THE ACADEMY CORNER

No. 11

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

Bernoulli Trials 1997

Christopher Small
Faculty of Mathematics
University of Waterloo
Waterloo, Ontario

At the University of Waterloo, we recently held a new mathematics competition called the **Bernoulli Trials**. The Bernoulli Trials came about as the result of a brainstorming session between Ian Goulden, Ken Davidson and myself. We felt that a winter competition was most desirable, and that it had to be fundamentally different from the competitions that the University of Waterloo holds in the Fall. So we decided to hold a double-knockout competition with "true" and "false" as the response categories for each round of the competition.

Students registered ahead of time so that we had an idea of the attendance figures. This gave us a chance to draw up a master list of participants, which we put on a transparency for the overhead projector. As the rounds of the competition progressed, the students were able to see their results posted on one of the screens.

Students were given a supply of "ballots" on which to place their "votes" for each round. At the beginning of each round, we posted a mathematical conjecture on the overhead projector. Ten minutes were allotted to working on each conjecture, and at the end of that time all students had to submit a completed, signed ballot with a "true" or "false" response. When all ballots were in, we announced the correct answer and a hint for solving the problem. (There was no time for a full solution during the competition.) The next five minutes were used to record the ballots. During this time, the students were able to stretch or chat about the problem and help themselves to food provided at the back of the room. Then it was on to the next round and the next conjecture. The double-knockout format meant that everyone had a chance to make one mistake without being eliminated from the competition. Those who were eliminated at any stage were encouraged to stay and cheer on the remaining students.

In order to make the event work, we needed to have many more questions available than we actually used. The questions varied from quite easy to extremely difficult. Our plan was to adjust the level of difficulty based upon the performance of the students. The hope was that in the early stages we could keep the students interested without knocking them out too quickly. Over all, we were quite satisfied with the results. The competition started shortly after 9:00 a.m., and were finished for a late lunch around 1:00 p.m.

There were 24 students participating. The final two participants, first year students, Richard Hoshino and Soroosh Yazdani, who both lasted a full 15 rounds, were declared co-winners of the competition. Tied for third and fourth place were Kevin Hare and Frédéric Latour, who lasted 11 rounds. In fifth place was David Kennedy, who lasted 10 rounds.

The prizes were given on the spot at the end of the competition. In keeping with the name Bernoulli Trials, the prizes were handed out on the spot at the end of the competition in batches of 100 coins each: quarters for third, fourth and fifth places, loonies for second place, and toonies for first. In a spirit of generosity, Richard Hoshino and Soroosh Yazdani invited everyone to lunch with the prize money they had earned.

THE BERNOULLI TRIALS

True or False?

1. The point
$$(x, y) = \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}, \frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$
is outside the closed curve $x = 1 + 5\cos\theta$, $y = -1 + 5\sin\theta$, where $\theta \in [0, 2\pi)$.

2. Let p and q be real numbers with $p^{-1} - q^{-1} = 1$ and $0 < p \leq 1/2$. Then
$$p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \dots \;=\; q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \frac{1}{4}q^4 + \dots$$

3. There are at least two positive integers n for which $2^n - 1$ and $2^n + 1$ are primes.

4. When expressed in the decimal system, $1000!$ ends in exactly 249 zeros. That is, $1000!$ has the form $n \times 10^{249}$, where n is not divisible by 10.

5. For any set $M \subset R^3$, there exists a plane $\sigma \subset R^3$ such that either $M \cap \sigma$ has infinitely many points or $M \cap \sigma$ is empty.

6. Suppose Johann Bernoulli tosses 1997 fair coins and Jakob Bernoulli tosses 1996 fair coins.

   The probability that Johann has more heads than Jakob is more than $1/2$.

7. Let $x^{(p)}$ be defined by $x^{(1)} = x$ and $x^{(p+1)} = x^{x^{(p)}}$.

   Then $\lim_{x \to 0+} x^{(1997)} = 1$.

8. If the elementary symmetric functions of a set of real numbers are all positive, then the real numbers themselves must be positive.

9. Let $a, b, c > 0$. Then $\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} > 2(a + c - b)$.

10. There does not exist a Fibonacci number with a decimal representation ending in four zeros.

11. $$\sum_{k=1}^{25} \frac{k}{k^4 + k^2 + 1} = \frac{325}{651}$$

12. There exists a unique pair of positive integers n, m with $n > m$ such that $1997 = n^2 + m^2$.

13. $$\int_0^{\pi/2} \frac{x\,dx}{1 + \sin x} = \frac{1}{\sqrt{2}}$$

14. The equation $\sqrt[7]{x} - \sqrt[5]{x} = \sqrt[3]{x} - \sqrt{x}$ has exactly three solutions in nonnegative real x.

15. For some $a > 0$, let $f : R \to R$ satisfy

    $$f(x + a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2}$$

    for all x.

    Then f is periodic.

In the next issue, we will give the hints and the answers!

THE OLYMPIAD CORNER

No. 182

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

For your problem solving pleasure over the next few weeks, we give three contests which were collected and forwarded to us by Richard Nowakowski, when he was Canadian Team Leader to the IMO in Hong Kong.

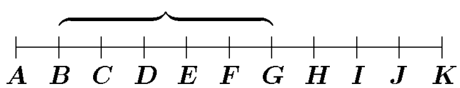
SWEDISH MATHEMATICS CONTEST 1993

Final

November 20

1. The integer x is such that the sum of the digits of $3x$ is the same as the sum of the digits of x. Prove that 9 is a factor of x.

2. A railway line is divided into 10 sections by the stations A, B, C, D, E, F, G, H, I, J and K. The distance between A and K is 56 km. A trip along two successive sections never exceeds 12 km. A trip along three successive sections is at least 17 km. What is the distance between B and G?



3. Assume that a and b are integers. Prove that the equation $a^2 + b^2 + x^2 = y^2$ has an integer solution x, y if and only if the product ab is even.

4. To each pair of real numbers a and b, where $a \neq 0$ and $b \neq 0$, there is a real number $a * b$ such that

$$a * (b * c) = (a * b) * c$$

$$\text{and} \qquad a * a = 1.$$

Solve the equation $x * 36 = 216$.

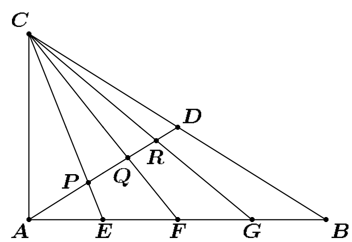
5. A triangle with perimeter $2p$ has the sides a, b and c. If possible, a new triangle with the sides $p - a$, $p - b$ and $p - c$ is formed. The process is then repeated with the new triangle. For which original triangles can the process be repeated indefinitely?

6. Let a and b be real numbers and let $f(x) = (ax + b)^{-1}$. For which a and b are there three distinct real numbers x_1, x_2, x_3 such that $f(x_1) = x_2$, $f(x_2) = x_3$ and $f(x_3) = x_1$?

DUTCH MATHEMATICAL OLYMPIAD
Second Round
September 1993

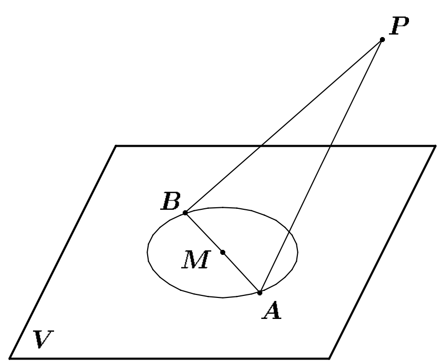
1. Suppose that $V = \{1, 2, 3, \ldots, 24, 25\}$.
Prove that any subset of V with 17 or more elements contains at least two distinct numbers the product of which is the square of an integer.

2. Given is a triangle ABC, $\angle A = 90^\circ$. D is the midpoint of BC, F is the midpoint of AB, E the midpoint of AF and G the midpoint of FB. AD intersects CE, CF and CG respectively in P, Q and R. Determine the ratio $\frac{PQ}{QR}$.



3. A series of numbers is defined as follows: $u_1 = a$, $u_2 = b$, $u_{n+1} = \frac{1}{2}(u_n + u_{n-1})$ for $n \geq 2$. Prove that $\lim_{n\to\infty} u_n$ exists. Express the value of the limit in terms of a and b.

4. In a plane V a circle C is given with centre M. P is a point not on the circle C.



(a) Prove that for a fixed point P, $AP^2 + BP^2$ is a constant for *every* diameter AB of the circle C.

(b) Let AB be any diameter of C and P a point on a fixed sphere S not intersecting V. Determine the point(s) P on S such that $AP^2 + BP^2$ is minimal.

5. $P_1, P_2, \ldots, P_{11}$ are eleven distinct points on a line. $P_iP_j \leq 1$ for every pair P_i, P_j. Prove that the sum of all (55) distances P_iP_j, $1 \leq i < j \leq 11$ is smaller than 30.

PROBLEMS OF THE 11th BALKAN MATHEMATICAL OLYMPIAD

Novi Sad, Yugoslavia

May 8–14, 1994 (Time: 4.5 hours)

1. [*Cyprus*] Given an acute angle $\angle XAY$ and a point P in its interior, construct a straight line through P and intersecting the sides AX and AY at the points B and C respectively, so that the triangle ABC has area equal to $(AP)^2$.

(Success rate: $26.6\overline{6}\%$)

2. [*Greece*] Show that the polynomial

$$x^4 - 1994x^3 + (1993 + m)x^2 - 11x + m, \quad m \in \mathbb{Z}$$

has at most one integral root. (Success rate: $39.6\overline{6}\%$)

3. [*Romania*] Compute (for $n \geq 2$)

$$\max\left\{ \sum_{k=1}^{n-1} |\alpha_{k+1} - \alpha_k| \;\middle|\; \begin{array}{r} (\alpha_k)_{k=1}^{n} \quad \text{is a permutation} \\ \text{of the numbers } 1, 2, \ldots, n \end{array} \right\}.$$

(Success rate: 30%)

4. [*Bulgaria*] Find the smallest number $n > 4$ such that there is a set of n people with the following properties:

(i) any two people who know each other have no common acquaintances;

(ii) any two people who do not know each other have exactly two common acquaintances.

Note: Acquaintance is a symmetric relation. (Success rate: 19%)

OVERALL SUCCESS RATE: 28.83%; Number of Students: 30,
Mean: $11.3\overline{3}$, Median: 7.5, S.D.: 12.3.

Recently I made time to clean off my desktop and sort a backlog of paper that piled up over the last year while I was on secondment to a university committee and only in my office for a short time each week. There appeared a small cache of solutions to problems from the 1995 numbers of the Corner which I had put down in haste (and in error), letting them become covered by other layers of incoming material. Let me try to set the record straight by acknowledging the readers' efforts now. My sincere apologies! Miguel Amengual Covas, Cala Figuera, Spain, submitted solutions to problem 2 of Part I of the Turkish Mathematical Olympiad Committee Final Selection Test, for which a solution was given in the December 1996 number

[1996: 349–350], and to problem 2 of part II of that test, for which a solution was given [1996: 352–353]. Mansur Boase, student, St. Paul's School, London, England, submitted solutions to problems 1 and 2 of Part I of the Turkish Mathematical Olympiad Final Selection Test, for which solutions were given [1996: 349–350], [1996: 349–350] and for questions 1 and 3 of Part II, for which solutions were given [1996: 351–352], [1996: 353-354]. In another submission he gave answers to problems 1, 2 and 4 of the 1992 Dutch Mathematical Olympiad, for which solutions were given [1997: 13], [1997: 13–14] and [1997: 15–17], respectively. In a third letter he provided solutions to four problems of the 16th Austrian-Polish Mathematics Competition. Two solutions of these appeared earlier this year: problem 1 in [1997: 71–72] and problem 6 in [1997: 72–73]. In yet a fourth letter he gave solutions to three of the problems proposed to the jury of the 35th IMO but not used. A solution to number 3 was given in [1997: 135]. Finally a fat envelope of solutions by George L. Evagelopoulos to problems proposed but not used at the 35th IMO appeared. Below we give some of the solutions submitted by Mansur Boase but missed before. First back to problems of the 2nd Mathematical Olympiad of the Republic of China.

1. [1995: 80] *2nd Mathematical Olympiad of the Republic of China.*

A sequence $\{a_n\}$ of positive integers is defined by $a_n = [n + \sqrt{n} + (1/2)]$, $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. Determine the positive integers which belong to the sequence.

Solution by Mansur Boase, student, St. Paul's School, London, England.

Now $a_n = [n + \sqrt{n} + \frac{1}{2}]$, and $a_{n+1} = [n + 1 + \sqrt{n+1} + \frac{1}{2}]$ so

$$a_{n+1} - a_n = 2 \quad \text{if} \quad \sqrt{n+1} > \left[\sqrt{n+1}\right] + \frac{1}{2} = a + \frac{1}{2}$$

$$\underline{\text{and}} \quad \sqrt{n} < [\sqrt{n}] + \frac{1}{2} = a + \frac{1}{2} \text{ or } a - \frac{1}{2} \text{ if } n = a^2 - 1$$

and $a_{n+1} - a_n = 1$ otherwise.

For $a_{n+1} - a_n = 2$ we have $n + 1 > a^2 + a + \frac{1}{4}$ and $n < a^2 + a + \frac{1}{4}$, since for $a \geq 2$, $\sqrt{a^2 - 1} \not< a - \frac{1}{2}$. Thus $n + 1 = a^2 + a + 1$. Now

$$\left[a^2 + a + \sqrt{a^2 + a} + \frac{1}{2}\right] = a^2 + 2a$$

$$\left[a^2 + a + 1 + \sqrt{a^2 + a + 1} + \frac{1}{2}\right] = a^2 + 2a + 2.$$

Thus the only numbers left out are of the form $a^2 + 2a + 1 = (a + 1)^2$; that is, the squares. Thus all positive non-squares belong to the sequence.

Now to problems given in the September number.

3. [1995: 221] 16*th Austrian Polish Mathematical Competition*.

Let the function f be defined as follows:

If $n = p^k > 1$ is a power of a prime number p, then $f(n) := n + 1$.

If $n = p_1^{k_1} \cdots p_r^{k_r}$ $(r > 1)$ is a product of powers of pairwise different prime numbers, then $f(n) := p_1^{k_1} + \cdots + p_r^{k_r}$.

For every $m > 1$ we construct the sequence $\{a_0, a_1, \dots\}$ such that $a_0 = m$ and $a_{j+1} = f(a_j)$ for $j \geq 0$. We denote by $g(m)$ the smallest element of this sequence. Determine the value of $g(m)$ for all $m > 1$.

Solution by Mansur Boase, student, St. Paul's School, London, England.

It is evident that $g(2) = 2$, $g(3) = 3$, $g(4) = 4$, $g(5) = 5$, $g(6) = 6$, $g(7) = 7$.

Lemma. For $m > 6$, $g(m) > 6$.

Proof. Consider $m > 6$, and its associated sequence.

$$6 = 1 + 5 = 2 + 4.$$

Now $2+4$ is not permissible since $(2, 4) \neq 1$, and the only way 6 can be part of the sequence is if 5 occurs earlier.

$$5 = 1 + 4 = 2 + 3.$$

So 5 is only part of the sequence if 4 or 6 occur earlier.

$$4 = 1 + 3 = 2 + 2,$$

so 4 is part of the sequence only if 3 occurs earlier.

$$3 = 1 + 2,$$

so 3 is part of the sequence only if 2 occurs earlier, which is impossible with $a_0 > 6$.

Consider the next several values of $g(m)$. By calculation $g(8) = 7 = g(9) = g(10) = g(11) = g(12) = g(13) = g(14) = g(15) = g(16)$. We next prove by induction that if $m \geq 16$ then $g(m) = 7$. We may assume $m > 16$.

Observe that at least one of m, $m+1$, $m+2$, $m+3$, $m+4$, $m+5$ is not a power of a prime because it is congruent to $0 \bmod 6$. Letting n be the first value which is not a power of a prime $m \leq n \leq m + 5$. The result follows immediately from the following since $g(m) \leq g(n) \leq f(n)$.

Lemma. Suppose $n > 16$ is not a power of a prime. Then $f(n) < n-5$.
Proof. Write $n = ab$ with $1 < a < b$ and $(a, b) = 1$. Then $b \geq 5$. Now

$$a + b < ab - 5$$

is equivalent to $1 + \frac{a}{b} + \frac{5}{b} < a$ which is immediate unless $a = 2$ because $\frac{a}{b} < 1$, $\frac{5}{b} \leq 1$. But if $a = 2$ then $b > 8$ as $ab > 16$ and

$$1 + \frac{a}{b} + \frac{5}{b} < 1 + \frac{2}{8} + \frac{5}{8} < 2.$$

The weaker inequality $a + b < ab$ for $1 < a < b$ is immediate. It follows that

$$f(n) < a + b < ab - 5$$

if $n > 16$.

