Letter from the Editors

As announced in the November issue, 1997 will see an increase in the quantity of material devoted to high school mathematics, without a decrease in any of the present composition of ***CRUX***. We are pleased to announce that this will be effected through an amalgamation of ***CRUX MATHEMATICORUM*** with ***Mathematical Mayhem***. For those of you not familiar with it, ***Mayhem*** was started by a group of Canadian IMO Alumni, and aimed to be a "journal of high school and college level mathematics, written by and for students". It has run continuously for eight years, and has a circulation of about 130. Its previous editors have included the distinguished problemists Ravi Vakil (co-founder with Patrick Surry) and J.P. Grossman, who were the Deputy Leader and Leader of the Canadian delegation of the 1996 IMO. The present editors are Naoki Sato and Cyrus Hsia, who will be joining the ***CRUX with MAYHEM*** Editorial Board in January. ***Mayhem*** has published items such as expository articles, hints of problem solving, IMO problems and solutions, and three levels of problems — high school, advanced and challenge board — (set by the editors or contributors). By keeping communication alive amongst the Canadian IMO alumni, team members and "wanabees", ***Mayhem*** has played an important role in keeping the IMO spirit alive and well in Canada.

This amalgamation will give ***CRUX*** readers additional high school level material (if you have forgotten ***CRUX***'s mandate, please re-read the back inside cover), as well as giving ***Mayhem*** a wider and international exposure. ***CRUX with MAYHEM*** will have 64 pages per issue and will have problems to challenge all readers from interested high school students to senior undergraduate students. We hope that high school readers who will be new to ***CRUX with MAYHEM*** will find the range of problems useful in the classroom and stimulating.

If the issue that you habitually read is a library copy, please ensure that your librarian is aware of this change. Since ***CRUX*** is the older journal, and taken by more libraries, the ***Mayhem*** editors have agreed that ***CRUX with MAYHEM*** will continue the volume numbering of ***CRUX*** (next year will see volume 23) as well as maintain the general external appearance. However, the table of contents will be moved to the outside back cover since it will be enlarged, and also to make space for the words ***Mathematical Mayhem*** on the front cover.

With all best wishes,

Bruce Shawyer
Editor-in-Chief

Graham Wright
Managing Editor

Lettre des rédacteurs

Comme nous l'avons annoncé dans le numéro de novembre, le ***CRUX*** sera davantage axé sur les mathématiques de niveau secondaire en 1997, sans pour autant que l'on touche à son contenu actuel. Il nous fait plaisir de vous annoncer que nous réussirons ce tour de force en fusionnant le ***CRUX MATHEMATICORUM*** et le ***Mathematical Mayhem***. Pour ceux d'entre vous qui ne connaîtraient pas le ***Mayhem***, sachez qu'il a été fondé par d'anciens membres de l'équipe canadienne à l'OIM s'étant donné pour mission de produire un périodique sur les mathématiques de niveau secondaire et collégial, conçu par et pour des étudiants. Ce magazine, qui paraît régulièrement depuis huit ans, est tiré à environ 130 exemplaires. Parmi ses anciens rédacteurs, mentionnons les excellents problémistes Ravi Vakil (cofondateur avec Patrick Surry) et J.P. Grossman, respectivement chef d'équipe adjoint et chef d'équipe de la délégation canadienne à l'OIM 1996. Les rédacteurs actuels, Naoki Sato et Cyrus Hsia, se joindront à l'équipe de rédaction du ***CRUX with Mayhem*** en janvier. Le ***Mayhem*** a entre autres publié des articles de fond, des trucs pour faciliter la résolution de problèmes et certains problèmes posés aux dernières OIM accompagnés. Les problèmes présentés sont classés selon trois niveaux de difficulté, déterminés par les rédacteurs ou les collaborateurs : secondaire, avancé et ≪challenge board≫. En permettant les échanges entre les anciens participants canadiens à l'OIM, les membres de l'équipe canadienne actuelle et les aspirants à ce titre, le ***Mayhem*** a grandement contribué à préserver l'esprit de l'OIM au Canada.

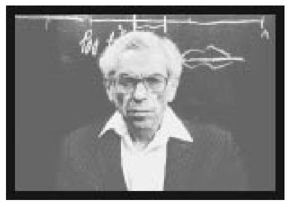
La fusion de ces deux magazines permettra aux lecteurs du ***CRUX*** d'avoir accès à une plus grande quantité d'articles sur les mathématiques de niveau secondaire (pour un rappel du mandat du ***CRUX***, relire la troisième de couverture) et procurera au ***Mayhem*** une visibilité accrue et internationale. Chaque numéro du ***CRUX with Mayhem*** comptera 64 pages et présentera des problèmes qui sauront susciter l'intérêt de tous les lecteurs, des élèves du secondaire aux étudiants en fin de bac. Nous espérons que les lecteurs du secondaire qui découvriront le ***CRUX with Mayhem*** y trouveront des problèmes stimulants et utiles pour leurs études.

Si vous lisez habituellement un exemplaire provenant d'une bibliothèque, vérifiez si votre bibliothécaire est au courant de cette fusion. Puisque le ***CRUX*** est le plus ancien des deux magazines et le plus commun dans les bibliothéques, les rédacteurs du ***Mayhem*** ont accepté que le ***CRUX with Mayhem*** suive la numérotation par volumes du ***CRUX*** (l'an prochain : volume no 23) et conserve en gros sa présentation extérieure. On déplacera toutefois la table des matières en quatrième de couverture, puisqu'elle sera agrandie et qu'il faudra faire de la place aux mots ≪***Mathematical Mayhem***≫ sur la page-titre.

Meilleurs voeux à tous!

Bruce Shawyer — Graham Wright
Rédacteur en chef — Directeur de rédaction

In memoriam — Pál Erdős



Pál Erdős — **1913-03-26 to 1996-09-20**

In appreciation of the "Prince of Problem Posers", Pál Erdős, who, sadly for those of us who had the privilege of knowing him, died while he was in Warsaw, attending a mathematics meeting.

It always seemed that Erdős was eternal; it's hard to realize that we won't hear his delightful language again. He influenced hundreds, probably thousands, of mathematicians, and each in an individual way. Anything one can say must be personal.

A major part of Erdős's genius was asking innumerable questions, and, most importantly, asking them of the right person. He seemed to know, better than you yourself, what problems you could solve. He gave the confidence that many of us needed to embark on research.

Erdős was the problem proposer par excellence. Almost anyone can ask questions that are impossibly difficult or are trivially easy. To achieve the balance between these extremes is given to few of us. Erdős's questions were always just right. Many have remained as outstanding, but important problems, but most have been attacked and partially, if not completely solved.

But he didn't only pose problems; he wrote more than 1500 papers. Those who didn't know him well thought that he just threw out ideas and got others to write for him. Certainly he had a phenomenal number of co-authors, but a good fraction of his papers were solo efforts, and, far from depending on his co-authors, he often wrote the paper himself and added on the other names, even on occasions when their contributions were comparatively minor. He opened up new areas of research in number theory, combinatorics and geometry. Notable are probabilistic number theory with Rényi & Turán and the partition calculus with Rado & Hajnal.

Perhaps 'bosses', 'slaves', 'captured', 'epsilons', 'noise', 'poison', 'the SF', 'The Book', 'the Sam & Joe show', 'vot vos this ven it vos alive?', 'is your brain open?' will remain in the vocabulary of a few of his many friends, but the words won't sound the same now that it's not Erdős saying them.

R.K. Guy

For the information of those readers who may not know who Pál Erdős was, here is some further information.

Erdős was highly regarded in the mathematical community as one of the most brilliant and eccentric mathematicians of the 20th century. His sole interest was in mathematics, resulting in him being only known to the mathematical community. He lived a very simple life, eschewing personal comfort and possessions. He considered money to be an encumbrance, and gave away, as prizes for solving mathematical problems, whatever he earned that was over and above his basic simple needs.

Erdős published over 1,500 papers, most of them jointly with other mathematicians. He gave generously of his ideas to everyone with whom he came into contact. He would ascertain your mathematical interests, and then proceed with "I have this problem, . . . ". *Over 4,000 mathematicians have published jointly with Erdős.*

Pál was born in Budapest into a Hungarian-Jewish family, the son of two mathematics teachers. While his father was a prisoner of war in Siberia for six years, his mother taught him at home. He took his doctorate in 1934 at the University of Budapest, and then went for further study in Manchester, England. The conditions in Europe made it unsafe for him to return to Hungary, and he moved to the United States. In the 1950's, the United States denied him re-entry, as he was suspected of being a Soviet sympathiser. He then spent much time in Israel. He was allowed re-entry to the United States in the 1960's, and his mother, then in her eighties, began to travel with him.

Erdős never saw the need to restrict himself to one institution, so he criss-crossed the world, inspiring mathematicians wherever he went. After his mother died in 1971, he immersed himself in his work with great enthusiasm, often spending 18 hours daily on mathematical problems. He was still following this active lifestyle when the great reaper called on him in Warsaw.

Algebraic Integers and Tensor Products of Matrices.

Shaun M. Fallat
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A complex number is called an *algebraic integer* if it is a zero of a monic (i.e., the coefficient of the term with the highest exponent is one) polynomial with integer coefficients. Let C denote the complex numbers. A set $S \subseteq C$ is called a *subring* if $1 \in S$, and for every $a, b \in S$, $a + b \in S$ and $a \cdot b \in S$. It is well-known that the set of algebraic integers forms a subring of the complex numbers (see [3]). The aim of the present note is to show that this result follows easily from basic facts about tensor products of matrices.

First we will need some notation. The set of $n \times n$ matrices over a subring R is denoted by $M_n(R)$. The notation $A = [a_{ij}]$ means the $(i, j)^{th}$ entry of A is a_{ij}. The set of polynomials with integer coefficients is denoted by $Z[x]$, where x is an indeterminate. Let $\det A$ denote the *determinant* of an $n \times n$ matrix A. If $A \in M_n(R)$, then $p_A(x) = \det(xI_n - A)$ is the *characteristic polynomial* of A, where I_n denotes the $n \times n$ identity matrix. Since the operations involved in computing the determinant of A are multiplication and addition, it is readily verified that if $A \in M_n(Z)$, then $p_A(x)$ is a monic polynomial in $Z[x]$. Recall that if λ is a zero of $p_A(x)$, then λ is called an *eigenvalue* of A.

We begin with an example for completeness.

Example 1. It is easy to check that $\sqrt{2}$, $\sqrt{3}$ are algebraic integers. For example, $\sqrt{2}$ is a zero of $x^2 - 2$. Therefore there must exist a polynomial in $Z[x]$, for which $\sqrt{2} \cdot \sqrt{3}$ is a root. Since $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$, it is easy to verify that $x^2 - 6$ is an example of a polynomial in $Z[x]$ with $\sqrt{6}$ as a root. Similarly, there must exist a polynomial in $Z[x]$, for which $\sqrt{2} + \sqrt{3}$ is a root. Consider

$$\begin{aligned} f(z) &= (x - (\sqrt{2} + \sqrt{3}))(x - (\sqrt{2} - \sqrt{3})) \\ &\quad \times (x - (-\sqrt{2} + \sqrt{3}))(x - (-\sqrt{2} - \sqrt{3})) \\ &= x^4 - 10x^2 + 1. \end{aligned}$$

Hence $x^4 - 10x^2 + 1$ is an example of a polynomial in $Z[x]$ with $\sqrt{2} + \sqrt{3}$ as a root.

The following result will establish the relationship between algebraic integers and matrices.

