Dissecting Squares into Similar Rectangles

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1 Introductory Remarks.

Karl Scherer and Martin Gardner [1] have proposed the following three-part problem. Cut a square into three similar pieces, where

(1) all three are congruent;

(2) exactly two are congruent;

(3) no two are congruent.

Note that (2) really consists of two sub-problems, where the congruent pieces are

(2a) smaller than the third;

(2b) larger than the third.

All their solutions to (2) are to (2a). It appears that, alas, (2b) is not to be. Also considered in [1] are related dissections of an equilateral triangle, but they do not concern us here.

We generalize the problem of Scherer and Gardner for the square as follows. Given any integer $m > 1$ and any of its 2^{m-1} compositions, or ordered partitions, $m = a_1 + a_2 + \cdots + a_n$, dissect a square into m similar pieces so that there are a_1 congruent pieces of the largest size, a_2 congruent pieces of the next largest size, and so on. In the original problem, $m = 3$ and the compositions are: (1) 3; (2a) $1 + 2$; (2b) $2 + 1$; (3) $1 + 1 + 1$.

Our main result is that the dissection problem always admits a solution using rectangular pieces if and only if the composition is not of the form $k+1$, where k is any positive integer. These solvable cases are covered by two constructions which are only slightly different.

2 Construction for the case $m = a_1 + a_2 + \cdots + a_n, a_n > 1$.

Suppose the composition is $m = a_1 + a_2 + \cdots + a_n$, $a_n > 1$. We start with a rectangle R and divide it into n rectangles $R_1, R_2, \ldots, R_n$ as follows. R_1 is the right half of R, R_2 is the bottom half of $R - R_1$, R_3 is the right half of $R - R_1 - R_2$, R_4 is the bottom half of $R - R_1 - R_2 - R_3$, and so on, except that $R_n = R - R_1 - R_2 - \cdots - R_{n-1}$. The dimensions of these rectangles will be adjusted later. Divide R_i into a_i rectangular pieces, using vertical lines if i is odd and horizontal lines if i is even. Let the horizontal and vertical dimensions of each piece in R_i be x_i and y_i respectively, as shown in Figure 1 for the case $6 = 1 + 3 + 2$.

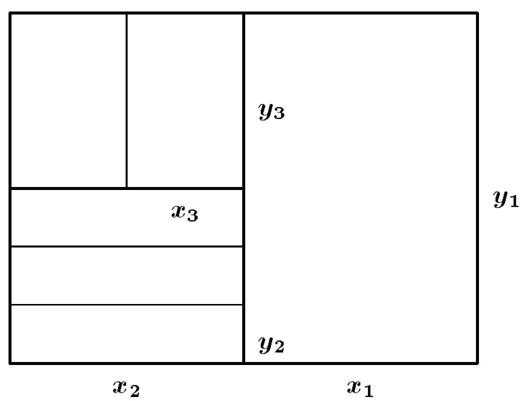


Figure 1

Next, we make all the pieces similar to one another by setting $y_i = x_i t$ for all i, where t is a given positive number. It follows from the assumption $a_n > 1$ that a piece in R_i is larger than a piece in R_j if $i > j$. We choose $x_n = 1$, so that $y_n = t$. If n is even, $y_{n-1} = a_n y_n = a_n t$ and $x_{n-1} = a_n$. If $n > 1$ is odd, $x_{n-1} = a_n x_n = a_n$ and $y_{n-1} = a_n t$.

Going backwards step by step on the basis of the recursive formulae $y_{i-1} = a_i y_i + y_{i+1}$ for i even and $x_{i-1} = a_i x_i + x_{i+1}$ for i odd, we can compute the dimensions of the pieces in $R_{n-2}, R_{n-3}, \ldots, R_1$ and R. Figure 2 is obtained by applying this process to Figure 1. Note that the x_i are always integers while the y_i are always integral multiples of t.

Finally, we want to find a t so that R is a square. In other words, t is a solution of $y_1 = a_1 x_1 + x_2$. Since y_1 is linear in t while $a_1 x_1 + x_2$ is a constant, $y_1 > a_1 x_1 + x_2$ for sufficiently large t. On the other hand, if $t = 1$, then the pieces will be square pieces, so that $y_1 = x_1 \le a_1 x_1 \le a_1 x_1 + x_2$. Since $m > 1$, either $a_1 > 1$ or $x_2 > 0$, so that $y_1 < a_1 x_1 + x_2$. It follows that there exists a real number $t > 1$ for which $y_1 = a_1 x_1 + x_2$.

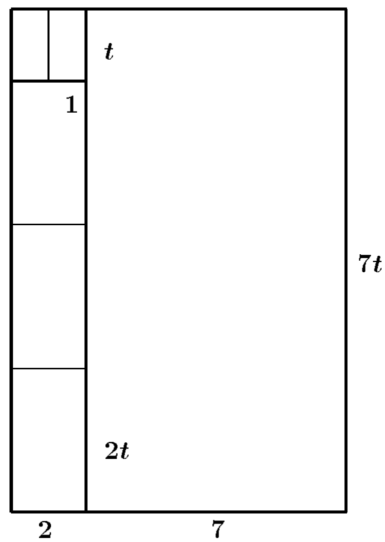


Figure 2

From Figure 2, we have $7t = 9$ or $t = \frac{9}{7}$. In general, since t is the solution of a linear equation with integral coefficients, it is a rational number. Hence we can apply a uniform magnification to make all pieces have integral sides. Figure 3 is a magnification of Figure 2 by a factor of 7.

3 Construction for the case $m = a_1 + a_2 + \cdots + a_n, a_n = 1$ and $n \geq 3$.

Suppose the composition is $m = a_1 + a_2 + \cdots + a_n$, $n \geq 3$ and $a_n = 1$. The first step is identical to the construction in the last Section. We make all the pieces similar to one another by setting $y_i = x_i t$, except that $x_i = y_i t$ for the last odd i. We shall prove that the equation $y_1 = a_1 x_1 + x_2$ has a real solution. Note that, as before, we have $y_1 < a_1 x_1 + x_2$ if $t = 1$.

Suppose $n \geq 4$ is even. Let $x_n = 1$ and $y_n = t$. Then $y_{n-1} = t, x_{n-1} = t^2, x_{n-2} = t^2 + 1$ and $y_{n-2} = t^3 + t$. By induction, x_i is of the form $at^2 + b$ while y_i is of the form $ct^3 + dt$ for all i, where a, b, c and d are integers dependent only on i. Thus $y_1 > a_1 x_1 + x_2$ for sufficiently large t.

Suppose $n \geq 3$ is odd. Let $x_n = t$ and $y_n = 1$. Then $x_{n-1} = t, y_{n-1} = t^2, y_{n-2} = t^2 + 1$ and $x_{n-2} = t + \frac{1}{t}$. By induction, x_i is of the form $at + \frac{b}{t}$ while y_i is of the form $ct^2 + d$ for all i, where a, b, c and d are integers dependent only on i. Thus $y_1 > a_1 x_1 + x_2$ for sufficiently large t.

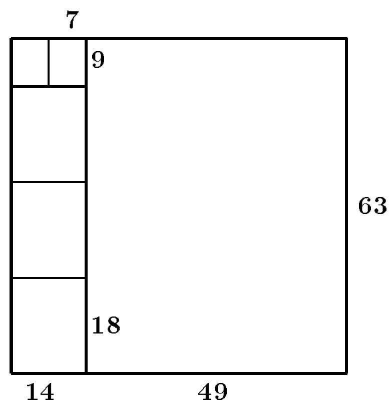


Figure 3

Figure 4 illustrates the cases $6 = 1 + 2 + 2 + 1$ and $6 = 2 + 3 + 1$. In many cases covered by this construction, we can also use pieces with integral sides in the same orientation. Figure 5 illustrates one such example, namely, $11 = 4 + 6 + 1$.

4 Impossibility proof for the case $m = k + 1$.

We now consider the remaining cases, compositions of the form $m = k + 1$, where k is any positive integer. We shall prove that the dissection problem is not solvable with rectangular pieces. Our approach is indirect. We assume that a solution exists, with a "1 by t" piece and k "s by st" pieces, where $t \geq 1$ and $s > 1$ are real numbers.

Let the side length of the square be ℓ. Consider a horizontal or vertical segment from one side to the other, not running along any side of a piece. Then $\ell = as + bst$ for some integers a and b if the segment does not cut the small piece. If it does, we have either $\ell = cs + dst + 1$ or $\ell = es + fst + t$ for some integers c, d, e and f.

From $as + bst = cs + dst = 1$, it follows that s is rational if and only if t is. We first dispose of the case where s and t are rational. Let $s = \frac{g}{h}$ and $t = \frac{i}{j}$, where g and h are relatively prime integers, as are i and j. Then $agj + bgi = cgj + dgi + hj = egj + fgi + hi$. Hence g divides hi and hj. Since g and h are relatively prime, g divides i and j. Since i and j are relatively prime, we must have $g = 1$. However, $s = \frac{1}{h} \leq 1$, which is a contradiction.

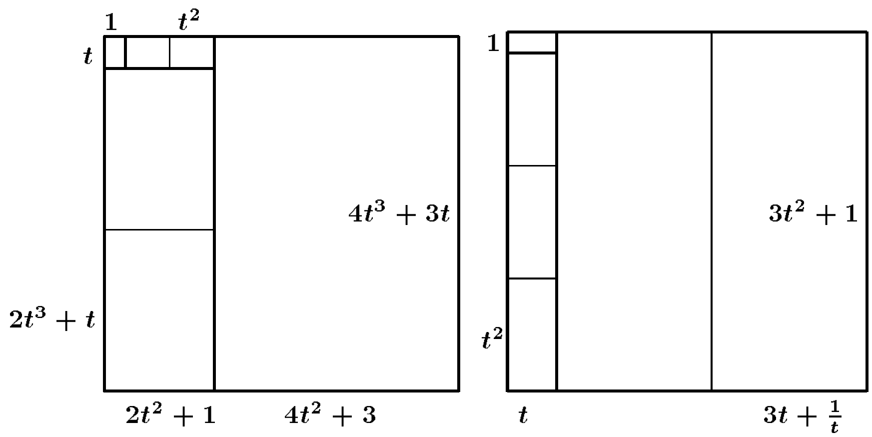


Figure 4

We now consider the case where s and t are irrational, so that $t > 1$. Consider a horizontal or vertical segment from one side of the square to the other, not running along any side of a piece and not cutting the small piece. As before, we have $\ell = as + bst$ for some integers a and b.

We make the important observation that a and b are independent of the choice of such a segment. If we have $\ell = a's + b'st$ for some integers $a' \neq a$ and $b' \neq b$, then $t = \frac{a-a'}{b'-b}$ will be rational. Moreover, since the segment can be horizontal or vertical, we must have $a > 0$ and $b > 0$.

We now use this to prove that the dissected square does not have a fault-line. This is defined as a segment which divides the square into two rectangles, each containing only complete pieces. Suppose to the contrary that there is a horizontal fault-line. Then one of the rectangles is dissected into only large pieces. Divide it into horizontal strips of width s from the top. Each strip contains b "st by s" pieces and a or $2a$ squares of side s, each of which contains parts of one or two s by st pieces. It follows that the height of the rectangle is equal to ps for some positive integer p.

We can rearrange the pieces within this rectangle so that all the horizontal ones are to the left and all the vertical ones are to the right. We then have $ps = qst$ for some positive integer q. However, t will then be rational, which is a contradiction. It follows that the dissected square has no fault-lines.

We now combine the large pieces into rectangles until the union of any two of them is not a rectangle. The small piece is not part of any rectangle. If this combination is not unique, we choose the one for which the number r of rectangles is minimum. If $r \leq 3$, there will be a fault-line. Hence $r \geq 4$.

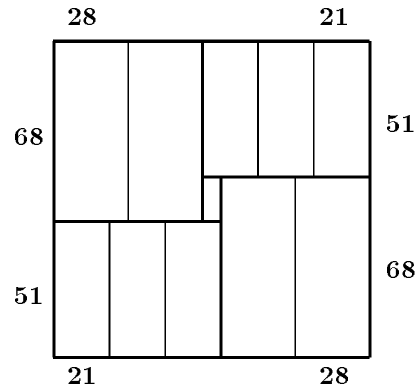


Figure 5

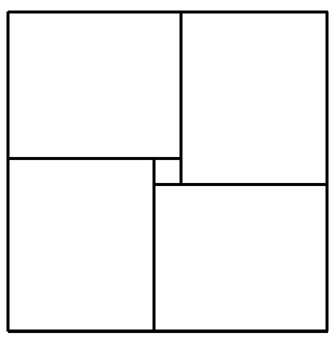


Figure 6

Suppose $r = 4$. The only possible configuration has the four rectangles at the corners and the small piece in the middle, as shown in Figure 6. Now each of the rectangles is dissected into large pieces all of which are in the same orientation. Since $\ell = as + bst$ with constants $a > 0$ and $b > 0$, two opposite rectangles must be dissected into a columns and b rows of s by st pieces, with the other two into a rows and b columns of st by s pieces. However, the small piece will then be a square, and $t = 1$ is a contradiction.

