LETTER FROM THE EDITORS

This issue sees a change in the Editorship of *CRUX MATHEMATICORUM*. After many successful years under the accomplished leadership of Bill Sands and Robert Woodrow, the editorial office has moved from the University of Calgary to Memorial University of Newfoundland. It is our intention to maintain the high standard that has led to *CRUX MATHEMATICORUM* having an excellent international reputation as one of the world's top problem solving journals. This high standard is a result of the submissions from the readership, and we encourage you all to continue to provide this. Please write to the new editor if you have suggestions for improvements.

Readers will be delighted to know that Robert Woodrow has agreed to continue as editor of the **Olympiad Corner**. However, he will soon be relinquishing the **Skoliad Corner**. Contributions intended for that section should be sent directly to the editor. Bill Sands will remain "in the background", and the new editor is indeed delighted to be able to call on Bill's sage advice whenever necessary.

There are a few changes in the works. We shall continue with the present format, which the readership appears to enjoy, but we shall change from 10 issues of 36 pages (360 pages per volume) to 8 issues of 48 pages (384 pages per volume). Although this has two fewer issues, the bonus is that there are 24 extra pages per year. This allows increased efficiency of printing and helps to control mailing costs. The new schedule will have issues for February, March, April, May, September, October, November and December. We feel that this fits in with the teaching year of the majority of the readership.

Since postal rates for mailing outside of Canada are significantly higher than within Canada, and as is customary for many publications, the Society has reluctantly adopted the policy that all subscription and other rates for subscribers with non-Canadian addresses must be paid in US funds. Although this policy was instituted in 1995, the Society has, until the present time, accepted payment in Canadian funds from any subscriber. However, effective January 1996, the Society will require payment in US funds where applicable. We trust that our non-Canadian subscribers will understand the necessity of this change in payment policy.

We are also considering how to make *CRUX* available electronically to its subscribers. One possibility would be to send postscript files to subscribers. Please communicate your thoughts to the Editor-in-Chief or to the Managing Editor (email: *gpwright@acadvm1.uottawa.ca*).

We are now encouraging electronic submission of material. Please email submissions to *cruxeditor@cms.math.ca*. We use LaTeX2e. Those of you who would like some of the technical details of the style use are asked to request this by email.

Bruce Shawyer, Editor-in-Chief *Graham Wright, Managing Editor*

LETTRE DES RÉDACTEURS

Le présent numéro marque l'entrée en fonction d'une nouvelle équipe à la direction de *CRUX MATHEMATICORUM*. Après de nombreuses années sous la direction experte de Bill Sands et de Robert Woodrow, le conseil de rédaction quitte l'Université de Calgary pour s'installer à l'Université Memorial de Terre-Neuve. Nous avons l'intention de maintenir le niveau de qualité qui a fait de *CRUX MATHEMATICORUM* une publication reconnue mondialement comme l'une des meilleures en résolution de problèmes. L'excellence de notre périodique tient à votre apport, chers lecteurs, et nous vous encourageons tous à poursuivre dans la même voie. Si vous avez des améliorations à proposer, veuillez en faire part au nouveau rédacteur en chef.

Vous serez enchanté d'apprendre que Robert Woodrow a accepté de conserver la direction de la chronique "**Olympiade**"; il délaissera toutefois la chronique "**Skoliad**". Si vous désirez contribuer à cette dernière, veuillez vous adresser au rédacteur en chef. Bill Sands, pour sa part, demeurera "en veilleuse"; fort heureusement, nous pourrons compter sur ses judicieux conseils au besoin.

Quelques changements sont à venir. Nous conserverons la présentation actuelle de la revue puisqu'elle semble plaire à nos lecteurs, mais nous produirons désormais huit numéros de 48 pages (env. 384 pages par volume) au lieu de dix numéros de 36 pages (env. 360 pages par volume). Vous recevrez donc deux numéros de moins par année, mais vous aurez droit en prime à 24 pages de plus. Cette formule s'avère plus avantageuse du point de vue de l'impression et des frais d'expédition. Le nouveau calendrier de publication prévoit la parution d'un numéro en février, mars, avril, mai, septembre, octobre, novembre et décembre. Nous pensons qu'il correspond au calendrier scolaire de la majorité de nos lecteurs.

Comme il lui en coûte beaucoup plus cher d'expédier ses publications à l'étranger qu'au Canada, la Société, à l'instar de nombreux autres éditeurs de revues, a été contrainte d'adopter une nouvelle politique; dorénavant, tous les abonnés dont l'adresse postale n'est pas au Canada devront régler leurs frais d'abonnement et autres en devises américaines. Même si cette politique avait été introduite en 1995, la Société a jusqu'à présent accepté les paiements en dollars canadiens qu'elle avait reçus de ses abonnés. Mais à partir de janvier 1996, elle exigera des paiements en dollars américains, le cas échéant.

Nos abonnés de l'extérieur du Canada comprendront qu'il nous était devenu nécessaire de modifier ainsi notre grille tarifaire.

Par ailleurs, nous songeons à distribuer *CRUX* par voie électronique. L'une des options envisagées serait l'envoi de fichiers postscript aux abonnés. Veuillez transmettre vos commentaires au rédacteur en chef ou au rédaction-gérant (courrier électronique : *gpwright@acadvm1.uottawa.ca*).

Nous encourageons désormais nos lecteurs à soumettre leurs contributions par courrier électronique à *cruxeditor@cms.math.ca*. Nous utilisons LaTeX2e. Si vous désirez obtenir certains détails techniques quant au style, faites-en la demande par courrier électronique.

Bruce Shawyer, Rédacteur en chef *Graham P. Wright, Rédacteur-gérant*

CONGRATULATIONS

CRUX would like to extend its collective congratulations to Professor Ron Dunkley, founder of the Canadian Mathematics Competitions, on his appointment as a Member of the Order of Canada.

GUIDELINES FOR ARTICLES

Articles for this section of *CRUX* should satisfy the following:

- have length of two to four pages, ideally (we have allowed up to six pages in exceptional circumstances);
- be of interest to advanced high school and first or second year university students;
- contain some new material that leads to further interesting questions (for this level);
- be well referenced as to origin of the problem and related material;
- not contain long involved formulas or expressions, that is, we like more elegant mathematics as opposed to that which involves tedious calculations and attention to detail. We really want to emphasize ideas and avoid many, many formulas. *CRUX* does not want to be too technical in its appeal, rather we want to have wide general interest.

— Denis Hanson

UNITARY DIVISOR PROBLEMS

K.R.S. Sastry

Suppose d is a (positive integral) divisor of a (natural) number n. Then d is a **unitary** divisor of n if and only if d and n/d are relatively prime, that is $(d, n/d) = 1$.

For example, 4 is a unitary divisor of 28 because $(4, 28/4) = (4, 7) = 1$. However, 2 is **not** a unitary divisor of 28 because $(2, 28/2) = (2, 14) = 2 \neq 1$. Our aim is to consider

(i) the unitary analogue of a number theory result of Gauss, and

(ii) the unitary extension of super abundant numbers.

THE NUMBER $d^*(n)$ AND THE SUM $\sigma^*(n)$ OF UNITARY DIVISORS of n.

To familiarize ourselves with the novel concept of unitary divisibility, the following table, adapted from the one in [2], is presented. It consists of $n, 1 \leq n \leq 12$; n's unitary divisors; $d^*(n)$, the number of unitary divisors of n; and $\sigma^*(n)$, the sum of unitary divisors of n.

TABLE

n	unitary divisors of n	$d^*(n)$	$\sigma^*(n)$
1	1	1	1
2	1, 2	2	3
3	1, 3	2	4
4	1, 4	2	5
5	1, 5	2	6
6	1, 2, 3, 6	4	12
7	1, 7	2	8
8	1, 8	2	9
9	1, 9	2	10
10	1, 2, 5, 10	4	18
11	1, 11	2	12
12	1, 3, 4, 12	4	20

From the above table it is clear that if $n = p_1^{\alpha_1}$ is the prime decomposition of n, then its unitary divisors are 1 and $p_1^{\alpha_1}$. So, $d^*(n) = 2$ and $\sigma^*(n) = 1 + p_1^{\alpha_1}$. If $n = p_1^{\alpha_1} p_2^{\alpha_2}$ is the prime decomposition of n, then its unitary divisors are 1, $p_1^{\alpha_1}, p_2^{\alpha_2}, p_1^{\alpha_1} p_2^{\alpha_2}$. So $d^*(n) = 4 = 2^2$ and $\sigma^*(n) = (1+p_1^{\alpha_1})(1+p_2^{\alpha_2})$.

Now it is a simple matter to establish the following results:
Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denote the prime decomposition of n.
Then

$$\begin{aligned} d^*(n) &= 2^k, &(1)\\ \sigma^*(n) &= (1+p_1^{\alpha_1})(1+p_2^{\alpha_2})\cdots(1+p_k^{\alpha_k}), &(2)\\ \text{If}\quad (m,n) &= 1, \quad \text{then} \quad \sigma^*(mn) = \sigma^*(m)\sigma^*(n). &(3)\end{aligned}$$

THE EULER ϕ FUNCTION AND A RESULT OF GAUSS.

The Euler ϕ function counts the number $\phi(n)$ of positive integers that are less than and relatively prime to n. Also, $\phi(1) = 1$ by definition. For example, $\phi(6) = 2$ because 1 and 5 are the only positive integers that are less than and relatively prime to 6. The following results are known [1]:

$$\text{If} \quad n = p_1^{\alpha_1}, \quad \text{then} \quad \phi(n) = (p_1 - 1)p_1^{\alpha_1 - 1}. \tag{4}$$

$$\text{If} \quad n = \prod_{i=1}^{k} p_i^{\alpha_i}, \quad \text{then} \quad \phi(n) = \prod_{i=1}^{k} \phi\left(p_i^{\alpha_i}\right). \tag{5}$$

$$\text{If} \quad (m,n) = 1, \quad \text{then} \quad \phi(mn) = \phi(m)\phi(n). \tag{6}$$

Let $D = \{d\colon d \text{ is a divisor of } n\}$. Then Gauss showed that $\sum \phi(d) = n$. For example, if $n = 12$, then $D = \{1, 2, 3, 4, 6, 12\}$ and $\sum \phi(d) = \phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(6) + \phi(12) = 1+1+2+2+2+4 = 12$. Analogously, if $D^* = \{d^*: d^* \text{ is a unitary divisor of } n\}$, then what is $\sum \phi(d^*)$? The answer is given by Theorem 1.

Theorem 1 *Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ denote the prime decomposition of n and $D^* = \{d^*: d^* \text{ is a unitary divisor of } n\}$. Then*

$$\sum \phi(d^*) = \prod_{i=1}^{k} [1 + \phi(p_i^{\alpha_i})] .$$

Proof: The unitary divisors of n are the 2^k elements, see (1), in the set

$$D^* = \{1; p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_k^{\alpha_k}; p_1^{\alpha_1} p_2^{\alpha_2}, \cdots, p_{k-1}^{\alpha_{k-1}} p_k^{\alpha_k}; \cdots; \prod_{i=1}^{k} p_i^{\alpha_i}\}.$$

Hence

$$\sum \phi(d^*) = \phi(1) + \sum_{i=1}^{k} \phi\left(p_i^{\alpha_i}\right) + \sum_{i<j} \phi\left(p_i^{\alpha_i} p_j^{\alpha_j}\right) + \cdots + \phi\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right).$$

On repeated applications of (5) and (6) we find that

$$\sum \phi(d^*) = 1 + \sum_{i=1}^{k} \phi\left(p_i^{\alpha_i}\right) + \sum_{i<j} \phi\left(p_i^{\alpha_i}\right) \phi\left(p_j^{\alpha_j}\right) + \cdots + \prod_{i=1}^{k} \phi\left(p_i^{\alpha_i}\right).$$

But the right-hand side expression in the above equation is precisely the expansion of $\prod_{i=1}^{k}[1 + \phi(p_i^{\alpha_i})]$. ■

For a numerical illustration, let $n = 108$. Then $D^* = \{1, 4, 27, 108\}$, $\sum \phi(d^*) = \phi(1) + \phi(4) + \phi(27) + \phi(108) = 1 + 2 + 18 + 36 = 57$ by actual count. Also $n = 108 = 2^2 \cdot 3^3$.
Therefore $\sum \phi(d^*) = [1 + \phi(2^2)][1 + \phi\,(3^3)] = [1 + 2][1 + 18] = 57$, from Theorem 1.

It is the converse problem that is more challenging to solve: Given a positive integer m, find the set N such that $N = \{n\colon \sum \phi(d^*) = m\}$. There is a solution n if and only if m has the form $\prod_{i=1}^{k}[1 + \phi(p_i^{\alpha_i})]$. To see this, let $m = 4$. Then there is no solution n such that $\sum \phi(d^*) = m$. This follows because if $n = p^\alpha$, then $1 + \phi(p^\alpha) = 4$ yields $(p-1)p^{\alpha-1} = 3$. If $p = 2$, then $2^{\alpha-1} = 3$ has no solution for a positive integer α. If p is odd, then $(p-1)p^{\alpha-1}$ is even and hence there is no solution of $(p-1)p^{\alpha-1} = 3$ for a positive integer α. We leave it as an exercise to show that $N = \phi$ if $m = 2^\alpha$ for $\alpha > 1$. It is easy to verify that if $m = p$ is an odd prime, then $N = \{p, 2p\}$. As another exercise, the reader may find N when $m = 1995$. It is an open problem to determine the integers m for which $N = \phi$.

