$.

Since $\angle ABO+\angle CBO = \angle ADO+\angle CDO$, subtracting gives $2\,\angle CBO = 2\,\angle CDO$ so $\angle CBO = \angle CDO$, and we are done.

Case (ii): $\angle ABO - \angle CBO = \angle CDO - \angle ADO$.

Since we know that $\angle ABO+\angle CBO = \angle CDO+\angle ADO$, adding gives $2\,\angle ABO = 2\,\angle CDO$ so $\angle ABO = \angle CDO$ and $\angle CBO = \angle ADO$.

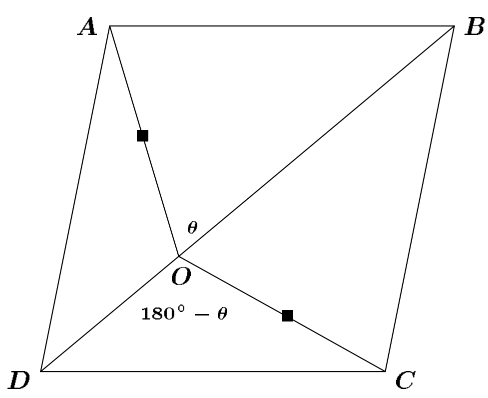
Substituting this into (1), it follows that $OA = OC$.

Also, $\angle ADO + \angle ABO = \angle CBO + \angle ABO = \angle ABC$.

Now, $\angle ABC = 180° - \angle BAD$ since $ABCD$ is a parallelogram.

Hence $\angle BAD + \angle ADO + \angle ABO = 180°$ so $\angle DOB = 180°$ and D, O, B are collinear.

We now have the diagram



Then $\angle COD + \angle BOC = 180°$, so $\angle BOC = \theta = \angle AOB$.

$\triangle AOB$ is congruent to $\triangle COB$ (SAS, OB is common, $\angle AOB = \angle COB$ and $AO = CO$), so $\angle ABO = \angle CBO$. Since also $\angle ABO = \angle CDO$ we conclude that $\angle CBO = \angle CDO$.

Since it is true in both cases, then $\angle CBO = \angle CDO$.

5. Write the sum

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k^3 + 9k^2 + 26k + 24},$$

in the form $\frac{p(n)}{q(n)}$, where p and q are polynomials with integer coefficients.

Solution by Sabin Cautis, Earl Haig Secondary School, North York, ON.

We first note that

$$k^3 + 9k^2 + 26k + 24 = (k+2)(k+3)(k+4).$$

Let $S(n) = \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k^2 + 9k^2 + 26k + 24}$.

Then

$$\begin{aligned} S(n) &= \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!(k+2)(k+3)(k+4)} \\ &= \sum_{k=0}^{n} \left(\frac{(-1)^k (n+4)!}{(k+4)!(n-k)!} \right) \times \left(\frac{k+1}{(n+1)(n+2)(n+3)(n+4)} \right). \end{aligned}$$

Let

$$T(n) = (n+1)(n+2)(n+3)(n+4)S(n) = \sum_{k=0}^{n} \left((-1)^k \binom{n+4}{k+4} (k+1) \right).$$

Now, for $n \geq 1$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \qquad (*)$$

since

$$(1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \ldots + (-1)^n \binom{n}{n} = 0.$$

Also

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i\binom{n}{i}i &= \sum_{i=1}^{n}(-1)^i\frac{i\cdot n!}{i!\cdot(n-i)!}+(-1)^0\cdot\frac{0\cdot n!}{0!\cdot n!}\\ &= \sum_{i=1}^{n}(-1)^i\frac{n!}{(i-1)!(n-i)!}\\ &= \sum_{i=1}^{n}(-1)^i n\binom{n-1}{i-1}\\ &= n\sum_{i=1}^{n}(-1)^i\binom{n-1}{i-1}\\ &= -n\sum_{i=1}^{n}(-1)^{i-1}\binom{n-1}{i-1}.\end{aligned}$$

Substituting $j = i - 1$, (*) shows that

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}i = -n\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j} = 0. \qquad (**)$$

Hence

$$\begin{aligned}T(n) &= \sum_{k=0}^{n}(-1)^k\binom{n+4}{k+4}(k+1)\\ &= \sum_{k=0}^{n}(-1)^{k+4}\binom{n+4}{k+4}(k+1)\\ &= \sum_{k=-4}^{n}(-1)^{k+4}\binom{n+4}{k+4}(k+1)\\ &\quad -\left(-3+2(n+4)-\binom{n+4}{2}\right).\end{aligned}$$

Substituting $j = k + 4$, gives

$$\begin{aligned}T(n) &= \sum_{j=0}^{n+4}(-1)^j\binom{n+4}{j}(j-3)-\left(2n+8-3-\frac{(n+4)(n+3)}{2}\right)\\ &= \sum_{j=0}^{n+4}(-1)^j\binom{n+4}{j}j\\ &\quad -3\sum_{j=0}^{n+4}(-1)^j\binom{n+4}{j}-\frac{1}{2}(4n+10-n^2-7n-12)\end{aligned}$$

The first two terms are zero because of results $(*)$ and $(**)$ so

$$T(n) = \frac{n^2+3n+2}{2}.$$

Then

$$\begin{aligned} S(n) &= \frac{T(n)}{(n+1)(n+2)(n+3)(n+4)} \\ &= \frac{n^2+3n+2}{2(n+1)(n+2)(n+3)(n+4)} \\ &= \frac{(n+1)(n+2)}{2(n+1)(n+2)(n+3)(n+4)} \\ &= \frac{1}{2(n+3)(n+4)}. \end{aligned}$$

Therefore

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k^3+9k^2+26k+24} = \frac{1}{2(n+3)(n+4)}.$$

That completes the *Corner* for this issue. Send me Olympiad contests and your nice solutions and generalizations for use in the *Corner*.

Congratulations!

The following contributors to ***CRUX with MAYHEM*** distinguished themselves at the 38th IMO held in Argentina:

Name	Country	Award
Florian Herzig	Austria	Gold Medal
Sabin Cautis	Canada	Bronze Medal
Adrian Chan	Canada	Silver Medal
Byung Kyu Chun	Canada	Silver Medal
Mansur Boase	United Kingdom	Silver Medal
Toby Gee	United Kingdom	Bronze Medal
Carl Bosley	USA	Gold Medal

CRUX with MAYHEM offers hearty congratulations to these students. We trust that their participation in solving problems in ***CRUX with MAYHEM*** was a contributory factor in their successes.

BOOK REVIEWS

Edited by ANDY LIU

The Puzzle Arcade, by **Jerry Slocum**,
published in 1996 by Klutz, 2121 Staunton Court, Palo Alto, CA 94306, USA.,
ISBN# 0-888075-851-4, coil bound, 48 pages, US$20 plus handling.

Quantum Quandaries, edited by **Timothy Weber**,
published in 1996 by the National Science Teachers' Association,
1840 Wilson Boulevard, Arlington, VA 22201-3000, USA.,
ISBN# 0-87355-136-2, softcover, 208 pages, US$7.95 plus handling.

Reviewed by Andy Liu, University of Alberta.

These two books are the most delightful additions to the literature of popular mathematics recently, and should appeal to everyone.

Jerry Slocum, a retired executive in the aircraft industry, has a collection of mechanical puzzles numbering over 22000. *The Puzzle Arcade* features a very small portion of this amazing trove. The book is profusely illustrated with striking colours, and comes with several pockets containing either complete puzzles or equipment needed for others. Apart from the standard tangram-like puzzles, matchstick puzzles, topological entanglement puzzles, optical illusions, mazes and word puzzles, there are many unlikely to be familiar to the readers.

The readers are asked to put together two identical polyhedral pieces to form a familiar object. This is perhaps the most annoying of the puzzles with only two pieces. A three-piece puzzle features Sam Loyd's tantalizing ponies. There is also a puzzle with four or five pieces. With four, the readers can form the back of a playing card, but it takes all five to come up with the King of Hearts. The list goes on and on, and the readers simply have to see the book to relish it. Hints and answers are provided. The publisher has done a fantastic job packing so much into a relatively compact offering. Purchased separately, the puzzles will cost many times more than this bargain of a book!

Timothy Weber of the NSTA is the managing editor of the magazine *Quantum*. It is the sister publication of the Russian magazine *Kvant*, a Russian word which means quantum. About two-thirds of the material in *Quantum* are translated from *Kvant*, the rest being original contributions in English. About two-thirds of the material are mathematical, the rest being in physics. It is published six times a year by Springer-Verlag New York Inc., P.O.Box 2485, Secaucas, NJ 07096-2485, USA.. Student subscription at US$18 per volume is a steal.

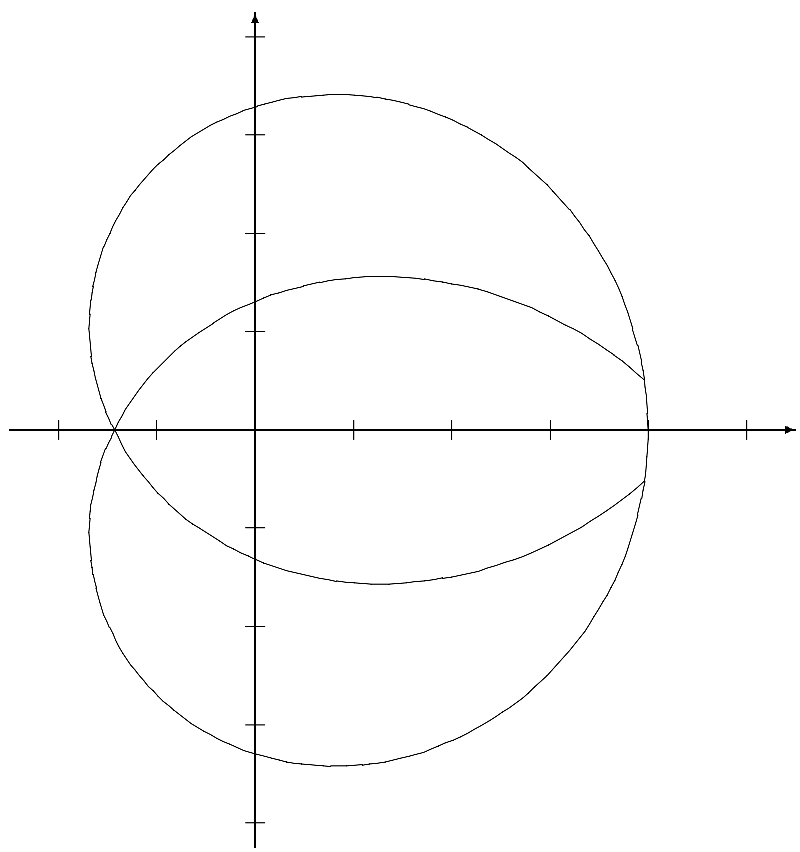
One of the most popular columns in *Quantum* is the **Brainteasers**. There are five in each issue, usually four in mathematics and one in physics. Many are presented in whimsical style, and there are always the most delightful cartoon illustrations. *Quantum Quandaries*, a pocket-sized book, contains the first 100 Brainteasers and their solutions. Each Brainteaser is on one page, with its solution overleaf for easy reference. Additional cartoon illustrations are provided by Sergey Ivanov. The only regret is that the book is in black-and-white, whereas the original presentation in the magazine was in full colour.

We conclude this review with a sample Brainteaser. Solve the alphametic equation

USA+USSR=PEACE

where letters stand for different digits if and only if they are different.

What are the parametric equations, and the range of the parameter, of the following curve?



A Fermat-Fibonacci Collaboration

K.R.S. Sastry

Fermat begins

Fermat the mischievous proposed the impossible: Find three squares in arithmetic progression (AP), the common difference (cd) being a square, [Dickson [1] pp. 435]. Throughout this paper we deal with integer squares but, we find that, what is impossible for squares is possible for triangles: 6, 21, 36 are triangular numbers in AP with cd of 15 also being triangular! Let us see how we can generate triples of triangular numbers in AP whose cd is also triangular. Curiously, the supplier is Fibonacci!

We recall that the nth triangular number is given by $T_n = \dfrac{1}{2}n(n+1)$. We also note that $8T_n + 1 = (2n+1)^2$ is a perfect square, an odd square to be exact. Simply retrace the preceding steps to see that the converse, every odd square leads to a triangular number, holds. Furthermore, multiplying each term of an AP by 8 and then adding 1 to them results in a new AP, but the new cd is just eight times the old. Take an example:

$$AP\text{: } 2,\ 7,\ 12 \text{ results in the } AP\text{: } 17,\ 57,\ 97 \text{ with } cd\text{'s of } 5 \text{ and } 40. \qquad (1)$$

The upshot of all this is that the triangular numbers in AP can be made to depend on the square numbers in AP.

An expression for squares in AP

Suppose the squares x^2, y^2, z^2 are in AP. Then $x^2 + z^2 = 2y^2$. We may assume x, y, z are relatively prime in pairs and that $x < y < z$ holds. Hence x and z are both odd. Put $x = p - q$, $z = p + q$ where p and q are relatively prime integers having opposite parity. Then we have the famous Pythagorean relation $p^2 + q^2 = y^2$ whose primitive solutions are well known: $p = m^2 - n^2$, $q = 2mn$, $y = m^2 + n^2$. Here m and n are relatively prime integers with opposite parity. This yields

$$x = |m^2 - n^2 - 2mn|, \quad y = m^2 + n^2, \quad z = m^2 - n^2 + 2mn. \qquad (2)$$

The cd: $y^2 - x^2$, equals $4mn(m^2 - n^2)$. This cannot be a square in view of Fermat's proof of the fact $mn(m^2 - n^2)$, the area of the (Pythagorean) right-angled triangle with sides $m^2 - n^2$, $2mn$, $m^2 + n^2$ cannot be a square [Dickson [1] pp. 615-616].