Lemma 2. *Let λ be a complex number. Then λ is an algebraic integer if and only if λ is an eigenvalue of a square matrix with integer coefficients.*

Proof. If λ is an eigenvalue of $A \in M_n(Z)$, then $p_A(x)$ is a monic polynomial in $Z[x]$, and $p_A(\lambda) = 0$. Thus λ is an algebraic integer. To prove the converse, assume that $f(\lambda) = 0$, where $f(x)$ is a monic polynomial in $Z[x]$, say $f(x) = x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$. Let A be the companion matrix of $f(x)$, i.e.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -c_n & -c_{n-1} & \cdots & -c_2 & -c_1 \end{bmatrix}.$$

Expanding $p_A(x)$ by the first column and using a simple induction argument, it follows that $p_A(x) = f(x)$, and this fact can be found in [1; pp 230-231]. Therefore λ is an eigenvalue of A and $A \in M_n(Z)$. ∎

Definition 3. If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is a $p \times q$ matrix then the *Kronecker* or *tensor product* of A and B, denoted $A \otimes B$, is the $mp \times nq$ matrix defined as follows:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Note that if A and B have integer entries, then so does $A \otimes B$. From the above definition it is easy to verify that if $A, B, C, D \in M_n(R)$, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

Vectors in C^n may be identified with $n \times 1$ matrices over **C**, so the above definition may be used to define tensor products $x \otimes y$ with $x \in C^n$ and $y \in C^m$. The following proposition establishes a part of the well-known results that identify the eigenvalues of tensor products see, e.g. [2; pp 242-245]. We provide an easy proof for completeness.

Proposition 4. *Let $A \in M_n(Z)$, $B \in M_m(Z)$, let λ be an eigenvalue of A and let μ be an eigenvalue of B. Then*

(a) $\lambda\mu$ is an eigenvalue of $A \otimes B$,

(b) $\lambda + \mu$ is an eigenvalue of $(A \otimes I_m) + (I_n \otimes B)$.

Proof. There exist non-zero vectors $x \in C^n$ and $y \in C^m$ such that $Ax = \lambda x$ and $By = \mu y$. It follows immediately that

$$(A \otimes B)(x \otimes y) = \lambda\mu(x \otimes y),$$

and

$$((A \otimes I_m) + (I_n \otimes B))(x \otimes y) = (\lambda + \mu)(x \otimes y).$$

This proves the propostion. ■

Notice that if $A \in M_n(Z)$, $B \in M_m(Z)$, then $A \otimes B \in M_{mn}(Z)$. Hence, if two algebraic integers α, β are given, the tensor product is a useful tool for determining the existence of an integral matrix, for which $\alpha \cdot \beta$ or $\alpha + \beta$ is an eigenvalue. We summarize this in the following theorem.

Theorem 5. *The set of algebraic integers forms a subring of the ring of complex numbers.*

The proof follows directly from Lemma 2 and Proposition 4.

References

[1] K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall, New Jersey, 1971.

[2] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.

[3] H.L. Montgomery, I. Niven and H.S. Zuckerman, *An Introduction to the Theory of Numbers*, John Wiley and Sons, Inc. New York, 1991.

PHOTO PROBLEM

Can you identify this regular ***CRUX*** contributor?

THE SKOLIAD CORNER

No. 18

R.E. Woodrow

Last issue we gave the problems of the Canadian Mathematical Society Prize Exam for Nova Scotia. Here are solutions.

CANADIAN MATHEMATICAL SOCIETY PRIZE EXAM

Friday, April 26, 1996 — Time: 2.5 hours

1. (a) Solve $\sqrt{x+20} - \sqrt{x+1} = 1$.

(b) Try to solve $\sqrt[3]{x+20} - \sqrt[3]{x+1} = 1$.

Solution. (a) Set $A = \sqrt{x+20}$ and $B = \sqrt{x+1}$. We get $A^2 - B^2 = A + B$ as $A - B = 1$. So $x + 20 - (x+1) = 19 = A + B$. Now $2A = 20$, $A = 10$ and $x = 80$.

(b) Let $A = \sqrt[3]{x+20}$ and $B = \sqrt[3]{x+1}$ so $A - B = 1$. Multiplying by A^2+AB+B^2 gives $A^3-B^3 = A^2+AB+B^2$ or $A^2+AB+B^2 = 19$. As $A^2 - 2AB + B^2 = 1^2 = 1$ we obtain $3AB = 19 - 1 = 18$ or $AB = 6$. So $(x+20)(x+1) = A^3B^3 = 6^3 = 216$. This gives the quadratic equation $x^2 + 21x - 196 = 0$, or $(x+28)(x-7) = 0$ and $x = -28$ or $x = 7$.

2. Suppose a function is defined so that $f(xy) = f(x) + f(y)$.

(a) Show that if f is defined at 1, $f(1) = 0$.

(b) Similarly if f is defined at 0, its value at any x will be 0, so it is a "trivial" function.

(c) Show that if f is defined only for $\{1, 2, 3, \dots\}$ there are many non-trivial ways to define such an f (give an example).

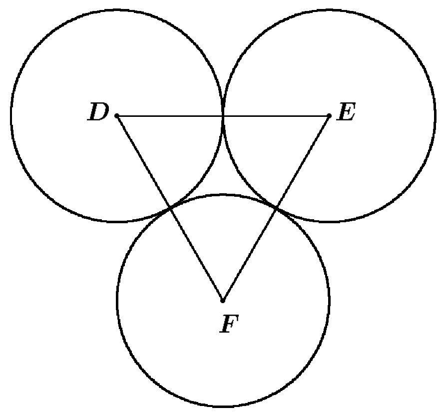
Solution. (a) With $x = 1 = y$ we obtain $f(1 \cdot 1) = f(1) + f(1)$, so $f(1) = 2f(1)$ giving $f(1) = 0$.

(b) Suppose that f is defined at 0. Then with $y = 0$ and x arbitrary we have $f(0) = f(x \cdot 0) = f(x) + f(0)$ and $f(x) = 0$.

(c) To see there are many such f with domain the natural numbers consider the natural logarithm, logarithms to base 10, and the following family of examples: Let S be a set of prime numbers. A given natural number can be written uniquely in the form $\prod P_i^{m_i}$ where $P_1, P_2, \dots$, lists the prime numbers and m_i is a non-negative integer. (Of course only finitely many of the m_i are non-zero. Define $f_s(n) = \sum m_i$.

3. Three circles of equal radii r all touch each other to enclose a three cornered concave area A. How big is the area of A?

Solution.



Let the centres of the circles be labelled D, E, F. Then DEF is an equilateral triangle with side length $2r$ and area $\sqrt{3}\,r^2$. The area A is the area of the triangle less the three sectors, each subtending an angle of $60°$ in a circle of radius r, so the area of A is

$$\sqrt{3}\,r^2 - 3\cdot\frac{1}{6}\pi r^2 = \left(\sqrt{3}-\frac{\pi}{2}\right)r^2.$$

4. Show that for all real numbers x,

(a) $x^4 \geq 4x - 3$.

(b) x^4 is not greater than $3x - 2$, even though it appears to be true.

x	-1	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
x^4	1	0	$.004$	0.12	$.063$	1	16	81
$4x-3$	-7	-3	-2	-1.67	-1	1	5	9
$3x-2$	-5	-2	-1.25	-1	$-.5$	1	4	7

Solution. (a) $x^4 \geq 4x - 3$ is the same as $x^4 - 4x + 3 \geq 0$.

Set $f(x) = x^4 - 4x + 3$. Now

$$\begin{aligned}
f(x) &= f((x-1)+1)\\
&= [(x-1)+1]^4 - 4[(x-1)+1] + 3\\
&= (x-1)^4 + 4(x-1)^3 + 6(x-1)^2\\
&\quad +4(x-1) + 1 - 4(x-1) - 4 + 3\\
&= (x-1)^4 + 4(x-1)^3 + 6(x-1)^2\\
&= (x-1)^2[(x-1)^2 + 4(x-1) + 6].
\end{aligned}$$

Now $(x-1)^2 \geq 0$ and the discriminant of X^2+4X+6 is $4^2-4\cdot 1\cdot 6 < 0$ so both factors of $f(x)$ are non-negative. The minimum value is 0 at $x = 1$, as required.

(b) Consider $x = 0.9$. Now $x^4 = 0.6561$ and $3x - 2 = 0.7$, so that $x^4 \not\geq 3x - 2$.

5. Two integers are called *equivalent*, written $x \sim y$ if they are divisible by the same prime numbers (*primes* are $2, 3, 5, 7, \ldots$) so $2 \sim 2 \sim 4$, $3 \sim 27$ but $2 \not\sim 3$.

(a) Show that $10 \sim 80$ but $10 \not\sim 90$.

(b) Prove that if $x \sim y$, then $x^2 \sim y^2$.

Solution. (a) $10 = 2 \times 5$, $80 = 2^4 \times 5$ so 10 and 80 are divisible by the same primes, namely 2 and 5.

(b) Suppose $x \sim y$. Consider any fixed prime p. If p divides x^2 then p must also divide p. But then if p divides x^2 it divides x and y, and thus y^2. Similarly if p divides y^2 it also divides x^2. Thus $x^2 \sim y^2$, as required.

6. We can describe certain fractions in terms of others all with bigger denominators (always in lowest terms). For instance $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$ and $\frac{2}{3} = \frac{1}{4} + \frac{1}{4} + \frac{1}{6}$ but $\frac{2}{3} = \frac{2}{6} + \frac{2}{6}$ doesn't work since $\frac{2}{6} = \frac{1}{3}$ and $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ doesn't since $2 < 3$.

(a) Can you write $\frac{1}{2}$ as a sum $\frac{1}{a} + \frac{1}{b}$ for integers $2 < a < b$?

(b) Try to write $\frac{1}{1996}$ as $\frac{1}{a} + \frac{1}{b}$ for integers $1996 < a < b$.

Solution. (a) $\dfrac{1}{2} = \dfrac{1}{3} + \dfrac{1}{6}$ and $2 < 3 < 6$.

(b) $\dfrac{1}{1996} = \dfrac{1}{998} \times \dfrac{1}{2} = \dfrac{1}{998} \times \left(\dfrac{1}{3} + \dfrac{1}{6}\right) = \dfrac{1}{2994} + \dfrac{1}{5998}$.

That completes the Skoliad Corner for this issue. Send me your contests, suggestions, and recommendations to improve this feature.

THE OLYMPIAD CORNER

No. 178

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin this Issue with the remaining problems proposed to the jury, but not used at the 36th International Olympiad held at Toronto, Ontario in July 1995. As always, I welcome your novel, nice solutions that differ from the "official" published solutions.

36th INTERNATIONAL MATHEMATICAL OLYMPIAD
Problems proposed to the jury but not used

Algebra

1. Let $\mathbb{R}$ be the set of real numbers. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ which simultaneously satisfies the following three conditions?
(a) There is a positive number M such that $-M \le f(x) \le M$ for all x.
(b) $f(1) = 1$.
(c) If $x \neq 0$, then

$$f\left(x+\frac{1}{x^2}\right) = f(x) + \left[f\left(\frac{1}{x}\right)\right]^2 .$$

2. Let n be an integer, $n \ge 3$. Let $x_1, x_2, \dots, x_n$ be real numbers such that $x_i < x_{i+1}$ for $1 \le i \le n-1$. Prove that

$$\frac{n(n-1)}{2}\sum_{i<j} x_i x_j > \left(\sum_{i=1}^{n-1}(n-i)x_i\right)\left(\sum_{j=2}^{n}(j-1)x_j\right).$$

Geometry

3. Let $A_1A_2A_3A_4$ be a tetrahedron, G its centroid, and A_1', A_2', A_3' and A_4' the points where the circumsphere of $A_1A_2A_3A_4$ intersects GA_1, GA_2, GA_3 and GA_4 respectively. Prove that

$$GA_1 \cdot GA_2 \cdot GA_3 \cdot GA_4 \le GA_1' \cdot GA_2' \cdot GA_3' \cdot GA_4'$$

and

$$\frac{1}{GA_1'}+\frac{1}{GA_2'}+\frac{1}{GA_3'}+\frac{1}{GA_4'} \le \frac{1}{GA_1}+\frac{1}{GA_2}+\frac{1}{GA_3}+\frac{1}{GA_4}.$$

4. O is a point inside a convex quadrilateral $ABCD$ of area S. K, L, M and N are interior points of the sides AB, BC, CD and DA respectively. If $OKBL$ and $OMDN$ are parallelograms, prove that $\sqrt{S} \geq \sqrt{S_1} + \sqrt{S_2}$, where S_1 and S_2 are the areas of $ONAK$ and $OLCM$ respectively.

5. Let ABC be a triangle. A circle passing through B and C intersects the sides AB and AC again at C' and B', respectively. Prove that BB', CC' and HH' are concurrent, where H and H' are the orthocentres of triangles ABC and $AB'C'$ respectively.

