Letter from the Editor

Changes to *CRUX*

Readers will be interested to learn about some of the changes that are in the works for ***CRUX***.

First, ***CRUX*** is now available as an **on-line** supplement for individual subscribers at

http://camel.math.ca/CRUX

This new service was instituted at the beginning of September, and is proving to be very popular. A survey of the traffic accessing ***CRUX*** **on-line** shows that many people are surfing to take a look, and as a result, we have several new subscriptions.

Information about each issue will be available to anyone who accesses the ***CRUX*** **on-line** site. As well as seeing the table of contents, a two page synopsis is also available. This gives a short summary of what is in each published article, as well as some highlights of what is in the various corners. One of the problems is given as a "free sample". This is to try to encourage more problem solvers to become involved.

When individual subscribers renew their subscription for 1997, they will receive information from the CMS Executive Office on how they can activate their account on CAMEL and gain access to ***CRUX*** **on-line**. In the future, ***CRUX*** **on-line** will be an extra free service to individuals with a normal (hard copy) subscription. As well, an **on-line only** service (no hard copy) will be available. The rates will be published in the next issue of ***CRUX***.

Since ***CRUX*** is copyrighted by the Canadian Mathematical Society, there are restrictions on the use of the **on-line** version (as there is for the printed version). Here is the statement that appears **on-line**:

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Also, in 1997, ***CRUX*** is proposing to increase the quantity of high school level materials in each issue. This will not involve any decrease in the quantity of each of the present sections. We are proposing to increase the issue size to 64 pages (at no increase in the subscription price over that planned for 1997 based on 48 pages per issue). Thus, subscribers, especially those who have connections with high schools, would get a lot more for their money. Watch for a further announcement in the next issue.

Bruce Shawyer
Editor-in-Chief

A Note on the Mean Value Theorem

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Every student of Calculus knows that given two distinct points A, B on the graph of a smooth function there is a point C on the arc of the curve joining A and B where the tangent to the graph is parallel to the chord joining A and B. Moreover, it is easy to construct examples where the number of such points C is either finite, countably infinite or uncountable; see the exercises.

The purpose of this note is to show that if the function is *convex*, then there are only two possibilities: namely, either there is a unique point or uncountably many such points.

We recall the definition of convexity. Geometrically it means that the portion of the graph joining any two points on it must lie below the chord joining them. Analytically, f is convex on an interval $[a, b]$, with $a < b$, if

$$f(x) \leq f(s) + \frac{f(t) - f(s)}{t - s}(x - s), \text{whenever } a \leq s \leq x \leq t \leq b.$$

The result we wish to establish depends on a lemma which we will deal with first.

Lemma 1 Suppose f is convex on $[a, b]$ and differentiable at $c \in (a, b)$. Then

$$f(c) + f'(c)(x - c) \leq f(x), \ \forall\, x \in [a, b].$$

Proof. To see this, suppose $c < t < x \leq b$. Then

$$f(t) \leq f(c) + \frac{f(x) - f(c)}{x - c}(t - c).$$

Hence

$$\frac{f(t) - f(c)}{t - c} \leq \frac{f(x) - f(c)}{x - c}$$

and so, letting $t \to c^+$, we deduce that

$$f'(c) \leq \frac{f(x) - f(c)}{x - c},$$

that is, $(x-c)f'(c) \leq f(x)-f(c)$ which proves the desired result for $x > c$. Next, if $a \leq x < t < c$, then

$$\begin{aligned} f(t) &\leq f(x) + \frac{f(c)-f(x)}{c-x}(t-x) \\ &= f(x) + \frac{f(c)-f(x)}{c-x}(t-c+c-x) \\ &= f(x) + \frac{f(c)-f(x)}{c-x}(t-c) + f(c) - f(x) \\ &= f(c) + \frac{f(x)-f(c)}{x-c}(t-c). \end{aligned}$$

Hence

$$\frac{f(t)-f(c)}{t-c} \geq \frac{f(x)-f(c)}{x-c}$$

and so, letting $t \to c^-$, we deduce that

$$f'(c) \geq \frac{f(x)-f(c)}{x-c},$$

that is, $(x-c)f'(c) \leq f(x)-f(c)$ which proves the desired result for $x < c$. This completes the proof of the lemma since its conclusion certainly holds when $x = c$.

This result means that the graph of $y = f(x)$ lies above the tangent line $y = f(c) + f'(c)(x-c)$.

Theorem 1 Let $-\infty < a < b < \infty$. Suppose f is continuous and convex on $[a, b]$, and differentiable on (a, b). Then the equation

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

has either a unique solution in (a, b) or infinitely many solutions.

Proof. The Mean Value Theorem guarantees that the solution set is non-empty. Suppose c_1, c_2 are two distinct solutions in (a, b), so that

$$f'(c_1) = \frac{f(b)-f(a)}{b-a} = f'(c_2).$$

We can and do suppose that $c_1 < c_2$. It follows from the lemma that

$$f(c_1) + f'(c_1)(c_2 - c_1) \leq f(c_2)$$

and

$$f(c_2) + f'(c_2)(c_1 - c_2) \leq f(c_1).$$

Hence

$$f'(c_1) = f'(c_2) = \frac{f(c_2) - f(c_1)}{c_2 - c_1}.$$

But, by convexity, the line joining the two points $(c_1, f(c_1)), (c_2, f(c_2))$, lies above the graph of f that is, we have

$$f(x) \leq f(c_1) + f'(c_1)(x - c_1), \ \forall x \in [c_1, c_2].$$

Applying the lemma once more, we deduce that

$$f(x) = f(c_1) + f'(c_1)(x - c_1), \ \forall \ x \in [c_1, c_2].$$

In other words, $f'(x) = (f(b) - f(a))/(b - a)$ for all $x \in [c_1, c_2]$. It follows that there are infinitely many points with this property as claimed.

Exercises

1. Concoct examples to show that the number of points on the arc of a smooth curve where the tangent to the curve is parallel to the chord joining the ends of the arc can be (i) finite; (ii) uncountable.

2. Show that, for every non-negative integer n, the equation

$$\tan x = \frac{x + n\pi}{3}$$

has a unique solution on the interval $[0, \pi/2)$.

3. Consider the function g defined on $[-1, 1]$ as follows:

$$g(x) = \begin{cases} x^3 \sin(1/x), & \text{if } 0 < |x| \leq 1, \\ 0, & \text{if } x = 0. \end{cases}$$

Show that g is continuous on $[-1, 1]$ and has a continuous derivative on $(-1, 1)$. Using the previous exercise, or otherwise, show that there is a countably infinite number of points on the graph of $y = g(x)$ where the tangent is parallel to the chord joining the points $(-1, g(-1))$ and $(1, g(1))$.

4. Deduce, and/or prove directly, that the function g is not convex on $[-1, 1]$.

5. Show that f is convex on $[a, b]$ if and only if the line segment joining any two points in the set

$$S = \{(x, y) : f(x) \leq y, a \leq x \leq b\}$$

is a subset of S.

6. Suppose f is continuous on $[a, b]$, differentiable on (a, b) and the tangent at any point on the graph of $y = f(x)$ lies below the graph. Show that f is convex on $[a, b]$. (This is a converse of the lemma.)

7. Suppose f is convex on $[a, b]$ and differentiable on (a, b). Show that f' is increasing on (a, b).

8. Suppose f is twice continuously differentiable on (a, b) and $f'' \geq 0$ on (a, b). Show that f is convex on $[a, b]$. (This provides a simple — and well-known — test for convexity.) [Hint: if $x, c \in (a, b)$, then

$$f(x) = f(c) + f'(c)(x - c) + \int_c^x (x - t) f''(t)\, dt.]$$

9. Suppose f is twice differentiable on (a, b) with $f'' > 0$ on (a, b). Show that the function

$$f^*(x) = \sup_{a<s<b} \{xs - f(s)\}$$

is convex on the range of f'. (The function f^* is called the Fenchel-Legendre transform of f.)

10. Determine the Fenchel-Legendre transforms of each of the following functions defined on $(0, \infty)$:

$$e^x,\ x\log x - x,\ 1/x,\ -2\sqrt{x},\ x^p/p,\ x^q/q\ (p > 1,\ q = p/(p-1)).$$

DID YOU KNOW···

— that the following numbers, containing only the digits 1, 4, 9, are all perfect squares:

$$1,\quad 4,\quad 9,\quad 49,\quad 144,\quad 441,\quad 1444,\quad 11449,\quad 44944,$$

$$991494144,\quad 4914991449,\quad 149991994944\ ?$$

Are there infinitely many such numbers?

— Neven Jurić

THE SKOLIAD CORNER

No. 17

R.E. Woodrow

As a problem set in this issue, we give the grade 12 level contest from Nova Scotia, written April 26, 1996. My thanks go to Professor Michael Nutt of Acadia University, Wolfville, Nova Scotia, for sending the contest to us for our use.

CANADIAN MATHEMATICAL SOCIETY PRIZE EXAM

Friday, April 26, 1996 — Time: 2.5 hours

1. (a) Solve $\sqrt{x+20} - \sqrt{x+1} = 1$.

(b) Try to solve $\sqrt[3]{x+20} - \sqrt[3]{x+1} = 1$.

2. Suppose a function is defined so that $f(xy) = f(x) + f(y)$.

(a) Show that if f is defined at 1, $f(1) = 0$.

(b) Similarly if f is defined at 0, its value at any x will be 0, so it is a "trivial" function.

(c) Show that if f is defined only for $\{1, 2, 3, \dots\}$ there are many non-trivial ways to define such an f (give an example).

3. Three circles of equal radii r all touch each other to enclose a three cornered concave area A. How big is the area of A?

4. Show that for all real numbers x,

(a) $x^4 \geq 4x - 3$.

(b) x^4 is not greater than $3x - 2$, even though it appears to be true.

x	-1	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
x^4	1	0	$.004$	0.12	$.063$	1	16	81
$4x-3$	-7	-3	-2	-1.67	-1	1	5	9
$3x-2$	-5	-2	-1.25	-1	$-.5$	1	4	7

5. Two integers are called equivalent, written $x \sim y$, if they are divisible by the same prime numbers (primes are $2, 3, 5, 7, \dots$) so $2 \sim 2 \sim 4$, $3 \sim 27$ but $2 \not\sim 3$.

(a) Show that $10 \sim 80$ but $10 \not\sim 90$.

(b) Prove that if $x \sim y$, then $x^2 \sim y^2$.

6. We can describe certain fractions in terms of others all with bigger denominators (always in lowest terms). For instance $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$ and $\frac{2}{3} = \frac{1}{4} + \frac{1}{4} + \frac{1}{6}$ but $\frac{2}{3} = \frac{2}{6} + \frac{2}{6}$ does not work since $\frac{2}{6} = \frac{1}{3}$ and $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ does not since $2 < 3$.

(a) Can you write $\frac{1}{2}$ as a sum $\frac{1}{a} + \frac{1}{b}$ for integers $2 < a < b$?

(b) Try to write $\frac{1}{1996}$ as $\frac{1}{a} + \frac{1}{b}$ for integers $1996 < a < b$.

Last issue we gave the problems of the Saskatchewan Senior Mathematics Competition. Here are the official solutions. Many thanks go to Professor Garreth Griffith, of the Department of Mathematics of the University of Saskatchewan, and long time contest organizer in Saskatchewan, for providing the problems and solutions.