Triangular numbers in AP whose cd is triangular

Suppose the triangular numbers

$$\begin{aligned} T_a &= \frac{1}{2}a(a+1), \\ T_b &= \frac{1}{2}b(b+1), \\ T_c &= \frac{1}{2}c(c+1), \end{aligned}$$

are in AP. Then

$$\begin{aligned} 8T_a+1 &= (2a+1)^2 = A^2, \\ 8T_b+1 &= (2b+1)^2 = B^2, \\ 8T_c+1 &= (2c+1)^2 = C^2, \end{aligned}$$

are three **odd** squares in AP. Hence from the expressions in (2)

$$A = |m^2 - n^2 - 2mn|, \quad B = m^2 + n^2, \quad C = m^2 - n^2 + 2mn. \qquad (3)$$

The cd of the AP squares A^2, B^2, C^2 is $4mn(m^2 - n^2)$. Therefore, from the remark in (1), the triangular numbers T_a, T_b, T_c in AP have cd:

$$\frac{1}{8}(4mn(m^2 - n^2)) = \frac{1}{2}mn(m^2 - n^2).$$

The very form of this cd suggests that we choose $m^2 - n^2 = mn \pm 1$. In that case the cd:

$$\frac{1}{2}mn(m^2 - n^2) = \frac{1}{2}mn(mn \pm 1)$$

would be triangular. Thus we seek, in natural numbers, the solution of the equation

$$m^2 - n^2 = mn \pm 1.$$

Fibonacci enters

Before solving the equation $m^2 - n^2 = mn \pm 1$, let us recall the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$$

given recursively by $F_1 = F_2 = 1$, $F_k = F_{k-1} + F_{k-2}$, $k > 2$. It is also given by Binet's formula

$$F_k = \frac{(1+\sqrt{5})^k + (1-\sqrt{5})^k}{2^k\sqrt{5}}, \quad k = 1, 2, 3, \cdots . \qquad (4)$$

Also, the Lucas sequence

$$1, 3, 4, 7, 11, 18, 29, \cdots$$

is likewise defined: $L_1 = 1$, $L_2 = 3$, $L_k = L_{k-1} + L_{k-2}$, $k > 2$. We leave it to the reader to deduce the following relations:

$$L_k = F_{k-1} + F_{k+1} = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k. \tag{5}$$

Now consider the solution of the equation $m^2 - n^2 = mn \pm 1$, put in the form $(2m-n)^2 - 5n^2 = \pm 4$.

The first few trial solutions of the equation $(2m-n)^2 - 5n^2 = -4$ are

$2m-n$	1	4	11	$\cdots$
n	1	2	5	$\cdots$
m	1	3	8	$\cdots$

It is already apparent that $2m - n = L_{2k-1}$, $m = F_{2k}$, $n = F_{2k-1}$, $k = 1, 2, 3, \cdots$. Direct calculations using (4) and (5) show that

$$L_{2k-1}^2 - 5F_{2k-1}^2 = -4, \quad k = 1, 2, 3, \cdots .$$

We leave the verification to the reader.

Likewise the first few trial solutions of the equation

$$(2m-n)^2 - 5n^2 = 4$$

are

$2m-n$	3	7	18	$\cdots$
n	1	3	8	$\cdots$
m	2	5	13	$\cdots$

This time we observe that $2m - n = L_{2k}$, $m = F_{2k+1}$, $n = F_{2k}$ and again we leave the verification that

$$L_{2k}^2 - 5F_{2k}^2 = 4, \quad k = 1, 2, \cdots$$

to the reader.

A numerical example

Choose the solution $m = 8$, $n = 5$. Then from (3), we have

$$\begin{aligned} 2a+1 &= A = |m^2 - n^2 - 2mn| = 41, \\ 2b+1 &= B = m^2 + n^2 = 89, \\ 2c+1 &= C = m^2 - n^2 + 2mn = 119. \end{aligned}$$

This yields $a = 20$, $b = 44$, $c = 59$, $T_a = 210$, $T_b = 990$, $T_c = 1770$ in AP whose cd, $780 = \frac{1}{2}(39)(40)$, is triangular too. The reader may have spotted a like parity solution $m = 5$, $n = 3$. This makes a, b, c non-integral, since A, B, C are even. This happens when $m = F_{6i-1}$, $n = F_{6i-2}$, $i = 1, 2, \cdots$, a fact easily shown by induction.

Fermat concludes

The next question to settle would be: Are there three pentagonal numbers in AP whose cd is pentagonal? The cases that have been settled are $n = 3$ and $n = 4$, instances of the general problem: For what values of n are there three n-gonal numbers in AP whose cd is n-gonal too? It may be recalled that the rth n-gonal number is

$$p(n, r) = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2}, n \geq 3, r = 1, 2, 3, \cdots .$$

For $n > 4$, is this Fermat's **newest** last problem?

Reference

[1] L.E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, NY, (1971), pp. 1-39, 435, 615-616.

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THE SKOLIAD CORNER

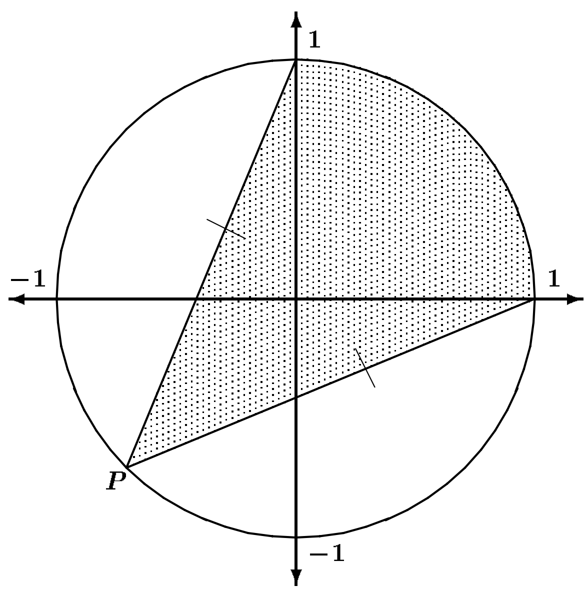
No. 23

R.E. Woodrow

This number we give the problems of the 1995 Prince Edward Island Mathematics Competition. Thanks go to Gordon MacDonald, University of Prince Edward Island, organizer of the contest, for supplying us with the problems, and the solutions which we will give next issue.

1995 P.E.I. Mathematics Competition

1. Find the area of the shaded region inside the circle in the following figure.



2. "I will be n years old in the year n^2", said Bob in the year 1995. How old is Bob?

3. Draw the set of points (x, y) in the plane which satisfy the equation $|x| + |x - y| = 4$.

4. An *autobiographical number* is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells you how many zeros are in the number, the second digit tells you how many 1's, the third digit tells you how many 2's, and so on. For example, $6,210,001,000$ is an autobiographical number. Find the smallest autobiographical number and prove that it is the smallest.

5. A solid cube of radium is floating in deep space. Each edge of the cube is exactly 1 kilometre in length. An astronaut is protected from its radiation if she remains at least 1 kilometre from the nearest speck of radium. Including the interior of the cube, what is the volume (in cubic kilometres) of space that is forbidden to the astronaut? (You may assume that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and the volume of a right circular cylinder of radius r and height h is $\pi r^2 h$.)

6. Which is greater, 999! or 500^{999}? (Where 999! denotes 999 factorial, the product of all the natural numbers from 1 to 999 inclusive.) Explain your reasoning.

Where do those errors come from?

Several readers wrote about the error in the solution of Problem 2(a) of the Manitoba Contest 1995 [1995: 219].

2. (a) Find two numbers which differ by 3 and whose squares differ by 63.

Corrections by Jamie Batuwantudawa, student, Fort Richmond Collegiate, Winnipeg, Manitoba; by Paul–Olivier Dehaye, Brussels, Belgium; and by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA.

The solution of the equations $6x + 9 = 63$ or $6x + 9 = -63$ is obvious: $x = 9$ or $x = -12$, and not $x = 7$ or $x = -8$ as given.

Next we give solutions for the problems of the Twelfth W.J. Blundon Contest, written February 22, 1995, [1997: 218–219].

THE TWELFTH W.J. BLUNDON CONTEST
February 22, 1995

1. (a) From a group of boys and girls, 15 girls leave. There are then left two boys for each girl. After this, 45 boys leave. There are then 5 girls for each boy. How many boys and how many girls were in the original group?

Solution. Call the number of boys b and the number of girls g. When 15 girls leave $b = 2(g - 15)$. After 45 boys leave $5(b - 45) = g - 15$. So $b = 2(5(b - 45))$, $b = 10b - 450$, $9b = 450$, $b = 50$; $g - 15 = 50/2 = 25$ and $g = 40$. Thus the total number of boys and girls was originally 90.

(b) A certain number of students can be accommodated in a hostel. If two students share each room then two students will be left without a room. If three students share each room then two rooms will be left over. How many rooms are there?

Solution. Let s be the number of students and r the number of rooms. At two students per room $2r = s - 2$. Consider three per room, then $3(r - 2) = s$. Now $2r = 3(r - 2) - 2$ so $r = 8$. There are 8 rooms (and 18 students).

2. How many pairs of positive integers (x, y) satisfy the equation

$$\frac{x}{19} + \frac{y}{95} = 1?$$

Solution. Note that $95 = 19 \times 5$. For $x = 1, 2, \ldots, 18$ we have

$$\frac{y}{95} = 1 - \frac{x}{19} = \frac{19 - x}{19} = \frac{95 - 5x}{95},$$

giving a solution. There are then 18 such pairs.

3. A book is to have 250 pages. How many times will the digit 2 be used in numbering the book?

Solution. Two is used once in each decade in the one's place for a total of 25 such occurrences. It is used 10 times in each hundred in the ten's place for another 30 occurrences. And it occurs 51 times in the hundred's place, for a grand total of $25 + 30 + 51 = 106$ times.

4. Without using a calculator

(a) Show that $\sqrt{7 + \sqrt{48}} + \sqrt{7 - \sqrt{48}}$ is a rational number.

Solution. Let $x = \sqrt{7 + \sqrt{48}} + \sqrt{7 - \sqrt{48}}$. Then

$$\begin{aligned} x^2 &= 7 + \sqrt{48} + 2\sqrt{7 + \sqrt{48}}\,\sqrt{7 - \sqrt{48}} + 7 - \sqrt{48} \\ &= 14 + 2\sqrt{49 - 48} = 16 \end{aligned}$$

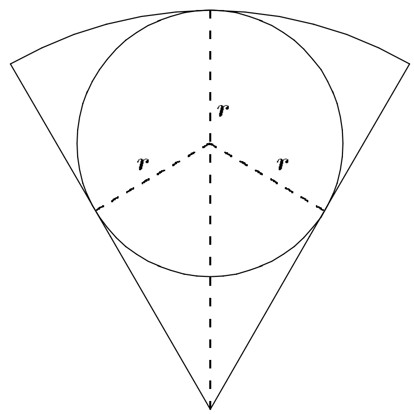
Thus $x = 4$, as $x > 0$.

(b) Determine the largest prime factor of 9919.

Solution. Considering the units digit we are led to try 7 as a factor, yielding $9919 = 7 \times 1417$. We are now led to try for a factor of 1417 ending in 3 or 9 less than 38, (since $38^2 = 1444$), and $1417 = 13 \times 109$. Since 109 is prime, the answer is 109.

5. A circle is inscribed in a circular sector which is one sixth of a circle of radius 1, and is tangent to the three sides of the sector as shown. Calculate the radius of the inscribed circle.

Solution.



Call the radius of the inscribed circle r. The central angle is $\frac{1}{6} \times 360° = 60°$, so each small triangle is a $30°, 60°, 90°$ right triangle. The hypotenuse, h, has length $h = r \sin 30° = \frac{r}{2}$. So $r + h = 1$ gives $\frac{3}{2}r = 1$, and $r = \frac{2}{3}$.

6. Determine the units digit of the sum

$$26^{26} + 33^{33} + 45^{45}.$$

Solution. The units digit is the same as for

$$6^{26} + 3^{33} + 5^{45}.$$

Now $6^n \equiv 6 \bmod 10$ for $n \geq 1$ and

$$3^n \equiv \begin{cases} 3 & n \equiv 1 \bmod 4 \\ 9 & n \equiv 2 \bmod 4 \\ 7 & n \equiv 3 \bmod 4 \\ 1 & n \equiv 4 \bmod 4 \end{cases}, \quad \text{so } 3^{33} \equiv 3 \bmod 10.$$

Finally $5^n \equiv 5 \bmod 10$ for $n \geq 1$. Hence

$$26^{26} + 33^{33} + 45^{45} \equiv 6 + 3 + 5 \equiv 4 \bmod 10$$

and the last digit is 4.

7. Find all solutions (x, y) to the system of equations

$$x + y + \frac{x}{y} = 19$$

$$\frac{x(x+y)}{y} = 60.$$

Solution. Now $\frac{x}{y} = \frac{60}{x+y}$ so

$$x + y + \frac{60}{x+y} = 19, \qquad (x+y)^2 - 19(x+y) + 60 = 0$$

and $x + y = 4$ or $x + y = 15$.