In the next section, we consider an extension of the concept of super-abundant numbers in the context of unitary divisors.

UNITARY SUPER ABUNDANT NUMBERS.

We call a natural number n **unitary** abundant if $\sigma^*(n)$ is greater than $2n$. For example, 150 is unitary abundant because $\sigma^*(150) = 312 > 2(150)$.

To extend the work of Erdös and Alaoglu [3], we call a natural number n **unitary** super abundant if $\frac{\sigma^*(n)}{n} \geq \frac{\sigma^*(m)}{m}$ for all natural numbers $m \leq n$. Theorem 2 shows that the product of the first k primes, $k = 1, 2, \cdots$ is a unitary super abundant number.

Theorem 2 Let p_k denote the k^{th} prime, $k = 1, 2, \cdots$. Then $n = p_1, p_2 \cdots p_k$ is a unitary super abundant number.

Proof: Consider the natural numbers of $m \leq n$. Then m belongs to one of the three groups described below.

I. m is composed of powers of primes $p_j \leq p_k$.

II. m is composed of powers of some primes $p_j \leq p_k$ and powers of some primes $p_\ell > p_k$.

III. m is composed of powers of primes $p_\ell \geq p_k$.

First of all we note that the total number of primes composing m in any of the three groups does not exceed k.

We now show that $\frac{\sigma^*(m)}{m} \leq \frac{\sigma^*(n)}{n}$ in all the above three cases.

Case I. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j}$ where some α's, except α_j, may be zero. From (2) we see that

$$\begin{aligned}
\frac{\sigma^*(m)}{m} &= \prod_{\substack{i=1 \\ \alpha_i \neq 0}}^{j} \left(\frac{1 + p_i^{\alpha_i}}{p_i^{\alpha_i}} \right) \\
&= \prod_{\substack{i=1 \\ \alpha_i \neq 0}}^{j} \left(1 + \frac{1}{p_i^{\alpha_i}} \right) \\
&\leq \prod_{\substack{i=1 \\ \alpha_i \neq 0}}^{j} \left(1 + \frac{1}{p_i} \right) \\
&\leq \prod_{i=1}^{k} \left(1 + \frac{1}{p_i} \right) \\
&= \frac{\sigma^*(n)}{n}. \qquad (7)
\end{aligned}$$

Case II. In this case $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j} p_{k+1}^{\alpha_{k+1}} \cdots p_\ell^{\alpha_\ell}$. Here too (7) holds. Some α's, except α_j and α_ℓ, may be zero. Furthermore,

$$\ell > k \Rightarrow 1 + \frac{1}{p_\ell} < 1 + \frac{1}{p_k} < 1 + \frac{1}{p_{k-1}} < \cdots < \frac{3}{2}. \qquad (8)$$

Again, from (2) and (3)

$$\begin{aligned}
\frac{\sigma^*(m)}{m} &= \prod_{\substack{i=1\\ \alpha_i\neq 0}}^{j} \left(\frac{1+p_i^{\alpha_i}}{p_i^{\alpha_i}}\right) \prod_{\substack{i=1\\ \alpha_i\neq 0}}^{\ell-k} \left(\frac{1+p_{k+i}^{\alpha_{k+i}}}{p_{k+i}^{\alpha_{k+i}}}\right)\\
&= \prod_{\substack{i=1\\ \alpha_i\neq 0}}^{j} \left(1+\frac{1}{p_i^{\alpha_i}}\right) \prod_{\substack{i=1\\ \alpha_i\neq 0}}^{\ell-k} \left(1+\frac{1}{p_{k+i}^{\alpha_{k+i}}}\right)\\
&\leq \prod_{i=1}^{k} \left(1+\frac{1}{p_i^{\alpha_i}}\right)\\
&\leq \prod_{i=1}^{k} \left(1+\frac{1}{p_i}\right)\\
&= \frac{\sigma^*(n)}{n},
\end{aligned}$$

on using (7) and (8).

Case III. In this case $m = p_k^{\alpha_k}\cdots p_\ell^{\alpha_\ell}$. Here too (7) holds. Some α's, except α_ℓ, may be zero. As in earlier cases

$$\begin{aligned}
\frac{\sigma^*(m)}{m} &= \prod_{\substack{i=0\\ \alpha_i\neq 0}}^{\ell-k} \left(1+\frac{1}{p_{k+i}^{\alpha_{k+i}}}\right)\\
&\leq \prod_{i=0}^{\ell-k} \left(1+\frac{1}{p_{k+i}}\right)\\
&\leq \prod_{i=1}^{k} \left(1+\frac{1}{p_i}\right)\\
&= \frac{\sigma^*(n)}{n} \quad \text{on using (8).} \quad \blacksquare
\end{aligned}$$

We now give an example of a number n to show that there are values of m less than n that can belong to any of the three groups described in Theorem 2.

Let $n = 210 = 2.3.5.7$. Then $\frac{\sigma^*(n)}{n} = \frac{96}{35}$.

I. $m = 120 = 2^3.3.5$ is less than n and belongs to Group **I.**

Here $\dfrac{\sigma^*(m)}{m} = \dfrac{9}{5} \leq \dfrac{\sigma^*(n)}{n}$ holds.

II. $m = 165 = 3.5.11$ is less than n and belongs to Group **II.**

Here $\dfrac{\sigma^*(m)}{m} = \dfrac{96}{55} \leq \dfrac{\sigma^*(n)}{n}$ holds.

III. $m = 143 = 11.13$ is less than n and belongs to Group **III.**

Here too, $\dfrac{\sigma^*(m)}{m} = \dfrac{168}{143} \leq \dfrac{\sigma^*(n)}{n}$ holds.

If we write $n_k = p_1 p_2 \cdots p_k,\ k = 1, 2, \cdots$, then we observe that

$$\begin{aligned} \frac{\sigma^*(n_{k+1})}{n_{k+1}} &= \frac{\sigma^*(n_k)}{n_k}\left(1 + \frac{1}{p_{k+1}}\right) \\ &> \frac{\sigma^*(n_k)}{n_k}, \quad k = 1, 2, \cdots . \end{aligned}$$

This above observation, coupled with the argument used in the proof of Theorem 2, implies Theorem 3.

Theorem 3 The only unitary super abundant numbers are

$$n_k, \quad k = 1, 2, 3, \cdots .$$

That is

$$2, 6, 30, 210, 2310, \cdots .$$

Acknowledgement.

The author thanks the referees for their suggestions.

References

[1] L.E. Dickson, *History of the Theory of Numbers*, Vol. I, Chelsea Publications (1971), 113–114.

[2] R.T. Hanson and L.G. Swanson, Unitary Divisors, *Mathematics Magazine*, **52** (September 1979), 217–222.

[3] R. Honsberg, *Mathematical Gems* I, MAA, 112–113.

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THE SKOLIAD CORNER

No. 11

R.E. Woodrow

This number marks the first anniversary of the Skoliad Corner. While I have not received a lot of correspondence about the column *per se*, what I have received has been positive. A good source of more entry level problems seems needed, and it would be nice to find a way of recognizing efforts by students at this level to solve the problems.

As a problem set this issue, we give the Sharp U.K. Intermediate Mathematical Challenge which was written by about 115,000 students February 2, 1995. Entrants had to be in the equivalent of School Year 11 or below for England and Wales. Questions 1–15 were worth 5 marks each and Questions 16–25 were worth 6 marks each. The allotted time was one hour. The use of calculators, rulers, and measuring instruments was forbidden. The contest was organized by the U.K. Mathematics Foundation with additional help from The University of Birmingham. Many thanks go to Tony Gardiner, University of Birmingham, for sending me this contest and others sponsored by the Foundation.

SHARP U.K. INTERMEDIATE MATHEMATICAL CHALLENGE

February 2, 1995

Time: 1 hour

1. Which of these divisions has a whole number answer?

A. $1234 \div 5$ **B.** $12345 \div 6$ **C.** $123456 \div 7$

D. $1234567 \div 8$ **E.** $12345678 \div 9$

2. The word "thirty" has six letters, and 6 is a factor of 30. How many of the numbers from one up to twenty have this curious property?

A. 14 **B.** 2 **C.** 3 **D.** 4 **E.** 5

3. René Descartes (1596–1650) published his most famous book *Discourse on Method* in 1637. To illustrate his "method" he included an appendix on *Geometry*, in which he introduced the idea of *cartesian coordinates* and showed how their use allowed one to solve many problems which had previously been unsolved. Descartes is best known for a sentence on which he based part of his philosophy. The famous sentence (in Latin) is "Cogito, ergo sum". This is usually translated into the five English words given below (in dictionary order). When they are written in the correct order to make Descartes' saying, which word is in the middle?

A. am **B.** I **C.** I **D.** therefore **E.** think

4. What is the sum of the first nine prime numbers?

A. 45 **B.** 78 **C.** 81 **D.** 91 **E.** 100

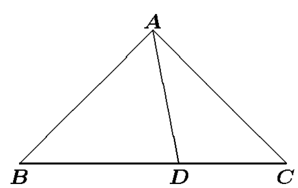
5. Three hedgehogs Roland, Spike and Percival are having a race against two tortoises Esiotrot and Orinoco. Spike (S) is 10 m behind Orinoco (O), who is 25 m ahead of Roland (R). Roland is 5 m behind Esiotrol (E), who is 25 m behind Percival (P). Which of the following words gives the order (first to fifth) at this point in the race?

A. POSER **B.** SPORE **C.** ROPES **D.** PORES **E.** PROSE

6. Nabil was told to add 4 to a certain number and then divide the answer by 5. Instead he first added 5 and then divided by 4. He came up with the "answer" 54. What should his answer have been?

A. 34 **B.** 43 **C.** 45 **D.** 54 **E.** 211

7. Find the value of the angle BAD, where $AB = AC$, $AD = BD$, and angle $DAC = 39°$.



A. 39° **B.** 45° **C.** 47° **D.** 51° **E.** 60°

8. A maths teacher who lives "opposite" (or is it "adjacent to"?) our local hospital claims they recently hung up a banner saying: "Selly Oak Hospital Can Always Handle the Odd Accident". In fact, between March 1993 and March 1994 the hospital handled 53, 744 accident and emergency cases. Approximately how many cases was this per day?

A. 90 **B.** 150 **C.** 250 **D.** 500 **E.** 1000

9. The diagonal of a square has length 4 cm. What is its area (in cm^2)?

A. 2 **B.** 4 **C.** $4\sqrt{2}$ **D.** 8 **E.** 16

10. If $x = 3$, which expression has a different value from the other four?

A. $2x^2$ **B.** $x^2 + 9x$ **C.** $12x$ **D.** $x^2(x-1)^2$ **E.** $2x^2(x-1)$

11. The triangle formed by joining the points with coordinates $(-2, 1)$, $(2, -1)$ and $(1, 2)$ is

A. scalene **B.** right-angled but not isosceles **C.** equilateral **D.** isosceles but not right-angled **E.** right-angled and isosceles

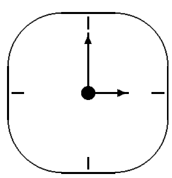
12. The number of students who sat the 1992 U.K. S.M.C. was $80,000$. The number who sat the 1993 U.K. S.M.C. was $105,000$. Which calculation gives the right percentage increase?

A. $\frac{80,000}{104,000} \times 100$ **B.** $\frac{80,000}{100} \times 105,000$ **C.** $\frac{105,000}{80,000} \times 100$ **D.** $\frac{(105,000-80,000)}{80,000} \times 100$ **E.** $\frac{(105,000-80,000)}{105,000} \times 100$

13. Professor Hardsum was very absent-minded. He kept forgetting his four-digit "PIN number"; without it he could not use his bank card to get cash out of the bank cash machines. Then one day he noticed that none of the digits is zero, the first two digits form a power of five, the last two digits form a power of two, and the sum of all the digits is odd. Now he no longer needs to remember the number because he can always work it out each time he needs it! What is the product of the digits of his PIN number?

A. 60 **B.** 120 **C.** 240 **D.** 480 **E.** 960

14. A stopped clock may be useless, but it does at least show the correct time twice a day. A "good" clock, which gains just one second each day, shows the correct time far less often. Roughly how often?



A. once every 60 days **B.** once every 72 days **C.** once every 360 days **D.** once every 12 years **E.** once every 120 years

15. What is the value of the fraction

$$1 + \frac{2}{1 + \frac{3}{1+4}}$$

when written as a decimal?

A. 1.5 **B.** 2.25 **C.** 2.5 **D.** 2.6 **E.** 3.5

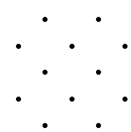
16. Each year around $750,000$ bottles of water are prepared for the runners in the London Marathon. Each bottle holds 200 ml. George (who finished last year in 3hr 39min) reckoned that of each bottle used "one quarter was more or less drunk, one half was sloshed all over the runner, and one quarter went straight into the gutter". If four fifths of the bottles prepared were actually used by the runners, roughly how many litres were "more or less drunk"?

