THE ACADEMY CORNER

No. 9

Bruce Shawyer

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In this issue, we feature a university undergraduate mathematics competition. We invite your solutions, especially from university students. Please send me your nice solutions.

Undergraduate Mathematics Competition

September 18, 1996

Answer as many questions as you can. Complete solutions carry more credit than scattered comments about many problems.

1. If n is any integer, show that $n^5 - n$ is divisible by 5.
2. A line l with slope $m = 2$ cuts the parabola $y^2 = 8x$ to form a chord.
   Find the equation of l if the midpoint of the chord lies on $x = 4$.
3. Show that three tangents can be drawn from the origin to the curve given by
   $$y = x^3 - 13x^2 + 10x - 36.$$
4. Prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$, for all positive integers n.
5. Show that
   $$\int_0^{\pi/2} \frac{\sin^{13} x}{\sin^{13} x + \cos^{13} x}\, dx = \frac{\pi}{4}.$$
6. In triangle ABC, $\angle B = 3\angle A$.
   If the sides opposite to the angles A, B, C have lengths a, b, c, respectively, prove that
   $$ac^2 = (b-a)^2\,(b+a).$$

THE OLYMPIAD CORNER

No. 180

R.E. Woodrow

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We begin this number of the Corner with the problems of the 3rd Mathematical Olympiad of the Republic of China (Taiwan). The contest was written April 14 and 15, 1994. Many thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Hong Kong for collecting these problems and many others.

3rd MATHEMATICAL OLYMPIAD OF THE REPUBLIC OF CHINA (Taiwan)

First Day — April 14, 1994

1. Let $ABCD$ be a quadrilateral with $\overline{AD} = \overline{BC}$ and $\angle A + \angle B = 120^\circ$. Three equilateral triangles $\triangle ACP$, $\triangle DCQ$ and $\triangle DBR$ are drawn on $\overline{AC}$, $\overline{DC}$ and $\overline{DB}$ away from $\overline{AB}$. Prove that the three new vertices P, Q and R are collinear.

2. Let a, b, c be positive real numbers, α be a real number. Suppose that

$$f(\alpha) = abc(a^\alpha + b^\alpha + c^\alpha),$$

$$g(\alpha) = a^{\alpha+2}(b + c - a) + b^{\alpha+2}(a - b + c) + c^{\alpha+2}(a + b - c).$$

Determine $|f(\alpha) - g(\alpha)|$.

3. Let a be a positive integer such that $(5^{1994} - 1) \mid a$. Show that the expression of the number a in the base 5 contains at least 1994 digits different from zero.

Second Day — April 15, 1994

4. Prove that there are infinitely many positive integers n having the following property: for every arithmetic progression $a_1, a_2, \ldots, a_n$ of integers with n terms, both the mean and standard deviation of the set $\{a_1, a_2, \ldots, a_n\}$ are integers. (Note: For any set $\{x_1, x_2, \ldots, x_n\}$ of real numbers, the mean of the set is defined to be the number

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

and the standard deviation of the set is defined to be the number

$$\sqrt{\frac{\sum(x_i - \overline{x})^2}{n}}.$$

5. Let $X = \{0, a, b, c\}$ and $M(X) = \{f \mid f : X \to X\}$ be the set of all functions from X into itself. Here, an addition table of X is given as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(1) If $S = \{f \in M(X) \mid f(x \oplus y \oplus x) = f(x) \oplus f(y) \oplus f(x),\ \forall x, y \in X\}$, determine the number of elements of S.
(2) If $I = \{f \in M(X) \mid f(x \oplus x) = f(x) \oplus f(x),\ \forall x \in X\}$, determine the number of elements of I.

6. For $-1 \leq x \leq 1$ define

$$T_n(x) = \frac{1}{2^n}\left[\left(x + \sqrt{1 - x^2}\right)^n + \left(x - \sqrt{1 - x^2}\right)^n\right]$$

(1) Prove that, for $-1 \leq x \leq 1$, $T_n(x)$ is a monic polynomial of degree n in the x-variable and the maximum value of $T_n(x)$ is $\frac{1}{2^{n-1}}$.
(2) Suppose that $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a monic polynomial with real coefficients such that for all x in $-1 \leq x \leq 1$, $p(x) > -\frac{1}{2^{n-1}}$. Prove that there exists x_* in $-1 \leq x \leq 1$ such that $p(x_*) \geq \frac{1}{2^{n-1}}$.

Last issue we gave Five Klamkin Quickies. Here we give his "Quick" solutions plus another 5 problems. Many thanks to Murray S. Klamkin, the University of Alberta.

ANOTHER FIVE KLAMKIN QUICKIES
October 21, 1996

6. Determine the four roots of the equation $x^4 + 16x - 12 = 0$.

7. Prove that the smallest regular n-gon which can be inscribed in a given regular n-gon is one whose vertices are the midpoints of the sides of the given regular n-gon.

8. If 31^{1995} divides $a^2 + b^2$, prove that 31^{1996} divides ab.

9. Determine the minimum value of

$$S = \sqrt{(a+1)^2 + 2(b-2)^2 + (c+3)^2} + \sqrt{(b+1)^2 + 2(c-2)^2 + (d+3)^2} + \sqrt{(c+1)^2 + 2(d-2)^2 + (a+3)^2} + \sqrt{(d+1)^2 + 2(a-2)^2 + (b+3)^2}$$

where a, b, c, d are any real numbers.

10. A set of 500 real numbers is such that any number in the set is greater than one-fifth the sum of all the other numbers in the set. Determine the least number of negative numbers in the set.

We will give the solutions to these in the next issue so you can have some fun looking for the answers. Next we give his solutions to the five Quickies we gave last issue.

FIVE KLAMKIN QUICKIES
October 21, 1996

1. For $x, y, z > 0$, prove that

(i) $1 + \dfrac{1}{(x+1)} \geq \left\{1 + \dfrac{1}{x(x+2)}\right\}^x$,

(ii) $[(x+y)(x+z)]^x[(y+z)(y+x)]^y[(z+x)(z+y)]^z \geq [4xy]^x[4yz]^y[4zx]^z$.

Solution. Both inequalities will follow by a judicious application of the weighted arithmetic-geometric mean inequality (W–A.M.–G.M.) which for three weights is

$$u^a v^b w^c \leq \left[\frac{au + bv + cw}{a+b+c}\right]^{a+b+c},$$

where $a, b, c, u, v, w \geq 0$.

(i) The inequality can be rewritten in the more attractive form

$$\left[1 + \frac{1}{x}\right]^x \leq \left[1 + \frac{1}{x+1}\right]^{x+1},$$

and which now follows by the W–A.M.–G.M.

$$\left[1 + \frac{1}{x}\right]^x \leq \left\{\frac{1 + x\left(1 + \frac{1}{x}\right)}{1+x}\right\}^{x+1} = \left[1 + \frac{1}{x+1}\right]^{x+1}.$$

(ii) Also, the inequality here can be rewritten in the more attractive form

$$\left[\frac{2x}{z+x}\right]^{z+x}\left[\frac{2y}{x+y}\right]^{x+y}\left[\frac{2z}{y+z}\right]^{y+z} \leq 1.$$

But this follows by applying the W–A.M.–G.M. to

$$1 = \sum[z+x]\left[\frac{2x}{z+x}\right] \div \sum[z+x].$$

2. If $ABCD$ is a quadrilateral inscribed in a circle, prove that the four lines joining each vertex to the nine point centre of the triangle formed by the other three vertices are concurrent.

Solution. The given result still holds if we replace the nine point centres by either the orthocentres or the centroids.

A vector representation is particularly à propos here, since (with the circumcentre O as an origin and F denoting the vector from O to any point F) the orthocentre H_a, the nine point centre N_a, the centroid G_a of $\triangle BCD$ are given simply by $H_a = B + C + D$, $N_a = (B + C + D)/2$, $G_a = (B + C + D)/3$, respectively, and similarly for the other three triangles. Since the proofs for each of the three cases are practically identical, we just give the one for the orthocentres. The vector equation of the line L_a joining A to H_a is given by $L_a = A + \lambda_a[B + C + D - A]$ where λ_a is a real parameter. By letting $\lambda_a = 1/2$, one point on the line is $[A + B + C + D]/2$ and similarly this point is on the other three lines. For the nine point centres, the point of concurrency will be $2[A + B + C + D]/3$, while for the centroids, the point of concurrency will be $3[A + B + C + D]/4$.

3. How many six digit perfect squares are there each having the property that if each digit is increased by one, the resulting number is also a perfect square?

Solution. If the six digit square is given by

$$m^2 = a \cdot 10^5 + b \cdot 10^4 + c \cdot 10^3 + d \cdot 10^2 + e \cdot 10 + f,$$

then

$$n^2 = (a+1)\cdot 10^5 + (b+1)\cdot 10^4 + (c+1)\cdot 10^3 + (d+1)\cdot 10^2 + (e+1)\cdot 10 + (d+1),$$

so that

$$n^2 - m^2 = 111,111 = (111)(1,001) = (3 \cdot 37)(7 \cdot 11 \cdot 13).$$

Hence,

$$n + m = d_i \quad \text{and} \quad n - m = 111,111/d_i$$

where d_i is one of the divisors of $111,111$. Since $111,111$ is a product of five primes it has 32 different divisors. But since we must have $d_i > 111,111/d_i$, there are at most 16 solutions given by the form $lm = \frac{1}{2}(d_i - 111,111/d_i)$. Then since m^2 is a six digit number, we must have

$$632.46 \approx 200\sqrt{10} < 2m < 2,000.$$

On checking the various divisors, there are four solutions. One of them corresponds to $d_i = 3 \cdot 13 \cdot 37 = 1,443$ so that $m = \frac{1}{2}(1,443 - 7 \cdot 11) = 683$ and $m^2 = 466,489$. Then, $466,489 + 111,111 = 577,600 = 760^2$. The others are given by the table

d_i	m	m^2	n^2	n
$3 \cdot 7 \cdot 37 = 777$	317	$100,489$	$211,600$	460
$3 \cdot 11 \cdot 37 = 1,221$	565	$319,225$	$430,336$	656
$7 \cdot 11 \cdot 13 = 1,001$	445	$198,025$	$309,136$	556

4. Let v_iw_i, $i = 1, 2, 3, 4$, denote four cevians of a tetrahedron $v_1v_2v_3v_4$ which are concurrent at an interior point P of the tetrahedron. Prove that

$$pw_1 + pw_2 + pw_3 + pw_4 \le \max v_iw_i \le \text{longest edge}.$$

Solution. We choose an origin, o, outside of the space of the tetrahedron and use the set of 4 linearly independent vectors $V_i = ov_i$ as a basis. Also the vector from o to any point q will be denoted by Q. The interior point p is then given by $P = x_1V_1 + x_2V_2 + x_3V_3 + x_4V_4$ where $x_i > 0$ and $\sum_i x_i = 1$. It now follows that $W_i = \frac{P - x_iV_i}{1 - x_i}$ (for other properties of concurrent cevians via vectors, see [1987: 274–275]) and then that

$$pw_i = \left|\frac{P - x_iV_i}{1 - x_i} - P\right| = \left|\frac{x_i(P - V_i)}{1 - x_i}\right| = \left|x_i \sum_j x_j \frac{V_j - V_i}{1 - x_i}\right|,$$

$$v_iw_i = \left|\frac{P - x_iV_i}{1 - x_i} - V_i\right| = \left|\frac{P - V_i}{1 - x_i}\right| = \left|\sum_j x_j \frac{V_j - V_i}{1 - x_i}\right|.$$

Summing

$$\sum_i pw_i = \sum_i \left|x_i \sum_j x_j \frac{V_j - V_i}{1 - x_i}\right| = \sum_i x_i(v_iw_i) \le \max_i v_iw_i,$$

and with equality only if v_iw_i is constant. Also,

$$v_iw_i \le \sum_{j \ne i} \left[\frac{x_j}{1 - x_i}\right] \max_r |V_r - V_i| = \max_r |V_r - V_i|.$$

Finally,

$$\sum pw_i \le \max_i v_iw_i \le \max_{r,s} |V_r - V_s|.$$

Comment: In a similar fashion, it can be shown that the result generalizes to n-dimensional simplexes. The results for triangles are due to Paul Erdős, Amer. Math. Monthly, Problem 3746, 1937, p. 400; Problem 3848, 1940, p. 575.

5. Determine the radius r of a circle inscribed in a given quadrilateral if the lengths of successive tangents from the vertices of the quadrilateral to the circle are a, a, b, b, c, c, d, d, respectively.

Solution. Let $2A$, $2B$, $2C$, $2D$ denote the angles between successive pairs of radii vectors to the points of tangency and let r be the inradius. Then

$$r = \frac{a}{\tan A} = \frac{b}{\tan B} = \frac{c}{\tan C} = \frac{d}{\tan D}.$$

Also, since $A + B + C + D = \pi$, we have $\tan(A + B) = \tan(C + D) = 0$, or

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} + \frac{\tan C + \tan D}{1 - \tan C \tan D} = 0,$$

so that

$$\frac{r(a + b)}{r^2 - ab} + \frac{r(c + d)}{r^2 - cd} = 0.$$

Finally,

$$r^2 = \frac{abc + bcd + cda + dab}{a + b + c + d}.$$

Now we turn our attention to solutions by our readers to problems given in the September 1995 number of the Corner where we gave the 16th Austrian Polish Mathematics Competition [1995: 221–222].

16TH AUSTRIAN POLISH MATHEMATICS COMPETITION

First Day — June 30, 1993

Time: 4.5 hours (individual competition)

1. Determine all natural numbers $x, y \geq 1$ such that $2^x - 3^y = 7$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Amengual's solution.

First of all we note that both numbers x and y must be even. Suppose to the contrary that one of the numbers is odd.

If x is odd, the number $2^x + 1$ (which we have from the factorization $A^x + 1 = (A + 1)(A^{x-1} - A^{x-2} + \cdots - A + 1)$ that $2^x + 1$) is a multiple of 3. Consequently $2^x - 3^y \equiv -1 \bmod 3$, while $7 \equiv 1 \bmod 3$, and the given equation is invalid $\bmod 3$.

If y is odd, we will use the modular technique as in the previous case, but this time modulo 8. We have $3^2 \equiv 1 \bmod 8$. It follows that $3^{2k+1} \equiv 3 \bmod 8$ for $k = 0, 1, 2, \ldots$.

Consequently $3^y + 7 \equiv 0 \bmod 8$ and since $2^x \equiv 3^y + 7$ we must have $x \leq 2$. If $x = 1$, we have $2 - 3^y = 7$, which is impossible. If $x = 2$, we have $2^2 - 3^y = 7$, which is also impossible.

Suppose that the numbers x and y are even. So $x = 2l$, $y = 2m$, with l and m natural numbers. The given equation can then be written in the form $(2^l + 3^m)(2^l - 3^m) = 7$, where $2^l + 3^m$ and $2^l - 3^m$ are natural numbers, which implies that $2^l + 3^m = 7$ and $2^l - 3^m = 1$. These two equations determine the values of l, m, namely $l = 2$, $m = 1$ for which we have $x = 4$, $y = 2$.

The only two natural numbers $x, y \geq 1$ such that $2^x - 3^y = 7$ are therefore $x = 4$ and $y = 2$.

5. Determine all real solutions x, y, z of the system of equations:

$$\begin{aligned} x^3 + y &= 3x + 4, \\ 2y^3 + z &= 6y + 6, \\ 3z^3 + x &= 9z + 8. \end{aligned}$$

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Panos E. Tsaoussoglou, Athens, Greece. We give the solution of Tsaoussoglou.

From $x^3 + y = 3x + 4$, we have

$$x^3 - 1 - 1 - 3x = 2 - y,$$

or

$$(x-2)(x+1)^2 = 2 - y. \tag{1}$$

From $2y^3 - 2 - 2 - 6y = 2 - z$, we have

$$2(y-2)(y+1)^2 = (2 - z), \tag{2}$$

and $3z^3 - 3 - 3 - 9z = 2 - x$ gives

$$3(z-2)(z+1)^2 = (2 - x), \tag{3}$$

so that

$$\begin{aligned} (x-2)(x+1)^2 &= -(y-2), \\ 2(y-2)(y+1)^2 &= -(z-2), \\ 3(z-2)(z+1)^2 &= -(x-2), \end{aligned}$$

and

$$(x-2)(y-2)(z-2)\left((x+1)^2(y+1)^2(z+1)^2 + \frac{1}{6}\right) = 0.$$

As the last factor is always positive for real x, y, z, we have

$$(x-2)(y-2)(z-2) = 0.$$

This gives at least one of $x = 2$, $y = 2$, $z = 2$. In conjunction with (1), (2) and (3) this gives the unique solution $x = y = z = 2$.