This completes the proof.

4. [1995: 221] 16*th Austrian Polish Mathematical Competition*.

The Fibonacci numbers are defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Let A and B be natural numbers such that A^{19} divides B^{93} and also B^{19} divides A^{93}. Prove: for all natural numbers $n \geq 1$ the number $(A^4 + B^8)^{F_{n+1}}$ is divisible by $(AB)^{F_n}$.

Solution by Mansur Boase, student, St. Paul's School, London, England.

Now $A^{19} \mid B^{93}$ and $B^{19} \mid A^{93}$. Thus any prime, p, which divides A must divide B, and vice versa.

Also if $p^{\alpha_1} \mid A$ then $p^{[(93/19)\alpha_1]}$ is the largest power of p which can divide B and vice versa. $(*)$

First we must prove that

$$AB \mid (A^4 + B^8)^2. \tag{1}$$

Let

$$A = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n},$$
$$B = p_1^{\beta_1} p_2^{\beta_2} \dots p_n^{\beta_n}.$$

Consider

$$p_1^{\alpha_1+\beta_1} \mid (p_1^{4\alpha_1} + p_1^{8\beta_1})^2 = p_1^{8\alpha_1} + p_1^{16\beta_1} + 2p_1^{4\alpha_1+8\beta_1}.$$

Because of $(*)$, $\beta_1 < 5\alpha_1$, and $\alpha_1 < 5\beta_1$, so $\alpha_1 + \beta_1 < 8\beta_1$, $16\beta_1$ and $4\alpha_1 + 8\beta_1$. From this (1) follows.

Next we argue that

$$(AB)^2 \mid (A^4 + B^8)^3 = A^{12} + B^{24} + \text{ a multiple of } (AB)^2. \tag{2}$$

So we must prove that $(AB)^2 \mid A^{12} + B^{24}$; that is

$$p_1^{2\alpha_1+2\beta_1} \mid p_1^{12\alpha_1} + p_1^{24\beta_1}.$$

Again because of $(*)$, $2\alpha_1 + 2\beta_1 < 2\alpha_1 + 2(5\alpha_1) = 12\alpha_1$ and $2\alpha_1 + 2\beta_1 < 24\beta_1$. Thus (2) holds.

We have therefore proved the assertion for $n = 1$ and $n = 2$.

Assume for a proof by induction that

$$(AB)^{F_k} \mid (A^4 + B^8)^{F_{k+1}}$$

and

$$(AB)^{F_{k+1}} \mid (A^4 + B^8)^{F_{k+2}}.$$

Multiplying, we get

$$(AB)^{F_k + F_{k+1}} \mid (A^4 + B^8)^{F_{k+1} + F_{k+2}},$$

that is

$$(AB)^{F_{k+2}} \mid (A^4 + B^8)^{F_{k+3}}.$$

Thus the statement is true for $n = k + 2$. The result follows by induction.

We conclude this "catch-up" part of the Corner with two solutions to problems posed to the jury but not used at the 35th IMO.

1. [1995: 299] *Problems Proposed But Not Used, 35th IMO in Hong Kong, Selected Problems*

M is a subset of $\{1, 2, 3, \ldots, 15\}$ such that the product of any three distinct elements of M is not a square. Determine the maximum number of elements in M.

Solution by Mansur Boase, student, St. Paul's School, London, England.

Let n denote the maximum number of elements in such a subset M. It is easy to check that there are no three elements in the following subset whose product is a square

$$M = \{3, 4, 5, 6, 7, 9, 10, 11, 13, 14\}.$$

Hence $n \geq 10$.

We shall prove that $n > 10$ is impossible. We split the problem into three cases: (i) $2 \in M$; (ii) $2 \notin M$, $3 \notin M$; (iii) $2 \notin M$, $3 \in M$.

Case (i). $2 \in M$.

We now list (x, y), $x, y \neq 2$, $1 \leq x < y \leq 15$ such that $2xy$ is a perfect square and hence we cannot have $x \in M$ and $y \in M$:

$$(1, 8),\ (3, 6),\ (4, 8),\ (5, 10),\ (6, 12),\ (7, 14)\ (8, 9).$$

In order to have $n > 10$ we can have at most one of 5 or 10 and one of 7 or 14. Thus $8 \notin M$. Now it follows that $6 \notin M$ as well. To have $n \geq 11$ we require $1, 4, 9 \in M$ but $1 \cdot 4 \cdot 9 = 36 = 6^2$, a contradiction.

Case (ii). $2 \notin M$, $3 \notin M$.

One member from each of the following triples could not belong to M:

$$(1,4,9),\ (7,8,14),\ (5,12,15).$$

Thus $n \leq 10$.

Case (iii). $2 \notin M$, $3 \in M$.

Each of the following triples must contain a number which does not belong to M:

$$(3,1,12),\ (3,4,12),\ (3,9,12),\ (3,6,8),\ (3,5,15),\ (1,4,9).$$

We must have $12 \notin M$, one of 6 or $8 \notin M$, $5 \notin M$ or $15 \notin M$ and 1 or 4 or $9 \notin M$. But as $2 \notin M$ we have at least 5 elements which do not belong.

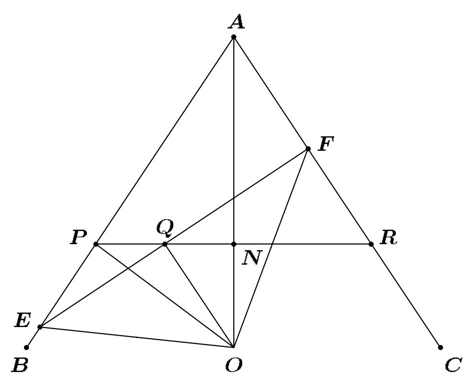
Thus $n = 10$.

6. [1995: 335] *Problems Proposed But Not Used, 35th IMO in Hong Kong, More Selected Problems*

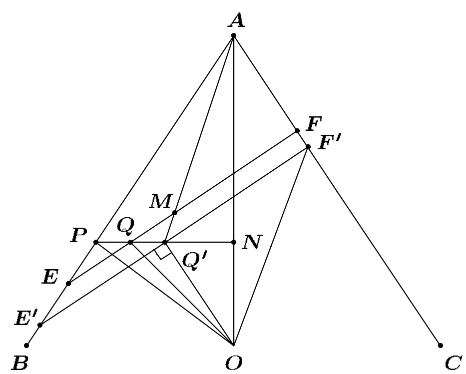
N is an arbitrary point on the bisector of $\angle BAC$. P and O are points on the lines AB and AN, respectively, such that $\angle ANP = 90° = \angle APO$. Q is an arbitrary point on NP, and an arbitrary line through Q meets the lines AB and AC at E and F respectively. Prove that $\angle OQE = 90°$ if and only if $QE = QF$.

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by George Evagelopoulos, Athens, Greece. We give the "official" solution which is very similar to that of Evagelopoulos.

First assume that $\angle OQE = 90°$. Extend PN to meet AC at R. Now $OEPQ$ and $ORFQ$ are cyclic quadrilaterals. Hence $\angle OEQ = \angle OPQ = \angle ORQ = \angle OFQ$. It now follows that we have $\triangle OEQ \equiv \triangle OFQ$ and $QE = QF$.



Now suppose that $\angle OQE \neq 90°$. Let the perpendicular through O to EF meet NP at $Q' \neq Q$. Draw the line through Q' parallel to EF, meeting the lines AB and AC at E' and F', respectively. Then $Q'E' = Q'F'$ as before. Let AQ' meet EF at $M \neq Q$. Then $ME = MF$ so that $QE \neq QF$. It follows that if $QE = QF$, then $\angle OQE = 90°$. (See next page for the diagram.)



To conclude this number of the Olympiad Corner we give solutions by the readers to problems given in the February 1996 number with the National Round of the 29th Spanish Mathematical Olympiad, 1992 [1996: 22–23].

1. At a party there are 201 people of five different nationalities. In each group of six, at least two people have the same age. Show that there are at least five people of the same country, of the same age and of the same sex.

Solutions by Mansur Boase, student, St. Paul's School, London, England; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; and by Derek Kisman, student, Queen Elizabeth High School, Calgary, Alberta. We give Boase's solution.

By the Pigeonhole Principle there are at least 41 of the same nationality and of these at least 21 must be of the same gender. Let these 21 ages be a_i, $1 \leq i \leq 21$.

Consider a_1, a_2, a_3, a_4, a_5, a_6 at least 2 must be equal. Without loss of generality, let $a_1 = a_2$.

Similarly with $a_3, a_4, \ldots, a_8$, without loss of generality, let $a_3 = a_4$. Continuing as above we can go as far as $a_{15} = a_{16}$. The sequence now reads

$$a_2 a_2 a_4 a_4 a_6 a_6 a_8 a_8 a_{10} a_{10} a_{12} a_{12} a_{14} a_{14} a_{16} a_{16} a_{17} a_{18} a_{19} a_{20} a_{21}$$

Consider $a_2 a_4 a_6 a_8 a_{10} a_{12}$. Two must be equal. Without loss of generality, let $a_2 = a_4$. Similarly we have from $a_6, \ldots, a_{16}$ we may assume that $a_6 = a_8$.

Now consider $a_2 a_6 a_{10} a_{12} a_{14} a_{16}$. If $a_2 = a_i$ or $a_6 = a_i$ for some other i then we have five equal terms and we are done.

Therefore we must have a pair equal among a_{10}, a_{12}, a_{14}, a_{16}; without loss of generality, let $a_{10} = a_{12}$. The sequence now reads

$$a_2 a_2 a_2 a_2 a_6 a_6 a_6 a_6 a_{10} a_{10} a_{10} a_{10} a_{14} a_{14} a_{16} a_{17} a_{18} a_{19} a_{20} a_{21}.$$

If any a_i, $17 \leq i \leq 21$ is among a_2, a_6, a_{10}, we are done.

Otherwise, consider $a_2 a_6 a_{10} a_{14} a_{16} x$ where x is any one of the last five terms. Each x must be equal to a_{14} or a_{16}. Without loss of generality, let three of them be equal to a_{14} (by the Pigeonhole Principle).

This provides five terms equal to a_{14}, and hence five with the same age, gender and nationality.

2. Given the number triangle

$$\begin{array}{ccccccccccccccc}
0 & & 1 & & 2 & & 3 & & 4 & \ldots\ldots & 1991 & & 1992 & & 1993 \\
& 1 & & 3 & & 5 & & 7 & & \ldots\ldots & & 3983 & & 3985 & \\
& & 4 & & 8 & & 12 & & & \ldots\ldots & & & 7968 & &
\end{array}$$

in which each number equals the sum of the two above it, show that the last number is a multiple of 1993.

Solutions by Mansur Boase, student, St. Paul's School, London, England; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Derek Kisman, student, Queen Elizabeth High School, Calgary, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. First, we give Kisman's solution.

Form a new triangle of numbers by adding each element of this triangle and its (horizontally flipped) reverse. It obeys the same addition rule as the original, and every element in the top row is 1993. Thus every element in the new triangle is a multiple of 1993. Let n be the last number in the new triangle. The last number of the original triangle is just $n/2$. Since it is an integer, it must also be a multiple of 1993.

Next we give Wang and Yu's generalization.

We show more generally that if the given row is $0, 1, 2, \ldots, n$, where n is a natural number, then the last number in the triangle is $n2^{n-1}$ and so it is a multiple of n. Note that the completed triangle will have $n+1$ rows. If we denote the ith row by R_i then R_i contains $n + 2 - i$ numbers, $i = 1, 2, \ldots, n+1$. Let $R_{i,j}$ denote the jth number in R_i, $j = 1, 2, \ldots, n+2-i$. We show by induction on i that

$$R_{i,j} = (i + 2j - 3)2^{i-2}. \qquad (*)$$

This is the case when $i = 1$ since $R_1 = 0, 1, 2, \ldots, n$ implies $R_{1,j} = j - 1$ and $(i + 2j - 3)2^{i-2} = (2j - 2)2^{-1} = j - 1$. Suppose $(*)$ holds for some $i \geq 1$. Then from the described construction, we have

$$\begin{aligned}
R_{i+1,j} &= R_{i,j} + R_{i,j+1} \\
&= (i + 2j - 3)2^{i-2} + (i + 2j - 1)2^{i-2} \\
&= (2i + 4j - 4)2^{i-2} \\
&= (i + 1 + 2j - 3)2^{i+1-2}
\end{aligned}$$

for $j = 1, 2, \ldots, n + 1 - i$, completing the induction. Letting $i = n + 1$ and $j = 1$ in $(*)$ we find that the last number in the triangle is $R_{n+1,1} = n2^{n-1}$.

3. Show that in any triangle, the diameter of the incircle is not bigger than the circumradius.

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution and remark.

Let a, b, c, r, R and Δ denote the side lengths, the inradius, the circumradius and the area of the given triangle, respectively. Then it is well known that $r = \dfrac{\Delta}{s}$ and $R = \dfrac{abc}{4\Delta}$ where $s = \dfrac{1}{2}(a + b + c)$ denotes the semiperimeter of the triangle. Hence $2r \le R$ is equivalent to $8\Delta^2 \le sabc$, which by virtue of Heron's formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

becomes
$$8(s-a)(s-b)(s-c) \le abc,$$
or
$$(b+c-a)(c+a-b)(a+b-c) \le abc. \tag{1}$$
To show (1), note that $a^2 - (b-c)^2 \le a^2$, $b^2 - (c-a)^2 \le b^2$, and $c^2 - (a-b)^2 \le c^2$. Multiplying, we obtain

$$(b+c-a)^2(c+a-b)^2(a+b-c)^2 \le a^2b^2c^2. \tag{2}$$

Since $b+c-a$, $c+a-b$ and $a+b-c$ are all positive, (1) follows from (2) by taking the square roots of both sides.

Clearly, equality holds if and only if $a = b = c$, that is, when the triangle is equilateral.

Remark. This is a classical problem which has been known for at least 230 years. See, for example. §5.1 of *Geometric Inequalities* by O. Bottema et al. In fact, the inequality $8(s-a)(s-b)(s-c) \le abc$ mentioned in the proof above can also be found in §1.3 of this book.

4. Show that each prime number p (different from 2 and from 5) has an infinity of multiples which can be written as $1111\ldots1$.

Solutions by Mansur Boase, student, St. Paul's School, London, England; by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; by Derek Kisman, student, Queen Elizabeth High School, Calgary, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Godin's solution.

It suffices to show that, for each prime p (other than 2 and 5), there exists a number n of the form $11\ldots1$, such that $p \mid n$. Since if $p \mid 11\ldots1$ (where the number is made up of k 1's), then p clearly divides all numbers of the form

$$(11\ldots1)\sum_{j=0}^{N} 10^{kj}$$

for any nonnegative integer which has the desired property.

By Fermat's Little Theorem, $10^{p-1} \equiv 1 \bmod p$ for all primes except 2 and 5.

Thus $p \mid (10^{p-1} - 1)$, but $10^{p-1} - 1 = 9(11\ldots1)$.

Thus for $p \neq 2, 3, 5$, $p3 \mid 111\ldots1$, where there are $(p-1)$ 1's in the number. It is easy to check that $3 \mid 111$ so we are done.

6. A game-machine has a screen in which the figure below is shown. At the beginning of the game, the ball is at the point S.

With each impulse from the player, the ball moves up to one of the neighbouring circles, with the same probability for each. The game is over when one of the following events occurs:

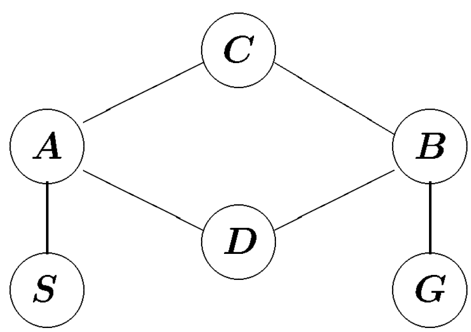
(1) The ball goes back to S, and the player loses.

(2) The ball reaches G, and the player wins.

Determine:

(a) The probability for the player to win the game.

(b) The mean time for each game.



Solutions by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario; and by Derek Kisman, student, Queen Elizabeth High School, Calgary, Alberta. We give Kisman's solution.

(a) From C or D any path to S has a corresponding exact opposite to G, so when at C or D the chance of winning is exactly $1/2$. From S we must go to A where we have a $1/3$ chance of losing and a $2/3$ chance of going to C or D, where there is a $1/2$ chance of winning.

So the probability is $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$.