Number Theory and Combinatorics

6. At a meeting of $12k$ people, each person exchanges greetings with exactly $3k+6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting?

7. Does there exist an integer $n > 1$ which satisfies the following condition? The set of positive integers can be partitioned into n non-empty subsets, such that an arbitrary sum of $n-1$ integers, one taken from each of any $n-1$ of the subsets, lies in the remaining subset.

8. Let p be an odd prime. Determine positive integers x and y for which $x \leq y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is non-negative and as small as possible.

Sequences

9. For positive integers n, the numbers $f(n)$ are defined inductively as follows: $f(1) = 1$, and for every positive integer n, $f(n+1)$ is the greatest integer m such that there is an arithmetic progression of positive integers $a_1 < a_2 < \cdots < a_m = n$ and

$$f(a_1) = f(a_2) = \cdots = f(a_m).$$

Prove that there are positive integers a and b such that $f(an+b) = n+2$ for every positive integer n.

10. Let $\mathbb{N}$ denote the set of all positive integers. Prove that there exists a unique function $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(m + f(n)) = n + f(m + 95)$$

for all m and n in $\mathbb{N}$. What is the value of $\sum_{k=1}^{19} f(k)$?

We now turn to readers' solutions to problems of the contest given in the Corner as the Turkish Mathematical Olympiad Committee Final Selection Test of April 4, 1993. This problem set was given in the April 1995 number of the corner [1995: 117-118].

TURKISH MATHEMATICAL OLYMPIAD COMMITTEE FINAL SELECTION TEST

April 4, 1993 — Part I

(Time: 3 hours)

1. Show that there is an infinite sequence of positive integers such that the first term is 16, the number of distinct positive divisors of each term is divisible by 5, and the terms of the sequence form an arithmetic progression. Of all such sequences, find the one with the smallest possible common difference between consecutive terms.

Solutions by Cyrus Hsia, student, University of Toronto, Toronto, Ontario; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Clearly the question should have been reworded as "of all..., find the one with the smallest *non-zero* common difference between consecutive terms." For otherwise the constant sequence $16, 16, 16, \dots$ is clearly the answer. Let d denote the common difference. If $d < 0$ then the terms of the sequence will eventually become negative. Hence assume that $d > 0$.

For natural numbers n, let $\tau(n)$ denote the number of positive divisors of n. Then it is well-known that if $n = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_k^{\alpha_k}$ denotes the prime power decomposition of n, then $\tau(n) = \prod_{i=1}^{k}(\alpha_i + 1)$. Furthermore, τ is a multiplicative function, i.e., if gcd $(n, m) = 1$ then $\tau(n \cdot m) = \tau(n) \cdot \tau(m)$.

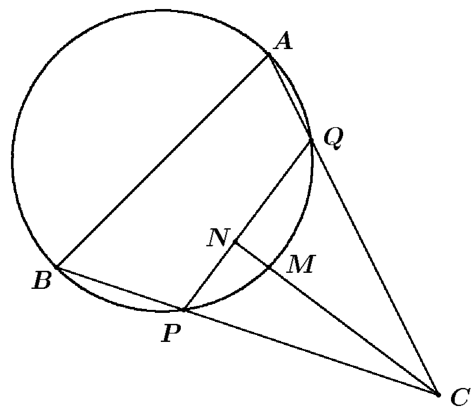
Therefore if $16 + d = 2^4(2q+1)$, where $q = 0, 1, 2, \dots$, then $5 | \tau(16 + d)$. To minimize d, or $16 + d$, we take $q = 1$ since $q = 0$ implies $d = 0$. Then $d = 3 \times 2^4 - 2^4 = 2^5$. To see that this value of d would yield an arithmetic progression with the desired property, note first that for all $m = 0, 1, 2, \dots$, $\tau(16 + md) = \tau(2^4 + 2^5 m) = \tau(2^4(2m+1)) \equiv 0 \bmod 5$. Finally, straightforward checking reveals that $5 \not| \, \tau(16 + t)$ for $t = 1, 2, \dots, 31$. Therefore, the required sequence is indeed $\{16 + 32m\}$, $m = 0, 1, 2, \dots$ with $d = 32$.

2. Let M be the circumcentre of an acute-angled triangle ABC, and assume the circle (BMA) intersects the segment $[BC]$ at P, and the segment $[AC]$ at Q. Show that the line CM is perpendicular to the line PQ.

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario.

Let triangle ABC have angles A, B, and C. $\angle AMC = 2\angle B$ in circle (ABC). Thus

$$\begin{aligned} \angle MCA &= \frac{180^\circ - 2B}{2} \quad \text{(in isosceles triangle } AMC\text{)} \\ &= 90^\circ - B; \\ \angle PQC &= \angle ABP \quad \text{(in circle } (AMB)\text{)} \\ &= B. \end{aligned}$$



Thus, extending CM to meet PQ in N we have $\angle QCN = \angle MCA = 90^\circ - B$ and $\angle NQC = \angle PQC = B$.

It follows that

$$\begin{aligned}\angle QNC &= 180^\circ - (\angle QCN + \angle NQC) \\ &= 180^\circ - (90^\circ - B + B) \\ &= 90^\circ.\end{aligned}$$

That is, $CM \perp PQ$.

3. Let $\{b_n\}$ be sequence of positive real numbers such that

$$\text{for each } n \geq 1, \quad b_{n+1}^2 \geq \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \cdots + \frac{b_n^2}{n^3}.$$

Show that there is a natural number K such that

$$\sum_{n=1}^{K} \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} > \frac{1993}{1000}.$$

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario.

$$\begin{aligned}&(1^3 + 2^3 + \cdots + n^3)(b_{n+1}^2) \\ &\geq (1^3 + 2^3 + \cdots + n^3)\left(\frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \cdots + \frac{b_n^2}{n^3}\right) \\ &\geq (b_1 + b_2 + \cdots + b_n)^2\end{aligned}$$

by the Cauchy–Schwartz inequality. Thus

$$\frac{b_{n+1}^2}{(b_1 + b_2 + \cdots + b_n)^2} \geq \frac{1}{1^3 + 2^3 + \cdots + n^3} = \left[\frac{2}{n(n+1)}\right]^2.$$

It follows that

$$\frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} \geq \frac{2}{n(n+1)}$$

since the sequence $\{b_n\}$ has only positive real terms. Thus

$$\begin{aligned}\sum_{n=1}^{K} \frac{b_{n+1}}{b_1+b_2+\cdots+b_n} &\geq \sum_{n=1}^{K} \frac{2}{n(n+1)} \\ &= 2\sum_{n=1}^{K} \frac{1}{n(n+1)} \\ &= 2\sum_{n=1}^{K} \left(\frac{1}{n}-\frac{1}{n+1}\right) \\ &= 2\left(1-\frac{1}{K+1}\right) \\ &= \frac{2K}{K+1}.\end{aligned}$$

By setting $K = 999$ we have

$$\sum_{n=1}^{999} \frac{b_{n+1}}{b_1+\cdots+b_n} \geq \frac{2(999)}{999+1} = \frac{1998}{1000} > \frac{1993}{1000},$$

as required.

April 4, 1993 — Part II

(Time: 3 hours)

1. Some towns are connected to each other by some roads with at most one road between any pair of towns. Let v denote the number of towns, and e denote the number of roads. Show that

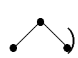
(a) if $e < v - 1$, then there are at least two towns such that it is impossible to travel from one to the other,

(b) if $2e > (v-1)(v-2)$, then travelling between any pair of towns is possible.

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario.

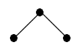
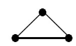
(a) The minimum value that e could be such that the graph with v towns is connected is $e = v - 1$.

We prove this by induction. For $v = 3$, we must have e at least 2 since $e = 1$ would only connect two towns.

(And $e = 2$ is possible, as shown ).

Suppose that the minimum for v towns is $e = v - 1$. Consider a connected situation with $v + 1$ towns. We can assume there is at least one town with at most one road connecting it to other towns otherwise $2e \geq 2(v+1)$ and $e \geq v + 1$. Removing that town (and its road to another town) must leave a connected situation with v towns. So $e-1 \geq v-1$ and $e \geq v$ as required. It is easy to see that one can construct a connected example with $v+1$ towns and $v+1-1=e$ edges, completing the induction.

(b) The statement is proved by induction. For $v = 3$, we have $3 > \frac{(v-1)(v-2)}{2} = 1$, so that $e \geq 2$.

With $e = 2$ we have  and $e = 3$  as the essential possibilities.

Suppose travelling between any pair of towns is possible for v towns if $2e > (v-1)(v-2)$. Now consider $v+1$ towns joined by $2e > (v)(v-1)$ roads. Consider a fixed town. If it is connected to the v remaining towns, we are done. Otherwise, let it be joined to $x \leq v-1$ towns. If the town and its roads are removed we are left with v towns and

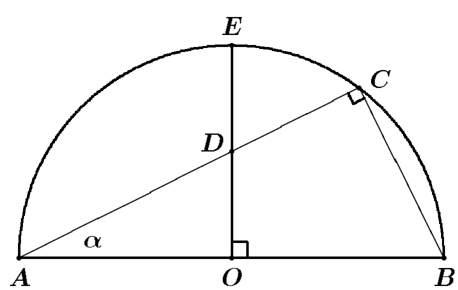
$$\begin{aligned} e - x > \frac{v(v-1)}{2} - x &\geq \frac{v(v-1)}{2} - (v-1) \\ &= \frac{(v-1)(v-2)}{2} \end{aligned}$$

roads, that is, the remaining v towns have more than $\frac{(v-1)(v-2)}{2}$ roads and are connected. If $x > 0$, the $(v+1)^{\text{st}}$ town is joined to some one of the remaining towns, and hence connected to all of them. If $x = 0$ then the v towns can have a maximum number of $\binom{v}{2} = \frac{v(v-1)}{2}$ roads. But $e > \frac{v(v-1)}{2}$, so $x \neq 0$.

Thus for v towns, $e > \frac{(v-1)(v-2)}{2}$ implies all towns are connected and one can travel between any pair of towns.

2. On a semicircle with diameter AB and centre O points E and C are marked in such a way that OE is perpendicular to AB, and the chord AC intersects the segment OE at a point D which is interior to the semicircle. Find all values of the angle $\angle CAB$ such that a circle can be inscribed into the quadrilateral $OBCD$.

Solutions by Cyrus Hsia, student, University of Toronto, Toronto, Ontario; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.



We set $OA = OB = 1$ and denote $\angle BAC = \alpha$. Then $OD = \tan\alpha$, $OB = 1$ and $BC = 2\sin\alpha$, $CD = 2\cos\alpha - \frac{1}{\cos\alpha}$.

For a circle to be inscribed in quadrilateral $OBCD$, we have

$$\begin{aligned} OB + CD &= OD + BC \\ 1 + 2\cos\alpha - \frac{1}{\cos\alpha} &= \tan\alpha + 2\sin\alpha \\ \cos\alpha + 2\cos^2\alpha - 1 &= \sin\alpha + 2\sin\alpha\cos\alpha \\ \cos\alpha + \cos 2\alpha &= \sin\alpha + \sin 2\alpha \\ 2\cos\frac{3}{2}\alpha\cos\frac{\alpha}{2} &= 2\sin\frac{3}{2}\alpha\cos\frac{\alpha}{2}. \end{aligned}$$

Since $\cos\dfrac{\alpha}{2} \neq 0$, we have

$$\begin{aligned} \tan\frac{3\alpha}{2} &= 1 \\ \frac{3\alpha}{2} &= \frac{\pi}{4} \\ \alpha &= \frac{\pi}{6}. \end{aligned}$$

3. Let $\mathbb{Q}^+$ denote the set of all positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$\text{for every } x, y \in \mathbb{Q}^+, \quad f\left(x + \frac{y}{x}\right) = f(x) + \frac{f(y)}{f(x)} + 2y.$$

Solutions by Cyrus Hsia, student, University of Toronto, Toronto, Ontario; and by Beatriz Margolis, Paris, France. We give the solution by Margolis.