6. Show: For all real numbers $a, b \geq 0$ the following chain of inequalities is valid

$$\begin{aligned} \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2 &\leq \frac{a + \sqrt[3]{a^2 b} + \sqrt[3]{a b^2} + b}{4} \\ &\leq \frac{a + \sqrt{ab} + b}{3} \leq \sqrt{\left(\frac{\sqrt[3]{a^2} + \sqrt[3]{b^2}}{2}\right)^3}. \end{aligned}$$

Also, for all three inequalities determine the cases of equality.

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and by Panos E. Tsaoussoglou, Athens, Greece. We give Tsaossoglou's solution.

1. $\left(\dfrac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leq \dfrac{\sqrt[3]{a^2}(\sqrt[3]{a}+\sqrt[3]{b})+\sqrt[3]{b^2}(\sqrt[3]{a}+\sqrt[3]{b})}{4}$
is equivalent to

$$(\sqrt{a}+\sqrt{b})^2 \leq (\sqrt[3]{a^2}+\sqrt[3]{b^2})(\sqrt[3]{a}+\sqrt[3]{b}),$$

which holds by the Cauchy inequality.

Let $\sqrt[6]{a}=A$, $\sqrt[6]{b}=B$, $(A^3+B^3)^2 \leq (A^4+B^4)(A^2+B^2)$.

2. $3(a+b)+3\sqrt[3]{ab}(\sqrt[3]{a}+\sqrt[3]{b}) \leq 4(a+\sqrt{ab}+b)$,
or equivalently

$$a+3\sqrt[3]{a^2b}+3\sqrt[3]{ab^2}+b \leq 2(a+2\sqrt{ab}+b),$$

or

$$(\sqrt[3]{a}+\sqrt[3]{b})^3 \leq 2(\sqrt{a}+\sqrt{b})^2,$$

or

$$\left(\frac{A^2+B^2}{2}\right)^3 \leq \left(\frac{A^3+B^3}{2}\right)^2,$$

(with A, B as above), a known inequality.

3. $\dfrac{a+\sqrt{ab}+b}{3} \leq \sqrt{\left(\dfrac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3}$.

With A and B as above this is equivalent to

$$\left(\frac{A^6+A^3B^3+B^6}{3}\right)^2 \leq \left(\frac{A^4+B^4}{2}\right)^3.$$

For this it is enough to prove that

$$\left(\frac{A^4+B^4}{2}\right)^3 - \left(\frac{A^6+A^3B^3+B^6}{3}\right)^2 \geq 0,$$

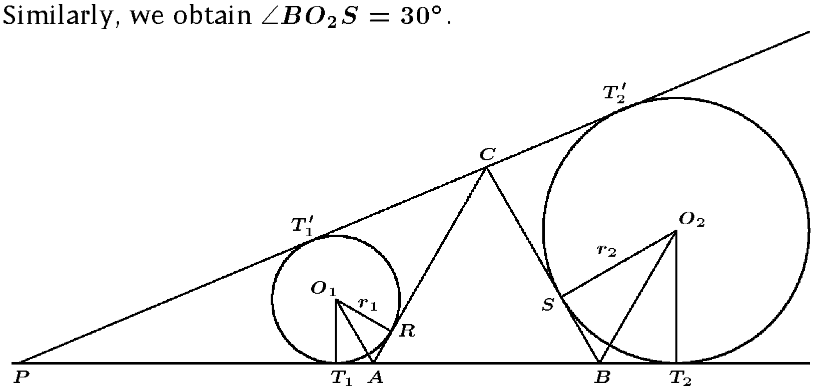
or

$$\begin{aligned}
&9(A^4+B^4)^3-8(A^6+A^3B^3+B^6)^2\\
&= A^{12}-16A^9B^3+27A^8B^4-24A^6B^6+27A^4B^8-16A^3B^9+B^{12}\\
&= (A-B)^4\left[A^8+4A^7B+10A^6B^2+4A^5B^3-2A^4B^4\right.\\
&\qquad \left.+4A^3B^5+10A^2B^6+4AB^7+B^8\right]\\
&\geq (A-B)^4(A^3B^3(A-B)^2) \geq 0.
\end{aligned}$$

9. Let $\triangle ABC$ be equilateral. On side AB produced, we choose a point P such that A lies between P and B. We now denote a as the length of sides of $\triangle ABC$; r_1 as the radius of incircle of $\triangle PAC$; and r_2 as the exradius of $\triangle PBC$ with respect to side BC. Determine the sum $r_1 + r_2$ as a function of a alone.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina. We give Amengual's solution and comment.

Looking at the figure, we see that $\angle T_1O_1R = 60°$ since it is the supplement of $\angle T_1AR = 120°$ (as an exterior angle for $\triangle ABC$). Hence, $\angle AO_1R = 30°$. Similarly, we obtain $\angle BO_2S = 30°$.



Since tangents drawn to a circle from an external point are equal, we have

$$\begin{aligned} T_1T_2 &= T_1A + AB + BT_2 \;=\; RA + AB + SB \\ &= r_1 \tan 30° + a + r_2 \tan 30° \;=\; \frac{r_1 + r_2}{\sqrt{3}} + a, \end{aligned}$$

and

$$T_1'T_2' = T_1'C + CT_2' = CR + CS = (a - RA) + (a - SB) = 2a - \frac{r_1 + r_2}{\sqrt{3}}.$$

Since common external tangents to two circles are equal, $T_1T_2 = T_1'T_2'$. Hence,

$$\frac{r_1 + r_2}{\sqrt{3}} + a = 2a - \frac{r_1 + r_2}{\sqrt{3}},$$

whence we find that

$$r_1 + r_2 = \frac{a\sqrt{3}}{2}.$$

Comment. This problem is identical to problem 2.1.11, page 25, of H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, (The Charles Babbage Research Centre, 1989).

We next give the solution to one problem of the VII Nordic Mathematical Contest.

2. [1995: 223] *VII Nordic Mathematical Contest*

A hexagon is inscribed in a circle with radius r. Two of its sides have length 1, two have length 2 and the last two have length 3. Prove that r is a root of the equation

$$2r^3 - 7r - 3 = 0.$$

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Equal chords subtend equal angles at the centre of a circle; if each of sides of length i subtends an angle α_i $(i = 1, 2, 3)$ at the centre of the given circle, then

$$2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 360^\circ,$$

whence

$$\frac{\alpha_1}{2} + \frac{\alpha_2}{2} = 90^\circ - \frac{\alpha_3}{2},$$

and

$$\cos\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) = \cos\left(90^\circ - \frac{\alpha_3}{2}\right) = \sin\frac{\alpha_3}{2}.$$

Next we apply the addition formula for the cosine:

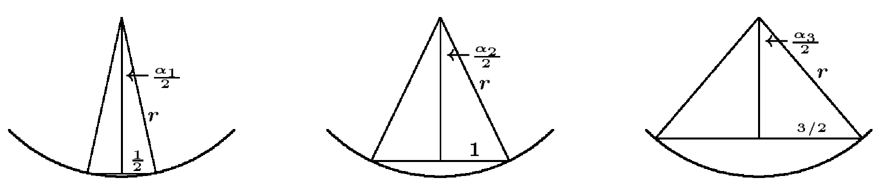
$$\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2} = \sin\frac{\alpha_3}{2}, \tag{1}$$

where (see figures)

$$\sin\frac{\alpha_1}{2} = \frac{1/2}{r}, \quad \cos\frac{\alpha_1}{2} = \frac{\sqrt{4r^2-1}}{2r};$$

$$\sin\frac{\alpha_2}{2} = \frac{1}{r}, \quad \cos\frac{\alpha_2}{2} = \frac{\sqrt{r^2-1}}{r};$$

$$\sin\frac{\alpha_3}{2} = \frac{3/2}{r}.$$



A B C

We substitute these expressions into (1) and obtain, after multiplying both sides by $2r^2$,

$$\sqrt{4r^2-1}\cdot\sqrt{r^2-1} - 1 = 3r.$$

Now write it in the form

$$\sqrt{(4r^2-1)(r^2-1)} = 3r+1,$$

and square, obtaining

$$(4r^2-1)(r^2-1) = 9r^2+6r+1,$$

which is equivalent to

$$r(2r^3-7r-3) = 0.$$

Since $r \neq 0$, we have

$$2r^3-7r-3 = 0,$$

which was to be shown.

To finish this number of the Corner we turn to two problems from the 32nd Ukrainian Mathematical Olympiad given in the October 1995 number of the Corner [1995 : 266].

32nd UKRAINIAN MATHEMATICAL OLYMPIAD

March 1992 — Selected Problems

2. (8) There are real numbers a, b, c, such that $a \geq b \geq c > 0$. Prove that

$$\frac{a^2-b^2}{c} + \frac{c^2-b^2}{a} + \frac{a^2-c^2}{b} \geq 3a-4b+c.$$

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Arslanagić.

From $a \geq b \geq c > 0$ we have

$$\frac{a+b}{c} \geq 2, \quad 0 < \frac{b+c}{a} \leq 2 \quad \text{and} \quad \frac{a+c}{b} \geq 1.$$

Now we get

$$\frac{a^2-b^2}{c} \geq 2(a-b), \quad \text{because } a \geq b;$$

$$\frac{c^2-b^2}{a} \geq 2(c-b), \quad \text{because } c \leq b;$$

and

$$\frac{a^2-c^2}{b} \geq a-c, \quad \text{because } a \geq c.$$

After addition of these inequalities, we have

$$\frac{a^2-b^2}{c} + \frac{c^2-b^2}{a} + \frac{a^2-c^2}{b} \geq 2(a-b)+2(c-b)+(a-c),$$

that is,

$$\frac{a^2-b^2}{c}+\frac{c^2-b^2}{a}+\frac{a^2-c^2}{b}\geq 3a-4b+c.$$

The equality holds if and only if $a=b=c>0$.

5. (10) Prove that there are no real numbers x, y, z, such that

$$x^2+4yz+2z=0,$$

$$x+2xy+2z^2=0,$$

$$2xz+y^2+y+1=0.$$

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by D.J. Smeenk, Zaltbommel, the Netherlands; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Label the three given equations (1), (2) and (3), (in that order), respectively. If $x=0$ or $z=0$, then from (3), $y^2+y+1=0$, which has no real solutions. Hence we may assume that $xz\neq 0$. From (1) and (2), we get $x^2=-2z(2y+1)$ and $2z^2=-x(2y+1)$, which, when multiplied gives $2x^2z^2=2xz(2y+1)^2$ or $xz=(2y+1)^2$. Substituting into (3) we get

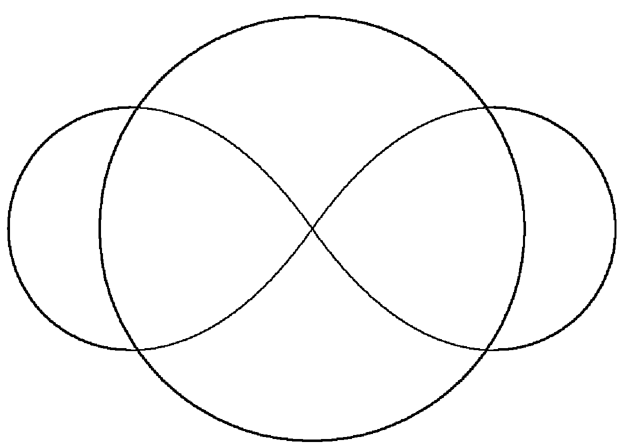
$$2(2y+1)^2+y^2+y+1=0$$

or

$$3y^2+3y+1=0,$$

which has no solution.

That completes the Corner for this issue. Olympiad season is upon us. Send me your Olympiads, as well as your nice solutions.



BOOK REVIEWS

Edited by ANDY LIU

Leningrad Mathematical Olympiads 1987–1991, by Dmitry Fomin and Alexey Kirichenko,
published by MathPro Press, Westford, MA, USA, 1994.
(Contests in Mathematics Series, vol. 1.), paperbound, 202 + xxii pages, ISBN 0–9626401–4–X, US$24.
Reviewed by ***József Pelikán,*** Eötvös Loránd University, Budapest, Hungary.

The regulations of the International Mathematical Olympiad (IMO) stipulate that a country can send a team to the IMO consisting of at most six students. Still, there was a *city* in the 1965 IMO which participated with *nine* students: five in the Russian team, two in the Israeli team, and one in each of the USA and German teams. This city was Leningrad, now known by its old name St. Petersburg again.

If a city has such an output of talented young mathematicians, you can imagine that the city olympiad organized there should be of an exceptionally high quality. And so it is. Let me briefly describe the system of the Leningrad Olympiads. Until 1989 Soviet schools comprised grades 1–10, and the competition was held for 5–10 graders. Since that time the Russian schools comprise grades 1–11, and the competition is held for 6–11 graders. The competition is organized in four rounds. The first round is at the school level in December and January and the second round held at the regional level (that is, in the 22 regions of Leningrad) in February, some 10,000 to 12,000 students taking part in the second round. Both of these rounds are written examinations. It is important to emphasize this, because the last two rounds — quite unusually for a mathematics competition — are oral examinations. This requires of course a huge number of able and devoted jurors, and few countries — let alone cities — can produce the number of mathematicians necessary for this task. In the third round (called "main round") about 90–130 students participate in each grade. In the final round (called somewhat misleadingly "elimination round" — this term normally means the initial phase of a competition, not the final one) which is held only for 9–11 (earlier 8–10) graders some 100 students take part altogether, although in 1991 there were only 34. The main round is held in February or March, and the elimination round in March. In the period 1962–1983 and also in 1991 the elimination round was used only to pick the city team at the All-Union Olympiad (this might explain the strange name), while in the period 1984–1990 the result of the competition was decided by the elimination round (not by the main round).

A few words about the earlier history of the Leningrad (and other) Olympiads in the Soviet Union. The Leningrad Olympiad started in 1934 (the Moscow Olympiad one year later). The All-Russia Olympiad started in 1961 and the All-Union (i.e. Soviet Union) Olympiad in 1967. These are respectable ages, although there are a few national mathematics contests which are much older (notably the Hungarian Eötvös — later Kürschák — competitions which started in 1894). But I fully agree with the authors that the Leningrad competition is quite unique in being an oral one. On the other hand I must contest another statement of the authors, namely that the Leningrad competition would be quite unique in having only new and original problems. In fact the majority of the competitions I know of have new and original problems.

Besides the main part containing the problems and solutions the book contains useful appendices: statistical data about the number of participants, the names of each year's winners, a glossary and the names of the authors of each problem. Also, there are valuable comments on the book by the publisher, Stanley Rabinowitz, and a very good evaluation of the reasons of the excellence of Soviet mathematics by Mark Saul.

To give a sample of the problems (and solutions) I give you two examples from the book. Both come from the 1991 Olympiad and were given to grade 11 students.

Problem A black pawn is placed in the top right square of an 8×8 chess-board. One may place a white pawn on any empty square of the board, having repainted all pawns in adjacent squares so that black pawns become white, and vice versa. (Two squares are called adjacent if they have a common vertex.) Can one place pawns in this manner so that all 64 squares of the board would be filled with white pawns?

Solution Connect the centres of each pair of adjacent free squares by a segment. Then, when we place the next white pawn on some square of the board, we erase all segments from the centre of this square. Thus, the number of erased segments coincides each time with the number of free adjacent squares for the square on which each white pawn is placed. To make a certain pawn be white at the end, it must be placed on a square having an even number of free adjacent squares because the number of repaintings for each pawn is equal to the number of pawns that will be placed on adjacent squares after this one. So, we conclude that each time we have to erase an even number of segments. But this is impossible because the initial number of segments is odd (we have lost at the beginning three segments caused by the black pawn in the corner) and the final number is even — it is equal to zero. Thus, we see that in the final position there is at least one black pawn.

The next problem has a rather unusual solution.

Problem One may perform the following two operations on a natural number:

(a) multiply it by any natural number;

(b) delete zeros in its decimal representation.

Prove that for any natural number n, one can perform a sequence of these operations that will transform n to a one-digit number.

Solution We use the following fact:

Lemma For any integer n that is not divisible by 2 or 5 one can find a number consisting of digits 1 only that is a multiple of n.

Proof Consider n numbers $1, 11, \ldots, 11\ldots1$. If one of these numbers is a multiple of n, we are done. Otherwise, their remainders modulo n can have $n-1$ possible values $1, \ldots, n-1$, and therefore (by the pigeonhole principle) at least two of those numbers have the same remainders modulo n. Then their difference is divisible by n and looks like $11\ldots1 \cdot 10^k$. But n is coprime with 10, and we conclude that the first factor is divisible by n.