A. $30,000$ **B.** $36,000$ **C.** $120,000$ **D.** $150,000$ **E.** $190,000$

17. Augustus Gloop eats x bars of chocolate every y days. How many bars does he get through each week?

A. $\frac{7x}{y}$ **B.** $\frac{7y}{x}$ **C.** $7xy$ **D.** $\frac{1}{7xy}$ **E.** $\frac{x}{7y}$

18. How many squares can be formed by joining four dots in the diagram?



A. 4 **B.** 5 **C.** 9 **D.** 11 **E.** 13

19. Timmy Riddle (no relation) never gives a straightforward answer. When I asked him how old he was, he replied: "If I was twice as old as I was eight years ago, I would be the same age as I will be in four years time". How old is Timmy?

A. 12 **B.** 16 **C.** 20 **D.** 24 **E.** 28

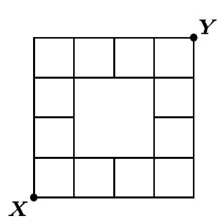
20. Ivor Grasscutter's lawn, which is circular with radius 20 m, needs returfing. Ivor buys the turf in 40 cm wide strips. What is the approximate total length he needs?

A. 300 m **B.** 600 m **C.** 1500 m **D.** 3000 m **E.** 6000 m

21. Last year's carnival procession was $1\frac{1}{2}$ km long. The last float set off, and finished, three quarters of an hour after the first float. Just as the first float reached us, young Gill escaped. She trotted off to the other end of the procession and back in the time it took for half the procession to pass us. Assuming Gill trotted at a constant speed, how fast did she go?

A. 3 km/h **B.** 4 km/h **C.** 5 km/h **D.** 6 km/h **E.** 7 km/h

22. In this 4 by 4 square you have to get from X to Y moving only along black lines. How many different shortest routes are there from X to Y?

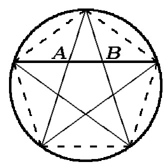


A. 18 **B.** 26 **C.** 28 **D.** 32 **E.** 34

23. When we throw two ordinary dice, the possible totals include all the numbers from 2 to 12. Suppose we have two dice with blank faces. If we mark the six faces of one dice $1, 2, 2, 3, 3, 4$ how should we mark the faces of the second dice so that all totals from 2 to 12 are possible, and each total has exactly the same probability of occurring as with two ordinary dice?

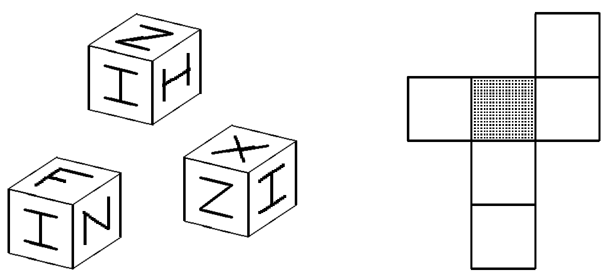
A.	B.	C.	D.	E.
$1, 2, 2, 7, 7, 8$	$1, 3, 4, 5, 6, 8$	$1, 4, 4, 5, 5, 8$	$1, 3, 4, 5, 7, 8$	$1, 2, 3, 6, 7, 8$

24. A regular pentagon is inscribed in a circle of radius p cm, and the vertices are joined up to form a pentagram. If AB has length 1 cm, find the area inside the pentagram (in cm^2).



A. p^2 **B.** $5p/2$ **C.** $5p^2/2$ **D.** $5p$ **E.** $5p^2$

25. The six faces of a made-up cube are labelled F, H, I, N, X, Z. Three views of the labelled cube are shown. The cube is then opened up to form the net shown here (the net has been turned so that the F is upright). What should be drawn (upright) in the shaded square?



A. H **B.** I **C.** N **D.** X **E.** Z

Last month we gave the problems of the Saskatchewan Senior Mathematics Contest 1994. This month we give the “official” solutions.

My thanks go to Gareth Griffith, University of Saskatchewan, longtime organizer of the contest, for supplying the problems and answers.

Saskatchewan Senior Mathematics Contest 1994

1. Solve the equation

$$1 + 68x^{-4} = 21x^{-2}.$$

Solution. The equation $1+68x^{-4} = 21x^{-2}$ is equivalent to $u^2 - 21u + 68 = 0$ where $u = x^2$. Since $(u-4)(u-17) = 0$, $x^2 = 4$ or 17. Therefore $x = \pm 2$ or $\pm\sqrt{17}$.

2. Find the number of divisors of $16\,128$ (including 1 and $16\,128$).

Solution. To begin with, consider a simpler example: Find the number of divisors of 12. The divisors are:

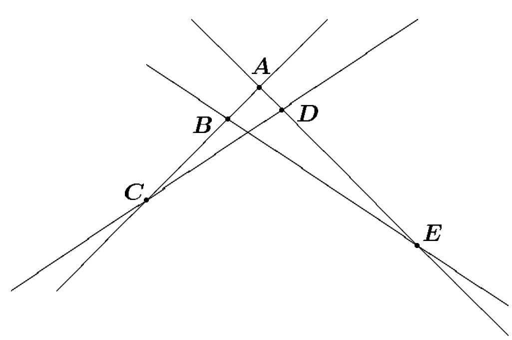
$$\begin{array}{ccc} 2^0 \times 3^0, & 2^1 \times 3^0, & 2^2 \times 3^0, \\ 2^0 \times 3^1, & 2^1 \times 3^1, & 2^2 \times 3^1. \end{array}$$

There are 6 divisors. We notice that we arrive at the number 6 as $(a+1)(b+1)$ where $12 = 2^a \cdot 3^b$. This results is true in general

$$16128 = 2^8 \cdot 3^2 \cdot 7.$$

The number of divisors is $(8+1)(2+1)(1+1) = 54$.

3. In the figure, lines ABC and ADE intersect at A. The points $BCDE$ are chosen such that angles CBE and CDE are equal. Prove that the rectangle whose sides have length AB and AC and the rectangle whose sides have length AD and AE are equal in area.



Solution. Since angle CBE = angle CDE, the points $BCDE$ are concyclic. (Converse of the theorem "angles in the same segment are equal".) (This is known as the theorem of Thales.) ABC and ADE are secants of this circle. Therefore $AB \cdot AC = AD \cdot AE$.

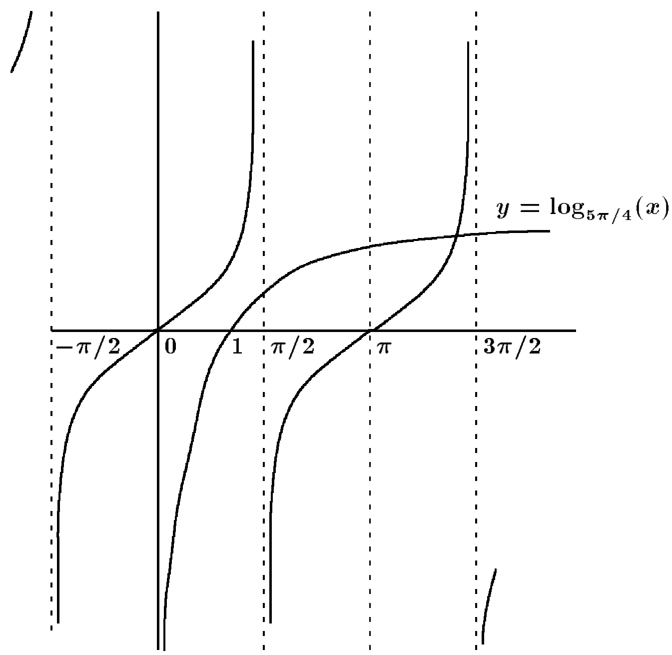
4. (a) State the domain and range of the functions

$$f(x) = \tan x \quad \text{and} \quad g(x) = \log_a x, \quad \text{where } a = \frac{5\pi}{4}.$$

(b) Determine the smallest value of x for which $\tan x = \log_{5\pi/4} x$.

Solution. (a) The domain of $\tan x$ is all the real numbers except for odd multiples of $\pi/2$. The range of $\tan x$ is all real numbers. The domain of $\log_a x$ is all positive real numbers. The range is the real numbers. This is true for all $a > 0$, $a \neq 1$, and thus is true for $a = \frac{5\pi}{4}$.

(b) Consider the graphs of $y = \tan x$, $y = \log_{(5\pi)/4} x$ for $0 < x < \frac{3\pi}{2}$.



$\log_{(5\pi)/2} x$ is not defined for $x \leq 0$.

The graphs indicate that the smallest value for x at which the graphs intersect lies between π and $\frac{3\pi}{2}$. But $\tan \frac{5\pi}{4} = 1$ and $\log_{(5\pi)/4} \frac{5\pi}{4} = 1$. Therefore this point is $x = \frac{5\pi}{4}$.

Note: for students who have studied calculus:

$$\tan x \neq \log_{(5\pi)/4} x \quad \text{for} \quad 0 < x < \frac{\pi}{2},$$

since (i) $\tan 1 > \tan \dfrac{\pi}{4} = 1$, $\log_{(5\pi)/4} 1 = 0$ and $\dfrac{5\pi}{4} > 3 > e$,

and (ii) $\sec^2 x > 1 > \dfrac{1}{x \ln \frac{5\pi}{4}}$ for $1 < x < \dfrac{\pi}{2}$.

5. (a) Prove that the system of equations

$$\begin{aligned} x + y &= 1 \\ x^2 + y^2 &= 2 \\ x^3 + y^3 &= 3 \end{aligned}$$

has no solution.

(b) Determine all values of k such that the system

$$\begin{aligned} x + y &= 1 \\ x^2 + y^2 &= 2 \\ x^3 + y^3 &= k \end{aligned}$$

has at least one solution.

Solution. (a) $\left.\begin{array}{ccccc} x & + & y & = & 1 \\ x^2 & + & y^2 & = & 2 \end{array}\right\}.$

Therefore $1 = (x+y)^2 = x^2 + 2xy + y^2 = 2 + 2xy$. Therefore $xy = -\frac{1}{2}$.

If $x^3 + y^3 = 3$ then

$$(x+y)(x^2 - xy + y^2) = (1)(2 - xy) = 3.$$

Therefore $xy = -1$. This is a contradiction and so the system has no solution.

(b) In order that the system admits at least one solution,

$$(x+y)(x^2 - xy + y^2) = (1)(2 - xy) = k.$$

Therefore $xy = 2 - k$ and $xy = -\frac{1}{2}$ so that $2 - k = -\frac{1}{2}$; $k = \frac{5}{2}$.

6. (Contributed by Murray Bremner, University of Saskatchewan.) This problem shows how we may find *all* solutions to the equation $X^2 + Y^2 = Z^2$ where X, Y and Z are positive integers. Such a solution (X, Y, Z) is called a *Pythagorean triple*. If (X, Y, Z) have no common factor (other than 1), we call (X, Y, Z) a *primitive* Pythagorean triple.

Part I

Remarks. **Part I** shows that if a, b are positive integers with no common factor and $a > b$ then $X = a^2 - b^2$, $Y = 2ab$, $Z = a^2 + b^2$ is a primitive Pythagorean triple.

(a) Let a, b be two positive integers with $a > b$. Show that $X = a^2 - b^2$, $Y = 2ab$, $Z = a^2 + b^2$ is a Pythagorean triple.

Solution. **Part I** (a).

$$\begin{aligned} X^2 + Y^2 &= (a^2 - b^2)^2 + 4a^2b^2 \\ &= a^4 - 2a^2b^2 + b^4 + 4a^2b^2 \\ &= a^4 + 2a^2b^2 + b^2 = Z^2, \end{aligned}$$

(b) Now assume that a, b have no common factor and not both are odd. Show that (X, Y, Z) in (a) is a primitive Pythagorean triple. (Hint: Suppose that X, Y, Z have a common factor, $p =$ some prime number. Then p divides $Z + X$ and $Z - X$. Note that $Z + X = 2a^2$ and $Z - X = 2b^2$. So what is p? But Z must be odd (why?) so p can't be 2).

Solution. **Part I** (b). Since a, b have no common factor, they cannot both be even. Since a, b are not both odd (given), one is odd and the other even. Therefore $Z = a^2 + b^2$ is odd. (The square of an odd number is odd; the square of an even number is even.) By the hint, if we suppose that there exists a prime p which divides X, Y and Z, p divides $Z + X = 2a^2$ and $Z - X = 2b^2$. But p cannot divide a and b (given) and so p divides 2. Therefore $p = 2$. But Z is odd. This contradiction implies that there does not exist such a prime p. Therefore, (X, Y, Z) is a primitive Pythagorean triple.

Part II

Remarks. **Part II** shows that every primitive Pythagorean triple arises this way (for suitable choice of a, b).

(a) Let (X, Y, Z) be any Pythagorean triple. Show that the point $(\frac{X}{Z}, \frac{Y}{Z})$ lies on the unit circle $x^2 + y^2 = 1$.

Solution. **Part II** (a). The point $(\frac{X}{Z}, \frac{Y}{Z})$ lies on the circle $x^2 + y^2 = 1$ if and only if $(\frac{X}{Z})^2 + (\frac{Y}{Z})^2 = 1$. This is true since (X, Y, Z) is a Pythagorean triple.