We show that the only solution is $f(x) = x^2$, $x \in \mathbb{Q}^+$.

Take $x = 1$, this gives

$$f(1 + y) = f(1) + \frac{f(y)}{f(1)} + 2y. \tag{1}$$

Now taking $x = y$ we get

$$f(y + 1) = f(y) + 1 + 2y. \tag{2}$$

From (1), and (2) we obtain

$$(f(1) - 1)\left(1 - \frac{f(y)}{f(1)}\right) = 0.$$

Now f can **not** be a constant function, so we have that

$$f(1) = 1. \tag{3}$$

By induction, from (2) and (3) we get

$$f(k) = k^2 \qquad k \in \mathbb{N}^+. \tag{4}$$

Therefore, if n, $k \in \mathbb{N}^+$, using (4), we obtain that

$$\begin{aligned} f\left(k + \frac{n}{k}\right) &= f(k) + \frac{f(n)}{f(k)} + 2n \\ &= k^2 + \frac{n^2}{k^2} + 2n = \left(k + \frac{n}{k}\right)^2. \end{aligned} \tag{5}$$

Let $x = \frac{m}{k} \in \mathbb{Q}^+$, and $N \in \mathbb{N}^+$. By (2), we have

$$\begin{aligned} f(x+N) - f(x) &= \sum_{j=0}^{N-1} [f(x+j+1) - f(x+j)] \\ &= \sum_{j=0}^{N-1} [1 + 2(x+j)] \\ &= N(2x+1) + (N-1)N \\ &= (x+N)^2 - x^2. \end{aligned}$$

Therefore $f(x+N) - (x+N)^2 = f(x) - x^2$, $N \in \mathbb{N}^+$, $x \in \mathbb{Q}^+$.
In particular, using (5), we obtain that

$$\begin{aligned} 0 = f\left(\frac{n}{k} + k\right) - \left(\frac{n}{k} + k\right)^2 &= f\left(\frac{n}{k}\right) - \left(\frac{n}{k}\right)^2 \\ &= f(x) - x^2, \end{aligned}$$

where $x = \frac{n}{k}$, with n, $k \in \mathbb{N}^+$.

Thus $f(x) = x^2$ for $x \in \mathbb{Q}^+$.

That completes the solutions we have on file and the Olympiad Corner for this issue. Send me your nice problem sets and solutions!

THE ACADEMY CORNER

No. 7

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In the May 1996 issue of ***CRUX*** [1996: 165], we printed the questions of the 1996 Memorial University Undergraduate Mathematics Competition, and invited readers to send in solutions. Here is a complete solution set.

MEMORIAL UNIVERSITY OF NEWFOUNDLAND UNDERGRADUATE MATHEMATICS COMPETITION

March, 1996

Solutions

1. Prove that if n is a positive integer, then $\dfrac{n^2+3n+1}{n^2+4n+3}$ is an irreducible fraction.

   Solution by Panos E. Tsaoussoglou, Athens, Greece [Shortened by the editors.]

   Assume that the fraction is reducible. Then there is a positive integer k such that $k|(n^2+3n+1)$ and $k|(n^2+4n+3)$.

   Then $k|(n^2+3n+1)-(n^2+4n+3)$ or $k|n+2$, so that there is a positive integer l such that $n+2=kl$.

   Now, $\dfrac{n^2+3n+1}{k} = nl + \dfrac{n+1}{k}$. The latter term is an integer only if $k|n+1$. Since $k|n+2$, this implies that $k=1$.

   This is a contradiction, proving the result.

2. A jar contains 7 blue balls, 9 red balls and 10 white balls. Balls are drawn at random one by one from the jar until either four balls of the same colour or at least two of each colour have been drawn. What is the largest number of balls that one may have to draw?

   Solution by Panos E. Tsaoussoglou, Athens, Greece.

   Assume that three balls of one colour are drawn, then three balls of an other colour are drawn, and then one ball of the third colour. The next ball (of any colour) satisfies the conditions of the problem.

   The answer is 8 balls.

3. Find all functions $u(x)$ satisfying $u(x) = x + \int_0^{\frac{1}{2}} u(t)\,dt$.

"Official Solution" from Maurice Oleson.

Since $\int_0^{\frac{1}{2}} u(t)\,dt$ is a constant, we have that $u(x) = x + c$ for some constant c.

Then $\int_0^{\frac{1}{2}} (t + c)\,dt = c$, which gives $c = \frac{1}{4}$.

Thus $u(x) = x + \frac{1}{4}$.

4. Show that $\left(\sqrt{5}+2\right)^{\frac{1}{3}} - \left(\sqrt{5}-2\right)^{\frac{1}{3}}$ is a rational number and find its value.

Solution by Panos E. Tsaoussoglou, Athens, Greece.

Let $k = \left(\sqrt{5}+2\right)^{\frac{1}{3}} - \left(\sqrt{5}-2\right)^{\frac{1}{3}}$. Then

$$\begin{aligned} k^3 &= \left(\sqrt{5}+2\right) - \left(\sqrt{5}-2\right) - 3\left(\left(\sqrt{5}+2\right)\left(\sqrt{5}-2\right)\right)^{\frac{1}{3}} \\ &\quad \times \left(\left(\sqrt{5}+2\right)^{\frac{1}{3}} - \left(\sqrt{5}-2\right)^{\frac{1}{3}}\right), \end{aligned}$$

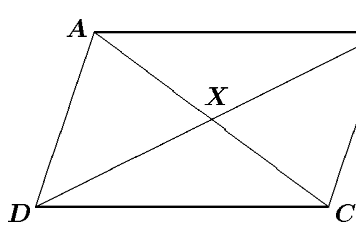
or

$$k^3 = 4 - 3k \qquad \text{or} \qquad (k^3 - 1) + 3(k - 1) = 0.$$

Thus $(k-1)(k^2+4k+4) = 0$, so that $k = 1$ (since $k > 0$).

5. In a quadrilateral $ABCD$ (vertices named in clockwise order), AC and BD intersect in X. You are given that $AB \parallel DC$, that AB is twice as long as CX and that AC is equal in length to DC. Show that AB and CD are equal in length (and hence $ABCD$ is a parallelogram).

Solution by Panos E. Tsaoussoglou, Athens, Greece.



We have $AB = 2CX$ and $AC = DC$.

Triangles $\triangle AXB$ and $\triangle DXC$ are similar, so that

$$\frac{AX}{CX} = \frac{AB}{DC}$$

so that

$$\frac{AX + CX}{CX} = \frac{AB + DC}{DC}.$$

Thus, we have

$$\begin{aligned} \frac{2DC}{AB} &= \frac{AB + DC}{DC} \\ 2DC^2 &= AB^2 + AB \cdot DC \\ 0 &= DC^2 - AB^2 + DC^2 - AB \cdot DC \\ &= (DC - AB)(AB + 2CD), \end{aligned}$$

so that $AB = DC$.

6. Prove that among any thirteen distinct real numbers it is possible to choose two, x and y, such that $0 < \dfrac{x - y}{1 + xy} < 2 - \sqrt{3}$.

"Official Solution" from Maurice Oleson.

We recognize the similarity of $\dfrac{x - y}{1 + xy}$ with $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$.

Since twelve fifteen degree angles make a straight angle, the pigeon-hole principle gives that $0 < \dfrac{x - y}{1 + xy} < \tan(15°)$.

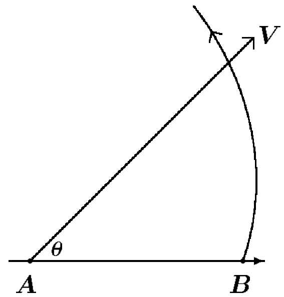
We note that

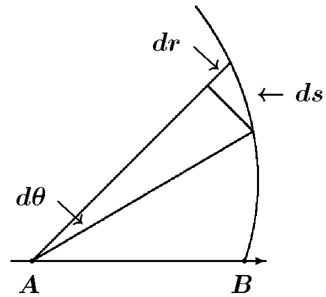
$$\begin{aligned} \tan^2(15°) &= \frac{\sin^2(15°)}{\cos^2(15°)} \\ &= \frac{1 - \cos(30°)}{1 + \cos(30°)} \\ &= \frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}} \\ &= \frac{2 - \sqrt{3}}{2 + \sqrt{3}} \\ &= (2 - \sqrt{3})^2, \end{aligned}$$

and the result follows.

7. A coastguard boat is hunting a bootlegger in a fog. The fog rises disclosing the bootlegger 4 miles distant and immediately descends. The speed of the boat is 3 times that of the bootlegger, and it is known that the latter will immediately depart at full speed on a straight course of unknown direction. What course should the boat take in order to overtake the bootlegger?

"Official Solution" from Maurice Oleson. We look at two diagrams:





The distance from A to B is 1 mile.

From $ds^2 = dr^2 + (r\,d\theta)^2$, $\frac{ds}{dt} = 3v$ and $\frac{dr}{dt} = v$, we get

$$9v^2 = v^2 + r^2\left(\frac{d\theta}{dt}\right)^2 .$$

Thus

$$\sqrt{8}v = r\frac{d\theta}{dt} = r\frac{d\theta}{dr}\cdot\frac{dr}{dt} = r\frac{d\theta}{dr}v,$$

yielding $\dfrac{dr}{r} = \dfrac{1}{\sqrt{8}}d\theta$. Hence $\ln(r) = \dfrac{\theta}{\sqrt{8}} + c$.

Since $\theta = 0$ when $r = 1$, we have $c = 0$, giving $r = e^{\theta/\sqrt{8}}$.

This is the spiral path that the coastguard must follow if she wishes to intercept the bootlegger.

PHOTO PROBLEM

Can you identify this regular ***CRUX*** contributor?

BOOK REVIEWS

Edited by ANDY LIU

The Universe in a Handkerchief by Martin Gardner,
published by Copernicus, an imprint of Springer-Verlag New York Inc., 1994,
ISBN 0-387-94673-X, x+158 pages, US$19.00.
Reviewed by ***Richard Guy****, University of Calgary.*

A book by Martin Gardner is always welcome, and most mathematicians and many other people are interested in Lewis Carroll. Although it is almost a century since he died, there is still much to learn about him. Morton Cohen's biography [2] only appeared last year and Edward Wakeling's edition of Lewis Carroll's diaries [4] is still in process of appearing. The present book contains a bibliography of more than thirty items: and there is little overlap with the nearly fifty items which relate to Carroll in John Fisher's book [3].

Many mathematicians are interested in word play of all sorts, and this makes up the major part of the book: Carroll's 'Doublets' or 'Word Links' are still a popular pastime: can you get from 'Bread' to 'Toast' with less than 21 links? But if you're looking for specifically mathematical items, then they'll need picking out for you. The gravity-operated train from *Sylvie and Bruno concluded*. Carroll's attachment to, if not obsession with, the number 42. The Butcher's piece of algebra in Fit 5 of *The Hunting of the Snark*. The quadratic equation in the poem in *Rhyme? and Reason?* The puzzle of the monkey and weight over a pulley. John Conway's "Doomsday Rule" for finding what day of the week any date falls on has its original inspiration from Lewis Carroll. Rhymes for remembering logarithms to seven decimal places. The Telegraph Cipher. The game of Arithmetical Croquet. Conjuring tricks depending on 'casting out the nines'. Probability paradoxes which anticipate the 'motor car and goats' debate of recent years. And mathematical recreations of all the traditional kinds, such as are listed in Rouse Ball [1], the first three editions of which appeared in Carroll's lifetime. It was probably in the fourth edition that Rouse Ball added a footnote to his geometrical fallacies, saying that 'they particularly interested Mr. C. L. Dodgson; see the *Lewis Carroll Picture Book*, London, 1899, pp. 264, 266, where they appear in the form in which I originally gave them.'

The following problem, from a letter by Carroll to Enid Stevens, was thought by Morton Cohen to be too ambiguously stated to have a precise answer:

> Three men, A, B and C, are to run a race of a quarter-of-a-mile. Whenever A runs against B, he loses 10 yards in every hundred; whenever B runs against C, he *gains* 10 yards in every hundred. How should they be handicapped?