SASKATCHEWAN SENIOR MATHEMATICS CONTEST

Wednesday, February 22, 1995

Time: 1.5 hours

1. They sell regular and jumbo sized orders of fish at Jerry's Fish & Chips Emporium. The jumbo order costs $(\frac{4}{3})$ times as much as a regular one and an order of chips costs $1.30.

Last Thursday, the Emporium was quite busy over the lunch break (11:30 am - 1:30 pm). Exactly thirteen jumbo sized orders of fish along with a quantity of regular orders of fish and chips were sold. $702.52 had been placed in the till.

During the period 1:30 - 4:30, business slackened off. 26 regular orders of fish were sold during this time. Four times as many regular orders had been sold during the lunch break. No jumbo portions were sold and the number of orders of chips declined to one fifth the number that had been sold during the lunch break. At 4:30 pm there was $850.46 in Jerry's till.

What is the price of a regular sized order of fish at Jerry's Emporium? [5 marks]

Solution. Let x be the price of a regular order of fish at Jerry's. Then, the cost of a jumbo order is $\frac{4x}{3}$. Let y be the number of orders of chips sold at lunchtime. Since 26 regular orders of fish were sold during the afternoon period, 104 were sold at lunch time. Also, $\frac{y}{5}$ orders of chips were sold during the afternoon. The revenue at lunchtime was

$$(1.30)y + 104x + 13\left(\frac{4x}{3}\right) = 702.52 \qquad (L)$$

and the revenue during the afternoon was

$$\begin{aligned}(1.30)\frac{y}{5} + 26x &= 850.46 - 702.52 \\ &= 147.94. \qquad \text{(PM)}\end{aligned}$$

To solve these two equations, consider

$$5(PM) - (L) = 130x - 104x - 13\left(\frac{4x}{3}\right) = 37.18$$

which is equivalent to

$$\frac{2x}{3} = 2.86 \quad \text{or} \quad x = 4.29.$$

2. $ABCD$ is a square with side of length s. A circle, centre A and radius r is drawn so that the arc of this circle which lies within the square divides the square into two regions of equal area. Write r as a function of s. [6 marks]

Solution. The area of the square is s^2. The area of the (whole) circle is πr^2. One quarter of this area lies within the square. This area is $\pi r^2/4$. Therefore

$$\frac{\pi r^2}{4} = \frac{s^2}{2} \quad \text{or} \quad r^2 = \frac{2s^2}{\pi}$$

so that

$$r = \sqrt{\frac{2}{\pi}}\, s.$$

(The $\pm$ sign is unnecessary since r and s are distances.)

3. (a) Solve the equation $3^y = 10^y$. [3 marks]

(b) Solve the equation $3^y = 10$. [3 marks]

(c) Write $t\log_8\sqrt{x} - 2\log_8 y$ as a single logarithm. [4 marks]

Solution. (a) Take (common) logs of both sides:

$$\log 3^y = \log 10^y$$

$$y\log 3 = y\log 10 = y.$$

$0 = y(1 - \log 3)$ so that the only solution is $y = 0$.

(b) Similarly, $y\log 3 = 1$ so that

$$y = \frac{1}{\log 3}.$$

If we use base 3 instead, then

$$y\log_3 3 = \log_3 10 \quad \text{or} \quad y = \log_3 10.$$

(The two answers are equal.)

(c)

$$\begin{aligned} 6\log_8\sqrt{x} - 2\log_8 y &= 6\log_8 x^{1/2} - 2\log_8 y \\ &= \log_8(x^{1/2})^6 - \log_8 y^2 \\ &= \log_8\left(\frac{x^3}{y^2}\right). \end{aligned}$$

4. Establish the identity $2\cot A = \cot\frac{A}{2} - \tan\frac{A}{2}$. [5 marks]

Solution.

$$\begin{aligned}
\text{The right side} &= \frac{\cos A/2}{\sin A/2} - \frac{\sin A/2}{\cos A/2} \\
&= \frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}} \\
&= \frac{\cos A}{\frac{1}{2}\sin A} = \frac{2\cos A}{\sin A} \\
&= 2\cot A = \text{ the left side.}
\end{aligned}$$

5. ABC is a triangle, right angled at C. Let a, b, c denote the lengths of the sides opposite angles A, B, C respectively. Given that $a = 1$, $\angle B = 75°$ and that $\tan 75° = 2 + \sqrt{3}$, express b and c in the form $p + q\sqrt{3}$ such that $p = 2$ or $\sqrt{2}$. [8 marks]

Solution. Since ABC is a right angled triangle, $\tan B = \frac{b}{a}$. Since $a = 1$, $b = 2 + \sqrt{3}$. By the theorem of Pythagoras,

$$\begin{aligned}
c^2 &= a^2 + b^2 = 1 + b^2 \\
&= 1 + (2 + \sqrt{3})^2 = 8 + 4\sqrt{3} = r(2 + \sqrt{3}).
\end{aligned}$$

Therefore $c = 2\sqrt{2 + \sqrt{3}}$. (As in problem 2, we dismiss the $\pm$ sign.) The problem now is to write $2\sqrt{2 + \sqrt{3}}$ as $p + q\sqrt{3}$ where $p = 2$ or $\sqrt{2}$. Square both sides

$$4(2 + \sqrt{3}) = p^2 + 3q^2 + 2pq\sqrt{3}.$$

If $p = 2$, then

$$\begin{aligned}
8 &= 4 + 3q^2 &(1)\\
\text{and}\quad 4\sqrt{3} &= 4q\sqrt{3}. &(2)
\end{aligned}$$

From (2), $q = 1$. This is not consistent with (1).

If $p = \sqrt{2}$, then

$$\begin{aligned}
8 &= 2 + 3q^2 &(3)\\
\text{and}\quad 4\sqrt{3} &= 2\sqrt{2}q\sqrt{3}. &(4)
\end{aligned}$$

From (4), $q = \sqrt{2}$ which is consistent with (3) so that

$$p = q = \sqrt{2} \quad \text{and} \quad c = \sqrt{2} + \sqrt{2}\sqrt{3} \quad \text{or} \quad \sqrt{2}(1 + \sqrt{3}).$$

Check!

$$\begin{aligned}
c^2 &= 2(1 + \sqrt{3})^2 \\
&= 2(4 + 2\sqrt{3}) \\
&= 4(2 + \sqrt{3}).
\end{aligned}$$

6. Determine the function $f(x)$ which satisfies all of the following conditions: [8 marks]

(i) $f(x)$ is a quadratic function.

(ii) $f(x+2) = f(x) + x + 2$.

(iii) $f(2) = 2$.

Solution. (i) Let $f(x) = ax^2 + bx + c$.

(ii) $f(x+2) = a(x+2)^2 + b(x+2) + c$ so that

$$a(x+2)^2 + b(x+2) + c = ax^2 + bx + c + x + 2$$

or

$$ax^2 + 4ax + 4a + bx + 2b + c = ax^2 + bx + c + x + 2$$

so that $4ax + 4a + 2b = x + 2$.

Since this is an identity,

$$4a = 1 \quad \text{and} \quad 4a + 2b = 2 \quad \text{so that} \quad a = \frac{1}{4} \text{ and } b = \frac{1}{2}.$$

(iii) $f(2) = 4a + 2b + c = 2$. Therefore $2 + c = 2$ so that $c = 0$. It follows that

$$f(x) = \frac{x^2}{4} + \frac{x}{2}.$$

7. Prove that if n is a positive integer (written in base 10) and that if 9 is a factor of n, then 9 is also a factor of the sum of the digits of n. [8 marks]

Solution. Let a be the units digit of n; let b be the tens digit of n; let c be the hundreds digit of n and so on. Then

$$\begin{aligned} n &= a + 10b + 100c + \cdots \\ &= a + (1+9)b + (1+99)c + \cdots \\ &= (a + b + c + \cdots) + 9b + 99c + \cdots \end{aligned}$$

Since 9 divides $9b + 99c + \cdots$ it follows that 9 divides n if and only if 9 divides $(a + b + c + \cdots)$ which is the sum of the digits of n.

That completes the Skoliad Corner for this issue. Send me your contests, your suggestions, and your recommendations to improve this feature.

THE OLYMPIAD CORNER

No. 177

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin this Issue with some of the problems proposed to the jury, but not used at the 36th International Olympiad held at Toronto, Ontario. I welcome your novel, nice solutions that differ from the "official" published solutions.

36th INTERNATIONAL MATHEMATICAL OLYMPIAD

Canadian Problems for Consideration by the International Jury

Algebra

1. Let a and b be non-negative integers such that $ab \geq c^2$, where c is an integer. Prove that there is a number n and integers $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots y_n$ such that

$$\sum_{i=1}^{n} x_i^2 = a, \quad \sum_{i=1}^{n} y_i^2 = b, \quad \text{and} \sum_{i=1}^{n} x_i y_i = c.$$

2. Let n be an integer, $n \geq 3$. Let $a_1, a_2, \ldots, a_n$ be real numbers, where $2 \leq a_i \leq 3$ for $i = 1, 2, \ldots, n$. If $s = a_1 + a_2 + \cdots + a_n$, prove that

$$\frac{a_1^2 + a_2^2 - a_3^2}{a_1 + a_2 - a_3} + \frac{a_2^2 + a_3^2 - a_4^2}{a_2 + a_3 - a_4} + \cdots + \frac{a_n^2 + a_1^2 - a_2^2}{a_n + a_1 - a_2} \leq 2s - 2n.$$

3. Let a, b and c be given positive real numbers. Determine all positive real numbers x, y and z such that

$$x + y + z = a + b + c$$

and

$$4xyz - (a^2x + b^2y + c^2z) = abc.$$

Geometry

4. Let A, B and C be non-collinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2$.

5. The incircle of ABC touches BC, CA and AB at D, E and F respectively. X is a point inside ABC such that the incircle of XBC touches BC at D also, and touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

6. An acute triangle ABC is given. Points A_1 and A_2 are taken on the side BC (with A_2 between A_1 and C), B_1 and B_2 on the side AC (with B_2 between B_1 and A) and C_1 and C_2 on the side AB (with C_2 between C_1 and B) so that

$$\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1.$$

The lines AA_1, BB_1, and CC_1 bound a triangle, and the lines AA_2, BB_2 and CC_2 bound a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

Number Theory and Combinatorics

7. Let k be a positive integer. Prove that there are infinitely many perfect squares of the form $n2^k - 7$, where n is a positive integer.

8. Let Z denote the set of all integers. Prove that, for any integers A and B, one can find an integer C for which $M_1 = \{x^2 + Ax + B : x \in \mathbb{Z}\}$ and $M_2 = \{2x^2 + 2x + C : x \in \mathbb{Z}\}$ do not intersect.

9. Find all positive integers x and y such that $x + y^2 + z^3 = xyz$, where z is the greatest common divisor of x and y.

Sequences

10. Does there exist a sequence $F(1), F(2), F(3), \ldots$ of non-negative integers which simultaneously satisfies the following three conditions?

(a) Each of the integers $0, 1, 2, \ldots$ occurs in the sequence.

(b) Each positive integer occurs in the sequence infinitely often.

