On the Sum of n Dice

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Introduction

An interesting problem can be found in Tucker [2, p. 421] which we rephrase as follows.

> Suppose n distinct fair dice are rolled and S is the sum of their faces. Show that the probability that 2 divides S is $\frac{1}{2}$.

There are several ways to solve this problem. One, which we will use, lends itself to generalization.

		Die Two					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Die	3	4	5	6	7	8	9
One	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Table 1. Possible Sums of Two Dice

The result is clear for one die, and easily verified for two dice. See Table 1. The thirty-six entries in Table 1 each have the same probability. Since there are eighteen even sums among the thirty-six entries, the result follows for $n = 2$. A sub-problem is, how to represent the sum of three, or more generally n, dice while maintaining the different frequencies for different sums. That is, the sum 3 is only obtained in one way using three dice, while the sum 15 can be obtained in ten ways with three dice. Fortunately, these frequencies are unnecessary for this solution.

In Table 2, we label the rows by the possible sums of $(n - 1)$ dice and the six columns by the possible values of the nth die. We note that, unlike the entries in Table 1, the entries in Table 2 are not equi-probable. However, the table does allow us to count the number of even sums among the 6^n possible outcomes from rolling n dice. For suppose there are f_{n+2} ways to obtain the sum $(n + 2)$ when $(n - 1)$ dice are rolled. Then by looking at the row labelled by $(n + 2)$, we can account for $3f_{n+2}$ even sums among the 6^n possible outcomes for n dice, since there are three even entries in this row. Since each row of the table consists of six consecutive integers, there

		Die n					
		1	2	3	4	5	6
	$n-1$	n	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$
	n	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$
Sum of	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$	$n+7$
$(n-1)$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$	$n+7$	$n+8$
dice							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$6n-7$	$6n-6$	$6n-5$	$6n-4$	$6n-3$	$6n-2$	$6n-1$
	$6n-6$	$6n-5$	$6n-4$	$6n-3$	$6n-2$	$6n-1$	$6n$

Table 2. Possible Sums of n Dice

are exactly three even entries in each row. Thus, we can sum the frequencies $(f_{n-1} + f_n + f_{n+2} + \dots + f_{6n-6})$ and multiply this sum by 3 to obtain the number of even sums among the 6^n possible outcomes. As the sum of the frequencies is 6^{n-1}, we have the probability that 2 divides S is

$$\frac{3 \cdot 6^{n-1}}{6^n} = \frac{1}{2}.$$

As mentioned above, this method generalizes easily. Let us first state two problems. Suppose n dice are rolled, and let S_n be the sum of their faces. Let $P(k|S_n)$ be the probability that k divides S_n.

1. Find values of k such that $P(k|S_n) = \frac{1}{k}$ for all n.

2. Find values of k and n for which $P(k|S_n) = \frac{1}{k}$.

Problem 1. Find values of k such that $P(k|S_n) = \frac{1}{k}$ for all n.

As mentioned, each row in Table 2 consists of six consecutive integers. Thus, in each row, two entries are divisible by 3, and one entry is divisible by 6.

It follows, in exactly the same manner as above, that $P(3|S_n) = \frac{1}{3}$ and $P(6|S_n) = \frac{1}{6}$. Clearly, $P(1|S_n) = 1$. Combining these observations with our first result, we have that, for all n, $P(k|S_n) = \frac{1}{k}$ when $k = 1, 2, 3$, and 6.

By considering $n = 1$ (or $n = 3$, if we wish to avoid the trivial case), we see that the only values of k that hold for all n are precisely $k = 1, 2, 3$, and 6.

Problem 2. Find values of k and n for which $P(k|S_n) = \frac{1}{k}$.

Let n and k satisfy $P(k|S_n) = \frac{1}{k}$. Since $6^n P(k|S_n)$ is an integer, it follows that k divides 6^n, that is, k is of the form $2^s 3^t$ for some non-negative integers s and t.

For $0 \le s,\ t \le 1$ we have $k = 1, 2, 3$, or 6. These values of k, as we have seen, are precisely the solutions to Problem 1. Let us consider $k = 4$.

We shall show that $P(4|S_n) = \frac{1}{4}$ if and only if $n \equiv 2 \pmod 4$. We note that in Table 2 each row does not contain the same number of multiples of 4. For this reason, we are led to consider another approach, in particular, the congruence classes depicted in Table 3.

	1	2	3	4	5	6
0	1	2	3	0	1	2
1	2	3	0	1	2	3
2	3	0	1	2	3	0
3	0	1	2	3	0	1

Table 3. Congruence Classes Modulo 4 for n Dice

We construct Table 3 in a similar fashion to Table 2, except that the congruence classes $0, 1, 2,$ and 3 (modulo 4) label the rows. The columns are still labelled by the face values of the nth die. The entry in the ith row and the jth column is the congruence class of the sum of an element from the congruence class of the ith row and the face value of the jth column.

Let $P(n, \equiv\ i)$ be the probability that the sum of n dice is congruent to $i \pmod 4$ for $i = 0, 1, 2, 3$. Note that $P(4|S_n)$ is precisely $P(n, \equiv\ 0)$. The following recurrence relations are a consequence of Table 3.

$$\begin{aligned}
6P(n, \equiv 0) &= 2P(n-1, \equiv 3) + 2P(n-1, \equiv 2) \\
&\quad + P(n-1, \equiv 1) + P(n-1, \equiv 0) \\
&= P(n-1, \equiv 3) + P(n-1, \equiv 2) + 1
\end{aligned}$$

$$\begin{aligned}
6P(n, \equiv 1) &= 2P(n-1, \equiv 3) + 2P(n-1, \equiv 0) \\
&\quad + P(n-1, \equiv 1) + P(n-1, \equiv 2) \\
&= P(n-1, \equiv 3) + P(n-1, \equiv 0) + 1
\end{aligned}$$

$$\begin{aligned}
6P(n, \equiv 2) &= 2P(n-1, \equiv 0) + 2P(n-1, \equiv 1) \\
&\quad + P(n-1, \equiv 2) + P(n-1, \equiv 3) \\
&= P(n-1, \equiv 0) + P(n-1, \equiv 1) + 1
\end{aligned}$$

$$\begin{aligned}
6P(n, \equiv 3) &= 2P(n-1, \equiv 1) + 2P(n-1, \equiv 2) \\
&\quad + P(n-1, \equiv 3) + P(n-1, \equiv 0) \\
&= P(n-1, \equiv 1) + P(n-1, \equiv 2) + 1
\end{aligned}$$

From these recurrence relations, we observe that

$$P(n, \equiv 0) + P(n, \equiv 2) = \tfrac{1}{2} \quad \text{and} \quad P(n, \equiv 1) + P(n, \equiv 3) = \tfrac{1}{2}.$$

With a little iteration, we obtain $P(n, \equiv 0) = \frac{323}{1296} + \frac{4}{1296}P(n-4, \equiv 2)$.

Replacing the last term by $(\frac{1}{2} - P(n-4, \equiv 0))$ we have,

$$P(n, \equiv 0) = \frac{1}{4} + \left[\frac{1}{1296} - \frac{4}{1296}P(n-4, \equiv 0)\right].$$

It follows that $P(4|S_n) = \frac{1}{4}$ if and only if $P(4|S_{n-4}) = \frac{1}{4}$. Now $P(4|S_2) = \frac{1}{4}$, but $P(4|S_m) \neq \frac{1}{4}$ if $m = 1, 3$, or 4.

Thus, $P(4|S_n) = \frac{1}{4}$ if and only if $n \equiv 2$ (mod 4).

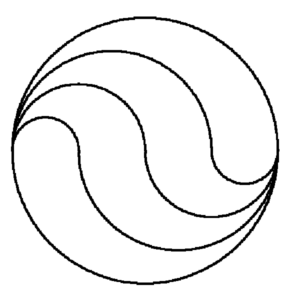
Conclusion

What about other possible k's $(k = 2^s \cdot 3^t)$ for which it may be that $P(k|S_n) = \frac{1}{k}$ for some n? To be honest, we don't know. We have checked those k's which are less than 54 for $n \leq 55$. For $k = 8$ and 9 respectively, the only values of $n \leq 55$ for which $P(k|S_n) = \frac{1}{k}$ are $n = 3$ and 2 respectively. For $n \leq 55$, $P(k|S_n) = \frac{1}{k}$ never holds for $12 < k < 54$. The last value for which we could report is $k = 12$. But rather than spoil the fun of the interested reader, we say no more.

References

1. Neff, J.D. Dice tossing and Pascal's triangle. *Two-Year College Mathematics Journal*, **13** (1982), pp. 311-314.
2. Tucker, A. *Applied Combinatorics,* 2nd Ed., John Wiley and Sons, 1984.

Here are three curves that divide a circle into four equal areas.



Any other nice examples?

THE SKOLIAD CORNER

No. 14

R.E. Woodrow

First this month, an observation by an astute reader, Derek Kisman, student, Queen Elizabeth High School, Calgary, who noticed that the figure we gave for question 25 of the Sharp U.K. Intermediate Mathematical Challenge, 1995, [1996: 14] is missing the crucial upright F in the net. It should be in the top right corner!

As a contest this month we give the European "Kangaroo" Mathematical Challenge, written Thursday 23 March, 1995. It is organized by the U.K. Mathematics Foundation and L'Association Européennes ≪Kangarou des Mathématiques≫. The contest is for students at about school year nine or below. This was written by about 5000 students in the U.K. My thanks go to Tony Gardiner of the U.K. Mathematics Foundation for sending the materials to Crux.

EUROPEAN "KANGAROO" MATHEMATICAL CHALLENGE

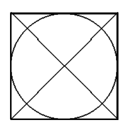
23 March 1995

1. $1 \times 9 \times 9 \times 5 - (1 + 9 + 9 + 5)$ makes:

A. 0 **B.** 381 **C.** 481 **D.** 429 **E.** 995

2. Which shape does not appear in this figure?

A. circle
B. square
C. right-angled triangle
D. isosceles triangle
E. equilateral triangle



3. The whole numbers from 1 to 1995 are alternately added and subtracted thus: $1 - 2 + 3 - 4 + 5 - 6 + \cdots + 1993 - 1994 + 1995$. What is the result?

A. 997 **B.** 1995 **C.** 998 **D.** 0 **E.** -997

4. What is the angle between the hour hand and the minute hand of a clock at 1.30?

A. 180° **B.** 120° **C.** 130° **D.** 150° **E.** 135°

5. C_1 is a circle of radius 6cm, C_2 is a circle of radius 8cm. José wants the two circles to touch tangentially. He knows that there are two possibilities for the distance between their centres. What are these two distances?

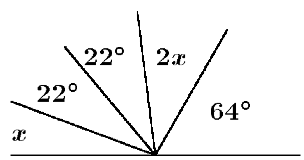
A. 3 and 4cm **B.** 2 and 8cm **C.** 2 and 14cm **D.** 6 and 8cm **E.** 6 and 14cm

6. A train 1km long is restricted to travel through a tunnel of length 1km at 1km/h. How long does it take the train to pass through the tunnel?

A. 1 hour **B.** 1h 30m **C.** 2 hours **D.** 3 hours **E.** 1/2 hour

7. The angle x in the figure is equal to:

A. $20°$
B. $22°$
C. $24°$
D. $26°$
E. $28°$

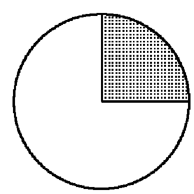


8. Which number is smallest?

A. 19^{95} **B.** 19×95 **C.** 1^{995} **D.** 199^5 **E.** 1995

9. What is the ratio of the perimeter of the shaded region to the circumference of the circle?

A. $\frac{3}{4}$ **B.** $\frac{4+\pi}{4\pi}$ **C.** $\frac{2\pi}{4+\pi}$ **D.** $\frac{4+\pi}{2\pi}$ **E.** $\frac{1}{4}$



10. After two successive 20% reductions, a coat costs \$320. What was its original price?

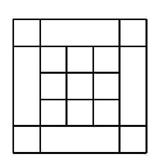
A. \$204 **B.** \$400 **C.** \$448 **D.** \$500 **E.** \$533

11. Nine people are sitting in a room; their average age is 25. In another room eleven people are gathered their average age is 45. If the two groups were to combine, what would their average age be?