Isn't the following a reasonable interpretation? For distances run in equal times, $A : B = 90 : 100$ and $B : C = 110 : 100$, so that

$$A : B : C = 99 : 110 : 100 = 396 : 440 : 400,$$

and B should start from scratch, with C at the 40 yard mark and A at 44 yards.

Something I learned while reviewing this book, though not while reading it, and which Canadians might be particularly interested in, is that Carroll, with such definitions as 'OBTUSE ANGER is that which is greater than Right Anger', anticipated the ideas in Stephen Leacock's *Boarding House Geometry*.

And where does the title come from? In *Sylvie and Bruno concluded* Mein Herr gives instructions, not in fact practicable in three dimensions, to Lady Muriel for sewing three handkerchiefs into the Purse of Fortunatus, which is a projective plane. "Whatever is *inside* that Purse, is *outside* it; and whatever is *outside* it, is *inside* it. So you have all the wealth of the world in that leetle Purse!"

References

[1] W. W. Rouse Ball & H. S. M. Coxeter, *Mathematical Recreations & Essays*, 12th edition, Univ. of Toronto Press, 1974.

[2] Morton Cohen, *Lewis Carroll: A Biography*, Knopf, New York, 1995.

[3] John Fisher (ed.), *The Magic of Lewis Carroll*, Penguin Books, Harmondsworth, 1975.

[4] Edward Wakeling (ed.), *Lewis Carroll's Diaries*, Vols. 1, 2 & 3., Lewis Carroll Society, Luton, England, 1993, 1994, 1995.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that* **1 June 1997**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2189. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The incircle of a triangle ABC touches BC at D. Let P and Q be variable points on sides AB and AC respectively such that PQ is tangent to the incircle. Prove that the area of triangle DPQ is a constant multiple of $BP \cdot CQ$.

2190. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the range of

$$\frac{\sin^2 A}{A} + \frac{\sin^2 B}{B} + \frac{\sin^2 C}{C}$$

where A, B, C are the angles of a triangle.

2191. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Find all positive integers n, that satisfy the inequality

$$\frac{1}{3} < \sin\left(\frac{\pi}{n}\right) < \frac{1}{2\sqrt{2}}.$$

2192. *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Let $\{a_n\}$ be a sequence defined as follows:

$$a_{n+1} + a_{n-1} = \left(\frac{a_2}{a_1}\right) a_n, \qquad n \geq 1.$$

Show that if $\left|\frac{a_2}{a_1}\right| \geq 2$, then $\left|\frac{a_n}{a_1}\right| \geq n$.

2193. *Proposed by Luis V. Dieulefait, IMPA, Rio de Janeiro, Brazil.*

(a) Prove that every positive integer is the difference of two relatively prime composite positive integers.

(b) Prove that there exists a positive integer n_0 such that every positive integer greater than n_0 is the sum of two relatively prime composite positive integers.

2194. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Prove or disprove that it is possible to find a triangle ABC and a transversal NML with N lying between A and B, M lying between$A$ and C, and L lying on BC produced, such that BC, CA, AB, NB, MC, NM, ML, and CL are all of integer length, and $NMCB$ is a cyclic inscriptable quadrilateral.

2195. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

A barrel contains $2n$ balls, numbered 1 to $2n$. Choose three balls at random, one after the other, and with the balls replaced after each draw.

What is the probability that the three-element sequence obtained has the properties that the smallest element is odd and that only the smallest element, if any, is repeated?

For example, the sequences 453 and 383 are acceptable, while the sequences 327 and 388 are not.

(NOTE: this problem was suggested by a final exam that I marked recently.)

2196. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Find all solutions of the diophantine equation

$$2(x + y) + xy = x^2 + y^2,$$

with $x > 0$, $y > 0$.

2197. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let n be a positive integer. Evaluate the sum:

$$\sum_{k=n}^{\infty} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}}.$$

2198. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

Prove that, if a, b, c are the lengths of the sides of a triangle,

$$(b-c)^2\left(\frac{2}{bc}-\frac{1}{a^2}\right)+(c-a)^2\left(\frac{2}{ca}-\frac{1}{b^2}\right)+(a-b)^2\left(\frac{2}{ab}-\frac{1}{c^2}\right) \;\geq\; 0,$$

with equality if and only if $a=b=c$.

2199. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*

Find the maximum value of c for which $(x+y+z)^2 > cxz$ for all $0 \leq x < y < z$.

2200. *Proposed by Jeremy T. Bradley, Bristol, UK and Christopher J. Bradley, Clifton College, Bristol, UK.*

Find distinct positive integers a, b, c, d, w, x, y, z, such that

$$z^2 - y^2 \;=\; x^2 - c^2 \;=\; w^2 - b^2 \;=\; d^2 - a^2$$

and

$$c^2 - a^2 \;=\; y^2 - w^2.$$

Bonus Problem for 1996

220A*. *Proposed by Ji Chen, Ningbo University, China.*

Let P be a point in the interior of the triangle ABC, and let $\alpha_1 = \angle PAB$, $\beta_1 = \angle PBC$, $\gamma_1 = \angle PCA$.

Prove or disprove that $\sqrt[3]{\alpha_1\beta_1\gamma_1} \leq \pi/6$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2030. [1995: 91, 1996: 132] *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

For which complex numbers s does the polynomial $z^3 - sz^2 + \overline{s}z - 1$ possess exactly three distinct zeros having modulus 1?

Editor's comment.

Readers who found #2030 interesting might want to look at "*What Is the Shape of a Triangle*" by Dana N. Mackenzie [NOTE DI MATEMATICA 13:2 (1993) 237-250; MATH REVIEWS 96c: 51029].

The author notes that an arbitrary triangle ABC is similar to a triangle whose vertices are represented by three complex numbers z_i with $|z_i| = 1$ and $z_1 \times z_2 \times z_3 = 1$.

He defines the **SHAPE** of the triangle to be $\sigma = z_1 + z_2 + z_3$.

It follows that the shape appears as the coefficient of z^2 in the cubic $z^3 - (\sigma)z^2 + (\bar{\sigma})z - 1$, whose zeros all lie on the unit circle.

His solution to the question, "*what are the possible values of σ?*" is essentially the solution to ***CRUX*** 2030 (appearing more than two years before it appeared in ***CRUX***). The author goes on to discuss properties of his "*shape invariant*"; quite surprisingly, during that discussion the Morley triangle of triangle ABC plays a key role.

2076. [1995: 278] *Proposed by John Magill, Brighton, England.*

		4
	AC	
	C	24

This is a multiplicative magic square, where the product of each row, column and diagonal has the same value, ABCD. Each letter represents a digit, the same digit wherever it appears, and each cell contains an integer. Complete the square by entering the correct numbers in each of the nine cells.

Solution by Toby Gee, student, the John of Gaunt School, Trowbridge, England.

Letting $AC = 10A + C$ and letting x be the value in the bottom left hand corner we have $24Cx = 4(10A + C)x$, implying $C = 2A$. If we further let z be the value of the upper left hand corner and y be that of the middle top row box, we also have $4yz = 24(10A + C)z = 24 \cdot 12Az$, which yields $y = 72A$. The product of the entries in the middle column is then $72A \cdot 12A \cdot 2A = 1728A^3$, which must be a four digit integer, whence $A = 1$ and we can complete the square as below:

6	72	4
8	12	18
36	2	24

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; POLLY KONTOPOULOU, student, Aristotle University of Thessaloniki, Greece; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; WOLFGANG GMEINER, Millstatt, Austria; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HRISTOS SARAGHIOTES, student, Aristotle University, Thessaloniki, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia; DAVID TASCIONE, student, St. Bonaventure University, New York; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

2087. [1995: 307] *Proposed by Toby Gee, student, the John of Gaunt School, Trowbridge, England.*

Find all polynomials f such that $f(p)$ is a power of two for every prime p.

Solution by Luis V. Dieulefait, IMPA, Rio de Janeiro, Brazil.

The only solutions are constant polynomials. Suppose instead that a polynomial f with $\deg(f) > 0$ were a solution. Since $f(x)$ takes positive values for every prime value of x, it is clear that $\lim_{x\to\infty} f(x) = \infty$. Also there is a real number x_0 such that $f(x)$ is increasing for $x > x_0$. If we enumerate the sequence of primes: $p_1, p_2, p_3, \ldots$ in increasing order, then

$$f(p_{i+1}) > f(p_i) \quad \text{for} \quad p_i > x_0. \qquad (*)$$

Lemma. Given a real number $r > 1$, every sufficiently large p_i satisfies $r \cdot p_i \geq p_{i+1}$.

Proof. Let us suppose, on the contrary, that there exist infinitely many primes p_j satisfying $p_{j+1} > r \cdot p_j$. Then $\pi(r \cdot p_j) - \pi(p_j) = 0$ where $\pi(x)$ is the number of primes less than or equal to x. This means that $\frac{\pi(r \cdot p_j)}{\pi(p_j)} = 1$ for infinitely many primes. Applying the prime number theorem, we see that

$$\lim_{i \to \infty} \frac{\pi(r \cdot p_i)}{\pi(p_i)} = \lim_{i \to \infty} \frac{r \cdot \ln(p_i)}{\ln(r \cdot p_i)} = r > 1.$$

This contradiction proves the lemma.

Let $d = \deg(f) > 0$. Choose $r > 1$ such that $r^d < 2$. From (*) above we see that $f(p_{i+1} \geq 2 \cdot f(p_i)$ for every i large enough. Applying the lemma we have $f(r \cdot p_i) > 2 \cdot f(p_i)$ for every i large enough (because f is increasing for such i). This implies $\lim_{i \to \infty} \frac{f(r \cdot p_i)}{f(p_i)} \geq 2$. But f is a polynomial of degree d, so we know that the limit above equals r^d and $r^d < 2$. This contradiction proves that no polynomial f with $\deg(f) > 0$ can solve our problem.

Then the only solutions are constant, and obviously we must have $f(x) = 2^n$, for n a non-negative integer.

Also solved by MANUAL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, California, USA; KEE-WAI LAU, Hong Kong; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

2089. [1995: 307] *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Let $ABCD$ be a trapezoid with $AB \parallel CD$, and let X be a point on segment AB. Put $P = CB \cap AD$, $Y = CD \cap PX$, $R = AY \cap BD$ and $T = PR \cap AB$. Prove that

$$\frac{1}{AT} = \frac{1}{AX} + \frac{1}{AB}\,.$$

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

Using Menelaus's Theorem on the triangle ABD, we get

$$\frac{AT}{TB} \cdot \frac{BR}{RD} \cdot \frac{DP}{PA} = 1.$$

Since $BR : RD = AB : DY$ and $DP : PA = DY : AX$, we obtain

$$\frac{TB}{AT} = \frac{AB}{DY} \cdot \frac{DP}{PA} = \frac{AB}{DY} \cdot \frac{DY}{AX} = \frac{AB}{AX},$$

whence

$$\frac{AB}{AT} = \frac{TB}{AT} + 1 = \frac{AB}{AX} + 1 \quad \text{or} \quad \frac{1}{AT} = \frac{1}{AX} + \frac{1}{AB},$$

as we wished.

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TIM CROSS, King Edward's School, Birmingham, England; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2091. [1995: 343] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Four points A, B, C, D are on a line in this order. We put $AB = a$, $BC = b$, $CD = c$. Equilateral triangles ABP, BCQ and CDR are constructed on the same side of the line. Suppose that $\angle PQR = 120°$. Find the relation between a, b and c.

I. *Solution by Jordi Dou, Barcelona, Spain.*

If the middle triangle is largest then b is twice the arithmetic mean of a and c; if smallest, b is half the harmonic mean of a and c.