(c) For any $n \geq 2$,

$$F(F(n^{163})) = F(F(n)) + F(F(361)).$$

11. For an integer $x \geq 1$, let $p(x)$ be the least prime that does not divide x, and define $q(x)$ to be the product of all primes less than $p(x)$. In particular, $p(1) = 2$. For x having $p(x) = 2$, define $q(x) = 1$. Consider the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = 1$ and

$$x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$$

for $n \geq 0$. Find all n such that $x_n = 1995$.

12. Suppose that $x_1, x_2, x_3, \ldots$ are positive real numbers for which

$$x_n^n = \sum_{j=0}^{n-1} x_n^j$$

for $n = 1, 2, 3, \ldots$. Prove that for all n,

$$2 - \frac{1}{2^{n-1}} \leq x_n < 2 - \frac{1}{2^n}.$$

Next we turn to the "official" results of the 37th IMO which was written in Mumbai, India, July 10–11, 1996. My source this year was Ravi Vakil, former star Olympian and this year's Canadian Team Leader. I hope that I have made no serious errors in compiling the results and transcribing names.

This year a total of 426 students from 75 countries took part. This is somewhat up from last year. Sixty-five countries sent teams of six (the number invited to participate in recent years). But there were ten teams of smaller size, two of five members; five of four, and one of each of sizes three, two and one.

The contest is officially an individual competition and the six problems were assigned equal weights of seven marks each (the same as the last fifteen IMOs for a maximum possible individual score of 42 and a total possible of 252 for a national team of six students). For comparison see the last fifteen IMO reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207], [1988: 193], [1989: 193], [1990: 193], [1991: 257], [1992: 263], [1993: 256], [1994: 243], and [1995: 267].

There was only one perfect score. The jury awarded first prize (Gold) to the thirty-five students who scored 28 or more. Second (Silver) prizes went to the sixty-six students with scores from 20 to 27, and third (Bronze) prizes went to the ninety-nine students with scores from 12 to 19. Any student who did not receive a medal, but who scored full marks on at least one problem, was awarded honourable mention. This year there were twenty-one honourable mentions awarded. The median score on the examination was 11.

Congratulations to the Gold Medalists.

Name	Country	Score
Ciprian Manolescu	Romania	42
Yu Seek Kong	Korea	39
Sug Woo Shin	Korea	38
Nikolai Dourov	Russia	37
Alexander Harry Saltman	U.S.A.	37
Ngo Dac Tuan	Vietnam	37

Name	Country	Score
Peter Burcsi	Hungary	36
Sachiko Nakajima	Japan	36
Serguei Norine	Russia	36
Christopher C. Chang	U.S.A.	36
Arend Boyer	Germany	35
Peter Frenkel	Hungary	35
Juliy Sannikov	Ukraine	34
Ivan Ivanov	Bulgaria	33
David Chkhaidze	Georgia	33
Constantin Chiscanu	Romania	33
David William Bibby	United Kingdom	33
Ngo Duc Duy	Vietnam	33
Serguei Chikh	Belarus	32
Chen Huayi	China	32
Gunther Vogel	Germany	31
Carl Bosley	U.S.A.	31
Michael Korn	U.S.A.	31
Nguyen Thai Ha	Vietnam	31
He Xuhua	China	30
Yan Jun	China	30
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Next we give the problems from this year's IMO Competition. Solutions to these problems, along with those of the 1995 USA Mathematical Olympiad will appear in a booklet entitled *Mathematical Olympiads* 1996 which may be obtained for a small charge from: Dr. W.E. Mientka, Executive Director, MAA Committee on H.S. Contests, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, 68588, USA.

37th INTERNATIONAL MATHEMATICAL OLYMPIAD

July 10–11, 1996 (Mumbai, India)

First Day — Time: 4.5 hours

1. Let $ABCD$ be a rectangular board with $|AB| = 20$, $|BC| = 12$. The board is divided into 20×12 unit squares. Let r be a given positive integer. A coin can be moved from one square to another if and only if the distance between the centres of the two squares is $\sqrt{r}$. The task is to find a sequence of moves taking the coin from the square which has A as a vertex to the square which has B as a vertex.

(a) Show that the task cannot be done if r is divisible by 2 or 3.

(b) Prove that the task can be done if $r = 73$.

(c) Can the task be done when $r = 97$?

2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incentres of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.

3. Let $S = \{0, 1, 2, 3, \dots\}$ be the set of non-negative integers. Find all functions f defined on S and taking their values in S such that

$$f(m + f(n)) = f(f(m)) + f(n) \quad \text{for all} \quad m, n \in S.$$

Second Day — Time: 4.5 hours

4. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. Find the least possible value that can be taken by the minimum of these two squares.

5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to ED, BC is parallel to FE and CD is parallel to AF. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{p}{2}.$$

6. Let n, p, q be positive integers with $n > p + q$. Let $x_0, x_1, \dots, x_n$ be integers satisfying the following conditions:

(a) $x_0 = x_n = 0$;

(b) for each integer i with $1 \leq i \leq n$,

$$\text{either } x_i - x_{i-1} = p \quad \text{or} \quad x_i - x_{i-1} = -q.$$

Show that there exists a pair (i, j) of indices with $i < j$ and $(i, j) \neq (0, n)$ such that $x_i = x_j$.

Although the IMO is officially an individual event, the compilation of team scores is unofficial, if inevitable. These totals and the prize awards are given in the following table.

Rank	Country	Score	Gold	Silver	Bronze	Total
1.	Romania	187	4	2	—	6
2.	U.S.A.	185	4	2	—	6
3.	Hungary	167	3	2	1	6
4.	Russia	162	2	3	1	6
5.	United Kingdom	161	2	4	—	6
6.	China	160	3	2	1	6
7.	Vietnam	155	3	1	1	5
8.	Korea	151	2	3	—	5
9.	Iran	143	1	4	1	6
10.	Germany	137	3	1	1	5
11.–12.	Bulgaria	136	1	4	1	6
11.–12.	Japan	136	1	4	—	5
13.	Poland	122	—	3	3	6
14.	India	118	1	3	1	5
15.	Israel	114	1	2	2	5
16.	Canada	111	—	3	3	6
17.	Slovakia	108	—	2	4	6
18.	Ukraine	105	1	—	5	6
19.	Turkey	104	—	2	3	5
20.	Taipei	100	—	2	3	5
21.	Belarus	99	1	1	2	4
22.	Greece	95	—	1	5	6
23.	Australia	93	—	2	3	5
24.	Yugoslavia	87	—	1	2	3
25.–26.	Italy	86	—	2	2	4
25.–26.	Singapore	86	1	—	3	4
27.	Hong Kong	84	—	1	4	5
28.	Czech Republic	83	—	2	1	3
29.	Argentina	80	—	1	3	4
30.	Georgia	78	1	—	2	3
31.	Belgium	75	—	—	4	4
32.	Lithuania	68	—	1	2	3
33.	Latvia	66	—	—	3	3
34.–35.	Armenia	63	—	—	1	1
34.–35.	Croatia	63	—	1	1	2
36.	France	61	—	2	—	2
37.–38.	New Zealand	60	—	—	3	3
37.–38.	Norway	60	—	—	3	3
39.–40.	Colombia	58	—	1	—	1
39.–40.	Finland	58	—	—	2	2
41.	Sweden	57	—	1	1	2
42.	Moldava (Team of 5)	55	—	—	2	2
43.	Austria	54	—	1	—	1
44.	Republic of South Africa	50	—	—	2	2
45.–46.	Mongolia	49	—	—	2	2
45.–46.	Slovenia	49	—	—	2	2
47.	Thailand	47	—	—	1	1
48.–51.	Denmark	44	—	—	2	2
48.–51.	Macao	44	—	—	1	1
48.–51.	Former Yugoslav Republic of Macedonia	44	—	—	2	2
48.–51.	Spain	44	—	—	—	—
52.	Brazil	36	—	—	—	—
53.–54.	Mexico	34	—	—	—	—
53.–54.	Sri Lanka	34	—	—	1	1

Rank	Country	Score	Gold	Silver	Bronze	Total
55.	Estonia	33	—	—	—	—
56.	Iceland	31	—	—	1	1
57.	Bosnia-Herzegovina (Team of 4)	30	—	—	1	1
58.	Azerbaijan	27	—	—	—	—
59.	The Netherlands	26	—	—	—	—
60.	Trinidad & Tobago	25	—	—	—	—
61.	Ireland	24	—	—	—	—
62.	Switzerland (Team of 4)	23	—	—	1	1
63.	Portugal	21	—	—	—	—
64.	Kazakhstan	20	—	—	—	—
65.	Morocco	19	—	—	1	1
66.	Cuba (Team of 1)	16	—	—	1	1
67.–68.	Albania (Team of 4)	15	—	—	—	—
67.–68.	Kyrgyztan	15	—	—	—	—
69.	Cyprus (Team of 5)	14	—	—	—	—
70.	Indonesia	11	—	—	—	—
71.	Chile (Team of 2)	10	—	—	—	—
72.–73.	Malaysia (Team of 4)	9	—	—	—	—
72.–73.	Turkmenistan (Team of 4)	9	—	—	—	—
74.	Philippines	8	—	—	—	—
75.	Kuwait (Team of 3)	1	—	—	—	—

This year the Canadian Team rose to 16th place from 19th last year and 24th the previous year. The Team members were:

Richard Hoshino	22	Silver
Derek Kisman	22	Silver
Saroosh Yazdani	22	Silver
Byung Kyu Chun	18	Bronze
Adrian Chan	14	Bronze
Sabin Cautis	13	Bronze

This is the first time every member of the Canadian Team was awarded a medal.

The Team Leader was Ravi Vakil, a former Canadian Team Member and now a graduate student; and P.J. Grossman was Deputy Leader, also a former Olympian and current graduate student.

The Romanian Team placed first this year. Its members were:

Ciprian Manolescu	42	Gold
Constantin Chiscanu	33	Gold
Adrian Dumitru Corouneanu	30	Gold
Stefan Rodu Niculescu	28	Gold
Dragos Ghioca	27	Silver
Nicolae Dragos Oprea	27	Silver

Congratulations to the Romanian Team!!

To finish this number of the Corner we discuss solutions sent in by the readers to problems of the 6th Korean Mathematical Olympiad, which we gave in the February 1995 number (6th Korean Mathematical Olympiad, Final Round, April 17–18, 1993, [1995: 45–46]).

1. Let there by a 9×9 array of white squares. Find the largest positive integer n satisfying the following property: There always remains either a 1×4 or a 4×1 array of white squares no matter how you choose n out of 81 white squares and colour them black.

Solutions by Mansur Boase, student, St. Paul's School, London, England; by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany; and by Solomon Golomb, University of Southern California. We give Golomb's solution and comments.

The *largest* positive integer n such that, no matter which n squares of a 9×9 board are blocked, it will still be possible to place a 1×4 rectangle ["straight tetromino"] on the board, either horizontally or vertically, is 19. We prove the following clearly equivalent proposition: "The *smallest* number of squares which must be blocked on a 9×9 board so that a straight tetromino will no longer fit is 20."