If $x + y = 4$, $\frac{x}{y} = 60$, we get $x = 60y$, and $61y = 4$, so $y = \frac{4}{61}$ and $x = \frac{240}{61}$.

If $x + y = 15$, $\frac{x}{y} = 5$ giving $y = 3$, $x = 12$.

The solutions for (x, y) are $(\frac{240}{61}, \frac{4}{61})$ and $(12, 3)$.

8. Find the number of different divisors of 10800.

Solution. Now $10800 = 9 \times 12 \times 100 = 2^4 \times 3^3 \times 5^2$. The factors must be of the form $2^a 3^b 5^c$ where $0 \le a \le 4$, $0 \le b \le 3$ and $0 \le c \le 2$, so the number of factors is $5 \times 4 \times 3 = 60$.

9. Show that $n^4 - n^2$ is divisible by 12 for any positive integer $n > 1$.

Solution. Now $n^4 - n^2 = (n-1)n^2(n+1)$. In any three consecutive numbers, at least one is divisible by 3. Also if n is even, 4 divides n^2, and if n is odd, both $n - 1$ and $n + 1$ are even so 4 divides $n^2 - 1$. In either case 4 divides $n^4 - n^2$ and we are done.

10. Two clocks now indicate the correct time. One gains a second every hour, and the other gains 3 seconds every 2 hours. In how many days will both clocks again indicate the correct time?

Solution. There is a bit of a problem about whether the clock is a 12 hour or 24 hour clock. Let us assume a 12 hour clock in each case. The first clock will gain 12 hours in $12 \times 60 \times 60$ hours, or 1800 days, which is the first time it will again tell the correct time. The second clock will gain 12 hours in $12 \times 60 \times 60 \div (\frac{3}{2})$ hours, or 1200 days. At the end of 3600 days they will both tell the correct time.

That completes the Corner for this number. I need suitable exam materials, and would welcome your suggestions for the evolution of this feature of ***Crux with Mayhem***.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Have, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Richard Hoshino (University of Waterloo), Wai Ling Yee (University of Waterloo), and Adrian Chan (Upper Canada College).

Shreds and Slices

Playful Palindromes

What are palindromes? They are any sort of objects which are the same when read backwards or forwards. An example of a palindromic word is **NOON**. An example of a palindromic sentence is

ABLE WAS I ERE I SAW ELBA,

accredited to Napoleon. Here we concentrate on palindromic numbers such as 12321, and henceforth the word *palindrome* will refer to palindromic numbers.

Quickly work out the following problems.

1. How many two digit palindromes are there?
2. How many three digit palindromes are there?
3. How many palindromes are there from 0 to 1998?
4. How many n-digit palindromes are there? (Consider the cases when n is even and odd.

Palindromic Sums

Ask a friend to pick a positive integer. Tell this friend to reverse the digits of the integer and add it to the original. If the sum is not a palindrome then take the sum's reverse and add it to itself. Continue doing this until the sum is a palindrome.

Examples

$$\begin{array}{r} 123 \\ +\ 321 \\ \hline 444 \end{array} \qquad \begin{array}{r} 139 \\ +\ 931 \\ \hline 1070 \\ +\ 0701 \\ \hline 1771 \end{array}$$

It is clear that if all the digits are less than 5 then the process terminates after the first step. The second example shows that a number with a digit greater than 5 can also have a palindromic sum.

Now try these examples:

1. 1990 2. 89 3. 196

Answers

1. 68486 2. 8813200023188 3. No one has yet found the answer!

Unsolved Problem

Is it true that every integer has a palindromic sum (that is, the process will eventually terminate)?

This is an example of where better algorithms and faster computers will not help. Even if a super-fast computer with terabytes of memory were to find the solution of 196, this leaves the problem open still as we do not know that ALL integers have a palindromic sum.

Perhaps for the eager reader, an attempt should be made to classify as much as possible those numbers which DO have palindromic sums. We have already made one such class of numbers, namely those whose digits are all less than 5. Are there any other classes of numbers that guarantee to give palindromic sums?

Best of the Web

We present here some web sites which may be of interest to the mathematical web surfer:

`http://www.math.princeton.edu/~kkedlaya/competitions.html`

Kiran Kedlaya's Math Competition Archive, with copies of old USAMO, IMO, and Putnam papers, and many links to other related sites.

`http://www.math.toronto.edu/mathnet/`

U of T Mathnet. A collection of material for high school students and teachers.

http://www.research.att.com/~njas/sequences/

Sloane's On-line Sequence Dictionary.

Thanks to Ben Chia, ex Cedarbrae C.I.

Decimal Expansion of Fractions

Cyrus Hsia

While programming on the computer, a couple of my friends, Peter Plachta and Jacob Mouka noticed some interesting patterns. One such pattern came about as follows: Stumped on a hard programming problem, they tried examining their output. Through the complex mess of numbers, they noticed that the decimal expansion of fractions, of the form $\frac{1}{p}$, where p is a prime number (other than 2 or 5) always had a repeating decimal pattern. Furthermore, this repeating pattern appeared immediately after the decimal point. For example, $\frac{1}{3} = 0.\overline{3}$. With many such examples generated on the computer, they conjectured that this must be the case for all primes p, $p \neq 2, 5$. In fact, their conjecture is correct. We will show this result and some interesting generalizations. But first let us give some definitions so that there will be no ambiguity in their meaning.

The decimal expansion of a fraction is of the form

$$0.a_1a_2\ldots a_nb_1\ldots b_m\ldots b_1\ldots b_m\ldots\,,$$

where the sequence $b_1\ldots b_m$ repeats indefinitely. We write a bar over it to indicate that it repeats, that is,

$$0.a_1a_2\ldots a_n\overline{b_1\ldots b_m}.$$

Definition 1.1. The *repeating part* in a decimal expansion is the number (that is, the sequence of digits $b_1\ldots b_m$) under the bar. The astute reader would notice that the bar could have been placed one decimal place over, as $0.a_1a_2\ldots a_nb_1\overline{b_2\ldots b_mb_1}$, or that a bar could have been placed over twice the above sequence and still give the same value. So to make this definition unambiguous, we will always mean the earliest and smallest such pattern after the decimal point. Finally, we will say that fractions, like $\frac{1}{2}$, that terminate do not have a repeating part.

Definition 1.2. With the above definition of the repeating part in mind, the *leading part* is the number (that is, the sequence of digits $a_1\ldots a_n$) immediately following the decimal point before the repeating part. Note that fractions may not have a leading part, as in $\frac{1}{3}$.

Finally, let us call a number *good* if it is relatively prime to 10, that is it is not divisible by 2 or 5. With these definitions, let us show that the conjecture above is true by first proving some simple results that we need. We will assume from now on that we are looking at the decimal expansions of $\frac{1}{p}$, where p is a good prime.

Lemma 1.3. The decimal expansion of $\frac{1}{p}$, where p is a good prime, must have a repeating part, that is, $\frac{1}{p} = 0.a_1a_2\ldots a_n\overline{b_1\ldots b_m}$, $m \geq 1$.

Proof. Left to the reader as an exercise.

Lemma 1.4. If $0.a_1a_2\ldots a_n\overline{b_1\ldots b_m}$ is the decimal expansion of a fraction, then $a_n \neq b_m$.

Proof. Left to the reader as an exercise.

Lemma 1.5. For a decimal expansion,

$$\begin{aligned} &0.a_1\ldots a_n\overline{b_1\ldots b_m} \\ &= \frac{10^{n-1}a_1 + 10^{n-2}a_2 + \cdots + a_n}{10^n} + \frac{10^{m-1}b_1 + 10^{m-2}b_2 + \cdots + b_m}{10^n(10^m-1)}, \end{aligned}$$

which we write

$$\frac{a_1\ldots a_n}{10^n} + \frac{b_1\ldots b_m}{10^n(10^m-1)}$$

in short form.

Proof. Left to the reader as an exercise.

Now with these lemmas we can prove the conjecture.

Conjecture 1.6. (Peter Plachta and Jacob Mouka)

The fraction $\frac{1}{p}$ in decimal notation consists only of a repeating part (that is, it has no leading part) if p is a good prime.

Proof. Assume on the contrary that $\frac{1}{p} = 0.a_1a_2\ldots a_n\overline{b_1\ldots b_m}$, where $n \geq 1$, so that there is at least one digit in the leading part.

Case I: $m = 0$, that is, the decimal expansion terminates. Then

$$\frac{1}{p} = 0.a_1\ldots a_n = \frac{a_1\ldots a_n}{10^n} \;\Rightarrow\; 10^n = p(a_1\ldots a_n) \;\Rightarrow\; p \mid 10^n,$$

but p is good, so this is not possible.

Case II: $m \geq 1$. Then

$$\frac{1}{p} = \frac{a_1\ldots a_n}{10^n} + \frac{b_1\ldots b_m}{10^n(10^m-1)}$$

(by Lemma 1.5). Multiplying both sides by $10^n(10^m-1)p$, we get

$$\begin{aligned} 10^n(10^m-1) &= p[(10^m-1)(a_1\ldots a_n) + b_1\ldots b_m] \\ &= p[10^m a_1\ldots a_n + b_1\ldots b_m - a_1\ldots a_n]. \end{aligned}$$

Since $n \geq 1$, $10 \mid 10^n(10^m - 1)$, so 10 divides the right hand side of the equation. In particular, since $m \geq 1$,

$$10 \mid (b_1 \ldots b_m - a_1 \ldots a_n) \;\Rightarrow\; 10 \mid (b_m - a_n).$$

Now $0 \leq a_n, b_m \leq 9 \Rightarrow a_n = b_m$, which by Lemma 1.4 is impossible.

Why does the fraction have to have a prime as its denominator? The prime is what makes the particular types of fractions of Conjecture 1.6 have no leading part. Then a natural question to ask is whether the converse holds, that is, given a decimal expansion with only a repeating part, is the denominator in its fractional representation a prime. To see that the converse does not hold, look at the decimal expansion of $\frac{1}{9}$, which is $0.\overline{1}$. It has only a repeating part but the denominator is not prime.

It is not the fact that the denominator is a prime that makes Conjecture 1.6 true as the following theorem will show.

Theorem 1.7. For all positive integers $N > 1$, $\frac{1}{N}$ in decimal notation has a repeating part but no leading part if and only if N is good.

First, we need a result that most of the readers will already be familiar with. In fact, it has popped up a number of times in olympiad and pre-olympiad type problems. Consider the numbers M_i consisting of i digits made up entirely of 1's, for example, $M_1 = 1$, $M_2 = 11$, $M_3 = 111$. Then we have:

Lemma 1.8. For any good positive integer N, N divides some M_i.

Proof. Consider the $N+1$ numbers $M_1, M_2, \ldots, M_{N+1}$ modulo N. Then by the Pigeonhole Principle, at least two of the M_i's are congruent, say $M_j \equiv M_k \pmod{N}$, where $j > k$. Then we have

$$\begin{aligned} M_j \equiv M_k \pmod{N} \quad &\Rightarrow \quad \underbrace{11\ldots1}_{j} \equiv \underbrace{11\ldots1}_{k} \pmod{N} \\ &\Rightarrow \quad \underbrace{11\ldots1}_{j-k}\underbrace{00\ldots0}_{k} \equiv 0 \pmod{N} \\ &\Rightarrow \quad \underbrace{11\ldots1}_{j-k} \equiv 0 \pmod{N}. \end{aligned}$$

This can be done since neither 2 nor 5 divides N. This means that M_{j-k} is divisible by N.

Proof of Theorem 1.7.
(Only if part) If $\frac{1}{N}$ has a repeating part but no leading part, then we have

$$\frac{1}{N} = 0.\overline{b_1 \ldots b_m} = \frac{10^{m-1}b_1 + \cdots + b_m}{10^m - 1}$$

by Lemma 1.5. Cross-multiplying, we get $10^m - 1 = N(10^{m-1}b_1 + \cdots + b_m)$. Thus N is good.

(If part) If N is good, then by Lemma 1.8 there is an $M = M_i$ divisible by N. Let $M = kN$, where k is an integer, $k \geq 1$. We have then

$$\frac{1}{N} = \frac{k}{M} = \frac{k}{\underbrace{11\ldots1}_{i}} = \frac{9k}{\underbrace{99\ldots9}_{i}} = \frac{9k}{10^i - 1}.$$

Hence,

$$\frac{1}{N} < 1 \implies k < M \implies 9k < 9M = 10^i - 1 < 10^i.$$

This means that $9k$ is at most an i-digit number; that is, $9k$ can be written as $b_1 \ldots b_i$, where some of the b_i's can be 0. Thus

$$\frac{1}{N} = \frac{b_1 \ldots b_i}{10^i - 1} = 0.\overline{b_1 \ldots b_i}$$

as required.

Now it is time for the readers to roll up their sleeves and tackle some problems.

Exercises

1.1 Prove Lemmas 1.3, 1.4, and 1.5.

1.2 (a) Show that any fraction of the form $\frac{a}{b}$ in lowest terms, where a and b are positive integers, $a < b$, also has no leading term if and only if b is good.

(b) Show that (a) holds even when $a \geq b$ and with the same conditions. Thus $\frac{a}{b} = a_1 a_2 \ldots a_n.\overline{b_1 \ldots b_m}$ if and only if b is good.

1.3 (Related exercise) Find the smallest positive integer n such that

$$\frac{n^3 + 9n^2 + 9n + 7}{1996}$$

in decimal notation terminates (that is, it has no repeating part, just a finite leading part).