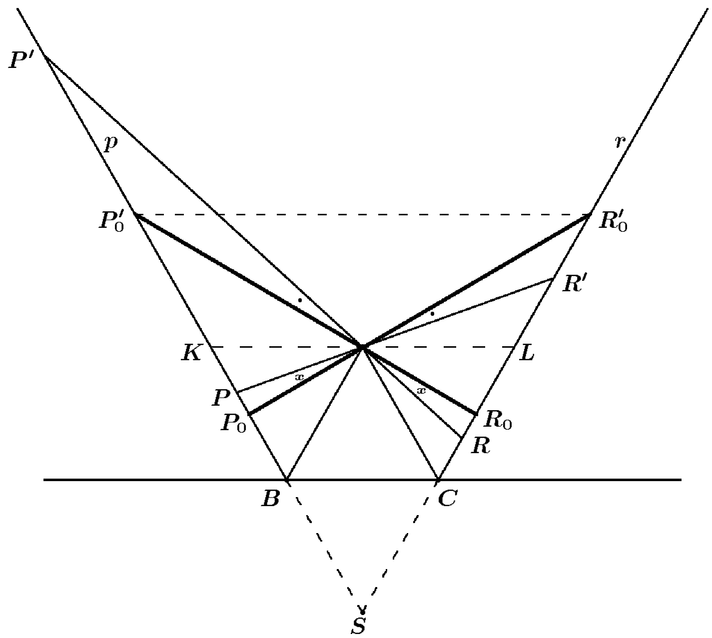
Proof: We suppose that BCQ is fixed. Let S be the image of Q under reflection in BC. P and R will lie on the prolongations of SB and SC, which we will call p and r respectively.

As in the figure, K, L are the points where the line through Q parallel to BC intersects p, r. P_0 is the foot of the perpendicular to p from Q, and R_0' is the point where that line meets r; analogously, R_0 is the foot of the perpendicular to r from Q, and P_0' is where that line meets p.

Any point P on the segment BK of p corresponds to a point R on the segment LC for which $\angle PQR = 120°$. From $\angle P_0QR_0 = 120°$ it follows that $\angle P_0QP = \angle R_0QR$ and $PP_0 = RR_0$. In this case $a = BP$, and $c = CR = PK$, so that the desired relation is

$$a + c = b.$$

Any point P' on p beyond K corresponds to a point R' on r beyond L for which $\angle R'QP' = 120°$. As before, because $\angle R_0'QP_0' = 120°$ it follows that $\angle P_0'QP' = \angle R_0'QR'$ and, therefore, that the pencils QP' and QR' are congruent so that there is a projectivity that takes P' to R'. [That is, $P' \barwedge P'Q \barwedge R'Q \barwedge R'$.] This projectivity takes P_∞' (the point at infinity of p) to L, and K to R_∞' (the point at infinity of r). Since cross ratios are preserved it follows that $KP' \cdot LR' = constant = KP_0' \cdot LR_0' = KQ \cdot LQ = b^2$. [If you prefer, assign the coordinates 0, 1, ∞ to the points K, P_0', P_∞ on p, and assign ∞, 1, 0 to their images — R_∞', R_0', L — on r. Then the projective



transformation that takes P' to R' takes the corresponding coordinate x to $1/x$.] Thus $(a-b)(c-b)=b^2$, or $ac=ba+cb$, or

$$\frac{1}{b}=\frac{1}{c}+\frac{1}{a} \text{ or } b=\frac{ca}{c+a}.$$

II. *Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*

[*Editor's comment:* Most solvers used either the sine law or the cosine law. Here is an example of the former for the case when the middle triangle is the smallest or, in Doster's words, when pentagon $APQRD$ is not convex.]

In this case, $\angle CQR = 60^\circ+\theta$, $\angle CRQ = 60^\circ-\theta$. By the law of sines we have

$$\frac{\sin\theta}{\sin(120^\circ-\theta)}=\frac{QB}{PB}=\frac{b}{a}, \quad \frac{\sin(60^\circ-\theta)}{\sin(60^\circ+\theta)}=\frac{QC}{RC}=\frac{b}{c}.$$

Since $\sin(120^\circ-\theta)=\sin(60^\circ+\theta)$ and $\cos(30^\circ-\theta)=\sin(60^\circ+\theta)$, we have

$$\frac{b}{a}+\frac{b}{c}=\frac{\sin\theta+\sin(60^\circ-\theta)}{\sin(60^\circ+\theta)}=\frac{2\sin 30^\circ\cos(30^\circ-\theta)}{\sin(60^\circ+\theta)}=1.$$

Therefore, $\dfrac{1}{b}=\dfrac{1}{a}+\dfrac{1}{c}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, Berlin, Germany; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; C. DIXON, Newcastle upon Tyne, England; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; J.K. FLOYD, Newnan, Georgia; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO CHRISTOV KUNCHEV, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; NEIL REID, Mississauga, Ontario; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands(2 solutions); PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. One solution was submitted without a name attached, and one solution was flawed. Several solvers dealt with only one case, but their method was sufficient to deal with the omitted case had they considered it.

A few solvers combined the cases in the pretty, but less informative relation

$$abc = (a-b)(c-b)(a+b).$$

2092. [1995: 343] *Proposed by K.R.S. Sastry, Dodballapur, India.*

I take a three-digit base-ten integer (in which the first digit is non-zero) and consider it as a number in a different base. If I convert this new number into base ten, I find that it is exactly twice the original number. In what base does this happen?

Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.

Let k be the desired base. Then

$$ak^2 + bk + c = 2(100a + 10b + c) \iff a(k^2 - 200) + b(k - 20) = c$$

with $a \in \{1, 2, \ldots, 9\}$ and $b, c \in \{0, 1, \ldots, 9\}$.

If $k \leq 14$ then $a(k^2 - 200) + b(k - 20) \leq -4a - 6b < 0$, which is false.

If $k \geq 16$ then $a(k^2 - 200) + b(k - 20) \geq 56a - 4b \geq 20$, which is also false.

If $k = 15$ then

$$25a - 5b = c \Rightarrow c \in \{0, 5\} \Rightarrow 5a - b \in \{0, 1\}.$$

So $k = 15$, and $(\overline{150})_{15} = 300$, $(\overline{145})_{15} = 290$, $(\overline{295})_{15} = 590$ are all the solutions.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; TIM CROSS, King Edward's School, Birmingham, England; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KATHLEEN E. LEWIS, SUNY Oswego. Oswego, NY, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; JOHN GRANT McLOUGHLIN, Okanagan University College, Kelowna, British Columbia; P. PENNING, Delft, the Netherlands; HRISTOS SARAGHIOTES, student, Aristotle University of Thessaloniki, Greece; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; LAWRENCE SOMER, Catholic University of America, Washington, DC; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There were four incorrect solutions.

Most solvers had variations on the above solution. Some did not find all the solutions, but were still able to identify the base. STONE asks the opposite question as well: what is the answer if you interchange the roles of 10 *and the unknown base, and provides the answer, which we will leave to the reader. Indeed, several of the incorrect solutions treated this problem instead of the one above. JANOUS and GRANT McLOUGHLIN both consider a generalization where the factor above is replaced by larger positive integers. JANOUS has written a short computer program to look for small examples and found many. GRANT McLOUGHLIN also observed that* 121 *was a solution for the factor* 4 *with base* 21, 441 *was a solution for the factor* 9 *with base* 31, *and* 961 *was a solution for the factor* 16 *with base* 41*; is this extendable? JANOUS further asks:*

1. Let n be an arbitrary natural number in base 10 representation $(n)_{10}$. Does there exist necessarily a further base $b \geq 11$ such that $(n)_b$ is an integer multiple of $(n)_{10}$?

2. Let n be a natural number satisfying the above question in the positive. Do there exist infinitely many such bases b? If "yes", what is their density $d(n)$ in the set of all natural numbers?

3. Does there exist for any base $b \geq 11$ a natural number n such that $(n)_b$ is an integer multiple of $(n)_{10}$?

2093⋆. [1995: 343] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, C be the angles (in radians) of a triangle. Prove or disprove that

$$(\sin A + \sin B + \sin C)\left(\frac{1}{\pi - A} + \frac{1}{\pi - B} + \frac{1}{\pi - C}\right) \leq \frac{27\sqrt{3}}{4\pi}\,.$$

Solution by Kee-Wai Lau, Hong Kong.

The inequality is true.

Let $X = \dfrac{\pi - A}{2}$, $Y = \dfrac{\pi - B}{2}$, $Z = \dfrac{\pi - C}{2}$, so that X, Y, Z are also the angles of a triangle. So, we have

$$\begin{aligned}
&(\sin A + \sin B + \sin C)\left(\frac{1}{\pi - A} + \frac{1}{\pi - B} + \frac{1}{\pi - C}\right) \\
= \;& 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)\left(\frac{1}{\pi - A} + \frac{1}{\pi - B} + \frac{1}{\pi - C}\right) \\
= \;& 2\sin X \sin Y \sin Z\left(\frac{1}{X} + \frac{1}{Y} + \frac{1}{Z}\right) \\
= \;& \frac{\sin X}{X}\,\frac{\sin Y}{Y}\,\frac{\sin Z}{Z}\left\{(X+Y+Z)^2 - (X^2+Y^2+Z^2)\right\} \\
\leq \;& \frac{\sin X}{X}\,\frac{\sin Y}{Y}\,\frac{\sin Z}{Z}\left\{(X+Y+Z)^2 - \frac{(X+Y+Z)^2}{3}\right\}
\end{aligned}$$

by the arithmetic mean inequality

$$\begin{aligned}
= \;& \frac{2\pi^2}{3}\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\,\frac{\sin Z}{Z} \\
= \;& \frac{2\pi^2}{3}\left(\sqrt[3]{\frac{\sin X}{X}\,\frac{\sin Y}{Y}\,\frac{\sin Z}{Z}}\right)^3 \\
\leq \;& \frac{2\pi^2}{3}\left(\frac{\dfrac{\sin X}{X} + \dfrac{\sin Y}{Y} + \dfrac{\sin Z}{Z}}{3}\right)^3, \qquad (1)
\end{aligned}$$

by the arithmetic mean–geometric mean inequality.

According to ***CRUX*** 1216 [1987: 53, 1988: 120], we have

$$\frac{\sin X}{X} + \frac{\sin Y}{Y} + \frac{\sin Z}{Z} \leq \frac{9\sqrt{3}}{2\pi}. \qquad (2)$$

The required inequality now follows immediately from (1) and (2).

Also solved by THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA and BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA. One incorrect solution was received.

2094. [1995: 343] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

The problemist Victor Thébault has noted (*American Mathematical Monthly* Vol. **66** (1959), p. 65) an interesting Pythagorean triangle in which the two perpendicular sides are integers having the same digits in reverse order, *viz.*, 88209 and 90288, with hypotenuse 126225.

(a) Can such a Pythagorean triangle be primitive?

(b) Find an example of a primitive Pythagorean triangle in which the hypotenuse and one other side are integers having the same digits in reverse order.

Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY.

(a) No, it cannot be primitive. If a and b have the same digits in reverse order, then they are congruent modulo 3. If they were not divisible by 3, they would be congruent to 1 or 2, so their squares would have to be congruent to 1 mod 3. Thus, if $c^2 = a^2 + b^2$, then c^2 would have to be congruent to 2 mod 3. But no perfect square is congruent to 2. Thus a and b are both divisible by 3 and so are not relatively prime.

(b) One example of such a triple is 33, 56, 65.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; RICHARD I. HESS, Rancho Palos Verdes, California, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; P. PENNING, Delft, the Netherlands; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer.

Solutions to part (a) only were received from JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; and to (b) only from CHARLES ASHBACHER, Cedar Rapids, Iowa, USA.

One reader pointed us to another reference to Thébault's type of Pythagorean triangle in the Reader Reflections section of the Mathematics Teacher [Vol. 82, No. 1, January 1989, p. 8]. The first additional such triple given there with legs 125928 *and* 829521 *is Pythagorean (but of course not primitive); however, the second pair of numbers,* 725068 *and* 860527 *cannot be the legs of a Pythagorean triangle since the unit digit of* $(725068)^2 + (860527)^2$ *is* 3, *while perfect squares can only have unit digits of* 0, 1, 4, 5, 6 *or* 9.

One other such triangle "with hypotenuse of 1164481*" (no legs given) is mentioned in this article. Is this really the hypotenuse of such a triangle and, if so, what are the legs?*

Hess provides another (non-primitive) triple of Thébault's type with legs 2328480 *and* 0848232.

The type of triple in (b) seems to be quite rare; Penning reports that "going up to numbers above 10000*, no others solutions were found". All primitive Pythagorean triples have the form* $(2mn, m^2-n^2, m^2+n^2)$*, where* m, n *are relatively prime and of opposite parity. Hess reports no other (b) type triples were found for* $n < m \leq 999$.