Now let n be an arbitrary natural number. Multiplying n by 2 and 5 and deleting zeros, we can transform n to a number that is coprime with 10. Then multiplying the result by the appropriate number, we can obtain a number containing only digits 1 in its decimal representation (we use the assertion of the lemma). Now the chain of the following operations leads to the desired result:

(a) Multiplying by 82, we obtain the number $911\ldots102$.

(b) Delete zero in the last obtained number and multiply it by 9. This gives the number $8200\ldots08$, which is transformed into 828.

(c) $828 \cdot 25 = 20700$.

(d) $27 \cdot 4 = 108$.

(e) $18 \cdot 5 = 90$, and we can obtain the single-digit number 9.

All in all this book is very well written, full of interesting problems and I warmly recommend it to anyone interested in mathematical competitions, or just in nice problems.

Folding the Regular Heptagon

Robert Geretschläger, Bundesrealgymnasium, Graz, Austria

Introduction

Ever since Greek antiquity, mathematicians have been considering constructions that can be done with straight-edge and compass only, the so-called Euclidean constructions. A number of famous problems, such as squaring the circle, trisecting angles and doubling the cube, were unsolvable for the Greeks, and later shown to be theoretically unsolvable by Euclidean methods. The reason for this is that only such problems that can be reduced algebraically to combinations of linear and quadratic equations are solvable in this sense. We now know that these three problems, as well as many others, cannot be represented by combinations of such equations.

One specific problem the Greeks attempted to solve in this way was the construction of regular n–gons for small n. They were successful in finding constructions for $n = 3, 4, 5, 6, 8, 10$ and 12, but not for $n = 7, 9$ or 11. Since 7 is the smallest n for which no construction could be found, it was of special interest why this particular problem should prove so stubborn. As it turned out, the construction of the regular heptagon by Euclidean methods is impossible for the same reason that angle trisection and doubling the cube are, in that each of these problems requires the graphic solution of an irreducible cubic equation in its algebraic representation.

As shown in *"Euclidean Constructions and the Geometry of Origami"* ([1]), all cubic equations can be solved graphically using elementary methods of origami[1]. This is especially interesting in light of the fact that regular n–gons are commonly used in the development of origami folding bases. A heptagon could conceivably find use in developing models of insects for instance, since six legs + one head = seven corners. In this article, I present a theoretically precise method of folding the regular heptagon from a square, derived from the results established in the above-mentioned article. The folding method is presented in standard origami notation, and the mathematical section is cross-referenced to the appropriate diagrams.

The Cubic Equation

The seven corners of a regular heptagon can be thought of as the seven solutions of the equation

$$z^7 - 1 = 0 \tag{1}$$

[1] Origami is, of course, the art of paper folding. For readers not yet familiar with this ancient art, but interested in becoming so, there is a large amount of introductory literature easily available. I was personally introduced to origami by the books of Robert Harbin ([2]). A fine introduction to the geometry of origami is the classic "Geometric Exercises in Paper Folding" by T. Sundara Row ([3]).

in the complex plane. This implies that the unit circle is the circumcircle of the heptagon, and that one corner of the heptagon is the point $z_1 = 1$ on the real axis (Fig. 1.1.). Since one solution of (1) is known, the other six are the roots of

$$\frac{z^7 - 1}{z - 1} = z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1 = 0. \tag{2}$$

For any specific z satisfying this equation, the conjugate $\overline{z}$ is also a solution, since the real axis is an axis of symmetry of the regular heptagon. Also, since

$$|z| = |\overline{z}| = 1,$$

we have $\overline{z} = \frac{1}{z}$. Therefore we can define

$$\zeta = z + \frac{1}{z} = z + \overline{z} = 2 \cdot \mathrm{Re}\, z. \tag{3}$$

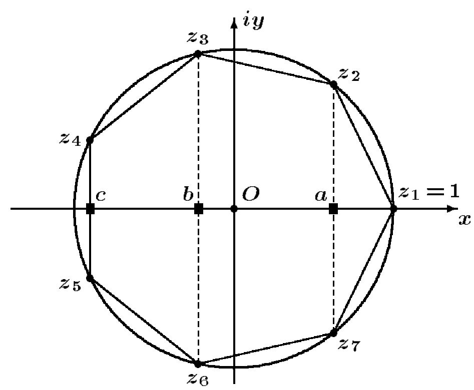


Fig. 1.1.

$$z_1 = 1 + i \cdot 0 \qquad z_2 = \cos\tfrac{2\pi}{7} + i \cdot \sin\tfrac{2\pi}{7} \qquad z_3 = \cos\tfrac{4\pi}{7} + i \cdot \sin\tfrac{4\pi}{7}$$
$$z_4 = \cos\tfrac{6\pi}{7} + i \cdot \sin\tfrac{6\pi}{7} \qquad z_5 = \overline{z_4} = \cos\tfrac{8\pi}{7} + i \cdot \sin\tfrac{8\pi}{7}$$
$$z_6 = \overline{z_3} = \cos\tfrac{10\pi}{7} + i \cdot \sin\tfrac{10\pi}{7} \qquad z_7 = \overline{z_2} = \cos\tfrac{12\pi}{7} + i \cdot \sin\tfrac{12\pi}{7}$$
$$a = \cos\tfrac{2\pi}{7} = \mathrm{Re}\, z_2 = \mathrm{Re}\, z_7 \qquad b = \cos\tfrac{4\pi}{7} = \mathrm{Re}\, z_3 = \mathrm{Re}\, z_6$$
$$c = \cos\tfrac{6\pi}{7} = \mathrm{Re}\, z_4 = \mathrm{Re}\, z_5$$

Dividing by z^3, we see that equation (2) is equivalent to

$$z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} = 0$$

since 0 is not a root, and since

$$\begin{aligned}
\zeta^3 &= \left(z + \frac{1}{z}\right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \\
&= z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right) = z^3 + \frac{1}{z^3} + 3\zeta \\
\Longleftrightarrow \zeta^3 - 3\zeta &= z^3 + \frac{1}{z^3}
\end{aligned}$$

and

$$\begin{aligned}\zeta^2 &= \left(z+\frac{1}{z}\right)^2 = z^2+2+\frac{1}{z^2}\\ \Longleftrightarrow \zeta^2-2 &= z^2+\frac{1}{z^2},\end{aligned}$$

substituting yields

$$\begin{aligned}\left(z^3+\frac{1}{z^3}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)+1 &= 0\\ \Longleftrightarrow \quad \zeta^3-3\zeta+\zeta^2-2+\zeta+1=0&\\ \Longleftrightarrow \quad \zeta^3+\zeta^2-2\zeta-1=0.&\end{aligned}$$

From (3) ,we see that each root of the equation

$$\zeta^3+\zeta^2-2\zeta-1=0 \tag{4}$$

is real, and is equal to twice the common real component of two conjugate complex solutions of (1). It is therefore possible to find the six complex roots of (1) in the complex plane by finding the roots of (4), taking half their values, finding the straight lines parallel to the imaginary axis and at precisely these distances from it, and finally finding the points of intersection of these parallel lines with the unit circle. We shall now proceed to utilize these steps in folding the regular heptagon.

A Step-by-step Description of the Folding Process

As is usually the case in origami, we assume a square of paper to be given. We consider the edge-to-edge folds in step 1 as the x– and y–axes of a system of cartesian coordinates, and the edge-length of the given square as four units. The mid-point of the square is then the origin $M(0,0)$, and the end-points of the folds have the coordinates $(-2,0)$ and $(2,0)$, and $(0,-2)$ and $(0,2)$ respectively. For readers not familiar with origami notation, it should be mentioned that dashed lines represent so-called "valley" folds (folding up), and dot-dashed lines represent so-called "mountain" folds (folding down). Thin lines represent visible creases in the paper generated by previous folds.

As shown in [1], the solutions of the cubic equation

$$x^3+px^2+qx+r=0$$

are the slopes of the common tangents of the parabolas p_1 and p_2 defined by the foci

$$F_1\left(-\frac{p}{2}+\frac{r}{2},\frac{q}{2}\right) \qquad \text{and} \qquad F_2\left(0,\frac{1}{2}\right)$$

and directrices

$$l_1: x=-\frac{p}{2}-\frac{r}{2} \qquad \text{and} \qquad l_2: y=-\frac{1}{2}$$

respectively.

The solutions of (4) can therefore be obtained by finding the common tangents of the parabolas with foci

$$F_1(-1,-1) \qquad \text{and} \qquad F_2\left(0,\frac{1}{2}\right)$$

and directrices

$$l_1 : x = 0 \qquad \text{and} \qquad l_2 : y = -\frac{1}{2}$$

respectively. Since the slope of the common tangents is not altered by translating the parabolas parallel to the y–axis we can, for convenience, use

$$F_1\left(-1,-\frac{1}{2}\right) \qquad \text{and} \qquad F_2(0,1)$$

and

$$l_1 : x = 0 \qquad \text{and} \qquad l_2 : y = 0.$$

This is precisely what is done in steps 2 to 5. F_1 is the point A, and F_2 is the point B. The fold in step 4 is then the only common tangent of the parabolas with positive slope, and thus twice the real component of the solutions of (1) which lie to the right of the imaginary axis and are not equal to 1. In other words, the slope of this fold is $2 \cdot \cos\frac{2\pi}{7}$. Step 4, by the way, is the only step that cannot be replaced by a straight-edge and compass construction.

In steps 6 to 8, the unit-length is then transferred in such a way that point E in step 8 has y–coordinate $-2 \cdot \cos\frac{2\pi}{7}$. Since the distance from M to point 1 in step 9 is 2 units, the distances from M to points 2 and 7 are also 2 units, and so points 7, 1 and 2 are three consecutive corners of the regular heptagon. (We assume that point 1 with coordinates $(0,-2)$ is the first corner, and continue from there.)

Step 10 thus yields two sides of the heptagon, and steps 11 to 13 yield the remaining sides of the heptagon by making use of its radial symmetry, until finally step 14 shows us the completed regular heptagon. The folding process is shown at the end of the article.

Conclusion

Unlike other regular n–gons with small n, the regular heptagon is not very common in popular culture or graphics. Apart from the seven-sided star one comes across in astrology, the heptagon does not seem to show up much in public, unlike its close relatives. We come across the octagon at many a street corner, and the pentagon and hexagon can be seen on most soccer balls, just to name a few. I do not know if this (relatively) easy generation of the regular heptagon will lead to its mass popularization, but an ardent Heptagonist can certainly dream.

It should be mentioned that a similar folding method for the regular heptagon is described in the article "*Draw of a Regular Heptagon by the*

Folding" by Benedetto Scimemi ([4]) in the relatively hard to find Proceedings of the First International Meeting of Origami Science and Technology. (I have only recently gained access to a copy myself.) This volume offers a great many ideas for further research for anyone interested in the geometry of origami, and is certainly worth searching for.

References

[1] R. Geretschläger, *Euclidean Constructions and the Geometry of Origami*, Mathematics Magazine, **68** No. 5 December 1995, pp. 357-371

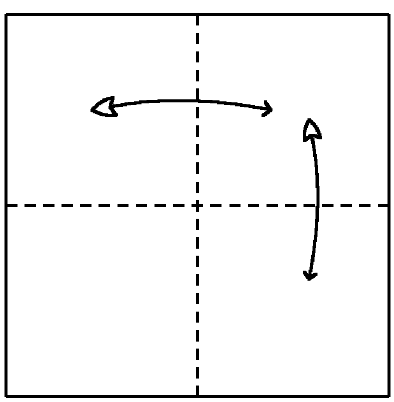
[2] R. Harbin, *Origami, The Art of Paper-Folding, Vols. 1-4,* Hodder Paperbacks, Norwich (1968)

[3] T. Sundara Row, *Geometric Exercises in Paper Folding*, Dover Publications, Inc., Mineola, NY (1966) reprint of 1905 edition

[4] B. Scimemi, *Draw of a Regular Heptagon by the Folding*, Proceedings of the First International Meeting of Origami Science and Technology, Ferarra, Italy (1989)

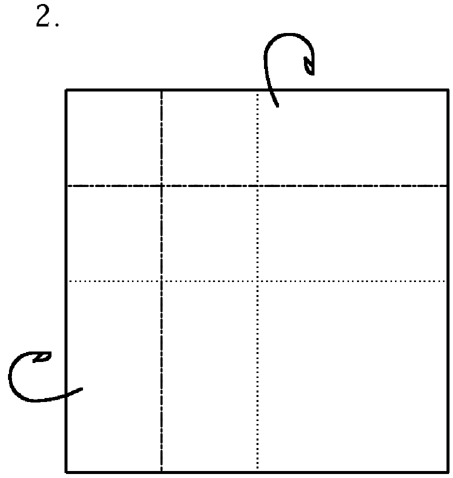
The Folding Process

1.



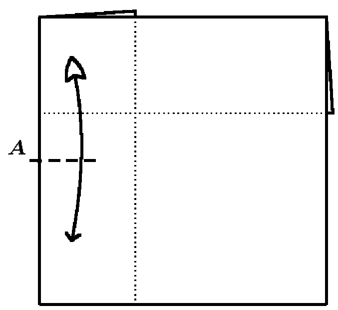
Fold and unfold twice.

2.



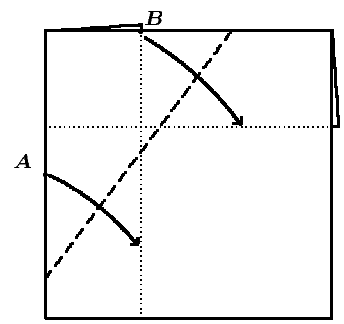
Fold back twice.

3.



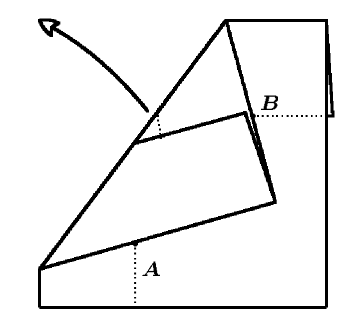
Fold and unfold, making a crease mark at point ***A*** (bisecting the side).

4.



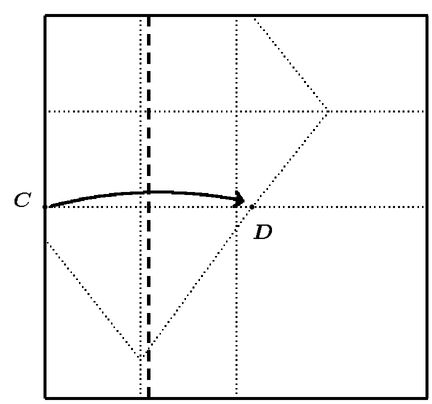
Fold such that ***A*** and ***B*** come to lie on the creases.

5.



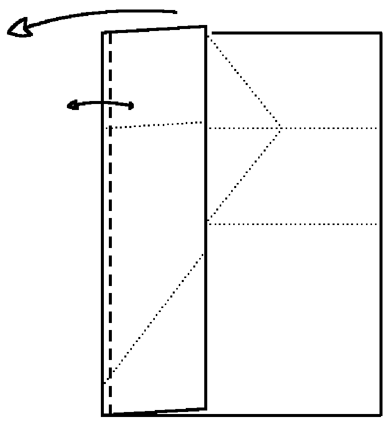
Unfold everything.

6.



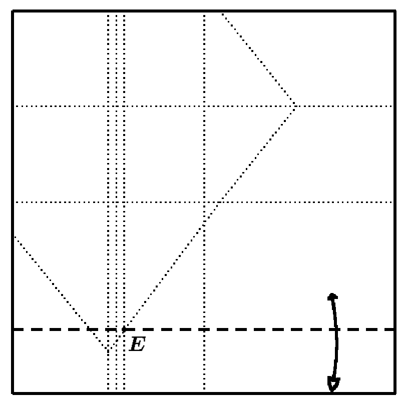
Fold ***C*** to ***D***.

7.



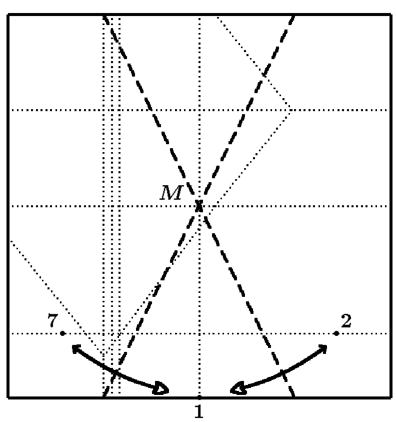
Fold and unfold both layers at crease, then unfold everything.

8.