First here is a configuration with 20 blocking squares such that no straight tetromino will fit:

		X				X		
	X				X			
X				X				X
			X				X	
		X				X		
	X				X			
X				X				X
			X				X	
		X				X		

Clearly, a blocking configuration must involve at least two excluded squares in each row and in each column. (One excluded square cannot block a row or column of length nine.) This already shows that $2 \times 9 = 18$ excluded squares are required, to block all *rows*. But consider the two (or more) excluded squares in the bottom row. Their *columns* each require two *additional* excluded squares to block those columns. So all the columns have at least two excluded squares, but at least two columns must have at least three excluded squares, for a total of at least $7 \times 2 + 2 \times 3 = 20$ excluded squares.

Comment. Methods for this type of problem are described in Chapter 3: "*Where Pentominoes Will Not Fit*," in my book **Polyominoes**. (Original edition, Charles Scribner's Sons, 1965; expanded, revised edition, Princeton University Press, 1994.)

2. Let ABC be a triangle with $\overline{BC} = a$, $\overline{CA} = b$, $\overline{AB} = c$. Find the point P for which

$$a \cdot \overline{AP}^2 + b \cdot \overline{BP}^2 + c \cdot \overline{CP}^2$$

is minimal, and find the minimum.

Solutions by Šefket Arslanagić, Berlin, Germany; and by Panos E. Tsaoussoglou, Athens, Greece. We give the solution of Arslanagić.

We have (e.g. pages 278 and 280 of [1]):

Let P be a point in the plane of a triangle ABC and M be an arbitrary point in space. Then $\overrightarrow{MP} = (\sum x_i \overrightarrow{MA})/(\sum x_i)$, where x_1, x_2, x_3 are real numbers, and summations are taken cyclically. The following generalization of the well-known Leibnitz identity is valid:

$$\left(\sum x_i\right)^2 \overline{MP}^2 = \left(\sum x_i\right) \sum x_i \overline{MA}^2 - \sum a^2 x_2 x_3.$$

For $P = I$, (where I is the incentre of ΔABC), we get (because we can take $\sum x_i = 2s$, $\sum a^2 x_i x_j = abc \cdot 2s$)

$$\sum a\overline{MA}^2 = 2s\overline{MI}^2 + abc,$$

and as a consequence we have the following inequality,

$$\sum a\overline{MA}^2 \geq abc,$$

and equality holds only for $M = I$.

From this take $M = P$,

$$a\overline{AP}^2 + b\overline{BP}^2 + c\overline{CP}^2 \geq abc$$

and

$$\min(a\overline{AP}^2 + b\overline{BP}^2 + c\overline{CP}^2) = abc$$

with $P = I$.

Reference

[1] D.S. Mitrinović, Y.E. Pećarić and V. Volenec, *Recent Advances in Geometric Inequalities*.

3. Find the smallest positive integer x for which

$$\frac{7x^{25} - 10}{83}$$

is an integer.

Solutions by Mansur Boase, student, St. Paul's School, London, England; Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany; and by Stewart Metchette, Gardena, California, USA. We give the solution of Boase.

We want $7x^{25} \equiv 10 \bmod 83$, or $7x^{25} = 83a + 10$.
So $83a + 10 \equiv 0 \bmod 7$ whence $a \equiv 3 \bmod 7$ and $x^{25} \equiv 37 \bmod 83$.
Now 83 is prime and congruent to $3 \bmod 5$, so there is only one solution to this congruence $\bmod 83$.

$$(x^5)^5 \equiv 37 \bmod 83$$
$$x^5 \equiv 16 \bmod 83$$
$$x \equiv 69 \bmod 83.$$

The *smallest* such x is 69.

4. An integer is called a Pythagorean number if it is the area of a right triangle whose sides are of integral lengths, say $x, y, z \in N$ such that $x^2 + y^2 = z^2$. Prove that for each positive integer n ($n > 12$), there exists a Pythagorean number between n and $2n$.

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany. We give the solution of Engelhaupt.

The smallest Pythagorean numbers are

x	3	6	5	9	8
y	4	8	12	12	15
z	5	10	13	15	17
Pythagorean Number	6	24	30	54	60

Thus for $n = 13$ to 23 we have 24 in the interval $[n, 2n]$,
for $n = 24$ to 29 we have 30 in the interval $[n, 2n]$, and
for $n = 30$ to 53 we have 54 in the interval $[n, 2n]$.

Now the numbers $6k^2$, with $k \in \mathbb{N}$ are Pythagorean numbers ($x = 3k$, $y = 4k$). Observe that $\dfrac{6(k+1)^2}{6k^2} < 2$ if $2k + 1 < k^2$, or $k > 2$.
Henceforth, $k \geq 3$.
For $n = 6k^2$ to $6(k+1)^2 - 1$ we have $6(k+1)^2$ lies in the interval $[n, 2n]$, for example
for $n = 54$ to 95, 96 lies in $[n, 2n]$.
for $n = 96$ to 149, 150 lies in $[n, 2n]$;
for $n = 150$ to 215, 216 lies in $[n, 2n]$;
and so on.

5. Let n be a given natural number. Find all the continuous functions $f(x)$ satisfying:

$$\binom{n}{0} f(x) + \binom{n}{1} f\left(x^2\right) + \binom{n}{2} f\left(x^{2^2}\right) + \cdots + \binom{n}{n-1} f\left(x^{2^{n-1}}\right) + \binom{n}{n} f\left(x^{2^n}\right)$$

*

Solution by the editor.

We prove by induction that $f(x) \equiv 0$ if f is a continuous function satisfying the condition (*). With $n = 0$ there is nothing to prove as the condition becomes $f(x) = 0$ for all x. For $n = 1$, assume that $f(x) + f(x^2) = 0$ for all x with f continuous. Now $f(-x) = -f(x^2) = f(x)$ so f is an even function and it suffices to prove the result for $x \geq 0$. From $f(0) = -f(0)$ and $f(1) = -f(1)$ we get $f(0) = 0 = f(1)$. So consider a fixed value of $x > 0$. Now $\lim_{k\to\infty} x^{1/k} = 1$ from which we obtain $\lim_{m\to\infty} x^{1/2^m}$ $= 0$. It is easy to see that $f(x^{1/2^m}) = (-1)^m f(x)$ for $m \geq 0$. However, $\lim_{m\to\infty} f(x^{1/2^m}) = f(1) = 0$. From this it follows that $f(x) = 0$, establishing the result for $n = 1$.

Now assume f is a continuous function satisfying

$$\sum_{j=0}^{n+1} \binom{n+1}{j} f(x^{2^j}) = 0 \quad \text{for all } x.$$

Set

$$g(x) = \sum_{l=0}^{n} \binom{n}{l} f(x^{2^l}).$$

Now

$$\begin{aligned} g(x) + g(x^2) &= \sum_{l=0}^{n} \binom{n}{l} f(x^{2^l}) + \sum_{l=0}^{n} \binom{n}{l} f((x^2)^{2^l}) \\ &= \sum_{l=0}^{n} \binom{n}{l} f(x^{2^l}) + \sum_{l=0}^{n} \binom{n}{l} f(x^{2^{l+1}}) \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} f(x^{2^j}) = 0 \end{aligned}$$

since $\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$.

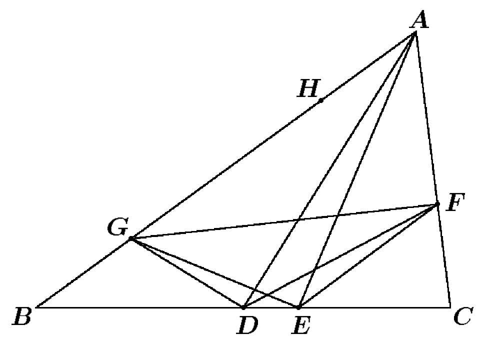
But then $g(x)$ is a continuous function satisfying $g(x) + g(x^2) = 0$ for all x. By the case $n = 1$, $g(x) = 0$ for all x, i.e.

$$g(x) = \sum_{l=0}^{n} \binom{n}{l} f(x^{2^l}) = 0 \text{ for all } x.$$

By the induction hypothesis $f(x) = 0$ for all x and the induction is complete.

6. Let ABC be a triangle with $\overline{BC} = a$, $\overline{CA} = b$ and $\overline{AB} = c$. Let D be the mid-point of the side BC, and let E be the point on BC for which the line segment AE is the bisector of angle A. Let the circle passing through A, D, E intersect with the sides CA, AB at F, G respectively. Finally let H be the point on AB for which $\overline{BG} = \overline{GH}$, i.e. $\overline{BH} = 2\overline{BG}$. Prove that the triangles EBH and ABC are similar and then find the ratio $\frac{\Delta EBH}{\Delta ABC}$ of these areas.

Solution by Mansur Boase, St. Paul's School, London, England.



Now $\angle EAF = \angle EGF = \angle FDE$ and $\angle GAD = \angle GFD = \angle GED$. Thus $\Delta GED \sim \Delta ABD$ since two angles are the same. Thus

$$\frac{BE}{AB} = \frac{BG}{BD} = \frac{GE}{AD} \quad \text{and} \quad \frac{BE}{AB} = \frac{BH}{BC} = \frac{HE}{AC}.$$

Thus $\Delta BHE \sim \Delta ABC$.

The ratio $\frac{\Delta BHE}{\Delta ABC} = \left(\frac{BE}{C}\right)^2$. By the sine rule

$$\frac{BE}{\sin\frac{A}{2}} = \frac{AE}{\sin B}$$

$$\frac{a - BE}{\sin\frac{A}{2}} = \frac{AE}{\sin C}.$$

Therefore $BE \sin B = (a - BE)\sin C$, and so

$$\begin{aligned} BE &= \frac{a\sin C}{\sin B + \sin C} = \frac{a\frac{c}{2R}}{\frac{b}{2R} + \frac{c}{2R}} \\ &= \frac{ac}{b+c}, \end{aligned}$$

and

$$\left(\frac{BE}{c}\right)^2 : 1 = \left(\frac{ac}{c(b+c)}\right)^2 : 1 = \frac{a^2}{(b+c)^2}.$$

That completes the solutions we have and the Olympiad Corner for this issue. Send me your contests and nice solutions!

THE ACADEMY CORNER

No. 5

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

The **Mathematics Archives** on the World Wide Web.

Everyone knows that there is a plethora of information on the World Wide Web. I was directed recently to the **Mathematics Archives: Lessons, Tutorials and Lecture Notes** at

`http://archives.math.utk.edu/tutorials.html`

This contains over 140 links to mathematical assistance. The whole is too large to mention here, but I mention a few of the links that could be of use to ***CRUX*** readers.

1. `http://www.lance.colostate.edu/auth/lessons/1/`
   Aristotle University of Thessaloniki

   Lesson on the Newton-Raphson method

2. `http://www.math.uga.edu/~andrew/Binomial/index.html`
   Arithmetic Properties of Binomial Coefficients

   Binomial Coefficients is an on-line dynamic survey, available on the World Wide Web, and accessible from various mathematical sites and e-journals. It is an edited document, in HTML, and can and will be edited as new developments arise. Content is at the discretion of the editors. Editors are Andrew Granville and Richard Witt.

3. `http://www2.ncsu.edu/math/Projects/MA141Manual/Contents.html`
   CALCULUS I WITH MAPLE V, North Carolina State University

4. `http://www.math.ilstu.edu/~day/305S96.html`
   Combinatorics: Topics for K-8 Teachers

   Information and materials from a course taught by Roger Day at Illinois State University.