2095. [1995: 344] *Proposed by Murray S. Klamkin, University of Alberta.*

Prove that

$$a^x(y-z) + a^y(z-x) + a^z(x-y) \geq 0$$

where $a > 0$ and $x > y > z$.

Solution: a shortened form of that submitted by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

The solution is an immediate consequence of the convexity of $f(x) = a^x$, $x > 0$, $x \in \mathbb{R}$.

In Mitrinović's "Elementary Inequalities" (North-Holland, Holland, 1964), pages 22–33, in the section on Jensen's Inequality, we have the following geometric interpretation — an equivalent characterization of the convexity of a function:

> *Consider* x_1, x_2, $x_3 \in [\alpha, \beta]$*, where* $x_1 < x_2 < x_3$*, and the corresponding functional values* $f(x_1)$, $f(x_2)$, $f(x_3)$*. The area of the triangle with coordinates* $(x_1, f(x_1))$, $(x_2, f(x_2))$, $(x_3, f(x_3))$*, is given by* $\pm P$*, where*
>
> $$P = \frac{1}{2}\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix}.$$
>
> *Then* f *is convex if* $P > 0$*, and* f *is concave if* $P < 0$.

We write $z = x_1$, $y = x_2$, $x = x_3$ and $f(x) = a^x$, $x > 0$, $x \in \mathbb{R}$, and the result follows.

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MANSUR BOASE, student, St. Paul's School, London, England; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; ROBERT GERETSCHLÄGER, *Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta, NY, USA; CORY PYE, student, Memorial University, St. John's, Newfoundland; JOEL SCHLOSBERG; student, Hunter College, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; JOHANNES WALDMANN, Friedrich–Schiller–Universität, Jena, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.*

Klamkin comments that the problem is given in S. Barnard and J.M. Child's "Higher Algebra" (MacMillan, London, 1949, p. 226) without the constraint condition $x > y > z$. He notes (as observed in the highlighted solution) that the inequality is reversed if $x > z > y$.

Some of our "senior" solvers knew the more general result, whereas our "junior" solvers tended to prove the result ab initio.

Janous asks how this result can be generalized to four or more variables. [Ed: I will leave it thus and so challenge our readers to generalize it themselves. Please send me your "nice" generalizations.]

2096. [1995: 344] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle $A_1A_2A_3$ has circumcircle Γ. The tangents at A_1, A_2, A_3 to Γ intersect (the productions of) A_2A_3, A_3A_1, A_1A_2 respectively in B_1, B_2, B_3. The second tangent to Γ through B_1, B_2, B_3 touches Γ at C_1, C_2, C_3 respectively. Show that A_1C_1, A_2C_2, A_3C_3 are concurrent.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

By Pascal's theorem, B_1, B_2, B_3 are collinear. The polars of B_1, B_2, B_3 with respect to Γ are therefore concurrent. A_1C_1 is the polar of B_1, A_2C_2 that of B_2, and A_3C_3 that of B_3. Hence A_1C_1, A_2C_2, A_3C_3 are concurrent at the pole of $B_1B_2B_3$

Comment by Chris Fisher.

To avoid invoking Pascal's theorem, you can assume, without loss of generality, that the given triangle is equilateral (because the projective group fixing a conic is transitive on the triangles), in which case, the problem reduces to an easy observation.

Solved using projective geometry by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; and JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA. Solved without using projective geometry by FRANCISCO BELLOT ROSADO, *I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HOE TECK WEE, student, Hwa Chong Junior College, Singapore: and the proposer.*

2097. [1995: 344] *Proposed by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.*

Let n be a positive integer and p a prime number. Prove that

$$p^n(p^n-1)(p^{n-1}-1)\dots(p^2-1)(p-1)$$

is divisible by $n!$.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

The desired result actually holds for all positive integers p.

Let $A_p(n)$ denote the given product. From a well-known formula, we have $n! = \prod_{q \text{ prime}} q^{e_q}$, where $e_q = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{q^j} \right\rfloor$.
[Ed: Seiffert called this "de Polignac's formula".]

Let q be a prime. Since $e_q \le \left\lfloor \sum_{j=1}^{\infty} \frac{n}{q^j} \right\rfloor = \left\lfloor \frac{n}{q-1} \right\rfloor =: a_q$, it suffices to show that q^{a_q} divides $A_p(n)$. This is clearly true if q divides p. Otherwise, q divides $p^{k(q-1)}-1$ for all $k \in \mathbb{N}$ by Fermat's Little Theorem. However, $\prod_{k=1}^{a_q} \left(p^{k(q-1)}-1\right)$ is a factor of $A_p(n)$ since $a_q(q-1) \le n$. Our claim follows.

Also solved by MANSUR BOASE, student, St. Paul's School, London, England; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

Most submitted solutions were similar to the one given above. Besides Seiffert, only Chronis and Hess noted that the statement holds for all positive integers. Janous remarked that the result can be sharpened to: $n! | A_p(n-1)$, *or equivalently,* $(n+1)! | A_p(n)$.

2098. [1995: 344] *Proposed by John Magill, Brighton, England.*

At the conclusion of our first inter-species soccer tournament, in which each team played each of the others once, the scoresheets, prepared by the Zecropians and the Valudians, were, respectively,

	Won	Drawn	Lost	Goals against	Goals for	Points
Zecropia	c	b	b	cc	ffh	d
Earth Station	b	b	c	fbe	ff	b
Valudia	f	b	f	ah	db	c

	Won	Drawn	Lost	Goals against	Goals for	Points
Valudia	p	x	p	pxq	ppr	r
Zecropia	r	x	x	mm	mxr	q
Earth Station	x	x	r	rqp	pq	x

Each scoresheet is equivalent to the other in that both give the correct values. Each, however, is in the fixed-base positional number systems of those who prepared the scoresheets, each base being less than 10 and greater than 1. Both Valudians and Zecropians use the same operations of addition, subtraction, division and multiplication, and rules of manipulation, as are used by Earth. I have substituted letters for the symbols originally used. Each letter represents a digit, the same digit wherever it appears. Two points were awarded for a win and one for a draw.

As the answer to this puzzle, state the total number of goals scored by each team, and the total number of goals scored against each team — in the base 10 number system.

Solution by Kathleen E. Lewis, SUNY Oswego. Oswego, NY, USA.

Looking at the scoresheet prepared by the Zecropians, we see that Earth won b games and drew b games and so should have $3 \times b$ points. Since Earth has b points, b must be zero. Each team played two games, so $c = 2$ and $f = 1$. Zecropia won two games and got d points so $d = 4$. On the Valudians' scoresheet, we get the corresponding values: $x = 0, p = 1, r = 2$, and $q = 4$.

Let s denote the base used by the Zecropians and t the base used by the Valudians. Comparing the goals columns on the two scoresheets, we find that the number of goals scored for Earth Station is equal to $(11)_s$ and $(14)_t$, so $s + 1 = t + 4$. Comparing the goals scored for Valudia, we see that $(40)_s = (112)_t$, so $t^2 + t + 2 = 4s = 4(t + 3)$. Therefore $t = 5$ and $s = 8$.

Zecropia had $(22)_8 = 18$ goals scored against it, so $(\mathrm{mm})_5 = 18$, and thus m is 3. The number of goals scored for Zecropia equals $(302)_5 = 77$, so $(11h)_8 = 77$, and h must be 5. The number of goals against Valudia comes out to $(104)_5 = 29$, so $(a5)_8 = 29$, and $a = 3$.

The scoresheet prepared by the Earthlings will therefore look like:

	Won	Drawn	Lost	Goals against	Goals for	Points
Earth Station	0	0	2	71	9	0
Valudia	1	0	1	29	32	2
Zecropia	2	0	0	18	77	4

Also solved by MANSUR BOASE, student, St. Paul's School, London, England; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; JEFFREY K. FLOYD, Newnan, Georgia, USA; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; HRISTOS SARAGHIOTES, student, Aristotle University of Thessaloniki, Greece; and the proposer.

Geretschläger suggests that Earth Station's dismal performance could be "an early example of the Prime Directive at work, as in not interfering (too much) between other worlds' goalposts"! Another explanation might be found in the bases 5 *and* 8*, which suggest that Valudians probably have a total of* 5 *toes and Zecropians a total of* 8 *toes. Assuming a constant number of toes per foot, we could conclude that Valudians most likely have five (one-toed) legs and that Zecropians may very well have four or even eight legs, which would make both teams a real handful (so to speak) for the underlegged Earthlings.*

2099. [1995: 345] *Proposed by* ***Proof****, Warszawa, Poland.*

The tetrahedron T is contained inside the tetrahedron W. Must the sum of the lengths of the edges of T be less than the sum of the lengths of the edges of W?

Solution by Nikolai Dolbilin, Fields Institute, Toronto, Ontario and the Steklov Mathematical Institute, Moscow, Russia.

Let W have a "small" triangular base and a summit that is "far" from its base. Let T have two vertices inside W that are "close" to the base, and two inside W that are "close" to the summit.

Then T has four "long" edges while W has only three "long" edges. The parameters can therefore be adjusted so that T has a longer edge-sum.

Also solved by ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; VICTOR OXMAN, University of Haifa, Haifa, Israel; GOTTFRIED PERZ, Pestalozzi-gymnasium, Graz, Austria; and the proposer.

2100. [1995: 345] *Proposed by Iliya Bluskov, student, Simon Fraser University, Burnaby, BC.*

Find 364 five–element subsets $A_1, A_2, \dots, A_{364}$ of a 17-element set such that $|A_i \cap A_j| \leq 3$ for all $1 \leq i < j \leq 364$.

Solution by the proposer.

Without loss of generality, let $B = \{0, 1, 2, \ldots, 16\}$ be the 17–element set. Let $X^{(5)}$ be the family of all 5–element subsets of B. Then we have $\left|X^{(5)}\right| = \binom{17}{5}$. For each $k = 0, 1, 2, \ldots, 16$, define

$$X_k^{(5)} = \left\{\{a_1, a_2, a_3, a_4, a_5\} \in X^{(5)} \,\middle|\, \sum_{i=1}^{5} a_i \equiv k \pmod{17}\right\}.$$

Clearly, the classes $X_k^{(5)}$, $k = 0, 1, 2, \ldots, 16$, are disjoint, and hence, by the pigeon-hole principle, at least one of the classes must contain no less than $\binom{17}{5}/17 = 364$ "elements".

It remains to show that $|A_i \cap A_j| \leq 3$ for all distinct 5–element subsets A_i and A_j which belong to the same class. Clearly, it suffices to show that $|A_i \cap A_j| \neq 4$. This is obvious since, if $A_i = \{a_1, a_2, a_3, a_4, a_5\}$ and $A_j = \{b_1, b_2, b_3, b_4, b_5\}$ are such that $a_i = b_i$ for $i = 1, 2, 3, 4$, then $\sum_{i=1}^{5} a_i \equiv \sum_{i=1}^{5} b_i \pmod{17}$ would imply that $a_5 \equiv b_5 \pmod{17}$, which is impossible unless $a_5 = b_5$. This completes the proof.

Also solved by Kee-Wai Lau, Hong Kong, who exhibited a family of 365 5–*element subsets satisfying the given condition. He obtained this example with the help of a* PC *and a program using* **GW BASIC**.

Proposer's remarks. This is not a new problem, but (hopefully) it is not too well-known. In general, if we let M denote the maximum possible cardinality of a family $\mathcal{F}$ of m–element subsets of an n–element set, with the condition that $|A_i \cap A_j| \leq m - 2$ for all $A_i, A_j \in \mathcal{F}$ such that $A_i \neq A_j$, then using the same argument, one can easily see that $M \geq \left\lceil \frac{\binom{n}{m}}{n} \right\rceil$, where $\lceil x \rceil$ denotes the least integer greater than or equal to x. This result is due to Graham and Sloane [1]. For the present case ($n = 17$, $m = 5$), the estimate has been improved to $M \geq 424$ [2]. On the other hand, it is known that $M \leq 476$. Finding a structure with $M = 476$ would resolve a long-standing open question in design theory: the existence of a Steiner system $S(4, 5, 17)$.