(b) To find the mean we sum the products of the number of moves and the probability of getting that number. The number of moves to win or lose is always even. For a given number of moves $2n$, we can find the probability of lasting exactly that long:

1 move is entering from S to A, probability 1
$n-1$ moves are switching from A or B to C or D, probability $\frac{2}{3}$
$n-1$ moves are switching from C or D to A, or B, probability 1
1 move is moving from A to B or S or G, probability $\frac{1}{3}$.

Thus for $2n$ moves the probability is

$$\frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \left(\frac{2}{3}\right)^{n} \cdot \frac{1}{2}.$$

Now, the infinite sum to find the mean is thus

$$\begin{aligned}
&2 \cdot \left(\frac{2}{3}\right)^{1} \cdot \frac{1}{2} + 4 \cdot \left(\frac{2}{3}\right)^{2} \cdot \frac{1}{2} + \cdots + 2n\left(\frac{2}{3}\right)^{n} \cdot \frac{1}{2} + \cdots \\
&\quad = 1 \cdot \left(\frac{2}{3}\right) + 2 \cdot \left(\frac{2}{3}\right)^{2} + 3 \cdot \left(\frac{2}{3}\right)^{3} + \cdots + n\left(\frac{2}{3}\right)^{n} + \cdots \\
&\quad = \left(\frac{2}{3} + \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{3} + \cdots\right) + \frac{2}{3}\left(\frac{2}{3} + \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{3} + \cdots\right) \\
&\qquad\quad + \left(\frac{2}{3}\right)^{2}\left(\frac{2}{3} + \left(\frac{2}{3}\right)^{2} + \cdots\right) + \cdots \\
&\quad = \left(\frac{2}{3}\right)\frac{1}{1-\frac{2}{3}} + \left(\frac{2}{3}\right)^{2}\frac{1}{1-\frac{2}{3}} + \left(\frac{2}{3}\right)^{3}\frac{1}{1-\frac{2}{3}} + \cdots \\
&\quad = 3\left(\frac{2}{3} + \left(\frac{2}{3}\right)^{2} + \cdots\right) \\
&\quad = 3 \cdot \left(\frac{2}{3}\right)\frac{1}{1-\frac{2}{3}} = 6.
\end{aligned}$$

So the mean time is 6.

That completes the issue of the Corner. Please send me your nice solutions along with your national and regional Olympiads.

BOOK REVIEWS

Edited by ANDY LIU

Cien Problemas de Matemáticas: Combinatoria, Álgebra, Geometría,
One Hundred Mathematics Problems: Combinatorics, Algebra, Geometry
by Francisco Bellot Rosado and María Ascensión López Chamorro,
published by Instituto de Ciencias de la Educación, Universidad de Valladolid,
ISBN 84-7762-405-4, 1994, softcover, 146+ pages.
reviewed by ***María Falk de Losada***.

There are very few problem-solving books written in Spanish and even fewer good problem-solving books written in other languages and translated into Spanish. One notable exception has been several Russian books published in Spanish by MIR. However, Francisco Bellot Rosado and María Ascensión López Chamorro have recently addressed that need with their book *Cien Problemas de Matemáticas: Combinatoria, Álgebra y Geometría* and they have selected their problems from several important sources. From the IMO, the OME (Spanish Mathematics Olympiad) and the OIM (Olimpiada Iberoamericana de Matemáticas) there are both problems that have appeared and also problems that were proposed but unused. From *Mathesis* and *Gazeta Matematica* we find many classical problems that every interested problem-solver should know. From several other competitions and journals there is a wide-ranging selection of problems. These were, in general, unavailable to the Spanish-speaking student previously.

Many of the solutions come from the authors themselves, while others are those given by Spanish participants in the IMO, OIM and still others come from a variety of sources. It is a truism that it is important for students working in the area of problem solving to develop confidence in themselves and their own abilities. For students from countries that do not have a time-honoured tradition in mathematics competitions and serious problem-solving, as is true in many Spanish-speaking countries, having access to solutions given by their own fellow students brings the possibility of success closer and helps to build such confidence.

Folding the Regular Nonagon

Robert Geretschläger, Bundesrealgymnasium, Graz, Austria

Introduction

In the March 1997 of *Crux* [1997: 81], I presented a theoretically precise method of folding a regular heptagon from a square of paper using origami methods in an article titled *"Folding the Regular Heptagon"*. The method was derived from results established in *"Euclidean Constructions and the Geometry of Origami"* [2], where it is shown that all geometric problems that can be reduced algebraically to cubic equations can be solved by elementary methods of origami. Specifically, the corners of the regular heptagon were thought of as the solutions of the equation

$$z^7 - 1 = 0$$

in the complex plane, and this equation was then found to lead to the cubic equation

$$\zeta^3 + \zeta^2 - 2\zeta - 1 = 0,$$

which was then discussed using methods of origami. Finally, a concrete method of folding the regular heptagon was presented, as derived from this discussion.

In this article, I present a precise method of folding the regular nonagon from a square of paper, again as derived from results established in [2]. However, as we shall see, the sequence of foldings used is quite different from that used for the regular heptagon. As for the heptagon, the folding method is once again presented in standard origami notation, and the mathematical section cross-referenced to the appropriate diagrams.

Angle Trisection

For any regular n-gon, the angle under which each side appears as seen from the mid-point is $\frac{2\pi}{n}$. Specifically, for $n = 9$, the sides of a regular nonagon are seen from its mid-point under the angle $\frac{2\pi}{9}$. As is well known, this angle cannot be constructed by Euclidean methods. Three times this angle (or $\frac{2\pi}{3}$) can, however, and we note that it would be possible to construct a regular nonagon by Euclidean methods, if it were possible to trisect an arbitrary angle, or at least the specific angle $\frac{2\pi}{3}$. If this were the case, all that we would have to do would be to construct an equilateral triangle, trisect the angles from its mid-point to its corners, and intersect these with the triangle's circumcircle. Unfortunately however, as generations of mathematicians have been forced to accept (although there are a few hold-outs still out there), angle trisection by Euclidean methods of straight-edge and compass is impossible, as is the construction of a regular nonagon.

The underlying reason for the impossibility of angle trisection by Euclidean methods is the fact that straight-edge and compass constructions only allow the solution of problems that reduce algebraically to linear or quadratic equations. Angle trisection, however, involves the irreducible cubic equation

$$x^3 - \frac{3}{4}x - \frac{1}{4}\cos 3\alpha = 0,$$

which derives from the well established fact (see for instance [1]) that

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$$

For the specific case at hand, where $3\alpha = \frac{2\pi}{3}$, the cubic equation in question is

$$x^3 - \frac{3}{4}x + \frac{1}{8} = 0.$$

As shown in [2], the solutions of this equation are the slopes of the common tangents of the parabolas p_1 and p_2, whereby p_1 is defined by its focus

$$F_1\left(\tfrac{1}{16}, -\tfrac{3}{8}\right)$$

and its directrix

$$\ell_1 : x = -\frac{1}{16},$$

and p_2 is defined by its focus

$$F_2\left(0, \tfrac{1}{2}\right)$$

and its directrix

$$\ell_2 : y = -\frac{1}{2}.$$

(It is not too difficult to prove that this is indeed the case. Interested readers may like to try their hand at doing the necessary calculations themselves.) Finding the common tangents of two parabolas defined by their foci and directrices is quite straight-forward in origami, as it merely means making one fold, which places two specific points (the foci) onto two specific lines (the corresponding directrices). By this method, we will therefore now show how to fold a regular nonagon.

A Step-by-step Description of the Folding Process

As is usually the case in origami, we assume a square of paper to be given. We consider the edge-to-edge folds in step 1 as the x– and y–axes of a system of cartesian coordinates, and the edge-length of the given square as two units. The mid-point of the square is then the origin $M(0,0)$, and the end-points of the folds have the coordinates $(-1,0)$ and $(1,0)$, and $(0,-1)$ and $(0,1)$ respectively.

Steps 1 through 8 yield the foci and directrices of the parabolas discussed at the end of the second section of this article. The point C is the focus F_1 of parabola p_1, B is the focus F_2 of parabola p_2, and the creases onto which these two points are folded in step 9 are the directrices ℓ_1 and ℓ_2. Since the coordinates involved are all arrived at by halving certain line segments, it is quite easy to see that this is indeed the case.

The fold made in step 9 is then a common tangent of the parabolas, and its slope is therefore $\cos\frac{2\pi}{9}$. (This step, by the way, is the only one that cannot be replaced by Euclidean constructions.) The point of steps 10 to 13 is then to find the horizontal line represented by the equation $y = -\cos\frac{2\pi}{9}$. This is the horizontal fold through point E, as the distance between the vertical folds through points D and E is equal to 1.

We then obtain corners 2 and 9 of the nonagon (assuming the point with coordinates $(0, -1)$ to be corner 1) on this horizontal line by folding the unit length onto this line from mid-point M in step 14. Step 15 therefore yields the first two sides of the nonagon, and steps 16 and 17 complete the fold, making use of both the radial symmetry of the figure, and its axial symmetry with respect to the vertical line $M1$. Step 18, finally, shows us the completed regular nonagon.

The Folding Process

The diagrams follow the bibliography.

Conclusion

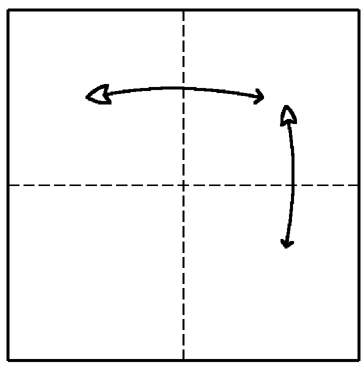
In the conclusion of *Folding the Regular Heptagon*, I declared myself as an ardent Heptagonist. I have no qualms or reservations about declaring myself an equally ardent Nonagonist now. Perhaps I will find some similar-minded folk out there willing to join me in my quest of popularizing these heretofore sadly neglected polygons.

References

[1] B. Bold, *Famous Problems of Geometry and How to Solve Them*, Dover Publications, Inc., Mineola, NY (1969).

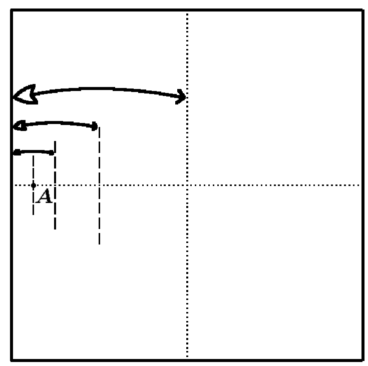
[2] R. Geretschläger, *Euclidean Constructions and the Geometry of Origami*, Mathematics Magazine, Vol. 68, No. 5 December 1995.

1.



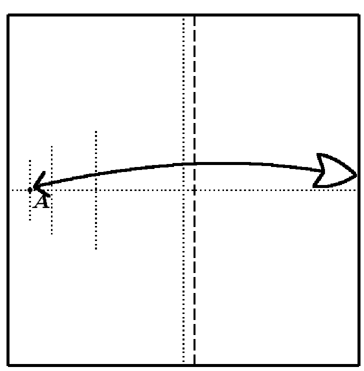
Fold and unfold twice.

2.



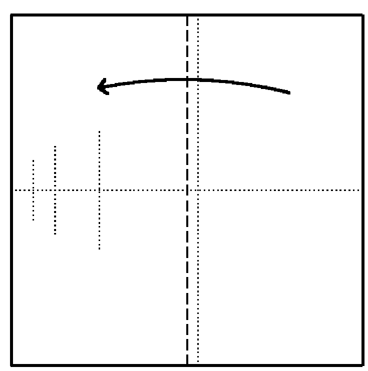
Fold and unfold three times,
making crease marks each time.
final crease yields point ***A***.

3.



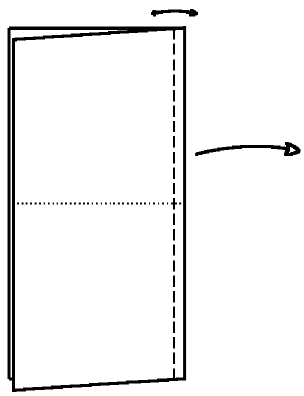
Fold edge to point ***A*** and unfold.

4.



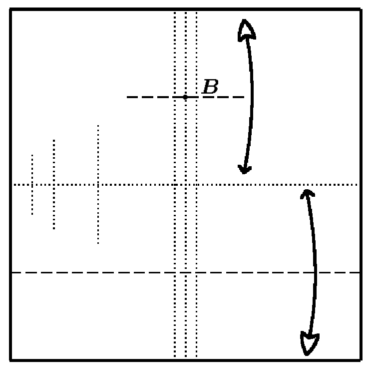
Refold edge to edge.

5.



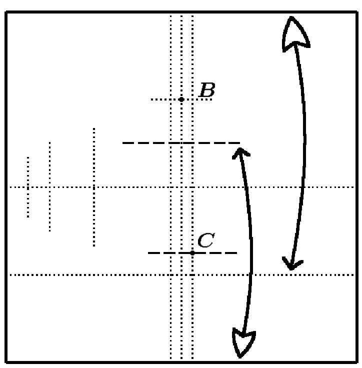
Fold and unfold both layers
at crease, then unfold.

6.



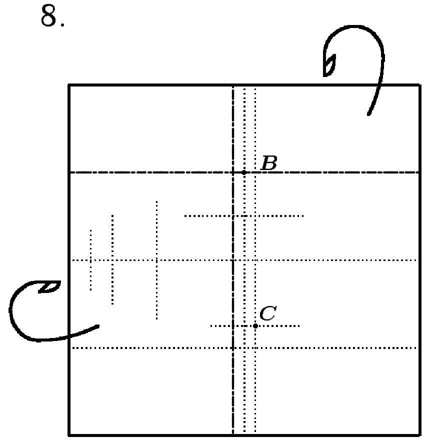
Fold and unfold twice, making
crease mark at point B.

7.



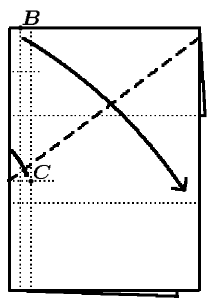
Fold upper edge to crease, unfold,
then fold lower edge to new crease,
making crease mark at point C.

8.



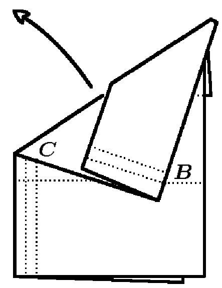
Mountain fold along creases.

9.



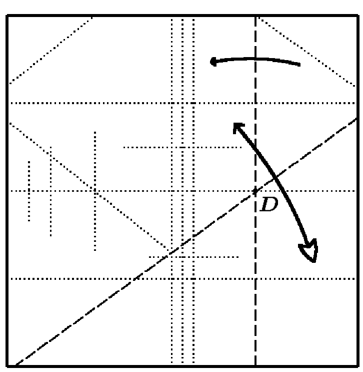
Fold so that $\boldsymbol{B}$ comes to lie on crease, and previous fold on $\boldsymbol{C}$.

10.



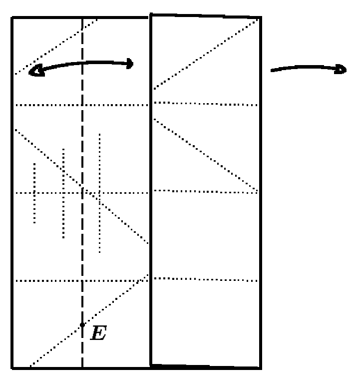
Unfold everything.

11.



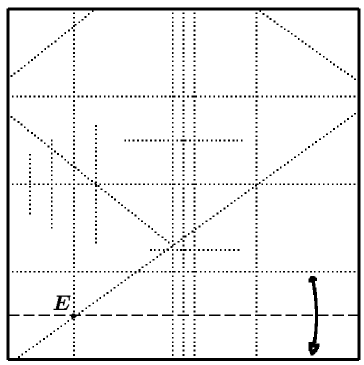
Fold along crease and unfold, then fold vertically through point $\boldsymbol{D}$.

12.



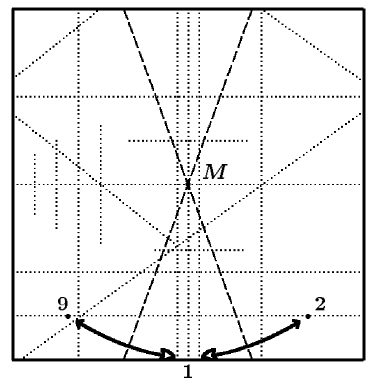
Fold edge to edge and unfold everything.

13.



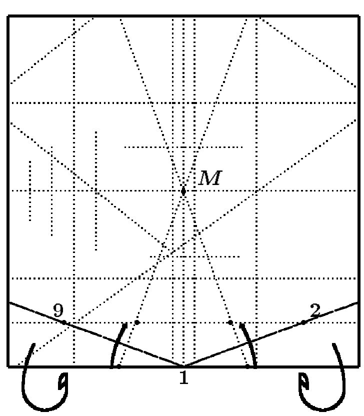
Fold horizontally through point E and unfold.