Suppose $r \geq 5$. Then there is a rectangle at a corner of the square which is not adjacent to the small piece. At least one of its two sides within the square is the union of the sides of at least two other rectangles. If the vertical side is not, as illustrated in Figure 7, then the horizontal side must be.

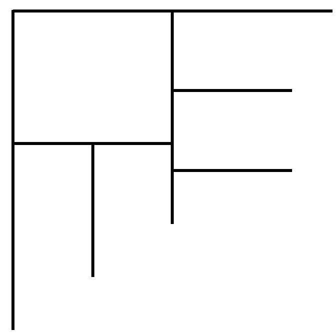


Figure 7

Now this corner rectangle and those immediately below it are dissected into large pieces in the same orientation within each. In fact, it must be the same in all of them, as otherwise we can show that t is rational. Now we can expand the corner rectangle until it swallows up a rectangle below it. However, this reduction in r contradicts its minimality assumption.

5 Concluding Remarks.

We conjecture that for the compositions $m = k + 1$, the dissection problem is not solvable even if non-rectangular polygonal pieces are permitted. This is true for the simplest case, namely, $2 = 1 + 1$. We give a proof using an indirect argument.

Suppose that a solution exists. Clearly, the smaller polygon P cannot contain two opposite corners of the square S. Hence the larger polygon P' contains a side of S. We claim that this is the longest sides of P'. Otherwise, the longest side will be inside S, and it separates P' from P. Hence it will also be the longest side of P, and the two polygons are in fact congruent. This contradiction justifies the claim. It follows that the longest side of P is shorter than a side of S. Now P and P' have the same number of sides inside S, but P' has more sides than P on the boundary of S. Hence P and P' cannot be similar.

On the other hand, Figure 8 shows a "solution" using fractal-like pieces.

For the cases covered by our two constructions, we have many solutions using polygonal pieces that are not rectangles. Figure 9 illustrates one such example, namely, $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$, based on a solution to a problem in [2].

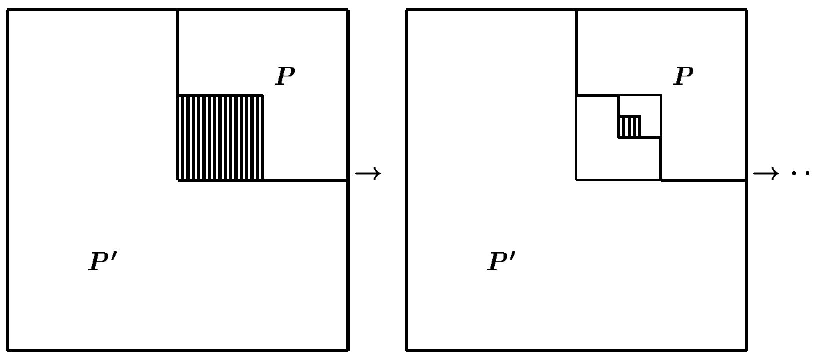


Figure 8

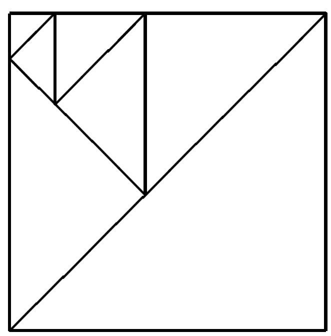


Figure 9

6 References:

[1] Martin Gardner, *Six Challenging Dissection Tasks*, Quantum, Volume 4, Issue 5 (1994) 26–27.

[2] Peter Taylor, "International Mathematics Tournament of the Towns (1980-1984)", Australian Mathematics Trust, Canberra (1993) 106.

THE SKOLIAD CORNER

No. 16

R.E. Woodrow

As a contest this issue, we give the Saskatchewan Senior Mathematics Contest. This contest was written Wednesday, February 22, 1995. My thanks go to Garreth Griffith, Mathematics Department, The University of Saskatchewan, long time organizer of the contests in Saskatchewan, for permission to use the contest and for the solutions we shall give next issue.

1995 SASKATCHEWAN SENIOR MATHEMATICS CONTEST

February 22, 1995 — Time: 1.5 hours

1. They sell regular and jumbo sized orders of fish at Jerry's Fish & Chips Emporium. The jumbo order costs $(\frac{4}{3})$ times as much as a regular one and an order of chips costs $1.30.

Last Thursday, the Emporium was quite busy over the lunch break (11:30 am - 1:30 pm). Exactly thirteen jumbo sized orders of fish along with a quantity of regular orders of fish and chips were sold. $702.52 had been placed in the till.

During the period 1:30 - 4:30, business slackened off. 26 regular orders of fish were sold during this time. Four times as many regular orders had been sold during the lunch break. No jumbo portions were sold and the number of orders of chips declined to one fifth the number that had been sold during the lunch break. At 4:30 pm there was $850.46 in Jerry's till.

What is the price of a regular sized order of fish at Jerry's? [5 marks]

2. $ABCD$ is a square with side of length s. A circle, centre A and radius r is drawn so that the arc of this circle which lies within the square divides the square into two regions of equal area. Write r as a function of s. [6 marks]

3. (a) Solve the equation $3^y = 10^y$. [3 marks]

(b) Solve the equation $3^y = 10$. [3 marks]

(c) Write $t\log_8\sqrt{x} - 2\log_8 y$ as a single logarithm. [4 marks]

4. Establish the identity $2\cot A = \cot\frac{A}{2} - \tan\frac{A}{2}$. [5 marks]

5. ABC is a triangle, right angled at C. Let a, b, c denote the lengths of the sides opposite angles A, B, C respectively. Given that $a = 1$, $\angle B = 75°$ and that $\tan 75° = 2 + \sqrt{3}$, express b and c in the form $p + q\sqrt{3}$ such that $p = 2$ or $\sqrt{2}$. [8 marks]

6. Determine the function $f(x)$ which satisfies all of the following conditions: [8 marks]

(i) $f(x)$ is a quadratic function.

(ii) $f(x + 2) = f(x) + x + 2$.

(iii) $f(2) = 2$.

7. Prove that if n is a positive integer (written in base 10) and that if 9 is a factor of n, then 9 is also a factor of the sum of the digits of n. [8 marks]

Last number we gave the problems of the American Invitational Mathematics Examination. These problems and their solutions are copyrighted by the Committee on the American Mathematical Competitions of the Mathematical Association of America and may not be reproduced without permission. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68588–0322.

Solutions to the 1996 A.I.M.E.

1.	200	**2.**	340	**3.**	044	**4.**	166	**5.**	023
6.	049	**7.**	300	**8.**	799	**9.**	342	**10.**	159
11.	276	**12.**	058	**13.**	065	**14.**	768	**15.**	777

That completes the Skoliad Corner for this issue. Please send me suitable contest materials, your students' nice solutions, and comments or advice for the future of the Corner.

THE OLYMPIAD CORNER

No. 176

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin this number with the problems of the 10th Iberoamerican Mathematical Olympiad, written September 26–27, 1995 at Valparaiso (Chile). My thanks go to Professor Francisco Bellot Rosado of Valladolid, Spain for sending me an English translation of the problems.

10th IBEROAMERICAN MATHEMATICAL OLYMPIAD

September 26–27, 1995 (Valparaiso, Chile)

FIRST DAY — Time: 4.5 hours

1. (*Brazil*). Determine all the possible values of the sum of the digits of all the perfect squares.

2. (*Spain*). Let n be an integer bigger than 1. Determine the real numbers $x_1, x_2, \ldots x_n \geq 1$, and $x_{n+1} > 0$, such that the following conditions are simultaneously fulfilled:

(a) $\sqrt{x_1} + \sqrt[3]{x_2} + \cdots + \sqrt[n+1]{x_n} = n \cdot \sqrt{x_{n+1}}$

(b) $\dfrac{x_1 + x_2 + \cdots + x_n}{n} = x_{n+1}$.

3. (*Brazil*). Let r and s be two orthogonal straight lines, not belonging to the same plane. Let AB be their common perpendicular, with $A \in r$ and $B \in s$. (Note that the plane which contains B and r is perpendicular to s). Consider the sphere with diameter AB. The points $M \in r$, and $N \in s$, are variable, with the condition that MN is tangent to the sphere at some point T. Find the locus of T.

SECOND DAY — Time: 4.5 hours

4. (*Argentina*). Coins are situated on an $m \times m$ board. Each coin situated on the board "dominates" all the cells of the row ($\leftrightarrow$), the column ($\updownarrow$) and the diagonal ($\nwarrow\!\searrow$) to which the coin belongs. Note that the coin does not "dominate" the diagonal ($\swarrow\!\nearrow$). Determine the smallest number of coins which must be placed in order that all the cells of the board be dominated.

5. (*Spain*). The inscribed circumference in the triangle ABC is tangent to BC, CA and AB at D, E and F, respectively. Suppose that this circumference meets AD again at its mid-point X, that is, $AX = XD$. The lines XB and XC meet the inscribed circumference again at Y and Z, respectively. Show that $EY = FZ$.

6. (*Chile–Brazil*). Let $\mathbb{N} = \{1, 2, 3, \dots\}$. A function $f : \mathbb{N} \to \mathbb{N}$ is called *circular* if for each $p \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $n \le p$ such that

$$f^n(p) = \underbrace{f(f(\dots f}_{n \text{ times}}(p))\dots) = p.$$

The function f has *repulse degree* k, $0 < k < 1$, if for each $p \in \mathbb{N}$, $f^i(p) \neq p$ for all $i \le [k \cdot p]$, in which $[x]$ is the integer part of x. Determine the biggest repulse degree that can be reached by a circular function.

The next contest we present in its official language. Here is your opportunity to brush up your French. (Of course we will accept your nice solutions in either English or French!) The questions are to the mini and maxi finals of the 19th Belgian Mathematical Olympiad, written in 1994. My thanks go to Richard Nowakowski, Canadian Team Leader at the 35th IMO in Hong Kong, for collecting this contest (and others) and forwarding them to the Olympiad Corner for its use.

DIX-NEUVIEME OLYMPIADE MATHEMATIQUE BELGE

Mini Finale 1994

1. Combien de nombres enteiers naturels de trois chiffres non nuls distincts (en base 10) sont primiers avec 10?

2. En prenant comme sommets les points d'intersection des côtés prolongés d'un hexagone régulier, Jean obtient un nouvel hexagone régulier. Il applique ensuite la même construction à ce nouvel hexagone, et recommence de même ... Combien de fois Jean doit-il effectuer cette construction pour que l'aire du dernier hexagone construit dépasse 1994 fois l'aire de l'hexagone initial?

3. Trente-huit lampes numérotées de l à 38 sont disposées en cercle autour d'une lampe centrale numérotée 0. Ces lampes forment des groupes de quatre:

$$\{0,1,2,3\},\ \{0,3,4,5\},\ \{0,5,6,7\},$$
$$\{0,7,8,9\}, \dots,\ \{0,35,36,37\},\ \{0,37,38,1\}$$

deux opérations seulement sont réalisables:

(α) éteindre les quatre lampes d'un même groupe;

(β) changer l'ètat de chacune des lampes d'un même groupe (c'est-à-dire, une lampe allumée est éteinte, une éteinte est allumée).

Tout état initial des 39 lampes est-il transformable par une suite de telles opératious en

(a) l'état où toutes les lampes sont allumées?

(b) l'état où seule la lampe numéro 0 est allumée?

4. Sur un terrain plat et carré de 32 ares (ou 3, 200 mètres carrés) dont les côtés sont orientés NO-SE et NE-SO se trouve une villa rectangulaire de 16 mètres sur 20 mètres, dont les quatre façades font face aux quatre points cardinaux. Le centre de la villa coïncide avec le centre du terrain. Le reste du terrain est aménagé en pelouse. Quelle fraction de la pelouse est constituée de points d'où sont visibles deux façades de la villa?

Maxi Finale 1994

1. Un pentagone plan covexe a deux angles droits non adjacents. Les deux côtés adjacents au premier angle droit out des longueurs égales. Les deux côtés adjacents au second angle droit ont des longueurs égales. En remplaçant par leur point milieu les deux sommets du pentagone situés sur un seul côté de ces angles droits, nous formons un quadrilatère. Ce quadrilatère admet-il nécessairement un angle droit?

2. Des lampes en nombre $2n$ (avec $n \geq 2$) et numérotées de 1 à $2n$ sont disposées en cercle autour d'une lampe centrale numérotée 0. Ces lampes forment des groupes de quatre:

$$\{0,1,2,3\},\ \{0,3,4,5\},\ldots\ \{0,2k-3,2k-2,2k-1\},\ \{0,2k-1,2k,2k+1\},$$
$$\{0,2k+1,2k+2,2k+3\},\ldots,\ \{0,2n-1,2n,1\}$$

et deux opérations seulement sont réalisables:

(α) éteindre les quatre lampes d'un même groupe;

(β) changer l'état de chacune des lampes d'un même groupe (c'est-à-dire, une lampe allumée est éteinte, une éteinte est allumée).

Pour quelles valeurs de n tout état initial des $2n+1$ lampes est-il transformable par une suite de telles opérations en

(a) l'état où toutes les lampes sont allumées?