References

[1] R.L. Graham and N.J.A. Sloane, Lower Bounds for Constant Weight Codes, *IEEE Transactions on Information Theory*, IT–26, No. 1 (1980), 37–43.

[2] A.E. Brouwer, J.B. Shearer, N.J.A. Sloane and W.D. Smith, A New Table of Constant Weight Codes, *IEEE Transactions on Information Theory*, Vol. 6, No. 6 (1990), 1334–1380.

Problemist of the Year!

In the index to volume 22, you will find a listing of all those who have participated in the **Solutions** section of ***CRUX*** in 1996. Based on these statistics, we are pleased to declare

CHRISTOPHER J. BRADLEY

as ***CRUX*** **Problemist of the Year for 1996**.

A close runner up is

WALTHER JANOUS

The following (in alphabetical order) deserve an *Honourable Mention* since each has participated in at least one third of the solutions in 1996:

CARL BOSLEY	*RICHARD I. HESS*
VÁCLAV KONEČNÝ	*KEE-WAI LAU*
P. PENNING	*HEINZ-JÜRGEN SEIFFERT.*

We offer our thanks and congratulations to all our participants.

Readers will note that we have now published solutions to all problems up to 2100.

YEAR END FINALE

How a year can fly by! It only seems like yesterday that I was persuaded to take on the mammoth task of being Editor-in-Chief of ***CRUX***. Yet it has been a job of great satisfaction, getting to know better many of the world's keenest mathematical problem solvers.

CRUX has gone **on-line**! This is an exciting development and has led to renewed interest and new subscribers. Thanks are due to LOKI JORGENSON, NATHALIE SINCLAIR, and the rest of the team at SFU who are responsible for this. As mentioned earlier in this issue, ***CRUX*** and ***Mathematical Mayhem*** have joined forces. I am delighted to welcome NAOKI SATO and CYRUS HSIA to the reconstituted Editorial Board. I have known both these young men for many years and have had the privilege or working with them when they were on Canada's IMO team.

There are many people that I wish to thank most sincerely for particular contributions. First and foremost is BILL SANDS. Bill is of such value to me and to the continuance of ***CRUX*** that his code name in my computer files is *CRUXWISE*. As well, I thank most sincerely, CATHY BAKER, ROLAND EDDY, CHRIS FISHER, BILL SANDS, JIM TOTTEN, and EDWARD WANG, for their regular yeoman service in assessing which solutions should be highlighted; DENIS HANSON, HARVEY ABBOTT, DOUG FARENICK, CHRIS FISHER, ALLEN HERMAN, MURRAY KLAMKIN, JOANNE MCDONALD, JUDI MCDONALD, RICHARD MCINTOSH, DIETER RUOFF, JIM TOMKINS, and MICHAEL TSATSOMEROS, for ensuring that we have quality articles; ANDY LUI, RICHARD GUY, KATHERINE HEINRICH, CLAUDE LAFLAMME, MURRAY KLAMKIN, MARCIN KUCZMA, and JACK MACKI, for ensuring that we have quality book reviews, ROBERT WOODROW (and JOANNE LONGWORTH), who carries the heavy load of two corners, one somewhat new and the other of long standing, and RICHARD GUY for sage advice whenever necessary. The quality of these people are vital parts of what makes ***CRUX*** what it is. Thank you one and all.

As well, I would like to give special thanks to our retiring Associate Editor, COLIN BARTHOLOMEW, for keeping me from printing too many typographical errors; and my colleagues, ERIC JESPERS, P.P. NARAYANASWAMI, MAURICE OLESON, MIKE PARMENTER and DONALD RIDEOUT for their occasional sage advice. I have also been helped by some Memorial University students, RON HAYNES, KIM PENDERGAST and CORY St. CROIX, as well as a WISE Summer student, CINDY HISCOCK. The staff of the Department of Mathematics and Statistics at Memorial University deserve special mention for their excellent work and support: ROS ENGLISH, MENIE FRENCH, WANDA HEATH, LEONCE MORRISSEY and KAY SCOTT; as well as the computer and networking expertise of RANDY BOUZANE. Not to mention GRAHAM WRIGHT, Managing Editor, would be a travesty. Graham has kept so much on the right track. He is a pleasure to work with. The CMS's $\TeX$ Editor, MICHAEL DOOB has been very helpful in ensuring that the printed master copies are up to the standard required for the U of T Press who continue to print a fine product. Finally, I would like to express real and heartfelt thanks to my Head of Department, BRUCE WATSON, and the Dean of Science of Memorial University, ALAN LAW, without whose support and understanding, I would not be able to try to do the job of Editor-in-Chief.

Last but not least, I send my thanks to you, the readers of ***CRUX***. Without you, ***CRUX*** would not be what it is. Keep those contributions and letters coming in. I do enjoy knowing you all.

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Polly Kontopoulou: 2076;
Jisho Kotani: 2042, 2053;
Hiroshi Kotera: 2026;
Marcin E. Kuczma: 2010, 2018, 2028, 2048, 2057, 2080;
Mitko Christov Kunchev: 2055, 2056, 2067, 2068, 2069, 2091;
Sai C. Kwok: 2032, 2064, 2068;
Jari Lappalainen: 2041;
Kee-Wai Lau: 2010, 2011, 2016, 2020, 2021, 2022, 2027, 2029, 2031, 2032, 2034, 2035, 2037, 2050, 2054, 2056, 2060, 2062, 2064, 2065, 2068, 2070, 2072, 2073, 2074, 2075, 2076, 2077, 2078, 2079, 2080, 2082, 2087, 2088, 2089, 2090, 2091, 2093, 2095, 2097, 2100;
Thomas Leong: 2064, 2068, 2070;
Katheleen E. Lewis: 2034, 2042, 2045, 2081, 2092, 2094, 2098;
David Lindsey: 2081;
Andy Liu: 2070;
Gary MacGillivray: 2085;
Maria Ascensión López Chamorro: 2008, 2016, 2035, 2038, 2045, 2050, 2051, 2053, 2056, 2059, 2063, 2071;
John Magill: 2066, 2076, 2098;
David E. Manes: 2007, 2016, 2020, 2066, 2068, 2081, 2083, 2084, 2088, 2090, 2092, 2094, 2095, 2098;
Beatriz Margolis: 2020, 2050, 2062, 2068;
J.A. McCallum: 2009, 2014, 2016, 2022, 2050, 2066, 2068, 2074, 2076, 2081;
John Grant Mcloughlin: 2066, 2092;
Bill Meisel: 2022;
Stewart Metchette: 2022, 2074;
P. Molenbroek: 2017;
Pieter Moree: 2007;
Vedula N. Murty: 2023, 2056, 2060, 2064, 2067, 2068;
J.E. Oliver: 2016;
Victor Oxman: 2031, 2032, 2036, 2038, 2067, 2099;
Michael Parmenter: 2022, 2042, 2052, 2058;
Dan Pedoe: 2017;
P. Penning: 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2027, 2030, 2035, 2038, 2040, 2042, 2043, 2045, 2047, 2055, 2059, 2060, 2071, 2072, 2074, 2075, 2076, 2079, 2081, 2082, 2083, 2086, 2088, 2089, 2091, 2094, 2096;
Gottfried Perz: 2007, 2016, 2017, 2032, 2036, 2038, 2041, 2042, 2045, 2055, 2060, 2061, 2063, 2064, 2066, 2067, 2068, 2069, 2075, 2076, 2081, 2096, 2099;
Waldemar Pompe: 2008, 2013, 2023, 2047, 2050, 2061, 2063, 2064, 2067, 2069, 2070, 2089;
Bob Prielipp: 2015, 2064, 2068, 2083, 2093;
***Proof*:** 2099;
Cory Pye: 2060, 2062, 2066, 2068, 2081, 2088, 2095;
Stanley Rabinowitz: 2046, 2056, 2065, 2074, 2083;
Neol Reid: 2091;
Juan-Bosco Romero Márquez: 2068, 2078, 2095;
Jawad Sadek: 2030;
Cristóbal Sánchez–Rubio: 2035, 2038, 2043, 2062, 2063, 2064, 2067, 2068, 2069, 2079, 2081, 2088;
Bill Sands: 1823, 2009;
Hristos Saraghiotes: 2076, 2088, 2092, 2098;
K.R.S. Sastry: 2012, 2022, 2033, 2042, 2052, 2062, 2068, 2072, 2081, 2092;
Joel Schlossberg: 2091, 2092, 2094, 2095, 2096, 2097;
Robert P. Sealy: 2009, 2074, 2080, 2088;
Harry Sedinger: 2070;
Heinz-Jürgen Seiffert: 2009, 2012, 2016, 2020, 2028, 2030, 2031, 2032, 2034, 2034, 2037, 2038, 2045, 2050, 2052, 2060, 2062, 2064, 2065, 2067, 2068, 2070, 2072, 2073, 2074, 2076, 2077, 2078, 2081, 2083, 2084, 2085, 2088, 2090, 2092, 2095, 2097;
Toshio Seimiya: 2008, 2011, 2017, 2021, 2027, 2031, 2033, 2038, 2041, 2043, 2044, 2047, 2051, 2055, 2059, 2061, 2063, 2067, 2069, 2069, 2071, 2073, 2075, 2082, 2083, 2089, 2091;
Catherine Shevlin: 2025;
Koji Shoda: 2026;
Ashish Kr. Singh: 2008, 2021, 2027, 2034, 2043, 2045, 2047, 2048, 2062, 2064, 2067, 2068, 2069;
D.J. Smeenk: 2008, 2010, 2017, 2019, 2021, 2027, 2031, 2033, 2035, 2041, 2043, 2047, 2055, 2069, 2071, 2075, 2079, 2082, 2089, 2091, 2096;
Digby Smith: 2032, 2050, 2052, 2056, 2068, 2072, 2076, 2078, 2083;
Lawrence Somer: 2012, 2092;
Frederick Stern: 2040;
David R. Stone: 2016, 2022, 2040, 2042, 2045, 2050, 2072, 2074, 2076, 2081, 2092, 2094, 2097;
Moshe Stupel: 2067;
M.V. Subbarao: 2034;
David Tascione: 2076;
Panos E. Tsaoussoglou: 2014, 2016, 2022, 2031, 2032, 2036, 2038, 2042, 2044, 2045, 2050, 2056, 2058, 2060, 2062, 2064, 2067, 2068, 2071, 2072, 2074, 2076, 2081, 2082, 2083, 2084, 2091, 2095;
Sofiya Vasina: 2028;
David C. Vella: 2081;
Dan Velleman: 2006;
John Vlachakis: 2016, 2020;
Stan Wagon: 2006, 2050;
Johannes Waldmann: 2064, 2073, 2095;
Edward T.H. Wang: 2016, 2032, 2064, 2068, 2078;
Hoe Teck Wee: 2008, 2010, 2021, 2022, 2023, 2027, 2028, 2031, 2032, 2038, 2041, 2042, 2048, 2054, 2059, 2071, 2080, 2096;
Chris Wildhagen: 2007, 2008, 2012, 2014, 2016, 2022, 2032, 2038, 2045, 2050, 2060, 2062, 2064, 2066, 2068, 2070, 2072, 2074, 2076, 2077, 2080, 2081, 2087, 2088, 2090, 2095, 2097;
Susan Schwartz Wildstrom: 2042, 2050, 2066, 2068;
Kenneth M. Wilke: 2062, 2066, 2068, 2070, 2072, 2074, 2076, 2081, 2092;
John B. Wilker: 2030;
Kenneth S. Williams: 2097;
Ana Witt: 2063;
Aram A. Yagubyants: 2043, 2063, 2086;
Paul Yiu: 2037, 2040, 2042, 2060, 2065, 2067, 2068, 2070, 2072, 2074, 2094;
Joseph Zaks: 2077;
Dong Zhou: 2039;

Austin Academy Problem Solvers: 2021;
Science Academy Problem Solvers: 2022, 2027, 2032, 2035, 2038, 2040;
Skidmore College Problem Group: 2070;
University of Arizona Problem Solving Lab: 2026;

***anonymous*:** 2064, 2065, 2067, 2068, 2069, 2071, 2077.

We would like to identify ***anonymous***! Thank you.