14.



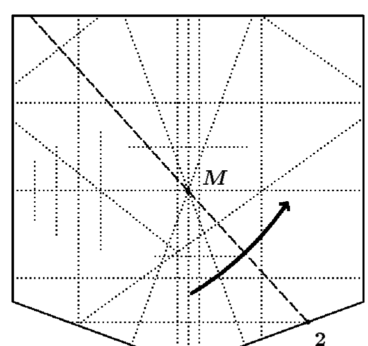
Fold through M such that point 1 lies on crease, unfold a repeat on other side (points 1, 2, 9 are corners of the nonagon).

15.



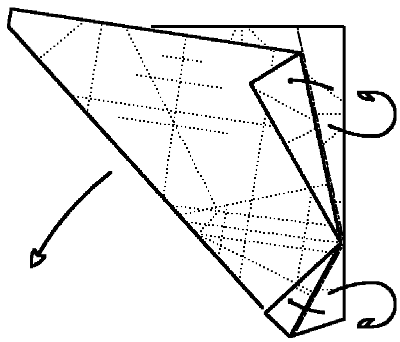
Fold back twice such that marked points come to lie on each other, resulting folds are sides of the nonagon.

16.



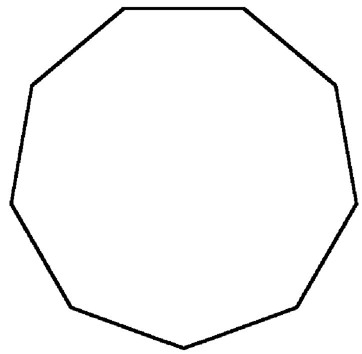
Fold through M and 2.

17.

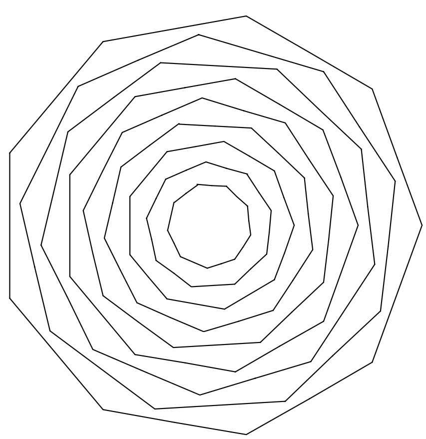


Mountain fold lower layer using edges of upper layer as guide lines. Resulting folds are two more sides of the nonagon. Open up fold from step 16 and repeat steps 16 and 17 on left side, then fold through M and 2 once more. New folds are new guide lines. Repeating process completes the nonagon.

18.



The finished nonagon.



THE SKOLIAD CORNER

No. 22

R.E. Woodrow

We begin this number by giving the problems of the Twelfth W.J. Blundon Contest, which was written February 22, 1995. This contest is prepared at the Memorial University of Newfoundland and is sponsored by the Canadian Mathematical Society.

THE TWELFTH W.J. BLUNDON CONTEST
February 22, 1995

1. (a) From a group of boys and girls, 15 girls leave. There are then left two boys for each girl. After this, 45 boys leave. There are then 5 girls for each boy. How many boys and how many girls were in the original group?

(b) A certain number of students can be accommodated in a hostel. If two students share each room then two students will be left without a room. If 3 students share each room then two rooms will be left over. How many rooms are there?

2. How many pairs of positive integers (x, y) satisfy the equation

$$\frac{x}{19} + \frac{y}{95} = 1?$$

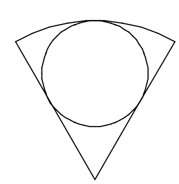
3. A book is to have 250 pages. How many times will the digit 2 be used in numbering the book?

4. Without using a calculator

(a) Show that $\sqrt{7+\sqrt{48}} + \sqrt{7-\sqrt{48}}$ is a rational number.

(b) Determine the largest prime factor of 9919.

5. A circle is inscribed in a circular sector which is one sixth of a circle of radius 1, and is tangent to the three sides of the sector as shown. Calculate the radius of the inscribed circle.



6. Determine the units digit of the sum

$$26^{26} + 33^{33} + 45^{45}.$$

7. Find all solutions (x, y) to the system of equations

$$x + y + \frac{x}{y} = 19$$

$$\frac{x(x+y)}{y} = 60.$$

8. Find the number of different divisors of 10800.

9. Show that $n^4 - n^2$ is divisible by 12 for any positive integer $n > 1$.

10. Two clocks now indicate the correct time. One gains a second every hour, and the other gains 3 seconds every 2 hours. In how many days will both clocks again indicate the correct time?

Last issue we gave the 1995 Manitoba Mathematical Contest. Here are the solutions.

THE MANITOBA MATHEMATICAL CONTEST 1995

For Students in Grade 12

Wednesday, February 22, 1995 — Time: 2 hours

1. (a) If a and b are real numbers such that $a + b = 3$ and $a^2 + ab = 7$ find the value of a.

Solution. $a^2 + ab = a(a + b) = a \cdot 3 = 7$ so $a = 7/3$.

(b) Noriko's average score on three tests was 84. Her score on the first test was 90. Her score on the third test was 4 marks higher than her score on the second test. What was her score on the second test?

Solution. $\frac{90+s+(s+4)}{3} = 84$ so $2s + 94 = 3 \times 84 = 252$, $2s = 158$ and the score on the second test was 79.

2. (a) Find two numbers which differ by 3 and whose squares differ by 63.

Solution. Let the smaller number be x. The larger number is then $x+3$. Now $|(x + 3)^2 - x^2| = 63$. Thus $|x^2 + 6x + 9 - x^2| = 63$ or $|6x + 9| = 63$. Now $6x + 9 = 63$ **OR** $6x + 9 = -63$. Thus $x = 7$ **OR** $x = -8$.

(b) Find the real number which is a root of the equation:

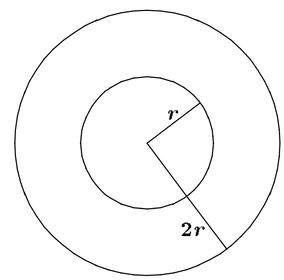
$$27(x - 1)^3 + 8 = 0.$$

Solution. $27(x - 1)^3 + 8 = 0$ is equivalent to $(3(x - 1))^3 = -8 = (-2)^3$. As -8 has only one real cube root $3(x - 1) = -2$ so $x = 1/3$.

3. (a) Two circles lying in the same plane have the same centre. The radius of the larger circle is twice the radius of the smaller circle. The area of the region between the two circles is 7. What is the area of the smaller circle?

Solution. Let the radius of the smaller circle be r. Then the radius of the larger circle, $R = 2r$. The areas of the circles are πr^2 and πR^2 respectively. Thus the area of the region between them is

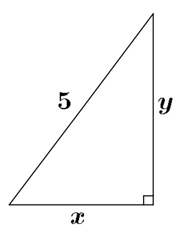
$$\begin{aligned} \pi R^2 - \pi r^2 &= 7, \\ \text{thus } \pi(2r)^2 - \pi r^2 &= 7, \\ (4-1)\pi r^2 &= 7, \\ \pi r^2 &= \frac{7}{3}. \end{aligned}$$



The area of the smaller circle is $7/3$.

(b) The area of a right triangle is 5. Also, the length of the hypotenuse of this triangle is 5. What are the lengths of the other two sides?

Solution.



Let the lengths of the two legs be x and y, with $x \geq y$. Then $\frac{1}{2}xy = 5$ and $x^2 + y^2 = 5^2 = 25$ from the formula for the area, and Pythagoras. So $2xy = 20$ and $(x+y)^2 = x^2+y^2+2xy = 45$, $(x-y)^2 = 5$, giving $x + y = 3\sqrt{5}$, $x - y = \sqrt{5}$, so $x = 2\sqrt{5}$ and $y = \sqrt{5}$. The lengths of the other two sides are $\sqrt{5}$ and $2\sqrt{5}$.

4. (a) The parabola whose equation is $8y = x^2$ meets the parabola whose equation is $x = y^2$ at two points. What is the distance between these two points?

Solution. For the intersection points $8y = x^2$ and $x = y^2$ giving $8y = (y^2)^2$ or $y(y^3 - 8) = 0$, so $y = 0$ and $x = 0$ **OR** $y = 2$ and $x = 4$. The intersection points are $(0, 0)$ and $(4, 2)$. The distance is $\sqrt{4^2 + 2^2} = 2\sqrt{5}$.

(b) Solve the equation $3x^3 + x^2 - 12x - 4 = 0$.

Solution.

$$\begin{aligned} 3x^3 + x^2 - 12x - 4 &= 0, \\ x^2(3x+1) - 4(3x+1) &= 0, \\ (3x+1)(x^2-4) &= 0. \end{aligned}$$

The three solutions are $x = -2, -1/3, 2$.

5. (a) Find the real number a such that $a^4 - 15a^2 - 16 = 0$ and $a^3 + 4a^2 - 25a - 100 = 0$.

Solution.

$$\begin{aligned} a^4 - 15a^2 - 16 &= 0, \\ (a^2 - 16)(a^2 + 1) &= 0. \end{aligned}$$

Now for **real** a, $a^2 + 1 \neq 0$ so $a^2 - 16 = 0$ and $a = \pm 4$.

Substituting these values in the other equation gives:

$$\text{for } a = -4, \quad (-4)^3 + 4(-4)^2 - 25) - 4) - 100 = 0,$$

and $a = -4$ is a solution;

$$\text{for } a = 4, \quad (4^3 + 4(4)^2 - 25(4) - 100 = 138 - 200 \neq 0.$$

The only real solution is $a = -4$.

(b) Find all positive numbers x such that $x^{x\sqrt{x}} = (x\sqrt{x})^x$.

Solution. Now

$$x^{x\sqrt{x}} = (x\sqrt{x})^x, \quad \text{so} \quad x^{x\sqrt{x}} = (x^{3/2})^x = x^{\frac{3}{2}x}.$$

Equating exponents:

$$x\sqrt{x} = \frac{3}{2}x, \quad \text{so} \quad \sqrt{x} = \frac{3}{2} \quad \text{and} \quad x = \left(\frac{3}{2}\right)^2 = \frac{9}{4},$$

since $x > 0$.

6. If x, y and z are real numbers prove that

$$(x|y| - y|x|)(y|z| - z|y|)(x|z| - z|x|) = 0.$$

Solution. Consider the left hand side. If any of x, y, z equal 0, say x then $(x|y| - y|x|) = (0 - 0) = 0$, and the equation holds. If none of x, y, z are zero then two must have the same sign. Without loss of generality we may consider the two cases $x > 0$ and $y > 0$ and $x < 0$, $y < 0$.

If $x, y > 0$ then $x|y| - y|x| = xy - yx = 0$.

If $x, y < 0$ then $x|y| - y|x| = x(-y) - y(-x) = -xy + xy = 0$.

In any case the left hand side has one of the terms 0, so the product is 0 and the identity holds.

7. x and y are integers between 10 and 100. y is the number obtained by reversing the digits of x. If $x^2 - y^2 = 495$ find x and y.

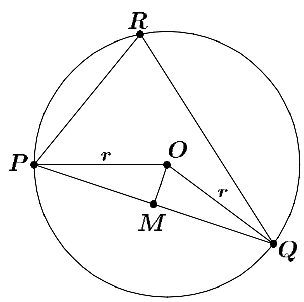
Solution. Let the digits of the numbers be a, b, with $x = 10a + b$ and $y = a + 10b$ (so $9 \geq a > b \geq 1$). Now

$$\begin{aligned} x^2 - y^2 &= (10a + b)^2 - (a + 10b)^2 \\ &= ((10a + b) + (a + 10b))((10a + b) - (a + 10b)) \\ &= 11(a + b) \cdot 9(a - b) \\ &= 99(a^2 - b^2) = 495. \end{aligned}$$

Thus $a^2 - b^2 = (a + b)(a - b) = 5$ and $a + b = 5$, $a - b = 1$ so $a = 3$, $b = 2$. The numbers x and y are 32 and 23, respectively.

8. Three points P, Q and R lie on a circle. If $PQ = 4$ and $\angle PRQ = 60°$ what is the radius of the circle?

Solution.

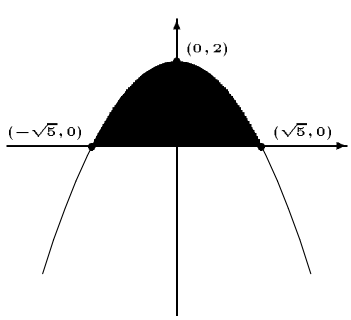


Let the radius of the circle be denoted by r, and the centre O. Now $\angle POQ = 2\angle PRQ = 120°$. Let the bisector of PQ be at M. Now $OM \perp MQ$ and $\angle MOQ = \dfrac{1}{2}\angle POQ = 60°$, so $\dfrac{MQ}{OQ} = \dfrac{2}{r} = \sin 60° = \dfrac{\sqrt{3}}{2}$.

Thus $r = \dfrac{4}{\sqrt{3}}$. The radius is $\dfrac{4\sqrt{3}}{3}$.

9. Three points are located in the finite region between the x-axis and the graph of the equation $2x^2 + 5y = 10$. Prove that at least two of these points are within a distance 3 of each other.

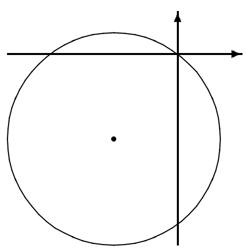
Solution.



Consider the rectangles R_1, R_2 with vertices $(-\sqrt{5}, 2)$, $(-\sqrt{5}, 0)$, $(0, 0)$, $(0, 2)$ and $(0, 2)$, $(0, 0)$, $(\sqrt{5}, 0)$, $(\sqrt{5}, 2)$ respectively. Every point of the region lies in one of the two boxes (or both). The diameter of each box is $\sqrt{(\sqrt{5})^2 + 2^2} = 3$. If three points in the region are given, two must lie in one of R_1 or R_2 by the Pigeonhole Principle and cannot be further than 3 units apart.

10. Three circles pass through the origin. The centre of the first circle lies in the first quadrant, the centre of the second circle lies in the second quadrant, and the centre of the third circle lies in the third quadrant. If P is any point that is inside all three circles, show that P lies in the second quadrant.

Solution.



The key is to consider the third circle. If its centre is in the third quadrant and it passes through the origin then the only point in common with the first quadrant is the origin itself, so no point inside could be in the first quadrant. Similarly no point inside the first circle can be in the third quadrant and no point in the second circle lies in the fourth quadrant. The result is immediate.

That completes the Skoliad Corner for this issue. Please send me contest material, submissions, and suggestions for the future direction of this feature.

Sports Writer's Math Oddity

In the issue of *Sports Illustrated* dated March 10, 1997, we find a long article about the coach of the University of Kansas Men's Basketball Coach. We quote:

> *Williams is straighter than an Arrow shirt, so square that he's divisible by four, and cornier than a corncob pipe.*

Pity that the writer could not even get this little bit of math right!

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, University of Toronto, Toronto, ON Canada M5S 1A1. The electronic address is

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Richard Hoshino (University of Waterloo), Wai Ling Yee (University of Waterloo), and Adrian Chan (Upper Canada College).

Shreds and Slices

Fibonacci Residues

Almost everyone is familiar with the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ..., where the first two terms are 0, 1, and each subsequent term is the sum of the previous two terms. This sequence, for those who don't know it, is very important and comes up in many problems and solutions unexpectedly.

Now consider each term of the sequence modulo 2. We obtain the sequence 0, 1, 1, 0, 1, 1, ... This is not very interesting, as we know that the only values of integers modulo 2 are 0 and 1, so we expect something like this which eventually repeats. Notice that the sequence that repeats (namely 0, 1, 1) has three terms, and that the pairs $(0,1)$, $(1,0)$, and $(1,1)$ appear in the sequence, but the pair $(0,0)$ does not.

Likewise, if we consider the sequence modulo 3, we get the following sequence:

$$0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, \ldots .$$

The only values this sequence can have are 0, 1, and 2. This sequence repeats after eight terms. It contains all pairs $(0,1)$, $(0,2)$, ..., $(2,2)$, except $(0,0)$.

At this point, it might be reasonable to make a conjecture. Is it true that upon taking the Fibonacci sequence modulo n, the sequence repeats after $n^2 - 1$ terms and contains all the pairs $(0,1)$, $(0,2)$, ..., $(n-1, n-1)$, except the term $(0,0)$? Maybe this is just true for prime integers n.