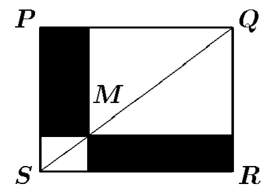
A. 70 **B.** 36 **C.** 35 **D.** 32 **E.** 20

12. How many squares are there in this figure?

A. 13
B. 14
C. 19
D. 21
E. 23



13. The quadrilateral $PQRS$ is a rectangle; M is any point on the diagonal SQ. What can one say for sure about the two shaded regions?

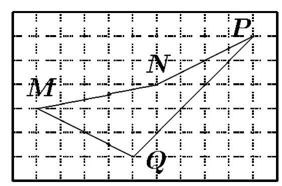


A. the upper area is larger
B. the lower area is larger
C. they always have equal areas
D. the two areas are equal only if M is the midpoint of SQ
E. insufficient information

14. A metallic disc of diameter 20cm weighs 2.4kg. From it one cuts out a disc of diameter 10cm. What does it weigh?

A. 1.2kg **B.** 0.8kg **C.** 0.6kg **D.** 0.5kg **E.** 0.4kg

15. What is the area (in unit squares) of the quadrilateral $MNPQ$?

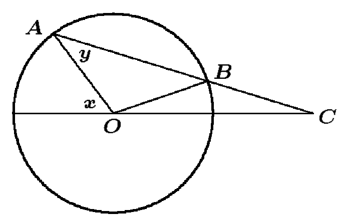


A. 9
B. 10
C. 11
D. 12
E. 13

16. Six hundred and twenty-five students enter a 100m competition. The track has five lanes. At the end of each race the winner survives, while the other four are eliminated. How many races are needed to determine the champion sprinter?

A. 98 **B.** 106 **C.** 125 **D.** 126 **E.** 156

17. In the figure O is the centre of the circle and $OA = BC$. Which of the following relations holds?



A. $2x = 3y$ **B.** $x = 2y$ **C.** $x = y$
D. $x + y = 90°$ **E.** $x + 2y = 180°$

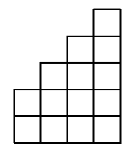
18. What is the sum of all the digits in the number $10^{95} - 95$?

A. 6 **B.** 29 **C.** 108 **D.** 663 **E.** 842

19. In a group of pupils, 40% of the pupils have poor eyesight; 70% of these wear glasses, while the other 30% wear contact lenses. If there are 21 pairs of glasses, which of these assertions is true?

A. 45 pupils have poor eyesight
B. 30 pupils have good eyesight
C. there are 100 pupils in the group
D. 10 pupils wear contact lenses
E. none of the other assertions is true

20. One has to arrange four pawns in this figure so that each column contains one pawn, and each row contains at most one pawn. How many different arrangements are possible?



A. 64 **B.** 28 **C.** 16 **D.** 8 **E.** 4

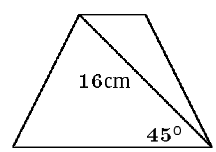
21. Let $\boxed{X} = X^4$ and $X \mid Y = X + Y$. Then $\boxed{\boxed{2} \mid \boxed{2}}$ equals:

A. 3×2^4 **B.** 2^9 **C.** 2^{12} **D.** 2^{20} **E.** 2^3

22. Bruce buys three ostriches, seven koalas and one kangaroo. Darren buys four ostriches, ten koalas and one kangaroo. Sheila buys one ostrich, one koala and one kangaroo. Bruce pays 3150 Australian dollars and Darren pays 4200 Australian dollars. How many Australian dollars must Sheila pay?

A. 1700 **B.** 1650 **C.** 1200 **D.** 1050 **E.** 950

23. The diagonal of an isosceles trapezium is 16cm long, and makes an angle of 45° with the base. What is the area of the trapezium?



A. 64cm^2 **B.** 96cm^2 **C.** 128cm^2
D. more information required **E.** 256cm^2

24. Each positive whole number which (in base 10) can be written as a string of 1's and 2's only is called *simple*. For example 22121 and 2222 are simple; 1021 is not simple. How many simple numbers are there less than one million?

A. 63 **B.** 62 **C.** 127 **D.** 128 **E.** 126

25. What is the maximum possible number of points of intersection one can get with eight circles?

A. 16 **B.** 32 **C.** 38 **D.** 44 **E.** 56

Next we give solutions to the contest given last issue, The Eleventh W.J. Blundon Contest.

THE ELEVENTH W.J. BLUNDON CONTEST

February 23, 1994

1. (a) The lesser of two consecutive integers equals five more than three times the larger integer. Find the two integers.

Solution. Let the integers be n, $n+1$. Then $n = 3(n+1) + 5$ so $n = -4$.

(b) If $4 \leq x \leq 6$ and $2 \leq y \leq 3$, find the minimum values of $(x-y)(x+y)$.

Solution. If $4 \leq x \leq 6$ and $2 \leq y \leq 3$, then $(x-y)(x+y) = x^2 - y^2 \geq 4^2 - 3^2 = 16 - 9 = 5$.

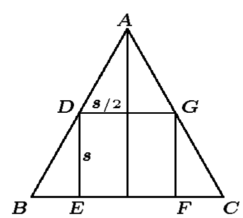
2. A geometric sequence is a sequence of numbers in which each term after the first can be obtained from the previous term by multiplying by the same fixed constant, called the **common ratio**. If the second term of a geometric sequence is 12 and the fifth term is $81/2$, find the first term and the common ratio.

Solution. Let the first term be a, and the ratio r. Then the sequence is $a, ar, arr = ar^2, ar^3, ar^4, ar^5, \ldots$, with the nth term being ar^{n-1}. So $ar = 12$ and $ar^4 = 81/2$. Dividing gives

$$r^3 = \frac{ar^4}{ar} = \frac{81/2}{12} = \frac{81}{24} = \frac{27}{8},$$

so $r = \frac{3}{2}$. Since $ar = a \cdot \frac{3}{2} = 12$, $a = 8$.

3. A square is inscribed in an equilateral triangle. Find the ratio of the area of the square to the area of the triangle.



Solution. Let the equilateral triangle ABC and the square $DEFG$ be as shown. Now $\angle ADG = \angle ABC = \angle AGD = \angle ACB = 60°$ so ADG is an equilateral triangle with side length s, the length of the side of the square. If DE is superimposed over GF a third equilateral triangle is formed with height s.

Now the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$, while the area of an equilateral triangle with *height* s is $\frac{1}{\sqrt{3}}s^2$. So the area of the square is s^2 and the area of ABC is

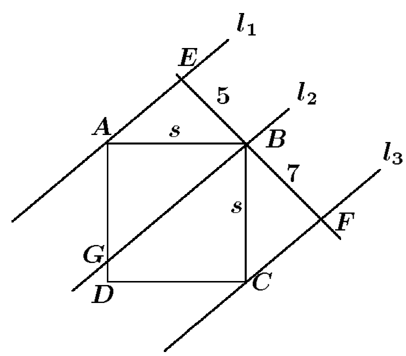
$$\frac{\sqrt{3}}{4}s^2 + s^2 + \frac{s^2}{\sqrt{3}} = \left(1 + \frac{\sqrt{3}}{4} + \frac{1}{\sqrt{3}}\right)s^2.$$

The ratio of the area of the square to the area of the triangle is then

$$\frac{1}{1+\frac{7}{4\sqrt{3}}} = \frac{4\sqrt{3}}{4\sqrt{3}+7} = \frac{4\sqrt{3}}{-48+49}(-4\sqrt{3}+7) = 28\sqrt{3} - 48.$$

4. $ABCD$ is a square. Three parallel lines l_1, l_2 and l_3 pass through A, B and C respectively. The distance between l_1 and l_2 is 5 and the distance between l_2 and l_3 is 7. Find the area of $ABCD$.

Solution. Let the line perpendicular to l_2 through B meet l_1 at E and l_2 at F respectively. Let l_2 meet AD at G. Then $\angle BAE = \angle CBF$ and $\angle AEB = 90° = \angle BFC$ so ΔBAE is similar to ΔBCF. If the side length is s this gives



$$\frac{s}{\sqrt{s^2 - 5^2}} = \frac{s}{7}$$

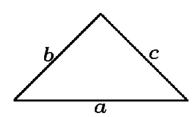
so $s^2 = 5^2 + 7^2 = 25 + 49 = 74$.

5. The sum of the lengths of the three sides of a right triangle is 18. The sum of the squares of the lengths of the three sides is 128. Find the area of the triangle.

Solution.

$$a^2 + b^2 = c^2$$

$$a + b + c = 18$$



and $$a^2 + b^2 + c^2 = 128.$$

So $2c^2 = 128$, $c^2 = 64$ and $c = 8$. Thus $a + b = 10$, $a^2 + b^2 = 64$, so

$$\begin{aligned} 2ab &= (a+b)^2 - (a^2 + b^2) \\ &= 100 - 64 = 36 \end{aligned}$$

The area $\frac{1}{2}ab = \frac{2ab}{4} = \frac{36}{4} = 9$.

6. A palindrome is a word or number that reads the same backwards and forwards. For example, 1991 is a palindromic number. How many palindromic numbers are there between 1 and 99, 999 inclusive?

Solution. Let us find the number n_k of the properly k digit palindromic numbers. To be a k digit number the first digit must be one of $1, 2, 3, \ldots, 9$. The answer now depends whether k is even or odd.

Now $n_1 = 9$. If $k > 1$ is odd, $k = 2l + 1$, $l \geq 1$ then $n_k = 9 \times 10^l$. If k is even, $k = 2l$, $l \geq 1$ then $n_k = 9 \times 10^{l-1}$. For the answer we want

$$n_1 + n_2 + n_3 + n_4 + n_5 = 9 + 9 + 90 + 90 + 900 = 1098.$$

7. A graph of $x^2 - 2xy + y^2 - x + y = 12$ and $y^2 - y - 6 = 0$ will produce four lines whose points of intersection are the vertices of a parallelogram. Find the area of the parallelogram.

Solution. Note that $x^2 - 2xy + y^2 - x + y = 12$ is equivalent to

$$(x - y)^2 - (x - y) - 12 = 0 \quad \text{or}$$

$$((x-y)-4)(x-y+3)=0$$

Thus its graph is the two parallel lines $x-y-4=0$ and $x-y+3=0$.

Also $y^2-y-6=0$ is equivalent to $(y-3)(y+2)=0$, giving two horizontal lines $y-3=0$ and $y+2=0$. The sides of the parallelogram lie on these lines. It has base 7 and height 5. (The distance between the horizontal lines.)

8. Determine the possible values of c so that the two lines $x-y=2$ and $cx+y=3$ intersect in the first quadrant.

Solution. For the curves to intersect in the first quadrant we need $x\geq 0$, $y\geq 0$. Adding the equations gives $(c+1)x=5$, so we require $c+1>0$, or $c>-1$. Substitution of $x=\frac{5}{c+1}$ into $x-y=2$ gives

$$y=\frac{5}{c+1}-2=\frac{3-2c}{c+1}$$

and we need $3-2c\geq 0$, so $c\leq\frac{3}{2}$. The intersection is in the first quadrant for $-1<c\leq\frac{3}{2}$.

9. Consider the function $f(x)=\frac{cx}{2x+3}$, $x\neq -3/2$. Find all values of c, if any, for which $f(f(x))=x$.

Solution.

$$f(f(x))=\frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right)+3}\quad\text{for } x\neq\frac{-3}{2}$$

and $f(x)\neq -3/2$, i.e. $cx/(2x+3)\neq -3/2$, $x\neq -9/(6+2c)$. For these excluded values $f(f(x))=x$ gives

$$\frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right)+3}=x$$

so

$$c^2x=x(2cx+3(2x+3))$$
$$c^2x=2cx^2+6x^2+9x$$

so equating coefficients of x, x^2, etc. (since the polynomials are equal for infinitely many values of x) gives $c^2=9$ and $2c=6$, so $c=3$.