1.4 (Related exercise) Write a computer program to find the repeating part in a fraction of the form $\frac{1}{p}$, where p is a prime.

To finish off, fractions with denominators that have factors of 2 and 5 still need to be considered. In fact, it turns out that there is a nice relationship between the number of factors of 2 and 5 in the denominator and the number of terms in the leading part of its decimal representation.

Lemma 2.1. If N is of the form $2^k M$, where M is good, then $\frac{1}{N}$ has k terms in its leading part, and it has a repeating part if $M > 1$.

Proof. If $M = 1$ then it is obvious that $\frac{1}{2^k} = \frac{5^k}{10^k}$ has k terms in the leading part and no repeating part. If $M > 1$, then we have

$$\frac{1}{N} = \frac{1}{2^k M} = \frac{5^k}{10^k M} = \frac{1}{10^k}\left(\frac{5^k}{M}\right).$$

By Exercise 1.2, we have $\frac{5^k}{M} = a_1 \ldots a_n.\overline{b_1 \ldots b_m}$. Dividing through by 10^k and noting $\frac{1}{N}$ is less than 1, we must have $n < k$. Now once we show that $a_n \neq b_m$, then we are finished, as then

$$\frac{1}{N} = \frac{1}{10^k}\left(\frac{5^k}{M}\right) = 0.\underbrace{0 \ldots 0 a_1 \ldots a_n}_{k \text{ terms}}\overline{b_1 \ldots b_m}.$$

If $a_n = b_m$ then

$$\frac{5^{k-1}}{2M} = \frac{1}{10}\left(\frac{5^k}{M}\right) = a_1 \ldots a_{n-1}.\overline{a_n b_1 \ldots b_{m-1}},$$

contradicting Exercise 1.2 that a fraction has a repeating part if and only if the denominator is good.

Lemma 2.2. Similarly, if N is of the form $5^k M$, where M is good, then $\frac{1}{N}$ has k terms in its leading part, and it has a repeating part if $M > 1$.

Proof. Similar (in fact identical) to the proof of Lemma 2.1.

Now for the grand finale...

Theorem 2.3. If N is of the form $2^k 5^l M$, where again M is good, then the fraction $\frac{1}{N}$ has $\max(k, l)$ terms in the leading part, and if $M > 1$ then it also has a repeating part.

Proof. Once again, this just takes from material shown before. If k or l equals 0, then the result becomes Lemma 2.2 or 2.1. If $k = l$, then the result is immediate, and in fact the leading part has $k = l$ 0's. If $k > l > 1$, then we have $\frac{1}{2^k 5^l M} = \frac{1}{10^l 2^{k-l} M}$. By Lemma 2.1, we have $k - l$ terms in the leading part of $\frac{1}{2^{k-1} M}$. Dividing by 10^l just shifts the decimal point l places to the left giving $(k - l) + l = k$ terms in the leading part (and similarly l terms for $l > k > 1$).

More Exercises

2.1 (Exploration) In our long analysis, we have shown (not on purpose), that the length of the leading part can be determined. However, the length of the repeating part was never discussed. Determine any relationships between the numbers in the fraction and the length of the repeating part in its decimal representation.

2.2 (a) If we had 6 fingers (3 on each hand?), then perhaps our system of counting would be in base 6. In general, if the base was pq, where p

and q are distinct primes, then show an analogous result to the above theorems.

(b) Do the same for any base B.

2.3 (More down to earth problem) Of the numbers 1995, 1996, $\ldots$, 1999, which number has the greatest number of terms in the leading part of its reciprocal? With this, we hope you will be a leading part in this year!

Acknowledgements

Peter Plachta and Jacob Mouka, 3rd year students, University of Toronto at Scarborough, for the conjecture.

An Ambivalent Sum

Naoki Sato

A conditionally convergent series $\sum a_n$ is a series which converges, but such that the series $\sum |a_n|$ diverges. One of the most famous examples of such a series, which appears in textbooks everywhere, is the following:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \;=\; \ln 2.$$

A basic, but unusual property of conditionally convergent series is that we may rearrange the terms, so that the sum comes out to be any value we wish (see *Calculus*, 3rd Ed., M. Spivak, pp. 476-8), so we must be careful when manipulating the terms in determining the sum. For example, a question of the 1954 Putnam asks to verify that

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots \;=\; \frac{3}{2}\ln 2.$$

Same terms, different sum. These results suggest a generalization, and the form of the alternating harmonic series makes for an elegant analysis.

For positive integers r and s, let

$$\begin{aligned} S(r,s) \;=\; & \underbrace{1 + \frac{1}{3} + \cdots + \frac{1}{2r-1}}_{r \text{ terms}} - \underbrace{\frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2s}}_{s \text{ terms}} \\ & + \underbrace{\frac{1}{2r+1} + \frac{1}{2r+3} + \cdots + \frac{1}{4r-1}}_{r \text{ terms}} - \underbrace{\frac{1}{2s+2} - \frac{1}{2s+4} - \cdots - \frac{1}{4s}}_{s \text{ terms}} \\ & + \cdots. \end{aligned}$$

A first step in summing $S(r,s)$ will be looking at its partial sums, so let $S_n(r,s)$ be the sum of the first $n(r+s)$ terms of $S(r,s)$. Assume for the moment that $r > s$. Then,

$$\begin{aligned}
S_n(r,s) &= \left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2s}\right)\\
&+\left(\frac{1}{2r+1}+\frac{1}{2r+3}+\cdots+\frac{1}{4r-1}\right)\\
&-\left(\frac{1}{2s+2}+\frac{1}{2s+4}+\cdots+\frac{1}{4s}\right)+\cdots\\
&+\left(\frac{1}{2(n-1)r+1}+\frac{1}{2(n-1)r+3}+\cdots+\frac{1}{2nr-1}\right)\\
&-\left(\frac{1}{2(n-1)s+2}+\frac{1}{2(n-1)s+4}+\cdots+\frac{1}{2ns}\right)\\
&= \left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}-\frac{1}{2}-\frac{1}{4}-\cdots-\frac{1}{2r}\right)\\
&+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2r}-\frac{1}{2}-\frac{1}{4}-\cdots-\frac{1}{2s}\\
&+\left(\frac{1}{2r+1}+\frac{1}{2r+3}+\cdots+\frac{1}{4r-1}\right.\\
&\quad\left.-\frac{1}{2r+2}-\frac{1}{2r+4}-\cdots-\frac{1}{4r}\right)\\
&+\frac{1}{2r+2}+\frac{1}{2r+4}+\cdots+\frac{1}{4r}-\frac{1}{2s+2}-\frac{1}{2s+4}\\
&\quad-\cdots-\frac{1}{4s}+\cdots\\
&+\left(\frac{1}{2(n-1)r+1}+\frac{1}{2(n-1)r+3}+\cdots+\frac{1}{2nr-1}\right.\\
&-\left.\frac{1}{2(n-1)r+2}-\frac{1}{2(n-1)r+4}-\cdots-\frac{1}{2nr}\right)\\
&+\frac{1}{2(n-1)r+2}+\frac{1}{2(n-1)r+4}+\cdots+\frac{1}{2nr}\\
&-\frac{1}{2(n-1)s+2}-\frac{1}{2(n-1)s+4}-\cdots-\frac{1}{2ns}\\
&= A_{2nr}+\frac{1}{2ns+2}+\frac{1}{2ns+4}+\frac{1}{2ns+6}+\cdots+\frac{1}{2nr}\\
&= A_{2nr}+\frac{1}{2}\left(\frac{1}{ns+1}+\frac{1}{ns+2}+\frac{1}{ns+3}+\cdots+\frac{1}{nr}\right),
\end{aligned}$$

where

$$A_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n+1}\frac{1}{n}.$$

We may estimate a more general sum by integrals. Since $y = \frac{1}{x}$ is decreasing on $(0, \infty)$, for $0 < d < a$,

$$\int_a^{a+md} \frac{1}{x}\,dx < \frac{d}{a} + \frac{d}{a+d} + \cdots + \frac{d}{a+(m-1)d} < \int_{a-d}^{a+(m-1)d} \frac{1}{x}\,dx$$

so that

$$\begin{aligned}\frac{1}{d}\ln\left(\frac{a+md}{a}\right) &< \frac{1}{a} + \frac{1}{a+d} + \cdots + \frac{1}{a+(m-1)d} \\ &< \frac{1}{d}\ln\left(\frac{a+(m-1)d}{a-d}\right)\end{aligned}$$

(if you are not sure why these inequalities are true, draw a graph).

So, plugging in $a = ns + 1$, $d = 1$, $m = n(r - s)$, we obtain

$$\ln\left(\frac{nr+1}{ns+1}\right) < \frac{1}{ns+1} + \frac{1}{ns+2} + \cdots + \frac{1}{nr} < \ln\left(\frac{r}{s}\right)$$

$$\Longrightarrow \lim_{n\to\infty}\left(\frac{1}{ns+1} + \frac{1}{ns+2} + \cdots + \frac{1}{nr}\right) = \ln\left(\frac{r}{s}\right).$$

Note that we could have also used the well-known result

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = \gamma,$$

where γ is Euler's constant.

Finally, we have that

$$S(r,s) = \lim_{n\to\infty} S_n(r,s) = \ln 2 + \frac{1}{2}\ln\left(\frac{r}{s}\right) = \frac{1}{2}\ln\left(\frac{4r}{s}\right).$$

The case $r < s$ is argued similarly (the case $r = s$ is obvious), and we obtain the same value.

We make some observations:

(1) The value $S(r, s)$ depends only on the ratio r/s, as can be predicted.

(2) The case $r = 1$, $s = 4$, produces the intriguing-looking formula

$$1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \cdots = 0.$$

(3) The set $\{S(r,s)\} = \{\frac{1}{2}\ln(\frac{4r}{s}) : r, s \in \mathbb{Z}^+\}$ is dense in $\mathbb{R}$.

(4) Going back to a result mentioned above, to rearrange the terms of a conditionally convergent series so that it sums to a given value, say S, we use the following algorithm: Assuming $S \geq 0$, we successively add the first positive terms of the series until we go over S; then we add the first negative terms of the series until we go under S, and so on (if $S < 0$, we merely start with the negative terms). In fact, this is the idea behind the proof. The nature of the conditionally convergent series guarantees this algorithm produces a series with sum S.

It has been observed that if we try this with the alternating harmonic series, the number of positive terms and negative terms taken each time becomes constant. We can see this is because of (3); that is, any real number can be approximated by a sum $S(r, s)$.

J.I.R. McKnight Problems Contest 1978

The J.I.R. McKnight Problems Contest is a problem solving contest in Scarborough, Ontario. It began in 1975 in honour of one of Scarborough's finest Mathematics teachers and continues today as a scholarship paper for senior students. This paper, although tailored to students taking OAC (Ontario Academic Credit) Mathematics courses, contains many problems that are accessible to all high school students. With some ingenuity, most problems reveal some very beautiful results and we hope our readers will enjoy this as well.

1. Verify that $13! = 112296^2 - 79896^2$.
2. Show that, for any real numbers x, y, and z,
$$(x + y)z \leq \frac{(x^2 + y^2)}{2} + z^2.$$
3. Sum to k terms the series whose nth term is
$$\frac{n^4 + 2n^3 + n^2 - 1}{n^2 + n}.$$
4. Show that in order that the quadratic function
$$3x^2 + 2pxy + 2y^2 + 2ax - 4y + 1$$
may be resolved into factors linear in x and y, p must be a root of the equation $p^2 + 4ap + 2a^2 + 6 = 0$.
5. Find the sum of all 3–digit odd numbers that can be made using the digits 1, 3, 5, 7, 8, 9, where no repeated digits are used in any number.

6. Find a cubic function $f(x)$ with the following properties:

   (a) It is a curve passing through $(1, 1)$.

   (b) The tangent to the curve at $(1, 1)$ has slope 1.

   (c) The sum of the roots of its corresponding equation is $\frac{1}{5}$.

   (d) The sum of the squares of the roots of its corresponding equation is $\frac{121}{25}$.

1984 Swedish Mathematics Olympiad

Qualifying Round

1. Solve the following system of equations (in 5 unknowns):

$$\begin{aligned} ab &= 1, \\ bc &= 2, \\ cd &= 3, \\ de &= 4, \\ ea &= 6. \end{aligned}$$

2. Show that if n is an odd natural number, then

$$n^{12} - n^8 - n^4 + 1$$

   is divisible by 2^9.

3. Find all positive numbers x such that

$$x^{8-3x} > x^7.$$

4. ABC is an isosceles triangle with $AB = AC$. Choose a point D on the side AB, and a point E on AC produced through C, such that $AD + AE = AB + AC$. Show that $DE > BC$.

5. The numbers 1, 2, $\ldots$, 9, are placed in a 3×3 grid, so that no row, no column and neither of the two major diagonals contains a sequence of numbers ordered by size (either increasing or decreasing). Show that the number in the central square must be odd.

6. A woman, who is not yet 100 years old, and one of her grandchildren have the same birthday. For six years in a row, the woman's age was a multiple of her grandchild's age. How old was the woman on the sixth of these birthdays?

Final Round

1. Let A and B be two arbitrary points inside a circle C. Show that there always exists a circle through A and B which lies completely inside C.

2. The squares in a 3×7 grid are coloured either blue or yellow. Consider all rectangles of $m \times n$ squares in this grid, where $2 \leq m \leq 3$, $2 \leq n \leq 7$. Show that at least one of these rectangles has all four of its corner squares the same colour.