(b) Let the slope of the line l which joins $(-1, 0)$ to $(\frac{X}{Z}, \frac{Y}{Z})$ be b/a where a, b are positive integers with no common factor and $a > b$. Find the points of intersection of the line l and the unit circle in terms of a, b to show that

$$\frac{X}{Z} = \frac{a^2 - b^2}{a^2 + b^2}, \qquad \frac{Y}{Z} = \frac{2ab}{a^2 + b^2}.$$

Solution. **Part II** (b). The line l, through $(-1, 0)$ and with slope $\frac{b}{a}$ has equation $y = \frac{b}{a}(x + 1)$. l intersects the circle at those points for which

$$x^2 + \left[\frac{b}{a}(x + 1)\right]^2 = 1 \quad \text{or} \quad (a^2 + b^2)x^2 + 2b^2 x + b^2 - a^2 = 0$$

$$\text{or} \quad x^2 + \frac{2b^2}{a^2 + b^2}x + \frac{b^2 - a^2}{a^2 + b^2} = 0 \quad \text{since} \quad a^2 + b^2 \neq 0.$$

$$(x + 1)\left(x + \frac{b^2 - a^2}{a^2 + b^2}\right) = 0.$$

Therefore $x = -1$ or $x = \frac{a^2 - b^2}{a^2 + b^2}$. So $\frac{X}{Z} = \frac{a^2 - b^2}{a^2 + b^2}$.
Further $y = \frac{b}{a}\left[\frac{a^2 - b^2 + a^2 + b^2}{a^2 + b^2}\right] = \frac{2ab}{a^2 + b^2}$ and so $\frac{Y}{Z} = \frac{2ab}{a^2 + b^2}$ as required.

(c) If (X, Y, Z) is a primitive Pythagorean triple and if a, b are not both odd, show that $X = a^2 - b^2$, $Y = 2ab$, $Z = a^2 + b^2$.

Solution. **Part II** (c). Since (X, Y, Z) is a Pythagorean triple, it follows from parts (a) and (b) there exist a, b such that

$$\frac{X}{Z} = \frac{a^2 - b^2}{a^2 + b^2}, \qquad \frac{Y}{Z} = \frac{2ab}{a^2 + b^2}.$$

Further, since a, b are not both odd, $a^2 + b^2$ and $a^2 - b^2$ are both odd.

If $a^2 - b^2$ and $a^2 + b^2$ have a common factor then that factor would be a factor of their sum and difference, namely $2a^2$, $2b^2$. Note that 2 is not a common factor and therefore since (X, Y, Z) is a *primitive* triple a, b have no common factor. Thus

$$X = a^2 - b^2, \quad Y = 2ab, \quad Z = a^2 + b^2.$$

(d) If (X, Y, Z) is a primitive Pythagorean triple and if a, b *are both odd*, $a > b$, we let $r = \frac{1}{2}(a + b)$, $s = \frac{1}{2}(a - b)$.

(i) Prove that r, s are positive integers, that $r > s$, that r, s have no common factor (other than 1) and that r, s are not both odd.

(ii) Let $X' = 2(2rs)$, $Y' = 2(r^2 - s^2)$, $Z' = 2(r^2 + s^2)$. Show that $X = X'/2$, $Y = Y'/2$, $Z = Z'/2$. (Thus $X = 2rs$, $Y = r^2 - s^2$, $Z = r^2 + s^2$).

Solution. **Part II** (d). (i) $r = \frac{1}{2}(a + b)$, $s = \frac{1}{2}(a - b)$ and a, b are both odd. The sum and difference of two odd numbers is even and one half of an even number is an integer. Since $a > b > 0$ (given), r, s are positive integers.

$$r - s = b > 0 \quad \text{and therefore} \quad r > s.$$

Suppose that r, s have a common factor, $p > 1$. Then p divides $r + s = a$ and $r - s = b$. But a, b have no common factor. Therefore neither do r, s. Suppose that r, s are both odd. Then $r + s$ is even. But $r + s = a$ which is odd. Therefore, r, s are not both odd.

(ii) Note that

$$\begin{aligned}
X' &= 2(2rs) = (a + b)(a - b) = a^2 - b^2,\\
Y' &= 2(r^2 - s^2) = 2\left[\tfrac{1}{4}(a^2 + 2ab + b^2 - a^2 + 2ab - b^2)\right] = 2ab,\\
Z' &= 2(r^2 + s^2) = 2\left[\tfrac{1}{4}(a^2 + 2ab + b^2 + a^2 - 2ab + b^2)\right] = a^2 + b^2.
\end{aligned}$$

Therefore (X', Y', Z') is a Pythagorean triple, but it is not primitive since 2 divides X', Y' and Z'. However $X = \frac{X'}{2}$, $Y = \frac{Y'}{2}$, $Z = \frac{Z'}{2}$ is primitive (given) and thus $X = 2rs$, $Y = r^2 - s^2$, $Z = r^2 + s^2$.

For the sake of interest, we list the Pythagorean triples that result from small values of a, b $(a > b)$:

a	b	$a^2 - b^2$	$2ab$	$a^2 + b^2$	Remarks
2	1	3	4	5	Surely, the most famous Pythagorean triple
3	2	5	12	13	
3	1	8	6	10	Both a, b are odd; see case d(ii)
4	3	7	24	25	
4	2	12	16	20	Since a, b are both even, the triple cannot be primitive. The corresponding primitive triple is the case $a = 2$, $b = 1$.
4	1	15	8	17	
5	4	9	40	41	
5	3	16	30	34	Both a, b are odd; see case d(ii)
5	2	21	20	29	
5	1	24	10	26	Case d(ii) again

That completes the Skoliad Corner for this month. Please send me your pre-Olympiad contests and suggestions for future directions for this feature of *Crux*.

Historical Titbit

Taken from a 1894 Ontario Public School textbook.

Show how to dissect a rhombus, and re-assemble the pieces to make a rectangle.

Now this is quite easy, so we shall restate the problem:

Given a rhombus with diagonals of length a and b, show how to dissect the rhombus, and re-assemble the pieces to make a rectangle of maximum possible area.

Call this maximum area Δ. Let $0 < \Lambda < \Delta$. Show how to dissect the rhombus, and re-assemble the pieces to make a rectangle of area Λ.

In each case, the dissection should be with the minimum possible number of pieces.

THE OLYMPIAD CORNER

No. 171

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

Another year has passed, and we begin the 1996 volume of *Crux Mathematicorum* with a change in the number of issues as well as in the Editorship. This is the first number of the Corner which I will submit that does not fall under the critical gaze of Bill Sands who stepped down as Editor-in-Chief at the end of December. I want to express my particular gratitude to Bill for his many hours of hard work and devotion to the publication over ten years. It is also time to welcome the new Editor-in-Chief, with whom I look forward to working over the next years.

Before launching into new material, let us pause to thank those who contributed to the Corner in 1995. My particular thanks go to Joanne Longworth whose layout skills with LaTeX enhance the readability of the copy. It is also time to thank those who have contributed problem sets, solutions, comments, corrections, and criticisms over the past year. Among our contributors we have:

Miquel Amenguel Covas	Federico Ardila M.	Šefket Arslanagić
Seung-Jin Bang	Gerd Baron	Bruce Bauslaugh
Francisco Bellot Rosado	Christopher Bradley	Himadri Choudhury
Tim Cross	George Evagelopoulos	Tony Gardiner
Murray Grant	Gareth Griffith	Georg Gunther
Walther Janous	Geoffrey Kandall	Derek Kisman
Murray Klamkin	Joseph Ling	Andy Liu
Sam Maltby	Beatriz Margolis	Stewart Metchette
John Morvay	Waldemar Pompe	Bob Prielipp
Toshio Seimiya	Michael Selby	Bruce Shawyer
D.J. Smeenk	Daryl Tingley	Jim Totten
Panos E. Tsaoussoglou	Stan Wagon	Edward T.H. Wang
Martin White	Chris Wildhagen	C.S. Yogananda

Thank you all (and anyone I've left out by accident).

As an Olympiad set this number, we give the problems of the 29th Spanish Mathematical Olympiad (National Round). My thanks go to Georg

Gunther, Sir Wilfred Grenfell College and Canadian Team leader at the 34th I.M.O. at Istanbul, Turkey; and to Francisco Bellot Rosada, Valladolid, Spain, for collecting and sending me a copy of the contest.

29th SPANISH MATHEMATICAL OLYMPIAD (National Round)

Madrid, February 26–27, 1992

FIRST DAY (Time: 4.5 hours)

1. At a party there are 201 people of 5 different nationalities. In each group of six, at least two people have the same age. Show that there are at least 5 people of the same country, of the same age and of the same sex.

2. Given the number triangle

$$\begin{array}{ccccccccccccccccc}
0 & & 1 & & 2 & & 3 & & 4 & \ldots\ldots & 1991 & & 1992 & & 1993 \\
 & 1 & & 3 & & 5 & & 7 & & \ldots\ldots & & 3983 & & 3985 & \\
 & & 4 & & 8 & & 12 & & & \ldots\ldots & & & 7968 & &
\end{array}$$

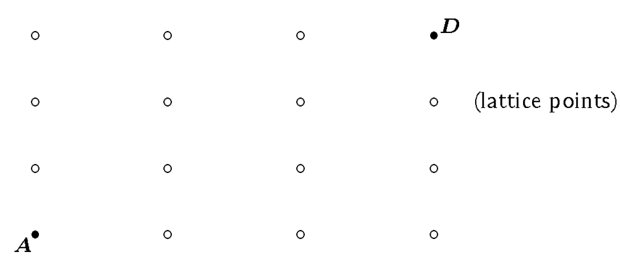
in which each number equals the sum of the two above it, show that the last number is a multiple of 1993.

3. Show that in any triangle, the diameter of the incircle is not bigger than the circumradius.

SECOND DAY (Time: 4.5 hours)

4. Show that each prime number p (different from 2 and from 5) has an infinity of multiples which can be written as $1\ 1\ 1\ 1\ \ldots\ 1$.

5. Given 16 points as in the figure



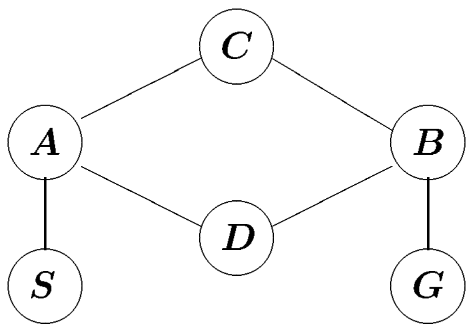
in which the points A and D are marked, determine, for all the possible manners, two other points B, C given that the six distances between these four points should be different. In this set of 4 points,

(a) How many figures of four points are there (with the condition above)?

(b) How many of them are non-congruent?

(c) If each point is represented by a pair of integers (X_i, Y_i), show that the sum $|X_i - X_j| + |Y_i - Y_j|$, extended to the six pairs AB, AC, AD, BC, BD, CD, is constant.

6. A game-machine has a screen in which the figure below is showed. At the beginning of the game, the ball is in the point S.



With each impulse from the player, the ball moves up to one of the neighbouring circles, with the same probability for each. The game is over when one of the following events occurs:

(1) The ball goes back to S, and the player loses.

(2) The ball reaches G, and the player wins.

Determine:

(a) The probability for the player to win the game.

(b) The mean time for each game.

Next I'd like to give six Klamkin Quickies. Many thanks to Murray Klamkin, University of Alberta, for creating them for us. The answers will appear in the next number.

SIX KLAMKIN QUICKIES

1. Which is larger

$$(\sqrt[3]{2} - 1)^{1/3} \qquad \text{or} \qquad \sqrt[3]{1/9} - \sqrt[3]{2/9} + \sqrt[3]{4/9}?$$

2. Prove that

$$3 \times \min\left\{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}, \frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right\} \geq (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

where a, b, c are sides of a triangle.

3. Let $\omega = e^{i\pi/13}$. Express $\frac{1}{1-\omega}$ as a polynomial in ω with integral coefficients.

4. Determine all integral solutions of the simultaneous Diophantine equations $x^2 + y^2 + z^2 = 2w^2$ and $x^4 + y^4 + z^4 = 2w^4$.

5. Prove that if the line joining the incentre to the centroid of a triangle is parallel to one of the sides of the triangle, then the sides are in arithmetic progression and, conversely, if the sides of a triangle are in arithmetic progression then the line joining the incentre to the centroid is parallel to one of the sides of the triangle.

6. Determine integral solutions of the Diophantine equation

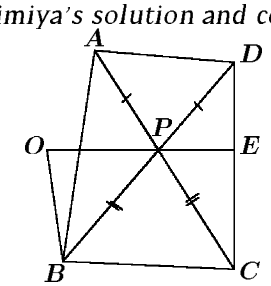
$$\frac{x-y}{x+y} + \frac{y-z}{y+z} + \frac{z-w}{z+w} + \frac{w-x}{w+x} = 0$$

(joint problem with Emeric Deutsch, Polytechnic University of Brooklyn).

To finish this number of the *Corner*, we turn to readers' solutions to problems from the May 1994 number of the Corner and the Final Round of the 43rd Mathematical Olympiad (1991–92) in Poland [1994: 129–130].

1. Segments AC and BD intersect in point P so that $PA = PD$, $PB = PC$. Let O be the circumcentre of triangle PAB. Prove that lines OP and CD are perpendicular.

Solutions by Joseph Ling, University of Calgary; by Toshio Seimiya, Kawasaki, Japan; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Seimiya's solution and comment.