10. Two numbers are such that the sum of their cubes is 5 and the sum of their squares is 3. Find the sum of the two numbers.

Solution. Let the numbers be x and y. We are given $x^3+y^3=5$ and $x^2+y^2=3$.

Let $x+y=A$.

From $(x+y)^3=x^3+3x^2y+3xy^2+y^3=x^3+y^3+3xy(x+y)$ we have $A^3=6+3xyA$.

From $(x+y)^2 = x^2 + y^2 + 2xy$ we have $A^2 = 3 + 2xy$ or $xy = \frac{1}{2}(A^2 - 3)$.

So we get $A^3 = 5 + \frac{3}{2}(A^2 - 3)A$ or

$$\frac{1}{2}A^3 + \frac{9}{2}A + 5 = 0 \quad \text{and} \quad A^3 + 9A + 10 = 0$$

this gives $A = -1$.

That completes the Skoliad Corner for this issue. Send me your contest materials, comments, suggestions, and solutions.

Historical Titbit

Taken from a 1950's University Scholarship Paper.

A box contains forty-eight chocolates all exactly similar in appearance.

There are four of each of twelve different sorts.

How many must you take out to be certain of having at least one of each of five different sorts?

Suppose that, having taken this number out, you eat four of them, and find that they are all of one sort.

How many must you now put back to be certain that the box contains at least two of each of eight different sorts?

THE OLYMPIAD CORNER

No. 174

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a first Olympiad set this number we give the third Team Competition, the Baltic Way 1994. The contest was written in Vilnius, Lithuania. Teams from Denmark, St. Petersburg, Poland, Latvia, Iceland, Lithuania, Estonia and Sweden participated. My thanks go to Georg Gunther for collecting this problem set when he was Canadian Team Leader to the IMO at Istanbul.

MATHEMATICAL TEAM CONTEST "BALTIC WAY — 92" Vilnius, 1992 — November 5–8

1. Let p, q be two consecutive odd prime numbers. Prove that $p + q$ is a product of at least 3 positive integers > 1 (not necessarily different).

2. Denote by $d(n)$ the number of all positive divisors of a positive integer n (including 1 and n). Prove that there are infinitely many n such that $\frac{n}{d(n)}$ is an integer.

3. Find an infinite non-constant arithmetic progression of positive integers such that each term is neither a sum of two squares, nor a sum of two cubes (of positive integers).

4. Is it possible to draw a hexagon with vertices in the knots of an integer lattice so that the squares of the lengths of the sides are six consecutive positive integers?

5. Given that $a^2 + b^2 + (a + b)^2 = c^2 + d^2 + (c + d)^2$, prove that $a^4 + b^4 + (a + b)^4 = c^4 + d^4 + (c + d)^4$.

6. Prove that the product of the 99 numbers $\frac{k^3-1}{k^3+1}$, $k = 2, 3, \ldots, 100$, is greater than $\frac{2}{3}$.

7. Let $a = \sqrt[1992]{1992}$. Which number is greater:

$$\left.a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}\right\}1992 \quad \text{or} \quad 1992?$$

8. Find all integers satisfying the equation

$$2^x \cdot (4-x) = 2x+4.$$

9. A polynomial $f(x) = x^3 + ax^2 + bx + c$ is such that $b < 0$ and $ab = 9c$. Prove that the polynomial has three different real roots.

10. Find all fourth degree polynomials $p(x)$ such that the following four conditions are satisfied:

(i) $p(x) = p(-x)$, for all x,
(ii) $p(x) \geq 0$, for all x,
(iii) $p(0) = 1$,
(iv) $p(x)$ has exactly two local minimum points x_1 and x_2 such that $|x_1 - x_2| = 2$.

11. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Show that there exists one and only one function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the following conditions:

(i) If $0 < q < \frac{1}{2}$ then $f(q) = 1 + f\left(\frac{q}{1-2q}\right)$.
(ii) If $1 < q \leq 2$ then $f(q) = 1 + f(q-1)$.
(iii) $f(q) \cdot f(\frac{1}{q}) = 1$ for all $q \in \mathbb{Q}^+$.

12. Let $\mathbb{N}$ denote the set of positive integers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijective function and assume that there exists a finite limit

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = L.$$

What are the possible values of L?

13. Prove that for any positive $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \frac{4n^2}{\sum_{i=1}^{n}(x_i + y_i)^2}$$

holds.

14. There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from the other one. Prove that there is a town such that all the remaining towns can be reached from it.

15. Noah has 8 species of animals to fit into 4 cages of the ark. He plans to put species in each cage. It turns out that, for each species, there are at most 3 other species with which it cannot share the accommodation. Prove that there is a way to assign the animals to their cages so that each species shares with compatible species.

16. All faces of a convex polyhedron are parallelograms. Can the polyhedron have exactly 1992 faces?

17. Quadrangle $ABCD$ is inscribed in a circle with radius 1 in such a way that one diagonal, AC, is a diameter of the circle, while the other diagonal, BD, is as long as AB. The diagonals intersect in P. It is known that the length of PC is $\frac{2}{5}$. How long is the side CD?

18. Show that in a non-obtuse triangle the perimeter of the triangle is always greater than two times the diameter of the circumcircle.

19. Let C be a circle in the plane. Let C_1 and C_2 be nonintersecting circles touching C internally at points A and B respectively. Let t be a common tangent of C_1 and C_2, touching them at points D and E respectively, such that both C_1 and C_2 are on the same side of t. Let F be the point of intersection of AD and BE. Show that F lies on C.

20. Let $a \leq b \leq c$ be the sides of a right triangle, and let $2p$ be its perimeter. Show that $p(p-c) = (p-a)(p-b) = S$ (the area of the triangle).

As a second Olympiad set to give you recreation over the summer months, we give the 8th Iberoamerican Mathematical Olympiad written September 14–15, 1993 in Mexico.

8th IBEROAMERICAN MATHEMATICAL OLYMPIAD
September 14–15, 1993 (Mexico)

First Day — 4.5 hours

1. (*Argentina*) Let $x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots$ be all the palindromic natural numbers, and for each i, let by $y_i = x_{i+1} - x_i$. How many distinct prime numbers belong to the set $\{y_1, y_2, y_3, \ldots\}$?

2. (*Mexico*) Show that for any convex polygon of unit area, there exists a parallelogram of area 2 which contains the polygon.

3. (*Mexico*) Let $\mathbb{N}^* = \{1, 2, 3, \ldots\}$. Find all the functions

$$f : \mathbb{N}^* \to \mathbb{N}^*$$

such that

(i) If $x < y$, then $f(x) < f(y)$

(ii) $f(yf(x)) = x^2 \cdot f(xy)$, for all x, y belonging to $\mathbb{N}^*$.

Second Day — 4.5 hours

4. (*Spain*) Let ABC be an equilateral triangle, and Γ its incircle. If D and E are points of the sides AB and AC, respectively, such that DE is tangent to Γ, show that

$$\frac{AD}{DB} + \frac{AE}{EC} = 1.$$

5. (*Mexico*) Let P and Q be distinct points of the plane. We denote $m(PQ)$ the perpendicular bisector of the segment PQ. Let S be a finite subset of the plane, with more than one element, which satisfies the following properties:
(i) If P and Q are distinct points of S, then $m(PQ)$ intersects S.
(ii) If P_1Q_1, P_2, Q_2 and P_3Q_3 are three distinct segments with extreme points belonging to S, then no point of S belongs simultaneously to the three lines $m(P_1Q_1)$, $m(P_2Q_2)$, $m(P_3Q_3)$.

Determine the number of possible points of S.

6. (*Argentina*) Two non-negative integer numbers, a and b, are "cuates" (friends in Mexican) if the decimal expression of $a+b$ is formed only by 0's and 1's. Let A and B be two infinite sets of non-negative integers, such that B is the set of all the numbers which are "cuates" of all the elements of A, and A is the set of all the numbers which are "cuates" of all the elements of B. Show that in one of the sets A or B there are infinitely many pairs of numbers x, y such that $x - y = 1$.

We now turn to the readers' comments and solutions to problems given in the December 1994 number of the *Corner* and the Nordic Mathematical Contest, 1992 [1994: 277].

1. [1994: 277] Determine all real numbers x, y, z greater than 1, satisfying the equation

$$x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1} = 2\left(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2}\right).$$

Solutions by Šefket Arslanagić, Berlin, Germany; by Cyrus C. Hsia, student, Woburn Collegiate Institute, Toronto; by Chandan Reddy, Rochester, Michigan; Michael Selby, University of Windsor, Windsor, Ontario; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Hsia's solution.

For $a > 1$, $a \in \mathbb{R}$ we have the Arithmetic Mean–Geometric Mean inequality

$$a - 1 + \frac{a+2}{a-1} \geq 2\sqrt{a+2},$$

with equality if and only if $a - 1 = \frac{a+2}{a-1}$, or $a^2 - 3a - 1 = 0$, giving $a = (3 + \sqrt{13})/2$, since $a > 1$.

For each of $a = x, y, z$ we have this same result and adding them gives

$$x + y + z - 3 + \frac{x+2}{x-1} + \frac{y+2}{y-1} + \frac{z+2}{z-1} \geq 2(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2})$$

whence

$$x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1} \geq 2(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2})$$

with equality if and only if $x = y = z = (3+\sqrt{13})/2$. Since there is equality, the unique solution is $x = y = z = (3 + \sqrt{13})/2$.

2. [1994: 277] Let n be an integer greater than 1 and let $a_1, a_2, \ldots, a_n$ be n different integers. Prove that the polynomial $f(x) = (x - a_1)(x - a_2)\ldots(x - a_n) - 1$ is not divisible by any polynomial of positive degree less than n and with integer coefficients and leading coefficient 1.

Solutions by Cyrus C. Hsia, Woburn Collegiate Institute, Toronto, Ontario; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Chandan Reddy; Rochester, Michigan; and by Michael Selby, University of Windsor, Windsor, Ontario. We give the solution sent in by Selby and Klamkin's comment.

Suppose $p(x)$ divides $f(x)$, where $1 \leq \deg p < n$, and p is monic with integral coefficients. Therefore $f(x) = p(x)q(x)$ where $q(x)$ is also monic, with integral coefficients and $1 \leq \deg q < n$.

Now $p(a_i)q(a_i) = -1$, $i = 1, 2, \ldots, n$. Since $p(a_i)$, $q(a_i)$ are integers, $p(a_i) = 1$ and $q(a_i) = -1$ or $p(a_i) = -1$ and $q(a_i) = 1$ for each $i = 1, 2, \ldots, n$.

In either case $p(a_i) + q(a_i) = 0$ for $i = 1, 2, \ldots, n$.

Consider $p(x) + q(x)$. This is a polynomial of positive degree less than n, since both $p(x)$ and $q(x)$ are monic with degree less than n. However $p(a_i) + q(a_i) = 0$ for n distinct values, but has positive degree less than n, an impossibility. Therefore no such $p(x)$ exists.

[Editor's Note.] Klamkin (whose solution was similar) points out that the problem is well-known.

3. [1994: 277] Prove that among all triangles with given incircle, the equilateral one has the least perimeter.

Solutions by Šefket Arslanagić, Berlin, Germany; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Chandon Reddy, Rochester, Michigan. We first give Reddy's solution.

Let the three sides be a, b, c. Then the inradius r times the semiperimeter is the area $A = rs$, $r = 1$ so, using Heron's formula

$$\sqrt{s(s-a)(s-b)(s-c)} = s$$

or

$$\sqrt{(s-a)(s-b)(s-c)} = \sqrt{s}$$

so

$$s = (s-a)(s-b)(s-c). \tag{1}$$

Also

$$s = (s-a) + (s-b) + (s-c). \tag{2}$$

We minimize the perimeter and therefore s by the AM–GM inequality and (1) and (2)

$$\sqrt{(s-a)(s-b)(s-c)} \leq \frac{(s-a)+(s-b)+(s-c)}{3}$$

The RHS is minimized when we have equality, that is $s-a=s-b=s-c$, and $a=b=c$.

