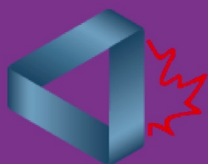


The top portion of the cover features a photograph of the Aurora Borealis (Northern Lights) in shades of green and blue, swirling across a dark night sky. Below the lights, the dark silhouettes of a forest of evergreen trees are visible against the horizon.

Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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EDITORIAL

I use AI daily, in a variety of ways. I use it as a souped-up Google to look up terms and search for content, as a language tutor to create exercises for my French practice, to soften my Slavic emails. I recently used it extensively to plan a trip across Quebec, getting suggestions for places to visit and possible itineraries to plan. Of course, it is not perfect: it makes mistakes and produces outdated information (such as ferry schedules). But it is a huge time saver because it aggregates information that I would otherwise have to look up separately, one piece at a time.

What I don't use it for is doing math[†] I simply don't have much use for it. The problem solving I do, I do for enjoyment. Why would I hand my hobby over to AI to enjoy instead? I would like AI to speed up mundane tasks, automate boring and repetitive activities and add efficiency where necessary or desired. (Let's be clear, though: it has to be desired, because time I enjoy wasting is not wasted.) But problem solving is a way to hone your skills, play with puzzles and exercise your brain. We wouldn't put a robot on a treadmill and claim that we had gotten a cardiovascular workout. Similarly, giving a *Cruz* problem to AI and asking it for a solution isn't really benefiting the problem solver at all. So why the rant? Because AI *is* getting better at math and in the last two issues of *Cruz* we have seen submissions that were clearly not just AI-inspired, but more AI than human. This entirely misses the point of a problem-solving journal, where the process of problem solving *is* the point. AI is obviously changing the way we work, but let's use it for good, not as a replacement for the joy of doing things ourselves.

On a similar note, starting with the next issue, we will no longer count solutions that rely heavily on computer algebra systems (such as Maple and Mathematica) as correct. The principle is the same as it has always been for students of the discipline: you need to show *your work* for a solution to count as your own.

We survived graphing calculators. We survived algebraic software. We will survive AI. As an avid knitter, I am certain that no industrial alternative can replace the human desire to make things with our own hands and brains.

Kseniya Garaschuk

[†] I mean actually doing the math itself, because I do use it for example for typesetting: AI is very useful for turning Word or PDF files into L^AT_EX, with remarkably few errors requiring manual correction.

MATHEMATTIC

No. 78

The problems in this section are intended for students at the secondary school level.

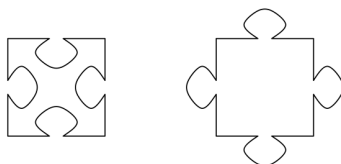
Click here to submit solutions, comments and generalizations to any problem in this section.

*To facilitate their consideration, solutions should be received by **December 15, 2026**.*

MA386.

- a) Show that the maximum range of an aeroplane is extended by a factor of $1/3$ by a second identical support plane which sets off at the same time to provide in-air refuelling. (The support plane must return safely to the starting point.)
- b) Now suppose there are two identical support planes which can instantaneously refuel each other or the original plane as required. The support planes set off at the same time as the original plane and both must return safely to the starting point. By how much can the maximum range of the plane be extended?

MA387. A simple jigsaw puzzle has 225 pieces and is made from a 15×15 board where each square has area 1. All the internal (i.e. non-edge) pieces are of one of the two types shown and these types alternate along each row and column. If the ratio of the area of all the edge-pieces together to the area of the whole puzzle is $\frac{169}{675}$, how many pieces of each type are there? What is the ratio of the two pieces shown?



MA388. A large board has 1000 + signs and 999 – signs written on it. Any two symbols can be deleted provided they are replaced as follows:

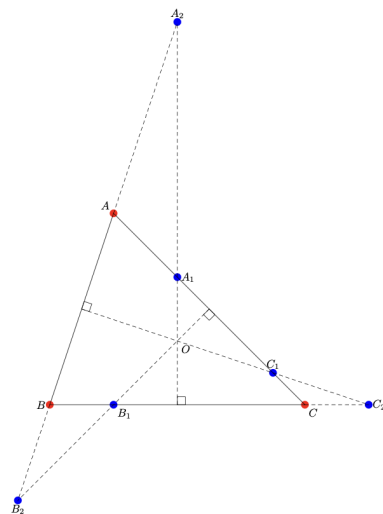
- if the deleted symbols are the same, they are replaced by a +
- if the deleted symbols are different they are replaced by a –

Repeat this process until there is only one symbol left. Which symbol is it and why?

MA389. Let n be a three-digit number and let m be the number obtained by reversing the order of the digits in n . Suppose that $m \neq n$ and that $n + m$ and $n - m$ are both divisible by 7. Find all such pairs n and m .

MA390. *Proposed by Michael Friday.*

A non-right scalene triangle ABC has circumcenter O . The right bisector of BC meets sides CA, AB at A_1, A_2 ; that of CA meets AB, BC at B_2, B_1 ; and that of AB meets BC, CA at C_2, C_1 . What is the probability that a triangle chosen from the set of triangles formed from the points $A, B, C, A_1, B_1, C_1, A_2, B_2, C_2$ has O as its orthocenter?



.....

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

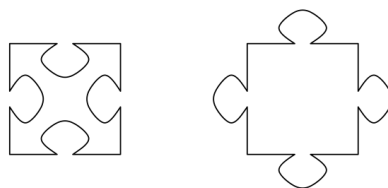
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 décembre 2026**.

MA386.

- a) Montrez que la distance maximale qu'un avion peut parcourir est augmentée d'un facteur de $1/3$ lorsqu'un deuxième avion de soutien identique décolle en même temps que lui pour lui fournir du ravitaillement en vol. (L'avion de soutien doit pouvoir revenir en toute sécurité à son point de départ.)
- b) Supposons maintenant qu'il y ait deux avions de soutien identiques qui peuvent se ravitailler mutuellement ou ravitailler l'avion initial, selon les besoins, de manière instantanée. Les deux avions de soutien décollent en même temps que l'avion initial et doivent tous deux pouvoir revenir en toute sécurité à leur point de départ. De combien la distance maximale que l'avion peut parcourir peut-elle être augmentée?

MA387. Un casse-tête simple comporte 225 pièces et est constitué d'un plateau de 15×15 cases, chaque case ayant une aire de 1. Toutes les pièces intérieures (c'est-à-dire les pièces qui ne sont pas sur le bord) sont de l'un des deux types illustrés, et ces deux types alternent le long de chaque ligne et de chaque colonne. Si le rapport de l'aire totale des pièces situées sur le bord à l'aire totale du casse-tête est $\frac{169}{675}$, combien y a-t-il de pièces de chaque type? Quel est le rapport des aires des deux pièces illustrées?



MA388. Un grand tableau porte 1000 signes $+$ et 999 signes $-$. On peut effacer deux signes quelconques, à condition de les remplacer selon les règles suivantes :

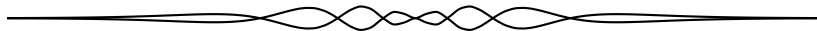
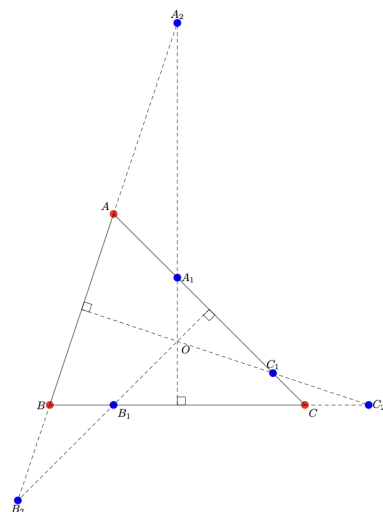
- si les deux signes effacés sont identiques, on les remplace par un signe $+$;
- si les deux signes effacés sont différents, on les remplace par un signe $-$.

On répète ce processus jusqu'à ce qu'il ne reste plus qu'un seul signe. Quel est ce signe et pourquoi ?

MA389. Soit n un nombre à trois chiffres et soit m le nombre obtenu en inversant l'ordre des chiffres de n . Supposons que $m \neq n$ et que $n + m$ et $n - m$ soient tous deux divisibles par 7. Déterminez toutes les paires (n, m) satisfaisant ces conditions.

MA390. *Soumis par Michael Friday.*

Soit ABC un triangle scalène non rectangle de centre du cercle circonscrit O . La médiatrice de BC coupe les côtés CA et AB en A_1 et A_2 ; celle de CA coupe AB et BC en B_2 et B_1 ; et celle de AB coupe BC et CA en C_2 et C_1 . Quelle est la probabilité qu'un triangle choisi parmi l'ensemble des triangles formés à partir des points $A, B, C, A_1, B_1, C_1, A_2, B_2, C_2$ ait O pour orthocentre ?



MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2026: 52(3), p. 111–112.

Note from the Editor. In this issue, we seem to have encountered many incomplete submissions, resulting from either missed cases or missing justifications. Please take care in considering all relevant information and potential problem conditions when writing up your solution and, as always, show your work.

MA361. *Proposed by Nguyen Van An.*

There are 1001 boxes placed around a circle, labeled $B_1, B_2, \dots, B_{1001}$ clockwise. Initially box B_i contains i balls ($i = 1, 2, \dots, 1001$). Fix a positive integer k . A move consists of choosing k balls from k different nonempty boxes (one ball per chosen box) and, for each chosen ball that is in B_n , moving it into B_{n+1} (with the convention $B_{1002} \equiv B_1$). The game ends when all boxes contain the same number of balls.