Because $PA = PD$, $PB = PC$, and $\angle APB = \angle DPC$, we get $\Delta PAB \equiv \Delta PDC$, so that

$$\angle BAP = \angle CDP. \tag{1}$$

At least one of $\angle PAB$ and $\angle PBA$ is acute, so we may assume without loss of generality that $\angle PAB$ is acute. Since O is the circumcentre of ΔPAB we get $OB = OP$ and $\angle BOP = 2\angle BAP$, so that

$$\angle OPB = 90^\circ - \frac{1}{2}\angle BOP = 90^\circ - \angle BAP. \tag{2}$$

Let E be the intersection of OP with CD. Then

$$\angle EPD = \angle OPB. \tag{3}$$

From (1), (2) and (3) we have

$$\angle EPD = 90^\circ - \angle CDP.$$

Thus $\angle EPD + \angle EDP = \angle EPD + \angle CDP = 90^\circ$. Therefore $OP \perp CD$.

Comment: Generally if A, B, C, D are concyclic, we have $OP \perp CD$ and this theorem is an extension of Brahmagupta's theorem.

2. Determine all functions f defined on the set of positive rational numbers, taking values in the same set, which satisfy for every positive rational number x the conditions

$$f(x+1) = f(x) + 1 \quad \text{and} \quad f(x^3) = (f(x))^3.$$

Solution by Edward T.H. Wang and by Siming Zhan, Wilfrid Laurier University, Waterloo, Ontario.

Let $\mathbb{N}$ and $\mathbb{Q}^+$ denote the set of positive integers. and the set of positive rational numbers, respectively. We show that $f(x) = x$, for all $x \in \mathbb{Q}^+$, is the only function satisfying the given conditions. First of all, by the first condition and an easy induction we see that $f(x+n) = f(x) + n$, for all $x \in \mathbb{Q}^+$, and for all $n \in \mathbb{N}$. Now for arbitrary $\frac{p}{q} \in \mathbb{Q}^+$, where $p, q \in \mathbb{N}$, we have

$$\begin{aligned} f\left(\left(\frac{p}{q} = q^2\right)^3\right) &= f\left(\frac{p^3}{q^3} + 3p^2 + 3pq^3 + q^6\right) \\ &= f\left(\left(\frac{p}{q}\right)^3\right) + 3p^2 + 3pq^3 + q^6. \end{aligned} \qquad (1)$$

On the other hand

$$\begin{aligned} f\left(\left(\frac{p}{q} + q^2\right)^3\right) = f\left(\left(\frac{p}{q} + q^2\right)\right)^3 &= \left(f\left(\frac{p}{q}\right) + q^2\right)^3 \\ &= \left(f\left(\frac{p}{q}\right)\right)^3 + 3\left(f\left(\frac{p}{q}\right)\right)^2 q^2 + 3\left(f\left(\frac{p}{q}\right)\right)^2 \cdot q^4 + q^6. \end{aligned} \quad (2)$$

Letting $t = f(\frac{p}{q})$ and comparing (1) and (2), we get, since $f(\frac{p^3}{q^3}) = (f(\frac{p}{q}))^3$, $p(p+q^3) = q^2t^2 + q^4t$ or $q^2t^2 + q^4t - p(p+q^3) = 0$ or $(qt-p)(qt+p+q^3) = 0$. Since $qt + p + q^3 > 0$, we must have $t = \frac{p}{q}$, i.e. $f(\frac{p}{q}) = \frac{p}{q}$, and we are done.

3. Prove that the inequality

$$\sum_{n=1}^{r}\left(\sum_{m=1}^{r}\frac{a_m a_n}{m+n}\right) \geq 0$$

holds for any real numbers $a_1, a_2, \ldots, a_r$. Find conditions for equality.

Solutions by Seung-Jin Bang, Seoul, Korea; by Joseph Ling, University of Calgary; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Ling's solution.

Consider the polynomial

$$p(x) = \sum_{n=1}^{r} \left(\sum_{m=1}^{r} a_m a_n x^{m+n-1} \right).$$

Then

$$\begin{aligned} xp(x) = \sum_{n=1}^{r} \sum_{m=1}^{r} a_m a_n x^{m+n} &= \left(\sum_{m=1}^{r} a_m x^m \right) \left(\sum_{n=1}^{r} a_n x^n \right) \\ &= \left(\sum_{i=1}^{r} a_i x^i \right)^2 \geq 0, \end{aligned}$$

for all $x \in \mathbb{R}$.

In particular $p(x) \geq 0$ for all $x \geq 0$. Hence

$$\begin{aligned} 0 \leq \int_0^1 p(x)\,dx &= \sum_{n=1}^{r} \left(\sum_{m=1}^{r} \frac{a_m a_n}{m+n} x^{m+n} \Big]_0^1 \right) \\ &= \sum_{n=1}^{r} \sum_{m=1}^{r} \frac{a_m a_n}{m+n} \end{aligned}$$

The inequality is strict unless $xp(x) \equiv 0$, that is $a_1 = a_2 = \cdots = a_r = 0$.
[*Editor's Remark.* All three solutions involved the integral. Can you furnish a nice solution avoiding the calculus?]

4. Define the sequence of functions $f_0, f_1, f_2, \ldots$ by

$$\begin{aligned} f_0(x) &= 8 \quad \text{for all } x \in \mathbb{R}, \\ f_{n+1}(x) &= \sqrt{x^2 + 6f_n(x)} \quad \text{for } n = 0, 1, 2, \ldots \text{ and for all } x \in \mathbb{R}. \end{aligned}$$

For every positive integer n, solve the equation $f_n(x) = 2x$.

Solutions by Seung-Jin Bang, Seoul, Korea; and by Chris Wildhagen, Rotterdam, The Netherlands. We use Bang's comment and solution.

This problem is the same as problem 2 of the 19th Annual U.S.A. Mathematical Olympiad (which appeared in *Math. Magazine* 64, 3 (1991), pp. 211–213.)

Since $f_n(x)$ is positive, $f_n(x) = 2x$ has only positive solutions. We show that, for each n, $f_n(x) = 2x$ has a solution $x = 4$. Since $f_1(x) = \sqrt{x^2 + 48}$, $x = 4$ is a solution of $f_2(x) = 2x$. Now $f_{n+1}(4) = \sqrt{4^2 + 6f_n(4)}$ $= \sqrt{4^2 + 6 \cdot 8} = 8 = 2 \cdot 4$, which completes the inductive step. Next, induction on n gives us that for each n, $\frac{f_n(x)}{x}$ decreases as x increases in $(0, \infty)$. It follows that $f_n(x) = 2x$ has the unique solution $x = 4$.

6. Prove that, for every natural k, the number $(k^3)!$ is divisible by $(k!)^{k^2+k+1}$.

Solution by Chris Wildhagen, Rotterdam, The Netherlands.

Applying the well known fact that $(ab)!$ is divisible by $(a!)^b \cdot b!$ yields $(k^3)! = (k \cdot k^2)!$ is divisible by $(k!)^{k^2} \cdot (k^2)!$ and $(k^2)! = (k \cdot k)!$ is divisible by $(k!)^{k+1}$ from which the required result follows immediately.

The astute reader will notice we did not have a solution on file from the readership to problem 5. There's your challenge!

That completes the *Olympiad Corner* for this issue. The Olympiad season is fast approaching. Please send me your national and regional contests.

Introducing the new Editor-in-Chief

This issue of *CRUX* marks the change of the Editor-in- Chief from Bill Sands and Robert Woodrow to Bruce Shawyer. For those of you who do not know Bruce, here is a short profile:

Born:	Kirkcaldy, Scotland [1]
Educated	Kirkcaldy High School University of St. Andrews, Scotland
Employment	University of Nottingham, England University of Western Ontario, Canada Memorial University of Newfoundland, Canada (Head of Department, 1985–91)
Visiting Positions	Universität Ulm, Germany University of St. Andrews
Service	Canadian Mathematical Society Team Leader, Team Canada at IMO 1987 and 1988 IMO95, Principal Organiser
Research	Summability of Series and Integrals Approximation Theory LaTeX macros for the picture environment
Home Page	http://www.math.mun.ca/~bshawyer

[1] *Kirkcaldy, an industrial town north of Edinburgh, is the birthplace of Sir Sandford Fleming, who invented* time zones, *and of Adam Smith, the Father of* economics.

THE ACADEMY CORNER

No. 1

Bruce Shawyer

All communications about this column should be sent to Professor Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

With this issue, we start a new corner, which, for want of a better name, is being called "the Academy Corner". It will be concerned, in particular, with problem solving at the undergraduate level. This can take the form of a competition, a "problem of the week", a course in problem solving, etc. Your submissions and comments are welcome!

Many Universities hold their own undergraduate Mathematics Competitions. We invite subscribers to send us information about them for publication here. We also solicit nice solutions to these problems. Please send them to the Editor-in-Chief.

We shall begin with the competition with which the editor is most familiar: the Undergraduate Mathematics Competition at Memorial University of Newfoundland. This competition is used as primary information for choosing Memorial's team for the Atlantic Provinces Council on the Sciences Mathematics Competition (more on that in a later issue).

Memorial University Undergraduate Mathematics Competition 1995

Time allowed — 3 hours

1. Find all integer solutions of the equation $x^4 = y^2 + 71$.
2. (a) Show that $x^2 + y^2 \geq 2xy$ for all real numbers x, y.

   (b) Show that $a^2 + b^2 + c^2 \geq ab + bc + ca$ for all real numbers a, b, c.
3. Find the sum of the series

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{99}{100!}.$$

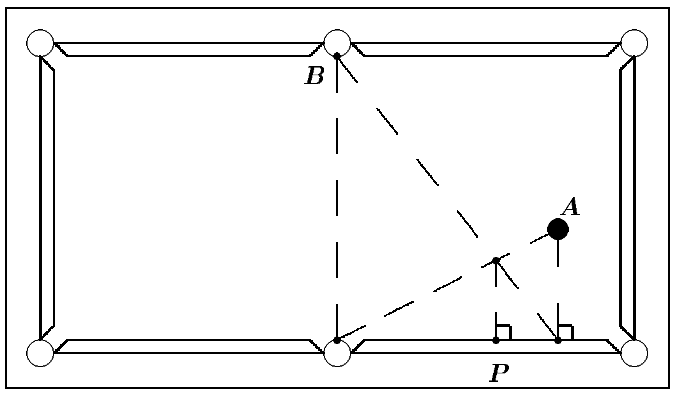
4. If a, b, c, d are positive integers such that $ad = bc$, prove that $a^2 + b^2 + c^2 + d^2$ is never a prime number.

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy
$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$$
for all real numbers x, y.

6. Assume that when a snooker ball strikes a cushion, the angle of incidence equals the angle of reflection.

   For any position of a ball A, a point P on the cushion is determined as shown.

   Prove that if the ball A is shot at point P, it will go into the pocket B.



This concludes the first Academy Corner.

Mathematical Literacy

1. Who thought that the binary system would convince the Emperor of China to abandon Buddhism in favour of Christianity?

2. Who asked which king for one grain of wheat for the first square of a chess board, two grains for the second square, four grains for the third square, and so on?

3. In which well known painting, by whom, does a Magic Square appear?

4. Where was bread cut into "Cones, Cylinders, Parallelograms, and several other Mathematical Figures"?

5. Which mathematician said: "Philosophers count about two-hundred and eighty eight views of the sovereign good"?

Answers will be given in a subsequent issue.

BOOK REVIEWS

Edited by ANDY LIU

The Monkey and the Calculator (Le singe et la calculatrice), softcover, ISBN 2-909737-07-1, 128 pages, 48 French francs (plus mailing).

Aladdin's Sword (Le sabre d'Aladin), softcover, ISBN 2-909737-08-X, 128 pages, 48 French francs (plus mailing).

Both published in 1995 by Production et Organisation du Loisir Educatif (POLE), 31 avenue des Gobelins, 75013 Paris, France.

Reviewed by **Claude Laflamme**, *University of Calgary*.

These two delightful little pocket books each contain a selection of almost 100 puzzles from the eighth International Championship of Mathematical Puzzles. This is an annual competition in four stages and in seven categories from elementary to advanced, open to anyone and administered by FFJM, the French Federation of Mathematical Games. These books are numbers 14 and 15 respectively in a series taken from these competitions and published by POLE, and most of these books are still in print.

The first book is from the Junior High level and the second from the Grand Public competition, slightly more sophisticated. Nevertheless, no specialized knowledge is required for the competitions and appropriately the statements of these puzzles are all very elementary; but they puzzle your puzzler to the point that once you have read one of the problems, you cannot leave the book until you figure out a solution. Now this is a serious competition and a mere solution is usually not enough, the most elegant and complete one is the favourite!

These puzzles should be of interest to anyone looking for some "fun brain gymnastics" (*gymnastique intellectuelle*), or "neurobics" as I have seen somewhere in the books. I can see here a great resource for teachers trying to interest students with some challenging but elementary and accessible problems. Here is a sample from the second book, called "A Numismatic Coincidence":

> By multiplying the number of pieces in her coin collection by 1994, the clever Miss Maths arrives at a product the sum of the digits of which is exactly equal to the number of pieces in her coin collection.
>
> **How many pieces does Miss Maths have in her coin collection?**

All the problems are translated into English, a job very well done even if this means that some problems require a new answer, such as the problem asking us to fill in the blank with a number (written out in words) making the following sentence true (a hyphen doesn't count as a letter):

"This sentence has ___________ letters."

There are two answers in English, and only one (and it is different) for the French version.

The problem solutions are in French only but you will not be puzzled here, even a small diagram or a few numbers and the solution jumps at you. Apart from the puzzles themselves, you will find a few advertisements for other mathematics magazines, even a puzzling one on music and mathematics, as well as for camps and Summer Schools. You will also find details regarding the competition itself.