(b) l'état où seule la lampe nuiméro 0 est allumée?

3. Existe-t-il une numérotation des arêtes d'un cube par douze nombres naturels consécutifs telle que

(a) la somme des nombres attribués aux arêtes aboutissant en un sommet soit toujours la même?

(b) la somme des nombres attribués aux arêtes d'une face soit toujours la même?

4. Le plan contient il 1994 points (distincts) non tous alignés tels que la distance entre deux quelconques d'entre eux soit un nombre entier?

As promised last issue, we now give the "official" solutions to problems of the 1996 Canadian Mathematical Olympiad. My thanks go to Daryl Tingley, Chair of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society, for forwarding the problems and solutions. Problems with communications during summer break did not allow us to incorporate the novel solutions of some contest participants. Hopefully next year we will be on track earlier and have time to solicit permission to use the submitted solutions.

1996 CANADIAN MATHEMATICAL OLYMPIAD

1. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}\,.$$

Solution. If $f(x) = x^3 - x - 1 = (x-\alpha)(x-\beta)(x-\gamma)$ has roots α, β, γ standard results about roots of polynomials give $\alpha + \beta + \gamma = 0$, $\alpha\beta + \alpha\gamma + \beta\gamma = -1$, and $\alpha\beta\gamma = 1$.

Then

$$S = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = \frac{N}{(1-\alpha)(1-\beta)(1-\gamma)}$$

where the numerator simplifies to

$$\begin{aligned} N &= 3 - (\alpha+\beta+\gamma) - (\alpha\beta+\alpha\gamma+\beta\gamma) + 3\alpha\beta\gamma \\ &= 3 - (0) - (-1) + 3(1) \\ &= 7. \end{aligned}$$

The denominator is $f(1) = -1$ so the required sum is -7.

2. Find all real solutions to the following system of equations. Carefully justify your answer.

$$\begin{cases} \dfrac{4x^2}{1+4x^2} &= y \\[2ex] \dfrac{4y^2}{1+4y^2} &= z \\[2ex] \dfrac{4z^2}{1+4z^2} &= x \end{cases}$$

Solution. For any t, $0 \leq 4t^2 < 1 + 4t^2$, so $0 \leq \dfrac{4t^2}{1+4t^2} < 1$. Thus x, y and z must be non-negative and less than 1.

Observe that if one of x y or z is 0, then $x = y = z = 0$.

If two of the variables are equal, say $x = y$, then the first equation becomes

$$\frac{4x^2}{1+4x^2} = x.$$

This has the solution $x = 0$, which gives $x = y = z = 0$ and $x = \dfrac{1}{2}$ which gives $x = y = z = \dfrac{1}{2}$.

Finally, assume that x, y and z are non-zero and distinct. Without loss of generality we may assume that either $0 < x < y < z < 1$ or $0 < x < z < y < 1$. The two proofs are similar, so we do only the first case.

We will need the fact that $f(t) = \dfrac{4t^2}{1+4t^2}$ is increasing on the interval $(0, 1)$.

To prove this, if $0 < s < t < 1$ then

$$\begin{aligned} f(t) - f(s) &= \frac{4t^2}{1+4t^2} - \frac{4s^2}{1+4s^2} \\ &= \frac{4t^2 - 4s^2}{(1+4s^2)(1+4t^2)} \\ &> 0. \end{aligned}$$

So $0 < x < y < z \Rightarrow f(x) = y < f(y) = z < f(z) = x$, a contradiction.

Hence $x = y = z = 0$ and $x = y = z = \dfrac{1}{2}$ are the only real solutions.

Alternate Solution. Notice that x, y and z are non-negative. Adding the three equations gives

$$x + y + z = \frac{4z^2}{1+4z^2} + \frac{4x^2}{1+4x^2} + \frac{4y^2}{1+4y^2}.$$

This can be rearranged to give

$$\frac{x(2x-1)^2}{1+4x^2} + \frac{y(2y-1)^2}{1+4y^2} + \frac{z(2z-1)^2}{1+4z^2} = 0.$$

Since each term is non-negative, each term must be 0, and hence each variable is either 0 or $\frac{1}{2}$. The original equations then show that $x = y = z = 0$ and $x = y = z = \frac{1}{2}$ are the only two solutions.

Alternate Solution. Notice that x, y, and z are non-negative. Multiply both sides of the inequality

$$\frac{y}{1+4y^2} \geq 0$$

by $(2y-1)^2$, and rearrange to obtain

$$y - \frac{4y^2}{1+4y^2} \geq 0,$$

and hence that $y \geq z$. Similarly, $z \geq x$, and $x \geq y$. Hence, $x = y = z$ and, as in Solution 1, the two solutions follow.

Alternate Solution. As for solution 1, note that $x = y = z = 0$ is a solution and any other solution will have each of x, y and z positive.

The arithmetic-geometric mean inequality (or direct computation) shows that $\frac{1+4x^2}{2} \geq \sqrt{1 \cdot 4x^2} = 2x$ and hence $x \geq \frac{4x^2}{1+4x^2} = y$, with equality if and only if $1 = 4x^2$, that is, $x = \frac{1}{2}$. Similarly, $y \geq z$ with equality if and only if $y = \frac{1}{2}$ and $z \geq x$ with equality if and only if $z = \frac{1}{2}$. Adding $x \geq y$, $y \geq z$ and $z \geq x$ gives $x + y + x \geq x + y + z$. Thus equality must occur in each inequality, so $x = y = z = \frac{1}{2}$.

3. We denote an arbitrary permutation of the integers $1, \ldots, n$ by $a_1, \ldots, a_n$. Let $f(n)$ be the number of these permutations such that
(i) $a_1 = 1$;
(ii) $|a_i - a_{i+1}| \leq 2$, $i = 1, \ldots, n-1$.
Determine whether $f(1996)$ is divisible by 3.

Solution. Let $a_1, a_2, \ldots, a_n$ be a permutation of $1, 2, \ldots, n$ with properties (i) and (ii).

A crucial observation, needed in Case II (b) is the following: If a_k and a_{k+1} are consecutive integers (i.e. $a_{k+1} = a_k \pm 1$), then the terms to the right of a_{k+1} (also to the left of a_k) are either all less than both a_k and a_{k+1} or all greater than both a_k and a_{k+1}.

Since $a_1 = 1$, by (ii) a_2 is either 2 or 3.

CASE I: Suppose $a_2 = 2$. Then $a_3, a_4, \ldots, a_n$ is a permutation of $3, 4, \ldots, n$. Thus $a_2, a_3, \ldots, a_n$ is a permutation of $2, 3, \ldots, n$ with $a_2 = 2$ and property (ii). Clearly there are $f(n-1)$ such permutations.

CASE II: Suppose $a_2 = 3$.
(a) Suppose $a_3 = 2$. Then $a_4, a_5, \ldots, a_n$ is a permutation of $4, 5, \ldots, n$ with $a_4 = 4$ and property (ii). There are $f(n-3)$ such permutations.
(b) Suppose $a_3 \geq 4$. If a_{k+1} is the first even number in the permutation then, because of (ii), $a_1, a_2, \ldots, a_k$ must be $1, 3, 5, \ldots, 2k-1$ (in that order). Then a_{k+1} is either $2k$ or $2k-2$, so that a_k and a_{k+1} are consecutive integers. Applying the crucial observation made above, we deduce that $a_{k+2}, \ldots, a_n$ are all either greater than or smaller than a_k and a_{k+1}. But 2 must be to the right of a_{k+1}. Hence $a_{k+2}, \ldots, a_n$ are the even integers less than a_{k+1}. The only possibility then, is

$$1, 3, 5, \ldots, a_{k-1}, a_k, \ldots, 6, 4, 2.$$

Cases I and II show that
$f(n) = f(n-1) + f(n-3) + 1, \quad n \geq 4.$ $(\star)$
Calculating the first few values of $f(n)$ directly gives

$$f(1) = 1,\ f(2) = 1,\ f(3) = 2,\ f(4) = 4,\ f(5) = 6.$$

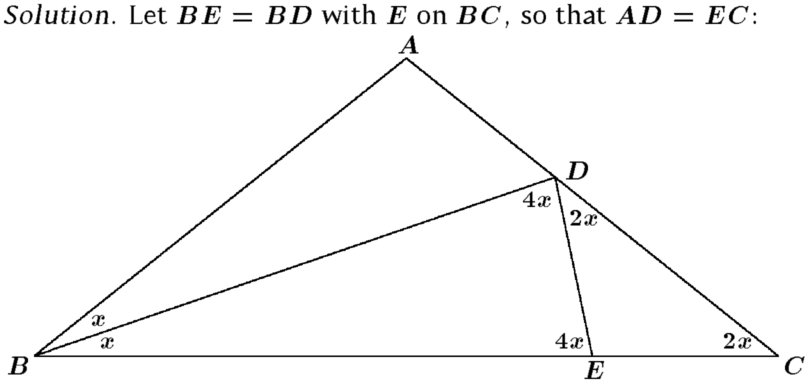
Calculating a few more $f(n)$'s using $(\star)$ and mod 3 arithmetic, $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 1$, $f(5) = 0$, $f(6) = 0$, $f(7) = 2$, $f(8) = 0$, $f(9) = 1$, $f(10) = 1$, $f(11) = 2$. Since $f(1) = f(9)$, $f(2) = f(10)$ and $f(3) = f(11)$ mod 3, $(\star)$ shows that

$$f(a) = f(a \bmod 8),\ \bmod 3,\ a \geq 1.$$

Hence $f(1996) \equiv f(4) \equiv 1 \pmod 3$ so 3 does not divide $f(1996)$.

4. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

Solution. Let $BE = BD$ with E on BC, so that $AD = EC$:



By a standard theorem, $\dfrac{AB}{CB} = \dfrac{AD}{DC}$; so in $\triangle\ CED$ and $\triangle\ CAB$ we have a common angle and

$$\frac{CE}{CD} = \frac{AD}{CD} = \frac{AB}{CB} = \frac{CA}{CB}.$$

Thus, $\triangle CED \sim \triangle CAB$, so that $\angle CDE = \angle DCE = \angle ABC = 2x$.

Hence $\angle BDE = \angle BED = 4x$, whence $9x = 180°$ so $x = 20°$.

Thus $\angle A = 180° - 4x = 100°$.

Alternate Solution. Apply the law of sines to $\triangle ABD$ and $\triangle BDC$ to get

$$\frac{AD}{BD} = \frac{\sin x}{\sin 4x} \quad \text{and} \quad 1 + \frac{AD}{BD} = \frac{BC}{BD} = \frac{\sin 3x}{\sin 2x}.$$

Now massage the resulting trigonometric equation with standard identities to get

$$\sin 2x\,(\sin 4x + \sin x) = \sin 2x\,(\sin 5x + \sin x).$$

Since $0 < 2x < 90°$, we get

$$5x - 90° = 90° - 4x\ ,$$

so that $\angle A = 100°$.

5. Let $r_1, r_2, \ldots, r_m$ be a given set of m positive rational numbers such that $\sum_{k=1}^{m} r_k = 1$. Define the function f by $f(n) = n - \sum_{k=1}^{m}[r_k n]$ for each positive integer n. Determine the minimum and maximum values of $f(n)$. Here $[x]$ denotes the greatest integer less than or equal to x.

Solution. Let

$$\begin{aligned} f(n) &= n - \sum_{k=1}^{m}[r_k n] \\ &= n\sum_{k=1}^{m} r_k - \sum_{k=1}^{m}[r_k n] \\ &= \sum_{k=1}^{m}\{r_k n - [r_k n]\}. \end{aligned}$$

Now $0 \le x - [x] < 1$, and if c is an integer, $(c + x) - [c + x] = x - [x]$.

Hence $0 \le f(n) < \sum_{k=1}^{m} 1 = m$. Because $f(n)$ is an integer, $0 \le f(n) \le m - 1$.

To show that $f(n)$ can achieve these bounds for $n > 0$, we assume that $r_k = \dfrac{a_k}{b_k}$ where a_k, b_k are integers; $a_k < b_k$.

Then, if $n = b_1 b_2 \ldots b_m$, $(r_k n) - [r_k n] = 0$, $k = 1, 2, \ldots, m$ and thus $f(n) = 0$.

Letting $n = b_1 b_2 \ldots b_n - 1$, then

$$\begin{aligned} r_k n &= r_k(b_1 b_2 \ldots b_m - 1) \\ &= r_k\{(b_1 b_2 \ldots b_m - b_k) + b_k - 1)\} \\ &= \text{integer} + r_k(b_k - 1). \end{aligned}$$

This gives

$$
\begin{aligned}
r_k n - [r_k n] &= r_k(b_k - 1) - [r_k(b_k - 1)] \\
&= \frac{a_k}{b_k}(b_k - 1) - \left[\frac{a_k}{b_k}(b_k - 1)\right] \\
&= \left(a_k - \frac{a_k}{b_k}\right) - \left[a_k - \frac{a_k}{b_k}\right] \\
&= \left(a_k - \frac{a_k}{b_k}\right) - (a_k - 1) \\
&= 1 - \frac{a_k}{b_k} = 1 - r_k.
\end{aligned}
$$

Hence

$$f(n) = \sum_{k=1}^{m}(1 - r_k) = m - 1.$$

Next we give reader solutions to problems of the *Second Stage Exam of the* 10*th Iranian Mathematical Olympiad* [1995: 9–10].