- (a) For $k = 1$, show how to finish the game.
- (b) For $k = 1$, prove that the number of moves to finish the game is always divisible by 1001.
- (c) For $2 \leq k \leq 1001$, is it always possible to finish the game? Explain and give a method when it is possible.

There were 4 submissions, 1 of which was complete and correct. We present the solution by Rad Sarar Anando.

(a) $k = 1$: How to finish the game.

First note that the total number of balls is

$$1 + 2 + \dots + 1001 = \frac{1001 \cdot 1002}{2} = 1001 \cdot 501.$$

Hence at the end, each box must contain exactly 501 balls.

A simple method: Let c_i be the number of balls in box B_i . As long as there exists a box with more than 501 balls, choose the smallest index n such that $c_n > 501$, and move one ball from that box to the next box (clockwise).

The reason behind this method: If $c_n > 501$ and $c_{n+1} < 501$, then one excess ball reaches a deficient box in one step. If $c_{n+1} \geq 501$ as well, the ball still moves one step forward, but the total “distance” that is number of steps that excess balls have to move through (a nonnegative integer) strictly decreases. Since this distance never increases and cannot become negative, eventually every deficient box is filled. Thus the game finishes.

(b) $k = 1$: Prove that the number of moves is always divisible by 1001.

Let $c_i(t)$ be the number of balls in box B_i at move t , where $0t \geq 0$. Define

$$S(t) = \sum_{i=1}^{1001} i \cdot c_i(t) \pmod{1001}.$$

Initially,

$$S(0) = \sum_{i=1}^{1001} i^2 = \frac{1001 \cdot 1002 \cdot 2003}{6}.$$

The numerator contains the factor 1001, so $S(0) \equiv 0 \pmod{1001}$. In the final state, every box contains 501 balls:

$$S_f = 501 \sum_{i=1}^{1001} i = 501 \cdot \frac{1001 \cdot 1002}{2} \equiv 0 \pmod{1001}.$$

Now examine one move. Moving a ball from B_n to B_{n+1} (with $1 \leq n \leq 1000$) changes S by $(n+1) - n = +1$. For $n = 1001$ (moving from B_{1001} to B_1), the change is $1 - 1001 = -1000 \equiv +1 \pmod{1001}$. Thus each single move increases S by exactly 1 modulo 1001.

After m moves, we have $S(m) \equiv S(0) + m \equiv m \pmod{1001}$. At termination $S(m) = S_f \equiv 0$, so $m \equiv 0 \pmod{1001}$. Hence the total number of moves is always a multiple of 1001.

(c) For $2 \leq k \leq 1001$: Is it always possible to finish?

Case 1: $k = 1001$. Every move takes one ball from each box. Box B_n loses one ball (to B_{n+1}) and gains one ball (from B_{n-1}). Hence every box's count remains unchanged. The initial configuration $(1, 2, \dots, 1001)$ is not the target $(501, \dots, 501)$, so the game can never finish. Thus for $k = 1001$ it is impossible.

Case 2: $2 \leq k \leq 1000$. We show that the game is always possible by simulating the $k = 1$ strategy inside a k -move.

Run the terminating algorithm for $k = 1$ from part (a). Whenever that algorithm would move one ball from B_a to B_{a+1} , we perform a k -move as follows:

- Include B_a as one of the chosen boxes.
- Choose the remaining $k - 1$ boxes as follows: if possible, pick boxes that have at least 2 balls (to avoid emptying them prematurely).
- If all other boxes have exactly 1 ball, then allow them to become empty in this move — those balls were not scheduled to move in the $k = 1$ simulation anyway, and the overall progress (reduction of excess) remains unaffected.

Since $k \leq 1000$, and the total number of balls is large, we can always find such $k - 1$ boxes (a brief case analysis or induction on moves confirms this).

Thus every step of the $k = 1$ algorithm can be realized by a valid k -move, and the game finishes after finitely many moves.

Summary for (c): Possible for $1 \leq k \leq 1000$, impossible for $k = 1001$.

MA362. *Proposed by Neculai Stanciu.*

Determine the five-digit numbers $abcde$ and $xyztu$ with distinct digits 1, 2, 3, 7 and 8 such that $4 \cdot abcde = xyztu$.

We received 14 submissions, 4 of which were correct and complete. We present the solution by The Ring Lords.

Using the divisibility rule for 4, the last two digits tu of the number $xyztu$ must be one of 28, 72, 32, or 12.

For the leading digit x , only 7 and 8 are possible, since otherwise dividing by 4 would result in a number with fewer than five digits.

From this, we obtain the following 12 candidates for $xyztu$:

71328, 73128, 81372, 83172, 87132, 81732, 78132, 71832, 83712, 87312, 73812, 78312.

Each of these numbers is then divided by 4. Only for 71328 and 87132 does the quotient consist solely of the digits 1, 2, 3, 7, 8.

Thus, the two solution pairs are:

$$(\overline{abcde}, \overline{xyztu}) = (17832, 71328)$$

and

$$(\overline{abcde}, \overline{xyztu}) = (21783, 87132).$$

MA363. The number 123456789101112...484950 has 91 digits. Wipe some of its digits off so that the largest possible number with the sum of its digits equal to 200 remains.

Originally from the World Federation of National Mathematics Competitions, First International Conference, Problem 32.

There were 8 submissions, only 1 of them was complete and correct. We present the solution by Jianlin Zhang, modified the the editor.

The sum of all the original digits is

$$5(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) + 5 + 10 + 20 + 30 + 40 = 330.$$

The sum of the removed digits is 130.

The fewer digits removed, the larger the remaining number. The minimum number of digits removed is 17:

$$130 = 5(9 + 8 + 7) + 6 + 4 = 5(9 + 8 + 7) + 5 + 5 = 5(9 + 8) + 4(7) + 2(6) + 5.$$

The corresponding larger numbers are as follows. For $5(9 + 8 + 7) + 6 + 4$:

12356101112131415161112021222324252622230313233343536333404142434445444450.

No other possibilities need to be considered: to maximize the number remaining after deleting the digits, we would like to delete a digit as close as possible to the beginning of the number, since that digit will be replaced by its neighbour to the right, which is larger. Since the digits of the original number are increasing (at the beginning), you'd like to delete as small a digit as possible. The smallest digit than can be removed is 4, because the largest sum of 16 digits you can get is

$$5(9 + 8 + 7) + 6 = 126.$$

MA364. Consider a rectangular block with dimensions $m \times n \times r$, where m, n, r are positive integers and $m < n < r$. After the surfaces of the block are painted red, the block is cut into cubes with sides 1 unit. If the sum of the number of the cubes with no red surface and with two red surfaces minus the number of the cubes with one red surface equals 1991, find the values of m, n , and r .

Originally from the World Federation of National Mathematics Competitions, First International Conference, Problem 41.

We received 10 solutions, only 1 of which was correct and complete. We present this distinct solution, submitted by The Ring Lords.

We count cubes according to how many red faces they have.

Case 1: $m = 1$. If $m = 1$, the block becomes a $1 \times n \times r$ prism. The front and back faces are painted, so every small cube lies on at least two painted faces.

Hence:

- Number of cubes with 0 red faces: 0
- Number of cubes with 1 red face: 0
- Number of cubes with 2 red faces: $(n - 2)(r - 2)$

Thus the condition becomes

$$0 + (n - 2)(r - 2) - 0 = 1991,$$

so

$$(n - 2)(r - 2) = 1991.$$

Factorizing:

$$1991 = 1 \cdot 1991 = 11 \cdot 181.$$

Since $r > n > m > 0$, only positive increasing factorizations are valid. The only admissible factor pairs are:

$$(n - 2, r - 2) = (1, 1991) \quad \text{and} \quad (n - 2, r - 2) = (11, 181).$$

Thus:

$$(m, n, r) \in \{(1, 3, 1993), (1, 13, 183)\}.$$

Case 2: $m > 1$. Note that for $m = 2$, we have $a = 0$ and the equation becomes

$$-2bc + 4b + 4c = 1991,$$

which has no integral solutions since the left hand side is even.

For the rest of the values, let

$$a = m - 2, \quad b = n - 2, \quad c = r - 2.$$

Then we classify cubes into 3 cases:

- 0 red faces (interior): abc
- 1 red face (face interiors excluding edges and corners): $2ab + 2ac + 2bc$
- 2 red faces (edges excluding corners): $4a + 4b + 4c$

Thus the expression becomes:

$$abc - (2ab + 2ac + 2bc) + (4a + 4b + 4c) = 1991.$$

Rewriting:

$$abc - 2ab - 2ac - 2bc + 4a + 4b + 4c = 1991.$$

We now factor:

$$(a - 2)(b - 2)(c - 2) = abc - 2ab - 2ac - 2bc + 4a + 4b + 4c - 8.$$

Hence:

$$(a - 2)(b - 2)(c - 2) = 1991 - 8 = 1983.$$

Now factor $1983 = 3 \cdot 661$. Since $r > n > m > 1$, we require $c > b > a > 0$. Thus, the only positive increasing factorization is valid:

$$(a - 2, b - 2, c - 2) = (1, 3, 661).$$

So $(a, b, c) = (3, 5, 663)$, and hence $(m, n, r) = (5, 7, 665)$.

In total there are exactly three solutions:

$$(m, n, r) \in \{(1, 3, 1993), (1, 13, 183), (5, 7, 665)\}.$$

Editor's Comments. Almost all submissions gave a unique solution of $(5, 7, 665)$. One other submission considered the case of $m = 1$, but made a small error resulting in an incomplete solution set.