3. Show that if a and b are positive numbers, then

$$\left(\frac{a+1}{b+1}\right)^{b+1} \geq \left(\frac{a}{b}\right)^{b}.$$

4. Find positive integers p and q such that all the roots of the polynomial

$$(x^2 - px + q)(x^2 - qx + p)$$

are positive integers.

5. Solve the system of equations

$$\begin{aligned} a^3 - b^3 - c^3 &= 3abc, \\ a^2 &= 2(a+b+c), \end{aligned}$$

where a, b, and c are natural numbers.

6. The numbers $a_1, a_2, \ldots, a_{14}$ are positive integers. It is known that

$$\sum_{i=1}^{14} a_i = 6558.$$

Show that the numbers $a_1, a_2, \ldots, a_{14}$ consist of the numbers $1, 2, \ldots, 7$ each taken twice.

Swedish Mathematics Olympiad Solutions

1983 Final Round

1. The positive integers are added in groups as follows: 1, $2+3$, $4+5+6$, $7+8+9+10$, and so on. What is the sum of the nth group?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, WI, USA.

Let $T_k = k(k+1)/2$ be the kth triangular number. Then the n numbers in the nth group are $T_{n-1}+1$, $T_{n-1}+2$, ... , $T_{n-1}+n = T_n$. Using the formula

$$S = \frac{n(a+l)}{2},$$

where S is the sum of a finite arithmetic progression, a is the first term, l is the final term, and n is the number of terms in the progression, the desired sum is

$$\frac{n[(T_{n-1}+1+T_n)]}{2} = \frac{n\left(\frac{(n-1)n}{2}+1+\frac{n(n+1)}{2}\right)}{2} = \frac{n^3+n}{2}.$$

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
Ravi Vakil	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from the previous issue be submitted by 1 October 1997, for publication in the issue 5 months ahead; that is, issue 2 of 1998. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Problems

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H225. In Cruxmayhemland, stamps can only be bought in two denominations, p and q cents, both of which are at least 31 cents. It is known that if p and q are relatively prime, the largest value that cannot be created by these two stamps is $pq - p - q$. For example, when $p = 5$ and $q = 3$, one can affix any postage that is higher than $15 - 5 - 3$, or 7 cents, but not 7 cents itself. The governor of Cruxmayhemland tells you that 1997 is the largest value that cannot be created by these stamps. Find all possible pairs of positive integers (p, q) with $p > q$.

H226. In right-angled triangle ABC, with BC as hypotenuse, $AB = x$ and $AC = y$, where x and y are positive integers. Squares $APQB$, $BRSC$, and $CTUA$ are drawn externally on sides AB, BC, and CA respectively. When QR, ST, and UP are joined, a hexagon is formed. Let K be the area of hexagon $PQRSTU$.

(a) Prove that K cannot equal 1997. (HINT: Try to find a general formula for K.)

(b) Prove that there is only one solution (x, y) with $x > y$ so that $K = 1998$.

H227. The numbers 2, 4, 8, 16, $\ldots$, 2^n are written on a chalkboard. A student selects any two numbers a and b, erases them, and replaces them by their average, namely $(a+b)/2$. She performs this operation $n-1$ times until only one number is left. Let S_n and T_n denote the maximum and minimum possible value of this final number, respectively. Determine a formula for S_n and T_n in terms of n.

H228. Verify that the following three inequalities hold for positive reals x, y, and z:

(i) $x(x-y)(x-z) + y(y-x)(y-z) + z(z-x)(z-y) \geq 0$ (this is known as Schur's Inequality),

(ii) $x^4 + y^4 + z^4 + xyz(x+y+z) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$,

(iii) $9xyz + 1 \geq 4(xy + yz + zx)$, where $x + y + z = 1$.

(Can you derive an ingenious method that allows you to solve the problem without having to prove all three inequalities directly?)

Advanced Problems

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A201. Consider an infinite sequence of integers $a_1, a_2, \ldots, a_k, \ldots$ with the property that every m consecutive numbers sum to x and every n consecutive numbers sum to y. If x and y are relatively prime then show that all numbers are equal.

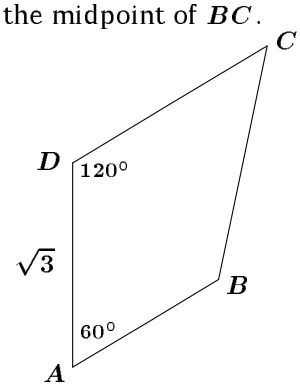
A202. Let ABC be an equilateral triangle and Γ its incircle. If D and E are points on AB and AC, respectively, such that DE is tangent to Γ, show that

$$\frac{AD}{DB} + \frac{AE}{EC} = 1.$$

(*8th Iberoamerican Mathematical Olympiad, Mexico '93*)

A203. Let $S_n = 1 + a + a^a + \cdots + a^{a^{\cdots^a}}$, where the last term is a tower of $n-1$ a's. Find all positive integers a and n such that $S_n = na^{\frac{S_{n-1}}{n}}$.

A204. Given a quadrilateral $ABCD$ as shown, with $AD = \sqrt{3}$, $AB + CD = 2AD$, $\angle A = 60°$ and $\angle D = 120°$, find the length of the line segment from D to the midpoint of BC.



Challenge Board Problems

Editor: Ravi Vakil, Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544–1000 USA <vakil@math.princeton.edu>

C73. *Proposed by Matt Szczesny, 4th year, University of Toronto.*

The sequence $\{a_n\}$ consists of positive reals, such that $\sum_{n=1}^{\infty} a_n$ diverges.

Show that $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges, where s_n is the nth partial sum, that is, $s_n = a_1 + a_2 + \cdots + a_n$.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 March 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

Correction

Problem 2235 [1997: 168] was incorrectly attributed to Walther Janous, Ursulinengymnasium, Innsbruck, Austria, when it was in fact proposed by D.J. Smeenk, Zaltbommel, the Netherlands. The editor apologizes for this error.

Solutions submitted by FAX

There has been an increase in the number of solutions sent in by FAX, either to the Editor-in-chief's departmental FAX machine in St. John's, Newfoundland, or to the Canadian Mathematical Society's FAX machine in Ottawa, Ontario. While we understand the reasons for solvers wishing to use this method, we have found many problems with it. The major one is that hand-written material is frequently transmitted very badly, and at times is almost impossible to read clearly. We have therefore adopted the policy that we will no longer accept submissions sent by FAX. We will, however, continue to accept submissions sent by email or regular mail. We do encourage email. Thank you for your cooperation.

2251. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

In the plane, you are given a circle (but not its centre), and points A, K, B, D, C on it, so that arc AK = arc KB and arc BD = arc DC.

Construct, using only an unmarked straightedge, the mid-point of arc AC.

2252. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Prove that the nine-point circle of a triangle trisects a median if and only if the side lengths of the triangle are proportional to its median lengths in some order.

2253. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle and I_b, I_c are the excentres of $\triangle ABC$ relative to sides CA, AB respectively.

Suppose that $I_bA^2 + I_bC^2 = BA^2 + BC^2$ and that $I_cA^2 + I_cB^2 = CA^2 + CB^2$.

Prove that $\triangle ABC$ is equilateral.

2254. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle with $AB = AC$. Let D be the point on side AC such that $CD = 2AD$. Let P be the point on the segment BD such that $\angle APC = 90°$.

Prove that $\angle ABP = \angle PCB$.

2255. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let P be an arbitrary interior point of an equilateral triangle ABC.

Prove that $|\angle PAB - \angle PAC| \geq |\angle PBC - \angle PCB|$.

2256. *Proposed by Russell Euler and Jawad Sadek, Department of Mathematics and Statistics, Northwest Missouri State University, Maryville, Missouri, USA.*

If $0 < y < x \leq 1$, prove that $\dfrac{\ln(x) - \ln(y)}{x - y} > \ln\left(\dfrac{1}{y}\right)$.

2257. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

The diagonals AC and BD of a convex quadrilateral $ABCD$ intersect at the point O. Let OK, OL, OM, ON, be the altitudes of triangles $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, $\triangle DAO$, respectively.

Prove that if $OK = OM$ and $OL = ON$, then $ABCD$ is a parallelogram.

2258. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

In a right-angled triangle ABC (with $\angle C = 90°$), D lies on the segment BC so that $BD = AC\sqrt{3}$. E lies on the segment AC and satisfies $AE = CD\sqrt{3}$. Find the angle between AD and BE.

2259. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let X, Y, Z, be the projections of the incentre of $\triangle ABC$ onto the sides BC, CA, AB respectively. Let X', Y', Z', be the points on the incircle diametrically opposite to X, Y, Z, respectively. Show that the lines AX', BY', CZ', are concurrent.

2260. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

Let n be a positive integer and $x > 0$. Prove that

$$(1+x)^{n+1} \geq \frac{(n+1)^{n+1}}{n^n}\,x.$$

2261. *Proposed by Angel Dorito, Geld, Ontario.*

Assuming that the limit exists, find

$$\lim_{N\to\infty}\left(1+\frac{2+\frac{N+\dots}{1+\dots}}{N+\frac{1+\dots}{2+\dots}}\right),$$

where every fraction in this expression has the form

$$\frac{a+\frac{b+\dots}{c+\dots}}{b+\frac{c+\dots}{a+\dots}}$$

for some cyclic permutation a, b, c of 1, 2, N.

[*Proposer's comment*: this problem was suggested by Problem 4 of Round 21 of the International Mathematical Talent Search, *Mathematics and Informatics Quarterly*, Vol. 6, No. 2, p. 113.]

2262. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Consider two triangles $\triangle ABC$ and $\triangle A'B'C'$ such that $\angle A \geq 90°$ and $\angle A' \geq 90°$ and whose sides satisfy $a > b \geq c$ and $a' > b' \geq c'$. Denote the altitudes to sides a and a' by h_a and h'_a.

Prove that (a) $\dfrac{1}{h_a h'_a} \geq \dfrac{1}{bb'} + \dfrac{1}{cc'}$, (b) $\dfrac{1}{h_a h'_a} \geq \dfrac{1}{bc'} + \dfrac{1}{b'c}$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2145. [1996: 170] *Proposed by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.*

Prove that $\prod_{k=1}^{n}\left(ak+b^{k-1}\right) \leq \prod_{k=1}^{n}\left(ak+b^{n-k}\right)$ for all $a,\, b > 1$.

Editor's composite solution based on the nearly identical solutions submitted by almost all the solvers.

Note that if n is odd, then both sides have the positive factor

$$\left(\frac{n+1}{2}\right) a + b^{\frac{n-1}{2}}.$$

Hence the given inequality is equivalent to

$$\begin{aligned}&\prod_{k=1}^{\lfloor n/2\rfloor}\left(ka+b^{k-1}\right)\left((n-k+1)a+b^{n-k}\right)\\ \leq\ &\prod_{k=1}^{\lfloor n/2\rfloor}\left(ka+b^{n-k}\right)\left((n-k+1)a+b^{k-1}\right).\end{aligned}$$

Therefore, it suffices to show that, for all $k = 1,\, 2,\, \ldots,\, \lfloor \frac{n}{2} \rfloor$,

$$\begin{aligned}&\left(ka+b^{k-1}\right)\left((n-k+1)a+b^{n-k}\right)\\ \leq\ &\left(ka+b^{n-k}\right)\left((n-k+1)a+b^{k-1}\right). \qquad (1)\end{aligned}$$

After expanding and cancelling equal terms, (1) becomes

$$(n-2k+1)ab^{k-1} \ \leq\ (n-2k+1)ab^{n-k}. \qquad (2)$$

We have $n - 2k + 1 \geq n - 2\left(\frac{n}{2}\right) + 1 = 1 > 0$, and $b^{n-2k+1} \geq 0$, which implies $b^{k-1} \leq b^{n-k}$. Thus, (2) follows immediately.

Solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; L. RICE, University of Toronto Schools, Toronto, Ontario; KRISTIAN SABO, student, *Osijek, Croatia; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.*

Several solvers noted that the given conditions can be relaxed to $a > 0$ (or even $a \geq 1$) and $b \geq 1$. This is obvious from the proof above.

2146. [1996: 171] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $AB > AC$, and the bisector of $\angle A$ meets BC at D. Let P be an interior point on the segment AD, and let Q and R be the points of intersection of BP and CP with sides AC and AB respectively. Prove that $PB - PC > RB - QC > 0$.

Solution by the proposer.

Let E be a point on AB such that $AE = AC$. Then we get $\triangle AEP \equiv \triangle ACP$, so that $\angle AEP = \angle ACP$ and $PE = PC$. Since $\angle BEP = 180° - \angle QCP > 180° - \angle ACB > \angle ABC > \angle EBP$, we have $PB > PE = PC$. Because $AE = AC < AB$, we obtain $\angle ABP < \angle AEP = \angle ACP$. Since $\angle BAP = \angle PAC$, we have $\angle APR = \angle ACP + \angle PAC > \angle ABP + \angle BAP = \angle APQ$.

Let F be a point on AB such that $\angle APF = \angle APQ$, then $\angle APF < \angle APR$, so F is between A and R, and $\triangle APF \equiv \triangle APQ$, so that $AF = AQ$, $PF = PQ$ and $\angle AFP = \angle AQP$. Therefore, $AR > AF = AQ$.

In addition, $\angle ARP = \angle RPB + \angle RBP = \angle RPB + \angle ABQ$ and $\angle PFR = \angle PQC = \angle BQC = \angle BAC + \angle ABQ$.