Next we give Klamkin's comments and alternative approaches.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. This is a well-known result. It follows immediately from the isoperimetric theorem for polygons, that is, of all n-gons with given perimeter, the regular one has the maximum area and, dually, of all polygons of given area, the regular one has the least perimeter. For triangles we have the inequality

$$\frac{p^2}{F} \geq \frac{p_0^2}{F_0} = 12\sqrt{3}$$

where p and F denote the perimeter and area of a general triangle and p_0, F_0 correspond to the same for an equilateral triangle. Since $F = rp/2$ (r = inradius), we have

$$p \geq 6\sqrt{3}\, r$$

and with equality **if and only if** the triangle is equilateral.

For another proof in terms of the angles A, B, C of the triangle, it is equivalent to establishing the known triangle inequality

$$2r\left(\frac{\cot A}{2}+\frac{\cot B}{2}+\frac{\cot C}{2}\right) \geq 6\sqrt{3}\, r$$

It also follows that of all n-gons circumscribed to a given circle, the regular one has the least perimeter (and area as well).

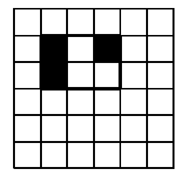
Comment: These results generalize to simplexes. In particular for the tetrahedron, we have the following known inequalities, where E = the sum of the 6 edges $\sum E_i$, F = the sum of the areas of the 4 faces $\sum F_i$, V = the volume and r = the inradius,

$$E \geq 6\prod E_i^{1/6} \geq k_1 V^{1/3}, \qquad (1)$$

$$F \geq 4\prod F_i^{1/4} \geq k_2 V^{2/3}, \qquad (2)$$

and there is equality if the tetrahedron is regular so that the constants k_1, k_2 are determined by taking $E_i = 1$. Then, $F_i = \sqrt{3}/4$ and $V = \sqrt{2}/12$. On multiplication of (1) and (2), we get $EF \geq k_1 k_2 V = \frac{k_1 k_2 r F}{3}$ or that $E \geq \frac{k_1 k_2 r}{3}$. This proves a 3-dimensional extension of the given result.

4. [1994: 277] Peter has a great number of squares, some of them are black, some are white. Using these squares, Peter wants to construct a square, where the edge has length n, and with the following property: The four squares in the corners of an arbitrary subrectangle of the big square, must never have the same colour. How large a square can Peter build?



$n = 6$

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

That the answer is 4×4 is given in the solution of problem #1 of the U.S.A. 1976 Mathematical Olympiad [1].

Reference

[1] M.S. Klamkin, U.S.A. Mathematical Olympiads 1972–1986, M.A.A., Washington, D.C. 1988, pp. 93–94.

To complete this number we given two comments by Murray Klamkin about earlier solutions.

3. [1994: 279; 1993: 131] The abscissa of a point which moves in the positive part of the axis Ox is given by $x(t) = 5(t+1)^2 + a/(t+1)^5$, in which a is a positive constant. Find the minimum a such that $x(t) \geq 24$ for all $x \geq 0$.

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

A simpler non-calculus solution is given by applying the AM–GM inequality, that is,

$$\frac{5(t+1)^2 + 2\left\{\frac{\frac{1}{2}a}{(t+1)^5}\right\}}{[5+2]} \geq \left(\frac{a^2}{4}\right)^{1/7}.$$

Then we have

$$7\left(\frac{a^2}{4}\right)^{1/7} \geq 24,$$

so that

$$\min a = 2\left(\frac{24}{7}\right)^7.$$

5. [1994: 281; 1993: 132]
For each natural number n, let $(1+\sqrt{2})^{2n+1} = a_n + b_n\sqrt{2}$ with a_n and b_n integers.

(a) Show that a_n and b_n are odd for all n.
(b) Show that b_n is the hypotenuse of a right triangle with legs

$$\frac{a_n + (-1)^n}{2} \quad \text{and} \quad \frac{a_n - (-1)^n}{2}.$$

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

A more direct solution of part (a) is as follows:

$$\begin{aligned} a_n &= \frac{(\sqrt{2}+1)^{2n+1} - (\sqrt{2}-1)^{2n+1}}{2} \\ &= \binom{2n+1}{1} 2^n + \binom{2n+1}{3} s^{n-1} + \cdots + 1, \\ b_n &= \frac{(\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1}}{2\sqrt{2}} \\ &= 2^n + \binom{2n+1}{2} 2^{n-1} + \cdots + \binom{2n+1}{2n}. \end{aligned}$$

That completes the Corner for this issue. Have a good summer – spend some time solving problems and send me your nice solutions as well as Olympiad Contest materials.

Book wanted!

Bruce Shawyer would like to purchase a copy of the out-of-print book:

On Mathematics and Mathematicians (Memorabilia Mathematica)

by Robert Edouard Moritz.

Dover Edition published 1942.

Originally published as:

Memorabilia Mathematica or The Philomath's Quotation-Book.

Original Edition published 1914.

Anyone willing to part with a copy please send him details. Thank you.

THE ACADEMY CORNER

No. 3

Bruce Shawyer

All communications about this column should be sent to Professor Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

Spring is approaching in parts of Canada, and, as undergraduate students in North America prepare for final examinations, some find time to try their skills on some mathematics problems. Memorial University offers some modest prizes for the best students. How do you compare? Please send me your best solutions to these problems.

MEMORIAL UNIVERSITY OF NEWFOUNDLAND UNDERGRADUATE MATHEMATICS COMPETITION

March, 1996

1. Prove that if n is a positive integer, then $\dfrac{n^2+3n+1}{n^2+4n+3}$ is an irreducible fraction.

2. A jar contains 7 blue balls, 9 red balls and 10 white balls. Balls are drawn at random one by one from the jar until either four balls of the same colour or at least two of each colour have been drawn. What is the largest number of balls that one may have to draw?

3. Find all functions $u(x)$ satisfying $u(x) = x + \int_0^{\frac{1}{2}} u(t)dt$.

4. Show that $(\sqrt{5}+2)^{\frac{1}{3}} - (\sqrt{5}-2)^{\frac{1}{3}}$ is a rational number and find its value.

5. In a quadrilateral $ABCD$ (vertices named in clockwise order), AC and BD intersect in X. You are given that $AB \parallel DC$, that AB is twice as long as CX and that AC is equal in length to DC. Show that AB and CD are equal in length (and hence $ABCD$ is a parallelogram).

6. Prove that among any thirteen distinct real numbers it is possible to choose two, x and y, such that $0 < \dfrac{x-y}{1+xy} < 2-\sqrt{3}$.

7. A coast guard boat is hunting a bootlegger in a fog. The fog rises disclosing the bootlegger 4 miles distant and immediately descends. The speed of the boat is 3 times that of the bootlegger, and it is known that the latter will immediately depart at full speed on a straight course of unknown direction. What course should the boat take in order to overtake the bootlegger?

Historical Titbit

Taken from a 1950's University Scholarship Paper.

A, C, B are collinear points.

Prove that there is one and only one point D such that $(ACBD)$ is harmonic.

The following result is stated in a book on geometry:

> if the pencils $O(ACBD)$ and $O'(ACBD)$ are harmonic and if A, C, B are collinear, then D lies on the line ACB.

Give an example to show that this result is not always true.

What alteration is required in order to make it true?

Prove the result after making this alteration.

BOOK REVIEWS

Edited by ANDY LIU

All the Math That's Fit to Print, by Keith Devlin,
published by the Mathematical Association of America, 1994,
ISBN 0-88385-515-1, paperbound, 330+ pages, US$29.50.
Reviewed by **A. Sharma**, *University of Alberta.*

In the foreword to his famous book *Mathematical Snapshots*, Steinhaus says, "My purpose is neither to teach in the usual sense of the word, nor to amuse the reader with some charades. One fine day it happened that I was asked the question: 'You claim to be a mathematician, well what does one do all day when one is a mathematician?' " Steinhaus conceived the idea of his book as he tried to explain to his questioner a few geometric problems solved and unsolved while sitting on a bench in a public park. Keith Devlin's book is meant for a much wider audience in a completely different milieu. It is also not meant to teach or to amuse the reader with riddles, although it succeeds eminently in doing a bit of both. As he explains in the preface, it is meant for "anyone who regularly reads a serious newspaper and has some interest in matters scientific, mathematical or is just curious".

This very interesting book, with its catching title is a collection of 143 articles written by the author for the Manchester Guardian over a period of eight years from 1983 to 1990. From letters received by him, the author claims that his audience was a mixed bunch: "Students at schools in their early teens, retired people in their nineties, from prison inmates to executives in the computer industry, from truckers to school teachers, both men and women".

The articles in this book are chronologically arranged under fancy headings some of which we sample here: *Seven-up, A Farey Story, The Vertical Confusion, World's Most Wanted Number, Rabbit Pi, Biblical Fingers Get Stuck Into Pi, Rabbits Do It By Numbers*, and so on. Each article can be read and enjoyed independently of the others, although cross references to allied topics are given almost everywhere. It is a book which can be enjoyed at one's leisure and with a pencil and a piece of paper can give hours of delight and education.

Devlin follows the style of his mentor Martin Gardner. He is informal and states results when he chooses and sketches a proof if it is short and well within the reach of a layman (see Article 15). Article 19, *The measure of all things*, begins with the news about the discovery that a long standing conjecture in number theory called Merten's conjecture is false. The author slowly and carefully explains the Möbius function, and tells the reader what Merten's conjecture is, and who proved that it is false, while giving a bird's eye view of the result proved.

Article 25 is just half a page, but it tells about the Institute for Pi Research at Emporia University. The Institute, we learn, is campaigning for the value $\pi = 3$ to be given equal time with the more conventional value in state schools.

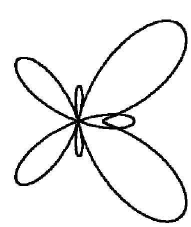
I personally found the Article 29, *Question Time*, very interesting, since I spent hours working out some of its problems, specially questions 11, 12, 13 and 15. I was particularly pleased to read in Article 99 about D.R. Kaprekar whom I knew while in India. There were no computers in those days, but I have witnessed his phenomenal abilities at doing long multiplication and other operations without pen and paper almost instantly at several meetings of the Indian Mathematical Society.

It would be surprising if a book of almost 320 pages had no misprints. I point out some minor ones which I came across:

- p. xi, line 2 from bottom — ready should be read;
- p. 27, column 2, line 6 from top — 33 should be 3;
- p. 46, column 1, line 15 from top — the should be to;
- p. 71, column 1, line 10 from top — 70-71 should be 72-73;
- p. 322, column 1, line 3 from bottom — savaged should be salvaged.

The range of topics covered in these articles is very impressive - Fermat Primes, Carmichael numbers, Palindromic numbers, information, artificial intelligence, Hilbert's tenth problem, Bieberbach's conjecture, Banach-Tarski paradox, cartography and many more. It is a book which should be read piecemeal — an article or two at a time, and as Devlin says, "it is a book for delving". I am sure this book will soon be in most libraries of schools and colleges for the benefit of both teachers and students.

Do you know the equation of this curve?



PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that* **1 December 1996**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in LaTeX, preferably in LaTeX2e*). *Graphics files should be in epic format, or plain postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2138. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

ABC is an acute angle triangle with circumcentre O. AO meets the circle BOC again at A', BO meets the circle COA again at B', and CO meets the circle AOB again at C'.

Prove that $[A'B'C'] \geq 4[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ.

2139. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Point P lies inside triangle ABC. Let D, E, F be the orthogonal projections from P onto the lines BC, CA, AB, respectively. Let O' and R' denote the circumcentre and circumradius of the triangle DEF, respectively. Prove that

$$[ABC] \geq 3\sqrt{3R'}\sqrt{R'^2 - (O'P)^2},$$

where $[XYZ]$ denotes the area of triangle XYZ.

2140. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Determine the quartic $f(x) = x^4 + ax^3 + bx^2 + cx - c$ if it shares two distinct integral zeros with its derivative $f'(x)$ and $abc \neq 0$.