MA365. $ABCD$ is a parallelogram in which $\angle ABC$ is acute and AD is a tangent to the circle ABC . Prove that

$$BD^2 = AB^2 + 2BC^2.$$

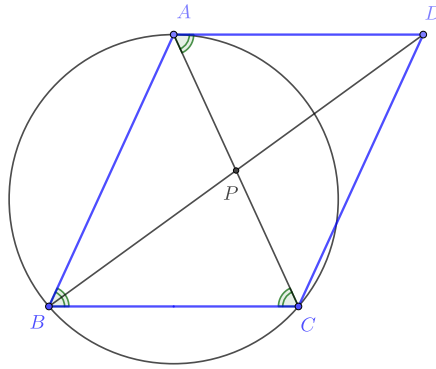
Originally from the World Federation of National Mathematics Competitions, First International Conference, Problem 198.

We received 14 submissions of which 11 were correct and complete. We present two solutions.

Solution 1, by Kousik Sett.

Since AD is tangent to the circumcircle of $\triangle ABC$, by the tangent-chord theorem we get

$$\angle CAD = \angle ABC.$$



From the fact that $ABCD$ is a parallelogram and hence $AD \parallel BC$ we get

$$\angle CAD = \angle ACB.$$

Therefore, $\angle ABC = \angle ACB$, so $\triangle ABC$ is isosceles with $AB = AC$.

Let P be the intersection point of AC and BD . Diagonals in a parallelogram bisect each other, so P is the midpoint of both AC and BD . Since AP is a median in $\triangle ABD$ we can apply Apollonius's Theorem to get

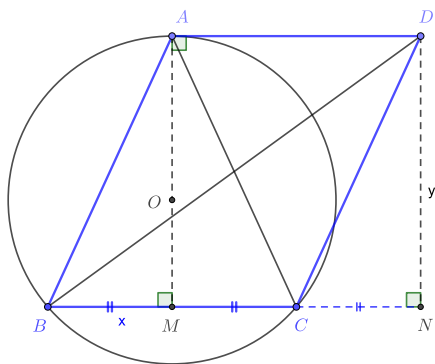
$$AB^2 + AD^2 = 2(AP^2 + BP^2).$$

Substituting $BP = \frac{BD}{2}$, $AP = \frac{AC}{2} = \frac{AB}{2}$ and $AD = BC$ (opposite sides in a parallelogram), we get

$$AB^2 + BC^2 = 2 \left(\frac{AB^2}{4} + \frac{BD^2}{4} \right),$$

which simplifies to $AB^2 + 2BC^2 = BD^2$, as desired.

Solution 2, by Corneliu Manescu-Avram.



Denote the circumcenter of $\triangle ABC$ by O and the midpoint of side BC by M . O is the intersection of the perpendicular bisectors, so $OM \perp BC$. We are given that AD is tangent to the circumcircle, whence $OA \perp AD$. Since $AD \parallel BC$ it follows that A , O and M are collinear; that is, the median AM is also an altitude, and $\triangle ABC$ is isosceles with $AB = AC$.

Denote by N the foot of the perpendicular from D to the line through BC . Then $DN \parallel AM$, so $\triangle DNC$ and $\triangle AMB$ have all sides parallel; since we also have $DC = AB$ it follows that $\triangle DNC \cong \triangle AMB$, and in particular $CN = BM$ and $DN = AM$.

Let $AM = y$ and $BM = x$. Then

$$\begin{aligned} BC &= 2BM = 2x, \\ CN &= BM = x, \\ DN &= AM = y, \\ BN &= BC + CN = BC + BM = 2x + x = 3x. \end{aligned}$$

The Pythagorean Theorem in $\triangle DNB$ gives us

$$BD^2 = BN^2 + ND^2 = (3x)^2 + y^2 = 9x^2 + y^2.$$

The Pythagorean Theorem in $\triangle AMB$ gives us

$$AB^2 = AM^2 + BM^2 = y^2 + x^2.$$

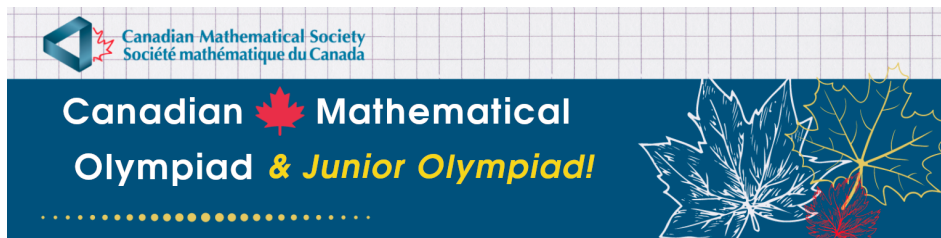
Therefore $AB^2 + 2BC^2 = (y^2 + x^2) + 2(2x)^2 = 9x^2 + y^2 = BD^2$, as desired.



Competition Highlights

Canadian Mathematical Olympiad and Junior Olympiad

by Paweł Prałat and James Rickards



The Canadian Mathematical Olympiad (CMO) is an annual, invitational, proof-based competition for Canadian students. It is Canada's premier national advanced mathematics competition. Students attempt to solve 5 problems in three hours, with each problem graded on a scale from 0 to 7. In 2020, the CMS introduced the Canadian Junior Mathematical Olympiad (CJMO), also by invitation only, a variant specifically for students in grade at most 10. These 3-hour competitions are held each March at a selected time and date (by default, the second Thursday of March). All official participants write at the same time and are proctored by their local school faculty or staff. For more information visit <https://cms.math.ca/competitions/cmo/>.

The CMO is an important contest for students with international aspirations, as a good performance leads to the Canadian Team Selection Test, and then onto the International Mathematical Olympiad (IMO) itself. Qualification for the C(J)MO is primarily via the Canadian Open Mathematics Challenge (COMC), an open contest written in late October.

The 2026 CMO was written by 95 students, and the CJMO was written by 18 students. Four Canadian provinces were represented, with the number of contestants as follows:

CMO: AB (11), BC (21), ON (44), QC (8)

CJMO: AB (3), BC (1), ON (10), QC (2)

(note that Canadian citizens residing outside of Canada can also officially write the C(J)MO, accounting for the discrepancy in numbers).

Grading for both contests went relatively smoothly, with a team of 15 mathematicians, including professors, students, and former contestants, contributing their time. The top score on the CMO was 31, achieved by Ansh Agarwal, and the mean score was 11.1. The Matthew Brennan Award for best solution went to Perry Dai for his solutions to both Problem 2 and Problem 5. In Problem 2, he proved a bound better than anticipated, and on Problem 5, he incorporated ideas from algebraic number theory as part of his solution. The top score on the

CJMO was 30, achieved by Haruka Kimura, and the mean score was 17.2. A full breakdown of the marks assigned problem by problem is in Table 1.

Score	P1	P2	P3	P4	P5	Score	P1	P2	P3	P4	P5
7	60	9	22	8	0	7	7	15	7	1	0
6	12	2	3	3	0	6	2	0	3	0	0
5	0	0	4	0	1	5	0	0	0	0	1
4	0	0	7	0	0	4	7	1	1	0	2
3	1	1	1	0	1	3	0	0	1	0	0
2	0	15	40	22	0	2	0	0	0	0	7
1	4	14	5	4	0	1	0	0	1	0	2
0	18	54	13	58	93	0	2	2	5	17	6
Avg	5.25	1.28	3.24	1.28	0.08	Avg	4.94	6.06	4.17	0.39	1.61

(a) CMO

(b) CJMO

Table 1 – C(J)MO score breakdown by problem.

Problem 2 on the CMO (which doubled as problem 4 on the CJMO) turned out to be unexpectedly tricky. It has a straightforward solution, that was likely overlooked by most students searching for more clever ideas!

Problem 2. *There are n types of coins in Wario's gold mine. Each coin of the i th type is worth d_i cents, where $d_1, \dots, d_n$ are distinct positive integers. A positive integer D is denoted lucky if the following holds: For each positive integer k , any collection of coins (containing any number of coins of each type) with a total value of exactly kD cents can be split into k groups, each worth D cents. Does a lucky number necessarily exist?*

For a basic example, imagine $n = 3$, $(d_1, d_2, d_3) = (4, 5, 6)$, and $D = 20$. If $k = 3$, the problem is asking if any combination of 4, 5, 6 cent coins making 60¢ can be split into 3 piles each totalling 20¢.

Playing around further, imagine that we had 4 of each coin. This can be solved by splitting into

$$2 \cdot 4\text{¢} + 2 \cdot 6\text{¢}, \quad 2 \cdot 4\text{¢} + 2 \cdot 6\text{¢}, \quad 4 \cdot 5\text{¢}.$$

However, we can soon realize that we were pretty tightly constrained. In fact, the only way to make out 20¢ using at least one 6¢ coin is to make $2 \cdot 4\text{¢} + 2 \cdot 6\text{¢}$. So, for example, if we had a collection of ten 6¢ coins, we immediately see that we cannot split this collection into 20¢ piles, as each pile will necessarily be a multiple of 6¢, hence 20 is not lucky in this example.

We can see an immediate generalization: if any d_i does not divide D , then D is not lucky. The proof is simple: take $k = d_i$, and consider a collection of D coins each worth $d_i\text{¢}$. Any sub-collection has value being a multiple of d_i , and thus cannot have value D , contradiction.

The above observation rules out many numbers D , but not everything! In particular, it tells us that in searching for a lucky D , we require $\text{lcm}(d_1, d_2, \dots, d_n) \mid D$.

If this holds, then our special case of having only one type of coin in the collection now works: we can indeed always split this into equal piles of value D , since each d_i divides D evenly.