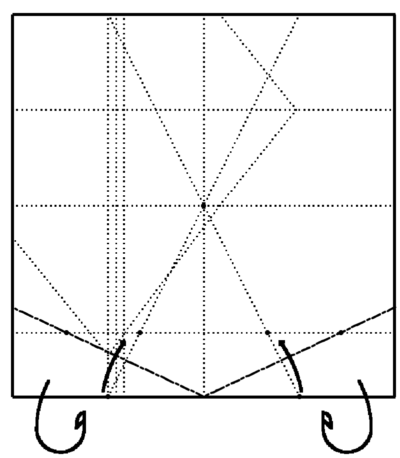
Fold horizontally through E, then unfold.

9.



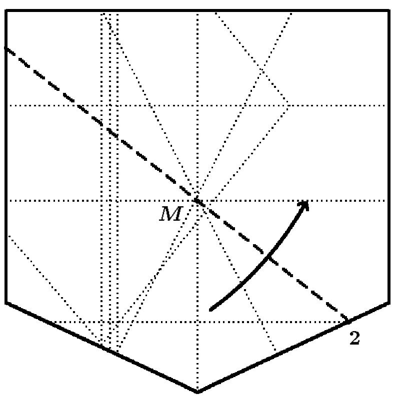
Fold through M, such that 1 lies on crease, resulting in 2 and 7 (M is the mid-point of the heptagon, 1, 2 and 7 are corners).

10.



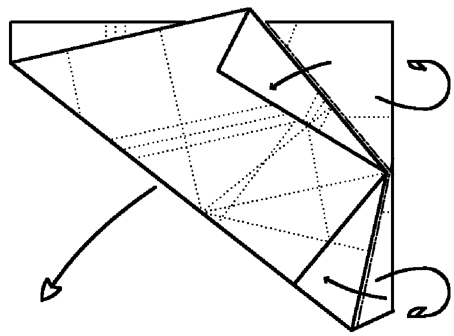
Fold back twice, so that the marked points come to lie on one another; resulting folds are first two sides of the heptagon.

11.



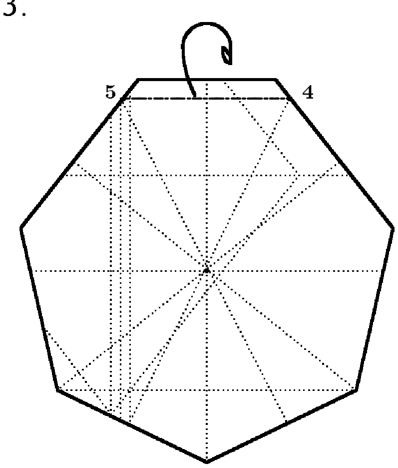
Fold through M and 2.

12.



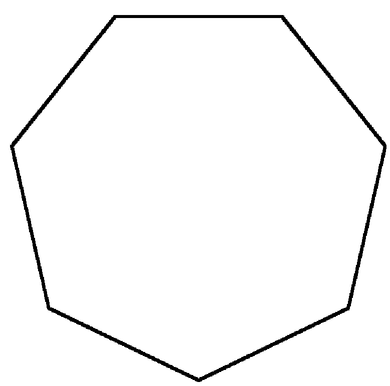
Fold back lower layers using edges of upper layer as guidelines; resulting folds are two more sides of the heptagon; open up fold from step 12 and repeat 11 and 12 on left side.

13.



Fold back final edge of the heptagon through 4 and 5.

14.



The finished heptagon.

THE SKOLIAD CORNER

No. 20

R.E. Woodrow

This issue we give the problems of the Mathematical Association National Mathematics contest, written November 18, 1994. The contest was written by about 30,000 students in the United Kingdom. My thanks go to Tony Gardiner, School of Mathematics, University of Birmingham for sending me the contest.

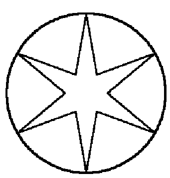
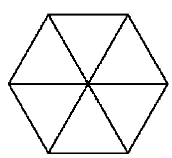
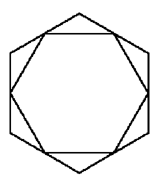
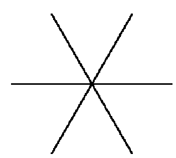
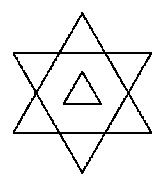
THE MATHEMATICAL ASSOCIATION NATIONAL MATHEMATICS CONTEST 1994

Friday, November 18, 1994 — Time: 90 minutes

1. The average of x and $8x$ is 18. What is the value of x?

A. $1\frac{1}{2}$ **B.** 2 **C.** 4 **D.** $4\frac{1}{2}$ **E.** 9.

2. Which of the following does not have six lines of symmetry?

A. fig 2a **B.** fig 2b **C.** fig 2c **D.** fig 2d **E.** fig 2e.

3. I write out the numbers from 1 up to 30 *in words*. If N denotes the number of times I write the letter "**n**", M denotes the number of times I write the letter "**m**", and C denotes the number of times I write the letter "**c**", then $N + M + C$ equals

A. 27 **B.** 28 **C.** 29 **D.** 30 **E.** 31.

4. Which of the following five numbers has a prime factor in common with exactly one of the other four numbers?

A. 91 **B.** 52 **C.** 39 **D.** 35 **E.** 24.

5. $ABCD$ is a quadrilateral with $AB = AD = 25$ cm, $CB = CD = 52$ cm and $DB = 40$ cm. How long in AC in cm?

A. 32.5 **B.** 48 **C.** 52 **D.** 60 **E.** 63.

6. The number of pounds of pickled peppers that Peter Piper purchased for £59 is equal to the number of pounds Peter would pay for two hundred and thirty six pounds of peppers. How much would he pay for twenty pounds of pickled peppers?

A. £5 **B.** £10 **C.** £20 **D.** £40 **E.** £80.

7. Which expression has the smallest value when $x = -0.5$?

A. $2^{1/x}$ **B.** $\frac{-1}{x}$ **C.** $\frac{1}{x^2}$ **D.** 2^x **E.** $\frac{1}{\sqrt{-x}}$.

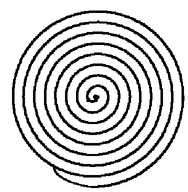
8. Over an average lifetime in the UK, roughly how many times does a person's heart beat?

A. 4×10^7 **B.** 5×10^7 **C.** 2×10^8 **D.** 3×10^9 **E.** 2×10^{10}.

9. What is the sum of the reciprocals of the first six triangular numbers 1, 3, 6, 10, etc.?

A. 10 **B.** $\frac{12}{7}$ **C.** 56 **D.** $\frac{3}{2}$ **E.** $\frac{49}{21}$.

10. A rope 15 m long and 5 cm in diameter is coiled in a flat spiral as shown. What is the best estimate for the diameter of the "circle" (in cm)?

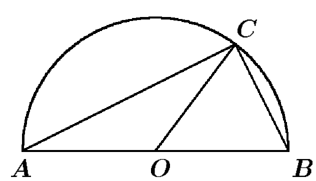


A. 10 **B.** 100 **C.** 150 **D.** 200 **E.** 300.

11. If $a \otimes b = \frac{(ab+a+b+1)}{a}$, then $19 \otimes 94$ equals

A. 95 **B.** 100 **C.** 208 **D.** 1882 **E.** 1994.

12. The diagram shows a semicircle with radius 1 cm and with centre O. If C is an arbitrary point on the semicircle, which of the following statements may be false?



A. $\angle ACB$ is a right angle. **B.** $\triangle OAC$ is isosceles.
C. the area of $\triangle ABC$ is ≤ 1 cm^2 **D.** $\triangle AOC$ is equal in area to $\triangle OBC$
E. $AO^2 + OB^2 = AC^2 + BC^2$.

13. A giant marrow in my garden weighed 50 pounds and was 98% water. Suppose that during a rainy day it absorbed water so that it became 99% water. What would its new weight be (in pounds)?

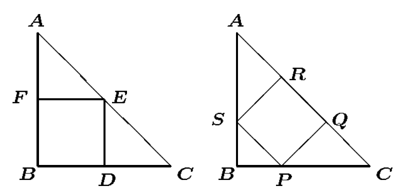
A. 50.01 **B.** 50.5 **C.** 98 **D.** 99 **E.** 100.

14. A solid cuboid has edges of lengths a, b, c. What is its surface area?

A. $(a+b+c)^2-(a^2+b^2+c^2)$ **B.** abc **C.** $2(a^2+b^2+c^2)$
D. $(a+b+c)^2$ **E.** $ab+bc+ca$.

15. Given two copies of an isosceles right-angled triangle ABC, squares $BDEF$ and $PQRS$ are inscribed in different ways as shown. What is the ratio

$$\frac{\text{area } PQRS}{\text{area } BDEF}?$$



A. $\frac{8}{9}$ **B.** $\frac{2}{3}$ **C.** 1 **D.** $\sqrt{\frac{2}{3}}$ **E.** $\frac{9}{8}$.

16. What is the last digit of $1994^{(1995+1996+1997+1998+1999+2000)}$?

A. 0 **B.** 2 **C.** 4 **D.** 6 **E.** 8.

17. When two dice are thrown the probability that the total score is a multiple of 2 is $\frac{1}{2}$. For how many other values of n is it true that, when two dice are thrown, the probability that the total score is a multiple of n is equal to $\frac{1}{n}$?

A. 1 **B.** 2 **C.** 3 **D.** 4 **E.** 5.

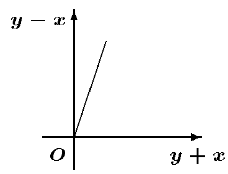
18. How many digits are there in the smallest number which is composed entirely of fives (eg. 5555) and which is divisibly by 99?

A. 9 **B.** 10 **C.** 18 **D.** 36 **E.** 45.

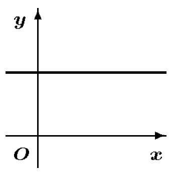
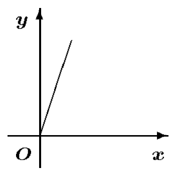
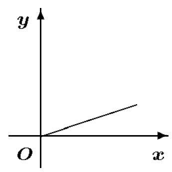
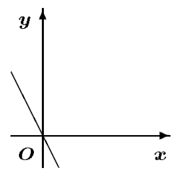
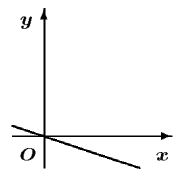
19. The price of a secondhand car is displayed (in pounds) on four cards on the windscreen. Each card shows one digit. If the card with the thousands digit blew off in the wind, the apparent price of the car would drop to one forty-ninth of the intended value. What number is on that card?

A. 5 **B.** 6 **C.** 7 **D.** 8 **E.** 9.

20. The graph of $y - x$ against $y + x$ is as shown.



The same scale has been used on each axis. Which of the following shows the graph of y against x?

A. fig 20a **B.** fig 20b **C.** fig 20c **D.** fig 20d **E.** fig 20e.

21. Which is smallest?

A. $5 + 6\sqrt{7}$ **B.** $7 + 6\sqrt{5}$ **C.** $6 + 5\sqrt{7}$ **D.** $7 + 5\sqrt{6}$ **E.** $6 + 7\sqrt{5}$.

22. A train leaves London at 0600 and arrives in Newcastle at 0930. Another train leaves Newcastle at 0700 and arrives in London at 0930. If both used the same route and each travelled at a constant speed, at what time would they meet?

A. $0757\frac{1}{2}$ **B.** $0802\frac{1}{2}$ **C.** $0807\frac{1}{2}$ **D.** $0827\frac{1}{2}$ **E.** more information required.

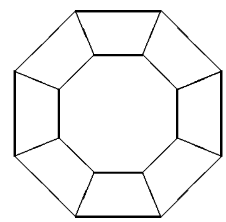
23. The triangle ABC has a right-angle at A. The hypotenuse BC is trisected at M and at N so that $BM = MN = NC$. If $AM = x$ and $AN = y$, then MN is equal to

A. $\frac{x+y}{2}$ **B.** $\frac{\sqrt{(y^2-x^2)}}{2}$ **C.** $\sqrt{(y^2 - x^2)}$ **D.** $\frac{\sqrt{(x^2+y^2)}}{3}$ **E.** $\sqrt{\frac{x^2+y^2}{5}}$.

24. If Susan drives to work at x mph she will be one minute late; if she drives at y mph she will be one minute early. How far does she drive to work (in miles)?

A. $\frac{yx}{30(y-x)}$ **B.** $\frac{2yx}{y-x}$ **C.** $\frac{x+y}{y-x}$ **D.** $\frac{x+y}{2}$ **E.** $\frac{x+y}{60(y-x)}$.

25. The octagonal figure is obtained by fitting eight congruent isosceles trapezia together. If the three shorter sides of each trapezium have length 1, how long is each outer edge?



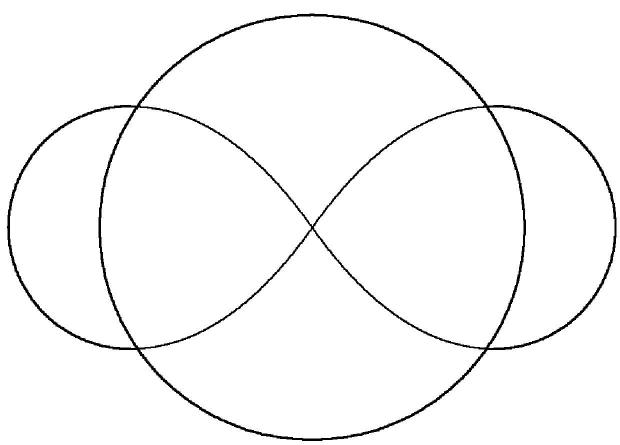
A. $1+\sqrt{2}$ **B.** $\frac{1+\sqrt{2}}{2}$ **C.** $\sqrt{2}$ **D.** 2

E. $1+\sqrt{2-\sqrt{2}}$.

Last number we gave the 1994 Nat West U.K. Junior Mathematical Challenge. It was written Tuesday, April 26, 1994. Here are the answers.

1. 455	**2**. B	**3**. E	**4**. A	**5**. A
6. B	**7**. B	**8**. D	**9**. D	**10**. A
11. B	**12**. E	**13**. A	**14**. B	**15**. E
16. E	**17**. C	**18**. C	**19**. D	**20**. E
21. A	**22**. D	**23**. D	**24**. C	**25**. C

That completes the Skoliad Corner for this issue. Send me your suitable contests and solutions. Also send me any suggestions for improvement of this feature.



MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, University of Toronto, Toronto, ON Canada M5S 1A1. The electronic address is

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Richard Hoshino (University of Waterloo), Wai Ling Yee (University of Waterloo), and Adrian Chan (Upper Canada College).

Matrix Exponentials: An Introduction

Donny Cheung
student, University of Waterloo
Waterloo, Ontario.

Let us start with the well-known power series for the exponential function,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

which works for all $x \in \mathbb{C}$. Likewise, we will define the exponential function for matrices (with complex entries) as

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \frac{M^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

for all $n \times n$ matrices M.

Matrix Exponentials Really *Do* Exist

First of all, we must show that this sum actually converges. Otherwise, the answer we get would be meaningless, and that would be bad. To start us off, we will define the **norm** of an $n \times n$ matrix, which we will denote $|A|$, as follows:

$$|A| = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right).$$

That is, for each row, take the sum of the absolute values of the components in that row, and the norm will be the largest of these row sums.

Lemma 1 For $n \times n$ matrices A and B,

$$|AB| \leq |A||B|.$$

Proof We label the entries of A and B in the standard way, a_{ij} and b_{ij}, respectively.

For any row in A, define the row sum as $\sum_{j=1}^{n} |a_{ij}|$. Since $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, the row sum of row i in AB is

$$\begin{aligned}
\sum_{j=1}^{n} \left| \sum_{k=1}^{n} a_{ik}b_{kj} \right| &\leq \sum_{j=1}^{n} \sum_{k=1}^{n} |a_{ik}b_{kj}| = \sum_{k=1}^{n} \sum_{j=1}^{n} |a_{ik}|\,|b_{kj}| \\
&= \sum_{k=1}^{n} \left(|a_{ik}| \sum_{j=1}^{n} |b_{kj}| \right) \leq \sum_{k=1}^{n} \left(|a_{ik}|\,|B| \right) \\
&= |B| \sum_{k=1}^{n} |a_{ik}| \leq |A||B|.
\end{aligned}$$

Now, $|AB|$, the maximum row sum, is still a row sum, thus $|AB| \leq |A||B|$. ■

Lemma 2 For $n \times n$ matrices A and B,

$$|A + B| \leq |A| + |B|.$$

The proof is left as an exercise to the reader.

Theorem 1 The sum e^M converges for all $n \times n$ matrices M.