5. `ftp://ftp.utirc.utoronto.ca/pub/ednet/maths/model_html/ExpMath.html`
   Experiments in Mathematics Using Maple

   A Computer Lab Book For High Schools and Home Computers by C. T. J. Dodson and E. A. Gonzalez, University of Toronto.

6. http://millbrook.lib.rmit.edu.au/fractals/exploring.html
   Exploring Chaos and Fractals

   Exploring Chaos and Fractals is an electronic textbook which includes full text, work sheets, sound, video and animation. Parts of the material have been placed on a Web server as an experiment in electronic publishing of hypertext based material.

7. http://math.holycross.edu/~davids/fibonacci/fibonacci.html
   The Fibonacci Numbers

8. http://william-king.www.drexel.edu/top/class/game-toc.html
   Game Theory: An Introductory Sketch

9. http://www.geom.umn.edu/docs/education/institute91/
   Geometry and the Imagination

   Notes and handouts for an innovative geometry course developed at Princeton and the Geometry Center by John Conway, Peter Doyle, Jane Gilman and Bill Thurston.

10. http://www.utm.edu:80/departments/math/graph/
    Graph Theory : Tutorials

    This is the home page for a series of short interactive tutorials by Chris K. Caldwell introducing the basic concepts of graph theory. They are designed with the needs of future high school teachers in mind.

11. http://www.intergalact.com/threedoor/threedoor.html

    The Three Door Puzzle

12. http://www.wam.umd.edu/~krc/numtheory/prime.html
    Prime Numbers

    These pages were written by Kevin Coombes as an experiment to aid students to better understand proofs of theorems. From an email message: "It seems to me that one of the main difficulties that students of (higher) mathematics encounter stems from the insistence on getting all the logical prerequisites set up before trying to explain anything interesting. (Of course, this style of presentation goes back at least as far as Euclid.)"

13. http://www.math.ilstu.edu/~day/326S96.html
    Technology Tools for Secondary School Mathematics

    Materials developed by Roger Day at Illinois State University.

Acknowledgment: Many of the descriptions were taken from and/or modified from the documents from the listed site.

BOOK REVIEWS

Edited by ANDY LIU

Five Hundred Mathematical Challenges
by Edward J. Barbeau, Murray S. Klamkin and William O.J. Moser.
Published by The Mathematical Association of America.
softcover, 227+ pages, ISBN # 0-88385-619-4, US $ 29.50.
Reviewed by **Marcin E. Kuczma**, *University of Warsaw.*

The authors' names alone are a sufficient recommendation of this book, which is a must for your book-shelf — unless you have already the previously published collection of booklets, some 20 years ago, by the same authors, containing the same problems. Actually, even in that case, it is still worth having the new book, since it is a revised and expanded version. Solutions are presented with greater care and elegance. Multiple solutions are often supplied, with extensions and remarks providing profound insight into the nature of the problems. There are also new bibliographical references.

I could have stopped the review right here. However, I should perhaps say some more about the book than just express my enchantment with it.

The title is a little bit misleading. Why so? The word "Challenges" calls for reflection. The readers may be relieved to learn that some of the problems presented are no more difficult than very simple "puzzlers" often found in non-mathematical journals. Here is an example.

Problem 471 *Ma and Pa and brother and me, The sum of our ages is eighty-three, Six times Pa's is seven times Ma's, And Ma's is three times me—no fractions please.*

Another frequent type is:

Without using a calculator tell which one of the two numbers (expressed in terms of horrifying root combinations) *is greater.*

I am sure the readers will welcome with much pleasure such exercises, which provide healthy entertainment. They bring a refreshing effect as they appear, from time to time, amidst heavier artillery: the more or less typical competition problems.

As mentioned, the book arose from a series of earlier publications. It must be kept in mind that in those years, publications of that type were rare, at least in America. Problem solving competitions had not yet expanded into a "branch of industry" as they are today. Students were not acquainted with any off-curricular techniques. That explains the presence of many problems which are immediately reducible to direct applications of, say, Jensen's Inequality, Menelaus' Theorem, the Factor Theorem, the Pigeonhole Principle, congruence considerations, and so on. Nowadays, those techniques are taught at every training course. Any student trying to tackle the olympiad or other serious mathematics competitions must be familiar with them, and this takes the challenge somewhat away.

A few problems can be quite "challenging" to a high school student, while being standard at the first-year college level. Examples are one-sided or two-sided estimates of the partial sums of some typical series or of expressions involving factorials and binomial coefficients; rate of growth: polynomial versus logarithmic or exponential; and maximization of functions by multivariable calculus methods. I would hesitate to call any one of those questions a "challenge".

What has been said so far should by no means be construed as criticism. Even if we agree that some trick has become "standard", this fact is not necessarily reflected in the standard problem literature. The book under review is a treasury of such training material.

About a quarter of the book consists of problems at the olympiad level—true challenges. I must confess to having real difficulties with many of them, such as the following. **Problem 432** Show that five or more great circles on a sphere, no three of which are concurrent, determine at least one spherical polygon having 5 or more sides.

With some of the problems I had no trouble only because they were well-known to me. Again, this is no criticism! Problems *do* circulate in the literature. I am quite sure that several problems were actually born and first published many years ago, in one of those booklets by the authors. They may have since been repeatedly reused in other publications and contests, to reappear in the present edition.

Several problems have been reprinted, with reference given, from other journals. Special mention must be made about those taken from the issues of *Math. Gazette* in the previous century(!)—a precious gift to every problem collectioner.

The authors write in the Preface: " *... we make no claim for the originality of most of the problems; we acknowledge our debt to the unsung creators ...* ". Too modest! Many of the problems have without any doubt been created by the authors of the book. The majority of those truly challenging and beautiful problems come from these domains: algebra of polynomials, in one or more variables; sophisticated inequalities, often involving symmetric forms; and smart combinatorial reasonings. Each one of the authors has left his own imprint.

The correlation between "challenging" and "beautiful" is well-known to every problemist; nice problems are usually the difficult ones. Now, it is a pleasant feature of this book that it also contains relatively many problems of medium or even less than medium difficulty, which nevertheless have more than average charm, sometimes revealed only after arriving at the solution. Here are some examples.

Problem 28
A boy lives in each of n houses on a straight line. At what point should the n boys meet so that the sum of the distances that they walk from their houses is as small as possible?
(*Two solutions are provided, contrasting a common sense argument against an analytic one.*)

Problem 292
Let $f(x)$ be a non-decreasing function of a real variable. Let c be any real number. Solve the equation $x = c - f(x + f(c))$.

Problem 327
Let three concentric circles be given such that the radius of the largest is less than the sum of the radii of the two smaller. Construct an equilateral triangle whose vertices lie one on each circle.

Problem 445
Prove that if the top 26 cards of an ordinary shuffled deck contain more red cards than there are black cards in the bottom 26, then there are in the deck at least three consecutive cards of the same colour.

This sample reflects my taste; another reader may point out other problems she/he considers particularly nice. While it is impossible that all 500 problems should be equally appealing to everybody, it is rather certain that everybody shall find in this book enough items matching her/his taste, needs and "sense of pleasure" accurately.

With a certain dose of pride, I can say that I was able to solve most of the problems I tried. And with a certain dose of shame I must say that, in most cases, I failed to find solutions as elegant as those included in the book. For many problems, different approaches are shown. The neatness of presentation must be emphasised. The solutions are clear and concise. Occasionally, a rigorous argument is preceded by a heuristic one.

The solution to **Problem 327** (see above) ends with a "rider": *Given equilateral triangle ABC such that $PA = 3$, $PB = 4$, $PC = 5$ for an interior point P, find AB*. Anyone who solved the problem (or has read and understood the solution) is invited to tackle this new one, employing the similar lines of reasoning, and yet revealing some new and unexpected features. Such "riders" follow the solutions to many problems. Here is another example.

Problem 380
Prove that the function $f(x, y) = \frac{1}{2}(x + y)(x + y + 1) + x$ is a one-to-one map from the set of lattice points (x, y) with $x, y \geq 0$, other than $(0, 0)$, onto the set N of positive integers.

Too easy? or well-known? Try the "rider", then: *For each $n \in N$ establish the existence of a polynomial $F_n(x_1, \ldots, x_n)$ such that $F_n\colon\ N^n \to N$ is a bijection.*

Problem 447
If m and n are positive integers, show that $m^{-1/n} + n^{-1/m} > 1$.

Known? Perhaps, but consider the intriguing "rider" which I very much recommend to every reader: *Can you give a non-calculus* (!) *proof that* $x^y + y^x > 1$ *for* $x, y > 0$?

These "riders" show extensions (sometimes far-reaching), teach analogies, and provide further training to learn the techniques from the given solutions. Their value cannot be over-stated.

Five hundred is a huge number (in fact, there are altogether some six hundred problems, since the "riders" might be counted separately). In such a voluminous text, small mistakes are unavoidable. The solution to **Problem 177** (a functional equation) misses certain discontinuous functions. According to the statement of **Problem 365**, the father's age has to be a perfect square and a prime number, at the same time. A mistake in sign occurs in the statement of the Inclusion-Exclusion Principle. Minor typos are not entirely absent, but they are rare enough not to affect the impression that the book has been prepared and edited with great care.

The book ends with a Tool Chest section. It is a handy help for the readers. For authors of problem collections, this is a model example of how such a section ought to be written.

In conclusion, I repeat that this book is a must.

Rider:

To buy or not to buy?

(Answer on page ??.)

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that **1 May 1997**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or plain postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

Correction to 2137

2137. [1996: 124] *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Three circles of (equal) radius t pass through a point T, and are each inside triangle ABC and tangent to two of its sides. Prove that:

(i) $t = \dfrac{rR}{R+2}$, [**NB**: r instead of 2]

(ii) T lies on the line segment joining the centres of the circumcircle and the incircle of ΔABC.

2177. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral, with P the intersection of its diagonals and M the mid-point of AD. MP meets BC at E. Suppose that $BE : EC = (AB)^2 : (CD)^2$. Characterize quadrilateral $ABCD$.

2178. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

If A, B, C are the angles of a triangle, prove that

$$\begin{aligned}\sin A \sin B \sin C &\leq 8\left(\sin^3 A \cos B \cos C + \sin^3 B \cos C \cos A\right. \\ &\qquad \left. + \sin^3 C \cos A \cos B\right) \\ &\leq 3\sqrt{3}\left(\cos^2 A + \cos^2 B + \cos^2 C\right).\end{aligned}$$

2179. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For real numbers $a \geq -1$, we consider the sequence
$F(a) := \{(1+\frac{1}{n})^{\sqrt{n(n+a)}}, n \geq 1\}$.
Determine the sets D, respectively I, of all a, such that $F(a)$ strictly decreases, respectively increases.

2180. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Prove that if $a > 0, x > y > z > 0, n \geq 0$ (natural), then

1. $a^x(yz)^n(y-z) + a^y(xz)^n(z-x) + a^z(xy)^n(x-y) \geq 0$,
2. $a^x \cosh x(y-z) + a^y \cosh y(z-x) + a^z \cosh z(x-y) \geq 0$.

2181. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove that the product of eight consecutive positive integers cannot be the fourth power of any positive integer.