Considering the case where there are coins of multiple types seems trickier, as we may potentially have to combine different values in non-trivial ways. A natural line of thought is to consider the famous Frobenius coin problem: given a collection of infinitely many coins of values $a_1, a_2, \dots, a_n$ where $\gcd(a_1, \dots, a_n) = 1$, what is the largest value that you cannot make exactly using these coins? For example, in the $n = 2$ case, the answer is $a_1 a_2 - a_1 - a_2$. With this intuition, it feels like if D is large enough, then this should help us find the combinations. In essence, we should have enough “room” to play around with combining coins to always make multiples of D . The $\gcd(a_1, \dots, a_n) = 1$ condition is a small issue, but that is taken care of our divisibility condition that $d_i \mid D$ for all i .

While the intuition is reasonable and it does lead to the correct answer, it does not seem to lead to an actual *solution*. One of the main issues is that we do not have control over how many of each coin there is: we allow the pile to contain *any* collection of the coins totalling $kD\mathfrak{c}$. Back to the drawing board!

A key to breaking this barrier is going back to the simplest case of having all coins of the same type. While this does not always need to be true, what if it is close? For example, if you had at least D/d_1 of the coin worth $d_1\mathfrak{c}$, then you could make a group of $d_1\mathfrak{c}$ coins worth $D\mathfrak{c}$, and remove those from the collection. You could then repeat this process: as long as there was some i for which you had at least D/d_i coins of value $d_i\mathfrak{c}$, then you could make a pile worth $D\mathfrak{c}$, and induct backwards. At the final step (that would eventually correspond to the base case of a proof by induction) you would be left with some assortment of the leftover coins, but they would necessarily be worth $D\mathfrak{c}$, and the problem would be solved.

If we cannot do the above, then we have less than D/d_i coins worth $d_i\mathfrak{c}$, hence their total value is less than $D\mathfrak{c}$, and thus the total collection is worth less than $nD\mathfrak{c}$. This provides us a solution when $k \geq n$, but does not help us with $2 \leq k \leq n - 1$. The final clever idea is realizing that, since we have full control over D , why don't we just make D larger, and repeat this idea? For example, if D was replaced by nD , then we could repeat this idea n times, making n piles worth $D\mathfrak{c}$ (each comprised of only a single type of coin), and combine those into our group.

Indeed, this idea does work, and leads to lucky numbers. Formally, consider $d = \text{lcm}(d_1, d_2, \dots, d_n)$ and $D = nd$. We claim (by induction) that any collection worth $kD\mathfrak{c}$ can be split as desired. The base case of $k = 1$ is trivial, so assume that $k \geq 2$. Assume that there are n_i coins worth $d_i\mathfrak{c}$, so that $kD = \sum_{i=1}^n n_i d_i$. Using only coins worth $d_i\mathfrak{c}$ to make piles of value $d\mathfrak{c}$, we form

$$\left\lfloor \frac{n_i}{d/d_i} \right\rfloor \geq \frac{n_i d_i}{d} - 1$$

piles. Thus, the total number of piles (across all i) that are worth exactly $d\mathfrak{c}$ is at

least

$$\sum_{i=1}^n \left(\frac{n_i d_i}{d} - 1 \right) = \frac{kD}{d} - n \geq 2n - n = n,$$

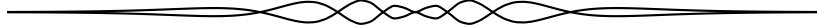
so we can form n of them, combine these piles into one group worth $nd = D$ cents, and remove this from the collection. By induction, we can repeat, proving that D is lucky.

While we have classified some lucky numbers, one can do better! In Perry Dai's solution, he was able to prove that $D = d_1 d_2 \cdots d_n$ is lucky. If our values d_i are pairwise co-prime, this value is significantly smaller than the one we used in the above proof. For that solution, see the official CMO solutions on the CMS website.

Finally, it is natural to ask for the smallest lucky number given each input: call this L . In our above commentary we have noted that

$$\text{lcm}(d_1, d_2, \dots, d_n) \leq L \leq d_1 d_2 \cdots d_n,$$

which does fully solve the problem when the d_i 's are pairwise co-prime. We leave the remaining cases as an open problem for the reader!



OLYMPIAD CORNER

No. 446

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

*To facilitate their consideration, solutions should be received by **December 15, 2026**.*

OC796. Let $P(x)$ be a polynomial of degree 5 and let a and b be nonzero real numbers. Suppose that the remainder of $P(x)$ when divided by $x^3 + ax + b$ is equal to the remainder of $P(x)$ when divided by $x^3 + ax^2 + b$. Determine the value of $a + b$.

OC797. A rectangle $ABCD$ is inscribed in a circle ω with center O . A line ℓ passes through O and intersects the segments BC and AD at E and F , respectively. Let K and L be the intersection points of ℓ with ω , where the points occur on ℓ in the order K, E, F, L . The tangents to ω at K and L intersect the line CD at M and N , respectively. Prove that the points E, F, M, N lie on a common circle.

OC798. Let k and n be natural numbers satisfying $k \leq 2n$. Prove that

$$2^{2n+2} + 2^{k+2} + 1$$

is the square of a natural number if and only if $k = n$.

OC799. The numbers $1, 2, \dots, 2024$ are initially written on a blackboard. Ana and Beto play the following game. On each of her turns, Ana chooses three numbers a, b, c written on the blackboard. On his turn, Beto erases these three numbers and writes in their place one of the numbers

$$a + b - c, \quad a - b + c, \quad -a + b + c,$$

at his choice. The game ends when only two numbers remain on the blackboard, so Ana can no longer make a move. If the sum of the two remaining numbers is divisible by 3, Beto wins; otherwise, Ana wins. Determine which player can guarantee a victory.

OC800. Let $M_2(\mathbb{Z})$ denote the set of 2×2 matrices with integer entries. Let $A \in M_2(\mathbb{Z})$ satisfy $A^2 + 5I = O$, where $I \in M_2(\mathbb{Z})$ and $O \in M_2(\mathbb{Z})$ denote

the identity matrix and the zero matrix, respectively. Prove that there exists an invertible matrix $C \in M_2(\mathbb{Z})$, with $C^{-1} \in M_2(\mathbb{Z})$, such that either

$$CAC^{-1} = \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix}$$

or

$$CAC^{-1} = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}.$$

.....

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 décembre 2026**.*

OC796. Soit $P(x)$ un polynôme de degré 5 et soient a et b des nombres réels non nuls. Supposons que le reste de la division de $P(x)$ par $x^3 + ax + b$ soit égal au reste de la division de $P(x)$ par $x^3 + ax^2 + b$. Déterminez la valeur de $a + b$.

OC797. Un rectangle $ABCD$ est inscrit dans un cercle ω de centre O . Une droite ℓ passe par O et coupe les segments BC et AD en E et F , respectivement. Soient K et L les points d'intersection de ℓ avec ω , les points étant disposés sur ℓ dans l'ordre K, E, F, L . Les tangentes à ω en K et en L coupent la droite CD en M et N , respectivement. Montrez que les points E, F, M et N sont cocycliques.

OC798. Soient k et n des nombres naturels satisfaisant $k \leq 2n$. Montrez que

$$2^{2n+2} + 2^{k+2} + 1$$

est le carré d'un nombre naturel si et seulement si $k = n$.

OC799. Les nombres $1, 2, \dots, 2024$ sont initialement écrits au tableau. Ana et Beto jouent au jeu suivant. À chacun de ses tours, Ana choisit trois nombres a, b, c écrits au tableau. À son tour, Beto efface alors ces trois nombres et les remplace par l'un des trois nombres suivants, qu'il choisit librement :

$$a + b - c, \quad a - b + c, \quad -a + b + c.$$

Le jeu se termine lorsqu'il ne reste plus que deux nombres au tableau, de sorte qu'Ana ne peut plus jouer. Si la somme des deux nombres restants est divisible

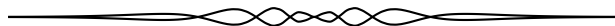
par 3, Beto gagne; sinon, Ana gagne. Déterminez lequel des deux joueurs peut garantir la victoire.

OC800. Soit $M_2(\mathbb{Z})$ l'ensemble des matrices 2×2 à coefficients entiers. Soit $A \in M_2(\mathbb{Z})$ telle que $A^2 + 5I = O$, où $I \in M_2(\mathbb{Z})$ et $O \in M_2(\mathbb{Z})$ désignent respectivement la matrice identité et la matrice nulle. Montrez qu'il existe une matrice inversible $C \in M_2(\mathbb{Z})$, dont l'inverse C^{-1} appartient également à $M_2(\mathbb{Z})$, telle que

$$CAC^{-1} = \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix}$$

ou

$$CAC^{-1} = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}.$$



OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2026: 52(3), p. 124–126.

OC771. Determine the smallest constant c such that the inequality

$$(x+y)^2(x^2+y^2+c) + (1-xy)^2 \geq 0$$

holds for all real numbers x and y . For which values of x and y does equality hold for this smallest constant c ?

Originally from the Austrian Mathematical Olympiad 2024, Final Round, Problem 1.

We received 15 submissions, of which 13 were correct and complete. We present 3 solutions.

Solution 1, by Oliver Geupel.

We show that the smallest constant with the required property is $c = -1$ and that equality holds for $c = -1$ if and only if $x = y = -3^{-1/2}$ or $x = y = 3^{-1/2}$.

Let

$$f(c, x, y) = (x+y)^2(x^2+y^2+c) + (1-xy)^2.$$

On the one hand, it holds

$$f\left(c, 3^{-1/2}, 3^{-1/2}\right) = \frac{4}{3} \left(\frac{2}{3} + c\right) + \frac{4}{9} = \frac{4}{3}(1+c),$$

which is negative for $c < -1$. On the other hand, we have

$$f(-1, x, y) = (x^2 + xy + y^2 - 1)^2 + (x - y)^2 \geq 0,$$

where the equality holds if and only if $x = y$ and $3x^2 = 1$. Hence the result.