Before plunging headlong into any proofs, we should at least check the validity of the conjectures with more values of n, and it turns out that they aren't true. Let $r(n)$ be the length of the repeating part of the Fibonacci sequence modulo n, known as the period. Here are the values of $r(n)$ for the first few positive integers n.

n	$r(n)$	n	$r(n)$	n	$r(n)$	n	$r(n)$	n	$r(n)$	n	$r(n)$	n	$r(n)$	n	$r(n)$
2	3	8	12	14	48	20	60	26	84	32	48	38	18	44	30
3	8	9	24	15	40	21	16	27	72	33	40	39	56	45	120
4	6	10	60	16	24	22	30	28	48	34	36	40	60	46	48
5	20	11	10	17	36	23	48	29	14	35	80	41	40	47	32
6	24	12	24	18	24	24	24	30	120	36	24	42	48	48	24
7	16	13	28	19	18	25	100	31	30	37	76	43	88	49	112

We see that our conjecture was wrong. In fact, it appears that

$$r(2^m) = 3 \times 2^{m-1}, \quad r(3^m) = 8 \times 3^{m-1}, \quad \text{and} \quad r(5^m) = 4 \times 5^m.$$

Can this be proved? Are there patterns for other numbers other than power of primes? Is there a closed form for the function $r(n)$ in terms of the factorization of n?

Perhaps to answer this question we must look at other repeating sequences other than that generated by $(0, 1)$ modulo n. Let us call the repeating sequence generated by (a, b) modulo n the residue-cycle of (a, b), and denote it by the symbol $\langle a, b\rangle_n$. For example, the residue-cycle $\langle 0, 1\rangle_4$ is {0, 1, 1, 2, 3, 1}, which we can verify has length six. There are also three other residue-cycles: $\langle 0, 2\rangle_4$ = {0, 2, 2}, $\langle 0, 3\rangle_4$ = {0, 3, 3, 2, 1, 3} and the trivial sequence $\langle 0, 0\rangle_4$ = {0}. For $n = 4$, all possible residue-cycles are $\langle 0, 0\rangle_4$, $\langle 0, 1\rangle_4$, $\langle 0, 2\rangle_4$, and $\langle 0, 3\rangle_4$. Note $\langle 1, 1\rangle_4 = \langle 0, 1\rangle_4$, since these sequences are understood to be cyclic, so any pair of consecutive terms in a residue-cycle can generate it.

Some observations:

(1) $\langle a, b\rangle_n$ need not be $\langle b, a\rangle_n$. For example, $\langle 1, 3\rangle_4$ and $\langle 3, 1\rangle_4$ are different residue-cycles.

(2) It might seem that the residue-cycles form a group structure under the operation $\langle a, b\rangle_n + \langle c, d\rangle_n = \langle a + c, b + d\rangle_n$, where addition of $a + c$ and $b + d$ is taken modulo n. This is not true. For example, assuming such a group operation could exist, consider the residue-cycles $\langle 0, 1\rangle_5$ = {0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1} = $\langle 0, 2\rangle_5$ and $\langle 1, 3\rangle_5$ = {1, 3, 4, 2}.

We have then $\langle 0, 1\rangle_5 + \langle 0, 1\rangle_5 = \langle 0, 2\rangle_5 = \langle 0, 1\rangle_5$, and $\langle 0, 1\rangle_5 = \langle 1, 2\rangle_5$ so $\langle 0, 1\rangle_5 + \langle 0, 1\rangle_5 = \langle 0, 1\rangle_5 + \langle 1, 2\rangle_5 = \langle 1, 3\rangle_5$, contradiction.

(3) For any n, if there are k distinct residue-cycles, with the i^{th} one containing l_i elements, then we have the relation $\sum_{i=1}^{k} l_i = n^2$.

As a corollary of this fact, we see that the length of the residue-cycle $\langle 1, 1\rangle_n$ is at most $n^2 - 1$.

(4) If a residue-cycle has length one, it must be $\langle 0, 0\rangle_n$ for some n.

We see that answering the question of what the length of the residue-cycle generated by $(0, 1)$ modulo n is, is a specific case of knowing what all the possible lengths are.

Given n, we may list the lengths of the distinct residue-cycles modulo n.

n	lengths of residue-cycles modulo n
2	1, 3
3	1, 8
4	1, 3, 6, 6
5	1, 4, 20
7	1, 16, 16, 16
8	1, 3, 6, 6, 12, 12, 12, 12

Is there a pattern in these numbers? Do the residue-cycles for a given n form some kind of structure? (group, ring, etc.)

Acknowledgements

Anand Govindarajan, First Year student, University of Toronto at Scarborough, for conjecturing that $r(n)$ is related to the factorization of n.

What is the next term?

Naoki Sato
student, University of Toronto
Toronto, Ontario

There are many puzzles in which the first few terms of a sequence are given, and the next term is asked to be determined. Supposedly, there will be a unique rule for the sequence. A similar type of problem appears in mathematics, where we are also told that the sequence is given by a polynomial. Since polynomials have nice properties, so do these sequences, some of which we will explore here. First, we will lay down a few important and motivating results.

Result 1. Given any distinct $n+1$ reals, $x_0, x_1, \ldots, x_n$, and any $n+1$ reals, $y_0, y_1, \ldots, y_n$, there exists a unique polynomial p of degree at most n, such that $p(x_i) = y_i$, $0 \le i \le n$.

Proof. Assume $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$, for some constants c_i. The condition $p(x_i) = y_i$, $0 \le i \le n$, is equivalent to the following linear system in the variables c_i:

$$\begin{aligned}
c_n x_0^n + c_{n-1}x_0^{n-1} + \cdots + c_1 x_0 + c_0 &= y_0, \\
c_n x_1^n + c_{n-1}x_1^{n-1} + \cdots + c_1 x_1 + c_0 &= y_1, \\
&\vdots \\
c_n x_n^n + c_{n-1}x_n^{n-1} + \cdots + c_1 x_n + c_0 &= y_n.
\end{aligned}$$

The matrix $\left(x_i^j\right)_{0\leq i,j\leq n}$ has non-zero determinant, since it is Vandermonde's determinant. By Cramer's Rule, there is a unique solution for the c_i, and hence for p. ∎

The next result is simply a corollary, sometimes called the Identity Theorem for polynomials.

Result 2. If two polynomials of degree at most n agree at $n+1$ points, then they are identical.

This is interesting stuff, but what does it have to do with extrapolating sequences? In these problems, we will fit a polynomial to the given data. By the Identity Theorem, this polynomial will be the one we seek. What is particularly nice about this method is that we will not have to appeal to the Lagrange Interpolation Formula or any other tedious equations to actually find the polynomial we seek (note that the first result only assures us of the existence of a polynomial p; little mention is made of what the coefficients might be).

Problem 1. Let p be the polynomial of least degree satisfying $p(k) = 1/(k+1)$ for $k = 0, 1, \ldots, n$. Find $p(n+1)$.

Solution. Let $q(x) = (x+1)p(x) - 1$. Then $q(k) = 0$ for $0 \leq k \leq n$, so $q(x) = cx(x-1)\cdots(x-n)$ for some constant c. Note $q(-1) = -1$, but

$$q(-1) = c(-1)(-2)\cdots(-n-1) = c\cdot(-1)^{n+1}(n+1)!,$$

so $c = (-1)^n/(n+1)!$, and

$$q(x) = (-1)^n x(x-1)\cdots(x-n)/(n+1)!,$$

giving

$$q(n+1) = (n+2)p(n+1) - 1 = (-1)^n,$$

so that

$$p(n+1) = (1+(-1)^n)/(n+2).$$

Hence,

$$p(n+1) = \begin{cases} 2/(n+2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Problem 2. Let p be the polynomial of least degree satisfying $p(k) = 1$ for $k = 0, 1, \ldots, n-1$, and $p(n) = 0$. Find $p(n+1)$.

Solution. Let $q(x) = p(x) - 1$, so that

$$q(x) = cx(x-1)(x-2)\cdots(x-(n-1))$$

for some constant c. Then

$$q(n) = c\cdot n! = p(n) - 1 = -1,$$

giving

$$c = -1/n!,$$

so that

$$p(n+1) = q(n+1) + 1 = c \cdot (n+1)! + 1 = -n - 1 + 1 = -n.$$

Problem 3. Let p be the polynomial of least degree satisfying $p(k) = 2^k$, for $k = 0, 1, \ldots, n$. Find $p(n+1)$.

Solution. Let

$$q(x) = \binom{x}{0} + \binom{x}{1} + \cdots + \binom{x}{n}.$$

(Note that $\binom{x}{k}$ is simply a polynomial in x, and that upon expanding, it is easy to see that $\binom{x}{k} = 0$ for $0 \le x < k$, where x is an integer.) It is easy to verify that $q(k) = 2^k$ for $k = 0, 1, \ldots, n$. Hence, $p \equiv q$. Furthermore,

$$\begin{aligned} p(n+1) &= \binom{n+1}{0} + \binom{n+1}{1} + \cdots + \binom{n+1}{n} \\ &= 2^{n+1} - \binom{n+1}{n+1} = 2^{n+1} - 1. \end{aligned}$$

Sometimes though, these methods are not sufficient. We turn to another method, one that is probably well-known among people who have played around with sequences.

Given a polynomial p, we list $p(0), p(1), p(2), \ldots$, in sequence. Under this row, write the sequence of differences between consecutive terms, and then repeat. We thus form a *difference table*.

For example, for

$$p(x) = (2x^3 - 15x^2 + 19x + 6)/6,$$

the table will be:

1		2		0		−3		−5		−4		2		15		...
	1		−2		−3		−2		1		6		13		...	
		−3		−1		1		3		5		7		...		
			2		2		2		2		2		...			

$$\vdots$$

We can in fact do the same for any function, but only polynomials have this property:

Result 3. The difference table of p will eventually have a constant row if and only if p is a polynomial.

To see this, we only need to realize that a row represents a polynomial of degree k if and only if the row below represents a polynomial of degree $k-1$. One direction is trivial. To see the other direction, we use a telescoping sum, the same that is used to find the sum of the k^{th} powers of 1, 2, $\ldots$, n (which is in fact a polynomial of degree $k+1$). Since a constant is simply a polynomial of degree 0, the result follows.

Now, several nice properties follow from this result. First, if the value of a polynomial p, of degree at most n, is an integer for $n+1$ consecutive integers, then $p(x)$ is an integer for all integers x. Note that it is not necessary for p to have integer coefficients (as seen in the example above). Second, if we label the first row 0, and the second 1, etc., then we find that the degree of the polynomial is the index of the first row to become constant. In the above example, p is a cubic, and indeed the third row is the first to become constant. We will exploit this property in extrapolating sequences. (Note: In the example, the first terms in each row, reading down, are 1, 1, -3, and 2. A nice result in this subject says that

$$p(x) = 1\binom{x}{0} + 1\binom{x}{1} + (-3)\binom{x}{2} + 2\binom{x}{3}.$$

A proof will not be provided here, but should not be too hard to construct.)

Problem 4. Let p be the polynomial of least degree satisfying $p(k) = (-1)^k$, for $k = 0, 1, \ldots, n$. Find $p(n+1)$.

Solution. Suppose n is odd. Writing out the difference table,

$$\begin{array}{ccccccccc}
1 & & -1 & & 1 & & -1 & \cdots & -1 \\
& -2 & & 2 & & -2 & \cdots & -2 & \\
& & 4 & & -4 & \cdots & -4 & & \\
& & & & \cdots & & & & \\
& & & 2^{n-1} & & -2^{n-1} & & & \\
& & & & -2^n & & & &
\end{array}$$

We know that the last row, the nth row, is constant. Hence, the table can be extended. See Figure 1 on page 230.

Hence, $p(n+1) = -2^{n+1} + 1$. Note that this is simply the sum of the rightmost elements of the first difference table. Similarly, for n even, $p(n+1) = 2^{n+1} - 1$.

Problem 5. Let F_n denote the n^{th} Fibonacci number. Let p be the polynomial of least degree satisfying $p(k) = F_k$, for $k = 0, 1, \ldots, n$. Calculate $p(n+1)$.

$$
\begin{array}{ccccccccccc}
1 & & -1 & & 1 & & -1 \;\ldots\; -1 & & -2^{n+1}+1 \\
& -2 & & 2 & & -2 \;\ldots\; -2 & & -2^{n+1}+2 & \\
& & 4 & & -4 \;\ldots\; -4 & & -2^{n+1}+4 & & \\
& & & & \ldots & & & & \\
& & & 2^{n-1} & & -2^{n-1} & -2^n - 2^{n-1} & & \\
& & & & -2^n & -2^n & & &
\end{array}
$$

Figure 1.

Solution. Forming the difference table,

$$
\begin{array}{ccccccccccc}
F_0 & & F_1 & & F_2 & & F_3 & \cdots & F_n \\
& F_{-1} & & F_0 & & F_1 & \cdots & F_{n-2} & \\
& & F_{-2} & & F_{-1} & \cdots & F_{n-4} & & \\
& & & & \cdots & & & & \\
& & & F_{-n+1} & & F_{-n+2} & & & \\
& & & & F_{-n} & & & &
\end{array}
$$

Then $p(n+1)$ is simply the sum of the right-most values, or

$$F_{-n} + F_{-n+2} + \cdots + F_{n-2} + F_n.$$

Now, $F_{-k} = (-1)^{k+1} F_k$ for all k. Therefore, if n is even, all the terms will cancel, and the sum is 0. If n is odd, the sum is

$$
\begin{aligned}
2(F_1 + F_3 + \cdots + F_n) &= 2(F_2 - F_0 + F_4 - F_2 + \cdots + F_{n+1} - F_{n-1}) \\
&= 2(F_{n+1} - F_0) = 2F_{n+1}.
\end{aligned}
$$

Problems

A. Find the monic polynomial f of minimal degree such that

$$f(x) \equiv 0 \pmod{100}.$$

B. Find the value of the polynomial p of least degree satisfying the following:

(a) $p(k) = a^k$, $k = 0, 1, \ldots, n$,

(b) $p(k) = \binom{n}{k}$, $k = 0, 1, \ldots, n$.

(Note that part (a) generalizes Problems 2, 3, and 4.)

C. Let p be a polynomial of degree at most n. Show that

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} p(k) = 0.$$

Hint: Use the difference table, or realize that the values $p(0)$, $p(1)$, $\dots$, $p(n+1)$ must satisfy a recurrence relation. Which one?

D. Assume p is a polynomial of degree at most n such that $p(k^2)$ is an integer for $k = 0, 1, \dots, n$. Show that $p(k^2)$ is an integer for all integers k.

Euler's and DeMoivre's Theorem

Soroosh Yazdani
student, University of Waterloo
Waterloo, Ontario

A fundamental relation in mathematics is given by Euler's Theorem, which states:

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad (1)$$

Using the Taylor expansion, one can check the identity as follows:

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

A result, known as Demoivre's Theorem immediately follows:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

We will use this result to derive several identities quickly and elegantly.

Now using the fact that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, we get:

$$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta.$$

Adding this result with (1) we get:

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \implies \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta.$$

Similarly,

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta.$$

Combining these identities, we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}.$$

This means that we can write trigonometric functions as pure exponential equations. From this we can prove many identities.

Problem: Prove that

$$2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \sin\alpha + \sin\beta.$$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \\
&= 2\left(\frac{e^{i\frac{\alpha+\beta}{2}} - e^{-i\frac{\alpha+\beta}{2}}}{2i}\right)\left(\frac{e^{i\frac{\alpha-\beta}{2}} + e^{-i\frac{\alpha-\beta}{2}}}{2}\right) \\
&= \frac{e^{i\frac{\alpha+\beta}{2}}e^{i\frac{\alpha-\beta}{2}} + e^{i\frac{\alpha+\beta}{2}}e^{-i\frac{\alpha-\beta}{2}} - e^{-i\frac{\alpha+\beta}{2}}e^{i\frac{\alpha-\beta}{2}} - e^{-i\frac{\alpha+\beta}{2}}e^{-i\frac{\alpha-\beta}{2}}}{2i} \\
&= \frac{e^{i\alpha} + e^{i\beta} - e^{-i\alpha} - e^{-i\beta}}{2i} \\
&= \sin\alpha + \sin\beta \\
&= \text{R.H.S.}
\end{aligned}$$

Problems

1. Prove $2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2}) = \cos\alpha + \cos\beta$.
2. Prove $2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2}) = \cos\beta - \cos\alpha$.
3. Prove $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Here is another neat problem: What is the expansion of $\cos n\theta$? We will need the Binomial Theorem.

$$
\begin{aligned}
\cos n\theta &= \tfrac{1}{2}(e^{in\theta} + e^{-in\theta}) \\
&= \tfrac{1}{2}\left((e^{i\theta})^n + (e^{-i\theta})^n\right) \\
&= \tfrac{1}{2}\left((\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\right) \\
&= \frac{1}{2}\left(\sum_{j=0}^{n}\binom{n}{j}(\cos^{n-j}\theta)(i^j)(\sin^j\theta)\right. \\
&\qquad \left. + \sum_{j=0}^{n}\binom{n}{j}(\cos^{n-j}\theta)(-i)^j(\sin^j\theta)\right) \\
&= \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}(\cos^{n-j}\theta\sin^j\theta)\left(i^j + (-i)^j\right) \\
&= \binom{n}{0}\cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \cdots
\end{aligned}
$$

Some readers will notice at this point that we got the real part of $(\cos\theta + i\sin\theta)^n$. Although we could have come to the same conclusion by equating real and imaginary parts of the expansion, I thought that this path might be a bit more interesting. A similar expression for $\sin n\theta$ exists, which is left as an exercise for the reader to derive.