1. In the right triangle ABC ($A = 90°$), let the internal bisectors of B and C intersect each other at I and the opposite sides D and E respectively. Prove that the area of quadrilateral $BCDE$ is twice the area of the triangle BIC.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Christopher J. Bradley, Clifton College, Bristol, UK; Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany; Cyrus Hsia, student, University of Toronto, Toronto, Ontario; Joseph Ling, University of Calgary, Calgary, Alberta; and by Dieter Ruoff, Department of Mathematics and Statistics, The University of Regina, Regina, Saskatchewan. We give the solution of Covas.

If b and c are the legs, a the hypotenuse, s the semiperimeter and r the inradius of the given right triangle then it is known that $r = s - a$.

The area of such a triangle is $bc/2$. On the other hand, the area of any triangle is sr. Setting the two expressions equal we have

$$bc = 2sr = 2s(s - a) \tag{1}$$

We see that (see figure 1.1)

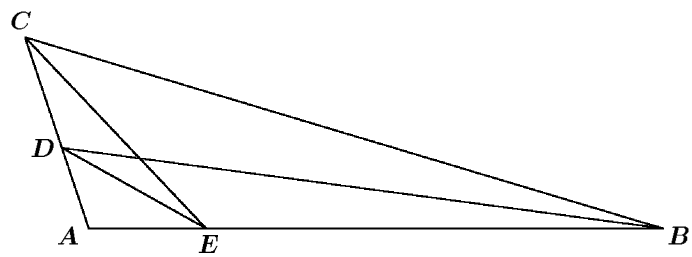


Figure 1.1

$$\text{area of quadrilateral } BCDE = \text{area } (\triangle ABC) - \text{area } (\triangle AED). \quad (2)$$

Since $AE = bc/(a+b)$ and $AD = bd/(a+c)$ we have Area $(\triangle AED) = \frac{1}{2}\frac{bc}{a+b} \cdot \frac{bc}{a+c}$. Substituting, using (2), we have

$$\begin{aligned} &\text{area of quadrilateral } BCDE \\ &= \frac{1}{2}bc - \frac{1}{2}\frac{bc}{a+b} \cdot \frac{bc}{a+c} \\ &= \frac{1}{2}bc\left(1 - \frac{bc}{(a+b)(a+c)}\right) \\ &= \frac{1}{2}bc\frac{a(a+b+c)}{(a+b)(a+c)} = \frac{abcs}{(a+b)(a+c)}. \quad (3) \end{aligned}$$

Since $(a+b)(a+c) = a(a+b+c) + bc = 2as + 2s(s-a) = 2s^2$ we can write (3) in the form

$$\text{Area of quadrilateral } BCDE = \frac{abc}{2s}.$$

Finally, we substitute $2rs$ for bc from (1), simplify, and obtain Area of quadrilateral $BCDE = ar = 2(\text{area of } \triangle BIC)$, which was to be proved.

Editor's Note. Both Ling and Ruoff generalized the result proving more. Here is Ruoff's generalization.

Let $\triangle ABC$ be a triangle with angles 2α, 2β, 2γ, D the intersecting point of the angle bisector at B and AC, E the intersecting point of the angle bisector at C and AB, and I the incentre of $\triangle ABC$. Then

$$|\triangle BCI| \gtreqless \frac{1}{2}|BCDE| \quad \text{iff } \alpha \gtreqless \frac{\pi}{4}. \quad (1)$$

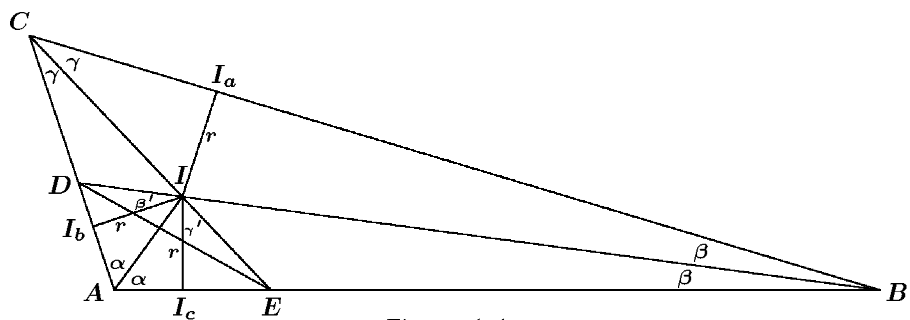


Figure 1.1a

Proof. Let I_a, I_b, I_c be the projections of I to BC, CA, AB respectively and note the LHS of (1) is equivalent to

$$|\triangle BII_c| + |\triangle CII_c| \gtreqless |BICDEB|,$$

$$|BICAB| - |II_bAI_c| \gtreqless |BICAB| - |\triangle ADE|$$

and

$$|\triangle ADE| \gtreqless |II_bAI_c|. \tag{2}$$

The essence and heuristic departure point of the following is the proof of

$$|\triangle BCI| = \frac{1}{2}|BCDE| \tag{1*}$$

respectively

$$|\triangle ADE| = |II_bAI_c| \tag{2*}$$

for triangles with right angle at A ($\alpha = \frac{\pi}{4}$). In this case II_cAI_c is obviously a square, $\triangle ADE$ a right triangle, $\angle I_cID = \beta' = \beta$, $\angle I_cIE = \gamma' = \gamma$, consequently $\beta' = \frac{\pi}{4} - \gamma'$, and (2*) becomes for $II_b = II_c = r$.

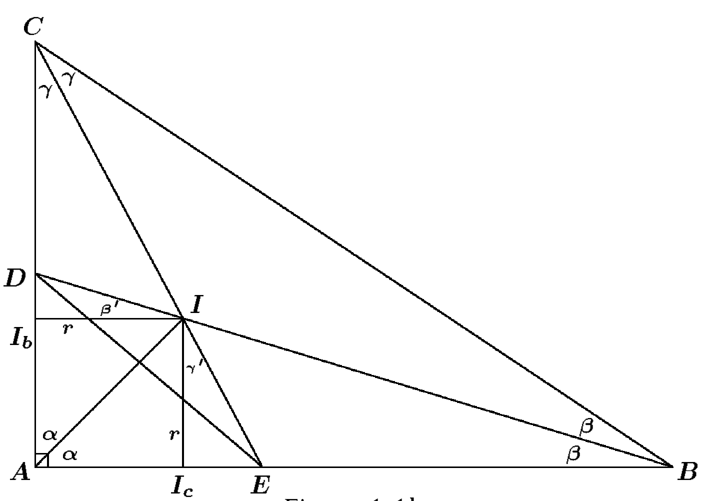


Figure 1.1b

$$\frac{r^2}{2}(1+\tan\gamma')(1+\tan(\frac{\pi}{4}-\gamma'))=r^2, \tag{3*}$$

which holds because of the trigonometric identity

$$(1+\tan x)(1+\tan(\frac{\pi}{4}-x))=2.$$

Returning to the general case, we first assume that I_c lies between A and E, and I_b between A and D, in which case (2) amounts to

$$\frac{r^2}{2}(\cot\alpha+\tan\gamma')(\cot\alpha+\tan\beta')\cdot\sin^2\alpha \gtreqless r^2\cot\alpha \tag{3}$$

respectively to

$$(1+\tan\alpha\tan\gamma')(1+\tan\alpha\tan\beta')\cdot\cos^2\alpha \gtreqless 1.$$

Multiplying this equation by $\dfrac{1}{\cos^2\alpha}=1+\tan^2\alpha$, we obtain

$$\frac{\tan\gamma'+\tan\beta'}{1-\tan\gamma'\cdot\tan\beta'}=\tan(\gamma'+\beta') \gtreqless \tan\alpha. \tag{4}$$

The angle sum in the quadrilateral $AEID$ is

$$2\alpha+(2\beta+\gamma)+(\gamma'+\pi-2\alpha+\beta')+(2\gamma+\beta)=2\pi \tag{5}$$

and hence

$$\begin{aligned}\gamma'+\beta' &= \pi-3(\beta+\gamma) = \pi-3\left(\frac{\pi}{2}-\alpha\right)\\ &= 3\alpha-\frac{\pi}{2} \gtreqless \alpha \text{ for } \begin{cases}\alpha>\frac{\pi}{4}\\ \alpha=\frac{\pi}{4}\\ \alpha<\frac{\pi}{4}\end{cases}.\end{aligned} \tag{6}$$

The conditions on the RHS of (6) determine the signs of (4) and one by one those of (3), (2) and (1), q.e.d.

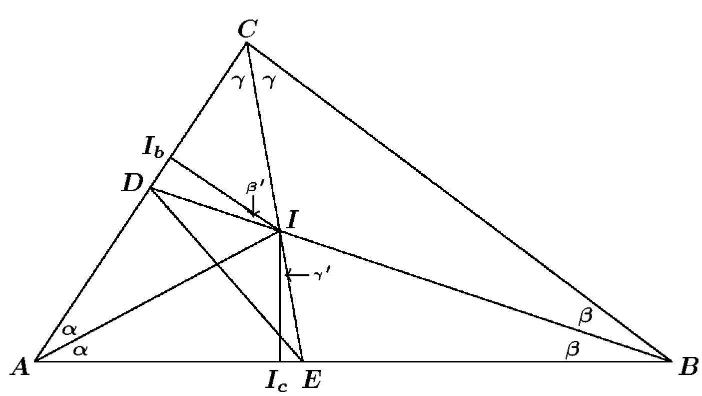


Figure 1.1c

If I_c lies between A and E, but D between A and I_b then, in (3) and in the following, β' has to be multiplied by -1, and (4), (5), (6) turn into

$$\tan(\gamma' - \beta') \gtreqless \tan\alpha, \qquad (4^*)$$

$$2\alpha + (2\beta + \gamma) + (\gamma' + \pi - 2\alpha - \beta') + (2\gamma + \beta) = 2\pi \qquad (5^*)$$

and

$$\gamma' - \beta' = 3\alpha - \frac{\pi}{2} \gtreqless \alpha \text{ for } \begin{cases} \alpha > \frac{\pi}{4} \\ \alpha = \frac{\pi}{4} \\ \alpha < \frac{\pi}{4} \end{cases}, \qquad (6^*)$$

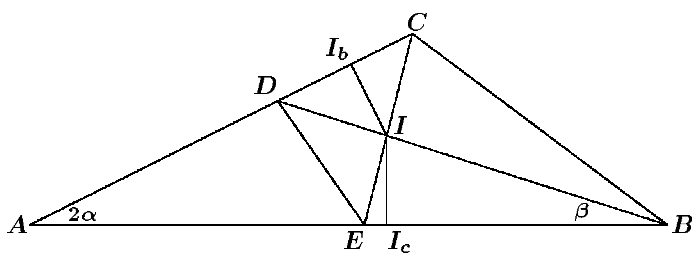


Figure 1.1d

from which, however, the same conclusions result as before. Since $2\alpha + \beta < \frac{\pi}{2}$, the only de facto applicable condition is $\alpha < \frac{\pi}{4}$. Also, if E lies between A and I_c and D between A and I_b, $\alpha < \frac{\pi}{4}$ is the only applicable condition; that the $<$-sign holds in (2) follows directly from the figure.

2. Given the sequence $a_0 = 1$, $a_1 = 2$, $a_{n+1} = a_n + \frac{a_{n-1}}{1+(a_{n-1})^2}$, $n > 1$, show that $52 < a_{1371} < 65$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Note first that "$n > 1$" in the statement should have been "$n \geq 1$" for the problem to be correct. We show that in general,

$$\sqrt{2n+1} \leq a_n \leq \sqrt{3n+2} \text{ for all } n \geq 0. \qquad (1)$$

Since when $n = 1371$, $\sqrt{2n+1} = \sqrt{2743} \approx 52.374$ and $\sqrt{3n+2} = \sqrt{4115} \approx 64.148$, $52 < a_{1371} < 65$ would follow. To establish (1), we first show by induction that

$$a_n = a_{n-1} + \frac{1}{a_{n-1}} \text{ for all } n \geq 1. \qquad (2)$$

This is clearly true for $n = 1$ since $a_1 = 2 = a_0 + \frac{1}{a_0}$. Suppose (2) holds for some $n \geq 1$. Then

$$a_n = \frac{(a_{n-1})^2 + 1}{a_{n-1}} \Rightarrow \frac{1}{a_n} = \frac{a_{n-1}}{1 + (a_{n-1})^2}$$

and thus, from the given recurrence relation, we get $a_{n+1} = a_n + \frac{1}{a_n}$, completing the induction. Since clearly $a_n > 0$ for all n, we see from (2) that the sequence $\{a_n\}$ is strictly increasing. In particular, $\frac{1}{a_{n-1}^2} \leq 1$ for all $n \geq 1$ and so from $a_n^2 = a_{n-1}^2 + 2 + \frac{1}{a_{n-1}^2}$ we get

$$a_{n-1}^2 + 2 < a_n^2 \leq a_{n-1}^2 + 3 \text{ for all } n \geq 1. \tag{3}$$

Now we use (3) and induction to establish (1). The case when $n = 0$ is trivial since $a_0 = 1 < \sqrt{2}$. Suppose (1) holds for some $n \geq 0$. Then by (3),

$$a_{n+1} \leq \sqrt{a_n^2 + 3} \leq \sqrt{3n + 2 + 3} = \sqrt{3(n+1) + 2}$$

and

$$a_{n+1} > \sqrt{a_n^2 + 2} \geq \sqrt{2n + 1 + 2} = \sqrt{2(n+1) + 1}$$

and our proof is complete.