2182. *Proposed by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.*

Many ***CRUX*** readers are familiar with the card game "Crazy Eights", of which there are many variations. We define the game of "Solo Crazy Eights" in the following manner:

We are given a standard deck of 52 cards, and are dealt k of these at random, $1 \leq k \leq 52$. We then attempt to arrange these k cards according to three rules:

1. Any card can be chosen as the first card of a sequence;
2. A card can be succeeded by any card of the same suit, or the same number, or by any eight;
3. Anytime in the sequence that an eight appears, any suit can be "called", and the succeeding card must be either of the called suit, or another eight. (This means that, in effect, any card can follow an eight).

The game is won if all dealt cards can be ordered into a sequence according to rules 1–3. If no such sequence is possible, the game is lost.

What is the largest value of k for which it is possible to lose the game?

2183. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Suppose that A, B, C are the angles of a triangle and that k, l, $m \geq 1$. Show that:

$$\begin{aligned} 0 &< \sin^k A \sin^l B \sin^m B \\ &\leq k^k l^l m^m S^{\frac{S}{2}} \left((Sk^2+P)^{-\frac{k}{2}}\right)\left((Sl^2+P)^{-\frac{l}{2}}\right)\left((Sm^2+P)^{-\frac{m}{2}}\right), \end{aligned}$$

where $S = k + l + m$ and $P = klm$.

2184. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let n be a positive integer and let a_n denote the sum

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}.$$

Prove that the sequence $\{a_n : n \geq 0\}$ is periodic.

2185. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Notice that

$$2^2 + 4^2 + 6^2 + 8^2 + 10^2 = 4 \cdot 5 + 5 \cdot 6 + 6 \cdot 7 + 7 \cdot 8 + 8 \cdot 9;$$

that is, the sum of the first n (in this case 5) even positive squares is equal to the sum of some n consecutive products of consecutive pairs of positive integers.

Find another value of n for which this happens.

(NOTE: this problem was suggested by a final exam that I marked recently.)

2186. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

Let a, b, c respectively denote the lengths of the sides BC, CA, AB of triangle ABC. Let G denote the centroid, let I denote the incentre, let R denote the circumradius, r denote the inradius, and let s denote the semi-perimeter.

Prove that

$$GI^2 = \frac{1}{9(a+b+c)} \Big((a-b)(a-c)(b+c-a)$$

$$+ (b-c)(b-a)(c+a-b) + (c-a)(c-b)(a+b-c)\Big).$$

Deduce the (known) result

$$GI^2 = \frac{1}{9}\left(s^2 + 5r^2 - 16Rr\right).$$

2187. *Proposed by Syd Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is easy to show that the maximum number of bishops that can be placed on an 8×8 chessboard, so that no two of them attack each other, is 14.

(a) Prove or disprove that in any configuration of 14 non-attacking bishops, all the bishops must be on the boundary of the board.

(b) Describe all of the configurations with 14 non-attacking bishops.

2188. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Suppose that a, b, c are the sides of a triangle with semi-perimeter s and area Δ. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{s}{\Delta}.$$

DID YOU KNOW···

— that the nine consecutive integers

$$2, 3, 4; \quad 5, 6, 7; \quad 8, 9, 10,$$

partitioned into three groups of three as shown, can be permuted inside each group to form three squares

$$324 = 18^2, \qquad 576 = 24^2, \qquad 1089 = 33^2\ ?$$

Is there another sequence of nine consecutive positive integers with this property?

— K. R. S. Sastry

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1940. [1994: 108; 1995: 107; 1995: 206] *Proposed by Ji Chen, Ningbo University, China.*

Show that if $x, y, z > 0$,

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}\,.$$

In ***CRUX*** [1995: 206], the editor asked *'if anyone finds a "nice" solution to this problem* [***CRUX*** [1994:108; 1995: 107]], *the editor would be interested to see it.'*

Comment by Vedula N. Murty, Andhra University, Visakhapatnam, India. Without loss of generality, assume that $0 < x \leq y \leq z$, and let $a = \frac{x+y}{2}$, $b = \frac{x+z}{2}$, and $c = \frac{y+z}{2}$. Then we have

$$0 < a \leq b \leq c, \qquad (1)$$

and the inequality proposed in ***CRUX*** [1994: 108] is equivalent to

$$(2bc + 2ca + 2ab - a^2 - b^2 - c^2)\left(\tfrac{1}{a^2} + \tfrac{1}{b^2} + \tfrac{1}{c^2}\right) \;\geq\; 9. \qquad (2)$$

We now proceed to establish (2). For this, we note that a, b and c represent the side lengths of a triangle. The left side of (2) is identical to

$$9 + (b-c)^2\left(\tfrac{2}{bc} - \tfrac{1}{a^2}\right) + (c-a)^2\left(\tfrac{2}{ca} - \tfrac{1}{b^2}\right) + (a-b)^2\left(\tfrac{2}{ab} - \tfrac{1}{c^2}\right) \;\geq\; 0.$$

Hence, (2) is established by proving

$$\begin{aligned} bc(b-c)^2(2a^2 - bc) + ca(c-a)^2(2b^2 - ca)& \\ + ab(a-b)^2(2c^2 - ab) \;\geq\; 0.& \qquad (3)\end{aligned}$$

To prove (3), we note that (1) implies that either

$$0 \;\leq\; 2a^2 - bc \;\leq\; 2b^2 - ca \;\leq\; 2c^2 - ab \qquad (4)$$

or

$$2a^2 - bc \;<\; 0 \;<\; 2b^2 - ca \;\leq\; 2c^2 - ab. \qquad (5)$$

If (4) holds, then (3) is true, and we are done.

If (5) holds, we note that the sum of the first two terms of (3) is non-negative, and hence that (3) is true.

2068. [1995: 235] *Proposed by Šefket Arslanagić, Berlin, Germany.*
Find all real solutions of the equation

$$\sqrt{17+8x-2x^2}+\sqrt{4+12x-3x^2}=x^2-4x+13.$$

Solution by Toby Gee, student, the John of Gaunt School, Trowbridge, England.

We have

$$\begin{aligned}
&\sqrt{17+8x-2x^2}+\sqrt{4+12x-3x^2}\\
&= \sqrt{25-2(x-2)^2}+\sqrt{16-3(x-2)^2}\\
&\le \sqrt{25}+\sqrt{16} = 9 \le 9+(x-2)^2\\
&= x^2-4x+13
\end{aligned}$$

with equality throughout if and only if $x-2=0$. Thus $x=2$.

Also solved by SEUNG-JIN BANG, Seoul, Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; PAUL BRACKEN, University of Waterloo; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; TIM CROSS, King Edward's School, Birmingham, England; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; JEFFREY K. FLOYD, Newnan, Georgia, USA; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO CHRISTOV KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; SAI CHONG KWOK, San Diego, CA, USA; KEE-WAI LAU, Hong Kong; THOMAS LEONG, Staten Island, NY, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; BEATRIZ MARGOLIS, Paris, France; J.A. MCCALLUM, Medicine Hat, Alberta; VEDULA N. MURTY, Andhra University, Visakhapatnam, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; CORY PYE, student, Memorial University, St. John's, Newfoundland; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; K.R.S. SASTRY, Dodballapur, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ASHSIH KR. SINGH, student, Kanpur, India; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, the Netherlands; SUSAN SCHWARTZ WILDSTROM, Kensington, Maryland, USA;

KENNETH M. WILKE, Topeka, Kansas, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; an anonymous solver; and the proposer.

2072. [1995: 277] *Proposed by K.R.S. Sastry, Dodballapur, India.*
Find positive integers x, y, u, v such that

$$x^2 + y^2 = u^2 \qquad \text{and} \qquad x^2 - xy + y^2 = v^2.$$

(Equivalently, find a right-angled triangle with integral sides x, y surrounding the right angle and a triangle with sides x, y surrounding a 60° angle, and with the third side an integer in both cases.)

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA. We need only consider x and y relatively prime, in which case $x^2 + y^2$ and $x^2 - xy + y^2$ are also relatively prime. These are squares if and only if their product is a square. It is convenient to consider the possibility that the second equation admit negative integral values of x as well (in that case $|x|$ and y surround a 120° angle instead). Writing $s = \frac{x}{y}$, we consider the rational points of the quartic curve

$$t^2 = (s^2 + 1)(s^2 - s + 1) = s^4 - s^3 + 2s^2 - s + 1.$$

This corresponds to the elliptic curve

$$v^2 = u^3 + 2u^2 - 3u - 6.$$

The quartic curve has an obvious rational point $(0, 1)$, which corresponds to a rational point $P = (2, 2)$ on the elliptic curve. By standard procedures, we determine the multiples nP, $2 \le n \le 8$. These are rational points on elliptic curves with corresponding rational points on the quartic curve. For $n = 2, 3, \ldots$ we have rational points on the quartic curve with

$$s = \frac{15}{8}, -\frac{1768}{2415}, \ldots$$

with corresponding values of x, y, u, and v given below. We separate those with different signs of x. The two solutions corresponding to $n = 2, 3$ can be found in Dickson's *history of the Theory of Numbers*, vol.2, p. 481.

n	2	4	6	7
x	15	8109409	101477031226926255	4676030077060796052820312
y	8	10130640	422390893185635192	1382348542917116969367345
u	17	12976609	434409546986238833	4876078831979879983187537
v	13	9286489	381901401745295077	4160798170065530232858973

Corresponding to $n = 3, 5, 8$ we have solutions with negative x (corresponding to triangles with a 120° angle):

n	3	5	8
$-x$	1768	498993199440	227124445985970945806894399956799
y	2415	136318711969	654056791401866496244333771257120
u	2993	517278459169	692369699180530962038852086430401
v	3637	579309170089	792419135769606228834850391429041

Remark. It is, however, not possible to find two triangles with integral sides, one with sides x, y surrounding a $60°$ angle, another with sides x, y surrounding a $120°$ angle, and with the third side an integer in both cases. This is because the quadratic forms $x^2 + xy + y^2$ and $x^2 - xy + y^2$ cannot be simultaneously made squares. See Dickson, *ibid*.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; JEFFREY FLOYD, Newnan, Georgia; ROBERT GERETSCHLÄGER, Bundes-realgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There was one incorrect solution. Most solvers simply gave the solution corresponding to $n = 2$ in YIU's solution above; several found sufficient conditions first.

2078*. [1995: 278] *Proposed by Šefket Arslanagić, Berlin, Germany.*
Prove or disprove that

$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} \leq \sqrt{c(ab+1)}$$

for $a, b, c \geq 1$.

Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.

Let $a - 1 = x^2$, $b - 1 = y^2$, $c - 1 = z^2$, where $x, y, z \geq 0$. Then the inequality to be proved becomes

$$x + y + z \leq \sqrt{(z^2+1)[(x^2+1)(y^2+1)+1]}.$$

From the Cauchy–Schwarz inequality,

$$x + y = x \cdot 1 + 1 \cdot y \leq \sqrt{(x^2+1)(y^2+1)}$$

and

$$z + \sqrt{(x^2+1)(y^2+1)} \le \sqrt{z^2+1}\sqrt{(x^2+1)(y^2+1)+1},$$

and we are done.