Solution 2, by Brian Beasley.

We show that the smallest such constant is $c = -1$.

Let $f_c(x, y) = (x+y)^2(x^2+y^2+c) + (1-xy)^2$. Then

$$f_c(x, y) = (x^2 + xy + y^2 + c)^2 - cx^2 - 2xy - cy^2 - c^2 + 1,$$

so for $c = -1$ we obtain

$$f_{-1}(x, y) = (x^2 + xy + y^2 - 1)^2 + (x - y)^2.$$

Thus the original inequality holds for all real numbers x and y when $c = -1$, with equality if and only if $x^2 + xy + y^2 = 1$ and $x = y$, which occurs precisely when

$$(x, y) = \pm \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right).$$

To show that c cannot be smaller than -1 , we calculate

$$f_c \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right) = \frac{4}{3}(1 + c) < 0$$

for $c < -1$.

Solution 3, by Michel Bataille.

If the inequality holds for all real numbers x and y , then taking $x = y = \frac{1}{\sqrt{3}}$, we must have

$$\frac{4}{3} \left(\frac{2}{3} + c \right) + \frac{4}{9} \geq 0,$$

that is, $c \geq -1$.

Now, consider

$$f(x, y) = (x + y)^2(x^2 + y^2 - 1) + (1 - xy)^2.$$

It is readily checked that

$$f(x, y) = ((x + y)^2 + 1 - xy)^2 - 3(x + y)^2.$$

Since $(x + y)^2 \geq 4xy$, we have $(x + y)^2 + 1 - xy \geq \frac{3(x+y)^2}{4} + 1 > 0$, hence

$$f(x, y) \geq \left(\frac{3(x + y)^2}{4} + 1 \right)^2 - 3(x + y)^2 = \left(\frac{3(x + y)^2}{4} - 1 \right)^2 \geq 0. \quad (1)$$

This completes the proof that -1 is the smallest value of c for which the given inequality holds. Equality holds in (1) calls for

$$((x+y)^2+1-xy)^2-3(x+y)^2 = \left(\frac{3(x+y)^2}{4} + 1 \right)^2 - 3(x+y)^2 = \left(\frac{3(x+y)^2}{4} - 1 \right)^2 = 0,$$

hence

$$(x + y)^2 = \frac{4}{3} \quad \text{and} \quad \left(\frac{7}{3} - xy \right)^2 - 4 = 0.$$

The latter gives $xy = \frac{1}{3}$ (since $xy = \frac{13}{3}$ does not satisfy $4xy \leq (x + y)^2 = \frac{4}{3}$). As a result,

$$xy = \frac{1}{3} \quad \text{and} \quad x + y = \frac{2}{\sqrt{3}} \quad \text{or} \quad -\frac{2}{\sqrt{3}}$$

so that

$$(x, y) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \quad \text{or} \quad (x, y) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right).$$

Conversely, it is readily checked that equality holds when (x, y) is one of these two pairs.

Thus, equality holds if and only if $(x, y) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ or $(x, y) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

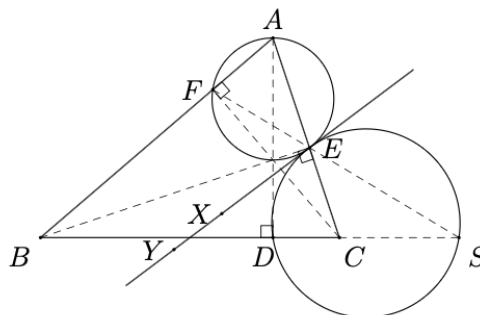
OC772. Let ABC be an acute triangle with $AB > AC$. Let D , E and F denote the feet of its altitudes on BC , AC and AB , respectively. Let S denote the intersection of lines EF and BC . Prove that the circumcircles k_1 and k_2 of the two triangles AEF and DES touch in E .

Originally from the Austrian Mathematical Olympiad 2024, Final Round, Problem 2.

We received 11 submissions, of which 10 were correct and complete. We present 3 solutions.

Solution 1, by Chikara Tsugawa.

All angles are directed modulo π . Let X be a point on the tangent to (AEF) at E . Let Y be a point on the tangent to (DES) at E .



Then,

$$\angle FEX = \angle FAE = \angle BAE.$$

On the other hand,

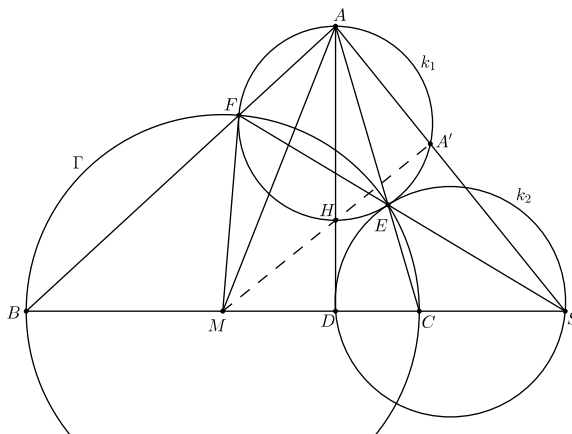
$$\begin{aligned} \angle FEY &= \angle SEY = \angle SED + \angle DEY = \angle SED + \angle DSE = \angle SDE = \angle BDE \\ &= \angle BAE \quad (\text{since } A, B, D, E \text{ are concyclic}) \end{aligned}$$

Thus, $\angle FEX = \angle FEY$. Therefore, E , X , Y are collinear. This completes the proof.

Remark. The proof gives the following more general statement:

Let A , B , D , E be concyclic. Let F be a point on the line AB , and let S be the intersection of EF and BD . Then the circumcircles (AEF) and (DES) are tangent at E .

Solution 2, by Michel Bataille.



Let M be the midpoint of BC , H the orthocenter of $\triangle ABC$ and let Γ be the circle with diameter BC . Since E, F are on Γ the point H of intersection of BE, CF is on the polar of S with respect to Γ . Since AH is perpendicular to BC , this polar is AH and consequently, D is the harmonic conjugate of S with respect to B, C . Since ABC is acute, the point D is between B and C , hence S is not. Also $AB > AC$ implies that B, M, D, C, S are in this order.

Let $\mathbf{I}$ be the inversion with center S transforming B into C . We have $\mathbf{I}(E) = F$ (since $\mathbf{I}(\Gamma) = \Gamma$), hence $\mathbf{I}(k_1) = k_1$. From $\frac{SC}{SB} = \frac{DC}{DB}$, we deduce

$$SC \cdot SB - SD \cdot SC = SB \cdot SD - SB \cdot SC,$$

hence

$$2SB \cdot SC = SD(SB + SC) = 2SD \cdot SM.$$

It follows that $\mathbf{I}(D) = M$ so that $\mathbf{I}(k_2) = MF$.

Since $\mathbf{I}$ preserves tangency, it suffices to prove that the line MF is tangent to k_1 .

Note that k_1 is the circle with diameter AH and $\angle(HA, HF) = \angle(BC, BA)$ (since $HA \perp BC$ and $HF \perp BA$). Since $MB = MF$, we have

$$\angle(FA, FM) = \angle(FB, FM) = \angle(BM, BF) = \angle(BC, BA)$$

and therefore

$$\angle(HA, HF) = \angle(FA, FM),$$

which proves that FM is tangent to k_1 .

Note. H is also the orthocenter of the triangle AMS .

Proof. The point $A' = \mathbf{I}(A)$ is on k_1 , hence $HA' \perp AS$. Also, the circle with diameter SA inverts into the line MA' , hence $MA' \perp SA$. Therefore MA' and HA' coincide and MH is the altitude from M of $\triangle AMS$. The result follows.

Solution 3, by Theo Koupelis.

Let H be the orthocenter of $\triangle ABC$. The points A, F, H, E are concyclic because

$$\angle HFA = \angle HEA = 90^\circ.$$

Let EK be the tangent to the circle $(AFHE)$ at point E such that

$$\angle HEK = \angle HFE = \angle HAE.$$

The points H, E, C, D are also concyclic because $\angle HDC = \angle HEC = 90^\circ$. Thus,

$$\angle HED = \angle HCD = \angle BAD.$$

Triangle ABC is acute with $AB > AC$, and thus $\angle B < \angle C$; therefore, $\angle BAD > \angle DAC$, and the point S is on the ray $\overrightarrow{BC}$ outside the triangle. Thus,

$$\angle KED = \angle HED - \angle HEK = \angle BCF - \angle CFS = \angle DSE.$$

Therefore, EK is also tangent to the circle (DES) .

OC773. Initially, the numbers $1, 2, \dots, 2024$ are written on a blackboard. Trixi and Nana play a game, taking alternate turns. Trixi plays first. The player whose turn it is chooses two numbers a and b , erases both, and writes their (possibly negative) difference $a - b$ on the blackboard. This is repeated until only one number remains on the blackboard after 2023 moves. Trixi wins if this number is divisible by 3, otherwise Nana wins. Which of the two has a winning strategy?

Originally from the Austrian Mathematical Olympiad 2024, Final Round, Problem 3.

We received 4 submissions, of which 2 were correct and complete. We present the solution by Oliver Geupel.

We will establish a winning strategy for the second player, Nana.

For $0 \leq n \leq 2023$, let a_n, b_n and c_n be the numbers of integers on the blackboard that are congruent to 0, 1 and 2, respectively, modulo 3, after the n th move. We have $(a_0, b_0, c_0) = (674, 675, 675)$. For $r, s \in \{0, 1, 2\}$, let (r, s) denote the type of move where numbers congruent to r and $s \pmod{3}$ are removed and a number congruent to $r - s \pmod{3}$ is written. If the $(n + 1)$ st move is (r, s) , the result is as follows.