3. There is a river with cities on both of its sides. Some boat lines connect these cities in such a way that each line connects a city of one side to a city on the other side, and each city is joined exactly to k cities on the other side. One can travel between every two cities. Prove if one of the boat lines is cancelled, one can travel between every two cities.

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario.

Originally, one can travel between every two cities. If we consider cities as vertices and boat lines as edges on a graph, then this graph is connected. We must show that if an edge is removed, then the graph is still connected.

We interpret cities on the two sides of the river as a bipartite graph since each line connects a city of one side to a city on the other side. As well, each vertex has k edges. If we count the number of edges for each vertex on one side we are counting all the vertices because each edge has exactly one end on that side. Thus, originally we have ak edges (where a is the number of vertices on one of the sides). This shows that we must have a vertices on both sides.

Now suppose, on the contrary, that by removing an edge we have two disjoint graphs. Suppose it is divided into parts U, V, X, Y where $U \cup V$ and $X \cup Y$ are the vertices of the two sides and the remaining edges are between U and X, and V and Y respectively. Further, since one edge was removed, let U and V have $k|U| - 1$ and $k|V|$ incident edges respectively. If X has $k|X| - 1$ incident edges, that would mean that originally the whole graph was not connected. Therefore X has $k|X|$ edges remaining and Y has $k|Y| - 1$ edges remaining. This is impossible! $k|U| - 1 \neq k|V|$ unless $k = 1$. If $k = 1$, $a = 1$ or else we would have disjoint lines!

Now there are cities on both sides, so $a > 1$, so $k > 1$, and it is impossible to have two disjoint bipartite graphs by removing one edge.

4. Prove that for each natural number t, 18 divides

$$A = 1^t + 2^t + \cdots + 9^t - (1 + 6^t + 8^t).$$

Remarks by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany; and by Stewart Metchette, Gardena, California, USA.

There must be a misprint.

For $t = 1$, $A = 1 + 2 + 3 + \cdots + 9 - (1 + 6 + 8) = 30$,

and $t = 2$, $A = 1 + 2^2 + 3^2 + \cdots + 9^2 - (1 + 6^2 + 8^2) = 285 - 101 = 184$.

Both are not divisible by 18.

5. In the triangle ABC we have $A \le 90°$ and $B = 2C$. Let the internal bisector of C intersect the median AM (M is the mid-point of BC) at D. Prove that $\angle MDC \le 45°$. What is the condition for $\angle MDC = 45°$?

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

It suffices to show that $\tan(\angle MDC) \le 1$.

If $\triangle ABC$ has sides a, b, c, in the usual order, then the condition $B = 2C$ is equivalent to the condition $b^2 = c(c + a)$ (see this journal [1976: 74] and [1984: 287]). (1)

We introduce a Cartesian frame with origin at B and x-axis along BC:

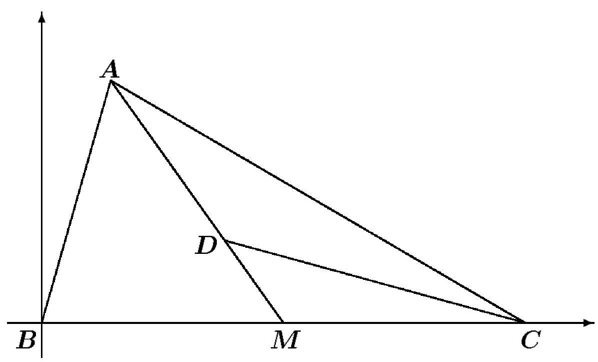


Figure 5

The coordinates of A are $(c \cdot \cos B, c \cdot \sin B)$, the coordinates of C are $(a, 0)$, and those of M are $(a/2, 0)$.

The internal angle bisector of C has slope

$$m_1 = \tan(180° - C/2) = -\tan(C/2)$$

and the slope of the median AM is

$$m_2 = \frac{c \cdot \sin B}{c \cdot \cos B - a/2}. \tag{2}$$

The law of cosines gives

$$b^2 = c^2 + a^2 - 2ca \cdot \cos B.$$

Substituting for b^2 from (1), we obtain

$$c(c+a) = c^2 + a^2 - 2ca \cdot \cos B,$$

and hence

$$c \cdot \cos B = (a-c)/2.$$

Substituting this into (2), we obtain

$$m_2 = -2 \cdot \sin B.$$

Since $B = 2C$ and $\sin 2C = 2 \sin C \cos C$, this equation may be rewritten as

$$m_2 = -4 \sin C \cos C.$$

Using the formula for the tangent of the angle between two lines, we get

$$\tan(\angle MDC) = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\tan(C/2) + 4 \sin C \cos C}{1 + 4(\tan(C/2)) \cdot \sin C \cos C}. \quad (3)$$

We express $\sin C$ and $\cos C$ in terms of $\tan(C/2)$. Putting $t = \tan(C/2)$, we get

$$\sin C = \frac{2t}{(1+t^2)}, \quad \cos C = \frac{(1-t^2)}{(1+t^2)}.$$

When these are substituted into (3), it becomes

$$\tan(\angle MDC) = \frac{-t + \frac{8t(1-t)^2}{(1+t^2)^2}}{1 + \frac{8t^2(1-t^2)}{(1+t^2)^2}} = \frac{-t(1+t^2)^2 + 8t(1-t^2)}{(1+t^2)^2 + 8t^2(1-t^2)}.$$

We now prove that $\tan(\angle MDC) \leq 1$. This holds if and only if

$$8t(1-t^2) - t(1+t^2)^2 \leq 8t^2(1-t^2) + (1+t^2)^2,$$

or, equivalently,

$$8t(1-t^2) - 8t^2(1-t^2) \leq (1+t^2)^2 + t(1+t^2)^2,$$

or

$$8t(1-t^2)(1-t) \leq (1+t^2)^2(1+t).$$

Dividing both sides by the positive number $1+t$, we get

$$8t(1-t)^2 \leq (1+t^2)^2,$$

equivalent to

$$8t - 16t^2 + 8t^3 \leq 1 + 2t^2 + t^4,$$

which is indeed true since

$$t^4 - 8t^3 + 18t^2 - 8t + 1 = (t^2 - 4t + 1)^2 \geq 0.$$

Equality $\angle MDC = 45^\circ$ occurs if and only if $t^2 - 4t + 1 = 0$, where $t = \tan(C/2)$. This is satisfied when $t = 2 - \sqrt{3}$, i.e. $C = 30^\circ + 360^\circ k$; or $t = 2 + \sqrt{3}$, i.e. $C = 150^\circ + 360^\circ k$ $(k = \dots, -2, -1, 0, 1, 2, \dots)$.

The only acceptable value for C is 30°.

We conclude that $\angle MDC = 45^\circ$ iff $A = 90^\circ$, $B = 60^\circ$, $C = 30^\circ$.

6. Let X be a non-empty finite set and $f : X \to X$ a function such that for all x in X, $f^p(x) = x$, where p is a constant prime. If $Y = \{x \in X : f(x) \neq x\}$, prove that the number of elements of Y is divisible by p.

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution and remarks.

For each $x \in Y$ consider that *orbit*, $B(x)$, of x defined by

$$B(x) = \{x, f(x), f^2(x), \dots, f^{p-1}(x)\}.$$

We claim that all the elements of $B(x)$ are distinct.

Suppose not. Then let j be the *least* positive integer such that $f^i(x) = f^j(x)$ for some integer i with $0 \leq i < j \leq p - 1$. (We define $f^i(x) = x$ if $i = 0$.) Then

$$x = f^p(x) = f^{p-j}(f^j(x)) = f^{p-j}(f^i(x)) = f^{p-j+i}(x)$$
$$\Rightarrow f^{j-i}(x) = f^{j-i}(f^{p-j+i}(x)) = f^p(x) = x.$$

Since $0 < j - i \leq j$, we must have $i = 0$ and thus $f^j(x) = x$. Now let $p = qj + r$, where q, r are integers with $q > 0$ and $0 \leq r < j$. Clearly $f^j(x) = x$ implies $f^{qj}(x) = x$ and hence

$$f^r(x) = f^r(f^{qj}(x)) = f^p(x) = x.$$

Since $r < j$, we must have $r = 0$ and thus $p = qj$. Since $f(x) \neq x$, $j > 1$. On the other hand, since $j < p$, $q > 1$. Hence p is a composite, a contradiction. Therefore, $f^i(x) \neq f^j(x)$ for all $i = 0, 1, 2, \dots, p-1$, we see that $B(x) \subset Y$.

Next we show that the orbits of two elements of Y are either *disjoint* or *identical*. Let $x, y \in Y$ and suppose $B(x) \cap B(y) \neq \emptyset$. Then $f^l(x) = f^k(y)$ for some integers l and k, with $0 \leq l \leq k \leq p - 1$. Hence

$$y = f^k(y) = f^{p-k}(fk(y)) = f^{p-k}(f^l(x)) = f^{p-k+l}(x),$$

which show that $y \in B(x)$. It then follows that $B(x) = B(y)$. Therefore Y can be partitioned into disjoint orbits each having cardinality p and the result follows.

Remarks. (1) Actually, the result still holds even when $p = 1$ since in this case $Y = \emptyset$ and thus $|Y| = 0$. (2) The result need not hold if p is composite. A counterexample is given by $X = \{1, 2, 3, 4, 5, 6\}$, $f(1) = 2$, $f(2) = 1$, $f(3) = 4$, $f(4) = 5$, $f(5) = 6$, and $f(6) = 3$. In this case, $p = 4$ and $Y = X$, $|Y| = 6$.

That completes the Olympiad Corner for this issue. Send me your contests and nice solutions.

THE ACADEMY CORNER

No. 5

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In the February 1996 issue, we gave the first set of problems in the Academy Corner. Here we present solutions to the last three questions, as sent in by Šefket Arslanagić, Berlin, Germany.

Memorial University Undergraduate Mathematics Competition 1995

4. If a, b, c, d are positive integers such that $ad = bc$, prove that $a^2 + b^2 + c^2 + d^2$ is never a prime number.

   We will give a generalization: If a, b, c, d, r, s are positive integers such that $rad = sbc$, prove that $r(a^2 + d^2) + s(b^2 + c^2)$ is never a prime number.

   Solution. All positive integers greater than or equal to 2 can be written as products of finitely many prime numbers. Therefore $rad = sbc = p_1 \ldots p_\ell$ (p_i, p_j not necessarily different) are the prime factors of r, a, d and similarly of factors s, b, c. Also, there exist positive integers $x_{11}, x_{12}, \ldots, x_{33}$ such that

$$r = x_{11}x_{12}x_{13} = y_1; \quad a = x_{21}x_{22}x_{23} = y_2; \quad d = x_{31}x_{32}x_{33} = y_3;$$

$$s = x_{11}x_{21}x_{31} = z_1; \quad b = x_{12}x_{22}x_{32} = z_2; \quad c = x_{13}x_{23}x_{33} = z_3.$$

   We write this as

$$\begin{array}{cccl} x_{11} & x_{12} & x_{13} & \rightarrow \prod = r = y_1 \\ x_{21} & x_{22} & x_{23} & \rightarrow \prod = a = y_2 \\ x_{31} & x_{32} & x_{33} & \rightarrow \prod = d = y_3 \\ \downarrow & \downarrow & \downarrow & \\ \prod = s & \prod = b & \prod = c & \\ = z_1 & = z_2 & = z_3 & \end{array}$$

Let

$$x_{iji} = \prod_{\substack{px \in y_i \\ px \in z_j \\ k = 1, \ldots, \ell}} px$$

Now, we get

$$\begin{aligned}
&r(a^2 + s^2) + s(b^2 + c^2) \\
&= x_{11}x_{12}x_{13}(x_{21}^2x_{22}^2x_{23}^2 + x_{31}^2x_{32}^2x_{33}^2) \\
&\quad +x_{11}x_{21}x_{31}(x_{12}^2x_{22}^2x_{32}^2 + x_{13}^2x_{23}^2x_{33}^2 \\
&= x_{11}x_{12}x_{13}x_{21}^2x_{22}^2x_{23}^2 + x_{11}x_{12}x_{13}x_{31}^2x_{32}^2x_{33}^2 \\
&\quad +x_{11}x_{21}x_{31}x_{12}^2x_{22}^2x_{32}^2 + x_{11}x_{21}x_{31}x_{12}^2x_{23}^2x_{33}^2 \\
&= x_{11}x_{12}x_{21}x_{22}^2(x_{13}x_{21}x_{23}^2 + x_{12}x_{31}x_{32}^2) \\
&\quad +x_{11}x_{13}x_{31}x_{33}^2(x_{12}x_{31}x_{32}^2 + x_{12}x_{21}x_{23}^2) \\
&= (x_{11}x_{12}x_{21}x_{22}^2 + x_{11}x_{13}x_{31}x_{33}^2) \\
&\quad \times(x_{13}x_{21}x_{23}^2 + x_{12}x_{31}x_{32}^2).
\end{aligned}$$

Thus, $r(a^2 + d^2) + s(b^2 + c^2)$ is the product of two factors of positive integers, and is greater than or equal to 2, because $x_{ij} \geq 1$ for all i, j.