Also solved by MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WOLFGANG GMEINER, Millstatt, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, New Mexico, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

As several solvers pointed out, and as can be seen from the above proof, equality holds in the original inequality if and only if

$$a = 1 + t, \qquad b = 1 + \frac{1}{t}, \qquad c = 1 + \frac{1}{ab}$$

for some positive real number t.

2079. [1995: 278] *Proposed by Cristóbal Sánchez–Rubio, I. B. Penyagolosa, Castellón, Spain.*

An ellipse is inscribed in a rectangle. Prove that the contact points of the ellipse with the sides of the rectangle lie on the rectangular hyperbola which passes through the foci of the ellipse and whose asymptotes are parallel to the sides of the rectangle.

Solution by P. Penning, Delft, the Netherlands.

Let the ellipse be $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ with $a > b$. Suppose that m and m' are the slopes of the sides of the rectangle, so that $mm' = -1$. By symmetry, the centre of the hyperbola will coincide with the origin. The square of one half of the distance between the foci is $a^2 - b^2$. Thus, the equation of the hyperbola is

$$(mx - y)(m'x - y) \;=\; a^2 - b^2.$$

Let $x = a = cos(\alpha)$, $y = b\sin(\alpha)$ be any of the four points of contact of the rectangle with the ellipse. The tangent to the ellipse at this point is

$$\frac{a\cos(\alpha)}{a} + \frac{y\sin(\alpha)}{b} \;=\; 1.$$

Thus we have

$$m = \frac{-b}{a\tan(\alpha)}, \qquad m' = \frac{a\tan(\alpha)}{b},$$

$$mx - y = \frac{-b\cos^2(\alpha)}{\sin(\alpha)} - b\sin(\alpha) = \frac{-b}{\sin(\alpha)},$$

$$m'x - y = \frac{-(a^2 - b^2)\sin(\alpha)}{b}.$$

Substitution into the equation of the hyperbola shows that this point of contact lies on the hyperbola.

[Ed: This argument fails if the rectangle has sides parallel to the axes. Only Konečný pointed out that in this case, the hyperbola is degenerate and is, in fact, given by the axes.]

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; JORDI DOU, Barcelona, Spain; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2080. [1995: 278] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Find the number of maps f from S to S such that $f^{2080}(x) = x$ for every $x \in S$. (Here the superscript denotes iteration: $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(X))$ for all $n > 1$.)

Solution by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.

Perhaps not too surprisingly, knowing Marcin, there are **2080** such maps. We can see this in the following manner.

First of all, each f must be one-to-one in S, since there would otherwise exist an $x \in S$ such that there is no $y \in S$ with $f(y) = x$, which would contradict $f(f^{2079}(x)) = f^{2080}(x) = x$. f must therefore be a permutation of the elements of S.

We know that every permutation of a finite set can be expressed uniquely as the product of cyclic permutations with no common elements. The problem is therefore equivalent to finding the number of such products of cycles, such that the 2080-th iteration is the identity. In order for this to be the case, the length of each cycle must be a divisor of 2080. The lengths of such cycles can therefore be either 1, 2, 4 or 5. We now consider three cases.

(a) There exists a cycle of length five.

There are $\binom{7}{5}$ ways to select the five numbers in the cycle, and $4!$ ways to build a cycle of five given numbers. Since the remaining numbers can either build a cycle of length two, or two of length one, there are $\binom{7}{5} \cdot 4! \cdot 2 = 1008$ such maps.

(b) There exists a cycle of length four.

There are $\binom{7}{4}$ ways to select the four numbers in the cycle, and $3!$ ways to build a cycle of four given numbers. The remaining three numbers can either each build a cycle of length one, or there are three ways in which to choose two to build a cycle of length two, leaving the other to build a cycle of length one. There are therefore $\binom{7}{4} \cdot 3! \cdot 4 = 840$ such maps.

(c) The only other cycles that exist are of lengths one and two.

There is one such map with no cycle of length two (the identity). There are $\binom{7}{2} = 21$ such maps with precisely one cycle of length two. There are $\frac{1}{2} \cdot \binom{7}{2} \cdot \binom{5}{2} = 105$ such maps with precisely two cycles of length two (choosing 2 from 7, then 2 from the remaining 5, and then dividing by the number of repetitions of the chosen cycles). Finally there are $\frac{1}{3!} \cdot \binom{7}{2} \cdot \binom{5}{2} \cdot \binom{3}{2} = 105$ such maps with precisely three cycles of length two (for the analogous reason as for two such cycles).

Adding up, we have

$$1008 + 840 + 1 + 21 + 105 + 105 = 2080$$

maps with the desired property.

Also solved by HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. There were also six incorrect solutions sent in.

Sealy finds all positive integers n such that there are exactly n maps $f : S \longrightarrow S$ (with $S = \{1, \ldots, 7\}$) satisfying $f^n(x) = x$ for all $x \in S$. He gets:

$$n = 1, 351, 505, 721, 1072, 1225, 1575, 2080, 2800, 3312, 4320, 5040.$$

As Sealy remarks, "You now have proposed problems 2800, 3312, 4320 and 5040."! This list contains two obvious members, 1 and $5040 = 7!$, and the analogous numbers would appear in the corresponding list if we were to use $S = \{1, \ldots, N\}$ for an arbitrary positive integer N. How many numbers are in the list for N? What are the second–smallest and second–largest numbers in the list?

2081. [1995: 306] *Proposed by K.R.S. Sastry, Dodballapur, India.*

In base ten, if we write down the first double-digit integer (10) followed by the last two single-digit integers (8 and 9) we form a four-digit number (1089) which is a *perfect square* (33^2). What other bases exhibit this same property?

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

In base $b \geq 2$, N_b has the four digits $10(b-2)(b-1)$, so

$$N_b = b^3 + (b-2)b + (b-1) = (b-1)(b+1)^2,$$

which shows that N_b is a perfect square if and only if $b-1$ is a perfect square.

(Editor's note: a number of solvers ignored base $b = 2$, where $1001_2 = 9_{10} = (3_{10})^2 = (11_2)^2$.) Benito Muñoz, Fernãndez Moral, and Geretschläger all noted that if $b - 1 = m^2$, then $\sqrt{N_b}$ is the base b two-digit number with both digits equal to m; Vella remarked that in base 10 this $\sqrt{N_b}$ is just the product of m and $b+1$. For example, if $b = 5 = 2^2+1$, $\sqrt{N_5} = 22_5 = 12_{10}$.)

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; TIM CROSS, King Edward's School, Birmingham, England; LUIS V. DIEULEFAIT, IMPA, Rio de Janeiro, Brazil; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; JEFFREY K. FLOYD, Newnan, Georgia, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego. Oswego, NY, USA; DAVID LINDSEY, Austin Peay State University, TN, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; CRIST´ OBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; DAVID C. VELLA, Skidmore College, Saratoga Springs, NY, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There was one incorrect solution.

2082. [1995: 306] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $\angle A > 90°$, and AD, BE and CF are its altitudes (with D on BC, etc.). Let E' and F' be the feet of the perpendiculars from E and F to BC. Suppose that $2E'F' = 2AD + BC$. Find $\angle A$.

Solution by Kee-Wai Lau, Hong Kong.
As usual, let $a = BC$, $b = CA$ and $c = AB$. We have

$$AD = b\sin C, \text{ and } E'F' = a(1 - \sin^2 B - \sin^2 C),$$

where the latter equation comes from $E'F' = a - CF' - BE'$, with $CF' = BC - BF' = a - BF\cos B = a - a\cos^2 B = a\sin^2 B$, and, similarly, $BE' = a\sin^2 C$.

Since we are given $2E'F' = 2AD + BC$, by the sine law, we have

$$2\sin A(1 - \sin^2 B - \sin^2 C) = 2\sin B\sin C + \sin A.$$

Hence,

$$\sin A(1 - 2\sin^2 B - 2\sin^2 C) = 2\sin B\sin C,$$

and so

$$\sin A\left(2\cos(B - C)\cos(B + C) - 1\right) = \cos(B - C) - \cos(B + C),$$

or (since $\cos A = -\cos(B + C)$)

$$-\cos(B - C)(1 + 2\sin A\cos A) = \cos A + \sin A,$$

or

$$(\cos A + \sin A)\left(1 + (\cos A + \sin A)\cos(B - C)\right) = 0.$$

Since $\angle A$ is obtuse, we have

$$\begin{aligned} 1 + (\cos A + \sin A)\cos(B - C) &= 1 + \sqrt{2}\cos(A - 45°)\cos(B - C) \\ &> 1 + \sqrt{2}\left(\frac{-1}{\sqrt{2}}\right) = 0. \end{aligned}$$

Thus, $\cos A + \sin A = 0$, and therefore $\angle A = 135°$.

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; P. PENNING, Delft, the Netherlands; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

2083. [1995: 306] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

The numerical identity $\cos^2 14° - \cos 7°\cos 21° = \sin^2 7°$ is a special case of the more general identity $\cos^2 2x - \cos x\cos 3x = \sin^2 x$. In a similar manner, find a generalization for each of the following numerical identities:

(a) $\tan 55^\circ - \tan 35^\circ = 2\tan 20^\circ$;

(b) $\tan 70^\circ = \tan 20^\circ + 2\tan 40^\circ + 4\tan 10^\circ$;

(c)* $\csc 10^\circ - 4\sin 70^\circ = 2$.

[Ed: it was not easy to decide which submitted solution to highlight here, since the answers to the problem are not unique. We decided to give the proposer's solution to (a) and (b), and one of several submitted for (c).]

Solutions: (a) *and* (b) *by the proposer,* (c)* *by several solvers.*

(a) $\tan x - \tan(90^\circ - x) \;=\; 2\tan(2x - 90^\circ)$,

(b) $\tan(x + 60^\circ) = \tan(30^\circ - x) + 2\tan(60^\circ - 2x) + 4\tan(4x - 30^\circ)$,

(c)* $\csc(2x) - 4\sin(15^\circ - x)\sin(75^\circ - x)\csc(2x) \;=\; 2$.

[Ed: all solvers listed below solved parts (a) and (b). A $\star$ before a solver's name indicates that part (c) was also solved, while a † before a solver's name indicates that a restricted case of part (c) was also solved.]

Solved by **ŠEFKET ARSLANAGIĆ, Berlin, Germany;* **Manuel Benito Muñoz and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain;* **CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA;* **CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario;* **THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CURTIS COOPER, Central Missouri State University, Warrensburg, Missouri, USA;* **TIM CROSS, King Edward's School, Birmingham, England;* **DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario;* **RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; †VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; †DAVID E. MANES, State University of New York, Oneonta, NY, USA; †P. PENNING, Delft, the Netherlands;* **BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany;* **TOSHIO SEIMIYA, Kawasaki, Japan; †DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.*

2084. [1995: 306] *Proposed by Murray S. Klamkin, University of Alberta.*

Prove that

$$\cos\tfrac{B}{2}\cos\tfrac{C}{2} + \cos\tfrac{C}{2}\cos\tfrac{A}{2} + \cos\tfrac{A}{2}\cos\tfrac{B}{2} \geq 1 - 2\cos\tfrac{A}{2}\cos\tfrac{B}{2}\cos\tfrac{C}{2}\ ,$$

where A, B, C are the angles of a triangle.

Solution by the proposer.