Here is a problem that takes advantage of the exponential form of trigonometric equations.

Problem: Evaluate $\displaystyle\sum_{j=0}^{n}\sin j\theta$.

Solution:

$$
\begin{aligned}
\sum_{j=0}^{n}\sin j\theta &= \frac{1}{2i}\sum_{j=0}^{n}\left(e^{ij\theta} - e^{-ij\theta}\right) = \frac{1}{2i}\sum_{j=0}^{n}\left((e^{i\theta})^j - (e^{-i\theta})^j\right) \\
&= \frac{1}{2i}\left(\frac{(e^{i\theta})^{n+1} - 1}{(e^{i\theta} - 1)} - \frac{(e^{-i\theta})^{n+1} - 1}{(e^{-i\theta} - 1)}\right) \\
&= \frac{((e^{i\theta})^{n+1} - 1)(e^{-i\theta} - 1) - ((e^{-i\theta})^{n+1} - 1)(e^{i\theta} - 1)}{2i(e^{i\theta} - 1)(e^{-i\theta} - 1)} \\
&= \frac{(e^{in\theta} - e^{-i\theta} - e^{i(n+1)\theta} + 1) - (e^{-in\theta} - e^{i\theta} - e^{-i(n+1)\theta} + 1)}{2i(2 - e^{i\theta} - e^{-i\theta})} \\
&= \frac{\sin n\theta + \sin\theta - \sin(n+1)\theta}{2 - 2\cos\theta}.
\end{aligned}
$$

Now we can simplify this equation using our previous identities.

$$\begin{aligned}
&\frac{\sin n\theta + \sin\theta - \sin(n+1)\theta}{2 - 2\cos\theta} \\
&= \frac{(\sin n\theta + \sin(-(n+1)\theta)) + (\sin\theta + \sin 0)}{2(\cos 0 - \cos\theta)} \\
&= \frac{2\cos(n+\frac{1}{2})\theta\sin(-\frac{\theta}{2}) + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{4\sin\frac{\theta}{2}\sin\frac{\theta}{2}} \\
&= \frac{-\cos(n+\frac{1}{2})\theta + \cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \\
&= \frac{2\sin(\frac{n+1}{2})\theta\sin(\frac{n}{2})\theta}{2\sin\frac{\theta}{2}} \\
&= \frac{\sin(\frac{n+1}{2})\theta\sin(\frac{n}{2})\theta}{\sin\frac{\theta}{2}}.
\end{aligned}$$

We can also evaluate $\sum_{j=0}^{n} \cos j\theta$, which is left as an exercise for the reader.

The last identity that we will derive is the expansion of $\tan n\theta$. We have seen that $\tan\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$.

Therefore, $i\tan\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$. Now we will evaluate $i\tan n\theta$.

$$\begin{aligned}
i\tan n\theta &= \frac{e^{in\theta} - e^{-in\theta}}{e^{in\theta} + e^{-in\theta}} = \frac{\left(e^{i\theta}\right)^n - \left(e^{-i\theta}\right)^n}{\left(e^{i\theta}\right)^n + \left(e^{-i\theta}\right)^n} \\
&= \frac{\left(2e^{i\theta}\right)^n - \left(2e^{-i\theta}\right)^n}{\left(2e^{i\theta}\right)^n + \left(2e^{-i\theta}\right)^n} \\
&= \frac{\left(\frac{2e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)^n - \left(\frac{2e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right)^n}{\left(\frac{2e^{i\theta}}{e^{i\theta}+e^{-i\theta}}\right)^n + \left(\frac{2e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right)^n} \\
&= \frac{(1+i\tan\theta)^n - (1-i\tan\theta)^n}{(1+i\tan\theta)^n + (1-i\tan\theta)^n}
\end{aligned}$$

$$= \frac{\sum_{j=0}^{n} \binom{n}{j} (i \tan\theta)^j - \sum_{j=0}^{n} \binom{n}{j} (-i \tan\theta)^j}{\sum_{j=0}^{n} \binom{n}{j} (i \tan\theta)^j + \sum_{j=0}^{n} \binom{n}{j} (-i \tan\theta)^j}$$

$$= \frac{\sum_{j=0}^{n} \binom{n}{j} [(i \tan\theta)^j - (-i \tan\theta)^j]}{\sum_{j=0}^{n} \binom{n}{j} [(i \tan\theta)^j + (-i \tan\theta)^j]}$$

$$= \frac{i \left[\binom{n}{1} \tan^1\theta - \binom{n}{3} \tan^3\theta + \cdots \right]}{\binom{n}{0} \tan^0\theta - \binom{n}{2} \tan^2\theta + \cdots},$$

which implies that

$$\tan n\theta = \frac{\binom{n}{1} \tan^1\theta - \binom{n}{3} \tan^3\theta + \cdots}{\binom{n}{0} \tan^0\theta - \binom{n}{2} \tan^2\theta + \cdots}.$$

Mayhem Problems

The Mayhem Problems editors are:

Cyrus Hsia	*Mayhem Advanced Problems Editor,*
Richard Hoshino	*Mayhem High School Problems Editor,*
Ravi Vakil	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from the previous issue be submitted by 1 October 1997, for publication in the issue 5 months ahead; that is, issue 2 of 1998. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Problems — Solutions

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 **rhoshino@undergrad.math.uwaterloo.ca**

H208. In a triangle ABC with incentre I and angles $2a$, $2b$, and $2c$ at vertices A, B, and C respectively, and circumradius R, show that $CI = 4R \sin a \sin b$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI, USA.

To avoid confusion, let the angles a, b, and c be denoted by α, β, and γ. Then the result follows immediately from the following theorem:

Theorem. $r = 4R \sin\alpha\ \sin\beta\ \sin\gamma$, where r is the in-radius.

Proof. By the extended Sine Law,

$$\frac{a}{2R} = \sin 2\alpha = 2\sin\alpha\ \cos\alpha.$$

Also, $\sin\alpha = \dfrac{r}{AI}$ and $\cos\alpha = \dfrac{s-a}{AI}$ in the small triangle with vertices A, I and the point of tangency on side AB with the incircle.

Putting all this together gives

$$\sin^2\alpha = \frac{ar}{4R(s-a)},$$

and likewise we have similar results for the other two angles. Their product gives

$$\sin^2\alpha\ \sin^2\beta\ \sin^2\gamma = \frac{abcr^3}{(4R)^3(s-a)(s-b)(s-c)}.$$

Let K be the area of triangle ABC. Using the relations

$$\frac{abc}{4R} = K, \quad rs = K, \quad \text{and} \quad K = \sqrt{s(s-a)(s-b)(s-c)}$$

(Heron's formula), the result follows.

Armed with this, we use $\sin\gamma = \dfrac{r}{CI}$, so that

$$CI = \frac{r}{\sin\gamma} = \frac{4R\ \sin\alpha\ \sin\beta\ \sin\gamma}{\sin\gamma} = 4R\sin\alpha\ \sin\beta,$$

as required.

H209. In an acute triangle with angles a, b, and c, show that the following inequality holds:

$$\frac{\sin c}{\cos(a-b)} + \frac{\sin a}{\cos(b-c)} + \frac{\sin b}{\cos(c-a)} \geq \frac{\sin a + \sin b + \sin c}{\sin 2a + \sin 2b + \sin 2c}.$$

Solution by Wai Ling Yee, University of Waterloo, Waterloo, Ontario.

Without loss of generality, assume that $b \geq a$. Note that

$$\sin(a-\lambda)+\sin(b+\lambda) = \cos\lambda(\sin a+\sin b)-\sin\lambda(\cos a-\cos b). \tag{1}$$

We have $0 \leq \cos\lambda \leq 1$, $\sin\lambda \geq 0$, and $\cos a - \cos b \geq 0$. Thus, for $0 \leq \lambda \leq \pi/2$, we have that (1) is less than or equal to $\sin a + \sin b$.

Setting $\lambda = \pi/2 - b > 0$, we have

$$\sin(a+b-\pi/2)+\sin\pi/2 \leq \sin a+\sin b.$$

Since the triangle is acute, $a + b > \pi/2$ and

$$1 < \sin a+\sin b < \sin a+\sin b+\sin c.$$

Using this inequality and the Weighted AM-HM Inequality with

$$x_1 = \cos(b-c), \quad x_2 = \cos(c-a), \quad x_3 = \cos(a-b),$$

and weights

$$w_1 = \frac{\sin a}{\sum \sin a}, \quad w_2 = \frac{\sin b}{\sum \sin a}, \quad w_3 = \frac{\sin c}{\sum \sin a},$$

respectively, we get

$$x_1w_1+x_2w_2+x_3w_3 \geq \frac{1}{\frac{w_1}{x_1}+\frac{w_2}{x_2}+\frac{w_3}{x_3}},$$

so that

$$\frac{w_1}{x_1}+\frac{w_2}{x_2}+\frac{w_2}{x_3} \geq \frac{1}{w_1x_1+w_2x_2+w_3x_3}.$$

Let

$$\begin{aligned} D &= \sin(a+c-b)+\sin(b+c-a)+\sin(a+b-c) \\ &\quad +\sin(a+c-b)+\sin(b+c-a)+\sin(a+b-c). \end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\sin c}{\cos(a-b)}+\frac{\sin a}{\cos(b-c)}+\frac{\sin b}{\cos(c-a)}\\
&\geq \frac{(\sin a+\sin b+\sin c)^2}{\sin c\cos(a-b)+\sin a\cos(b-c)+\sin b\cos(c-a)}\\
&> \frac{\sin a+\sin b+\sin c}{\sin c\cos(a-b)+\sin a\cos(b-c)+\sin b\cos(c-a)}\\
&= \frac{2(\sin a+\sin b+\sin c)}{2\sin c\cos(a-b)+2\sin a\cos(b-c)+2\sin b\cos(c-a)}\\
&= \frac{2(\sin a+\sin b+\sin c)}{D}\\
&= \frac{2(\sin a+\sin b+\sin c)}{\sin 2b+\sin 2a+\sin 2c+\sin 2b+\sin 2a+\sin 2c}\\
&= \frac{\sin a+\sin b+\sin c}{\sin 2a+\sin 2b+\sin 2c}.
\end{aligned}$$

H210. A "shuffle" of $2n$ cards labelled a_1, a_2, ... , a_{2n} in that order consists of taking the first n cards and merging it with the last n so that a_1, a_2, ... , a_n will be in that order somewhere in the new pile and likewise for a_{n+1}, a_{n+2}, ... , a_{2n}. Prove that if we start with a pile of $2n$ cards labelled 1, 2, ... , $2n$, then it is possible after a finite number of shuffles to get the cards in the reverse order.

Solution. Define a "step" to be two shuffles defined in the problem. We will show by induction on k that after k steps, it is possible to place the cards a_{2n}, a_{2n-1}, ... , a_{2n-k+1} in the first k positions, the cards a_k, a_{k-1}, ... , a_1 in the last k positions, and leave the remaining $(n-2k)$ cards in their original positions. Call the position that the card labelled a_i originally occupies position i.

Base Case: $k=1$. We show that in one step (or two shuffles) cards a_{2n} and a_1 can be in the first and last positions respectively and the other cards left in their original positions. For the first shuffle, put a_1 at position n and a_2, a_3, ... , a_n at positions $n+2$, $n+3$, ... , $2n$ respectively, and a_{n+1}, a_{n+2}, ... , a_{2n-1} in positions 1, 2, ... , $n-1$ respectively, leaving card a_{2n} to go into position $n+1$. For the second shuffle just take the first n cards and put it after the last n cards to give the desired configuration. Diagrammatically, this is as follows:

$$\begin{gathered}
a_1\ a_2\ \ldots\ a_n\ a_{n+1}\ \ldots\ a_{2n-1}\ a_{2n}\\
\downarrow\\
a_{n+1}\ a_{n+2}\ \ldots\ a_{2n-1}\ a_1\ a_{2n}\ a_2\ \ldots\ a_n\\
\downarrow\\
a_{2n}\ a_2\ \ldots,\ a_n\ a_{n+1}\ \ldots\ a_{2n-1}\ a_1
\end{gathered}$$

Induction Step: Suppose after k steps, $1 \leq k < n$, that the hypothesis holds, that is, cards $a_{2n}, \ldots, a_{2n-k+1}$ and $a_k, \ldots, a_1$ are in positions $1, 2, \ldots, k$, and $2n-k+1, \ldots, 2n$ respectively and the other $(n-2k)$ cards are in their original positions. Now for the first shuffle on the $(k+1)^{\text{th}}$ step leave $a_{2n}, \ldots, a_{2n-k+1}$ and $a_k, \ldots, a_1$ in their respective positions. Place card a_{k+1} in position $n+1$ and cards $a_{k+2}, \ldots, a_n$ in positions $n+1, \ldots, 2n-k-1$. Then place cards $a_{n+1}, \ldots, a_{2n-k+1}$ in positions $k+1, \ldots, n-2$ and finally a_{2n-k} must go in position $n+1$. For the second shuffle again leave $a_{2n}, \ldots, a_{2n-k+1}$ and $a_k, \ldots, a_1$ alone. Place cards at positions $k+1, \ldots, n$ in positions $n+1, \ldots, 2n-k$ and those cards that were at position $n+1, \ldots, 2n-k$ down into positions $k+1, \ldots, n$. Diagrammatically, this is as follows:

$$\begin{array}{c}
a_{2n} \;\cdots\; a_{2n-k+1} \; a_{k+1} \;\cdots\; a_n \; a_{n+1} \;\cdots\; a_{2n-k} \; a_k \;\cdots\; a_1 \\
\downarrow \\
a_{2n} \;\cdots\; a_{2n-k+1} \; a_{n+1} \;\cdots\; a_{2n-k+1} \; a_{k+1} \; a_{2n-k} \; a_{k+2} \;\cdots\; a_{2n-k} \; a_k \;\cdots\; a_1 \\
\downarrow \\
a_{2n} \;\cdots\; a_{2n-k+1} \; a_{2n-k} \; a_{k+2} \;\cdots\; a_n \; a_{n+1} \;\cdots\; a_{2n-k-1} \; a_{k+1} \; a_k \;\cdots\; a_1
\end{array}$$

Induction is complete, so when $k = n$ we have all the cards in the reverse order and we are done.

Advanced Problems — Solutions

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada. M1G 1C3 **hsia@math.toronto.edu**

A184. Let A be a 3×3 matrix with rational entries such that $A^8 = I$. Show that $A^4 = I$.

Solution

Let $m_A(x)$ be the minimal polynomial of A (over the rationals). Then $m_A(x)$ is a polynomial with rational coefficients, with degree at most 3. We also know $m_A(x)$ divides $x^8 - 1$, whose factorization over the rationals is $(x-1)(x+1)(x^2+1)(x^4+1)$. The polynomial $m_A(x)$ must be a product of some of these factors, and $x^4 + 1$ cannot be one of them. Hence, $m_A(x)$ divides $(x-1)(x+1)(x^2+1) = x^4 - 1 \Rightarrow A^4 = I$.

*Ravi Vakil and Nicolas Guay both correctly pointed out that the original statement of the problem, which had "**real**" instead of "**rational**", was flawed, as seen in the counterexamples:*

$$A = \begin{pmatrix} \sqrt{2}-1 & 1 & 0 \\ \sqrt{2}-2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \sqrt{2} & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which they provided respectively.

Also solved by MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; DONNY CHEUNG, University of Waterloo, Waterloo, Ontario; EDWARD WANG, Wilfrid Laurier University, Waterloo, Ontario.

A188. Let $ABCD$ and $A'B'C'D'$ be squares with opposite orientation, with $A = A'$. Let M be the midpoint of DD'. Show that $AM \perp BB'$ and $2 \cdot AM = BB'$.

Solution by Wai Ling Yee, University of Waterloo, Waterloo, Ontario.