By the way, the titles refer to some of the more puzzling puzzles in the books!

Tournament of the Towns, 1980–1984, Questions and Solutions; published in 1993, softcover, 117+ pages, Australian $18.

Tournament of the Towns, 1989–1993, Questions and Solutions; published in 1994, softcover, 209+ pages, Australian $22.

Both edited by Peter J. Taylor, Australian Mathematics Trust, University of Canberra, P. O. Box 1, Belconnen, A.C.T. 2616, Australia.

Reviewed by **Murray S. Klamkin**, *University of Alberta.*

In my previous review [1992: 172] of the first book of this series, which are the problems and solutions for 1984–1989 (and which was published first), I had given a short description of the competition and references for further information on the competition. More complete information on this is given in the prefaces of the books.

This competition, as I said before, is one of the premier mathematics competitions in the world for secondary school students and contains many wonderful challenging problems (even to professional mathematicians). In view of my previous review, I just give another sampling of these problems.

Junior Questions

1. Construct a quadrilateral given its side lengths and the length joining the midpoints of its diagonal. [1983]

2. (a) A regular $4k$-gon is cut into parallelograms. Prove that among these there are at least k rectangles.
   (b) Find the total area of the rectangles in (a) if the lengths of the sides of the $4k$-gon equal a. [1983]

3. Prove that in any set of 17 distinct natural numbers one can either find five numbers so that four of them are divisible into the other or five numbers none of which is divisible into any other. [1983]

4. Given the continued fractions

$$a = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\cdots + \cfrac{1}{99}}}} \qquad \text{and} \qquad b = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\cdots + \cfrac{1}{99 + \cfrac{1}{100}}}}},$$

prove that $|a - b| < \dfrac{1}{99! \cdot 100!}$. [1990]

5. The numerical sequence $\{x_n\}$ satisfies the condition

$$x_{n+1} = |x_n| - x_{n-1}$$

for all $n > 1$. Prove that the sequence is periodic with period 9, i.e., for any $n \geq 1$, we have $x_n = x_{n+9}$. [1990]

6. There are 16 boxers in a tournament. Each boxer can fight no more often than once a day. It is known that the boxers are of different strengths, and the stronger man always wins. Prove that a 10 day tournament can be organized so as to determine their classification (put them in order of strength). The schedule of fights for each day is fixed on the evening before and cannot be changed during the day. [1991]

Senior Questions

1. We are given 30 non-zero vectors in 3 dimensional space. Prove that among these there are two such that the angle between them is less than $45°$. [1980]

2. Prove that every real positive number may be represented as a sum of nine numbers whose decimal representation consists of the digits 0 and 7. [1981]

3. A polynomial $P(x)$ has unity as coefficient of its highest power and has the property that with natural number arguments, it can take all values of form $2m$, where m is a natural number. Prove that the polynomial is of degree 1. [1982]

4. A square is subdivided into K^2 equal small squares. We are given a broken line which passes through the centres of all the smaller squares (such a broken line may intersect itself). Find the minimum number of links in this broken line. [1982]

5. Do there exist $1,000,000$ distinct positive numbers such that the sum of any collection of these numbers is never an exact square? [1989]

6. There are 20 points in the plane and no three of them are collinear. Of these points 10 are red while the other 10 are blue. Prove that there exists a straight line such that there are 5 red points and 5 blue points on either side of this line. [1990]

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$" $\times$ 11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that **1 September 1996**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in LaTeX, preferably in LaTeX2e*). *Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2101. *Proposed by Ji Chen, Ningbo University, China.*

Let a, b, c be the sides and A, B, C the angles of a triangle. Prove that for any $k \leq 1$,

$$\sum \frac{a^k}{A} \geq \frac{3}{\pi} \sum a^k,$$

where the sums are cyclic. [The case $k = 1$ is known; see item 4.11, page 170 of Mitrinović, Pečarić, Volenec, *Recent Advances in Geometric Inequalities*.]

2102. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with incentre I. Let P and Q be the feet of the perpendiculars from A to BI and CI respectively. Prove that

$$\frac{AP}{BI} + \frac{AQ}{CI} = \cot \frac{A}{2}.$$

2103. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle. Let D be the point on side BC produced beyond B such that $BD = BA$, and let M be the midpoint of AC. The bisector of $\angle ABC$ meets DM at P. Prove that $\angle BAP = \angle ACB$.

2104. *Proposed by K. R. S. Sastry, Dodballapur, India.*
In how many ways can 111 be written as a sum of three integers in geometric progression?

2105. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*

Find all values of λ for which the inequality

$$2(x^3+y^3+z^3)+3(1+3\lambda)xyz \geq (1+\lambda)(x+y+z)(yz+zx+xy)$$

holds for all positive real numbers x, y, z.

2106. *Proposed by Yang Kechang, Yueyang University, Hunan, China.*

A quadrilateral has sides a, b, c, d (in that order) and area F. Prove that

$$2a^2+5b^2+8c^2-d^2 \geq 4F.$$

When does equality hold?

2107. *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC is not isosceles nor equilateral, and has sides a, b, c. D_1 and E_1 are points of BA and CA or their productions so that $BD_1 = CE_1 = a$. D_2 and E_2 are points of CB and AB or their productions so that $CD_2 = AE_2 = b$. Show that $D_1E_1 \parallel D_2E_2$.

2108. *Proposed by Vedula N. Murty, Dover, Pennsylvania.*

Prove that

$$\frac{a+b+c}{3} \leq \frac{1}{4}\sqrt[3]{\frac{(b+c)^2(c+a)^2(a+b)^2}{abc}},$$

where $a, b, c > 0$. Equality holds if $a = b = c$.

2109. *Proposed by Victor Oxman, Haifa, Israel.*

In the plane are given a triangle and a circle passing through two of the vertices of the triangle and also through the incentre of the triangle. (The incentre and the centre of the circle are not given.) Construct, using only an unmarked ruler, the incentre.

2110. *Proposed by Jordi Dou, Barcelona, Spain.*

Let S be the curved *Reuleaux* triangle whose sides AB, BC and CA are arcs of unit circles centred at C, A and B respectively. Choose at random (and uniformly) a point M in the interior and let $C(M)$ be a chord of S for which M is the midpoint. Find the length l such that the probability that $C(M) > l$ is $1/2$.

2111. *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

Does there exist a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) satisfying the three conditions:

(i) $f(1996) = 1$;

(ii) for all primes p, every prime occurs in the sequence
$f(p)$, $f(2p)$, $f(3p)$, $\ldots$, $f(kp)$, $\ldots$ infinitely often; and

(iii) $f(f(n)) = 1$ for all $n \in \mathbb{N}$?

2112. *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Find a four-digit base-ten number $abcd$ (with $a \neq 0$) which is equal to $a^a + b^b + c^c + d^d$.

2113. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Prove the inequality

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left(\sum_{i=1}^{n}(a_i + b_i)\right)\left(\sum_{i=1}^{n} \frac{a_i b_i}{a_i + b_i}\right)$$

for any positive numbers $a_1, \ldots , a_n, b_1, \ldots , b_n$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1827. [1993: 78; 1994: 57; 1995: 54] *Proposed by Šefket Arslanagić, Trebinje, Yugoslavia, and D. M. Milošević, Pranjani, Yugoslavia.*

Let a, b, c be the sides, A, B, C the angles (measured in radians), and s the semi-perimeter of a triangle.

(i) Prove that

$$\sum \frac{bc}{A(s-a)} \geq \frac{12s}{\pi},$$

where the sums here and below are cyclic.

(ii)* It follows easily from the proof of *Crux* 1611 (see [1992: 62] and the correction on [1993: 79]) that also

$$\sum \frac{b+c}{A} \geq \frac{12s}{\pi}.$$

Do the two summations above compare in general?

IV. *Comment by Waldemar Pompe, student, University of Warsaw, Poland.*

Here we give a short demonstration of the equality

$$\sum \frac{bc}{s-a} = s + \frac{(4R+r)^2}{s}, \tag{1}$$

which has appeared on [1995: 55]. Since

$$\sum a = 2s, \qquad \sum bc = s^2 + r^2 + 4Rr, \qquad abc = 4sRr,$$

we get

$$\sum (s-b)(s-c) = 3s^2 - 4s^2 + \sum bc = r^2 + 4Rr.$$

Using the above equality we obtain

$$\sum \frac{1}{s-a} = \frac{r^2+4Rr}{(s-a)(s-b)(s-c)} = \frac{r^2+4Rr}{sr^2} = \frac{r+4R}{sr}.$$

Therefore

$$\begin{aligned}\sum \frac{bc}{s-a} &= abc \sum \frac{1}{a(s-a)} = 4Rr \sum \left(\frac{1}{a} + \frac{1}{s-a}\right) \\ &= 4Rr \left(\frac{s^2+r^2+4Rr}{4sRr} + \frac{r+4R}{sr}\right) \\ &= s + \frac{r^2+4Rr}{s} + \frac{4Rr+16R^2}{s} = s + \frac{(4R+r)^2}{s}.\end{aligned}$$

A (not quite as) short demonstration of (1) *has also been sent in by Toshio Seimiya, Kawasaki, Japan.*

2006. [1995: 20] *Proposed by John Duncan, University of Arkansas, Fayetteville; Dan Velleman, Amherst College, Amherst, Massachusetts; and Stan Wagon, Macalester College, St. Paul, Minnesota.*

Suppose we are given $n \geq 3$ disks, of radii $a_1 \geq a_2 \geq \cdots \geq a_n$. We wish to place them in some order around an interior disk so that each given disk touches the interior disk and its two immediate neighbours. If the given disks are of widely different sizes (such as 100, 100, 100, 100, 1), we allow a disk to overlap other given disks that are not immediate neighbours. In what order should the given disks be arranged so as to maximize the radius of the interior disk? [*Editor's note*. Readers may assume that for any ordering of the given disks the configuration of the problem exists and that the radius of the interior disk is unique, though, as the proposers point out, this requires a proof (which they supply).]

Solution by the proposers.

Let r be the radius of the central disk. First look at a single tangent configuration made up of the central disk, and two disks of radii x and y. The three centres form a triangle with sides $r + x$, $r + y$, and $x + y$; let $\theta = \theta_r(x, y)$ be the angle at the centre of the disk with radius r. Applying the law of cosines to this angle and simplifying gives

$$\theta_r(x, y) = \arccos\left(1 - \frac{2xy}{r^2 + ry + rx + xy}\right).$$

A routine calculation shows that the mixed partial derivative θ_{12} is given by

$$\theta_{12} = \frac{r^2}{2\sqrt{xy}(r^2 + rx + ry)^{3/2}};$$

therefore $\theta_{12} > 0$ (for $x, y > 0$). Integrating from a to $a + s$ and b to $b + t$ (where $s, t > 0$) yields:

$$\theta(a + s, b + t) + \theta(a, b) > \theta(a + s, b) + \theta(a, b + t). \qquad (1)$$

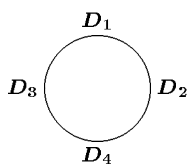
Note that equality occurs if and only if $s = 0$ or $t = 0$.

We may assume $n \geq 4$ and $a_1 \geq a_2 \geq \cdots \geq a_n$; let D_i denote the disk with radius a_i. And let $S(r)$ denote $\sum_{i=1}^{n} \theta_r(a_i, a_{i+1})$, the total angle made by the configuration; when $S < 2\pi$, then the disk of radius r is too large and the ring is not yet closed up.

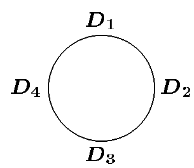
THEOREM. The largest inner radius r occurs by placing D_2 and D_3 on either side of D_1, then D_4 alongside D_2, D_5 alongside D_3, and so on around the ring.

Proof. By induction on n. We actually prove the stronger assertion that for any radius r, $S(r)$ for the configuration of the statement of the theorem is not less than $S(r)$ for any other configuration. This suffices, for if r is such that $S(r) = 2\pi$, then $S(r)$ for any other configuration is not greater than 2π.

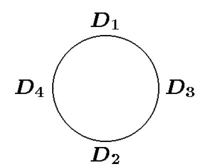
For $n = 4$ there are only three arrangements:



Arrangement I



Arrangement II



Arrangement III

for which (suppressing the subscripts r in the θ's)

$$S_I(r) = \theta(a_1, a_2) + \theta(a_2, a_4) + \theta(a_3, a_4) + \theta(a_1, a_3),$$

$$S_{II}(r) = \theta(a_1, a_2) + \theta(a_2, a_3) + \theta(a_3, a_4) + \theta(a_1, a_4),$$

$$S_{III}(r) = \theta(a_1, a_3) + \theta(a_2, a_3) + \theta(a_2, a_4) + \theta(a_1, a_4).$$

By (1), $S_I(r) \geq S_{II}(r) \geq S_{III}(r)$, which proves our assertion for this case.

For the general case, suppose $E_2, \ldots, E_n$ is an arrangement of $D_2, \ldots, D_n$, with b_i denoting the radius of E_i. Then $a_2 \geq b_2$ and we may also assume $a_3 \geq b_3$ (otherwise flip and relabel). Now it is sufficient, by the induction hypothesis, to show that

$$\theta(a_1, a_2) + \theta(a_1, a_3) - \theta(a_2, a_3) \geq \theta(a_1, b_2) + \theta(a_1, b_3) - \theta(b_2, b_3).$$

But (1) implies that

$$\theta(a_1, a_2) + \theta(a_1, a_3) + \theta(b_2, b_3) \geq \theta(a_1, a_3) + \theta(a_1, b_2) + \theta(a_2, b_3),$$

and this means it is sufficient to prove:

$$\theta(a_1, a_3) + \theta(a_2, b_3) \geq \theta(a_1, b_3) + \theta(a_2, a_3).$$

But this too is a consequence of (1). ■

Note. A similar argument shows that the *smallest* inner radius occurs for the configuration: $\ldots D_5 D_{n-3} D_3 D_{n-1} D_1 D_n D_2 D_{n-2} D_4 \ldots$.