Therefore, $r(a^2 + d^2) + s(b^2 + c^2)$ is never a prime number.

For $r = s + 1$, that is, $ad = bc$, the sum $a^2 + b^2 + c^2 + d^2$ is never a prime number.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$$

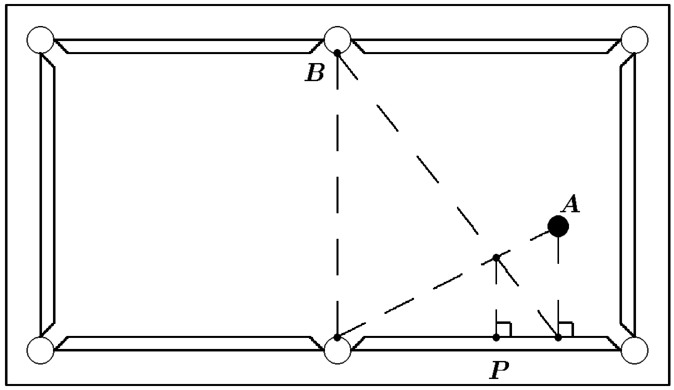
for all real numbers x, y.

Solution. This has already appeared in *CRUX* [1993: 41–42]. The problem was used in the *XLI Mathematics Olympiad in Poland*.

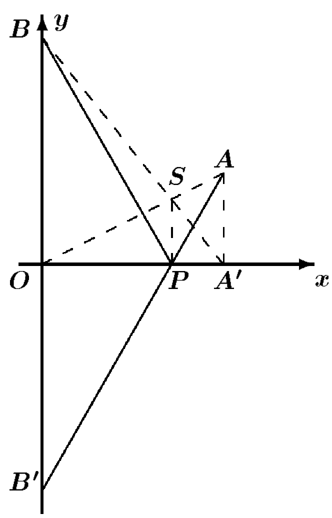
6. Assume that when a snooker ball strikes a cushion, the angle of incidence equals the angle of reflection.

For any position of a ball A, a point P on the cushion is determined as shown.

Prove that if the ball A is shot at point P, it will go into the pocket B.



Solution. We will use analytic geometry. We let the axes be $Ox \equiv OP$, $Oy = OB$, $Ox \perp Oy$; and give coordinates: $O(0,0)$, $A(a,b)$, $B(0,c)$, $A'(a,0)$; we have



OA: $y = \frac{b}{a}x$
BA': $y = -\frac{c}{a}x + c$,
$\{S\} = OA \cap BA'$, i,e,
$S\left(\frac{ac}{b+c}, \frac{bc}{b+c}\right)$ and $P\left(\frac{ac}{b+c}, 0\right)$
AP: $y = \frac{b+c}{a}x - c$,
that is, $B'(0, -c)$.
Therefore $|OB| = |OB'| = c$ and
$\Delta OBP \simeq \Delta OB'P$,
that is, $\angle BPO = \angle B'PO$.

Because $\angle B'PO = \angle APA'$,
we obtain $\angle BPO = \angle APA'$.

BOOK REVIEWS

Edited by ANDY LIU

Experience in Problem Solving - a W. J. Blundon Commemorative,
edited by R. H. Eddy and M. M. Parmenter.
Published by the Atlantic Provinces Council on the Sciences, 1994,
ISBN # 0-9698965-0-6, 82+ pages, hardbound, $25.00.
Reviewed by **Murray S. Klamkin**, *University of Alberta.*

This book is the culmination of efforts of its editors, whose idea it was to honour the late W. J. Blundon, by collecting his problem solving contributions to the mathematical literature. The preface contains a short bibliography of Blundon. *CRUX* readers before 1990 may remember some of his contributions.

The book consists of all (as far as its editors are aware) the published problems and/or solutions of Blundon and which have appeared in Crux, American Mathematical Monthly, Mathematics Magazine, College Mathematics Journal, Elemente der Mathematik, SIAM Review, and Nieuw Archief voor Wiskunde. The problems and solutions are given under four categories: Geometry, Geometric Inequalitites, Number Theory, and Miscellaneous.

Jack Blundon, whom I knew personally from my days on the Canadian Mathematical Olympiad Committee when he was the chairman, was a very personable man as well as mathematician, and was particularly fond of geometric inequalities (as can be seen from the corresponding section of the book). Further evidence of this is his several papers on this topic. In particular, his seminal paper *Inequalities associated with the triangle*, Canadian Mathematics Bulletin 8 (1963) 615-627, was a catalyst for a subsequent series of papers from around the world dealing with the theory for triangle inequalities, on which very little had been done before. He was also, as the editors note and I agree, an ardent disciple of elegance in mathematical exposition, often rewriting a proposal or solution several times in order to meet his exacting standards. As a problem section editor, this is something I would hope my contributors would also do.

As a very small sampling of his nice, easily understood proposals from each of the four sections and which did not appear in Crux, we have:

- If, in triangle ABC, we have

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \sqrt{3},$$

  prove that at least one angles of the triangle is $60°$.

- For any triangle (other than equilateral), with circumcentre O, incentre I and orthocentre H, let the angles have measures $\alpha \le \beta \le \gamma$. Prove that

  (1) $1 < \frac{OH}{IO} < 3$;

  (2) $0 < \frac{IH}{OH} < \frac{2}{3}$;

  (3) $0 < \frac{IH}{IO} < 1$ if $\beta < 60^\circ$, $IH = IO$ if $\beta = 60^\circ$,
  $1 < \frac{IH}{IO} < 2$ if $\beta > 60^\circ$.

- Find all solutions in integers of the equation

$$y^2 + y = x^4 + x^3 + x^2 + x.$$

- Find necessary and sufficient conditions on a, b and c in order that the system of equations

$$x + \frac{1}{x} = a, \quad y + \frac{1}{y} = b, \quad xy + \frac{1}{xy} = c$$

  has at least one solution.

Finally, it is to be noted that the editors, who were younger colleagues of Professor Blundon at Memorial University, have done a good job in compiling this nice collection of eighty-one problems and solutions. Not only did it remind me of results that I had forgotten and was looking for, it also suggested to me a number of new problems which I will be submitting to Crux!

[Ed.: The book is available, only from:

Atlantic Provinces Council on the Sciences
Memorial University of Newfoundland
P.O. Box 4200, St. John's, Newfoundland
Canada A1C 5S7

for CDN \$25 plus CDN \$3 for shipping and handling.]

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that* **1 April 1997**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX, *preferably in* LaTeX2e). *Graphics files should be in epic format, or plain postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2164. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let D be a point on the side BC of triangle ABC, and let E and F be the incentres of triangles ABD and ACD respectively. Suppose that B, C, E, F are concyclic. Prove that

$$\frac{AD+BD}{AD+CD} = \frac{AB}{AC}.$$

2165. *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

Given a triangle ABC, prove that there exists a unique pair of points P and Q such that the triangles ABC, PQC and PBQ are directly similar; that is, $\angle ABC = \angle PQC = \angle PBQ$ and $\angle BAC = \angle QPC = \angle BPQ$, and the three similar triangles have the same orientation. Find a Euclidean construction for the points P and Q.

2166. *Proposed by K.R.S. Sastry, Dodballapur, India.*

In a right-angled triangle, establish the existence of a unique interior point with the property that the line through the point perpendicular to any side cuts off a triangle of the same area.

2167. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove, **without the aid the differential calculus**, the inequality, that in a right triangle

$$\frac{a^2(b+c)+b^2(a+c)}{abc} \geq 2+\sqrt{2},$$

where a and b are the legs and c the hypotenuse of the triangle.

2168. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let P be a point inside a regular tetrahedron $ABCD$, with circumradius R and let R_1, R_2, R_3, R_4 denote the distances of P from vertices of the tetrahedron. Prove or disprove that

$$R_1R_2R_3R_4 \leq \frac{4}{3}R^4,$$

and that the maximum value of $R_1R_2R_3R_4$ is attained.

2169. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

AB is a fixed diameter of circle $\Gamma_1(0,R)$. P is an arbitrary point of its circumference. Q is the projection onto AB of P. Circle $\Gamma_2(P_1PQ)$ intersects Γ_1 at C and D. CD intersects PQ at E. F is the midpoint of AQ. $FG \perp CD$, where $G \in CD$. Show that:

1. $EP = EQ = EG$,
2. A, G and P are collinear.

2170. *Proposed by Tim Cross, King Edward's School, Birmingham, England.*

Find, with justification, the positive integer which comes next in the sequence $1411, 4463, 4464, 1412, 4466, 4467, 1413, 4469, \ldots$.

[Ed.: the answer is NOT 4470.]

2171. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let P be an arbitrary point taken on an ellipse with foci F_1 and F_2, and directrices d_1, d_2, respectively. Draw the straight line through P which is parallel to the major axis of the ellipse. This line intersects d_1 and d_2 at points M and N, respectively. Let P' be the point where MF_1 intersects NF_2.

Prove that the quadrilateral $PF_1P'F_2$ is cyclic.

Does the result also hold in the case of a hyperbola?

2172. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x, y, z \geq 0$ with $x+y+z=1$. For fixed real numbers a and b, determine the maximum $c = c(a,b)$ such that $a + bxyz \geq c(yz+zx+xy)$.

2173. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 2$ and $x_1, \ldots, x_n > 0$ with $x_1 + \ldots + x_n = 1$.

Consider the terms

$$l_n = \sum_{k=1}^{n} (1 + x_k) \sqrt{\frac{1 - x_k}{x_k}}$$

and

$$r_n = C_n \prod_{k=1}^{n} \frac{1 + x_k}{\sqrt{1 - x_k}}$$

where

$$C_n = (\sqrt{x-1})^{n+1} (\sqrt{n})^n / (n+1)^{n-1}.$$

1. Show $l_2 \leq r_2$.
2. Prove or disprove: $l_n \geq r_n$ for $n \geq 3$.

2174. *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Let A be an $n \times n$ matrix. Prove that if $A^{n+1} = 0$ then $A^n = 0$.

2175. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

The fraction $\frac{1}{6}$ can be represented as a difference in the following ways:

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}; \quad \frac{1}{6} = \frac{1}{3} - \frac{1}{6}; \quad \frac{1}{6} = \frac{1}{4} - \frac{1}{2}; \quad \frac{1}{6} = \frac{1}{5} - \frac{1}{30}.$$

In how many ways can the fraction $\frac{1}{2175}$ be expressed in the form

$$\frac{1}{2175} = \frac{1}{x} - \frac{1}{y},$$

where x and y are positive integers?

2176. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove that

$$\sqrt[n]{\prod_{k=1}^{n} (a_k + b_k)} \geq \sqrt[n]{\prod_{k=1}^{n} a_k} + \sqrt[n]{\prod_{k=1}^{n} b_k}$$

where $a_1, a_2, \ldots, a_n > 0$ and $n \in \mathbb{N}$.

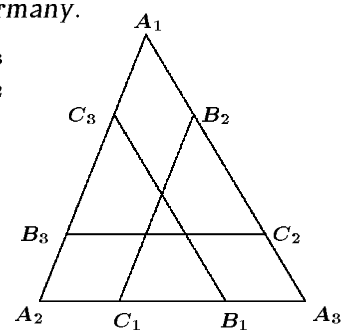
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1987. [1994: 250; 1995: 283-285] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In the figure, $B_2C_1 \| A_1A_2$, $B_3C_2 \| A_2A_3$ and $B_1C_3 \| A_3A_1$. Prove that B_2C_1, B_3C_2 and B_1C_3 are concurrent if and only if

$$\frac{A_1C_3}{C_3B_3} \cdot \frac{A_2C_1}{C_1B_1} \cdot \frac{A_3C_2}{C_2B_2} = 1.$$



II. *Solution by anonymous* (spotted on the internet by Waldemar Pompe, student, University of Warsaw, Poland).

Just apply Ceva's theorem to $\triangle B_1B_2B_3$.

Editor's comments by Chris Fisher.