If $\angle RPB = \angle BAC$, $\angle ARP = \angle PFR$, so $PR = PF = PQ$;

if $\angle RPB < \angle BAC$, $\angle ARP < \angle PFR$, so $PR > PF = PQ$;

if $\angle RPB > \angle BAC$, $\angle ARP > \angle PFR$, so $PR < PF = PQ$.

We consider 2 cases.

Case I: $\angle BAC \geq 90°$. Because $\angle RPB < \angle ARC < 90° \leq \angle BAC$, we have $PR > PQ$. Let G be a point on PR such that $PG = PQ$ and H a point on PB such that $PH = PC$. Then $\triangle PGH \equiv \triangle PQC$; thus we get $GH = QC$. As $\angle BRG > \angle RAC \geq 90°$, we get $GB > RB$, so $BH > GB - GH > RB - GH$. Hence

$$PB - PC = BH > RB - QC.$$

Case II: $\angle BAC < 90°$. Then there exists a point O on AD such that $\angle BOC = 180° - \angle BAC$.

(i) If O coincides with P, we have $\angle RPB = \angle BAC$ and so $PR = PQ$. We take a point J on PB such that $PJ = PC$. Then $\triangle PRJ \equiv \triangle PQC$, thus $RJ = QC$. Because $BJ > RB - RJ = RB - QC$, we have $PB - PC > RB - QC$.

(ii) If P is between A and O, then $\angle BPC < \angle BOC = 180° - \angle BAC$, so $\angle RPB = 180° - \angle BPC > \angle BAC$ and from above, $PR < PQ$. We take a point K on PQ such that $PK = PR$ and a point M on PC produced beyond C such that $PM = PB (> PC)$. Let T be the intersection of KM and QC. As $\triangle PKM \equiv \triangle PRB$, we get $KM = RB$ and $\angle PKM = \angle PRB$, so

$$\begin{aligned}\angle QKM &= \angle ARP = \angle RPB + \angle RBP = \angle RPB + \angle ABQ \\ &> \angle BAC + \angle ABQ = \angle BQC.\end{aligned}$$

Therefore, $\angle QKT > \angle KQT$, thus $KT < QT$. Hence

$$\begin{aligned}RB - QC &= KM - QC = (KT + TM) - (QT + TC) \\ &= (KT - QT) + (TM - TC) < TM - TC.\end{aligned}$$

As $TM - TC < CM = PB - PC$, we obtain $RB - QC < PB - PC$.

(iii) If P is between O and D, then $\angle BPC > \angle BOC = 180° - \angle BAC$, so $\angle RPB = 180° - \angle BPC < \angle BAC$. Thus, from the above, $PR > PQ$. We take points X and Y on PR and PB, respectively, such that $PX = PQ$ and $PY = PC$. Then $\triangle PXY \equiv \triangle PQC$, so $XY = QC$.

As $\angle BPC > \angle BOC = 180° - \angle BAC$, we have $\angle QPR + \angle RAQ > 180°$, thus $\angle ARP + \angle AQP < 180°$. Since $\angle AQP > \angle ARP$, we get $\angle ARP < 90°$, so $\angle BRP > 90°$; hence $RB < XB$.

Therefore, $YB > XB - XY > RB - XY = RB - QC$. Thus $PB - PC = YB > RB - QC$.

No other solutions were submitted.

2147. [1996: 171] *Proposed by Hoe Teck Wee, Singapore.*

Let S be the set of all positive integers x such that there exist positive integers y and m satisfying $x^2 + 2^m = y^2$.

(a) Characterize which positive integers are in S.

(b) Find all positive integers x so that both x and $x + 1$ are in S.

Solution by Thomas Leong, Staten Island, NY, USA.

(a) We show that $S = \{x : x = 2^r - 2^s, r > s \geq 0, r, s \in \mathbb{Z}\}$. First suppose that $x^2 + 2^m = y^2$ with x positive. Then $2^m = y^2 - x^2 = (y+x)(y-x)$, and so $y + x = 2^r$ and $y - x = 2^s$ where $r > s \geq 1$ ($s \neq 0$ since x and y are of the same parity). Thus $2x = 2^r - 2^s$, or $x = 2^{r-1} - 2^{s-1}$ where $r - 1 > s - 1 \geq 0$. Conversely, if $x = 2^r - 2^s$ with $r > s \geq 0$, then with $m = r + s + 2$ and $y = 2^r + 2^s$, we have $x^2 + 2^m = y^2$.

(b) The desired set is $\{x : x = 2^r - 2, r \geq 2 \text{ or } x = 2^r - 1, r \geq 1\}$. Now every odd number in S is of the form $2^r - 1$, $r \geq 1$. If $x + 1 \in S$ is odd, then $x + 1 = 2^r - 1$ and $x = 2^r - 2$ are both in S provided $r \geq 2$. If $x \in S$ is odd, then $x = 2^r - 1$ and $x + 1 = 2^r = 2^{r+1} - 2^r$ are both in S provided $r \geq 1$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, Washington, USA; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; GIOVANNI MAZZARELLO, Florence, Italy and IAN JUNE GARCES, Ateneo de Manila University, Manila, The Philippines; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; LAWRENCE SOMER, The Catholic University of America, Washington, DC; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There were 6 incorrect or incomplete solutions.

KONEČNÝ remarks the similarity between this problem and problem 13, page 162 of Introduction to Number Theory *by W.W. Adams and L.J. Goldstein, which claims that the Diophantine equation $x^2 - y^2 = m^k$ is solvable for x, y for any given $m, k \geq 3$. (In the present problem $m = 2$ and x and y are interchanged.)*

Several solvers described S in terms of the binary expansions of its members: namely any positive integer in S has a block of 1*s followed by a (possibly empty) block of* 0*s.*

2148. [1996: 171] *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Suppose that AD, BE and CF are the altitudes of triangle ABC. Suppose that L, M, N are points on BC, CA, AB, respectively, such that $BL = DC$, $CM = EA$, $AF = NB$.

Prove that:

1. the perpendiculars to BC, CA, AB at L, M, N, respectively are concurrent;

2. the point of concurrency lies on the Euler line of triangle ABC.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let H be the orthocentre and O the circumcentre. Reflect H in O to get a point H' on OH, and let L', M', N' be the feet of the perpendiculars from H' onto BC, CA and AB, respectively. If X, Y, Z denote the midpoints of BC, CA, AB, then $HD \parallel OX \parallel H'L'$ (and similarly for the other sides). As $HO = OH'$ we have $DX = XL'$ and, together with $XC = XD + DC = BL' + L'X = BX$, yields that $DC = BL'$. Hence from the definition of L and since L' is also an interior point of BC, we have $L = L'$ and analogously $M = M'$ and $N = N'$. The perpendiculars to the sides of the triangle LH', MH', and NH' intersect at H', which lies on the Euler line HO of $\triangle ABC$, as we wanted to prove.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; L. RICE, Woburn CI, Scarborough Ontario; TOSHIO SEIMIYA, Kawasaki, Japan; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. One solution was incomplete.

Several readers reduced the result to the theorem, attributed sometimes to Steiner and sometimes to Carnot, that appeared recently in CRUX with MAYHEM [1997: 122] *in an alternative solution to problem 2120. Although this approach leads to a nice solution here, it takes more work to set up the machinery than to solve the problem without it.*

2149. [1996: 171] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $ABCD$ be a convex quadrilateral and O is the point of the intersection of the diagonals AC and BD. Let $A'B'C'D'$ be the quadrilateral whose vertices, A', B', C', D', are the feet of the perpendiculars drawn from the point O to the sides BC, CD, DA, AB, respectively.

Prove that $ABCD$ is an inscribed (cyclic) quadrilateral if and only if $A'B'C'D'$ is a circumscribing quadrilateral ($A'B'$, $B'C'$, $C'D'$, $D'A'$ are tangents to a circle).

Solution by D.J. Smeenk, Zaltbommel, the Netherlands (with notation modified by the editor)

Assume $ABCD$ is cyclic. We denote:

$$\angle DBA = \angle DCA = \alpha; \quad \angle ACB = \angle ADB = \beta;$$
$$\angle BDC = \angle BAC = \gamma; \quad \angle CAD = \angle CBD = \delta.$$

[Because of the right angles at C' and D',] $AD'OC'$ is cyclic, so that $\angle OD'C' = \angle OAC' = \delta$. Likewise $BD'OA'$ is cyclic, so that $\angle A'D'O = \angle A'BO = \delta$. Thus $D'O$ is the bisector of $\angle A'D'C'$. Arguing in this manner we conclude that the angle bisectors of $A'B'C'D'$ all pass through O and, therefore, that this quadrilateral has an incircle.

For the converse assume that $A'B'C'D'$ has an incircle and obtain A, B, C, D as the appropriate intersection points of the lines through A' perpendicular to OA', through B' perpendicular to OB', etc. We are to show that $ABCD$ is cyclic and O is the point where AC and BD intersect.

We denote:

$$\angle OA'D' = \angle B'A'O = \alpha; \quad \angle OB'A' = \angle C'B'O = \beta;$$
$$\angle OC'B' = \angle D'C'O = \gamma; \quad \angle OD'C' = \angle A'D'O = \delta.$$

with

$$\alpha + \beta + \gamma + \delta = \pi. \tag{1}$$

$AD'OC'$ cyclic implies $\angle OAC' = \angle OD'C' = \delta$, and $\angle D'AO = \angle D'C'O = \gamma$.

In a similar way,

$$\angle C'DO = \beta \quad \text{and} \quad \angle ODB' = \gamma;$$
$$\angle B'CO = \alpha \quad \text{and} \quad \angle OCA' = \beta;$$
$$\angle A'BO = \delta \quad \text{and} \quad \angle OBD' = \alpha.$$

Consider the quadrangle $AOCD$. We have

$$\angle OAD + \angle ADC + \angle DCO + \angle COA = 2\pi;$$

that is,

$$\delta + (\beta + \gamma) + \alpha + \angle COA = 2\pi. \tag{2}$$

Equations (1) and (2) imply $\angle COA = \pi$, so that O lies on AC. Arguing analogously, O lies also on BD, so that O is the desired intersection point. Finally, opposite angles have the sum

$$\angle BAD + \angle DCB = (\gamma + \delta) + (\alpha + \beta) = \pi;$$

thus $ABCD$ is cyclic.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain (with two proofs of the "only if" portion); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK ("only if" portion); TOSHIO SEIMIYA, Kawasaki, Japan; SHAILESH SHIRALI, Rishi Valley School, India; and the proposer.

FLORIAN HERZIG, student, Perchtoldsdorf, Austria commented that the "only if" portion follows immediately from problem 3 in the 1990 Canadian Mathematical Olympiad, CRUX [1990; 161, 198].

Seimiya added a warning that the parenthetical comment appended to the statement of the problem actually leads to a different result which turns out to be not quite correct; if the lines $A'B'$, $B'C'$, $C'D'$, $D'A'$ are tangents to a circle then it is not always true that $ABCD$ is an inscribed quadrilateral. For a counterexample, let $\angle BAC + \angle BDC = \pi$, and $\angle BAC \neq \pi/2$. If T is the point where AB intersects CD, then T, A, O, D are concyclic, so that by Simson's theorem, B', C', D' are collinear. The incircle of $\triangle A'B'D'$ is tangent to the lines $A'B'$, $B'C'$, $C'D'$, $D'A'$; yet A, B, C, D are not concyclic.

2150. [1996: 171] *Proposed by Šefket Arslanagić, Berlin, Germany.*
Find all real solutions of the equation

$$\sqrt{1-x} = 2x^2 - 1 + 2x\sqrt{1-x^2}.$$

I. *Solution by Kee-Wai Lau, Hong Kong.*
By squaring both sides of the equation and simplifying we obtain

$$-x = 4x(2x^2-1)\sqrt{1-x^2}. \tag{1}$$

Clearly $x = 0$ is not a solution. So by (1) we obtain

$$-1 = 4(2x^2-1)\sqrt{1-x^2}. \tag{2}$$

Let $y = \sqrt{1-x^2}$. Then (2) becomes

$$\begin{aligned} 8y^3 - 4y - 1 &= 0 \\ \text{or} \quad (2y+1)(4y^2-2y-1) &= 0. \end{aligned}$$

Since y is positive, $y = \dfrac{1+\sqrt{5}}{4}$ and $x = \pm\sqrt{\dfrac{5-\sqrt{5}}{8}}$. For the negative value of x, both $2x^2 - 1$ and $2x\sqrt{1-x^2}$ are negative and so cannot be a solution. For $0 \leq x \leq 1$ the continuous function

$$\sqrt{1-x} - (2x^2-1) + 2x\sqrt{1-x^2}$$

is positive when $x = 0$ and negative when $x = 1$. Thus the function vanishes at least once in the interval $[0, 1]$. We conclude that the only solution of the equation is $x = \sqrt{\frac{5-\sqrt{5}}{8}}$.

II. *Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*

Squaring both sides of the given equation and simplifying the resulting equation yields

$$4(1-2x^2)\sqrt{1-x^2} = 1.$$

Since x lies between -1 and 1, we may set $x = \sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then we get $4(1-2\sin^2\theta)\sqrt{1-\sin^2\theta} = 1$, or equivalently, $4(2\cos^2\theta - 1)\cos\theta = 1$. Now the equivalent cubic equation $8\cos^3\theta - 4\cos\theta - 1 = 0$ can be rewritten as

$$(2\cos\theta + 1)(4\cos^2\theta - 2\cos\theta - 1) = 0.$$

Since $\cos\theta \ge 0$, we must have $\cos\theta = \frac{1+\sqrt{5}}{4}$. Thus $\sin\theta = \frac{\pm\sqrt{10-2\sqrt{5}}}{4}$. Since only the positive solution satisfies the original equation, we have $x = \frac{\sqrt{10-2\sqrt{5}}}{4}$. ($\theta$, incidentally, is $\pi/5$.)