A $90°$ rotation about A brings B' to D' and D to B. Therefore,

$$\overrightarrow{AB'} \cdot \overrightarrow{AD} = \overrightarrow{AD'} \cdot \overrightarrow{AB}.$$

Hence,

$$\begin{aligned} 2\overrightarrow{AM} \cdot \overrightarrow{BB'} &= \left(\overrightarrow{AD} + \overrightarrow{AD'}\right) \cdot \left(\overrightarrow{AB'} - \overrightarrow{AB}\right) \\ &= \overrightarrow{AD} \cdot \overrightarrow{AB'} - \overrightarrow{AD} \cdot \overrightarrow{AB} + \overrightarrow{AD'} \cdot \overrightarrow{AB'} - \overrightarrow{AD'} \cdot \overrightarrow{AB} \\ &= \overrightarrow{AD} \cdot \overrightarrow{AB'} - 0 + 0 - \overrightarrow{AD'} \cdot \overrightarrow{AB} = 0, \end{aligned}$$

and so, $AM \perp BB'$.

Now, $\left|\overrightarrow{AB'}\right| = \left|\overrightarrow{AD'}\right|$, $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AD}\right|$, and

$$\angle BAB' + \angle DAD' = 180°,$$

so that

$$\overrightarrow{AB} \cdot \overrightarrow{AB'} = -\overrightarrow{AD} \cdot \overrightarrow{AD'}.$$

Therefore,

$$\begin{aligned} 4\left|\overrightarrow{AM}\right|^2 &= 4\overrightarrow{AM} \cdot \overrightarrow{AM} = \left(\overrightarrow{AD} + \overrightarrow{AD'}\right) \cdot \left(\overrightarrow{AD} + \overrightarrow{AD'}\right) \\ &= \left|\overrightarrow{AD}\right|^2 + 2\overrightarrow{AD} \cdot \overrightarrow{AD'} + \left|\overrightarrow{AD'}\right|^2 \\ &= \left|\overrightarrow{AB}\right|^2 - 2\overrightarrow{AB} \cdot \overrightarrow{AB'} + \left|\overrightarrow{AB'}\right|^2 \\ &= \left|\overrightarrow{BB'}\right|^2. \end{aligned}$$

Hence, $2 \cdot AM = BB'$.

Also solved by MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain.

A189. Prove that no seven positive integers, not exceeding 24, can have sums of all subsets distinct.

Solution by Wai Ling Yee, University of Waterloo, Waterloo, Ontario.

Assume that it is possible to find a set of seven natural numbers, such that all sums of subsets are distinct. The numbers must be distinct, and hence there is a maximum value, x, and a minimum value, y. The set has $2^7 - 1 = 127$ non-empty subsets, and hence the difference between the largest sum and the smallest sum must be at least 126. The smallest sum must be y. The largest sum cannot exceed

$$x + (x-1) + (x-2) + (x-4) + (x-5) + (x-7) + y = 6x - 19 + y,$$

since there cannot be four consecutive terms of an arithmetic progression (first plus last equals sum of middle two terms). Therefore the difference between the largest and the smallest sum cannot exceed $6x - 19$. This takes its maximum value when $x = 24$. However, $6 \cdot 24 - 19 = 125 < 126$, contradiction. Therefore no seven positive integers, not exceeding 24, can have sums of all subsets distinct.

Also solved by EDWARD WANG, Wilfrid Laurier University, Waterloo, Ontario.

MATHEMATICAL
MATHEMATICA
MATHEMATIC
MAYHEMATI
MAYHEMAT
MAYHEMA
MAYHEM

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 December 1997**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2238. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

A four-digit number $\overline{abcd}$ is said to be *faulty* if it has the following property:

> The product of the two last digits c and d equals the two-digit number $\overline{ab}$, while the product of the digits $c-1$ and $d-1$ equals the two digit number $\overline{ba}$.

Determine all faulty numbers!

2239. *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

Suppose that $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, 3, \ldots, n\}$. The elements of these subsets are arranged in ascending order of magnitude. For i from 1 to r, let t_i denote the ith smallest element in the subset. Let $T(n, r, i)$ denote the arithmetic mean of the elements t_i.

Prove that $T(n, r, i) = i\,\dfrac{n+1}{r+1}$.

2240. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that $\frac{BD}{DC} \leq \frac{BF}{FA} \leq 1$ and $\frac{AE}{EC} \leq \frac{AF}{FB}$.

Prove that $[DEF] \leq \frac{[ABC]}{4}$ with equality if and only if two of the three points D, E, F, (at least) are mid-points of the corresponding sides.

Note: $[XYZ]$ denotes the area of triangle $\triangle XYZ$.

2241. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Triangle ABC ($AB \neq AC$) has incentre I and circumcentre O. The incircle touches BC at D. Suppose that $IO \perp AD$.

Prove that AD is a symmedian of triangle ABC. (The symmedian is the reflection of the median in the internal angle bisector.)

2242. *Proposed by K.R.S. Sastry, Dodballapur, India.*

$ABCD$ is a parallelogram. A point P lies in the plane such that

1. the line through P parallel to DA meets DC at K and AB at L,
2. the line through P parallel to AB meets AD at M and BC at N, and
3. the angle between KM and LN is equal to the non-obtuse angle of the parallelogram.

Find the locus of P.

2243. *Proposed by F.J. Flanigan, San Jose State University, San Jose, California, USA.*

Given $f(x) = (x - r_1)(x - r_2)\ldots(x - r_n)$ and $f'(x) = n(x - s_1)(x - s_2)\ldots(x - s_{n-1})$, $(n \geq 2)$, consider the harmonic mean h of the $n(n-1)$ differences $r_i - s_j$.

If $f(x)$ has a multiple root, then h is undefined, because at least one of the differences is zero.

Calculate h when $f(x)$ has no multiple roots.

2244. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle and D is a point on AB produced beyond B such that $BD = AC$, and E is a point on AC produced beyond C such that $CE = AB$. The perpendicular bisector of BC meets DE at P.

Prove that $\angle BPC = \angle BAC$.

2245. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Prove that $\dfrac{3^n + (-1)^{\binom{n}{2}}}{2} - 2^n$ is divisible by 5 for $n \geq 2$.

2246. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that G, I and O are the centroid, the incentre and the circumcentre of a non-equilateral triangle ABC.

The line through B, perpendicular to OI intersects the bisector of $\angle BAC$ at P.

The line through P, parallel to AC intersects BC at M.

Show that I, G and M are collinear.

2247. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Suppose that $n \geq 3$ is an odd natural number.

Show that the only polynomial $P \in \mathbb{R}[x]$ satisfying the functional equation:

$$(P(x+1))^n = (P(x))^n + \sum_{k=0}^{n-1} \binom{n}{k} x^k, \qquad \text{for all } x \in \mathbb{R},$$

is given by $P(x) = x$.

(b)$\star$ Suppose that $n \geq 1$ is a natural number.

Show that the only polynomial $P \in \mathbb{R}[x]$ satisfying the functional equation:

$$(P(x+1))^n = (P(x))^n + \sum_{k=0}^{n-1} \binom{n}{k} x^k, \qquad \text{for all } x \in \mathbb{R},$$

is given by $P(x) = x$.

(c)$\star$ Suppose that $n \geq 1$ is a natural number.

Show that the only polynomial $P \in \mathbb{R}[x]$ satisfying the functional equation:

$$P\left((x+1)^n\right) = (P(x))^n + \sum_{k=0}^{n-1} \binom{n}{k} x^k, \qquad \text{for all } x \in \mathbb{R},$$

is given by $P(x) = x$.

2248. *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Find the value of the sum $\sum_{k=1}^{\infty} \frac{d(k)}{k^2}$, where $d(k)$ is the number of positive integer divisors of k.

2249. *Proposed by K.R.S. Sastry, Dodballapur, India.*

How many distinct acute angles α are there such that

$$\cos\alpha \, \cos 2\alpha \, \cos 4\alpha \;=\; \tfrac{1}{8}\,?$$

2250. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a scalene triangle with incentre I. Let D, E, F be the points where BC, CA, AB are tangent to the incircle respectively, and let L, M, N be the mid-points of BC, CA, AB respectively.

Let l, m, n be the lines through D, E, F parallel to IL, IM, IN respectively.

Prove that l, m, n are concurrent.

Historical Titbit

Taken from a 1950's University Scholarship Paper.

1. A, B, C, D are four points on a circle.

   The tangent at B meets CD at L, the tangent at C meets AB at M, and AD meets BC at N.

   Prove that L, M, N are collinear.

2. If $ABCDEF$ are concyclic and $A(CDEF)$ is a harmonic pencil, prove that $B(CDEF)$ is also a harmonic pencil.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2129*. [1996: 123] *Proposed by Stanley Rabinowitz, Westford, Massachusetts, USA.*

For $n > 1$ and $i = \sqrt{-1}$, prove or disprove that

$$\frac{1}{4i} \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{4n} i^k \tan\left(\frac{k\pi}{4n}\right)$$

is an integer.

Solution by G.P. Henderson, Campbellcroft, Ontario; (modified slightly by the editor).

Let $S(n)$ denote the given sum. We show that $S(n)$ is always an integer. First we need a lemma:

Lemma. Let n be a positive integer, and let $g(n) = \sum_{j=1}^{n} \tan\left(\frac{4j-3}{4n}\right)$.

Then $g(n) = (-1)^{n+1} n$.

Proof. We first obtain an expansion of $\tan(n\theta)$. [Ed: see also [1997: 233].] Using the following well-known formulae:

$$\begin{aligned}
\sin(n\theta) &= \binom{n}{1}\sin\theta\cos^{n-1}\theta - \binom{n}{3}\sin^3\theta\cos^{n-3}\theta + \dots, & (1)\\
\cos(n\theta) &= \cos^n\theta - \binom{n}{2}\sin^2\theta\cos^{n-2}\theta + \binom{n}{4}\sin^4\theta\cos^{n-4}\theta - \dots. & (2)
\end{aligned}$$

Dividing both the numerator and the denominator of the fraction $\frac{(1)}{(2)}$ by $\cos^n\theta$, and setting $x = \tan\theta$, we get

$$\tan(n\theta) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials in x. Specifically, if n is odd, then

$$\begin{aligned}
P(x) &= \binom{n}{1}x - \binom{n}{3}x^3 + \dots + (-1)^{(n-1)/2}\binom{n}{n}x^n,\\
Q(x) &= 1 - \binom{n}{2}x^2 + \dots + (-1)^{(n-1)/2}\binom{n}{n-1}x^{n-1};
\end{aligned}$$

and if n is even, then the last terms of $P(x)$ and $Q(x)$ are

$$-(-1)^{n/2}\binom{n}{n-1}x^{n-1} \qquad \text{and} \qquad (-1)^{n/2}\binom{n}{n}x^n,$$

respectively.

Now consider the equation

$$\tan(n\theta) \;=\; 1 \;=\; \frac{P(x)}{Q(x)}. \tag{3}$$

The roots of $\tan(n\theta) = 1$ are clearly given by $\theta = (4j-3)/4n$; that is $x = \tan[(4j-3)/4n]$, $j = 1, 2, \ldots, n$. Thus $g(n)$ is simply the sum of the roots of $\tan(n\theta) = 1$.

On the other hand, the roots of $\dfrac{P(x)}{Q(x)} = 1$ are the roots of $P(x) - Q(x) = 0$. Hence, when n is odd, the sum of the roots is

$$\frac{-\left[\text{coefficient of } x^{n-1} \text{ in } (-Q(x))\right]}{[\text{coefficient of } x^{n} \text{ in } P(x)]} \;=\; n \;=\; (-1)^{n+1}n;$$

and when n is even, the sum of the roots is

$$\frac{\left[\text{coefficient of } x^{n-1} \text{ in } (P(x))\right]}{[\text{coefficient of } x^{n} \text{ in } -Q(x)]} \;=\; -n \;=\; (-1)^{n+1}n.$$

Therefore, $g(n) = (-1)^{n+1}n$. The proof of the lemma is complete.

Continuing, set $A = \{k \mid 1 \le k \le 4n,\ \gcd(k,n) = 1\}$. Note that if $k \in A$, then $4n - k \in A$ and $4n - k \neq k$. If k is even, then the sum of the terms corresponding to k and $4n-k$ is zero. If k is odd, then one of k and $4n-k$ is congruent to 1 and the other to -1, (all congruences are modulo 4, unless otherwise stated), and the corresponding terms are equal. Therefore

$$S(n) = \frac{1}{2}\sum_{k\in B}\tan\left(\frac{k\pi}{4n}\right),$$

where $B = \{k \mid 1 \le k \le 4n,\ \gcd(k,n) = 1 \text{ and } k \equiv 1 \pmod 4\}$.

Let $n = 2^m a$ where a is odd. If $a = 1$, then $n = 2^m$, $m \ge 1$, and $k \in B$ if and only if $k = 4j - 3$, $j = 1, 2, \ldots, n$. Hence

$$S(n) \;=\; \frac{g(n)}{2} \;=\; -\frac{n}{2},$$

which is an integer.

If $a \ge 3$, let $p_1, p_2, \ldots, p_r$, denote the distinct (odd) prime divisors of a, and suppose that $p_j \equiv b_j$, where $b_j = \pm 1$ for $j = 1, 2, \ldots, r$.

To get $S(n)$ from $g(n)$, we must subtract the terms with k divisible by at least one of $p_1, p_2, \ldots, p_r$.

Set $h = h(j_1, j_2, \ldots, j_s) = \displaystyle\sum_k \tan\left(\frac{k\pi}{4n}\right)$, where the summation is over all k such that $k \equiv 1$, and k is a multiple of $c = p_{j_1}p_{j_2}\ldots p_{j_s}$, where $1 \le s \le r$ and all the p's are distinct.

If $c \equiv 1$, then the values of k in the sum are $c, 5c, \ldots, (4d-3)c$, where $d = n/c$. Hence

$$\begin{aligned} h &= \sum_{j=1}^{d} \tan\left(\frac{(4j-3)\pi}{4d}\right) = g(d) = (-1)^{d+1}d \\ &= (-1)^{n+1}\frac{n}{p_{j_1}p_{j_2}\cdots p_{j_s}}, \end{aligned} \tag{4}$$

since $n = cd$ and c being odd, together imply that $d \equiv n \pmod{2}$.

If $c \equiv -1$, then the values of k in the sum are $3c, 7c, \ldots, (4d-1)c$ and

$$h = \sum_{j=1}^{d} \tan\left(\frac{(4j-1)\pi}{4d}\right).$$

Replacing j by $d - j + 1$, we find that

$$\begin{aligned} h &= -g(d) = -(-1)^{d+1}d \\ &= -(-1)^{n+1}\frac{n}{p_{j_1}p_{j_2}\cdots p_{j_s}}. \end{aligned} \tag{5}$$

Since

$$b_{j_1}b_{j_2}\ldots b_{j_s} = \begin{cases} 1 & \text{if} \quad c \equiv 1, \\ -1 & \text{if} \quad c \equiv -1, \end{cases}$$

the answers in formulae (4) and (5) can be combined in the formula:

$$h = (-1)^{n+1}\frac{n\, b_{j_1}b_{j_2}\ldots b_{j_s}}{p_{j_1}p_{j_2}\cdots p_{j_s}}.$$

It then follows from the Inclusion–Exclusion Principle that:

$$\begin{aligned} S(n) &= \frac{1}{2}\left(g(n) - \sum_{j=1}^{n} h(j) + \sum_{j_1<j_2} h(j_1, j_2) - \ldots\right) \\ &= \frac{(-1)^{n+1}n}{2}\left(1 - \sum_{j=1}^{n}\frac{b_j}{p_j} + \frac{b_{j_1}b_{j_2}}{p_{j_1}p_{j_2}} - \ldots\right) \\ &= \frac{(-1)^{n+1}n}{2}\left(1 - \frac{b_1}{p_1}\right)\left(1 - \frac{b_2}{p_2}\right)\ldots\left(1 - \frac{b_r}{p_r}\right) \\ &= \frac{(-1)^{n+1}n(p_1 - b_1)(p_2 - b_2)\ldots(p_r - b_r)}{2p_1p_2\ldots p_r} \end{aligned}$$

which is clearly an integer. (In fact, it is divisible by 2^{m+2r-1} since $p_j - b_j$ is divisible by 4.) This completes the proof.

Also solved by KURT GIRSTMAIR, University of Innsbruck, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA, and VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA.

Girstmair, using character coordinates, which play an important role in the study of cyclotomic fields, shows that

$$S(n) = \begin{cases} -T(n) & \text{if } n \text{ is even,} \\ T(n)/2 & \text{if } n \text{ is odd,} \end{cases}$$

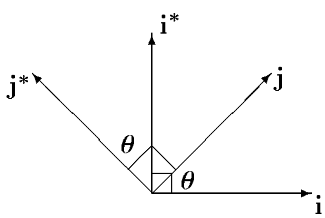
where $T(n) = n\prod \frac{1}{p} \prod_{p\equiv 1}(p-1) \prod_{p\equiv 3}(p+1)$ *with* p *running through all prime divisors of* n *and the congruences are taken modulo* 4. *It is not difficult to show that his answer is equivalent to the one given in the solution above.*

2133. [1996: 124] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Similar non-square rectangles are placed outwardly on the sides of a parallelogram π. Prove that the centres of these rectangles also form a non-square rectangle if and only if π is a non-square rhombus.