(r, s)	$(0, 0), (1, 0), (2, 0)$	$(0, 1)$	$(0, 2)$	$(1, 1)$	$(2, 2)$	$(1, 2)$	$(2, 1)$
a_{n+1}	$a_n - 1$	$a_n - 1$	$a_n - 1$	$a_n + 1$	$a_n + 1$	a_n	a_n
b_{n+1}	b_n	$b_n - 1$	$b_n + 1$	$b_n - 2$	b_n	$b_n - 1$	b_n
c_{n+1}	c_n	$c_n + 1$	$c_n - 1$	c_n	$c_n - 2$	c_n	$c_n - 1$

Nana makes the $(2k)$ th move ($1 \leq k \leq 1011$). And so her $(n + 1)$ st move (where $n + 1 = 2k$) is:

Case 1: $\mathbf{b_n + c_n = 0}$. Here the move $(0, 0)$ would be enforced, but we will subsequently see that this case will never occur.

Case 2: $\mathbf{b_n = 1}$. Since $a_n + b_n + c_n \geq 3$ we get either $(a_n, b_n, c_n) = (1, 1, 1)$ or $a_n \geq 2$ or $c_n \geq 2$. The respective moves are $(1, 0)$, $(0, 0)$ or $(2, 2)$.

Case 3: $\mathbf{b_n \geq 3, odd}$. The move is $(1, 1)$.

Case 4: $\mathbf{c_n odd}$. Similar to Cases 3 and 4 by symmetry $b \leftrightarrow c$.

Case 5: $\mathbf{b_n + c_n > 0; b_n \text{ and } c_n \text{ even}}$. Without loss of generality let $b_n > 0$. Since $a_n + b_n + c_n$ is odd, we obtain $a_n > 0$. The move is $(0, 1)$.

This completes the description of Nana's moves.

Let $P(k)$ denote the assertion that at least one of the two numbers b_{2k} and c_{2k} is odd and that at least one of b_{2k+1} and c_{2k+1} is greater than zero.

$P(0)$ holds because $b_0 = 675$ and $b_1 \geq 673$. Let us assume that $P(k)$ holds for some k with $0 \leq k \leq 1010$. We prove that this implies $P(k+1)$: Without loss of generality suppose that $b_{2k+1} > 0$. The $(2k+2)$ nd move is according to Nana's strategy. Case 1 does not apply since $b_{2k+1} > 0$. In case 2 we obtain $b_{2k+1} = 1$. In case 3 we get $b_{2k+2} = b_{2k+1} - 2$ which is odd. Case 4 is due to symmetry. In case 5 we have $c_{2k+2} = c_{2k+1} + 1$, which is odd. This completes the proof that b_{2k+2} or c_{2k+2} is odd. It remains to prove that b_{2k+3} or c_{2k+3} is positive. The table shows that moves $(0, 0)$, $(1, 0)$, $(2, 0)$, $(0, 1)$, $(0, 2)$, $(1, 1)$ and $(2, 2)$ yield b_{n+1} or c_{n+1} positive if b_n or c_n is odd. Moreover $(1, 2)$ is applicable only if $c_n > 0$ which ensures $c_{n+1} = c_n > 0$. Similarly for $(2, 1)$. Hence $P(k+1)$.

By recursion it follows $P(1011)$, which implies that one of b_{2023} and c_{2023} is positive. This proves that the last number on the blackboard is not divisible by 3.

OC774. Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u, v of vertices, let $d(u, v)$ denote the distance between u and v , that is, the number of edges in the shortest path in T that connects u with v .

Consider the sums

$$W(T) = \sum_{\substack{\{u,v\} \subseteq V(T) \\ u \neq v}} d(u, v) \quad \text{and} \quad H(T) = \sum_{\substack{\{u,v\} \subseteq V(T) \\ u \neq v}} \frac{1}{d(u, v)}.$$

Prove that

$$W(T) \cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

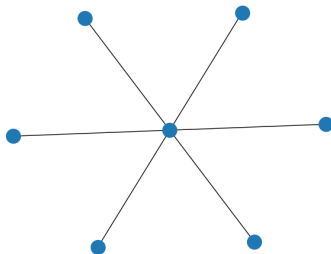
Originally from the IMC 2023 - Day 2, Problem 8.

We received 3 submissions, of which 2 were correct and complete. We present the solution by Theo Koupelis.

There are $N = n(n-1)/2$ distances between the n vertices in the tree, denote them by d_i , $i \in [1, N]$. Of those, $n-1$ are of length 1 because the tree contains no cycle and thus has $n-1$ edges. Without loss of generality, let $d_1 = d_2 = \dots = d_{n-1} = 1$, and $d_i \in [2, n-1]$, for $i = n, n+1, \dots, N$. Thus,

$$\begin{aligned} W(T) \cdot H(T) &= (n-1 + d_n + d_{n+1} + \dots + d_N) \left(n-1 + \frac{1}{d_n} + \frac{1}{d_{n+1}} + \dots + \frac{1}{d_N} \right) \\ &= (n-1)^2 + \sum_{i=n}^N d_i \cdot \sum_{i=n}^N \frac{1}{d_i} + (n-1) \sum_{i=n}^N \left(d_i + \frac{1}{d_i} \right) \\ &\geq (n-1)^2 + (N - (n-1))^2 + (n-1) \left(2 + \frac{1}{2} \right) (N - (n-1)) \\ &= \frac{(n-1)^3(n+2)}{4}, \end{aligned}$$

where we used the Cauchy-Schwarz inequality, and $d_i \geq 2$ because $d_i \in [2, n-1]$. Equality occurs when $d_i = 2$, for $i = n, n+1, \dots, N$, which is the case of the star graph:



OC775. Let V be the set of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$, differentiable on $(0, 1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0, 1)$ such that

$$f(\xi) + \alpha = f'(\xi).$$

Originally from the IMC 2023 - Day 2, Problem 7.

We received 9 submissions, of which 7 were correct and complete. We present 2 solutions.

Solution 1, by Kee-Wai Lau.

We show that for every $f \in V$, there exists some $\xi \in (0, 1)$ such that

$$f(\xi) + \alpha = f'(\xi)$$

if and only if

$$\alpha = \frac{1}{e-1}.$$

For any $\alpha \in \mathbb{R}$, let $g(x) = e^{-x}(f(x) + \alpha)$, so that

$$g(0) = \alpha, \quad g(1) = \frac{1 + \alpha}{e},$$

and

$$g'(x) = e^{-x}(f'(x) - f(x) - \alpha).$$

If $\alpha = \frac{1}{e-1}$, then $g(0) = g(1)$. By the mean value theorem, there exists $\xi \in (0, 1)$ such that

$$g'(\xi) = g(1) - g(0) = 0,$$

and so

$$f(\xi) + \alpha = f'(\xi).$$

If $\alpha \neq \frac{1}{e-1}$, let

$$f_1(x) = e^{x-1}((1 + \alpha - e\alpha)x + e\alpha) - \alpha,$$

so that $f_1 \in V$. It is easy to check that

$$f_1'(x) - f_1(x) - \alpha = e^{x-1}(1 + \alpha - \alpha e),$$

which never vanishes. Hence our claim.

Solution 2, by Michel Bataille.

We show that there is a unique α satisfying the desired property, namely $\alpha = \frac{1}{e-1}$.

The key remark is that the function $f_0 : x \mapsto f_0(x) = \frac{e^x - 1}{e - 1}$, which is obviously in V , satisfies

$$f_0'(x) - f_0(x) = \frac{e^x}{e - 1} - \frac{e^x - 1}{e - 1} = \frac{1}{e - 1} \quad \text{for all } x \in (0, 1). \quad (1)$$

Now, suppose that α exists. Then taking $f = f_0$, there exists some ξ in $(0, 1)$ such that $f_0'(\xi) - f_0(\xi) = \alpha$. From (1), $\alpha = \frac{1}{e-1}$. Thus, the only possible α is $\frac{1}{e-1}$.

Conversely, let f be any function in V . Define the function g by

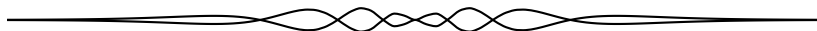
$$g(x) = e^{-x}(f(x) - f_0(x)).$$

Then, g is continuous on $[0, 1]$, differentiable on $(0, 1)$ with

$$\begin{aligned} g'(x) &= e^{-x}(f'(x) - f_0'(x)) - e^{-x}(f(x) - f_0(x)) \\ &= e^x(f'(x) - f(x) - (f_0'(x) - f_0(x))) \\ &= e^{-x}\left(f'(x) - f(x) - \frac{1}{e-1}\right) \quad (\text{from (1)}) \end{aligned}$$

for all $x \in (0, 1)$.

Since $g(1) = g(0) (= 0)$, Rolle's theorem shows that there exists ξ in $(0, 1)$ such that $g'(\xi) = 0$, meaning that $f'(\xi) - f(\xi) = \frac{1}{e-1}$. Thus, $\frac{1}{e-1}$ does satisfy the property required for α .



Evaluating A Class of Definite Integrals Using Symmetry of Integration

Neculai Stanciu

1 Introduction

A useful property for simplifying and solving a class of difficult definite integrals states that

$$I = \int_a^b f(x) dx = \int_a^b f(a+b-x) dx,$$

where f is a continuous function on the interval $[a, b]$.