Proof Using the two lemmas, we get

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \frac{M^4}{4!} + \cdots,$$

so that

$$\begin{aligned}
\left|e^M\right| &= |I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \frac{M^4}{4!} + \cdots| \\
&\leq 1 + |M| + \frac{|M|^2}{2!} + \frac{|M|^3}{3!} + \frac{|M|^4}{4!} + \cdots \\
&= e^{|M|}.
\end{aligned}$$

Since $|M|$ is a real number, $e^{|M|}$ is finite, and $|e^M|$ is bounded. Thus, e^M converges. ■

Exercises

1. Prove Lemma 2.

Some Linear Algebra Lingo

Recall that two $n \times n$ matrices A and B are **similar** if

$$B = C^{-1}AC$$

for some invertible $n \times n$ matrix C. (It is usually easier, when verifying that two matrices are similar, to show that $CB = AC$ instead.) Recall also that a **diagonal** matrix is a matrix with all its non-zero entries along the main diagonal. (But entries along the main diagonal are not necessarily non-zero). For example:

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

If a matrix A is similar to a diagonal matrix, we say that A is **diagonalizable**.

Calculating Matrix Exponentials

Suppose D is a diagonal matrix, with diagonal elements λ_1, λ_2, ..., λ_n. We have

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \quad D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix},$$

and

$$\begin{aligned} e^D &= I + D + \frac{D^2}{2!} + \cdots \\ &= \begin{pmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} + \cdots & 0 & \cdots & 0 \\ 0 & 1 + \lambda_2 + \frac{\lambda_2^2}{2!} + \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + \lambda_n + \frac{\lambda_n^2}{2!} + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix}. \end{aligned}$$

Jordan Matrices

A **Jordan block matrix** is an $n \times n$ matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

with λ's all down the main diagonal and 1's directly above them (except for the first column), and 0's everywhere else. Examples include

$$(5), \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}, \text{ and } \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

A **Jordan matrix** is an $n \times n$ matrix with Jordan blocks down the diagonal. An example containing our three examples above:

$$\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

Notice that all diagonal matrices are also Jordan matrices.

Exercises

1. Verify that

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}^k$$

$$= \begin{pmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{n-1}\lambda^{k-n+1} & \binom{k}{n}\lambda^{k-n} \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{n-2}\lambda^{k-n+2} & \binom{k}{n-1}\lambda^{k-n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & 0 & 0 & \cdots & 0 & \lambda^k \end{pmatrix}.$$

2. Compute e^B when B is a Jordan block matrix.
3. Compute e^J when J is a Jordan matrix.

So why did we just go through all that?

Here's the reason:

Theorem 2 For any $n \times n$ matrix A, there exists a Jordan matrix J which is similar, that is, $C^{-1}AC = J$ for some C.

J is known as a **Jordan canonical form** of A. Some matrices have more than one Jordan canonical form.

The proof is beyond the scope of this article, but can be found in any good advanced linear algebra textbook. But we *will* prove the following:

Theorem 3 If J is a Jordan canonical form of $n \times n$ matrix A, with $A = CJC^{-1}$, then $e^A = C(e^J)C^{-1}$.

Proof

$$\begin{aligned}
A^k &= (CJC^{-1})^k \\
&= \underbrace{(CJC^{-1})(CJC^{-1})\cdots(CJC^{-1})}_{k} \\
&= CJ(C^{-1}C)J(C^{-1}C)J(C^{-1}C)\cdots(C^{-1}C)JC^{-1} \\
&= CJ^kC^{-1}. \\
\text{Now, } e^A &= I + A + \frac{A^2}{2!} + \cdots \\
&= CIC^{-1} + CJC^{-1} + C\frac{J^2}{2!}C^{-1} + \cdots \\
&= C\left(I + J + \frac{J^2}{2!} + \cdots\right)C^{-1} \\
&= C(e^J)C^{-1}.
\end{aligned}$$

Thus, $e^A = C(e^J)C^{-1}$. ■

The process of finding a Jordan canonical form J of a matrix A is also beyond the scope of this article. Once again, I refer you to a good textbook. However, we have shown that it *is* possible to calculate e^A for any $n \times n$ matrix A.

And now for something less theoretical...

Here, we discuss an interesting application of matrix exponentials: first-order systems of linear differential equations. By that, we mean systems of the form:

$$\begin{aligned}
x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t), \\
x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t), \\
&\vdots \\
x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t).
\end{aligned}$$

Letting

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

we define the derivative $\mathbf{x}(t)$ as

$$\mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}.$$

We will also let A be the matrix of coefficients for the system. Now we can rewrite the system of equations as $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Exercises

1. Verify that $\mathbf{x}(t) = e^{At}\mathbf{c}$ is a solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ for every constant vector $\mathbf{c}$.

Further Exploration

This is where I become too lazy to show you all the neat stuff you can do with matrix exponentials, and where I encourage you, the reader, to explore on your own.

Exercises

1. We don't have to stop at matrix exponentials. We can define matrix equivalents for functions like $\sin(x)$ and $\cos(x)$ in a very similar fashion. What other types of functions can we extend to the matrices.
2. Prove that $\sin(A)\sin(A) + \cos(A)\cos(A) = I$ for any arbitrary $n \times n$ matrix A.
3. When is $e^{A+B} = e^A e^B$? The fact that matrix multiplication isn't commutative causes some problems.
4. As a corollary to the above problem, show that $e^{2A} = e^A e^A$.
5. Prove or disprove: there exist a 2×2 matrix A with real entries such that
$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

   [1996 Putnam, B4]

Open problems

1. As we have defined e^A, it is very easy to define x^A for $x \in \mathbb{R}$. Can we define B^A for $n \times n$ matrices A and B? What would their properties be? Would $A^{BC} = \left(A^B\right)^C$?
2. Can we define a $\log$ function? Would it have the similar properties to the real $\log$ function?
3. Can we solve other types of vector differential equations with matrix exponentials or (more generally) matrix functions like $\sin$ and $\cos$?

Mayhem Problems

A new year brings new changes and new problem editors. *Cyrus Hsia* now takes over the helm as *Mayhem Advanced Problems Editor*, with *Richard Hoshino* filling his spot as the *Mayhem High School Problems Editor*, and veteran *Ravi Vakil* maintains his post as *Mayhem Challenge Board Problems Editor*. Note that all correspondence should be sent to the appropriate editor — see the relevant section.

In this issue, you will find only solutions — the next issue will feature only problems. We intend to have problems and solutions in alternate issues.

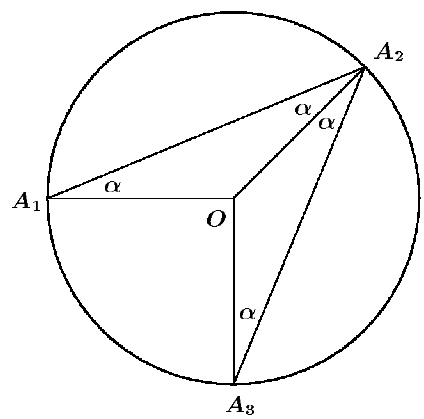
We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from the previous issue be submitted by 1 June 1997, for publication in the issue 5 months ahead; that is, issue 6. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions for others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Problems — Solutions

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 **rhoshino@undergrad.math.uwaterloo.ca**

H205. A circular billiard table is given with a cue ball at the circumference. It is shot at an angle of α to the line from the ball to the centre of the table. For what angles α will the ball come back to this point, assuming the ball keeps going indefinitely?

Solution by Samuel Wong, Mary Ward Catholic Secondary School, Toronto.



We let the ball start at A_1 as shown. Then $\angle OA_1A_2 = \angle A_1A_2O = \alpha$ (equal radii) and $\angle A_1OA_2 = 180° - 2\alpha$ (angles sum to $180°$). Since the angle of incidence equals the angle of reflection, we have $\angle A_3A_2O = \angle A_1A_2O = \alpha = \angle A_2A_3O$ by equal radii so $\angle A_3OA_2 = \angle A_2OA_1 = 180° - 2\alpha$. Thus, the angles at the centre are always equal. The cue ball returns to A_1 iff

$$k(180° - 2\alpha) = 360°c \implies k(90° - \alpha) = 180°c,$$

for some positive integers c and k. Since α must be real, $90° - \alpha$ must be either a rational or irrational number.

Case I: If $90° - \alpha$ is rational, it can be expressed in the form $\frac{m}{n}$, where m and n are positive, relatively prime integers. Thus $k \cdot \frac{m}{n} = 180c$. We may take $k = 180n$ and then $m = c$, so for all α rational, the cue ball will return to A_1.

Case II: If $90° - \alpha$ is irrational, let $x = 90° - \alpha$. Then $kx = 180°c \implies x = \frac{180°c}{k}$. But x is irrational, so we have a contradiction.

Also solved by MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain.

H206. For what values of n is an n-digit natural number uniquely determined from the sum and product of its digits?

Solution by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain.

The sum and product of the digits are two conditions, and so only two digits can be determined in general. This means $n = 1$ or 2, but when $n = 2$, the order cannot be determined (for example, consider 12 and 21), so the answer is $n = 1$.

H207. Is there a natural number n, such that $\phi(n) = p$, where p is an odd prime number?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI.

We shall show that there is NO positive integer n such that $\phi(n)$ is an odd prime number. This is an immediate consequence of the fact that $\phi(1) = 1 = \phi(2)$ and the result established below.

Theorem: If n is a positive integer, $n \geq 3$, then 2 divides $\phi(n)$.

Proof: Let n be a positive integer, $n \geq 3$.

Case I: $n = 2^a$, where a is an integer, $a \geq 2$. Then $\phi(n) = 2^{a-1}$, where $a - 1$ is a positive integer, so 2 divides $\phi(n)$.

Case II: n has an odd prime factor p. If the prime factorization of n is $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where p_k is odd, then

$$\phi(n) = p_1^{a_1-1} p_2^{a_2-1} \cdots p_k^{a_k-1}(p_1 - 1)(p_2 - 1) \cdots (p_k - 1).$$

Because p_k is odd, $p_k - 1$ is even so, 2 divides $\phi(n)$

Also solved by MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain, SAMUEL WONG, Mary Ward Catholic Secondary School, Toronto.

Advanced Problems — Solutions

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada. M1G 1C3 **hsia@math.toronto.edu**

A185. Let b_n be the highest power of 3 dividing $\binom{3^k}{n}$, $0 \leq n \leq 3^k$. Calculate $\displaystyle\sum_{n=0}^{3^k} \frac{1}{b_n}$.

Solution by Wai Ling Yee, student, University of Waterloo, Waterloo, Ontario.

The exponent of the highest power of 3 dividing $t!$ is

$$\left\lfloor \frac{t}{3} \right\rfloor + \left\lfloor \frac{t}{3^2} \right\rfloor + \left\lfloor \frac{t}{3^3} \right\rfloor + \cdots .$$

Thus, the exponent of the highest power of 3 dividing $\binom{3^k}{n} = \dfrac{3^k!}{n!(3^k-n)!}$ is

$$\sum_{j=1}^{k}\left(\left\lfloor\frac{3^k}{3^j}\right\rfloor - \left\lfloor\frac{n}{3^j}\right\rfloor - \left\lfloor\frac{3^k-n}{3^j}\right\rfloor\right).$$

Since $\dfrac{n}{3^j} + \dfrac{3^k-n}{3^j} = 3^{k-j}$, an integer,

$$\left\lfloor\frac{3^k}{3^j}\right\rfloor - \left\lfloor\frac{n}{3^j}\right\rfloor - \left\lfloor\frac{3^k-n}{3^j}\right\rfloor$$

can only take two possible values: 0 and 1.
When $3^j \mid n$, the value is 0. Otherwise, it is 1. Therefore, the value of b_n is 3^k divided by the highest power of 3 dividing n.

Consider the numbers divisible by $3^j, 0 \le j \le k-1$. Ignoring 0, we see that 3^{k-j} of the numbers are divisible by 3^j. These numbers can be written as $3^j \cdot (3m), 3^j \cdot (3m+1)$, or $3^j \cdot (3m+2)$. Therefore, there are $2 \cdot 3^{k-j-1}$ numbers in $\{1, 2, \ldots, 3^k\}$ for which 3^j is the highest power of 3 dividing them.

Each of the $2 \cdot 3^{k-j-1}$ numbers contributes $\dfrac{1}{3^{k-j}}$ to the sum $\displaystyle\sum_{n=0}^{3^k}\frac{1}{b_n}$ for a total of $\frac{2}{3}$. Therefore,

$$\sum_{n=0}^{3^k}\frac{1}{b_n} = \sum_{n=1}^{3^k-1}\frac{1}{b_n} + 2 = \frac{2}{3}k + 2.$$

Also solved by Edward Wang, Wilfred Laurier University, Waterloo, Ontario.

A186. Let a_n be the sequence defined by $a_1 = 1$, and

$$a_{n+1} = \frac{1 + a_1^2 + \cdots + a_n^2}{n}, \quad n \ge 1.$$

Prove that every a_n is an integer.

Solution

An April Fool's joke. It turns out that $a_1, a_2, \ldots, a_{43}$ are all integers, but a_{44} is not an integer.

[Ed: can you prove this?]

A187. Let $S_k(n)$ be the polynomial in n, such that $S_k(n) = \sum_{i=1}^{n} i^k$ for all positive integers n, e.g. $S_0(n) = n$, $S_1(n) = (n^2+n)/2$.
Prove that for $k \geq 1$, $n(n+1) \mid S_k(n)$.
Furthermore, prove that for k odd, $k \geq 3$, $n^2(n+1)^2 \mid S_k(n)$.

Solution by Wai Ling Yee, student, University of Waterloo, Waterloo, Ontario.

Note that $n(n+1) \mid S_1(n) = \dfrac{n(n+1)}{2}$. Assume that $n(n+1)$ divides all $S_k(n)$ up to $k = t$. Consider:

$$\begin{aligned}
(n+1)^{t+2} &= n^{t+2} + \binom{t+2}{1} n^{t+1} + \binom{t+2}{2} n^t + \cdots + \binom{t+2}{t+1} n + 1, \\
(n-1+1)^{t+2} &= (n-1)^{t+2} + \binom{t+2}{1}(n-1)^{t+1} + \binom{t+2}{2}(n-1)^t \\
&\quad + \cdots + \binom{t+2}{t+1}(n-1) + 1, \\
(n-2+1)^{t+2} &= (n-2)^{t+2} + \binom{t+2}{1}(n-2)^{t+1} + \binom{t+2}{2}(n-2)^t + \\
&\quad \cdots + \binom{t+2}{t+1}(n-2) + 1, \\
&\ \ \vdots \\
(2+1)^{t+2} &= 2^{t+2} + \binom{t+2}{1} 2^{t+1} + \binom{t+2}{2} 2^t + \cdots + \binom{t+2}{t+1} 2 + 1, \\
(1+1)^{t+2} &= 1^{t+2} + \binom{t+2}{1} 1^{t+1} + \binom{t+2}{2} 1^t + \cdots + \binom{t+2}{t+1} 1 + 1.
\end{aligned}$$

Adding all of the equations together, we have:

$$\sum_{i=2}^{n+1} i^{t+2} = \sum_{i=1}^{n} i^{t+2} + \sum_{i=0}^{t+1} \sum_{j=1}^{n} \binom{t+2}{t+2-i} j^i$$

if and only if

$$(n+1)^{t+2} - n - 1 = \binom{t+2}{1} S_{t+1}(n) + \sum_{i=1}^{t} \binom{t+2}{t+2-i} S_i(n).$$

By the induction hypothesis, we need only to prove that $n(n+1)$ divides $(n+1)^{t+2} - n - 1 = n(n+1)^{t+1}$, which is true.

By induction, $n(n+1) \mid S_k(n)$ for $k \geq 1$.