Let $x = \cos\frac{A}{2}$, $y = \cos\frac{B}{2}$, $z = \cos\frac{C}{2}$. The inequality is now equivalent to

$$1 \geq \frac{1-x}{1+x} + \frac{1-y}{1+y} + \frac{1-z}{1+z},$$

or

$$1 \geq \tan^2\frac{A}{4} + \tan^2\frac{B}{4} + \tan^2\frac{C}{4}.$$

Since $\tan^2\frac{x}{4}$ is convex for $x \in [0, \pi]$, the latter inequality follows by the majorization inequality. Note that

$$(\pi, 0, 0) \succ (A, B, C) \succ \left(\tfrac{\pi}{3}, \tfrac{\pi}{3}, \tfrac{\pi}{3}\right).$$

So there is equality only from the degenerate triangle of angles π, 0, 0. It also follows that

$$\tan^2\frac{A}{4} + \tan^2\frac{B}{4} + \tan^2\frac{C}{4} \geq 3\tan^2\frac{\pi}{12} = 21 - 12\sqrt{3}.$$

and with equality if and only if the triangle is equilateral.

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, State University of New York, Oneonta, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and PANOS E. TSAOUSSOGLOU, Athens, Greece.

2085. [1995: 307] *Proposed by Iliya Bluskov, student, Simon Fraser University, Burnaby, BC, and Gary MacGillivray, University of Victoria, B. C.*

Find a closed-form expression for the n by n determinant

$$\begin{vmatrix} n & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ -1 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix}.$$

Combination of solutions by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece, and Heinz-Jürgen Seiffert, Berlin, Germany.

Let D_n denote the determinant under consideration. We shall prove that

$$D_n = F_{2n-1} + F_{2n+1} - 2 \qquad \text{for all positive integers } n,$$

where (F_k) is the sequence of the Fibonacci numbers defined by $F_1 = F_2 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$.

Adding all rows to the first row and then adding all columns to the first column, we get that D_n equals the n by n determinant

$$\begin{vmatrix} 3 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix}.$$

By expanding this determinant in minors along the first row, we therefore have

$$\begin{aligned} D_n &= 3A_{n-1} - \left(A_{n-2} + (-1)^n(-1)^{n-2}\right) \\ &\quad +(-1)^{n+1}\left((-1)^{n-2} + (-1)^n A_{n-2}\right) \\ &= 3A_{n-1} - 2A_{n-2} - 2 \end{aligned}$$

for $n \geq 3$, where A_n is the n by n determinant

$$\begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix}.$$

By expanding this along the first row, we find the recurrence relation

$$A_n = 3A_{n-1} - A_{n-2}, \qquad n \geq 3.$$

Since $A_1 = 3 = F_4$ and $A_2 = 8 = F_6$, from the known identity

$$F_n = 3F_{n-2} - F_{n-4}$$

it follows that $A_n = F_{2n+2}$ for all $n \geq 1$. So the original determinant is equal to

$$\begin{aligned} D_n &= 3F_{2n} - 2F_{2n-2} - 2 \\ &= 2F_{2n} + (F_{2n+1} - F_{2n-1}) - 2(F_{2n} - F_{2n-1}) - 2 \\ &= F_{2n+1} + F_{2n-1} - 2 \end{aligned}$$

for $n \geq 3$. The cases $n = 1$ and $n = 2$ are done by direct computation $[D_1 = 1 = F_3 + F_1 - 2,\ D_2 = 5 = F_5 + F_3 - 2]$.

Also solved by MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposers.

Benito and Fernández give the answer in the form $D_n = L_{2n} - 2$, *where* L_n *is the* n^{th} *Lucas number, defined by* $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$. *They also point out that the answer could be written in yet another interesting form:*

$$D_n = \begin{cases} 5F_n^2 = L_n^2 - 4, & n \text{ even}, \\ L_n^2 = 5F_n^2 - 4, & n \text{ odd}. \end{cases}$$

In his article "The sequence 1 5 16 45 121 320 ... *in combinatorics", in the* Fibonacci Quarterly *Vol.* 13 (1975) *pp.* 51–55, *Kenneth Rebman considers the same sequence of integers defined by the determinant in this problem, and gives examples involving graphs and matrices where this sequence arises. He also evaluates a determinant which can easily be shown to be equal to the first determinant in the above solution.*

2086. *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

If the side AC of the spherical triangle ABC has length 120° (that is, it subtends an angle of 120° at the centre), prove that the median from B (that is, the arc of the great circle from B to the midpoint of AC) is bisected by the other two medians.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

This is a special case of a known result:

> *If* G *is the concurrency point of the three medians and* A' *is the midpoint of* BC *then* $\frac{\sin AG}{\sin GA'} = 2\cos\frac{a}{2}$, *etc.*

So, if $a = 120°$, then $AG = GA'$. (The other possibility $AG + GA' = 180°$ is ruled out since A and A' would then be antipodal, forcing triangle ABC to be degenerate.)

Proof. Let $\vec{A}$, $\vec{B}$, $\vec{C}$ denote unit vectors from the centre of the sphere to the vertices A, B, C, respectively. It now follows easily that

$$G = \frac{\vec{A} + \vec{B} + \vec{C}}{\left|\vec{A} + \vec{B} + \vec{C}\right|}$$

lies on each of the medians and hence is the point of concurrency. Also, $\vec{A'} = \dfrac{\vec{B} + \vec{C}}{\left|\vec{B} + \vec{C}\right|}$. Then

$$\sin AG = \frac{\left|\vec{A} \times \left(\vec{A}+\vec{B}+\vec{C}\right)\right|}{\left|\vec{A}+\vec{B}+\vec{C}\right|} = \frac{\left|\vec{A} \times \left(\vec{B}+\vec{C}\right)\right|}{\left|\vec{A}+\vec{B}+\vec{C}\right|},$$

$$\sin GA' = \frac{\left|\left(\vec{A}+\vec{B}+\vec{C}\right) \times \left(\vec{B}+\vec{C}\right)\right|}{\left|\vec{A}+\vec{B}+\vec{C}\right|\left|\vec{B}+\vec{C}\right|} = \frac{\left|\vec{A} \times \left(\vec{B}+\vec{C}\right)\right|}{\left|\vec{A}+\vec{B}+\vec{C}\right|\left|\vec{B}+\vec{C}\right|},$$

so that

$$\frac{\sin AG}{\sin GA'} = \left|\vec{B}+\vec{C}\right| = 2\cos\frac{a}{2}.$$

Also solved by MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, the Netherlands; and the proposer.

Benito and Fernañdez found the general result as an exercise in a nineteenth century text by J.A. Serret; they show it to be an easy application of spherical trigonometry.

2088. [1995: 307] *Proposed by Šefket Arslanagić, Berlin, Germany.*

Determine all real numbers x satisfying the equation

$$\left\lfloor \frac{2x+1}{3} \right\rfloor + \left\lfloor \frac{4x+5}{6} \right\rfloor = \frac{3x-1}{2},$$

where $\lfloor x \rfloor$ denotes the greatest integer $\le x$.

Solution by Luis Dieulefait, IMPA, Rio de Janeiro, Brazil.

Let $\mathbb{Z}$ denote the integers. Then, since $\lfloor(2x+1)/3\rfloor + \lfloor(4x+5)/6\rfloor = (3x-1)/2$ must be an integer,

$$x = \frac{2n+1}{3} \quad \text{for some } n \in \mathbb{Z}. \tag{1}$$

We will also use the fact that

$$\frac{2x+1}{3} + \frac{1}{2} = \frac{4x+5}{6}. \tag{2}$$

For every real number α we have:

$$\alpha - \lfloor\alpha\rfloor < \frac{1}{2} \qquad \Longleftrightarrow \qquad \left(\alpha+\frac{1}{2}\right) - \left\lfloor \alpha + \frac{1}{2}\right\rfloor \ge \frac{1}{2}$$

[that is, the fractional part of α is $< 1/2$ if and only if the fractional part of $\alpha + 1/2$ is $\ge 1/2$], so that

$$\lfloor\alpha\rfloor > \alpha - \frac{1}{2} \qquad \Longleftrightarrow \qquad \left\lfloor \alpha + \frac{1}{2}\right\rfloor \le \alpha.$$

From this, using $\lfloor y \rfloor \leq y$ we obtain

$$\lfloor \alpha \rfloor + \left\lfloor \alpha + \frac{1}{2} \right\rfloor \leq 2\alpha, \tag{3}$$

and using $\lfloor y \rfloor > y - 1$ we obtain

$$\lfloor \alpha \rfloor + \left\lfloor \alpha + \frac{1}{2} \right\rfloor > 2\alpha - 1, \tag{4}$$

both for every real number α. [*Editor's remark*: for example, to get (3) note that for any α either

$$\left\lfloor \alpha + \frac{1}{2} \right\rfloor \leq \alpha \qquad \text{and} \qquad \lfloor \alpha \rfloor \leq \alpha,$$

or

$$\lfloor \alpha \rfloor \leq \alpha - \frac{1}{2} \qquad \text{and} \qquad \left\lfloor \alpha + \frac{1}{2} \right\rfloor \leq \alpha + \frac{1}{2},$$

and adding gives (3) in either case.] Putting inequalities (3) and (4) together and using (2) gives, for $\alpha = (2x+1)/3$:

$$\frac{2(2x+1)}{3} - 1 < \left\lfloor \frac{2x+1}{3} \right\rfloor + \left\lfloor \frac{4x+5}{6} \right\rfloor \leq \frac{2(2x+1)}{3}\,.$$

Using the original equation, this is

$$\frac{4x-1}{3} < \frac{3x-1}{2} \leq \frac{4x+2}{3}\,. \tag{5}$$

From this we obtain $1 < x \leq 7$. Using (1) the only possibilities are

$$x = \frac{5}{3}\,, \frac{7}{3}\,, \frac{9}{3}\,, \ldots\,, \frac{21}{3}\,.$$

So this is exactly the set of all solutions.

Editorial comment. Alternatively, equations (3) and (4) imply that

$$\lfloor \alpha \rfloor + \left\lfloor \alpha + \frac{1}{2} \right\rfloor = \lfloor 2\alpha \rfloor$$

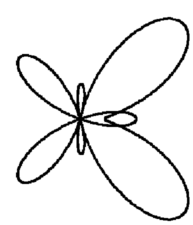
for all α, so the original equation becomes

$$\left\lfloor \frac{4x+2}{3} \right\rfloor = \frac{3x-1}{2}\,,$$

which is equivalent to (5) and (1) together. Benito and Fernández, Wildhagen and the proposer also gave solutions along these lines.

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, *Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; JEFFREY K. FLOYD, Newnan, Georgia, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta, NY, USA; P. PENNING, Delft, the Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; HRISTOS SARAGHIOTES, student, Aristotle University of Thessaloniki, Greece; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. As well, there were ten incorrect solutions sent in, most of which assumed that x had to be an integer.*

In the May 1996 issue of ***CRUX*** [1996: 168], we asked: "Do you know the equation of this curve?"



In the September 1996 issue of ***CRUX*** [1996: 288], we gave the hint: "it is known as the "butterfly"!"

The answer is $r = e^{\cos(\theta)} - 2\cos(4\theta) + \sin^5(\theta/12)$.

If you have any other "nice" curves, please send them to the editor.

(Answer to **Rider** on page ??:)

To buy or not to buy, that is not the question. **Piglet.**