1. Note that if we permit B_i to lie on the side of the triangle extended beyond A_{i-1} or A_{i+1} (as Bradley did in solution I) then Ceva's theorem fails to apply exactly when B_1, B_2, B_3 are collinear. Thus the alternative condition in the middle of page 284 (namely $u = pq/(p+q-1)$ in Bradley's notation) applies if and only if $B_1 \in B_2B_3$. As a consequence, Bradley's extended version of the problem can be restated as

$$\frac{A_1A_3}{C_3B_3} \cdot \frac{A_2C_1}{C_1B_1} \cdot \frac{A_3C_2}{C_2B_2} = 1 \text{ if and only if either}$$

$$B_2C_1, B_3C_2, \text{ and } B_1C_3 \text{ are concurrent or } B_1 \in B_2B_3.$$

2. It makes an amusing exercise to prove directly that $B_1 \in B_2B_3$ *if and only if* $C_1 \in C_2C_3$ (in the notation of Gülicher's problem).

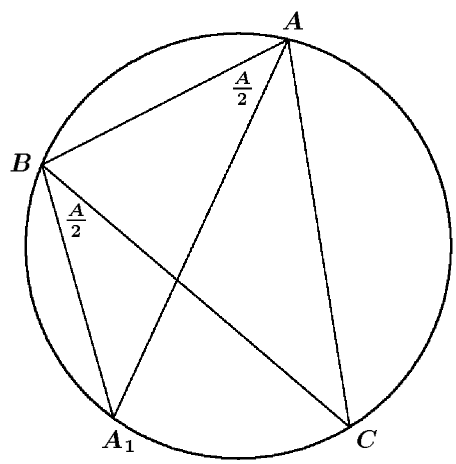
2067. [1995: 235] *Proposed by Moshe Stupel and Victor Oxman, Pedagogical Religious Girls' College "Shanan", Haifa, Israel.*

Triangle ABC is inscribed in a circle Γ. Let AA_1, BB_1 and CC_1 be the bisectors of angles A, B and C, with A_1, B_1 and C_1 on Γ. Prove that the perimeter of the triangle is equal to

$$AA_1 \cos\left(\tfrac{A}{2}\right) + BB_1 \cos\left(\tfrac{B}{2}\right) + CC_1 \cos\left(\tfrac{C}{2}\right) \ .$$

Solution by Miguel Angel Cabezón Ochoa, Logroño, Spain.

Let the internal bisector of $\angle BAC$ meet the circumcircle of $\triangle ABC$ again in A_1. Let R be the radius of the circumcircle.



From $\triangle BAA_1$, we have $\dfrac{AA_1}{\sin\left(B + \frac{A}{2}\right)} = 2R$, so that

$$AA_1 = 2R \sin\left(B + \tfrac{A}{2}\right).$$

Thus

$$\begin{aligned} AA_1 \cos\left(\tfrac{A}{2}\right) &= 2R \sin\left(B + \tfrac{A}{2}\right) \cos\left(\tfrac{A}{2}\right) \\ &= R\left(\sin(B + A) + \sin B\right) \\ &= R \sin C + R \sin B \ = \ \frac{b + c}{2}. \end{aligned}$$

Similarly

$$BB_1 \cos\left(\tfrac{B}{2}\right) \ = \ \frac{c + a}{2}; \qquad CC_1 \cos\left(\tfrac{C}{2}\right) \ = \ \frac{a + b}{2}.$$

Therefore

$$AA_1 \cos\left(\tfrac{A}{2}\right) + BB_1 \cos\left(\tfrac{B}{2}\right) + CC_1 \cos\left(\tfrac{C}{2}\right) \ = \ \frac{b + c}{2} + \frac{c + a}{2} + \frac{a + b}{2}.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; HIDETOSI FUKAGAWA, Gifu, Japan; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO CHRISTOV KUNCHEV, Rousse, Bulgaria; VEDULA N. MURTY, Andhra University, Visakhapatnam, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ASHSIH KR. SINGH, student, Kanpur, India; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposers. There was one anonymous solution.

Amengual Covas points out that this problem has already appeared in print, in **A Treatise on Plane Trigonometry** *by E.W. Hobson, Cambridge University Press, 2nd Edition, 1897, Example 2 on page 194.*

2069. [1995: 235] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

M is a variable point of side BC of triangle ABC. A line through M intersects the lines AB in K and AC in L so that M is the mid-point of segment KL. Point K' is such that $ALKK'$ is a parallelogram. Determine the locus of K' as M moves on segment BC.

I. *Essentially the same solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria; Waldemar Pompe, student, University of Warsaw, Poland; and Toshio Seimiya, Kawasaki, Japan.*

Suppose that the dilatation with centre A and ratio two transforms the points B, C and M to B', C' and N respectively. Since M is the mid-point of both KL and AN, we have that $AKNL$ is a parallelogram. By definition, $ALKK'$ is a parallelogram as well, so that K is the mid-point of $K'N$; also, $K'N$ and AC' are parallel. Define C'' to be the point such that A is the mid-point of $C'C''$. We conclude that, as M varies on BC, N varies on $B'C'$, so that the position of K' must vary on the segment $B'C''$.

Editor's comments:

1. Note that this solution provides an explicit construction of the points K and L.

2. Smeenk points out that K' helps in finding the position of M for which the corresponding segment KL has minimum length. ($KL = AK$;, and AK' has its minimum length when K' is the foot of the perpendicular from A to the line $B'C''$.)

II. *Essentially the same solution by Tim Cross, King Edward's School, Birmingham, England; Hidetosi Fukagawa, Gifu, Japan; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Mitko Christov Kunchev, Rousse, Bulgaria; Ashsih Kr. Singh, student, Kanpur, India.*

Take A to be the origin and set the vectors $\overrightarrow{AB} = \overrightarrow{B}$, etc. Then $\overrightarrow{M} = t\overrightarrow{B} + (1-t)\overrightarrow{C}$, where t varies from 0 to 1 as M moves from C to B. Suppose that $\overrightarrow{K} = \kappa\overrightarrow{B}$ and that $\overrightarrow{L} = \lambda\overrightarrow{C}$. Because M is the mid-point of KL, we have

$$\frac{\kappa\overrightarrow{B} + \lambda\overrightarrow{C}}{2} = t\overrightarrow{B} + (1-t)\overrightarrow{C},$$

so that $\kappa = 2t$ and $\lambda = 2(1-t)$ (since $\overrightarrow{B}$ and $\overrightarrow{C}$ are linearly independent). Since $ALKK'$ is a parallelogram, it follows that

$$\begin{aligned} \overrightarrow{K} &= \overrightarrow{K} - \overrightarrow{L} \\ &= t(2\overrightarrow{B} + (1-t)(-2\overrightarrow{C}). \end{aligned}$$

Thus the locus of K' is the segment joining $-2\overrightarrow{C}$ (where $t = 0$) to $2\overrightarrow{B}$ where $t = 1$).

Editor's comment: The algebra makes clear that it is not necessary to restrict M to the segment BC; as t ranges over the real numbers, K' moves along its line, while M moves along the line BC.

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer. One anonymous solution was received - see note at the start of this section. There was one incorrect solution received.

2070. [1995: 236] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

For which positive integers n is the Catalan number

$$\frac{1}{n+1}\binom{2n}{n}$$

odd?

I *Virtually identical solutions by Toby Gee, student, the John of Gaunt School, Trowbridge, England; Douglas E. Jackson, Eastern New Mexico University, Portales, New Mexico, USA; Thomas Leong, Staten Island, NY, USA; Andy Liu, University of Alberta, Edmonton, Alberta; Waldemar Pompe, student, University of Warsaw, Poland; and the proposer.*

It is well-known that the n^{th} Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$, satisfies the recurrence relation

$$C_{n+1} = C_0C_n + C_1C_{n-1} + \ldots + C_nC_0 \qquad n \geq 0.$$

We show that C_n is odd if and only if $n = 2^k - 1$ for some integer $k \geq 0$. We use induction on n.

Since $C_0 = 1$, the assertion is true for $n = 0$. Assume that among the numbers C_0, C_1, ..., C_n, only those with n of the form $2^k - 1$ are odd. If $n + 1$ is even, then

$$C_{n+1} = 2 \sum_{k=0}^{(n-1)/2} C_kC_{n-k},$$

which is even.

On the other hand, if $n + 1$ is odd, then

$$C_{n+1} = 2 \sum_{k=0}^{(n-1)/2} C_kC_{n-k} \; + \; C_{n/2}^2,$$

showing that C_{n+1} is odd if and only if $C_{n/2}$ is odd. But $C_{n/2}$ is odd if and only if $\frac{n}{2} = 2^k - 1$ for some integer $k \geq 0$. Thus C_{n+1} is odd if and only if $n + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1$, which completes the proof.

II *Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany.*

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the n^{th} Catalan number, $n = 1, 2, 3, \ldots$, and let $F(n)$ denote the highest power of 2 that divides n. Then it is well known that

$$F(n!) \; = \; \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \ldots .$$

From this, it is easy to see that $F(n!) \leq n - 1$, with equality if and only if $n = 2^k$ for some integer $k \geq 0$. [More generally, it is known, and easy to show, that if p is prime and $n = a_ra_{r-1}\ldots a_1a_0$ is the base-p representation of n, then

$$h = \frac{n - \sum_{i=0}^{r} a_i}{p - 1},$$

where $p^h \| n!$ [See, for example, Theorem 2.30 in *Elementary Introduction to Number Theory*, 3rd edition, by Calvin T. Long — Ed.]

Since

$$C_n = \frac{(2n)!}{(n+1)!\,n!} = \frac{2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{(n+1)!},$$

C_n is odd if and only if $F((n+1)!) = n$, which is true if and only if $n+1 = 2^k$ or $n = 2^k - 1$ for some integer $k \geq 1$. We can also allow $k = 0$ since $C_0 = 1$ is odd.

Solutions similar or equivalent to II *above were submitted by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; EMERIC DEUTSCH, Brooklyn, NY, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; SOLOMON W. GOLOMB, Univ of Southern California, Los Angeles, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SKIDMORE COLLEGE PROBLEM GROUP Saratoga Springs, NY, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, Kansas, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA (two solutions).*

Several solvers pointed out the well-known fact that the highest power of 2 *that divides* $\binom{2n}{n}$ *is equal to the number of ones in the binary representation of* n. *This is, of course, an immediate consequence of the more general fact mentioned in the editor's comment in solution* II *above.*

This problem is certainly not new. Deutsch supplied the reference: "The Parity of the Catalan Numbers via Lattice Paths" *by* Ömer Eğecioğlu, Fibonacci Quarterly, (**21**), *1983, 65–66. Godin gave the reference:* "Time Travel and other Mathematical Bewilderments" *by Martin Gardner, W.H. Freeman and Co. In fact, the result of this problem was also mentioned by Martin Gardner in his article* "Mathematical Games; Catalan number: an integer sequence that materializes in unexpected places", Scientific American, **234**(*6*), *1976, 120–125, but he did not indicate any proof.*

2071. [1995: 277] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

P is an interior point of an equilateral triangle ABC so that $PB \neq PC$, and BP and CP meet AC and AB at D and E respectively. Suppose that $PB : PC = AD : AE$. Find angle BPC.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We claim that $\angle BPC = 120°$. Let the side of the triangle be 1. Further, let $x = AE, y = AD, \eta = \angle BCE$ and $\varphi = \angle CBD$. By the law of cosines (for $\triangle ABD$ and $\triangle AEC$ respectively) we get (since $\cos 60^\circ = 1/2$): $BD = \sqrt{1+y^2-y}$ and $CE = \sqrt{1+x^2-x}$. Via the law of sines applied to

$$\triangle BCD: \quad \sin\varphi = \frac{1-y}{BD}\sin 60^\circ = \frac{1-y}{\sqrt{1+y^2-y}}\cdot\frac{\sqrt{3}}{2}$$

$$\triangle EBC: \quad \sin\eta = \frac{1-x}{CE}\sin 60^\circ = \frac{1-x}{\sqrt{1+x^2-x}}\cdot\frac{\sqrt{3}}{2}$$

$$\triangle BCP: \quad \frac{\sin\varphi}{\sin\eta} = \frac{PC}{PB} = \frac{x}{y} \text{ [by assumption for the problem!]}$$

Hence, $\frac{y-y^2}{\sqrt{1+y^2-y}} = \frac{x-x^2}{\sqrt{1+x^2-x}}$, that is (with $X = x - x^2$ and $Y = y - y^2$)

$$\frac{X}{\sqrt{1-X}} = \frac{Y}{\sqrt{1-Y}}. \qquad (1)$$

Here, since $0 < x, y < 1,\ 0 < X, Y \le 1/4$.

Now, (1) $\iff X^2(1-Y) = Y^2(1-X) \iff (X-Y)(XY-X-Y) = 0$. Thus, two cases are to be considered.

(a) $XY - X - Y = 0 \iff (X-1)(Y-1) = 1$, which is impossible.

(b) $X = Y \iff x - x^2 = y - y^2 \iff (y-x)(x+y-1) = 0 \iff y = 1 - x$ (since $x \neq y$).

Hence $\triangle ABD$ is congruent to $\triangle BCE \Rightarrow \angle ABD = \eta \Rightarrow \eta + \varphi = 60^\circ \Rightarrow \angle BPC = 180^\circ - (\eta + \varphi) = 120^\circ$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; P. PENNING, Delft, the Netherlands; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; one anonymous solver and the proposer. There were two incorrect solutions.