Now, $S_3(n) = \frac{n^2(n+1)^2}{4}$; therefore $n^2(n+1)^2$ divides $S_3(n)$. Assume that $n^2(n+1)^2$ divides $S_k(n)$ for all odd k up to $k = t-1$, t even. Consider:

$$\begin{aligned}
(n-1)^{t+2} &= n^{t+2} - \binom{t+2}{1} n^{t+1} + \binom{t+2}{2} n^t - \cdots - \binom{t+2}{t+1} n + 1, \\
(n-1-1)^{t+2} &= (n-1)^{t+2} - \binom{t+2}{1}(n-1)^{t+1} + \binom{t+2}{2}(n-1)^t \\
&\quad - \cdots - \binom{t+2}{t+1}(n-1) + 1,
\end{aligned}$$

$$\begin{aligned}
(n-2-1)^{t+2} &= (n-2)^{t+2} - \binom{t+2}{1}(n-2)^{t+1} + \binom{t+2}{2}(n-2)^t \\
&\quad - \cdots - \binom{t+2}{t+1}(n-2) + 1, \\
&\ \vdots \\
(2-1)^{t+2} &= 2^{t+2} - \binom{t+2}{1} 2^{t+1} + \binom{t+2}{2} 2^t - \cdots - \binom{t+2}{t+1} 2 + 1, \\
(1-1)^{t+2} &= 1^{t+2} - \binom{t+2}{1} 1^{t+1} + \binom{t+2}{2} 1^t - \cdots - \binom{t+2}{t+1} 1 + 1.
\end{aligned}$$

Subtracting the sum of all the equations in the above group from the sum of all the equations in the first group, we have:

$$\begin{aligned}
(n+1)^{t+2} + n^{t+2} - 1 &= 2\left[\binom{t+2}{1} S_{t+1}(n) + \binom{t+2}{3} S_{t-1}(n)\right. \\
&\quad \left. + \cdots + \binom{t+2}{t-1} S_3(n) + \binom{t+2}{t+1} S_1(n)\right].
\end{aligned}$$

By the induction hypothesis, we have to prove only that $n^2(n+1)^2$ divides

$$(n+1)^{t+2} + n^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1.$$

$$\begin{aligned}
n^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1 &= (n+1-1)^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1 \\
&= (n+1)^{t+2} - \cdots + \binom{t+2}{t}(n+1)^2 - \binom{t+2}{t+1}(n+1)^1 + 1 \\
&\quad - \binom{t+2}{t+1}(n^2+n) - 1 \\
&= (n+1)^{t+2} - \cdots + \binom{t+2}{t}(n+1)^2 - \binom{t+2}{t+1}(n^2+2n+1) \\
&= (n+1)^{t+2} - \cdots + \binom{t+2}{t}(n+1)^2 - \binom{t+2}{t+1}(n+1)^2.
\end{aligned}$$

Therefore $(n+1)^2$ divides $(n+1)^{t+2} + n^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1$.

$$\begin{aligned}
&(n+1)^{t+2} - \binom{t+2}{t+1}n - 1 \\
&= n^{t+2} + \cdots + \binom{t+2}{t}n^2 + \binom{t+2}{t+1}n + 1 - \binom{t+2}{t+1}n - 1 \\
&= n^{t+2} + \cdots + \binom{t+2}{t}n^2.
\end{aligned}$$

Therefore n^2 divides $(n+1)^{t+2} + n^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1$.

Therefore $n^2(n+1)^2$ divides $S_{t+1}(n)$.

By induction, $n^2(n+1)^2$ divides $S_k(n)$ for odd $k \geq 3$.

Challenge Board Problems — Solutions

Editor: Ravi Vakil, Department of Mathematics, One Oxford Street, Cambridge, MA, USA. 02138-2901 **ravi@math.harvard.edu**

We begin by dredging up an old favourite of ours. Well, maybe not so old – a solution appeared in [Mayhem 8: 4, 25: 1996]. We have since received a new solution that is worth printing.

C64. The numbers $x_1, x_2, \ldots, x_n$ are such that $x_1 + x_2 + \cdots + x_n = 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that there are two numbers among them whose product is no greater than $-1/n$.

(1991 Tournament of Towns)

Solution by Naoki Sato, student, University of Toronto, Toronto, Ontario.

Choose i such that $x_1 \leq x_2 \leq \cdots \leq x_i \leq 0 \leq x_{i+1} \leq \cdots \leq x_n$. Then

$$\begin{aligned} x_1^2 + x_2^2 + \ldots + x_i^2 &\leq x_1 \cdot x_1 + x_1 \cdot x_2 + \cdots + x_1 \cdot x_i \\ &= x_1(x_1 + x_2 + \cdots + x_i) \\ &= -x_1(x_{i+1} + \cdots + x_n) \\ &\leq -(n-i)x_1 \cdot x_n. \end{aligned}$$

Similarly,

$$\begin{aligned} x_{i+1}^2 + \cdots + x_n^2 &\leq x_n(x_{i+1} + \cdots + x_n) \\ &= -x_n(x_1 + x_2 + \cdots + x_i) \leq -ix_1 \cdot x_n. \end{aligned}$$

Thus $1 = x_1^2 + \cdots + x_n^2 \leq -nx_1 \cdot x_n$.

(As always, please send us new solutions to old problems.)

C68. *Proposed by Vin de Silva, student, Oxford University, Oxford, England.*

Let M be an $n \times n$ orthogonal matrix. (In other words, the rows are n vectors in n-space that are of length 1 and mutually perpendicular.) Let A be the $k \times k$ matrix in the upper-left corner of M, and let B be the $(n-k) \times (n-k)$ matrix in the lower-right corner of M. Prove that $\det A = \det B$.

Solution by Eric Wepsic, D.E.Shaw and Co., New York, NY, USA.

Let c_i be the i^{th} column of the orthogonal matrix Q, and let r_i be the i^{th} row. Let x_i be the i^{th} basis vector. Let M_1 be the top $k \times k$ submatrix of Q, and let M_2 be the bottom $(n-k) \times (n-k)$ submatrix. Then

$$\begin{aligned} \det M_1 &= c_1 \wedge c_2 \wedge \cdots \wedge c_k \wedge x_{k+1} \wedge \cdots \wedge x_n \\ &= Qx_1 \wedge Qx_2 \wedge \cdots \wedge Qx_k \wedge x_{k+1} \wedge \cdots \wedge x_n. \end{aligned}$$

As Q^t has determinant 1, this is equal to:

$$\begin{aligned} &Q^tQx_1 \wedge Q^tQx_2 \wedge \cdots \wedge Q^tQx_k \wedge Q^tx_{k+1} \wedge \cdots \wedge Q^tx_n \\ &= x_1 \wedge x_2 \wedge \cdots \wedge x_k \wedge r_{k+1} \wedge \cdots \wedge r_n \\ &= \det M_2. \end{aligned}$$

Solution by Sam Vandervelde, University of Chicago, Illinois, USA.

Let Q_1 be the matrix whose first k columns are the first k columns of Q, and whose remaining $n-k$ columns are the last $n-k$ columns of the identity matrix. Let Q_2 be the same sort of thing with the first k columns identical to I, and the last $n-k$ columns those of Q^t. Then an easy calculation (using $Q^tQ = I$, $Q^tI = Q^t$) shows that $Q^tQ_1 = Q_2$. Taking determinants of both sides yields the desired result.

The two solutions presented were in some sense identical, although the perspectives were different.

C69. Let Q be a polyhedron in $\mathbb{R}^3$. Let $\vec{n}_1, \ldots, \vec{n}_k$ and $A_1, \ldots, A_k$ be the normal vectors and areas of the faces of Q respectively. Prove that

$$\sum_{i=1}^{k} A_i \vec{n}_i = 0.$$

Solution by Miguel Carrión Álvarez, student, Universidad Complutense de Madrid, Spain.

We have $\sum_{i=1}^{k} A_i \vec{n}_i = \oint d\vec{s}$, where $d\vec{s}$ is the element of area (pointing outwards) and the integral is over the faces of the polyhedron. Then

$$\oint d\vec{s} = \left(\oint ds_x, \oint ds_y, \oint ds_z\right) = \left(\oint \vec{u}_x \cdot d\vec{s}, \oint \vec{u}_y \cdot d\vec{s}, \oint \vec{u}_z \cdot d\vec{s}\right),$$

for constant fields $\vec{u}_x$, $\vec{u}_y$, $\vec{u}_z$. We recall the divergence theorem: $\oint \vec{A} \cdot d\vec{s} = \int_V (\vec{\nabla} \cdot \vec{A}) dV$. But $\vec{u}_x$, $\vec{u}_y$, $\vec{u}_z$ are constant fields, so $\vec{\nabla} \cdot \vec{u}_i = 0$. Thus $\oint d\vec{s} = (0, 0, 0)$.

Comments.

1. *If ρ is the pressure field in a fluid, the total force on a (three-dimensional) object is*

$$F = -\oint \rho \, d\vec{s} = -\int (\vec{\nabla}\rho) dV.$$

   (Prove this!) In this case, we have constant pressure. What this result says is that a polyhedron sitting in a fluid with constant pressure from all sides (and no other forces) will not move.

2. *This is true in all dimensions.*

3. *Many other interesting results follow from this one, including the following variant of the Pythagorean theorem. Consider a tetrahedron $ABCD$ with AB, AC, AD mutually perpendicular. Then*

$$(ABC)^2 + (ACD)^2 + (ADB)^2 = (BCD)^2$$

   where the brackets denote the area of the triangle. This is also true in other dimensions. If you can think of any other interesting consequences (including other variants of Pythagoras), please send them in!

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 October 1997**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2214. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $n \geq 2$ be a natural number. Show that there exists a constant $C = C(n)$ such that for all real $x_1, \ldots, x_n \geq 0$ we have

$$\sum_{k=1}^{n} \sqrt{x_k} \leq \sqrt{\prod_{k=1}^{n} (x_k + C)}.$$

Determine the minimum $C(n)$ for some values of n.
[For example, $C(2) = 1$.]

2215*. *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*
Let P be a point inside a triangle ABC. It is known how to determine P such that $PA + PB + PC$ is a minimum (known as Fermat's Problem for Torricelli).

Determine P such that $PA + PB + PC$ is a maximum.

2216. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that $\lambda \geq 1$ is a natural number.

1. Determine the set of all λ's such that the diophantine equation $x^\lambda + y^2 = z^2$ has infinitely many solutions.

2.★ For any such λ, determine all solutions of this equation.

2217. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

(a) Prove that for every sufficiently large positive integer n, there are arithmetic progressions a_1, a_2, a_3 and b_1, b_2, b_3 of positive integers such that $n = a_1 b_1 + a_2 b_2 + a_3 b_3$.

(b) What happens if we require $a_1 = b_1 = 1$?

(This is a variation of problem 3 of the 1995/96 Alberta High School Mathematics Competition, Part II, which will appear in a future Skoliad Corner.)

2218. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Suppose that a, b, c are positive real numbers and that

$$abc = (a + b - c)(b + c - a)(c + a - b).$$

Clearly $a = b = c$ is a solution. Determine all others.

2219. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Show that there are an infinite number of solutions of the simultaneous equations:

$$\begin{aligned} x^2 - 1 &= (u + 1)(v - 1) \\ y^2 - 1 &= (u - 1)(v + 1) \end{aligned}$$

with x, y, u, v positive integers and $x \neq y$.

2220. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let V be the set of an icosahedron's twelve vertices, which can be partitioned into four classes of three vertices, each one in such a way that the three selected vertices of each class belong to the same face.

How many ways can this be done?

2221. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all members of the sequence $a_n = 3^{2n-1} + 2^{n-1}$, $(n \in \mathbb{N})$ which are the squares of any positive integer.

2222. *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Find the value of the continued root:

$$\sqrt{4+27\sqrt{4+29\sqrt{4+31\sqrt{4+33\sqrt{\ldots}}}}}.$$

NOTE: This was inspired by the problems in chapter 26 "*Ramanujan, Infinity and the Majesty of the Quattuordecillion*", pp 193–195, in "*Keys to Infinity*" by Clifford A. Pickover, John Wiley and Sons, 1995.

2223. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

We are given a bag with n identical bolts and n identical nuts, which are to be used to secure the n holes of a gadget.

The $2n$ pieces are drawn from the bag at random one by one. Throughout the draw, bolts and nuts are screwed together in the holes, but if the number of bolts exceeds the number of available nuts, the bolt is put into a hole until one obtains a nut, whereas if the number of nuts exceed the number of bolts, the nuts are piled up, one on top of the other, until one obtains a bolt.

Let L denote the discrete random variable which measures the height of the pile of nuts.

Find $\quad E[L] + E[L]^2$.

2224. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Point P lies inside triangle ABC. Triangle BCD is erected outwardly on side BC such that $\angle BCD = \angle ACP$ and $\angle CBD = \angle ABC$. Prove that if the area of quadrilateral $PBDC$ is equal to the area of triangle ABC, then triangles ACP and BCD are similar.

2225. *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

(a) For any positive integer n, prove that there exists a unique n-digit number

N such that:

(i) N is formed with only digits 1 and 2; and

(ii) N is divisible by 2^n.

(b) Can digits "1" and "2" in (a) be replaced by any other digits?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We have identified "anonymous" in the solutions of problems 2064, 2065, 2067, 2068, 2069, 2071 and 2077 as MARIA MERCEDES SÁNCHEZ BENITO, I.B. Luis Buñuel, Madrid, Spain.

The name of VICTOR OXMAN, University of Haifa, Haifa, Israel, should be added to the list of solvers of problem 2091. The name of ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick, should be added to the list of solvers to problem 2104.

2113. [1996: 35] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Prove the inequality

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right)$$

for any positive numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$.

I. *Solution by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

The given inequality follows from the easily verified identity:

$$\begin{aligned}&\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) - \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right)\\ &= \sum_{1\leq i<j\leq n}\frac{(a_ib_j-a_jb_i)^2}{(a_i+b_i)(a_j+b_j)}.\end{aligned}$$

II. *Solution by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.*

We know from Cauchy's inequality that

$$\left(\sum_{i=1}^{n}(a_i-b_i)\right)^2 \leq \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{(a_i-b_i)^2}{a_i+b_i}\right).$$

Therefore

$$\begin{aligned}
\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) &= \frac{1}{4}\left(\left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\right)^2 - \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\right)^2\right) \\
&= \frac{1}{4}\left(\left(\sum_{i=1}^{n}(a_i+b_i)\right)^2 - \left(\sum_{i=1}^{n}(a_i-b_i)\right)^2\right) \\
&\geq \frac{1}{4}\left(\left(\sum_{i=1}^{n}(a_i+b_i)\right)^2\right. \\
&\qquad \left. - \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{(a_i-b_i)^2}{a_i+b_i}\right)\right) \\
&= \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{(a_i+b_i)^2-(a_i-b_i)^2}{4(a_i+b_i)}\right) \\
&= \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right),
\end{aligned}$$

which completes the proof.

III. *Solution by Kee-Wai Lau, Hong Kong.*

For $n = 1$, the result is clear. Suppose that the inequality holds for $n = k$. Then for $n = k + 1$, we have

$$\begin{aligned}
&\left(\sum_{i=1}^{k+1} a_i\right)\left(\sum_{i=1}^{k+1} b_i\right) - \left(\sum_{i=1}^{k+1}(a_i+b_i)\right)\left(\sum_{i=1}^{k+1}\frac{a_ib_i}{a_i+b_i}\right) \\
&= \left(\sum_{i=1}^{k} a_i\right)\left(\sum_{i=1}^{k} b_i\right) - \left(\sum_{i=1}^{k}(a_i+b_i)\right)\left(\sum_{i=1}^{k}\frac{a_ib_i}{a_i+b_i}\right) \\
&\qquad + a_{k+1}\left(\sum_{i=1}^{k} b_i\right) + b_{k+1}\left(\sum_{i=1}^{k} a_i\right) \\
&\qquad\quad - (a_{k+1}+b_{k+1})\sum_{i=1}^{k}\frac{a_ib_i}{a_i+b_i} - \frac{a_{k+1}b_{k+1}}{a_{k+1}+b_{k+1}}\sum_{i=1}^{k}(a_i+b_i) \\
&\geq \sum_{i=1}^{k}\left(a_{k+1}b_i + b_{k+1}a_i - (a_{k+1}+b_{k+1})\frac{a_ib_i}{a_i+b_i}\right. \\
&\qquad\quad \left. - \frac{a_{k+1}b_{k+1}}{a_{k+1}+b_{k+1}}(a_i+b_i)\right) \qquad \text{by the induction hypothesis} \\
&= \frac{1}{a_{k+1}+b_{k+1}}\sum_{i=1}^{k}\frac{(a_{k+1}b_i - b_{k+1}a_i)^2}{a_i+b_i} \;\geq\; 0,
\end{aligned}$$

completing the induction.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; HAN PING DAVIN CHOR, Student, Cambridge, MA, USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; DAVID E. MANES, State University of New York, Oneonta, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. There were also one anonymous solution and one incorrect submission.

Klamkin remarked that this is a known inequality due to E.A. Milne [1], that came up in establishing an integral inequality. It also appeared as problem 67 in [2]. This was also noticed by Janous and Tsaoussoglou. Arslanagić, Herzig, Hess, Manes and Tsaoussoglou all pointed out that equality holds in the proposed inequality if and only if the vectors $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are linearly dependent. This is obvious from proof I *above.*

References

[1.] E.A. Milne, Note on Rosseland's Integral for the Stellar Absorption Coefficient, *Monthly Notices Royal Astronomical Soc.* **85** (1925) 979–984.

[2.] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, London, 1934, pp. 61–62.