2141. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$A_1A_2A_3A_4$ is a quadrilateral. Let B_1, B_2, B_3 and B_4 be points on the sides A_1A_2, A_2A_3, A_3A_4 and A_4A_1 respectively,
such that

$$A_1B_1 : B_1A_2 = A_4B_3 : B_3A_3 \quad \text{and} \quad A_2B_2 : B_2A_3 = A_1B_4 : B_4A_4.$$

Let P_1, P_2, P_3 and P_4 be points on B_4B_1, B_1B_2, B_2B_3
and B_3B_4 respectively, such that

$$P_1P_2 \| A_1A_2, \quad P_2P_3 \| A_2A_3 \quad \text{and} \quad P_3P_4 \| A_3A_4.$$

Prove that $P_4P_1 \| A_4A_1$.

2142. *Proposed by Victor Oxman, Haifa, Israel.*

In the plane are given an arbitrary quadrangle and bisectors of three of its angles. Construct, using only an unmarked ruler, the bisector of the fourth angle.

2143. *Proposed by B. M⋆⋆⋆y, Devon, Switzerland.*

My lucky number, 34117, is equal to $166^2 + 81^2$ and also equal to $159^2 + 94^2$, where $|166 - 159| = 7$ and $|81 - 94| = 13$; that is,

it can be written as the sum of two squares of positive integers in two ways, where the first integers occurring in each sum differ by 7 *and the second integers differ by* 13.

What is the smallest positive integer with this property?

2144. *Proposed by B. M⋆⋆⋆y, Devon, Switzerland.*

My lucky number, 34117, has the interesting property that $34 = 2 \cdot 17$ and $341 = 3 \cdot 117 - 10$, that is,

it is a $2N + 1$*-digit number* (in *base* 10) *for some* N, *such that*

(i) *the number formed by the first* N *digits is twice the number formed by the last* N, *and*

(ii) *the number formed by the first* $N + 1$ *digits is three times the number formed by the last* $N + 1$, *minus* 10.

Find another number with this property.

2145. *Proposed by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.*

Prove that $\displaystyle\prod_{k=1}^{n} \left(ak + b^{k-1}\right) \leq \prod_{k=1}^{n} \left(ak + b^{n-k}\right)$ for all a, $b > 1$.

2146. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $AB > AC$, and the bisector of $\angle A$ meets BC at D. Let P be an interior point on the segment AD, and let Q and R be the points of intersection of BP and CP with sides AC and AB respectively. Prove that $PB - PC > RB - QC > 0$.

2147. *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

Let S be the set of all positive integers x such that there exist positive integers y and m satisfying $x^2 + 2^m = y^2$.

(a) Characterize which positive integers are in S.

(b) Find all positive integers x so that both x and $x + 1$ are in S.

2148. *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Suppose that AD, BE and CF are the altitudes of triangle ABC. Suppose that L, M, N are points on BC, CA, AB, respectively, such that $BL = DC$, $CM = EA$, $AF = NB$.

Prove that:

1. the perpendiculars to BC, CA, AB at L, M, N, respectively are concurrent;

2. the point of concurrency lies on the Euler line of triangle ABC.

2149. *Proposed by Juan-Bosco Morero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $ABCD$ be a convex quadrilateral and O is the point of the intersection of the diagonals AC and BD. Let $A'B'C'D'$ be the quadrilateral whose vertices, A', B', C', D', are the feet of the perpendiculars drawn from the point O to the sides BC, CD, DA, AB, respectively.

Prove that $ABCD$ is an inscribed (cyclic) quadrilateral if and only if $A'B'C'D'$ is a circumscribing quadrilateral ($A'B'$, $B'C'$, $C'D'$, $D'A'$ are tangents to a circle).

2150. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Find all real solutions of the equation

$$\sqrt{1-x} = 2x^2 - 1 + 2x\sqrt{1-x^2}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2035. [1995: 130] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

If the locus of a point E is an ellipse with fixed foci F and G, prove that the locus of the incentre of triangle EFG is another ellipse.

Solution by the Science Academy Problem Solvers, Austin, Texas.

Let $F = (-c, 0)$ and $G = (c, 0)$ so that the locus of $E = (x, y)$ satisfies $GE + EF = 2a$ (where a is the length of the semimajor axis). Let I be the incentre and I' be the point where the incircle is tangent to FG (so that $II' \perp FG$). We have

$$FI' = a + c - GE = \frac{EF - GE}{2} + c$$

[since $s = a + c$ is half the perimeter of ΔEFG, and the tangents to the incircle from the vertices are equal in pairs and sum to $2s$]. If O is the origin then $FO = c$ and

$$\begin{aligned} OI' = FI' - c = \frac{EF - GE}{2} &= \frac{EF^2 - EG^2}{4a} \\ &= \frac{(x^2 + 2cx + c^2 + y^2) - (x^2 - 2cx + c^2 + y^2)}{4a} = \frac{cx}{a}. \end{aligned}$$

Since $r = II'$ is the radius of the incircle [and the area of ΔEFG is rs]

$$II' = \frac{\text{area}(EFG)}{a + c} = \frac{cy}{a + c}.$$

Thus, $I = \left(\frac{c}{a}x, \frac{c}{a + c}y\right)$. The mapping $(x, y) \to \left(\frac{c}{a}x, \frac{c}{a + c}y\right)$ is therefore an affine transformation that maps each point E of the given ellipse to the point I. Affine transformations map ellipses to ellipses so that I traces out an ellipse.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; JORDI DOU, Barcelona, Spain; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; CRISTÓBAL SÁNCHEZ-RUBIO, I.B. Penyagolosa, Castellón, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2040. [1995: 130] *Proposed by Frederick Stern, San Jose State University, San Jose, California.*

Let $a < b$ be positive integers, and let

$$t = \frac{2^a - 1}{2^b - 1}.$$

What is the relative frequency of 1's (versus 0's) in the binary expansion of t?

[Ed. My interpretation of the question asked is to find the ratio of the number of 1's to the number of 0's; most solvers also read it this way.

The proposer was the only solver to actually mention that the ratio we are to compute must be done asymptotically. I feel that should have been part of the problem statement.]

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

Let $x = .\overline{000\ldots01}$, where there are $b-1$ zeros (the pattern below the superbar repeats infinitely often). Then $(2^b-1)x = .111\ldots1 = 1$, so $x = 1/(2^b-1)$. Therefore, $(2^a-1)/(2^b-1) = .\overline{00\ldots01\ldots11}$, where there are $b-a$ zeros and a ones in every period, so there are a zeros for every $b-a$ ones when counted from the left.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; KEITH EKBLAW, Walla Walla, Washington, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, the Netherlands; Science Academy Problem Solvers, Austin, Texas, USA; DAVID STONE, Georgia Southern University, Statesboro, Georgia, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

Janous generalizes the problem to show that in base $z+1$ we have

$$\frac{(z+1)^a}{(z+1)^b - 1} = 000\ldots0zz\ldots z$$

where there are $b-a$ zeros and a copies of z when $a < b$.

2041. [1995: 157] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

P is an interior point of triangle ABC. AP, BP, CP meet BC, CA, AB at D, E, F respectively. Let M and N be points on segments BF and CE respectively so that $BM : MF = EN : NC$. Let MN meet BE and CF at X and Y respectively. Prove that $MX : YN = BD : DC$.

Solution by Jari Lappalainen, Helsinki, Finland.

First, we apply Menelaus' theorem to triangles CNY and YFM with line EB. Dividing the results, we get

$$\frac{-1}{-1} = \frac{\frac{CE}{EN} \cdot \frac{NX}{XY} \cdot \frac{YP}{PC}}{\frac{FB}{BM} \cdot \frac{MX}{XY} \cdot \frac{YP}{PF}} = 1$$

and substituting $CE : EN = FB : BM$ (which follows directly from $BM : MF = EN : NC$), we get

$$\frac{NX}{MX} = \frac{PC}{PF},$$

or equivalently

$$\frac{FC}{PF} = \frac{MN}{XM}. \tag{1}$$

In a similar way using Menelaus's theorem for triangles BXM and XNE with line FC, and substituting $MF : FB = NC : CE$, we get

$$\frac{EB}{PE} = \frac{NM}{YN}. \tag{2}$$

Finally, applying Ceva's theorem to triangle ABC and Menelaus's theorem to triangles CEP and BEA, with lines AB and CF respectively, we find

$$\frac{1}{-1 \times (-1)} = \frac{\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB}}{\frac{CA}{AE} \cdot \frac{EB}{BP} \cdot \frac{PF}{FC} \times \frac{BP}{PE} \cdot \frac{EC}{CA} \cdot \frac{AF}{FB}} = \frac{BD}{DC} \cdot \frac{PE}{EB} \cdot \frac{FC}{PF} = 1.$$

Using (1) and (2)

$$\frac{BD}{DC} \cdot \frac{YN}{NM} \cdot \frac{MN}{XM} = -\frac{BD}{DC} \cdot \frac{YN}{XM} = 1,$$

or $MX : YN = BD : DC$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; and the proposer.

2042. [1995: 157] *Proposed by Jisho Kotani, Akita, Japan, and K.R.S. Sastry, Dodballapur, India.*

If A and B are three-digit positive integers, let $A * B$ denote the six-digit integer formed by placing them side by side. Find A and B such that

$$A, \quad B, \quad B - A, \quad A * B \quad \text{and} \quad \frac{A * B}{B}$$

are all integer squares.

Solution by Kathleen E. Lewis, SUNY Oswego, New York, NY, USA.

Since A, $B - A$ and B are all integer squares, their square roots form a Pythagorean triple. Thus, the easiest place to look for numbers meeting the stated conditions is in the multiples of the $3, 4, 5$ triple. So, letting $A = 9t^2$ and $B = 25t^2$ for an integer t, we see that $B - A = 16t^2$,

$$A * B = 1000A + B = 9000t^2 + 25t^2 = 9025t^2 = (95t)^2$$

and

$$\frac{A * B}{B} = \frac{(95t)^2}{(5t)^2} = 19^2.$$

Any choice of t would yield A and B satisfying all the other conditions, but in order to make A and B three–digit numbers, t would have to be 4, 5 or 6, yielding values of

$$(144, 400), \qquad (225, 625), \qquad \text{and} \quad (324, 900)$$

for (A, B).

These are in fact the only possibilities. Suppose $A = x^2$, $B - A = y^2$ and $B = z^2$ satisfy the given conditions. Let $q = \gcd(x, y, z)$, $x' = x/q$, $y' = y/q$ and $z' = z/q$.

Then ${x'}^2 + {y'}^2 = {z'}^2$ and ${z'}^2$ divides $(A * B - B)/q^2 = 1000{x'}^2$. Since x' and z' are relatively prime, this means that ${z'}^2$ must divide 1000, so z' must be 1, 2, 5 or 10. By inspection, none of the others are possible values, so z' must be 5. We can also rule out the remaining case of $x' = 4$, $y' = 3$ and $z' = 5$ [that is, $A = 16t^2$ and $B = 25t^2$] since $(A * B)/t^2 = 16025$ is not a perfect square.

Editor's note. As some other solvers point out, the case $B - A = 0$ should be considered. This is the case $y' = 0$, $x' = z' = 1$, which is impossible because $A * B = 1001x^2$ is not a perfect square.

Also solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; J.K. FLOYD, Newnan, Georgia, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G.

HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; SUSAN SCHWARTZ WILDSTROM, Kensington, Maryland, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposers.

Most solvers found all three solutions.

The two proposers actually had sent the editor this problem, or something quite similar, independently and at almost the same time.

2043. [1995: 158] *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

What is the locus of a point interior to a fixed triangle that moves so that the sum of its distances to the sides of the triangle remains constant?

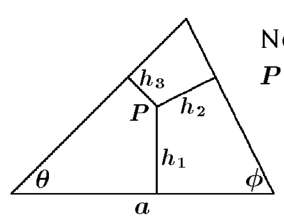
Editor's Comment.

The solution to the problem as stated in the proposal was quite easy using analytic/trigonometric arguments and this was the path chosen by most of the solvers. Some contributors, anxious no doubt to uphold the impeccable standards of CRUX, tried to do something more with the proposal - much to the delight of the Editor. One such example is to look at the locus for different triangles as portrayed in the featured solution by Hess. Just the observation that all points in the interior is the locus when the given triangle is equilateral can do wonders for the morale of the Editor. The second solution, while restricted to the interior of the triangle, did offer a respite from the analytic. Several solvers made comments of varying substance relating to the case when the point was exterior to the triangle. The third solution by Fritsch did the job nicely and was novel.