2073*. [1995: 277] *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let P be an interior point of an equilateral triangle $A_1A_2A_3$ with circumradius R, and let $R_1 = PA_1$, $R_2 = PA_2$, $R_3 = PA_3$. Prove or disprove that

$$R_1R_2R_3 \le \frac{9}{8}R^3.$$

Equality holds if P is the mid-point of a side. [Compare this problem with *Crux* 1895 [1995: 204].]

I *Solution by Kee-Wai Lau, Hong Kong.*

Without loss of generality we assume that $R = 2$, so that the length of each side of the triangle is $2\sqrt{3}$. Let $\angle A_1PA_2 = \theta, \angle A_2PA_3 = \phi$, so that $\angle A_1PA_3 = 2\pi - \theta - \phi$. Applying the cosine law to $\Delta A_1PA_2, \Delta A_2PA_3$ and ΔA_1PA_3, respectively, we obtain

$$\cos\theta = ({R_1}^2 + {R_2}^2 - 12)/(2R_1R_2),$$

$$\cos\phi = ({R_2}^2 + {R_3}^2 - 12)/(2R_2R_3) \text{ and}$$

$$\cos(\theta+\phi) = ({R_1}^2 + {R_3}^2 - 12)/(2R_1R_3).$$

Since $[\cos\theta\cos\phi - \cos(\theta+\phi)]^2 = (1-\cos^2\theta)(1-\cos^2\phi)$, we obtain after some simplification that

$$\begin{aligned} {R_1}^4 + {R_2}^4 + {R_3}^4 - {R_1}^2{R_2}^2 - {R_2}^2{R_3}^2 - {R_3}^2{R_1}^2 - & \\ 12({R_1}^2 + {R_2}^2 + {R_3}^2) + 144 &= 0. \end{aligned}$$

Thus

$${R_3}^2 = \frac{1}{2}(z + 24 \pm (12y - 3z^2)^{1/2}), \qquad (1)$$

where $y = {R_1}^2{R_2}^2, z = {R_1}^2 + {R_2}^2 - 12$ and $z^2 \le 4y$.

By considering $R_1 = R_2 = R_3 = 2$, we see that we should choose in (1) the $-$ sign from $\pm$. It follows that

$$f(y,z) := {R_1}^2{R_2}^2{R_3}^2 = \frac{1}{2}y[z + 24 - (12y - 3z^2)^{1/2}].$$

From $df/dz = 0$, we obtain $z = -\sqrt{y}$ and $f(y, -\sqrt{y}) = y(12 - 2\sqrt{y}) \le 64$.

Now at the boundary $z^2 = 4y$, we have $|R_1 - R_2| = \sqrt{12}$ or $R_1 + R_2 = \sqrt{12}$, so that $-6 \le z \le 0$. Hence $f = z^2(z+24)/8 \le 81$ and equality holds when $z = -6$ or P is the mid-point of a side. This proves the required inequality.

II *Solution by Manuel Benito Munoz and Emilio Fernández Moral, I.B. Sagasta, Logroño, Spain*

The proposed inequality holds (equality only on the mid-point of a side), as a particular case ($n = 3$) of the following:

$$R_1R_2\ldots R_n \le (1 + \cos^n\frac{\pi}{n}) \cdot R^n$$

when P is any point from a closed regular n-gon, $A_1A_2\ldots A_n$, with circumradius R and $R_i = PA_i$. (Equality holds only on the mid-point of a side.)

Without loss of generality, let us suppose that $R = 1$ and that the vertices of the n-gon are the points $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the complex plane, where $\alpha_k = e^{\frac{2\pi i}{n}(k-1)}$. Let $z = \rho e^{i\theta}$ represent the point P.

By the maximum modulus principle applied to the function $\Pi_{k=1}^{n}(z-\alpha_k)$, the modulus of that function, i.e., $PA_1 \cdot PA_2 \ldots PA_n = \Pi_{k=1}^{n}|z-\alpha_k|$, assumes its greatest value on the boundary of the n-gon.

Now, geometrical considerations permit us to avoid much analytical treatment and we conclude that the maximum value is attained on the mid-point of any side A_iA_{i+1}. Let P be a point on the side A_1A_2 of the n-gon; the vertices $A_3, \ldots, A_n$ of the n-gon are pairwise arranged symmetrically with respect to the mediatrix of the segment A_1A_2 (except for the single point $A_{\frac{n+1}{2}+1}$ when n is odd).

Let A_k, A_l be one of such pairs of vertices; if M is the mid-point of A_1A_2 we have, by the GA-means inequality and "shortest path principle", that

$$PA_k \cdot PA_l \leq \left(\frac{PA_k + PA_l}{2}\right)^2 \leq \left(\frac{MA_k + MA_l}{2}\right)^2 = MA_k \cdot MA_l$$

(the latter because $MA_k = MA_l$).

When n is even, as $PA_1 \cdot PA_2 \leq MA_1 \cdot MA_2$, we are done.

When n is odd, let A_m be the unpaired vertex and put $A_1A_2 = 2l$, $PM = x$ and $MA_m = a$. Therefore,

$$PA_1 \cdot PA_2 \cdot PA_m = (l^2 - x^2) \cdot \sqrt{a^2 + x^2},$$

and the maximum value of that function for $x \in [0, l]$ is attained at $x = 0$; that is, when $P = M$. (There are no other critical points because $a > l$.) So that, for every n, we have shown

$$PA_1 \cdot PA_2 \ldots PA_n \leq MA_1 \cdot MA_2 \ldots MA_n,$$

where M is the mid-point of a side.

Now finally

$$\begin{aligned}
&(PA_1 \cdot PA_2 \ldots PA_n)^2 \\
&\quad = \quad \Pi_{k=1}^{n}|z-\alpha_k|^2 = \Pi_{k=1}^{n}(z-\alpha_k) \cdot \Pi_{k=1}^{n}(\overline{z}-\overline{\alpha}_k) \\
&\quad = \quad (z^n - 1) \cdot (\overline{z}^n - 1) = \rho^{2n} - 2\rho^n \cos n\theta + 1,
\end{aligned}$$

and if P is the mid-point of A_1A_2 ($\rho = \cos\left(\frac{\pi}{n}\right), \theta = \frac{\pi}{n}$), that value is $\left(\cos\left(\frac{\pi}{n}\right)\right)^{2n} - 2\cos^n\left(\frac{\pi}{n}\right) \cdot \cos\pi + 1 = \left(\cos^n\left(\frac{\pi}{n}\right) + 1\right)^2$, as required.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WOLFGANG GMEINER, Bundesgymnasium Spittal/Drau, Spittal, Austria (also found the generalization of π); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta(with a generalization from triangles to tetrahedra); VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, *Japan; JOHANNES WALDMANN, Friedrich-Schiller-Universität, Jena, Germany; and the proposer.*

2074. [1995: 277] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

The number 3774 is divisible by 37, 34 and 74 but not by 77. Find another four-digit integer $abcd$ that is divisible by the two-digit numbers ab, ac, ad, bd and cd but is not divisible by bc.

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.

The two possible solutions are

$$1995 \qquad \text{and} \qquad 2184.$$

Let $N = abcd = 100 \cdot (ab) + cd$ be divisible by ab. Then cd must be divisible by ab as well; say, $cd = k \cdot (ab)$ where $k < 10$. Since N is divisible by cd, so is $100 \cdot (ab)$. It follows that k is a divisor of 100. As such, it must be one of 1, 2, 4, 5. Now, a direct computer search reveals that only 1995, 2184 and 3774 satisfy the requirement.

Note: apart from numbers of the form $aaaa$, 1155 is the only four-digit number divisible by all two-digit numbers "contained in it".

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; KEITH EKBLAW, Walla Walla, Washington, USA; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; JEFFREY K. FLOYD, Newnan, Georgia, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Gardena, California, USA; P. PENNING, Delft, the Netherlands; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

A bit over half of the solvers found both solutions, often by computer, as there does not appear to be a short way to do the problem by hand. The proposer also used a computer, but it seems to have let him down, as he thought that 1995 *was the only answer!*

Metchette also gave the "solution" 0315, *which works, but is not a true four-digit number. He wonders why all the solutions are multiples of* 3*; can any reader see an easy explanation?*

2075. [1995: 278] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK*

ABC is a triangle with $\angle A < \angle B < \angle C$, and I is its incentre. BCL, ACM, ABN are the sides of the triangle with L on BC produced, etc., and the points L, M, N chosen so that

$$\angle CLI = \frac{1}{2}(\angle C - \angle B), \quad \angle AMI = \frac{1}{2}(\angle C - \angle A), \quad \angle BNI = \frac{1}{2}(\angle B - \angle A).$$

Prove that L, M, N are collinear.

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

We have

$$\begin{aligned}\angle AIN &= 180^\circ - (\angle BAI + \angle BNI) \\ &= 180^\circ - \left(\frac{\angle A}{2} + \frac{\angle B - \angle A}{2}\right) \\ &= 180^\circ - \frac{\angle B}{2} = \angle NBI,\end{aligned}$$

which implies that triangles $\triangle ANI$ and $\triangle INB$ are similar, or

$$AN : NI : AI = NI : BN : BI$$

Hence it follows (via $AN \cdot BN = NI^2$) that

$$\frac{AN}{BN} = \frac{NI^2}{BN^2} = \frac{AI^2}{BI^2}.$$

Analogously we get

$$\frac{BL}{CL} = \frac{BI^2}{CI^2}, \quad \frac{CM}{AM} = \frac{CI^2}{AI^2}$$

and finally

$$\frac{AN}{BN} \cdot \frac{BL}{CL} \cdot \frac{CM}{AM} = \frac{AI^2}{BI^2} \cdot \frac{BI^2}{CI^2} \cdot \frac{CI^2}{AI^2} = 1$$

whence (by the converse of Menelaus' Theorem) M, N and L are collinear.

Perz also remarks that, more generally, L, M, N are collinear, if P is a point inside or outside $\triangle ABC$ and L, M, N are such that

$$\begin{aligned}\angle CLP &= \angle BCP - \angle PBC,\ \angle AMP \\ &= \angle CAP - \angle PCA,\ \angle BNI \\ &= \angle ABP - \angle PAB,\end{aligned}$$

where all angles are oriented angles.

Also solved by TOBY GEE, student, the John of Gaunt School, Trowbridge, England; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2077. [1995: 278] *Proposed by Joseph Zaks, University of Haifa, Israel.*

The determinant

$$\begin{vmatrix} z_1\overline{z_1} & z_1 & \overline{z_1} & 1 \\ z_2\overline{z_2} & z_2 & \overline{z_2} & 1 \\ z_3\overline{z_3} & z_3 & \overline{z_3} & 1 \\ z_4\overline{z_4} & z_4 & \overline{z_4} & 1 \end{vmatrix}$$

equals 0 if and only if the four complex numbers z_1, z_2, z_3, z_4 satisfy what simple geometric property?

Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA. [*modified slightly by the editor.*]

Let D denote the given determinant. If any two (or more) of the four complex numbers are equal, $z_1 = z_2$, say, then $D = 0$, and the points z_2, z_3, z_4 must be collinear (or concyclic). We thus assume that the four numbers are all distinct.

It is well-known that an equation of the circle passing through three non-collinear points (x_2, y_2), $(x_3, y_3$, (x_4, y_4), can be written as

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0.$$

Letting $z_1 = x + \imath y$ and $z_k = x_k + \imath y_k$ for $k = 2, 3, 4$, we have

$$D = \begin{vmatrix} z_1\overline{z_1} & x & \overline{z_1} & 1 \\ z_2\overline{z_2} & x_2 & \overline{z_2} & 1 \\ z_3\overline{z_3} & x_3 & \overline{z_3} & 1 \\ z_4\overline{z_4} & x_4 & \overline{z_4} & 1 \end{vmatrix} = -2\imath \begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{vmatrix}.$$

Thus $D = 0$ if and only if z_1 lies on the circle passing through the three points z_2, z_3 and z_4. There the sought property is that z_1, z_2, z_3, z_4 are either concyclic or collinear (which can be viewed as a degenerate case).

[Editor's note: This is certainly a known result. The references supplied by several readers are listed below. The number, if there is one, after a solver's name corresponds to the reference given by that solver.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain [3]; *CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; WOLFGANG GMEINER, Millstatt, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA* [2]; *WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany* [1]; *CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. One anonymous solution* [4] *and one incomplete solution were also received.*

References:

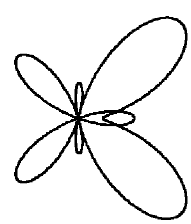
[1] Conway, J.B., *Functions of One Complex Variable*, Springer-Verlag, 2nd ed., 1978, p. 49, Proposition 3.10.

[2] Dodge, Clayton W., *Complex Numbers*, 1993, p. 135 and p. 199.

[3] Krzyz, *Problems in Complex Variable Theory*, Elsevier, PWN, 1971, P. 5 and p. 142.

[4] Schwerdtfeger, H., *Geometry for Complex Numbers*, Dover, NY, 1962.

In the May 1996 issue of ***CRUX*** [1996: 168], we asked "Do you know the equation of this curve?"



Here is a hint – it is known as the "butterfly"!