2114. [1996: 75] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a square with incircle Γ. A tangent ℓ to Γ meets the sides AB and AD and the diagonal AC at P, Q and R respectively. Prove that

$$\frac{AP}{PB} + \frac{AR}{RC} + \frac{AQ}{QD} = 1.$$

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, and Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain.

We suppose that the equation of Γ is $x^2 + y^2 = 1$, and the coordinates of A and C are $(1, 1)$ and $(-1, -1)$ respectively. Let $T(\cos t, \sin t)$ be the coordinates of the point of tangency of line ℓ with Γ, with $0 < t < \frac{\pi}{2}$.

Then the equation of PQ is

$$y \cdot \sin t + x \cdot \cos t = 1,$$

and the coordinates of the involved points are

$$R\left(\frac{1}{\sin t + \cos t}, \frac{1}{\sin t + \cos t}\right), P\left(\frac{1 - \cos t}{\sin t}\right), Q\left(\frac{1 - \sin t}{\cos t}, 1\right),$$

and so

$$\frac{AP}{PB} = \frac{\sin t + \cos t - 1}{\sin t - \cos t + 1},$$
$$\frac{AR}{RC} = \frac{\sin t + \cos t - 1}{\sin t + \cos t + 1},$$
$$\frac{AQ}{QD} = \frac{\cos t + \sin t - 1}{\cos t - \sin t + 1},$$

from which an easy calculation shows that

$$\frac{AP}{PB} + \frac{AQ}{QD} + \frac{AR}{RC} = 1.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; CLAUDIO ARCONCHER, Jundiaí, Brazil; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SAM BAETHGE, Science Academy, Austin, Texas, USA; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; TIM CROSS, King Edward's School, Birmingham, England; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; P. PENNING, Delft, the Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; D.J. SMEENK, Zaltbommel, the Netherlands; MELETIS VASILIOU, Elefsis, Greece; and the proposer.

2115. [1996: 75] *Proposed by Toby Gee, student, the John of Gaunt School, Trowbridge, England.*

Find all polynomials f such that $f(p)$ is a prime for every prime p.

Solution by Luis V. Dieulefait, IMPA, Rio de Janeiro, Brazil.

Let f be a polynomial such that $f(p)$ is prime for every prime value of p. We will prove that f is a constant polynomial (the constant being, of course, a prime number) or f is the identity; that is $f(x) = x$, for every integer x. Suppose that f is not the identity. Then $f(x) = x$ possesses only a finite number of solutions.

Therefore, there exist p, q primes, $p \neq q$ such that $f(p) = q$. Applying Dirichlet's Theorem, we know that the arithmetical progression $p + kq$, $k = 0, 1, 2, \ldots$, contains infinitely many primes. Besides, for every $k \geq 0$,

$f(p+kq) \equiv f(p) \equiv 0 \pmod{q}$, and so for the infinitely many primes r_i of the form $p+kq$ we must have $f(r_i) = q$. But f being a polynomial that takes the value q infinitely many times means that $f(x) = q$ for all x.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL JOSEPHY, Universidad de Costa Rica, San José, Costa Rica; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; and the proposer. There were 3 incorrect submissions.

2116. [1996: 75] *Proposed by Yang Kechang, Yueyang University, Hunan, China.*

A triangle has sides a, b, c and area F. Prove that

$$a^3b^4c^5 \geq \frac{25\sqrt{5}(2F)^6}{27}.$$

When does equality hold?

Several solvers pointed out that this proposal is just a special case of [1984: 19], proposed by M.S. Klamkin, which asks for the maximum value of $P \equiv \sin^\alpha A.\sin^\beta B.\sin^\gamma C$, where A, B, C are the angles of a triangle and α, β, γ are given positive numbers. The featured solution by Walther Janous [1985: 908] establishes the maximum to be

$$P_{max} = \left\{\frac{\alpha(\alpha+\beta+\gamma)}{(\alpha+\beta)(\alpha+\gamma)}\right\}^{\alpha/2} \cdot \left\{\frac{\beta(\alpha+\beta+\gamma)}{(\beta+\gamma)(\beta+\alpha)}\right\}^{\beta/2} \cdot \left\{\frac{\gamma(\alpha+\beta+\gamma)}{(\gamma+\alpha)(\gamma+\beta)}\right\}^{\gamma/2}.$$

In the humble opinion of the editor, nothing in the submissions added anything substantive to [1985: 908].

Solved by PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and PANOS E. TSAOUSSOGLOU, Athens, Greece.

2117. [1996: 76] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $AB > AC$, and the bisector of $\angle A$ meets BC at D. Let P be an interior point of the side AC. Prove that $\angle BPD < \angle DPC$.

Two solutions by Shailesh Shirali, Rishi Valley School, India.

[*Editor's comment*: Shirali strengthens the result by replacing the condition $AB > AC$ by the less stringent condition $\angle ABC < 90°$. (That seems to be sufficient for the inequality in his first proof: $90^\circ > \angle ABC > \angle PBC$ implies $\sin\angle ABC > \sin\angle PBC$.) Neither he nor the other solvers explicitly stated that they had stronger results.]

Solution I. Let E be the point where the internal bisector of $\angle P$ of $\triangle PBC$ meets side BC. Then

$$\frac{BE}{EC} = \frac{PB}{PC} = \frac{\sin\angle PCB}{\sin\angle PBC} > \frac{\sin\angle ACB}{\sin\angle ABC} = \frac{AB}{AC} = \frac{BD}{DC},$$

so that E lies between D and C. It follows that $\angle BPD < \angle DPC$.

[*Editor's further comment*: The above argument clearly continues to work for some positions of P on the side AC even when $\angle ABC \geq 90^\circ$. A set of conditions on P and $\triangle ABC$ under which $\angle BPD < \angle DPC$ is implicit in Shirali's second argument.]

Solution II. The set of points X in the plane of the triangle for which $\angle BXD = \angle DXC$ is Ω, the *circle of Apollonius through A with respect to B and C*. [Equivalently, Ω is the locus of points X for which $\dfrac{XB}{XC} = \dfrac{BD}{DC}$; it is the circle through A and D whose centre lies on BC; inversion in this circle fixes A and D, and interchanges B with C; see, for example, H.S.M. Coxeter, *Introduction to Geometry*, §6.6 pages 88-89.] When $AB > AC$ [so that $BD > DC$], the set of points X for which $\angle BXD < \angle DXC$ is the set of points interior to Ω. In this case the centre of Ω lies on the extension of BC beyond C, so the segment AC lies entirely within Ω. It follows that for all points P on AC, we have $\angle BPD < \angle DPC$. [When $AB < AC$, the centre of Ω lies on the part of BC beyond B, so that now we seek points P outside Ω; when $\angle B \leq 90^\circ$, the entire line segment AC lies outside Ω, which agrees with the first proof. When $\angle B$ is obtuse, then some points of AC lie inside Ω – for those points, the desired inequality no longer holds.]

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JORDI DOU, Barcelona, Spain; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; WALDEMAR POMPE, student, University of Warsaw, Poland; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrino, Greece; MELETIS VASILIOU, Elefsis, Greece; and the proposer.

2118. [1996: 76] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

The primitive Pythagorean triangle with sides 2547 and 40004 and hypotenuse 40085 has area 50945094, which is an 8–digit number of the form $abcdabcd$. Find another primitive Pythagorean triangle whose area is of this form.

Solution by the proposer.
There are three such primitive Pythagorean triangles:

	m	n	m^2-n^2	$2mn$	m^2+n^2	Area
(i)	146	137	2547	40004	40085	50945094
(ii)	146	9	21235	2628	21397	27902790
(iii)	105	32	10001	6720	12049	33603360

8–digit numbers of the form $abcdabcd$ can be factored as $10001\cdot N = 73\cdot 137\cdot N$ for an integer N in the range $1000 \le N < 10000$. The sides of a primitive Pythagorean triangle are given by m^2-n^2, $2mn$, m^2+n^2, with relatively prime integers m and n of different parity. The area of the triangle is $\triangle = mn(m-n)(m+n)$. Note that the factors m, n, $m+n$, and $m-n$ are pairwise relatively prime, and exactly one of them is even. These numbers are in the order

(i) $m+n > m > n > m-n$, or

(ii) $m+n > m > m-n > n$.

[Clearly, none of m, n or $m-n$ can be $10001r$. Also $m+n \ne 10001$; otherwise, $mn \ge (10000)(1)$ and $\triangle > 10^8$.] Each of 137 and 73 divides exactly one of the four numbers $m, n, m\pm n$. We list the twelve different types in the table below, in which h and k are positive integers.

Type	$m+n$	m	n	$m-n$	conditions
(1)	$2(137h)+73k$	$137h+73k$	$137h$	$73k$	k odd
(2)	$137h+2(73)k$	$137h+73k$	$73k$	$137h$	h odd
(3)	$137h+73k$	$73k$	$137h$	$73k-137h$	$h+k$ odd
(4)	$-137h+2(73)k$	$73k$	$-137h+73k$	$137h$	h odd
(5)	$73k$	$-137h+73k$	$137h$	$73k-2(137)h$	k odd
(6)	$73k$	$\frac{1}{2}(137h+73k)$	$\frac{1}{2}(-137h+73k)$	$137h$	$h\cdot k$ odd
(7)	$137h+73k$	$137h$	$73k$	$173h-73k$	$h+k$ odd
(8)	$2(137h)-73k$	$137h$	$137h-73k$	$73k$	k odd
(9)	$137h$	$137h-73k$	$73k$	$137h-2(73k)$	h odd
(10)	$137h$	$\frac{1}{2}(137h+73k)$	$\frac{1}{2}(137h-73k)$	$73k$	$h\cdot k$ odd
(11)	$137h$	$73k$	$137h-73k$	$-137h+2(73k)$	h odd
(12)	$73k$	$137h$	$-137h+73k$	$2(137)h-73k$	k odd

In Types (1) and (2), $\triangle > ((137h)(73k))^2 = (hk)^2(10001)^2 > 10^8$.

In each of Types (3), (4), (5), (6), note that $137h > 100$. This cannot be the smallest of the four numbers $m, n, m\pm n$, and must therefore be the *third* in order. The area of the triangle is greater than $(137h)^3\cdot 1$. We need therefore consider only h satisfying $(137h)^3 < 10^8$; $h < \frac{10^{\frac{8}{3}}}{137} \approx 3.388$; in other words, $h = 1, 2, 3$. In each of these types, the smallest number is a linear function $-p\,137h + q\,73k$, where each of p and q is 1, 2 or $\frac{1}{2}$.

Note that $-p\,137h + q\,73k < \frac{1}{h}73$ since the product of the two smallest measurements cannot exceed 10001.
It follows that $0 < -p\,137h + q\,73k < \frac{1}{h}73$. From this,

$$\frac{p}{q}\cdot\frac{137}{73}\cdot h < k < \frac{p}{q}\cdot\frac{137}{73}\cdot h + \frac{1}{qh}.$$

In each of Types (7), (8), (9), (10), either n or $m-n$ is $73k$, so $\triangle > (73k)^3$. In other words, $k \le 6$. Also, from $0 < p137h - q73k < \frac{1}{k}137$, we have

$$\frac{q}{p}\cdot\frac{73}{137}\cdot k < h < \frac{q}{p}\cdot\frac{73}{137}\cdot k + \frac{1}{pk}.$$

Type	p	q	possible (h,k)	relevant (h,k)	triangle
(3)	1	1	$(1,2)$, $(2,4)$	$(1,2)$	(i)
(4)	1	1	$(1,2)$, $(2,4)$	$(1,2)$	(ii)
(5)	2	1	$(1,4)$, $(2,8)$	none	
(6)	$\frac{1}{2}$	$\frac{1}{2}$	$(1,2)$, $(1,3)$, $(2,4)$, $(3,6)$	$(1,3)$	$m=178$, $n=41$
(7)	1	1	$(1,1)$	none	
(8)	1	1	$(1,1)$	$(1,1)$	$m=137$, $n=64$
(9)	1	2	$(2,1)$	none	
(10)	$\frac{1}{2}$	$\frac{1}{2}$	$(1,1)$, $(2,1)$, $(2,2)$,	$(1,1)$	(iii)
			$(2,3)$, $(3,5)$	$(3,5)$	$m=388$, $n=23$

Type (11). We consider (h,k) in either of the parallelograms

$$137h - 73k = 0, \quad 137h - 73k = 100,$$
$$-137h + 2(73k) = 0, \quad -137h + 2(73k) = 464;$$
$$137h - 73k = 0, \quad 137h - 73k = 464,$$
$$-137h + 2(73k) = 0, \quad -137h + 2(73k) = 100.$$

(Note: $464 = [\sqrt[3]{10^8}]$). These are the points

$$(h,k) = (1,1), (2,3), (t,t) \text{ for } t = 2,\ldots,8,$$

and we need only consider $(h,k) = (1,1)$ (since h is odd and $\gcd(h,k) = 1$). But this triangle, with $m = 73$, $n = 64$, is too small; it has area 5760576.

Type (12). We consider (h,k) in either of the parallelograms

$$-137h + 73k = 0, \quad -137h + 73k = 100,$$
$$2(137)h - 73k = 0, \quad 2(137)h - 73k = 464;$$

$$-137h + 73k = 0, \quad -137h + 73k = 464,$$

$$2(137)h - 73k = 0, \quad 2(137)h - 73k = 100.$$

These are the points

$$(h, k) = (1, 2), (1, 3), (2, 7), (3, 11), (2, 4), (3, 6),$$

and we need only consider $(h, k) = (1, 3), (2, 7)$ and (3,11). But the corresponding triangles are all too big.

Remark Enlarging the primitive triangle with $m = 73, n = 64$ by the factors 2, 3, 4, we obtain non-primitive Pythagorean triangles of areas 23042304, 51845184, and 92169216 respectively. Enlargements of the primitive triangles (i), (ii), (iii) all have areas exceeding 10^8. The non-primitive triangle from $m = 137$, $n = 73$ has area 13441344.

Both the additional triangles were also found by TIM CROSS, King Edward's School, Birmingham, England; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; and RICHARD I. HESS, Rancho Palos Verdes, California, USA.

The following readers each found one of the triangles: CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; SAM BAETHGE, Science Academy, Austin, Texas, USA; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; P. PENNING, Delft, the Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece.
There was one incorrect submission.

2119. [1996: 76] *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

(a) Show that for any positive integer $m \geq 3$, there is a permutation of m 1's, m 2's and m 3's such that

   (i) no block of consecutive terms of the permutation (other than the entire permutation) contains equal numbers of 1's, 2's and 3's; and

(ii) there is no block of m consecutive terms of the permutation which are all equal.

(b) For $m = 3$, how many such permutations are there?

Solution by P. Penning, Delft, the Netherlands.

(b) I found two:

$$aabbcbcca \quad \text{and} \quad abbcbccaa,$$

where a, b, c is any permutation of $1, 2, 3$. (In fact these are mirror-images of one another.) Thus there are $3! \cdot 2 = 12$ such permutations of 1's, 2's and 3's.

(a) To generate a solution for $m > 3$, start with three blocks of m equal terms, and to make it satisfy condition (ii) let the last term of each block change places with the first term of the next block (and do the same with the first and last block):

$$1\,\overbrace{22\ldots2}^{m-2}\,3\;2\,\overbrace{33\ldots3}^{m-2}\,1\;3\,\overbrace{11\ldots1}^{m-2}\,2.$$

Note that for $m = 3$ this permutation violates condition (i). [It is easy to check that it works whenever $m > 3$.—*Ed.*] Also note that if the permutation is considered as a cycle then there are solutions only if $m > 3$.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; JOEL SCHLOSBERG, student, Hunter College High School, New York, NY, USA; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer. Part (b) *only was solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA.*

Note that Penning's solution shows that for $m > 3$, condition (ii) *can be strengthened just a teensy bit to:*

(ii)′ *there is no block of $m - 1$ consecutive terms of the permutation which are all equal.*

However, there is an even better result. For any $m \geq 3$ there is a permutation of m 1's, m 2's and m 3's which satisfies condition (i) *and also*

(ii)* *there is no block of three consecutive terms of the permutation which are all equal.*

(*Thanks to expert colleague James Currie for suggesting examples of such permutations, which readers might enjoy finding for themselves!*)

2120. [1996: 76] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let $A_1A_3A_5$ and $A_2A_4A_6$ be non-degenerate triangles in the plane. For

$i = 1, \ldots, 6$ let ℓ_i be the perpendicular from A_i to line $A_{i-1}A_{i+1}$ (where of course $A_0 = A_6$ and $A_7 = A_1$). If ℓ_1, ℓ_3, ℓ_5 concur, prove that ℓ_2, ℓ_4, ℓ_6 also concur.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let the vector position of the vertex A_i relative to the point of concurrency of ℓ_1, ℓ_3, ℓ_5 be $\vec{a_i}$. Then $\vec{a_1}\cdot(\vec{a_2}-\vec{a_6}) = \vec{a_3}\cdot(\vec{a_4}-\vec{a_2}) = \vec{a_5}\cdot(\vec{a_6}-\vec{a_4}) = 0$. In particular, the sum is also zero, so

$$\vec{a_2}\cdot(\vec{a_1}-\vec{a_3}) + \vec{a_4}\cdot(\vec{a_3}-\vec{a_5}) + \vec{a_6}\cdot(\vec{a_5}-\vec{a_1}) = 0. \tag{1}$$

Now suppose that the perpendicular from A_2 to A_1A_3 meets the perpendicular from A_4 to A_3A_5 at B, with position vector $\vec{b}$. (These perpendiculars cannot be parallel.) Then

$$(\vec{b}-\vec{a_2})\cdot(\vec{a_1}-\vec{a_3}) = 0 \quad \text{and} \quad (\vec{b}-\vec{a_4})\cdot(\vec{a_3}-\vec{a_5}) = 0.$$

Adding gives $\vec{b}\cdot(\vec{a_1}-\vec{a_5}) = \vec{a_2}\cdot(\vec{a_1}-\vec{a_3}) + \vec{a_4}(\vec{a_3}-\vec{a_5})$, which, by (1), is equal to $-\vec{a_6}\cdot(\vec{a_5}-\vec{a_1})$.
Hence $(\vec{b}-\vec{a_6})\cdot(\vec{a_1}-\vec{a_5}) = 0$, which means that $BA_6 \perp A_1A_5$.