There were no other solutions sent in.

The problem was motivated by the special case of three pennies and two nickels, which was in (the late) Joe Konhauser's collection of problems.

2007. [1995: 20] *Proposed by Pieter Moree, Macquarie University, Sydney, Australia.*

Find two primes p and q such that, for all sufficiently large positive real numbers r, the interval $[r, 16r/13]$ contains an integer of the form

$$2^n, \quad 2^n p, \quad 2^n q, \quad \text{or} \quad 2^n pq$$

for some nonnegative integer n.

Solution by Peter Dukes, student, University of Victoria, B.C.

The prime numbers $p = 3$ and $q = 13$ solve the problem. To prove this, it suffices to find an increasing sequence of positive integers $s_1, s_2, \ldots$ such that each s_i is of one of the forms stated in the problem, and $s_{i+1}/s_i \leq 16/13$ for all $i = 1, 2, \ldots$. Then, given any real number $r \geq s_1$, the integer s_k (where $k = \min\{i \in \mathbb{Z}^+ : r \leq s_i\}$), is no more than $16r/13$. For if $s_k \notin [r, 16r/13]$, then $s_k/s_{k-1} \geq 16/13$, contrary to construction. Of course if $k = 1$ then $r = s_1$; so $s_1 \in [r, 16r/13]$. Now, consider the sequence

$$\{s_i\} : \ 24,\ 26,\ 32,\ 39, \qquad 48,\ 52,\ 64,\ 78,\ \ldots$$
$$\ldots,\ 3 \cdot 2^{m+3}, \quad 13 \cdot 2^{m+1}, \quad 2^{m+5}, \quad 3 \cdot 13 \cdot 2^m, \ \ldots .$$

It is a simple matter to verify that the ratio of consecutive terms s_{i+1}/s_i does not exceed $16/13$ for any $i \in \mathbb{Z}^+$. Thus, for all $r \geq 24$, the interval $[r, 16r/13]$ contains an integer of the form 2^n, $2^n p$, $2^n q$, or $2^n pq$. ■

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, British Columbia; ROBERT B. ISRAEL, University of British Columbia; DAVID E. MANES, State University of New York, Oneonta; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One other solution contained a correct pair of primes, but the editor could not decipher the proof that they worked.

Most solvers simply found a single pair of primes, some of which allow the constant $16/13$ to be replaced by a slightly smaller constant. As Engelhaupt, Hess and the proposer point out, the best you can do in this direction is that $16/13$ can be replaced by any real number greater than $\sqrt[4]{2}$. Hess says how: find primes "near" $2^{k_1+1/4}$ and $2^{k_2+1/2}$ for integers k_1 and k_2 (and he gives the example $p = 4871$, $q = 181$, which gets you within $.00013$ of $\sqrt[4]{2}$). More precisely, if

$$p = 2^{k_1+1/4} \cdot t_1 \qquad \text{and} \qquad q = 2^{k_2+1/2} \cdot t_2 \tag{1}$$

where k_1, k_2 are positive integers and t_1, t_2 are real numbers close to 1, then $2^{k_1+k_2} < p \cdot 2^{k_2} < q \cdot 2^{k_1} < pq$, and the ratios of consecutive terms are

$$\sqrt[4]{2} \cdot t_1, \quad \sqrt[4]{2} \cdot \frac{t_2}{t_1}, \quad \sqrt[4]{2} \cdot t_1, \quad \sqrt[4]{2} \cdot \frac{1}{t_1 t_2},$$

all of which can be made arbitrarily close to $\sqrt[4]{2}$. The reason (1) is possible is because (by the Prime Number Theorem) there is always a prime between n and nt for any given $t > 1$, if n is big enough. By increasing the number of primes allowed, with an analogous change in the kind of integers you want the interval to contain, the proposer can get the interval down to $[r, \mu r]$ for any given $\mu > 1$.

Subsequent to submitting the problem the proposer was able to generalize it. He writes:

For given $\mu > 1$ we claim that there exists squarefree D such that the consecutive integers $n_1, n_2, n_3, \ldots$ of the form $2^\alpha d$, with $d|D$ satisfy $n_{i+1}/n_i \leq \mu$ for all i sufficiently large. It suffices to find an infinite subsequence $\{m_i\}_{i=1}^{\infty}$ such that $m_{i+1}/m_i \leq \mu$ for all i sufficiently large. Let $3 = p_1 < p_2 < p_3 < \ldots$ be the sequence of odd consecutive primes. For $n \geq 1$ let p_{i_n} denote the smallest prime exceeding q^n and let p_{j_n} denote the greatest prime less than q^{n+1}. Using the Prime Number Theorem it follows that $\lim_{i\to\infty} \frac{p_{i+1}}{p_i} = 1$, so there exists n such that $p_i/p_{i+1} \leq \mu$ for $i = i_n, \ldots, j_n + 1$. Put $q_0 = 2^n$, $q_1 = p_{i_n}, q_2 = p_{1+i_n}, \ldots, q_s = p_{j_n}$ and $D = \prod_{i=1}^{s} q_i$. Put $m_{a+r(s+1)} = 2^r q_a$, for $0 \leq a \leq s$, $r \geq 0$. Notice that $1 < m_{i+1}/m_i \leq \mu$ for every $i \geq 1$. Thus the claim holds with $D = \prod_{i=1}^{s} q_i$.

Remark. A similar result holds with 2 replaced by an arbitrary prime.

2008. [1995: 20] *Proposed by Jun-hua Huang, The Middle School Attached To Hunan Normal University, Changsha, China.*

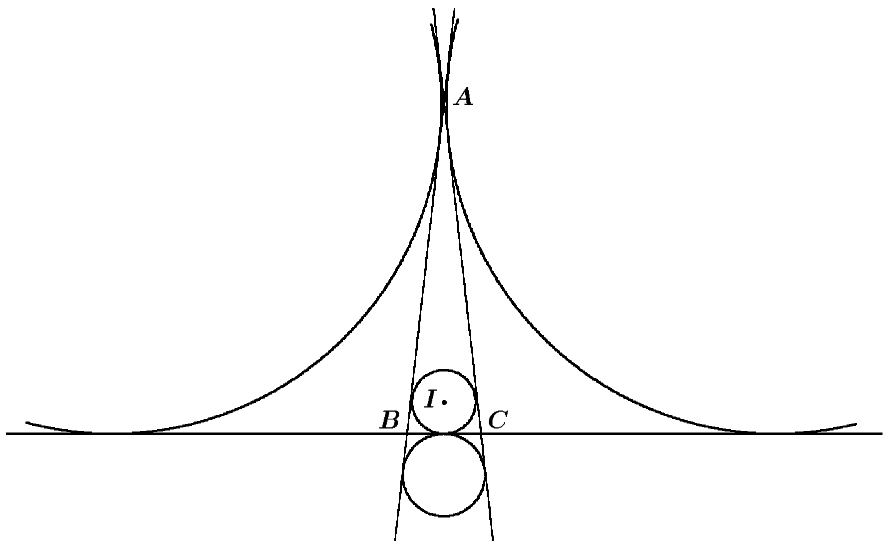
Let I be the incentre of triangle ABC, and suppose there is a circle with centre I which is tangent to each of the excircles of ΔABC. Prove that ABC is equilateral.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

We solve the problem assuming that the circle with centre I is tangent either externally or internally to all the excircles. Without this assumption the problem is not true; below we give a counterexample. (The triangle ABC in the figure has sides $2, 9, 9$, and of course is not equilateral.)

Assume first there is a circle $\mathcal{C}$ with centre I tangent externally to all the excircles [i.e., $\mathcal{C}$ does not contain the excircles — *Ed.*]. Then according to Feuerbach's theorem the circle $\mathcal{C}$ is the nine-point circle of ABC. Therefore $\mathcal{C}$ is also tangent to the incircle of ABC, but since I is the centre of $\mathcal{C}$, the circle $\mathcal{C}$ and the incircle of ABC must coincide. It follows that the triangle ABC is equilateral.

Now let $\mathcal{C}$ be the circle with centre I and tangent internally to all the excircles. Let DE and FG be the chords of $\mathcal{C}$ containing the segments AB and AC respectively. Since $\mathcal{C}$ and the incircle of ABC are concentric, the lines DE and FG are symmetric to each other with respect to the line AI. Therefore, since the excircles lying opposite to B and C are uniquely determined by the chords DE, FG and the circle $\mathcal{C}$, they also have to be symmetric



to each other with respect to the line AI. Thus $r_B = r_C$. Analogously we show that $r_C = r_A$, which implies that ABC has to be equilateral, as we wished to show. ∎

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; ASHISH KR. SINGH, Kanpur, India; D. J. SMEENK, Zaltbommel, The Netherlands; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one incorrect solution received.

Pompe was the only solver to find the counterexample. Everyone else either ignored the possibility, or (in a couple of cases at least) believed that it could not occur! Half the solvers considered both cases of tangency, that is, where the circle is tangent externally to all the excircles or internally to all of them, and the others considered only one.

2009. [1995: 20] *Proposed by Bill Sands, University of Calgary.*

Sarah got a good grade at school, so I gave her N two–dollar bills. Then, since Tim got a better grade, I gave him just enough five–dollar bills so that he got *more* money than Sarah. Finally, since Ursula got the best grade, I gave her just enough ten–dollar bills so that she got *more* money than Tim. What is the maximum amount of money that Ursula could have received? (This is a variation of problem 11 on the 1994 Alberta High School Mathematics Contest, First Part; see the January 1995 Skoliad Corner [1995: 6].)

Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.

The most Ursula could receive is $2N + 14$ dollars.

Sarah received $2N$ dollars and Tim received $2N+1$, $2N+2$, $2N+3$, $2N+4$ or $2N+5$ dollars depending on what residue class N belongs to modulo 5. But since Tim gets \$5 bills his amount is divisible by 5. Ursula will then receive either \$5 more than Tim (if Tim's amount is not divisible by 10) or \$10 more than Tim (if Tim's amount is divisible by 10). Clearly $2N+5$ is odd, thus not divisible by 10. So the maximum occurs when Tim receives $2N+4$ dollars, which means $2N \equiv 1 \bmod 5$, i.e. $N \equiv 3 \bmod 5$, and that maximum is $2N + 14$. ■

Also solved by HAYO AHLBURG, Benidorm, Spain; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; PAUL COLUCCI, student, University of Illinois; PETER DUKES, student, University of Victoria, B.C.; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; J.K. FLOYD, Newnan, Georgia; TOBY GEE, student, The John of Gaunt School, Trowbridge, England; RICHARD K. GUY, University of Calgary; DAVID HANKIN, John Dewey High School, Brooklyn, New York; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; J. A. MCCALLUM, Medicine Hat, Alberta; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JURGËN SEIFFERT, Berlin, Germany; and the proposer. Three other readers sent in solutions which the editor judges are not precise enough.

Here Ursula gets at most \$14 more than Sarah, which is 14/15 of the obvious maximum difference \$5 + \$10 = \$15. How small can this ratio be, if we replace the denominations \$2, \$5, \$10 by three other positive integers? (Choosing $N, 2N-1, 2N$ for large N gets it down arbitrarily close to 3/4.)

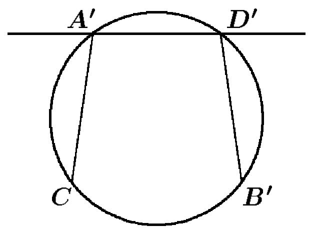
2010. [1995: 20] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

In triangle ABC with $\angle C = 2\angle A$, line CD is the internal angle bisector (with D on AB). Let S be the centre of the circle tangent to line CA (produced beyond A) and externally to the circumcircles of triangles ACD and BCD. Prove that $CS \perp AB$.

Composite of solutions by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore and Roland H. Eddy, Memorial University, St. John's, Newfoundland.

First, note that $\angle BCD = \angle CAD$, so that the line BC is tangent to the circumcircle of ΔACD. Next, perform an inversion with respect to C.

The line BC inverts into the line $B'C$, so that the circumcircle of ΔACD inverts into the line $A'D'$, which is parallel to $B'C$. The circumcircle of ΔBCD inverts into the line $B'D'$. The line AD inverts into the circumcircle of the quadrilateral $CA'D'B'$. Finally, the circle tangent to the line CA, and externally tangent to the circumcircles of ΔACD and ΔBCD inverts into a circle Γ, which is tangent to the line segments $A'C$, $A'D'$ and $B'D'$ as shown in the diagram.



Now, $\angle BCD = \angle ACD$, so that $\angle B'CD' = \angle A'CD'$. Thus $B'D' = A'D'$. Since $A'D'$ and $B'C$ are parallel, we have $A'D' = B'D' = A'C$. Hence, the circumcentre of the quadrilateral $A'D'B'C$ lies on the angle bisectors of $\angle CA'D'$ and $\angle A'D'B'$. These two angle bisectors intersect at the centre of Γ. Thus, quadrilateral $A'D'B'C$ and Γ are concentric. Let their common centre be O.