Proof. The formula is obtained through a simple change of variable (substitution):

$$x = u(t) = a + b - t, \quad u'(t) = -1, \quad u(a) = b, \quad u(b) = a, \quad dx = -dt,$$

so

$$I = \int_a^b f(x) dx = \int_b^a f(a+b-t)(-dt) = \int_a^b f(a+b-x) dx.$$

□

Since

$$2I = \int_a^b (f(x) + f(a+b-x)) dx,$$

then

$$I = \frac{1}{2} \int_a^b (f(x) + f(a+b-x)) dx.$$

A simple and familiar example of applying this fact is finding

$$\int_0^{\pi/2} \sin^2 x dx$$

without using antiderivatives. We get

$$\int_0^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} \sin^2 \left(0 + \frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \cos^2 x dx.$$

Since $\sin^2 x + \cos^2 x = 1$, we obtain

$$\int_0^{\pi/2} \sin^2 x dx + \int_0^{\pi/2} \cos^2 x dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2},$$

and hence

$$\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{4}.$$

Note that the transformation itself does not make the integrand simpler: $\sin^2 x$ is not simpler than $\cos^2 x$. This is typical. But for certain integrands and appropriate intervals, the two integrands are related by an algebraic identity and in special cases it may allow the integral to be evaluated without ever antidifferentiating.

Another example is the well-known Problem A3 from the 1980 Putnam, where the indefinite integral cannot be expressed in terms of elementary functions:

$$I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

Any reader who has never solved this one is invited to try it, using the hint above. Here are step-by-step instructions for using this property of integration.

- (i) Label your original integral as

$$I = \int_a^b f(x) \, dx$$

- (ii) Replace every x in the integrand $f(x)$ with $(a + b - x)$. The limits of integration, and the value of the integral, remain the same:

$$I = \int_a^b f(a + b - x) \, dx.$$

- (iii) Add the original I and the new modified I together to get $2I$:

$$2I = \int_a^b (f(x) + f(a + b - x)) \, dx.$$

- (iv) Evaluate the integral and divide the result by 2 to find the final value of I .

Here is another worked example. Suppose we wish to evaluate

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} \, dx.$$

- (i) Define

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} \, dx.$$

- (ii) Replace x with $(0 + \frac{\pi}{2} - x)$:

$$I = \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)} \, dx.$$

(iii) Next up, using identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \quad \text{and} \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x,$$

we get

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx.$$

(iv) Add the two together:

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2},$$

which yields

$$I = \frac{\pi}{4}.$$

This can be extended to a wider range of functions.

2 Applications

Problem 1. Compute

$$\int_{-a}^a \frac{\sin x + \tan x}{(b - \cos x)^m + \sin^{2n} x} dx,$$

where

$$a \in \left(0, \frac{\pi}{2}\right), \quad b \in [1, \infty), \quad m, n \in \mathbb{N}.$$

(Problem SP.054, Romanian Mathematical Magazine, Winter Edition, 2017).

Solution. Starting with the original integral I , let

$$t = u(x) = -x, \quad u'(x) = -1, \quad u(-a) = a, \quad u(a) = -a,$$

so

$$I = \int_a^{-a} \frac{\sin(-t) + \tan(-t)}{(b - \cos(-t))^m + \sin^{2n}(-t)} (-1) dt = - \int_{-a}^a \frac{\sin t + \tan t}{(b - \cos t)^m + \sin^{2n} t} dt = -I,$$

hence

$$2I = 0 \iff I = 0.$$

Problem 2. Compute

$$\int_a^b \frac{\sqrt[n]{x-a} (1 + \sqrt[n]{b-x})}{\sqrt[n]{x-a} + 2\sqrt[n]{-x^2 + (a+b)x - ab} + \sqrt[n]{b-x}} dx, \quad 0 < a < b.$$

(Problem 5226, *School Science and Mathematics Journal*, November, 2012).

Solution. Let

$$x = u(t) = a + b - t, \quad u'(t) = -1, \quad u(a) = b, \quad u(b) = a,$$

so

$$\begin{aligned} I &= \int_a^b \frac{\sqrt[n]{x-a} (1 + \sqrt[n]{b-x})}{\sqrt[n]{x-a} + 2\sqrt[n]{-x^2 + (a+b)x - ab} + \sqrt[n]{b-x}} dx \\ &= \int_a^b \frac{\sqrt[n]{b-t} (1 + \sqrt[n]{t-a})}{\sqrt[n]{b-t} + 2\sqrt[n]{(b-t)(t-a)} + \sqrt[n]{t-a}} dt. \end{aligned}$$

Therefore,

$$2I = \int_a^b \frac{\sqrt[n]{x-a} + 2\sqrt[n]{(b-x)(x-a)} + \sqrt[n]{b-x}}{\sqrt[n]{x-a} + 2\sqrt[n]{(b-x)(x-a)} + \sqrt[n]{b-x}} dx = \int_a^b dx = b - a.$$

Hence,

$$I = \frac{b-a}{2}.$$

Problem 3. Compute

$$\int_0^{\pi/2} \cos^2 x \left(\sin x \sin^2 \left(\frac{\pi}{2} \cos x \right) + \cos x \sin^2 \left(\frac{\pi}{2} \sin x \right) \right) dx.$$

Solution. We make the change

$$t = u(x) = \frac{\pi}{2} - x, \quad u'(x) = -1, \quad u(0) = \frac{\pi}{2}, \quad u\left(\frac{\pi}{2}\right) = 0$$

and obtain

$$I = \int_0^{\pi/2} \sin^2 x \left(\cos x \sin^2 \left(\frac{\pi}{2} \sin x \right) + \sin x \sin^2 \left(\frac{\pi}{2} \cos x \right) \right) dx,$$

therefore

$$\begin{aligned} 2I &= \int_0^{\pi/2} (\sin^2 x + \cos^2 x) \left(\cos x \sin^2 \left(\frac{\pi}{2} \sin x \right) + \sin x \sin^2 \left(\frac{\pi}{2} \cos x \right) \right) dx \\ &= \int_0^{\pi/2} \sin^2 \left(\frac{\pi}{2} \sin x \right) \cos x dx + \int_0^{\pi/2} \sin^2 \left(\frac{\pi}{2} \cos x \right) \sin x dx = A + B, \end{aligned}$$

where

$$A = \int_0^{\pi/2} \sin^2 \left(\frac{\pi}{2} \sin x \right) \cos x dx, \quad B = \int_0^{\pi/2} \sin^2 \left(\frac{\pi}{2} \cos x \right) \sin x dx.$$

We easily deduce that

$$\begin{aligned} A &= \int_0^{\pi/2} \sin^2 \left(\frac{\pi}{2} \sin x \right) \cos x \, dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin^2 \left(\frac{\pi}{2} \sin x \right) \left(\frac{\pi}{2} \cos x \right) \, dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin^2 t \, dt = \frac{1}{\pi} \int_0^{\pi/2} (1 - \cos 2t) \, dt = \frac{1}{2}, \end{aligned}$$

and analogously

$$B = \int_0^{\pi/2} \sin^2 \left(\frac{\pi}{2} \cos x \right) \sin x \, dx = \frac{1}{2}.$$

Hence,

$$2I = A + B = 1 \iff I = \frac{1}{2}.$$

Problem 4. Compute

$$I_n = \int_0^{\pi/2} \cos^2 x \left(\cos x \sin^{2n+1} \left(\frac{\pi}{2} \sin x \right) + \sin x \sin^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) \, dx.$$

Solution. We make the change

$$t = u(x) = \frac{\pi}{2} - x, \quad u'(x) = -1, \quad u(0) = \frac{\pi}{2}, \quad u\left(\frac{\pi}{2}\right) = 0$$

and we have

$$I_n = \int_0^{\pi/2} \sin^2 x \left(\sin x \sin^{2n+1} \left(\frac{\pi}{2} \cos x \right) + \cos x \sin^{2n+1} \left(\frac{\pi}{2} \sin x \right) \right) \, dx,$$

then

$$\begin{aligned} 2I_n &= \int_0^{\pi/2} (\sin^2 x + \cos^2 x) \left(\cos x \sin^{2n+1} \left(\frac{\pi}{2} \sin x \right) + \sin x \sin^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) \, dx \\ &= \int_0^{\pi/2} \sin^{2n+1} \left(\frac{\pi}{2} \sin x \right) \cos x \, dx + \int_0^{\pi/2} \sin^{2n+1} \left(\frac{\pi}{2} \cos x \right) \sin x \, dx \\ &= A_n + B_n, \end{aligned}$$

where

$$A_n = \int_0^{\pi/2} \sin^{2n+1} \left(\frac{\pi}{2} \sin x \right) \cos x \, dx, \quad B_n = \int_0^{\pi/2} \sin^{2n+1} \left(\frac{\pi}{2} \cos x \right) \sin x \, dx.$$

Therefore,

$$\begin{aligned} A_n &= \int_0^{\pi/2} \sin^{2n+1} \left(\frac{\pi}{2} \sin x \right) \cos x \, dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin^{2n+1} \left(\frac{\pi}{2} \sin x \right) \left(\frac{\pi}{2} \cos x \right) \, dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin^{2n+1} x \, dx, \end{aligned}$$

and analogously

$$B_n = \int_0^{\pi/2} \sin^{2n+1} \left(\frac{\pi}{2} \cos x \right) \sin x \, dx = \frac{2}{\pi} \int_0^{\pi/2} \sin^{2n+1} x \, dx = A_n.$$

Thus,

$$2I_n = A_n + B_n = 2 \cdot \frac{2}{\pi} \int_0^{\pi/2} \sin^{2n+1} x \, dx \iff I_n = \frac{2}{\pi} \int_0^{\pi/2} \sin^{2n+1} x \, dx \iff I_n = \frac{2}{\pi} J_n,$$

where

$$J_n = \int_0^{\pi/2} \sin^{2n+1} x \, dx,$$

which is easily integrated by parts to yield the well-known recurrence

$$(2n+1)J_n = 2nJ_{n-1}, \quad \forall n \in \mathbb{N},$$

with

$$J_0 = \int_0^{\pi/2} \sin x \, dx = 1.$$

Expanding this yields (writing $n!!$ for $n(n-2)\cdots(2 \text{ or } 1)$):

$$I_n = \frac{2}{\pi} J_n = \frac{2}{\pi} \frac{(2n)!!}{(2n+1)!!}.$$

Problem 5. Compute

$$\int_{-a}^a \left(\frac{\cos x}{1+e^x} - \sin x \ln(1+e^x) \right) dx,$$

where $a > 0$.