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California, USA.*

From the figure, where the largest side is the base, we have

$$\begin{aligned} h_1 &= y, \\ h_2 &= (a-x)\sin\phi - y\cos\phi, \\ h_3 &= x\sin\theta - y\cos\theta. \end{aligned}$$



Note:
$P = P(x, y)$

From $h_1 + h_2 + h_3 = k$ (a constant), we get

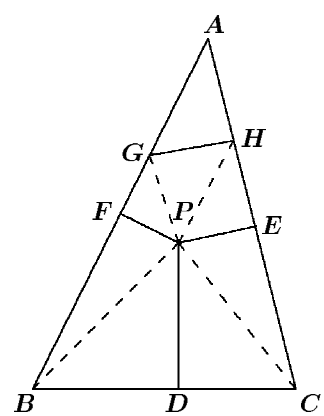
$$y(1 - \cos\phi - \cos\theta) = k - a\sin\phi + x(\sin\phi - \sin\theta).$$

There are several cases:

1. When $\phi = \theta = 60^\circ$, the whole interior of the triangle has $h_1+h_2+h_3 = a\sqrt{3}/2$.
2. When $\cos\theta + \cos\phi = 1$ but $\theta \neq \phi$, the locus is $x = \dfrac{k - a\sin\phi}{\sin\theta - \sin\phi}$.
3. When $\theta = \phi \neq 60^\circ$, the locus is $y = \dfrac{k - a\sin\phi}{1 - 2\cos\phi}$.
4. Otherwise, the locus is a straight line:
$$y = \frac{k - a\sin\phi}{1 - \cos\theta - \cos\phi} + \frac{x(\sin\phi - \sin\theta)}{1 - \cos\theta - \cos\phi}.$$

II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let P be a point interior to triangle ABC, so that the sum of its distances to the sides of triangle ABC is constant, and let D, E, F be the feet of the perpendiculars to BC, CA, AB respectively.
We assume that triangle ABC is non-equilateral and that BC is the least side.
We put $PD + PE + PF = k$, where k is a constant.



Let G, H be the points on the side AB, AC respectively, such that $BG = CH = BC$. Then

$$[PBC] + [PCH] + [PBG] = \tfrac{1}{2}a(PD + PE + PF) = \tfrac{1}{2}ak,$$

which is constant ($[XYZ]$ denotes the area of triangle XYZ).
Since the area of quadrilateral $BCHG$ is constant, so also is the area of triangle PGH. Therefore P lies on a fixed line ℓ, parallel to GH.
Hence the locus of P is the segment of ℓ contained within the triangle ABC.

III. *Solution by Rudolf Fritsch, Ludwig-Maximilians-Universität, München, Germany.*

Let ABC be a triangle in the real plane; without loss of generality we assume $a \geq b \geq c$. Then we may choose cartesian coordinates such that

$$A = (0,h)\,, \quad B = (-p,0)\,, \quad C = (q,0)$$

with $h,\ p,\ q > 0$. The signed distance of a point in the plane from a side of this triangle is taken positive if the point is in the same half plane as the vertex opposite to this side, thus, the signed distance of $P = (x, y)$ from the line BC is just y.

For the other sides of triangle we choose the following equations

$$\begin{aligned} AB &\equiv x \cdot \sin\beta - y \cdot \cos\beta + h \cdot \cos\beta = 0\,, \\ AC &\equiv -x \cdot \sin\gamma - y \cdot \cos\gamma + h \cdot \cos\gamma = 0\,. \end{aligned}$$

These equations are called the Hessian normal forms of the lines under consideration. The namesake is OTTO LUDWIG HESSE (born in Königsberg/East Prussia 1811/4/22, died in München 1874/8/4) although this form has already been used by Gauss in a paper published in 1810. The general idea is to normalize the line equation

$$g \equiv ux + vy + w = 0$$

by $u^2 + v^2 = 1$, $w \geq 0$ (or $w \leq 0$), the line g being unique if it does not pass through the origin. The advantage of this form is that for any point $P = (x, y)$ in the plane the expression

$$d(x, y) = ux + vy + w$$

gives the signed distance of P from g, positive if and only if P is on the same side of g as the origin (origin not on g).

Thus, the sum of the signed distances of $P = (x, y)$ from the reference triangle is

$$s(x, y) = x \cdot (\sin\beta - \sin\gamma) + y \cdot (1 - \cos\beta - \cos\gamma) + h \cdot (\cos\beta + \cos\gamma)\,.$$

Since this expression is linear, the equation $s(x, y) = k$ describes a line, for any $k \in R$. Since the slope is independent of k, all lines obtained in this way form a pencil of parallel lines. A distinguished member of this pencil is obtained by taking $k = 0$ giving the line connecting the points where the exterior angle bisectors meet the opposite sides.

Taking absolute distances we get buckled lines for the desired locus. If the problem is restricted to the interior points of the triangle ABC, then the result gives parallel line segments.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIELS BEJLEGAARD, Stavanger, Norway; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; P. PENNING, Delft, the Netherlands; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; ASHISH KR. SINGH, Kanpur, India; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Klamkin comments: since $a_1r_1 + a_2r_2 + a_3r_3 = 2\Delta$ *(where* r_i *is the distance to side* a_i*, and* Δ *is the area of the triangle), the constant sum of the distances must be bounded by*

$$\frac{2\Delta}{a_3} \geq r_1 + r_2 + r_3 \geq \frac{2\Delta}{a_1},$$

where it is assumed that $a_1 \geq a_2 \geq a_3$*. Otherwise, there are no points in the locus. This is also true, if, for example,* a_3 *is smaller than the other sides and the constant sum is* $\frac{2\Delta}{a_3}$*.*

2044. [1995: 158] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Suppose that $n \geq m \geq 1$ and $x \geq y \geq 0$ are such that

$$x^{n+1} + y^{n+1} \leq x^m - y^m.$$

Prove that $x^n + y^n \leq 1$.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

There is nothing to prove if $x = y$ since in this case the given inequality implies $x = y = 0$. Suppose $0 \leq y < x$, then $x^m - y^m > 0$. From $x^{n+1} \leq x^{n+1} + y^{n+1} \leq x^m - y^m \leq x^m$ and $n + 1 > m$, it follows that $x \leq 1$. Using $n - m \geq 0$ and $m \geq 1$, we now obtain

$$\begin{aligned}(x^n + y^n)(x^m - y^m) &= x^{n+m} - y^{n+m} - (xy)^m(x^{n-m} - y^{n-m}) \\ &\leq x^{n+m} \leq x^{n+1} \\ &\leq x^{n+1} + y^{n+1} \leq x^m - y^m.\end{aligned}$$

Dividing through by $x^m - y^m$ gives $x^n + y^n \leq 1$.

II. *Solution by the proposer.*

It is easy to see that $1 \geq x \geq y \geq 0$, and thus $x^m \leq x$, $xy^n \leq y^m$ and $x^n y \leq y$. Therefore,

$$\begin{aligned}(x^n + y^n)(x + y) &= x^{n+1} + y^{n+1} + x^n y + xy^n \\ &\leq x^m - y^m + y + y^m \leq x + y.\end{aligned}$$

Dividing through by $x + y$ [the case when $x + y = 0$ being trivial – Ed.] yields $x^n + y^n \leq 1$. Equality holds if and only if $x = 1$, $y = 0$.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; and PANOS E. TSAOUSSOGLOU, Athens, Greece.

Janous showed that if $1 \leq m \leq n-1$, *then the condition can be relaxed to* $x^{n+1}+y^{n+1} \leq x^m - y^{n-1}$. *Flanigan obtained the stronger result that if* $k = \lfloor \frac{n+1}{m} \rfloor$, *then* $x^{(k-1)m} + (k-1)y^{(k-1)m} \leq 1$ *if* $m|n+1$ *and* $x^{km} + ky^{km} \leq 1$ *if* $m \not| n+1$.

2045. [1995: 158] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Show that there are an infinite number of Pythagorean triangles (right-angled triangles with integer sides) whose hypotenuse is an integer of the form $3333\ldots3$.

Once again, our readers have been extremely inventive! Most provided a way to construct an infinite sequence (or more than one sequence) of suitable Pythagorean triples (a, b, c), where $a^2 + b^2 = c^2$. We summarize the results below.

I. *Solution by: Niels Bejlegaard, Stavanger, Norway; Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA; Christopher J. Bradley, Clifton College, Bristol, UK; Miguel Angel Cabezón Ochoa, Logroño, Spain; Toby Gee, student, the John of Gaunt School, Trowbridge, England; David Hankin, Hunter College Campus Schools, New York, NY, USA; Richard I. Hess, Rancho Palos Verdes, California, USA; Friend H. Kierstead Jr., Cuyahoga Falls, Ohio, USA; Kathleen E. Lewis, SUNY Oswego, NY, USA; Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain; P. Penning, Delft, the Netherlands; Gottfried Perz, Pestalozzigymnasium, Graz, Austria; David R. Stone, Georgia Southern University, Statesboro, USA; Panos E. Tsaoussoglou, Athens, Greece; and the proposer.*

First, note that neither 3 nor 33 is the hypotenuse of a Pythagorean triangle.

Let (a, b, c) be a Pythagorean triple, where $c = 33\ldots3$ is a k-digit integer, $k > 2$. Then (ma, mb, mc) is also a Pythagorean triple, for all $m = 10^k + 1, 10^{2k} + 10^k + 1, \ldots, \sum_{i=0}^{n} 10^{ik}, \ldots$ and

$mc = 33\ldots3$ has $2k, 3k, \ldots, (n+1)k, \ldots$ digits.

Each of the following triples can each be used in this way to generate an infinite sequence of triples:

$(108, 315, 333)$
$(660, 3267, 3333)$
$(7317, 32520, 33333)$
$(128205, 307692, 333333)$
$(487560, 3297483, 3333333)$
$(25114155, 21917808, 33333333)$.

II. *Solution by: Heinz-Jürgen Seiffert, Berlin, Germany.*

$$\left(\frac{2}{3}10^n(10^{2n}-1), \frac{1}{3}(10^{2n}-1)^2, \frac{1}{3}(10^{4n}-1)\right), \text{ where } n \in \mathrm{N},$$

gives an infinite sequence of Pythagorean triples with hypotenuse the $4n$-digit integer $33\ldots3$. Note that the first triple, $(660, 3267, 3333)$, is the same as one given above, but the rest of the sequence is different.

Also solved (in a non-constructive way) by ASHISH KR. SINGH, student, Kanpur, India; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; and CHRIS WILDHAGEN, Rotterdam, the Netherlands. There was one partial solution.

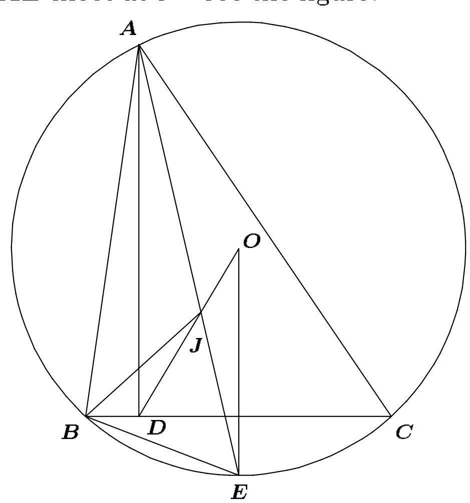
2047. [1995: 158] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

ABC is a non-equilateral triangle with circumcentre O and incentre I. D is the foot of the altitude from A to BC. Suppose that the circumradius R equals the radius r_a of the excircle to BC. Show that O, I and D are collinear.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

If $AB = AC$, the I, O lie on AD; hence O, I and D are collinear. Henceforth assume $AB \neq AC$.

Let the angle bisector of the angle $\angle A$ intersect the circumcircle of ABC at E and let DO and AE meet at J - see the figure.