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RUSSELL EULER, NW Missouri State University, Maryville, Missouri, USA; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; AMIT KHETAN, Troy, MI, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; BEATRIZ MARGOLIS, Paris, France; J.A. MCCALLUM, Medicine Hat, Alberta; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; L. RICE, University of Toronto Schools, Toronto, Ontario; KRISTIAN SABO, student, Osijek, Croatia; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Geor-

gia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; NISHKA VIJAY, student, Mount Allison University, Sackville, New Brunswick; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. There were 2 incorrect solutions.

Most solvers who used a trigonometric approach began immediately with the statement that $-1 \leq x \leq 1$, and then used either a sine or cosine substitution. This assumes that we are restricting our attention to an equation with real values, but we are told only that the variable x must be real-valued. Only HANKIN explicitly showed that there were no real-valued solutions to be found outside the above range.

2151. [1996: 217] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$\triangle ABC$ is a triangle with $\angle B = 2\angle C$. Let H be the foot of the perpendicular from A to BC, and let D be the point on the side BC where the excircle touches BC. Prove that $AC = 2(HD)$.

Solution by Cristóbal Sánchez–Rubio, I.B. Penyagolosa, Castellón, Spain.

Let $s = (a+b+c)/2$ be the semiperimeter of $\triangle ABC$; we have $b + CD = c + BD = s$, so that

$$DC = s - b = (a - b + c)/2.$$

Let C' be the image of A under the rotation about B through the angle $180° - \angle B$. Because $\triangle ABC'$ is isosceles, from the condition $\angle B = 2\angle C$ it follows that $\angle CAC'$ is also isosceles so that $HC = (a+c)/2$. Finally,

$$HD = HC - DC = \frac{a+c}{2} - \frac{a-b+c}{2} = \frac{b}{2} = \frac{AC}{2}.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; GEORGI DEMIZEV, Varna, Bulgaria, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; LUIZ A. PONCE, Santos, Brazil; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Many solvers used the fact that $\angle B = 2\angle C$ is equivalent to the condition $b^2 = c(c+a)$, which has appeared before in CRUX: [1976: 74], [1984: 278], *and* [1996: 265–267]. *For a recent problem concerning integer–sided triangles with $\angle B = 2\angle C$ see the solution to problem 578 in* The College Math J. **28**:*3 (May, 1997) 233–235. Further references are given there.*

2152. [1996: 217] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 2$ and $0 \leq x_1 \leq \ldots \leq x_n \leq \frac{\pi}{2}$ be such that $\sum_{k=1}^{n} \sin x_k = 1$.

Consider the set S_n of all sums $x_1 + \ldots + x_n$.

1. Show S_n is an interval.
2. Let l_n be the length of S_n. What is $\lim_{n\to\infty} l_n$?

I. *Solution by Florian Herzig, student, Perchtoldsdorf, Austria.*

1. First I prove that

$$n \arcsin \frac{1}{n} \leq x_1 + x_2 + \cdots + x_n \leq \frac{\pi}{2}.$$

Since the graph of $\sin x$ is concave down for $x \in [0, \pi/2]$, the chord joining the points $(0, \sin 0)$ and $(\pi/2, \sin \pi/2)$ lies below the graph. Hence [since this chord has slope $2/\pi$]

$$\frac{2x}{\pi} \leq \sin x \qquad \text{for all } x \in [0, \pi/2]$$

and we can deduce the right-hand side of the claim:

$$\frac{2}{\pi}(x_1 + x_2 + \cdots + x_n) \leq \sin x_1 + \sin x_2 + \cdots + \sin x_n = 1.$$

The left-hand side follows immediately from Jensen's inequality, since $\sin x$ is concave down for $x \in [0, \pi/2]$ and $0 \leq (x_1 + \cdots + x_n)/n < \pi/2$:

$$\frac{1}{n} = \frac{\sin x_1 + \sin x_2 + \cdots + \sin x_n}{n} \leq \sin\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right).$$

Next I will show that the set S_n of all sums $x_1 + x_2 + \cdots + x_n$ is the interval

$$\left[n \arcsin \frac{1}{n}, \frac{\pi}{2}\right].$$

First let $0 \leq x \leq y \leq \arcsin A$ be such that $\sin x + \sin y = A$, where $0 \leq A \leq 1$ is fixed. I prove that $x + y$ is a continuous function f depending on x only: for each value of x there is exactly one value of y such that all conditions are fulfilled, since

$$\sin y = A - \sin x \in [0, A] \qquad \text{if and only if} \qquad y = \arcsin(A - \sin x).$$

Thus $f(x) = x + \arcsin(A - \sin x)$. It is clear that f is continuous for $x \in [0, \arcsin A]$.

Now we consider the following process. Initially let $x_1 = \cdots = x_n = \arcsin(1/n)$ [so that $\sum \sin x_i = 1$]. In the first step keep $x_3, \ldots, x_n$ constant and vary x_1, x_2 such that $\sin x_1 + \sin x_2$ remains constant and x_1 becomes zero. Thereby $\sin x_1 + \sin x_2 + \cdots + \sin x_n$ remains constant (equal to 1) throughout and $x_1 + x_2 + \cdots + x_n$ varies continuously because of the previous result.

In the second step keep $x_1 = 0$, $x_4, \ldots, x_n$ constant and vary x_2 and x_3 as before, with $x_2 \to 0$. Again $x_1 + x_2 + \cdots + x_n$ varies continuously.

It is clear now how to vary the x_i step by step such that in the end $x_1 = x_2 = \cdots = x_{n-1} = 0$ and as a consequence $x_n = \pi/2$. As we see, $x_1 + \cdots + x_n$ has varied continuously from $n \arcsin(1/n)$ to $\pi/2$. Hence

$$S_n = \left[n \arcsin \frac{1}{n}, \frac{\pi}{2} \right].$$

2. By l'Hôpital's rule,

$$\begin{aligned} \lim_{n\to\infty} l_n &= \lim_{n\to\infty} \left(\frac{\pi}{2} - n \arcsin \frac{1}{n} \right) \\ &= \frac{\pi}{2} - \lim_{x\to 0} \frac{\arcsin x}{x} \\ &= \frac{\pi}{2} - \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}} \\ &= \frac{\pi}{2} - 1. \end{aligned}$$

II. *Solution by Thomas C. Leong, The City College of City University of New York, New York, NY, USA.*

Equivalently, we consider the set

$$Y = \{\mathrm{y} = (y_1, \ldots, y_n) \mid 1 \geq y_1 \geq \cdots \geq y_n \geq 0,\ y_1 + \cdots + y_n = 1\} \subset \mathbb{R}^n$$

and the image $f(Y)$ of Y under $f(\mathrm{y}) = \arcsin y_1 + \cdots + \arcsin y_n$. Note that $S_n = f(Y)$.

1. Since Y is a connected subspace of $\mathbb{R}^n$ and f is a continuous function, the image $f(Y) \subset \mathbb{R}$ is also connected, and we know that the only connected subspaces of $\mathbb{R}$ are intervals. Thus S_n is an interval.

2. Since $\arcsin x$ is convex in $[0, 1]$, we can use the majorization inequality. Since

$$\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) \prec (y_1, y_2, \ldots, y_n) \prec (1, 0, \ldots, 0)$$

[*Editor's note*: here $\mathbf{x} \prec \mathbf{y}$ means that $\mathbf{x}$ is majorized by $\mathbf{y}$; see for example page 45 and then Theorem 108, page 89 of Hardy, Littlewood and Pólya's *Inequalities*],

$$n \arcsin \frac{1}{n} \leq \arcsin y_1 + \cdots + \arcsin y_n \leq \arcsin 1 = \frac{\pi}{2},$$

with equality when $(y_1, \dots, y_n)$ is equal respectively to $(1/n, \dots, 1/n)$ and $(1, 0, \dots, 0)$. Thus $l_n = \frac{\pi}{2} - n\arcsin(1/n)$ and

$$\lim_{n\to\infty} l_n = \frac{\pi}{2} - \lim_{n\to\infty}\left(n\arcsin\frac{1}{n}\right) = \frac{\pi}{2} - \lim_{n\to\infty}\left(\frac{\arcsin(1/n)}{1/n}\right) = \frac{\pi}{2} - 1.$$

Also solved by KEITH EKBLAW, Walla Walla, Washington, USA; and RICHARD I. HESS, Rancho Palos Verdes, California, USA.

2153. [1996: 217] *Proposed by Šefket Arslanagić, Berlin, Germany.*

Suppose that $a, b, c \in \mathbb{R}$. If, for all $x \in [-1, 1]$, $|ax^2 + bx + c| \le 1$, prove that

$$|cx^2 + bx + a| \le 2.$$

Solution (virtually identical) by Florian Herzig, student, Perchtoldsdorf, Austria, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $f(x) = ax^2 + bx + c$ and $g(x) = cx^2 + bx + a$. Then by assumption $|a + b + c| = |f(1)| \le 1$, $|a - b + c| = |f(-1)| < 1$ and $|f(0)| \le 1$. Hence

$$\begin{aligned} |g(x)| &= \left|c(x^2-1) + (a+b+c)\frac{1+x}{2} + (a-b+c)\frac{1-x}{2}\right| \\ &\le |c||x^2-1| + |a+b+c|\frac{|1+x|}{2} + |a-b+c|\frac{|1-x|}{2} \\ &\le |x^2-1| + \frac{|1+x|}{2} + \frac{|1-x|}{2} \\ &= 1 - x^2 + \frac{1+x}{2} + \frac{1-x}{2} \\ &= 2 - x^2 \le 2 \end{aligned}$$

for all $x \in [-1, 1]$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; GEORGI DEMIZEV, Varna, Bulgaria, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the proposer. There was one incorrect solution.

Several solvers pointed out that this is not a new problem. Herzig cited the book International Mathematical Olympiad 1978–1985 *(published by the Mathematical Association of America, 1986). Chronis gave the reference* Selected Problems and Theorems in Elementary Mathematics — Arithmetic and Algebra *by D.O. Shklyarsky, N.N. Chentov, and I.M. Yaglom (exercise 304 on page 60). Demizev and Kunchev remarked that it can be found as problem #86 on page 35 of the book* Mathematics Competitions *by L. Davidov,* *V.Petnob, I. Tonov, and V. Chukanov (Sofia, 1977). Bellot Rosado mentioned that the same problem was proposed during the 2nd round of the 1996 Spanish Mathematical Olympiad.*

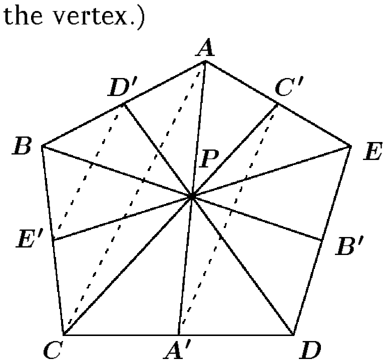
Janous remarked that the problem can be restated as "Prove that if $p(x)$ is a real polynomial of degree 2 satisfying the condition that $|p(x)| \leq 1$ for all $x \in [-1,1]$, then $|x^2 p(1/x)| \leq 2$ for all $x \in [-1,1]$." (Ed: $x^2p(1/x)$ is to be written as a polynomial before substituting values of x.) He then ventured the following conjecture:

Conjecture: *If $p(x)$ is a real polynomial with degree n such that $|p(x)| \leq 1$ for all $x \in [-1,1]$ then $|x^n p(1/x)| \leq 2^n$ for all $x \in [-1,1]$.*

Can any reader prove or disprove this?

2154. [1996: 217] *Proposed by K.R.S. Sastry, Dodballapur, India.*

In a convex pentagon, the medians are concurrent. If the concurrence point sections each median in the same ratio, find its numerical value. (A median of a pentagon is the line segment between a vertex and the midpoint of the third side from the vertex.)



I *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let the convex pentagon be $ABCDE$, and let A', B', C', D', E' be the midpoints of CD, DE, EA, AB, BC, respectively. We assume that AA', BB', CC', DD', EE' are concurrent at P, and that

$$\frac{AP}{PA'} = \frac{BP}{PB'} = \frac{CP}{PC'} = \frac{DP}{PD'} = \frac{EP}{PE'} = \lambda.$$

Then set $D'E' = 1$. Since D', E' are midpoints of AB, BC, respectively, we have $AC = 2D'E' = 2$, and $D'E' \parallel AC$. Since

$$\frac{DP}{PD'} = \frac{EP}{PE'} = \lambda,$$

we have $D'E' \parallel DE$ and $DE/D'E' = \lambda$, so that

$$DE = \lambda D'E' = \lambda. \tag{1}$$

Since $AP/PA' = CP/PC' = \lambda$, we have $AC/A'C' = AP/PA' = \lambda$, so that

$$A'C' = \frac{AC}{\lambda} = \frac{2}{\lambda}. \qquad (2)$$

Since $AC \parallel D'E'$ and $D'E' \parallel DE$, we get $AC \parallel DE$, and since A', C' are midpoints of CD, AE, respectively, we have

$$AC + DE = 2A'C'.$$

Therefore we have from (1) and (2)

$$2 + \lambda = \frac{4}{\lambda},$$

from which we have $\lambda^2 + 2\lambda - 4 = 0$. Thus we obtain $\lambda = \sqrt{5} - 1$.