Thus, C, O and S' (the inverse of S) are collinear, and so CS is perpendicular to AB. ∎

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; KEE-WAI LAU, Hong Kong; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

2012. [1995: 52] *Proposed by K. R. S. Sastry, Dodballapur, India.*

Prove that the number of primitive Pythagorean triangles (integer-sided right triangles with relatively prime sides) with fixed inradius is always a power of 2.

Solution by Carl Bosley, student, Topeka, Kansas

The formula for the inradius of a right triangle, $r = (a + b - c)/2$, where a, b are the legs and c is the hypotenuse, together with the formula $a = 2mn$, $b = m^2 - n^2$, $c = m^2 + n^2$, for the sides of the triangle, where m and n are relatively prime and not both odd, gives $r = n(m - n)$.

Let p be a prime which divides r. If $p = 2$, p can divide n but not $m - n$, since then m and n would either be both even or both odd. If p is not 2, then p must divide n or $m - n$, but cannot divide both, since m and n would have a common factor. Since each prime p other than 2 which divides r divides n or $m - n$, but not both, and each combination of choices produces a pair m, n which generates a right triangle, so if r has k distinct prime factors greater than 2, there are 2^k primitive Pythagorean triangles with inradius r. ∎

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; DAVID DOSTER, Choate Rose-

mary Hall, Wallingford, Connecticut; H. ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; TOBY GEE, student, The John of Gaunt School, Trowbridge, England; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, British Columbia; JAMSHID KHOLDI, New York, N. Y.; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; HEINZ–JURGËN SEIFFERT, Berlin, Germany; LAWRENCE SOMER, Catholic University of America, Washington, D. C.; CHRIS WILDHAGEN, Rotterdam, The Netherlands: and the proposer. There were seven incorrect solutions received, most of which did not handle the case $p = 2$ *correctly.*

Kholdi points out that the problem is solved on page 43 of Sierpiński's Pythagorean Triangles, *published by the Graduate School of Science, Yeshiva University, 1962.*

2013. [1995: 52] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Given is a convex n-gon $A_1A_2 \ldots A_n$ ($n \geq 3$) and a point P in its plane. Assume that the feet of the perpendiculars from P to the lines A_1A_2, A_2A_3, ... , A_nA_1 all lie on a circle with centre O.

(a) Prove that if P belongs to the interior of the n-gon, so does O.

(b) Is the converse to (a) true?

(c) Is (a) still valid for nonconvex n-gons?

Solution by Jerzy Bednarczuk, Warszawa, Poland.

(a) We show that (a) is true for a polygon that is not necessarily convex (so we will have treated (a) and (c) at the same time). Let us call the given circle C. We first see that when P is inside the n-gon, then it must also be inside C. The half-line on OP starting at P and going away from O intersects some side of the n-gon, say A_iA_{i+1}, at a point S. If P were exterior to C, then the circle with diameter PS would lie in the exterior of C. Since the latter circle is the locus of points B with $\angle PBS = 90°$, the foot of the perpendicular from P to A_iA_{i+1} would then lie outside of C, contrary to the definition of C. Also by definition, P cannot lie on C so we conclude that P must lie inside C as claimed. By way of contradiction we now assume that O lies outside of the given n-gon. Since P is inside, the segment OP would intersect some side of the n-gon, say A_jA_{j+1}, at some point T. Since P is also inside C, the circle whose diameter is PT would be contained inside C, which would force the foot of the perpendicular from P to A_jA_{j+1} to lie inside C contrary to its definition. We conclude that O lies inside the given n-gon as desired. ■

(b) No. We can find a counterexample for any $n > 2$! First draw a circle C and then choose any point P lying OUTSIDE of this circle. Next take n points $B_1, B_2, \ldots, B_n$ on the circle C and through each B_i draw the line perpendicular to the line PB_i. Those lines will form an n-gon which will contain O if you choose each B_i so that the half-plane determined by the line perpendicular to PB_i will contain all other B_j together with O. ∎

Also solved by the proposer.

2014. [1995: 52] *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Show that the polynomial

$$2\left(x^7+y^7+z^7\right)-7xyz\left(x^4+y^4+z^4\right)$$

has $x+y+z$ as a factor.

(b)* Is the remaining factor irreducible (over the complex numbers)?

I. *Solution to* (a) *by Jayabrata Das, Calcutta, India.*

Let $f(x,y,z) = 2\left(x^7+y^7+z^7\right)-7xyz\left(x^4+y^4+z^4\right)$. If we can show that $f(x,y,z)=z$ when $x+y+z=0$, we are done.

We know, for $x+y+z=0$, that $x^3+y^3+z^3=3xyz$. Thus

$$\begin{aligned} & x^7+y^7+z^7+x^3y^4+x^3z^4+y^3z^4+y^3x^4+z^3y^4+z^3x^4 \\ &\quad = \left(x^3+y^3+z^3\right)\left(x^4+y^4+z^4\right) \\ &\quad = 3xyz\left(x^4+y^4+z^4\right) \end{aligned}$$

so that

$$\begin{aligned} x^7+y^7+z^7 &= 3xyz\left(x^4+y^4+z^4\right) \\ &\qquad -x^3y^4-x^3z^4-y^3z^4-y^3x^4-z^3y^4-z^3x^4 \end{aligned}$$

Therefore

$$\begin{aligned} f(x,y,z) &= 2\left(x^7+y^7+z^7\right)-7xyz\left(x^4+y^4+z^4\right) \\ &= 6xyz\left(x^4+y^4+z^4\right) \\ &\quad -2\left(x^3y^4+x^3z^4+y^3z^4+y^3x^4+z^3y^4+z^3x^4\right) \\ &\quad -7xyz\left(x^4+y^4+z^4\right) \\ &= -xyz\left(x^4+y^4+z^4\right) \\ &\quad -2x^3y^3(x+y)-2y^3z^3(y+z)-2z^3x^3(z+x) \\ &= -xyz\left(x^4+y^4+z^4\right)+2x^3y^3z+2xy^3z^2+2x^3yz^3 \\ &= -xyz\left(x^4+y^4+z^4-2x^2y^2-2y^2z^2-2z^2x^2\right) \\ &= -xyz\left(\left(x^2+y^2+z^2\right)^2-4\left(x^2y^2+y^2z^2+z^2x^2\right)\right). \end{aligned}$$

Since $x^2+y^2+z^2=-2(xy+yz+zx)$, we now have that

$$\begin{aligned} f(x,y,z) &= -xyz\left(4\left(xy+yz+zx\right)^2-4\left(x^2y^2+y^2z^2+z^2x^2\right)\right) \\ &= -4xyz\left(2x^2yz+2xy^2z+2xyz^2\right) \\ &= -8xyz\left(xyz(x+y+z)\right) = 0. \quad ∎ \end{aligned}$$

II. *Solution to* (a) *by Cyrus Hsia, student, University of Toronto, Toronto, Ontario.*
Consider the sequence $a_n = x^n + y^n + z^n$. The characteristic equation with roots x, y, z, is

$$a^3 - Aa^2 + Ba - C = 0,$$

where $A = x + y + z$, $B = xy + yz + zx$ and $C = xyz$.

The sequence $\{a_n\}$ follows the recurrence relation:

$$a_{n+3} = A\, a_{n+2} - B\, a_{n+1} + C\, a_n \,.$$

Now, we have

$$\begin{aligned} a_0 &= x^0 + y^0 + z^0 = 3\,, \\ a_1 &= x^1 + y^1 + z^1 = A\,, \\ a_2 &= x^2 + y^2 + z^2 = (x+y+z)^2 - 2(xy+yz+zx) = A^2 - 2B \end{aligned}$$

From the recurrence relation, we see:

$$\begin{aligned} a_3 &= A\, a_2 - B\, a_1 + C\, a_0 \\ &= A^3 - 2AB - AB + 3C \\ &= A\, k_3 + 3C, \quad \text{where } k_3 \text{ is some term in } x,\ y \text{ and } z \end{aligned}$$

Similarly

$$\begin{aligned} a_4 &= A\, k_4 + 2B^2, \quad \text{where } k_4 \text{ is some term in } x,\ y \text{ and } z, \\ a_5 &= A\, k_5 - 5BC, \quad \text{where } k_5 \text{ is some term in } x,\ y \text{ and } z, \\ a_6 &= A\, k_6 - 2B^3 + 3C^2, \quad \text{where } k_6 \text{ is some term in } x,\ y \text{ and } z, \\ a_7 &= A\, k_7 + 7B^2C, \quad \text{where } k_7 \text{ is some term in } x,\ y \text{ and } z. \end{aligned}$$

Thus,

$$\begin{aligned} & 2\left(x^7 + y^7 + z^7\right) - 7xyz\left(x^4 + y^4 + z^4\right) \\ &\quad = 2a_7 - 7C\, a_4 \\ &\quad = a\left(A\, k_7 + 7B^2C\right) - 7C\left(A\, k_4 + 2B^2\right) \\ &\quad = A\, k \end{aligned}$$

where k is some term in x, y and z; that is,
$x + y + z$ divides $2\left(x^7 + y^7 + z^7\right) - 7xyz\left(x^4 + y^4 + z^4\right)$. ∎

Part (a) *was also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ADRIAN CHAN, Grade 8 student, Upper Canada College, Toronto, Ontario; TIM CROSS, Wolverley High School, Kidderminster, U. K.; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; J. A. MCCALLUM,* *Medicine Hat, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incorrect solution to part* (b) *was received.*

Other solvers of part (a) *made use of computer algebra, properties of roots, the substitution* $x = -y - z$, *or direct division. Some solvers pointed out that computer algebra failed to factorize the remaining factor. The editor* (*Shawyer*) *tried using* **DERIVE** *on a MSDOS 486 66MHZ computer. The factorisation stopped with no factors after 155 seconds.* **MAPLE** *also failed to find any factors. McCallum commented "An asterisk on a question of Klamkin's is equivalent to a* **DO NOT ENTER** *sign!"*

2015. [1995: 53 and 129 (Corrected)] *Proposed by Shi-Chang Chi and Ji Chen, Ningbo University, China.*

Prove that

$$\Big(\sin(A) + \sin(B) + \sin(C)\Big)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \geq \frac{27\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle.

I. *Solution by Douglass L. Grant, University College of Cape Breton, Sydney, Nova Scotia, Canada.*

If $A = B = C = \pi/3$, equality obtains. It then suffices to show that each factor has an absolute minimum at the point. Note that $C = \pi - (A + B)$. Let $S = \{(A, B) : A > 0,\ B > 0\ A + B < \pi\}$.

Let $f(A, B) = \dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{\pi - (A + B)}$. Then f is unbounded on S. So, if there is a unique critical point for f on S, it must be an absolute minimum.

Now, $F_A(A, B) = -\dfrac{1}{A^2} + \dfrac{1}{\Big(\pi - (A + B)\Big)^2} = 0$ implies that $\pi - (A + B) = A$. Similarly, $F_B(A, B) = 0$ implies that $\pi - (A + B) = B$, and so that $A = B$. Hence $A = B = \pi - (A + B) = \frac{\pi}{3}$.

Let $g(A, B) = \sin(A) + \sin(B) + \sin\Big(\pi - (A + B)\Big) = \sin(A) + \sin(B) + \sin(A + B)$. We now obtain that $0 = g_A(A, B) = \cos(B) + \cos(A + B)$, $0 = g_B(A, B) = \cos(A) + \cos(A + B)$, so that $\cos(A) = \cos(B)$. Since no two distinct angles in $(0, \pi)$ have equal cosines, we have that $A = B$.

Then $0 = \cos(A) + \cos(2A) = 2\cos^2(A) + \cos(A) - 1 = \Big(2\cos(A) - 1\Big)\Big(\cos(A) + 1\Big)$. Since $\cos(A)$ cannot have the value -1, it must then have value $\frac{1}{2}$, and so we have $A = B = C = \frac{\pi}{3}$. ∎

II. *Solution by the proposers.*
Let $y(x) = x^{-1/3}\cos(x)$ for $0 < x \le \frac{\pi}{2}$. Differentiating twice yields

$$\begin{aligned} x^{7/3}\sec(x).y''(x) &= \frac{2x\tan(x)}{3} + \frac{4}{9} - x^2 \\ &> \frac{2x}{3}\left(x + \frac{x^3}{3}\right) + \frac{4}{9} - x^2 \\ &= \frac{2}{9}\left(x^2 - \frac{3}{4}\right)^2 + \frac{23}{72} > 0\,. \end{aligned}$$

By the AM–GM inequality and the Jensen inequality, we have

$$\begin{aligned} \sum \sin(A) \times \sum \frac{1}{A} &= 4\prod \cos(A/2) \times \sum \frac{1}{A} \\ &\ge \frac{6\prod \cos(A/2)}{\prod \left(\frac{A}{2}\right)^{1/3}} \\ &\ge 6\left(\frac{\cos\left(\frac{\pi}{6}\right)}{\left(\frac{\pi}{6}\right)^{1/3}}\right)^3 = \frac{27\sqrt{3}}{2\pi}. \quad \blacksquare \end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and BOB PRIELIPP, University of Wisconsin–Oshkosh. Their solutions were based on known geometric inequalities. Also some readers wrote in after the initial publication of the problem, pointing out that the original result could not be true, and suggesting possible corrections. Thank you.