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; MELETIS VASILIOU, Elefsis, Greece; and the proposer.

Problem 2120 can be found in many reference texts. Dan Pedoe provides two proofs in his Geometry, A comprehensive Course *(Dover, 1998). Clerk Maxwell's remarkable treatment of the problem, reducing it to an observation about the radical axes of appropriate circle, appears in §28.4, page 115. Earlier, in §6.1, pages 35-37, Pedoe uses barycentric coordinates for a proof of a closely related problem involving* reciprocal triangles. *He comments that whole books have been written on the subject of reciprocal figures.*

Seimiya uses Steiner's theorem in his solutions (Jacob Steiner, *Gesammelte Werke* I, p. 189): A necessary and sufficient condition in order for ℓ_1, ℓ_3, ℓ_5 to be concurrent is

$$A_1A_2^2 + A_3A_4^2 + A_5A_6^2 = A_2A_3^2 + A_4A_5^2 + A_6A_1^2.$$

Since (by permuting the subscripts) this is also a necessary and sufficient condition for ℓ_2, ℓ_4, ℓ_6 to concur, our problem follows immediately. Arconcher refers to this condition as Carnot's Theorem, *but he provides no* *reference. Ardila discovered the theorem for himself and showed that it follows fairly quickly from Pythagoras' Theorem, so maybe we should refer to it as the Pythagoras-Ardila-Steiner-Carnot Theorem. Seimiya added that the problem may also be found in* Aref and Wernick, Problems and Solutions in Euclidean Geometry (*Dover*), *p. 55 problems 2.32, He includes two of his interesting generalizations that have appeared in Japanese, one in 1928 and one in 1967.*

Finally, Heuver found the problem as Ex. 7 of §54 of George Salmon's A Treatise on Conic Sections, *6th ed. His solution exploits Salmon's imaginative use of pencils of lines in Cartesian coordinates.*

2121. [1996: 76] *Proposed by Krzysztof Chelmiński, Technische Hochschule Darmstadt, Germany; and Waldemar Pompe, student, University of Warsaw, Poland.*

Let $k \geq 2$ be an integer. The sequence (x_n) is defined by $x_0 = x_1 = 1$ and

$$x_{n+1} = \frac{x_n^k + 1}{x_{n-1}} \qquad \text{for} \quad n \geq 1.$$

(a) Prove that for each positive integer $k \geq 2$ the sequence (x_n) is a sequence of integers.

(b) If $k = 2$, show that $x_{n+1} = 3x_n - x_{n-1}$ for $n \geq 1$.

(c)* Note that for $k = 2$, part (a) follows immediately from (b). Is there an analogous recurrence relation to the one in (b), not necessarily linear, which would give an immediate proof of (a) for $k \geq 3$?

I *Solution ((a) and (b) only) by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) We immediately get $x_2 = 2$ and $x_3 = 2^k + 1$. Now we use mathematical induction for the proof. Assume that $x_0, x_1, \ldots, x_n$ are all natural numbers. We must show that $x_{n+1} \in \mathbb{N}$. First we note that since $x_{n-2} \cdot x_n = x_{n-1}^k + 1$ it follows that x_{n-2} and x_{n-1} are relatively prime. Using $x_n = (x_{n-1}^k + 1)/x_{n-2}$ we infer that

$$x_{n+1} = \frac{x_n^k + 1}{x_{n-1}} = \frac{(x_{n-1}^k + 1)^k + x_{n-2}^k}{x_{n-2}^k x_{n-1}}.$$

Thus obviously x_{n-2}^k divides $N = (x_{n-1}^k + 1)^k + x_{n-2}^k$ since x_n is a natural number. Furthermore, modulo x_{n-1} we have:

$$N \equiv 1 + x_{n-2}^k = x_{n-3} \cdot x_{n-1} \equiv 0.$$

That is, x_{n-1} also divides N and we are done.

(b) Now,

$$x_{n+1} = \frac{x_n^2 + 1}{x_{n-1}} \iff x_{n-1} \cdot x_{n+1} - x_n^2 = 1.$$

That is, the sequence $\{y_n\} = \{x_{n-1} \cdot x_{n+1} - x_n^2\}$ is constant. Setting $y_{n+1} = y_n$ we have

$$\begin{aligned} x_n \cdot x_{n+2} - x_{n+1}^2 &= x_{n-1} \cdot x_{n+1} - x_n^2 \\ \iff \quad x_n(x_n + x_{n+2}) &= x_{n+1}(x_{n-1} + x_{n+1}) \\ \iff \quad \frac{x_n + x_{n+2}}{x_{n+1}} &= \frac{x_{n-1} + x_{n+1}}{x_n}. \end{aligned}$$

That is, the sequence $\{z_n\} = \{(x_{n-1}+x_{n+1})/x_n\}$ is constant. From $z_1 = 3$ we get $(x_{n-1} + x_{n+1})/x_n = 3$; that is, $x_{n+1} = 3x_n - x_{n-1}$ for all $n \geq 1$, as claimed.

II *Solution by Christopher J. Bradley, Clifton College, Bristol, UK.*

We first establish the following lemma:

Lemma. Suppose a and b have highest common factor 1 and $(a^k + b^k + 1)/ab$ is an integer (where k is an integer, $k \geq 2$), then
(i) $c = (b^k + 1)/a$ is an integer;
(ii) $(b^k + c^k + 1)/bc$ is an integer;
(iii) b and c have highest common factor 1.
Proof. Let $a^k + b^k + 1 = \lambda(a, b)ab$ where $\lambda(a, b)$ is an integer. Then since a divides $\lambda(a, b)ab - a^k$ it also divides $b^k + 1$; that is, $(b^k + 1)/a = c$ is an integer, which proves (i). Also

$$\frac{b^k + c^k + 1}{bc} = \frac{ac + c^k}{bc} = \frac{a + c^{k-1}}{b} = \frac{a^k + (b^k + 1)^{k-1}}{a^{k-1}b}.$$

Now $b^k + 1 = ac$ has a as a factor, so $a^k + (b^k + 1)^{k-1}$ is divisible by a^{k-1} and hence the numerator is divisible by a^{k-1}. Also $a^k + 1$ is divisible by b (from part (i)), so multiplying out the numerator by the Binomial Theorem we see that the numerator is divisible by b. But a and b have highest common factor 1, so the numerator is divisible by $a^{k-1}b$. Hence

$$\lambda(b, c) = \frac{b^k + c^k + 1}{bc}$$

is an integer, which proves (ii). Since $b^k + c^k + 1 = \lambda(b, c)bc$, if b and c have a common factor h, then h divides $b^k + c^k - \lambda(b, c)bc$; that is, h divides 1. Hence b and c have highest common factor 1, which proves (iii), and the lemma is proved.

We claim that the recurrence relation sought is

$$x_{n+1} = \lambda(x_{n-1}, x_n)x_n - x_{n-1}^{k-1} \qquad (*)$$

with $\lambda(x_{n-1}, x_n) = (x_n^k + x_{n-1}^k + 1)/(x_n x_{n-1})$, which is but one step away from $x_{n+1} = (x_n^k + 1)/x_{n-1}$. That $\lambda(x_{n-1}, x_n)$ is always an integer follows by induction, using the lemma, with the start of induction satisfied since $x_0 = x_1 = 1$ have highest common factor 1 and $\lambda(x_0, x_1) = 3$ is an integer. Relation $(*)$ then establishes that $\{x_n\}$ is a sequence of integers. This gives us parts (c) and (a). The term $(x_n^2 + x_{n-1}^2 + 1)/(x_n x_{n-1})$ is known from work on alternate terms of the Fibonacci sequence to be equal to 3 for all $n \geq 2$ and is also 3 for $n = 1$, which proves (b).

Parts (*a*) *and* (*b*) *together were also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; and the proposers. Part* (*b*) *alone was solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and TIM CROSS, King Edward's School, Birmingham, England. There was one incorrect solution.*

JANOUS comments that this was posed as a problem of the final round of the third Austrian Mathematical Olympiad held in 1972, and refers the interested reader to the book "Österreichische Matematik-Olympiaden" 1970-1989, *G. Baron & E. Windischbacher, Innsbruck 1990, problem 42.*

2122. [1996: 77] *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Little Sam is a unique child and his math marks show it. On four tests this year his scores out of 100 were all two-digit numbers made up of eight different non-zero digits. What's more, the average of these scores is the same as the average if each score is reversed (so 94 becomes 49, for example), and this average is an integer none of whose digits is equal to any of the digits in the scores. What is Sam's average?

I. *Solution by Mansur Boase, student, St. Paul's School, London, England.*

Let the four marks be $10a + b$, $10c + d$, $10e + f$ and $10g + h$. Then

$$10a + b + 10c + d + 10e + f + 10g + h = 10b + a + 10d + c + 10f + e + 10h + g,$$

so

$$9(a + c + e + g) = 9(b + d + f + h)$$

and

$$a + c + e + g = b + d + f + h. \tag{1}$$

The average of the four marks is therefore

$$\frac{10a + b + 10b + a + 10c + d + 10d + c + 10e + f + 10f + e + 10g + h + 10h + g}{4 \cdot 2}$$

$$= \frac{11(a + b + c + d + e + f + g + h)}{8}. \tag{2}$$

The average must consist only of k's and 0's, the digits *not* equal to a, b, c, d, e, f, g or h. Thus (2) is

$$\frac{11(1+2+\cdots+9-k)}{8} = \frac{11(45-k)}{8},$$

and so $8 \mid (45-k)$. Since $k \leq 9$, the only solution for k is $k = 5$, and therefore Sam's average is $11(40)/8 = 55$. An example of four such marks is $98, 76, 34, 12$.

II. *Solution by Tara McCabe, student, Mount Allison University, Sackville, New Brunswick.*

[*Editor's note*: McCabe first obtained equation (1) as above, using the same notation.] Since $a, \ldots, h$ are all different and non-zero, their total must lie between $1+\cdots+8 = 36$ and $2+\cdots+9 = 44$ inclusively. From (1),

$$\frac{36}{2} = 18 \leq a+c+e+g \leq 22 = \frac{44}{2}. \tag{3}$$

Letting the average be the two digit number xy,

$$\frac{10a+b+10c+d+10e+f+10g+h}{4} = 10x+y,$$

which by (1) means

$$11(a+c+e+g) = 4(10x+y).$$

Consequently, $a+c+e+g$ must be divisible by 4, and from (3), $a+c+e+g = 20$. Therefore $11(20) = 4(10x+y)$, and so $55 = 10x+y$. Little Sam has an average of 55.

A natural addition to this problem is to try to find Little Sam's four test scores ab, cd, ef and gh. We know that $a+c+e+g = 20$, and the problem becomes: how many sets of test scores are possible?

First, choose four digits from $\{1,2,3,4,6,7,8,9\}$ to be a, c, e and g (0 and 5 are not allowed). Notice that for a sum of 20 to be possible, two digits must be chosen from $\{1,2,3,4\}$ and two from $\{6,7,8,9\}$. There are $\binom{4}{2} = 6$ ways to choose two digits from $\{1,2,3,4\}$. The sum S_1 of these two digits is such that $3 \leq S_1 \leq 7$, and the only sum that can occur twice is 5. Similarly, there are 6 ways to choose two digits from $\{6,7,8,9\}$, the sum S_2 of these two digits is such that $13 \leq S_2 \leq 17$, and the only sum that can occur twice is 15. There are $2 \times 2 = 4$ combinations when $S_1 = 5$ (and $S_2 = 15$) and one combination for each other sum S_1. Therefore, there are eight ways to choose four digits from $\{1,2,3,4,6,7,8,9\}$ such that the sum is 20.

Once a, c, e and g are chosen, b, d, f and h are simply the four remaining digits. There are $4! = 24$ different ways to assign these four digits to be b, d, f and h. Therefore there are $8 \times 24 = 192$ different sets of four test scores possible for Little Sam. He's not such a unique child after all!

Also solved by SAM BAETHGE, Nordheim, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TIM CROSS, King Edward's School, Birmingham, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; NOEL EVANS and CHARLES DIMINNIE, Angelo State University, San Angelo, Texas, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, New York, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOHN GRANT MCLOUGHLIN, Okanagan University College, Kelowna, B.C.; P. PENNING, Delft, the Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; JOEL SCHLOSBERG, student, Hunter College High School, New York, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. Three other solvers did not prove that 55 *is the only possible answer. There was also one incorrect answer sent in.*

At the end of his solution, the proposer asks for the number of sets of test scores satisfying the problem, but only McCabe was able to read his mind and answer this too!

2123. [1996: 77] *Proposed by Sydney Bulman–Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is known (e.g., exercise 23, page 78 of Kenneth H. Rosen's *Elementary Number Theory and its Applications*, Third Edition) that every natural number greater than 6 is the sum of two relatively prime integers, each greater than 1. Find all natural numbers which can be expressed as the sum of *three* pairwise relatively prime integers, each greater than 1.

Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.

Since every number greater than 6 is the sum of two relatively prime integers, then each even number greater than 6 can be expressed as the sum of of two relatively prime **odd** numbers. If we add 2 to each of these expressions we see that every even number greater than 8 can be expressed as the sum of three pairwise relatively prime integers (two relatively prime odd integers and 2).

All odd numbers can be expressed in the form $18N + k$, where N is a non-negative integer and k is an odd number less than 18. Note that:

$$\begin{aligned}
18N+1 &= (6N-3)+(6N-1)+(6N+5) \quad \text{for } N \geq 1\\
18N+3 &= (6N-1)+(6N+1)+(6N+3) \quad \text{for } N \geq 1\\
18N+5 &= (6N-1)+(6N+1)+(6N+5) \quad \text{for } N \geq 1
\end{aligned}$$

$$\begin{aligned}
18N+7 &= (6N-1)+(6N+3)+(6N+5) &&\text{for } N\geq 1\\
18N+9 &= (6N+1)+(6N+3)+(6N+5) &&\text{for } N\geq 1\\
18N+11 &= (6N+1)+(6N+3)+(6N+7) &&\text{for } N\geq 1\\
18N+13 &= (6N+1)+(6N+5)+(6N+7) &&\text{for } N\geq 1\\
18N+15 &= (6N+3)+(6N+5)+(6N+7) &&\text{for } N\geq 0\\
18N+17 &= (6N+1)+(6N+7)+(6N+9) &&\text{for } N\geq 1
\end{aligned}$$

Clearly each of the pairs of terms in each expression is relatively prime, since if there is a number which divides each term in a pair it must divide the difference. The restriction on N ensures that each term is greater than 1. Putting all this together shows that the numbers which can be so written are 10, 12, 14, 15, 16, and all the natural numbers greater than 17.

[Ed: it is easily verified that all other integers do not have the desired property.]

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL JOSEPHY, Universidad de Costa Rica, San José, Costa Rica; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta, NY, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposers. There was 1 incorrect submission.

Most solvers used **mod** 18 *or* **mod** 12 *arithmetic to handle the odd values of* n. *MANES pointed out that both this problem and the problem in Rosen's book can be found in Sierpiński's book,* "250 Problems in Elementary Number Theory", *American Elsevier, New York, 1970, (problems 47 and 48) where they are both solved.*