As Florian Herzig and Václav Konečný both noted, the similar non-square rectangles must be placed "nicely", that is, they must either all have the longest side or all have the shortest side in common with the parallelogram. Otherwise, the conclusion of the problem does not hold!

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.



Let i, j be unit vectors making an angle θ with one another, and let i* and j* be the images of i and j under a 90° anticlockwise rotation.

Then

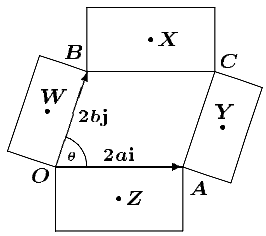
$$\mathbf{i}\cdot\mathbf{j} = \cos\theta \qquad \mathbf{i}^*\cdot\mathbf{i} = 0 \qquad \mathbf{i}^*\cdot\mathbf{j} = \sin\theta$$

$$(*)$$

$$\mathbf{i}^*\cdot\mathbf{j}^* = \cos\theta \qquad \mathbf{j}^*\cdot\mathbf{j} = 0 \qquad \mathbf{j}^*\cdot\mathbf{i} = \sin\theta$$

(The proof is trivial, but these relations will be much used in what follows.)

Let the parallelogram be $OACB$ with $\overrightarrow{OA} = 2a\mathbf{i}$ and $\overrightarrow{OB} = 2b\mathbf{j}$ and let W, X, Y, Z be the centres of the non-square similar rectangles (see the figure).



Those centred at X and Z have sides $2a, 2ka$ and the other two $2b, 2kb$, where $k \neq 1$, since they are non-square. Let the position vectors of W, X, Z relative to O be $\mathrm{w}, \mathrm{x}, \mathrm{z}$, respectively. Then it is easy to see that

$$\mathrm{w} = b\mathrm{j} + kb\mathrm{j}^*, \mathrm{x} = 2b\mathrm{j} + a\mathrm{i} = ka\mathrm{i}^*$$

$$\text{and} \qquad \mathrm{z} = a\mathrm{i} - ka\mathrm{i}^*.$$

Hence

$$\overrightarrow{ZW} = \mathrm{w} - \mathrm{z} = b\mathrm{j} + kb\mathrm{j}^* - a\mathrm{i} + ka\mathrm{i}^* \text{ and}$$

$$\overrightarrow{WX} = \mathrm{x} - \mathrm{w} = b\mathrm{j} + a\mathrm{i} + ka\mathrm{i}^* - kb\mathrm{j}^*.$$

From which, using relation $(*)$ repeatedly, we see that

$$\overrightarrow{ZW} \cdot \overrightarrow{WX} = (b^2 - a^2)(1 - k^2) \text{ and}$$

$$ZW^2 = WX^2 \text{ if and only if } 2ab\cos\theta(1 - k^2) = 0.$$

But $k \neq 1$, so ZW and WX are at right angles if and only if $a = b$ and $ZW = WX$ if and only if $\theta = 90°$.

So $ZWXY$ is a non-square rectangle if and only if $OACB$ is a non-square rhombus.

The proposer notes that this is related to two well-known results.

Napoleon's Theorem: The centres of equilateral triangles, placed on the sides of any triangle, form an equilateral triangle.

Thébault's Theorem: The centres of squares placed on any parallelogram form a square.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2141. [1996: 170] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

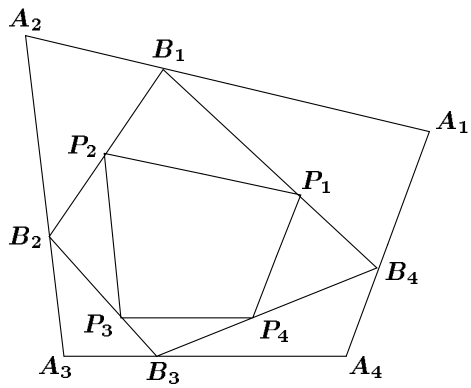
$A_1A_2A_3A_4$ is a quadrilateral. Let B_1, B_2, B_3 and B_4 be points on the sides A_1A_2, A_2A_3, A_3A_4 and A_4A_1 respectively, such that

$$A_1B_1 : B_1A_2 = A_4B_3 : B_3A_3 \quad \text{and} \quad A_2B_2 : B_2A_3 = A_1B_4 : B_4A_4.$$

Let P_1, P_2, P_3 and P_4 be points on B_4B_1, B_1B_2, B_2B_3 and B_3B_4 respectively, such that

$$P_1P_2 \| A_1A_2, \quad P_2P_3 \| A_2A_3 \quad \text{and} \quad P_3P_4 \| A_3A_4.$$

Prove that $P_4P_1 \| A_4A_1$.



Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

We denote:

$\overline{A_1A_2} = a_1,\ \overline{A_2A_3} = a_2,\ \overline{A_3A_4} = a_3,\ \overline{A_4A_1} = a_4;$
$\overline{A_1B_1} = \lambda a_1,\ \overline{B_1A_2} = \mu a_1,\ \lambda + \mu = 1;$
$\overline{A_2B_2} = \rho a_2,\ \overline{B_2A_3} = \tau a_2,\ \rho + \tau = 1;$
$\overline{A_3B_3} = \mu a_3,\ \overline{B_3A_4} = \lambda a_3;$
$\overline{A_4B_4} = \tau a_4,\ \overline{B_4A_1} = \rho a_4;$
$\angle A_4A_1A_2 = \alpha_1,\ \angle A_1A_2A_3 = \alpha_2,$
$\angle A_2A_3A_4 = \alpha_3,\ \angle A_3A_4A_1 = \alpha_4.$

The distance from P_1 and P_2 to A_1A_2 is d_1; from P_2 and P_3 to A_2A_3 is d_2; from P_3 and P_4 to A_3A_4 is d_3; from P_4 to A_4A_1 is d_4 and from P_1 to A_4A_1 is d_4'. It suffices to show: $d_4' = d_4$ which implies $P_4P_1 \| A_4A_1$.

Consider $\triangle B_1A_2B_2$. We see that $[B_1A_2P_2] + [A_2B_2P_2] = [B_1A_2B_2]$ or

$$\mu a_1 d_1 + \rho a_2 d_2 = \mu\rho a_1 a_2 \sin\alpha_2, \tag{1}$$

Similarly, for $\triangle B_2A_3B_3$, $\triangle B_3A_4B_4$ and $\triangle B_4A_1B_1$:

$$\tau a_2 d_2 + \mu a_3 d_3 = \mu\tau a_2 a_3 \sin\alpha_3, \tag{2}$$

$$\lambda a_3 d_3 + \tau a_4 d_4 = \lambda\tau a_3 a_4 \sin\alpha_4, \tag{3}$$

$$\rho a_4 d_4' + \lambda a_1 d_1 = \lambda\rho a_1 a_4 \sin\alpha_1. \tag{4}$$

Eliminating d_2 out of (1) and (2), we find:

$$\mu\tau a_1 d_1 - \mu\rho a_3 d_3 = \mu\rho\tau(a_1 a_2 \sin\alpha_2 - a_2 a_3 \sin\alpha_3). \tag{5}$$

Eliminating d_3 out of (3) and (5):

$$\lambda a_1 d_1 + \rho a_4 d_4 = \lambda\rho(a_1 a_2 \sin\alpha_2 - a_2 a_3 \sin\alpha_3 + a_3 a_4 \sin\alpha_4). \qquad (6)$$

We rewrite (4):

$$\lambda a_1 d_1 + \rho a_4 d_4' = \lambda\rho a_1 a_4 \sin\alpha_1. \qquad (7)$$

(6) and (7) imply

$$\rho a_4(d_4 - d_4') = \lambda\rho(a_1 a_2 \sin\alpha_2 + a_3 a_4 \sin\alpha_4) - \lambda\rho(a_2 a_3 \sin\alpha_3 + a_1 a_4 \sin\alpha_1). \qquad (8)$$

Then

$$2[A_1 A_2 A_3 A_4] = a_1 a_2 \sin\alpha_2 + a_3 a_4 \sin\alpha_4 = a_2 a_3 \sin\alpha_3 + a_1 a_4 \sin\alpha_4. \qquad (9)$$

(8) and (9) imply $d_4' = d_4$ and $P_4 P_1 || A_4 A_1$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK (for a parallelogram); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

2142. [1996: 170] *Proposed by Victor Oxman, Haifa, Israel.*

In the plane are given an arbitrary quadrangle and bisectors of three of its angles. Construct, using only an unmarked ruler, the bisector of the fourth angle.

Solution by Toshio Seimiya, Kawasaki, Japan.

$ABCD$ is a given quadrangle. We denote the bisectors of angles A, B, C, D, by a, b, c, d, respectively. We assume that a, b, c are given. We shall construct d using only an unmarked ruler.

Case I. $ABCD$ is not a parallelogram. We may assume that AB is not parallel to CD.

Construction: Let O be the intersection of AB, CD, and let P be the intersection of b and c. Join O and P, and let Q be the intersection of a and OP. Join D and Q; then DQ is d.

Proof: P is either the incentre or the excentre of $\triangle OBC$ [depending on whether or not the given quadrangle is convex and how its angles are arranged], so that OP is a bisector of $\angle BOC$. Thus Q is the excentre or incentre of $\triangle OAD$ so that QD bisects $\angle ADC$.

Case II. $ABCD$ is a parallelogram. If $ABCD$ is a rhombus then b coincides with d and is the bisector of $\angle D$. We shall assume that $ABCD$ is not a rhombus.

Construction. Let P be the intersection of b and c. Draw AC and BD, and let O be their intersection. Join P, O, and let Q be the intersection of PO with a. Join Q and D; then QD is the bisector of $\angle D$.

[*Editor's comment.* Although Seimiya provides a brief argument that his construction is correct, it simply proves that a parallelogram is symmetric about its centre, which can be left to the reader.]

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; NIELS BEJLEGAARD, Stavanger, Norway; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK (Case I only); and the proposer.

2143. [1996: 170] *Proposed by B. M$\star\star\star$y, Devon, Switzerland.*

My lucky number, 34117, is equal to $166^2 + 81^2$ and also equal to $159^2 + 94^2$, where $|166 - 159| = 7$ and $|81 - 94| = 13$; that is,

> *it can be written as the sum of two squares of positive integers in two ways, where the first integers occurring in each sum differ by* 7 *and the second integers differ by* 13.

What is the smallest positive integer with this property?

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let $N = a^2 + b^2 = (a - 13)^2 + (b + 7)^2$, where $a \geq 14$ and $b \geq 1$, be the smallest natural number having the desired property. Simplifying yields

$$a^2 + b^2 = a^2 - 26a + 169 + b^2 + 14b + 49,$$

then $13a = 7b + 109$ and finally

$$b = \frac{13a - 109}{7}.$$

As b is a natural number, $13a - 109 \equiv 0 \bmod 7$ which implies $a \equiv 3 \bmod 7$ [for example, since $109 \equiv 39 \bmod 7$]. Hence the smallest a is 17, and then $b = 16$ and

$$N = 17^2 + 16^2 = 4^2 + 23^2 = 545.$$

We get all solutions by setting $a = 10 + 7k$ for $k \geq 1$:

$$(10 + 7k)^2 + (3 + 13k)^2 = (7k - 3)^2 + (10 + 13k)^2 = 109(2k^2 + 2k + 1).$$

The solution mentioned in the proposal is obtained for $k = 12$.

From the above equation we see that 109 divides all solutions. And incidentally, the smallest solution which is also a square is

$$885^2 + 1628^2 = 872^2 + 1635^2 = 1853^2.$$

Also solved by SAM BAETHGE, Nordheim, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; IAN JUNE L. GARCES, Ateneo de Manila University, Manila, The Philippines, and GIOVANNI MAZZARELLO, Ferrovie *dello Stato, Firenze, Italy; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, Hunter College High School, New York, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Valley Catholic High School, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. Two other readers sent in incorrect solutions probably due to misunderstanding the problem.*

Janous notes that 545 *is a palindrome and wonders if there are any other palindromic solutions! Konečný replaces the digits* 5 *and* 4 *in the solution by the corresponding letters E and D respectively, and therefore wonders if the proposer's name is MEDEY. (It is not — in fact, each star represents two letters.)*

2144. [1996: 170] *Proposed by B. M⋆⋆⋆y, Devon, Switzerland.*

My lucky number, 34117, has the interesting property that $34 = 2 \cdot 17$ and $341 = 3 \cdot 117 - 10$; that is,

it is a $2N + 1$-digit number (in base 10) for some N, such that

(i) *the number formed by the first N digits is twice the number formed by the last N, and*

(ii) *the number formed by the first $N + 1$ digits is three times the number formed by the last $N + 1$, minus 10.*

Find another number with this property.

I *Solution by David Hankin, Hunter College High School, New York, NY, USA.*

Let A be the number formed by the first N digits of the required number, let B be the number formed by the last N digits, and let x be the middle digit. Then

$$A = 2B \qquad \text{and} \qquad 10A + x = 3(x \cdot 10^N + B) - 10.$$

From these, we obtain

$$17B = 3x \cdot 10^N - x - 10, \tag{1}$$

so $x + 10 \equiv 3x \cdot 10^N \bmod 17$. Since $A = 2B$ has N digits, $B < 5 \cdot 10^{N-1}$ and so $17B < 8.5 \cdot 10^N$. Thus by (1), $1 \leq x \leq 2$.

When $x = 1$, we have $3 \cdot 10^N \equiv 11 \bmod 17$, so $10^N \equiv 15 \bmod 17$. This is satisfied by $N = 2$, and since $10^{16} \equiv 1 \bmod 17$ [by Fermat's little theorem], it is satisfied by $N = 2 + 16k$ for non-negative k. Note that $x = 1$, $N = 2$ yields the lucky number referred to in the problem.

When $x = 2$, we have $6 \cdot 10^N \equiv 12 \bmod 17$, so $10^N \equiv 2 \bmod 17$. A little arithmetic yields $N = 10 + 16k$ this time, so the next smallest number with the required property is the one given by $x = 2$ and $N = 10$. These values yield $17B = 6 \cdot 10^{10} - 12$, from which $B = 3\ 529\ 411\ 764$, $A = 7\ 058\ 823\ 528$, and a lucky number of

$$705\ 882\ 352\ 823\ 529\ 411\ 764$$

(which coincidentally happens to be my lucky number too).

II *Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

[Godin first obtained equation (1) and that x must be 1 or 2, and then his solution, with Hankin's notation, continues as follows.]

When $x = 1$, we need to find a natural number of the form

$$B = \frac{3 \cdot 10^N - 11}{17}.$$

Now we can calculate

$$\frac{3}{17} = 0.\overline{1764705882352941},$$

and if we look in the long division for the first occurrence of a remainder of 11, we can construct a solution. It occurs after the first 7, which yields $B = 17$, $A = 34$ and thus the lucky number 34117 which is the proposer's lucky number. It is also the first in an infinite family of solutions. We can create new ones by appending a full period of the repeating decimal for $3/17$ to the left of the B of our last solution. Thus the next solution in this family has

$$B = 1764705882352941\ 17, \qquad A = 3529411764705882\ 34,$$

and thus is

$$352941176470588234\ 1\ 176470588235294117.$$

There is another family of solutions with $x = 2$, obtained the same way. In this case

$$B = \frac{6 \cdot 10^N - 12}{17} \qquad \text{and} \qquad \frac{6}{17} = 0.\overline{3529411764705882}$$

with a remainder of 12 occurring first at the second 4, so

$$B = 3529411764, \qquad A = 7058823528,$$

and the lucky number is

$$7058823528\ 2\ 3529411764.$$

Similarly this is the first solution in an infinite family of solutions. The others are obtained by appending a full period of $6/17$ to the left of the B of the last solution.

Also solved by SAM BAETHGE, Nordheim, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Valley Catholic High School, Beaverton, Oregon, USA; AMIT KHETAN, Troy, Michigan, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. One incorrect solution was sent in.

Notice from Godin's solution that the fractions with denominator 17 *all have the same digits in their repeating parts, just "cycled around". This is the same behaviour that readers will likely know from the fractions* $1/7$, $2/7$, *etc. In both cases the reason for this behaviour is that the repeating parts have the largest possible number of digits,* 16 *in the case of denominator* 17. *The consequence for this problem is that the solutions have "repeated blocks" in them, which some solvers remarked upon.*