OE is perpendicular to BC, so AD and OE are parallel. Therefore, since $R = r_a$

$$\frac{AJ}{JE} = \frac{AD}{OE} = \frac{h_a}{R} = \frac{h_a}{r_a} = \frac{2S}{ar_a} = \frac{2r_a(s-a)}{ar_a} = \frac{b+c-a}{a}, \qquad (1)$$

where h_a, S, s denote the altitude AD, the area, and the semiperimeter of ABC, respectively. To complete the solution it is enough to show that J is the incentre of ABC. Using Ptolemy's theorem on the quadrilateral $ACEB$, and the fact that $BE = CE$, we get

$$(AJ + JE)a = BE(b + c). \qquad (2)$$

Since (1) is equivalent to $(AJ + JE)a = JE(b + c)$, comparing it with (2) we obtain $JE(b + c) = BE(b + c)$, which gives $JE = BE$. Hence

$$\angle JBA + \angle BAJ = \angle BJE = \angle JBE = \angle CBJ + \angle CBE = \angle CBJ + \angle EAC.$$

Since $\angle BAJ = \angle EAC$, we get $\angle JBA = \angle CBJ$, which means that BJ bisects $\angle B$. And since AJ bisects $\angle A$, J is the incentre of ABC, as we wished to prove.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; ASHISH KR. SINGH, student, Kanpur, India; and the proposer.

Several solvers noted that the result was mentioned in the Editor's comments following the solution to problem 1918 (*CRUX, Vol. 21, No. 1*).

2046. [1995: 158] *Proposed by Stanley Rabinowitz, Westford, Massachusetts, USA.*

Find integers a and b so that

$$x^3 + xy^2 + y^3 + 3x^2 + 2xy + 4y^2 + ax + by + 3$$

factors over the complex numbers.

Solution by the proposer.

If the cubic is to factor in $\mathbb{C}[x, y]$, then one of the factors must be linear. Without loss of generality, we may assume this factor is of the form $x - py - q$ where p and q are complex numbers.

Substituting $x = py + q$ in the original cubic, we get a polynomial in y that must be identically 0. Thus each of its coefficients must be 0. This gives us the four equations:

$$\begin{aligned} 1 + p + p^3 &= 0 \\ 4 + 2p + 3p^2 + q + 3p^2q &= 0 \\ 3 + aq + 3q^2 + q^3 &= 0 \\ b + ap + 2q + 6pq + 3pq^2 &= 0. \end{aligned}$$

Solving these equations simultaneously, yields $a = 4$ and $b = 5$. (*The editor being a mere mortal needed Maple to verify this claim!*)

As a check we note that the resulting polynomial can be written as $(x+y+2)(y+1)^2+(x+1)^3$. The reducibility of this polynomial will not change if we let $x = X - 1$ and $y = Y - 1$. This produces the polynomial $X^3+XY^2+Y^3$. Letting $z = X/Y$ shows that this polynomial factors over $\mathbb{C}[x, y]$ since z^3+z+1 factors over $\mathbb{C}[z]$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California, USA. Rabinowitz also remarks that,

"except for a few exceptional cases, if $f(x, y)$ is a cubic polynomial in $\mathbb{C}[x, y]$, there will be unique complex constants a and b such that $f(x, y) + ax + by$ factors over $\mathbb{C}[x, y]$."

2049*. [1995: 158] *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let a tetrahedron $ABCD$ with centroid G be inscribed in a sphere of radius R. The lines AG, BG, CG, DG meet the sphere again at A_1, B_1, C_1, D_1 respectively. The edges of the tetrahedron are denoted a, b, c, d, e, f. Prove or disprove that

$$\frac{4}{R} \le \frac{1}{GA_1}+\frac{1}{GB_1}+\frac{1}{GC_1}+\frac{1}{GD_1} \le \frac{4\sqrt{6}}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right).$$

Equality holds if $ABCD$ is regular. (This inequality, if true, would be a three-dimensional version of problem 5 of the 1991 Vietnamese Olympiad; see [1994: 41].)

Solution to right hand inequality by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

We consider a generalization of the right hand inequality:

Let G and O be the centroid and circumcentre of an n-dimensional simplex $A_0A_1\ldots A_n$ inscribed in a sphere of radius R. Let the lines A_iG meet the sphere again in points A_i' : $i = 0, 1, \ldots, n$. If the edges are denoted by $e_j; j = 1, 2, \ldots, n(n+1)/2$, then

$$\sum_{i=0}^{n} \frac{1}{GA_i'} \le \sqrt{\frac{8(n+1)}{n^3}} \sum_{j=0}^{n(n+1)/2} \frac{1}{e_j}.$$

By the Power-of-a Point Theorem, we have $A_1'G \cdot A_iG = R^2 - OG^2$. By the power mean inequality, we have

$$\sum \frac{A_iG}{n+1} \le \left(\sum \frac{(A_iG)^2}{n+1}\right)^{\frac{1}{2}}.$$

So it suffices to prove the stronger inequality:

$$\left(\sum (A_iG)^2(n+1)\right)^{\frac{1}{2}} \leq \sqrt{\frac{8(n+1)}{n^3}}\left(R^2-(OG)^2\right)\sum \frac{1}{e_j}. \qquad (1)$$

It is known that

$$\sum (A_iG)^2 \;=\; (n+1)\left(R^2-(OG)^2\right) \qquad (2)$$

and

$$\left(R^2-(OG)^2\right) \;=\; \sum \frac{e_j^2}{(n+1)^2}. \qquad (3)$$

Using (2) and (3), we see that (1) becomes, after raising both sides to the power $2/3$,

$$\frac{n(n+1)}{2} \leq \left(\sum e_j^2\right)^{1/3}\left(\sum \frac{1}{e_j}\right)^{2/3},$$

and the result follows immediately from Hölder's inequality.

There is equality only if the simplex is regular.

Comment: Corresponding to the given left hand inequality, the analogous one for the simplex (and not as yet proved) is

$$\frac{n+1}{R} \leq \sum \frac{1}{A_i'G}$$

or, equivalently

$$R\sum A_iG \geq (n+1)\left(R^2-(OG)^2\right) = \sum (A_iG)^2.$$

Even more generally, I conjecture that, for $p \geq 1$, we have

$$\frac{2R\,(n^p+1)}{(n+1)\,(n^{p-1}+1)} \geq \frac{\sum (A_iG)^{p+1}}{\sum (A_iG)^p}.$$

Except for the case $p = 1$, there is no equality for a regular simplex, but for a degenerate one with n vertices coinciding at one end of a diameter and the remaining vertex at the other end of the diameter.

No other solutions were received.

2050. [1995: 158] *Proposed by Šefket Arslanagić, Berlin, Germany.*
Find all real numbers x and y satisfying the system of equations

$$2^{x^2+y} + 2^{x+y^2} = 128, \qquad \sqrt{x}+\sqrt{y} = 2\sqrt{2}.$$

Solution. Essentially identical solutions were submitted by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA; Christopher J. Bradley, Clifton College, Bristol, UK; Toby Gee, student, the John of Gaunt School, Trowbridge, England; David Hankin, Hunter College Campus Schools, New York, NY, USA; Cyrus Hsia, student, University of Toronto, Toronto, Ontario; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Kee-Wai Lau, Hong Kong; Waldemar Pompe, student, University of Warsaw, Poland; Panos E. Tsaoussoglou, Athens, Greece; and the proposer.

Squaring $\sqrt{x}+\sqrt{y}=2\sqrt{2}$, we obtain that

$$x+y=8-2\sqrt{xy}\geq 8-(x+y),$$

whence

$$x+y\geq 4. \tag{1}$$

Further

$$x^2+y^2\geq \tfrac{1}{2}(x+y)^2\geq 8. \tag{2}$$

From (1) and (2), we have

$$\begin{aligned} 64 &= \frac{2^{x^2+y}+2^{x+y^2}}{2} \\ &\geq 2^{\frac{1}{2}(x^2+y+x+y^2)} \geq 2^6 = 64. \end{aligned}$$

Thus $x+y=4$ and $x^2+y^2=8$. These are easily solved to obtain that $x=y=2$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; BEATRIZ MARGOLIS, Paris, France; J.A. MCCALLUM, Medicine Hat, Alberta; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, MARTHA BELL and JIM BRASELTON; Georgia Southern University, Statesboro, Georgia, USA; STAN WAGON, Macalester College, St. Paul, Minnesota, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and SUSAN SCHWARTZ WILDSTROM, Kensington, Maryland, USA. One other submission was received which assumed that x and y were integers.

Janous pointed out that the result can be generalized to:

Let $x_1,\ldots,x_n$ $(n\geq 2)$ be non-negative real numbers such that $x_1+\ldots+x_n=nw$, and let $b\neq 1$ be a positive real number.

Let $\alpha_1, \ldots, \alpha_n$ be real numbers, each greater than or equal to 1.
Then the equation

$$b^{x_1^{\alpha_1}+x_2^{\alpha_2}+\ldots+x_n^{\alpha_n}} + b^{x_2^{\alpha_1}+x_3^{\alpha_2}+\ldots+x_n^{\alpha_n}} + \ldots + b^{x_n^{\alpha_1}+x_1^{\alpha_2}+\ldots+x_{n-1}^{\alpha_n}}$$
$$= b^{w^{\alpha_1}+w^{\alpha_2}+\ldots+w^{\alpha_n}+\log_b(n)}$$

has as its only solution $x_1 = x_2 = \ldots = x_n = w$.

This can be proved by applications of the AM–GM and power mean inequalities.

2051. [1995: 202] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A convex quadrilateral $ABCD$ is inscribed in a circle Γ with centre O. P is an interior point of $ABCD$. Let O_1, O_2, O_3, O_4 be the circumcentres of triangles PAB, PBC, PCD, PDA respectively. Prove that the midpoints of O_1O_3, O_2O_4 and OP are collinear.

Combination of solutions by Jordi Dou, Barcelona, Spain and the proposer.

The result holds without restriction on the point P. The proof is in two steps.

Step 1. The result is trivially true when P is on Γ (and all the circumcentres coincide with O), so let P be a point off Γ and let Ω be the conic with foci O and P whose major axis has length R (the radius of Γ). Ω is an ellipse when P is interior to Γ and a hyperbola when P is exterior. Let Q be the intersection of O_1O_2 and OB. Because $O_1B = O_1P$ and $O_2B = O_2P$, we have that O_1O_2 is the perpendicular bisector of BP. Therefore $BQ = PQ$ and $\angle BQO_2 = \angle PQO_2$. When P is interior we conclude that $OQ + PQ = OQ + BQ = R$; when exterior, $OQ - PQ = OQ - BQ = R$. Thus, in either case, Q is a point on Ω. Moreover, as O_1O_2 is a bisector of $\angle OQP$, O_1O_2 is tangent to Ω at Q. Similarly O_2O_3, O_3O_4, and O_4O_1 are tangent to Ω. [*Editor's comment by Chris Fisher.* Dou refers to Γ as the *focal circle* of Ω. I was unable to confirm that terminology in any handy reference, but I did find the circle mentioned as the basis of a construction of a central conic by folding; see, for example, E.H. Lockwood, *A Book of Curves*: Draw a circle Γ on a sheet of paper and mark an arbitrary point P not on Γ. For any number of positions B on Γ fold P onto B and crease the paper. The creases (i.e. the perpendicular bisectors of PB) envelope a conic Ω.]

Step 2. By Newton's theorem the midpoints of O_1O_3, O_2O_4, and the centre of Ω (which is the midpoint of OP) are collinear as desired. As a bonus, also on that line is the midpoint of the segment joining $O_5 := O_1O_4 \cap O_2O_3$ to $O_6 := O_1O_2 \cap O_3O_4$. Here is a simple projective proof of this theorem. Consider the pencil of dual conics tangent to the sides of the complete quadrilateral $O_1O_2O_3O_4$. As a consequence of the Desargues' involution theorem (see, for example, Dan Pedoe, *A Course of Geometry for Colleges and Universities*, Theorem II, page 342), the poles of a line l with respect to the conics of the pencil lie on a line. In particular, when l is the line at infinity the line of poles is the line of centres of these conics. The centre of Ω is one such pole. Furthermore, the line of centres passes through the midpoints of the degenerate dual conics of the pencil (consisting of pairs of opposite vertices of the quadrilateral), namely O_1 and O_3, O_2 and O_4, O_5 and O_6.