II *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(We will use the same notation as used by Seimiya above. Each boldface letter represents the vector to that point from the origin.) Then we have $\mathbf{D}' = \frac{1}{2}(\mathbf{A} + \mathbf{B})$, $\mathbf{E}' = \frac{1}{2}(\mathbf{B} + \mathbf{C})$, etc. There must exist real numbers $\alpha, \beta > 0$, with $\alpha + \beta = 1$, such that

$$\mathbf{P} = \alpha\mathbf{A} + \beta\mathbf{A}' = \alpha\mathbf{B} + \beta\mathbf{B}' = \alpha\mathbf{C} + \beta\mathbf{C}' = \alpha\mathbf{D} + \beta\mathbf{D}' = \alpha\mathbf{E} + \beta\mathbf{E}'.$$

Now

$$\alpha\mathbf{D} + \beta\mathbf{D}' = \alpha\mathbf{A} + \beta\mathbf{A}' \iff 2\alpha\mathbf{D} + \beta(\mathbf{A} + \mathbf{B}) = 2\alpha\mathbf{A} + \beta(\mathbf{C} + \mathbf{D}); \ (3)$$
$$\alpha\mathbf{A} + \beta\mathbf{A}' = \alpha\mathbf{B} + \beta\mathbf{B}' \iff 2\alpha\mathbf{A} + \beta(\mathbf{C} + \mathbf{D}) = 2\alpha\mathbf{B} + \beta(\mathbf{D} + \mathbf{E}). \ (4)$$

Equation (4) can be simplified to

$$2\alpha\mathbf{A} + \beta\mathbf{C} = 2\alpha\mathbf{B} + \beta\mathbf{E} \quad \iff \quad 2\alpha(\mathbf{B} - \mathbf{A}) = \beta(\mathbf{C} - \mathbf{E}),$$

which implies that $AB \parallel EC$. Similarly,

$$BC \parallel AD, \quad CD \parallel BE, \quad DE \parallel CA, \quad EA \parallel DB. \qquad (5)$$

Now equation (3) can be written as:

$$\begin{aligned} 2\alpha\mathbf{D} + \beta\mathbf{A} - 2\alpha\mathbf{A} - \beta\mathbf{D} &= \beta\mathbf{C} - \beta\mathbf{B}, \\ (2\alpha - \beta)(\mathbf{D} - \mathbf{A}) &= \beta(\mathbf{C} - \mathbf{B}). \end{aligned}$$

On the other hand from (5) we have

$$2\alpha(\mathbf{C} - \mathbf{B}) = \beta(\mathbf{D} - \mathbf{A}).$$

Hence,

$$\mathbf{D} - \mathbf{A} = \frac{\beta}{2\alpha - \beta}(C - \mathbf{B}) = \frac{2\alpha}{\beta}(\mathbf{C} - \mathbf{B}),$$

whence we obtain

$$\begin{aligned} \frac{\beta}{2\alpha - \beta} &= \frac{2\alpha}{\beta} \\ 4\alpha^2 - 2\alpha\beta - \beta^2 &= 0 \\ 4\alpha^2 - 2\alpha(1-\alpha) - (1-\alpha)^2 &= 0 \\ 5\alpha^2 &= 1 \\ \alpha &= \frac{\sqrt{5}}{5}. \end{aligned}$$

Thus the ratio we are considering has the numerical value

$$\alpha : \beta = \alpha : (1 - \alpha) = 1 : (\sqrt{5} - 1)$$

Also solved by SAM BAETHGE, Nordheim, Texas, USA; MIHAI CIPU, Romanian Academy, Bucharest; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; GEORGI DEMIZEV, Varna, Bulgaria, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California, USA; D. KIPP JOHNSON, Valley Catholic High School, Beaverton, Oregon; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Seimiya also observes the following:

1. *If four medians are concurrent at a point P, then the fifth median also passes through P.*
2. *We assume that five medians are concurrent at a point P. If any three medians are divided at P into the same ratio $\lambda : 1$, then the other two medians are divided at P into the same ratio $\lambda : 1$.*
3. *This figure is an image of a regular pentagon by an affine transformation.*

2155. [1996: 218] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Prove there is no solution of the equation

$$\frac{1}{x^2} + \frac{1}{y^8} = \frac{1}{z^2}$$

in which y is odd and x, y, z are positive integers with highest common factor 1.

Find a solution in which $y = 15$, and x and z are also positive integers.

Solution by D. Kipp Johnson (modified slightly by the editor), Valley Catholic High School, Beaverton, Oregon, USA.

First, we find the general solution in $\mathbb{N}$ to the auxiliary equation

$$\frac{1}{x^2} + \frac{1}{t^2} = \frac{1}{z^2} \qquad \text{where } t \text{ is odd.} \tag{1}$$

(1) can be written as $z^2(x^2+t^2) = x^2t^2$, which shows that x^2+t^2 must be a square. From well–known results, we then have $x = k(2uv)$, $t = k(u^2-v^2)$, where $k \in \mathbb{N}$ is odd, and u, v are relatively prime integers of opposite parity with $u > v$. Substituting into (1) and simplifying gives

$$z(u^2+v^2) = 2kuv(u^2-v^2). \tag{2}$$

Since u^2+v^2 is odd, $z = 2w$ for some $w \in \mathbb{N}$ and (2) becomes

$$w(u^2+v^2) = kuv(u^2-v^2). \tag{3}$$

Since $(u,v) = 1$, $(u^2+v^2, uv) = 1$. Furthermore, $(u^2+v^2, u^2-v^2) = 1$ since if p is a prime such that $p|u^2+v^2$, u^2-v^2, then $p|2u^2$, $2v^2$ implies $p|(2u^2, 2v^2)$. But $(2u^2, 2v^2) = 2(u,v)^2 = 2$ and clearly $p \neq 2$ as u^2+v^2 is odd. This is a contradiction. Hence $u^2+v^2|k$. Letting $k = j(u^2+v^2)$, then (3) becomes

$$w = juv(u^2-v^2) \qquad \text{where } j \in \mathbb{N} \text{ is odd.}$$

Therefore, we may choose u, v subject to the stated conditions and arbitrarily odd $j \in \mathbb{N}$ to find the general solution of the original equation, which is given by:

$$x = 2juv(u^2+v^2), \quad y^4 = t = j(u^2-v^2)(u^2+v^2), \quad z = 2juv(u^2-v^2).$$

Since $(x,y,z) = 1$, we must have $j = 1$ and so $y^4 = u^4 - v^4$, which is impossible by Fermat's Last Theorem.

To find a solution in which $y = 15$, let

$$y^4 = 15^4 = 3^4 \times 5^4 = j(u-v)(u+v)(u^2+v^2). \tag{4}$$

Since $u-v$, $u+v$, and u^2+v^2 are pairwise relatively prime, and the left side of (4) contains only two distinct prime factors, $u - v = 1$ and (4) becomes

$$15^4 = j(2v+1)(2v^2+2v+1). \tag{5}$$

From (5) we see that $15^4 > 4v^3$ and so $v < 24$ resulting in $2v+1 < 49$. Now, $2v+1$ is odd and the only odd factors of 15^4 which are less than 49 are 3, 5, 9, 15, 25 and 45 with corresponding values of $v = 1$, 2, 4, 7, 12, and 22. A quick check shows that only in the case $v = 1$ is $2v^2+2v+1$ also a factor of 15^4. Thus $u = 2$ and (5) gives $j = 3^3 \times 5^3 = 3375$ resulting in the *unique* solution:

$$x = 67500, \quad y = 15, \quad z = 40500.$$

Also solved (both parts) by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, and the proposer. The second part of the problem was also solved by GEORGI DEMIZEV, Varna, Bulgaria, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria (jointly); HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; and DIGBY SMITH, Mount Royal College, Calgary, Alberta. All these solvers gave the same solution $(67500, 15, 40500)$. *However, most of their "proofs" erred in assuming that* $\gcd(x, y, z) = 1$ *implies that* x, y, z *are pairwise relatively prime. Furthermore, Johnson was the only one who gave a completely valid proof for the uniqueness of the solution when* $y = 15$, *though Janous found this to be the case by using* **DERIVE**.

Both Hess and Klamkin considered the given equation without the restriction that y *be odd and* $(x, y, z) = 1$. *Hess observed that an infinite family of solutions is given by* $x = 2mn(m^2+n^2)^4(m^2-n^2)^3$, $y = m^4-n^4$, *and* $z = 2mn(m^2+n^2)^3(m^2-n^2)^4$ *where* $m, n \in \mathbb{N}$ *with* $m > n$ *while Klamkin gave the family:* $x = 2^2 \times 3^3 \times 5^4 \times n^4$, $y = 15n$, $z = 2^2 \times 3^4 \times 5^3 \times n^4$. *Both formulas yield the solution given above when* $m = 2$ *and* $n = 1$.

2156. [1996: 218] *Proposed by Hoe Teck Wee, Singapore.*

$ABCD$ is a convex quadrilateral with perpendicular diagonals AC and BD. X and Y are points in the interior of sides BC and AD respectively such that

$$\frac{BX}{CX} = \frac{BD}{AC} = \frac{DY}{AY}\,.$$

Evaluate

$$\frac{BC \cdot XY}{BX \cdot AC}\,.$$

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

Let Z be the point in the interior of side AB such that $BZ : AZ = BX : CX$. Then it follows from

$$\frac{BZ}{AZ} = \frac{BX}{CX} = \frac{DY}{AY}\left(= \frac{BD}{AC}\right)$$

that XZ and AC as well as ZY and BD are parallel, which, considering $AC \perp BD$, implies that $\triangle XYZ$ is a right triangle with legs XZ and YZ. Thus

$$\frac{XZ}{ZY} = \frac{XZ}{AC}\cdot\frac{AC}{BD}\cdot\frac{BD}{ZY} = \frac{BZ}{AB}\cdot\frac{AZ}{BZ}\cdot\frac{AB}{AZ} = 1$$

[where $XZ/AC = BZ/AB$ and $BD/ZY = AB/AZ$ by similar triangles while $AC/BD = AZ/BZ$ by the definition of Z]. This means that $\triangle XYZ$

is an equilateral right triangle, whence

$$XY : ZX = \sqrt{2} : 1$$

and, consequently,

$$\frac{BC \cdot XY}{BX \cdot AC} = \frac{AC \cdot XY}{ZX \cdot AC} = \frac{XY}{ZX} = \sqrt{2}.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; GEORGI DEMIZEV, Varna, Bulgaria, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Bellot comments that facts about orthodiagonal quadrilaterals can be found in Jordan Tabov, Simple properties of the orthodiagonal quadrilaterals, Matematyka & Informatyka, **1**:1 (1991) 1–5. *Janous, using rectangular coordinates, shows that if the angle from AC to BD were ϕ, then*

$$\frac{BC \cdot XY}{BX \cdot AC} = 2\sin\frac{\phi}{2}.$$

2157. [1996: 218] *Proposed by Šefket Arslanagić, Berlin, Germany.*
Prove that $2^{1997 \cdot 1996} - 1$ is exactly divisible by 1997^2.

I. *Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

From the Fermat-Euler theorem we have $a^{\varphi(m)} \equiv 1 \pmod{m}$, for every a, m such that $\gcd(a, m) = 1$. We also have $\varphi(p^k) = p^{k-1}(p-1)$, when p is a prime number.

So $\varphi(1997^2) = 1997 \cdot 1996$ and $2^{\varphi(1997^2)} \equiv 1 (\mathrm{mod}\ 1)997^2$.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We have $\varphi(1997) = 1996 = 2^2 \cdot 499$.

Furthermore, $2^{449} \equiv 1585 (\mathrm{mod}\ 1)997$ and $2^{2 \cdot 449} \equiv -1 (\mathrm{mod}\ 1)997$.

Hence 2 is of order 1996 modulo 1997; that is

$$\alpha := 2^{1996} - 1 \equiv 0 \pmod{1997}.$$

On the other hand,

$$\alpha \equiv 672989 \pmod{1997^2}, \text{ that is, } \alpha \not\equiv 0 \pmod{1997^2}.$$

But

$$\begin{aligned}\beta &:= 2^{1997\cdot1996}-1=((2^{1996}-1)+1)^{1997}-1=(\alpha+1)^{1997}-1\\ &= 1+1997\alpha+\binom{1997}{2}\alpha^2+\binom{1997}{3}\alpha^3+\cdots+\alpha^{1997}-1\\ &= 1997\alpha+1997^3\cdot f,\end{aligned}$$

where f is an integer. Thus $\beta\equiv 0(\mathrm{mod}\ 1)997^2$, but $\beta\not\equiv 0(\mathrm{mod}\ 1)997^3$.

Also solved by MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; MIHAI CIPU, Romanian Academy, Bucharest, Romania; GEORGI DEMIZEV, Varna, Bulgaria, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; D. KIPP JOHNSON, Valley Catholic High School, Beaverton, Oregon; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; THOMAS LEONG, Staten Island, NY, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; CAN ANH MINH, University of California, Berkeley, California; SOLEDAD ORTEGA and JAVIER GUTIERRÉZ, students, Universidad de La Rioja, Logroño, Spain; YOLANDA PELLEJERO, student, Universidad de La Rioja, Logroño, Spain; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

It is interesting to note that only Janous and Manes addressed the question of "exact" divisibility; that is, they showed that no higher power of 1997 than 2 would work.