Also solved by MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain.

2053. [1995: 202] *Proposed by Jisho Kotani, Akita, Japan.*

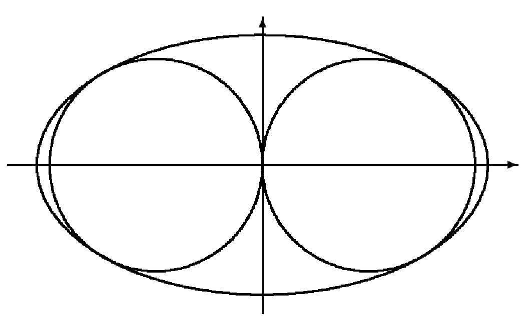
A figure consisting of two equal and externally tangent circles is inscribed in an ellipse. Find the eccentricity of the ellipse of minimum area.

Solution by David Hankin, Hunter College Campus Schools, New York, NY, USA.

[Editor's note: By symmetry, we have:

(a) the centre of the circumscribing ellipse, $\mathcal{E}$, must be the point of tangency of the two given circles;

(b) an axis of $\mathcal{E}$ passes through the centres of the two given circles.



All solvers assumed this, most without stating that they had done so.]

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Let the equations of the circles be $(x \pm r)^2 + y^2 = r^2$, and let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solving the first and third of these equations gives

$$x = \frac{-ar^2 \pm a\sqrt{a^2r^2 + b^4 - a^2b^2}}{b^2 - a^2}.$$

Therefore, at the two points of tangency (in the right half-plane), we have $a^2r^2 + b^4 - a^2b^2 = 0$. From this we get

$$a = \frac{b^2}{\sqrt{b^2 - r^2}}.$$

Since the area of the ellipse is given by $K = \pi ab$, we have

$$K = \frac{\pi b^3}{\sqrt{b^2 - r^2}}.$$

Therefore $\dfrac{dK}{db} = \dfrac{\pi b^2(2b^2 - 3r^2)}{(b^2 - r^2)^{3/2}}$. From this, we obtain that K is minimal when $b^2 = \dfrac{3r}{2}$. At this point $a^2 = 3b^2$, and from

$$e^2 = \frac{a^2 - b^2}{a^2},$$

we obtain that $e = \sqrt{\dfrac{2}{3}}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JORDI DOU, Barcelona, Spain; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; and the proposer. Two other submissions were received that were almost correct: they used an incorrect formula for the eccentricity.

2054. [1995: 202] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Are there any integral solutions of the Diophantine equation

$$(x + y + z)^3 = 9\left(x^2y + y^2z + z^2x\right)$$

other than $(x, y, z) = (n, n, n)$?

I. *Solution by Adrian Chan, student, Upper Canada College, Toronto, Ontario.*

No, there are no integral solutions other than $(x, y, z) = (n, n, n)$.

Without loss of generality, let $x \leq y$ and $x \leq z$. Let $y = x + a$ and $z = x + b$, where a and b are non-negative integers. Then the given equation becomes

$$(3x + a + b)^3 = 9\left(x^2(x + a) + (x + a)^2(x + b) + (x + b)^2x\right).$$

After expanding and simplifying, this is $(a+b)^3 = 9a^2b$, or

$$a^3 - 6a^2b + 3ab^2 + b^3 = 0. \qquad (1)$$

Let $a = kb$, where k is a rational number. Then (1) becomes

$$b^3(k^3 - 6k^2 + 3k + 1) = 0.$$

By the Rational Root Theorem, $k^3 - 6k^2 + 3k + 1 = 0$ does not have any rational roots. So, since k is rational, $k^3 - 6k^2 + 3k + 1 \neq 0$. Therefore $b = 0$ and $a = 0$, so $x = y = z$.

II. *Solution by the proposer.*

Letting $y = x + u$ and $z = x + v$, the equation reduces to

$$(u+v)^3 = 9u^2v \qquad (2)$$

where u and v are integers. We now show that the only solution to (2) is $u = v = 0$ so that $(x, y, z) = (n, n, n)$ is the only solution of the given equation. Letting $u + v = w$, (2) becomes

$$w^3 = 9u^2(w-u). \qquad (3)$$

Hence $w = 3w_1$ where w_1 is an integer, and (3) is $3w_1^3 = u^2(3w_1 - u)$. It follows that $u = 3u_1$ for some integer u_1, and we get

$$w_1^3 = 9u_1^2(w_1 - u_1).$$

Comparing this equation to (3), we see by infinite descent that the only solution to (3) is $u = w = 0$, which gives the negative result.

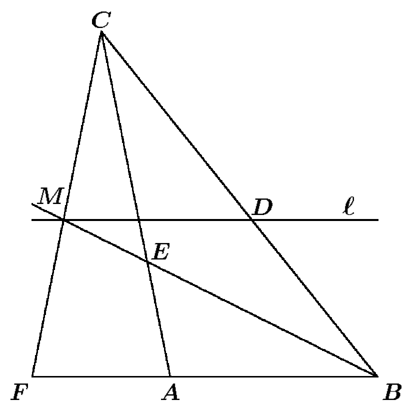
Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and HOE TECK WEE, student, Hwa Chong Junior College, Singapore. There was also one incorrect solution sent in.

2055. [1995: 202] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In triangle ABC let D be the point on the ray from B to C, and E on the ray from C to A, for which $BD = CE = AB$, and let ℓ be the line through D that is parallel to AB. If $M = \ell \cap BE$ and $F = CM \cap AB$, prove that

$$(BA)^3 = AE \cdot BF \cdot CD.$$

Solution by Toshio Seimiya, Kawasaki, Japan; essentially identical solutions were submitted by Jordi Dou, Barcelona, Spain; Mitko Christov Kunchev, Baba Tonka School of Mathematics, Rousse, Bulgaria; and Gottfried Perz, Pestalozzigymnasium, Graz, Austria.



By Menelaus' Theorem applied to triangle ACF and transverse line BEM, we have

$$\frac{AB}{BF} \cdot \frac{FM}{MC} \cdot \frac{CE}{EA} = -1.$$

Since $DM \| BF$, we have $\dfrac{FM}{MC} = \dfrac{BD}{DC}$, giving

$$\frac{AB}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1.$$

Hence we have

$$AE.BF.CD \;=\; -AB.BD.CE \;=\; BA.AB.AB \;=\; (BA)^3.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; P. PENNING, Delft, the Netherlands; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2057***.** [1995: 203] *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let P be a point inside an equilateral triangle ABC, and let R_a, R_b, R_c and r_a, r_b, r_c denote the distances of P from the vertices and edges, respectively, of the triangle. Prove or disprove that

$$\left(1+\frac{r_a}{R_a}\right)\left(1+\frac{r_b}{R_b}\right)\left(1+\frac{r_c}{R_c}\right) \ge \frac{27}{8}\,.$$

Equality holds if P is the centre of the triangle.

Solution by G.P. Henderson, Campbellcroft, Ontario. We will prove that the inequality is true.

We have
$$\sin(PAB) = \frac{r_c}{R_a}, \quad \sin(PAC) = \frac{r_b}{R_a}.$$
Therefore
$$\cos(A) = \frac{1}{2} = \sqrt{\left(1 - \frac{r_b^2}{R_a^2}\right)\left(1 - \frac{r_c^2}{R_a^2}\right)} - \frac{r_b r_c}{R_a^2}.$$
From this, we have
$$R_a = \frac{2}{\sqrt{3}}\sqrt{r_b^2 + r_b r_c + r_c^2},$$
and similar expressions for R_b and R_c.

We can assume, without loss of generality, that P is in the section of the triangle defined by $r_a \le r_b \le r_c$. Set $x = r_b/r_c$ and $y = r_a/r_c$, so that $0 \le y \le x \le 1$. The left side of the given inequality is now
$$\left(1 + \frac{\sqrt{3}y}{2\sqrt{x^2 + x + 1}}\right)\left(1 + \frac{\sqrt{3}x}{2\sqrt{y^2 + y + 1}}\right)\left(1 + \frac{\sqrt{3}}{2\sqrt{x^2 + xy + y^2}}\right). \qquad (1)$$
We will replace these factors by smaller quantities which do not involve square roots. [*Ed. This is a brilliant move. I recommend that the reader draws some graphs to see how effective this is. In fact* $\frac{6-x}{7+3x}$ *is "almost" the Čebyšev-Padé approximant of* $\frac{\sqrt{3}}{2\sqrt{x^2+x+1}}$, *which is* $\frac{6.08409 - 1.33375x}{7 + 2.53143x}$.]

We will show that, for $0 \le x \le 1$,
$$\frac{\sqrt{3}}{2\sqrt{x^2 + x + 1}} \ge \frac{6 - x}{7 + 3x}. \qquad (2)$$
This is equivalent to
$$(1 - x)^2\left(3 + 36x - 4x^2\right) = (1 - x)^2\left(4x(1 - x) + 32x + 3\right) \ge 0.$$
Similarly, we have, for $0 \le y \le 1$,
$$\frac{\sqrt{3}}{2\sqrt{y^2 + y + 1}} \ge \frac{6 - y}{7 + 3y}.$$
Replacing x by $y/x (\le 1)$ in (2), we have
$$\frac{\sqrt{3}}{2\sqrt{x^2 + xy + y^2}} \ge \frac{6x - y}{x(7x + 3y)}.$$
Using these three expression in (1), it is sufficient to prove that
$$\left(1 + \frac{y(6 - x)}{7 + 3x}\right)\left(1 + \frac{x(6 - y)}{7 + 3y}\right)\left(1 + \frac{6x - y}{x(7x + 3y)}\right) \ge \frac{27}{8}.$$

Now, this is equivalent to

$$f(x,y) = Py^3 + Qy^2 + Ry + S \geq 0,$$

where

$$\begin{aligned} P = P(x) &= 8(x-6)(x-3)(3x-1), \\ Q = Q(x) &= 56x^4 - 672x^3 + 663x^2 + 427x - 504, \\ R = R(x) &= -504x^4 + 35x^3 + 714x^2 - 273x - 392, \\ S = S(x) &= 1008x^4 + 423x^3 - 3493x^2 + 2352x. \end{aligned}$$

We will prove this by showing that f is a decreasing function of y for $0 \leq y \leq x$ [for fixed x] and that $f(x,x) \geq 0$. We find that

$$f(x,x) = 10x(1-x)^2\left(8x^3 - 124x^2 - 35x + 196\right) \geq 0.$$

[Note that $8 + 196 > 124 + 35$.]
It remains to show that

$$F(x,y) = \frac{\partial f}{\partial y} = 3Py^2 + 2Qy + R \leq 0. \tag{3}$$

First, we note that

$$\begin{aligned} Q &= -56x^3(1-x) - 616x(1-x)^2 - \left(569x^2 - 1043x + 504\right) < 0, \\ R &= -35x(1-x^2) - 238x - \left(504x^2 - 714x + 392\right) < 0. \end{aligned}$$

[Also, $P \leq 0$ for $0 \leq x \leq \frac{1}{3}$ and $P \geq 0$ for $\frac{1}{3} \leq x \leq 1$.] In equation (3), when $P \leq 0$, all three terms are negative, and so $F(x,y) < 0$. When $P > 0$, $F(x,y) \leq 0$ provided that

$$F(x,0) \leq 0 \quad \text{and} \quad F(x,x) \leq 0,$$

[since $\frac{\partial^2 F}{\partial y^2} > 0$].
The first of these terms is R, which is negative. The second is

$$\begin{aligned} &(1-x)\left(-184x^4 + 2336x^3 - 537x^2 - 1673x - 392\right) \\ &= (1-x)\left(-1184x^4 - 537x^2(1-x) - 1673x(1-x^2)\right. \\ &\qquad \left. -392(1-x^3) - 266x^3\right) \\ &\leq 0. \end{aligned}$$

Also solved by MARCIN E. KUCZMA, Warszawa, Poland.