Solution. Denote the original integral by I and make the following change of variable

$$x = u(t) = -t, \quad u'(t) = -1, \quad u(a) = -a, \quad u(-a) = a$$

and we obtain that

$$\begin{aligned} I &= \int_a^{-a} \left(\frac{\cos(-t)}{1+e^{-t}} - \sin(-t) \ln(1+e^{-t}) \right) (-1) \, dt \\ &= \int_{-a}^a \left(\frac{e^t \cos t}{1+e^t} + \sin t \ln \frac{1+e^t}{e^t} \right) dt \\ &= \int_{-a}^a \left(\frac{e^x \cos x}{1+e^x} - x \sin x + \sin x \ln(1+e^x) \right) dx. \end{aligned}$$

So,

$$\begin{aligned}
 2I &= \int_{-a}^a \left(\frac{\cos x}{1+e^x} + \frac{e^x \cos x}{1+e^x} - x \sin x - \sin x \ln(1+e^x) + \sin x \ln(1+e^x) \right) dx \\
 &= \int_{-a}^a (\cos x - x \sin x) dx \\
 &= x \cos x \Big|_{-a}^a \\
 &= a \cos a - (-a \cos(-a)) \\
 &= a \cos a + a \cos a \\
 &= 2a \cos a,
 \end{aligned}$$

hence

$$I = a \cos a.$$

Problem 6. Compute

$$\int_{-a}^a \ln(2^{\sin^3 x} + 3^{\sin^3 x}) \sin x dx,$$

where

$$a \in \left(0, \frac{\pi}{2}\right).$$

Solution. Let I be the original integral in which make the change

$$x = u(t) = -t, \quad u'(t) = -1, \quad u(a) = -a, \quad u(-a) = a$$

and we deduce that

$$\begin{aligned}
 I &= - \int_a^{-a} \ln(2^{\sin^3(-t)} + 3^{\sin^3(-t)}) \sin(-t) dt \\
 &= \int_{-a}^a \ln(2^{-\sin^3 t} + 3^{-\sin^3 t}) (-\sin t) dt \\
 &= - \int_{-a}^a \ln\left(\frac{2^{\sin^3 t} + 3^{\sin^3 t}}{6^{\sin^3 t}}\right) \sin t dt \\
 &= -I + \int_{-a}^a \ln 6^{\sin^3 t} \sin t dt \\
 &= -I + \ln 6 \int_{-a}^a \sin^4 t dt.
 \end{aligned}$$

Thus

$$2I = \ln 6 \int_{-a}^a \sin^4 x dx = \ln 6 \int_{-a}^a (\sin^2 x)^2 dx = \ln 6 \int_{-a}^a \frac{1}{4} (2 \sin^2 x)^2 dx.$$

Hence

$$\begin{aligned}
 2I &= \frac{\ln 6}{4} \int_{-a}^a (1 - \cos 2x)^2 dx \\
 &= \frac{\ln 6}{4} \left(\int_{-a}^a dx - 2 \int_{-a}^a \cos 2x dx + \int_{-a}^a \cos^2 2x dx \right) \\
 &= \frac{\ln 6}{4} \left(2a - \sin 2x \Big|_{-a}^a + \frac{1}{2} \int_{-a}^a (1 + \cos 4x) dx \right) \\
 &= \frac{\ln 6}{4} \left(2a - 2 \sin 2a + a + \frac{1}{2} \int_{-a}^a \cos 4x dx \right) \\
 &= \frac{\ln 6}{4} \left(3a - 2 \sin 2a + \frac{1}{4} \sin 4a \right).
 \end{aligned}$$

Equivalently,

$$2I = \frac{\ln 6}{16} (12a - 8 \sin 2a + \sin 4a).$$

Therefore

$$I = \frac{\ln 6}{32} (12a - 8 \sin 2a + \sin 4a).$$

3 Exercises

We invite the reader to practice using this property to solve the following exercises. Think you can do that?

(i) Compute

$$\int_a^b \frac{e^{f(x-a)}(f(x-a))^{1/c}}{e^{f(x-a)}(f(x-a))^{1/c} + e^{f(b-x)}(f(b-x))^{1/c}} dx,$$

where $a, b \in \mathbb{R}$, $a < b$, $c \in \mathbb{R}_+$, and $f : \mathbb{R} \rightarrow \mathbb{R}_+$ is a continuous function. (*Problem 5292, School Science and Mathematics Journal, February, 2014*).

(ii) Compute

$$I_n = \int_0^{\pi/2} x \left(\cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) \cos x + \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \sin x \right) dx.$$

(iii) Compute

$$\int_0^{\pi/2} \sin^2 x \left(\cos x \cos^2 \left(\frac{\pi}{2} \sin x \right) + \sin x \cos^2 \left(\frac{\pi}{2} \cos x \right) \right) dx.$$

(iv) Compute

$$\int_{-1}^1 \frac{1}{(x^2 + 2026)(1 + e^x)} dx.$$

Answers

$$(i) \frac{b-a}{2}$$

$$(ii) I_n = \frac{(2n)!!}{(2n+1)!!} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n-1)(2n+1)}$$

$$(iii) \frac{1}{2}$$

$$(iv) \frac{1}{\sqrt{2026}} \arctan \frac{1}{\sqrt{2026}}$$

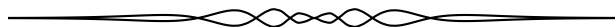
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PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **December 15, 2026**.



5171. *Proposed by Ziji Hu.*

Let $\mathcal{P}$ be a set of points in the Euclidean plane, initially empty. Let m and n be fixed positive integers. Two players, A and B , play a game by performing the following steps in each round:

- Player A adds m new points to $\mathcal{P}$ such that no three points in $\mathcal{P}$ are collinear. Player A then colours each of these new points using one of n distinct colours.
- Player B selects two points $u, v \in \mathcal{P}$ of the same colour, provided that neither u nor v has been chosen in any previous round. Player B then draws the straight line segment uv . No two segments drawn by Player B may intersect.

Prove that player B can always make a move in every round if and only if $m \geq n+2$.

5172. *Proposed by Vasile Cîrtoaje.*

Find the smallest positive value of the power k such that

$$\left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \right)^k \geq \frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}$$

for $n \geq 2$ and any nonnegative real numbers a_i with at most one $a_i < 1$ and $a_1 + a_2 + \cdots + a_n = n$.

5173. *Proposed by Michel Bataille.*

Let ABC be a triangle, not right-angled, inscribed in a circle Γ and let $BC = a, CA = b, AB = c, \angle BAC = \alpha, \angle CBA = \beta, \angle ACB = \gamma$. For any point P of Γ prove that

$$\frac{a^2 - PA^2}{bc \cos \beta \cos \gamma} + \frac{b^2 - PB^2}{ca \cos \gamma \cos \alpha} + \frac{c^2 - PC^2}{ab \cos \alpha \cos \beta} = 4.$$

5174. *Proposed by Tatsunori Irie.*

Let a, b, c be the side lengths of a triangle. Let R be the circumradius, r the inradius and S the area of the triangle. Prove that the following inequality holds:

$$\sqrt{3}Rr \geq \left(4\sqrt{3} - \sqrt{36 - \frac{a^2 + b^2 + c^2}{R^2}} \right) Rr \geq \frac{2}{3}S.$$

5175. *Proposed by Michael Friday.*

Let ABC be a triangle with circumcenter O . The tangent to the circumcircle of ABC at A meets side BC at P , while tangents at B and C meet at Q . Prove that the foot of the symmedian from A is the orthocenter of triangle OPQ .

5176. *Proposed by Nguyen Viet Hung.*

Given a triangle ABC with centroid G , area S and circumradius R . Denote respectively by R_a, R_b, R_c the circumradius of triangles BGC, CGA, AGB . Prove that

$$3R \leq R_a + R_b + R_c \leq R + \frac{a^3 + b^3 + c^3}{6S}.$$

5177. *Proposed by Nguyen Van Huyen.*

Let $n \geq 2$ and let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be non-negative real numbers. Prove that

$$2^n \prod_{i=1}^n (a_i^2 - a_i b_i + b_i^2) \geq \prod_{i=1}^n (a_i^2 + b_i^2) + (2^n - 1) \prod_{i=1}^n (a_i - b_i)^2.$$

5178. *Proposed by Ion Patrascu.*

Let ABC be a triangle with $\angle BAC = 110^\circ$ and $\angle ABC = 40^\circ$. The angle bisector of angle ABC intersects side AC at point D . Consider point E outside triangle ABC such that $CE = AC$, $\angle BCE = 70^\circ$, and points E and A are separated by line BC . The lines ED and BC intersect at point F . Prove that $BD = BF$.

5179. *Proposed by Michel Bataille.*

Let $(F_n)_{n \geq 0}$ denote the Fibonacci sequence where $F_0 = 0$, $F_1 = 1$ and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. Prove that

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{4F_{2n}}{F_{2n}^2 + 5} + \sum_{n=1}^{\infty} \tan^{-1} \frac{4F_{2n+1}}{F_{2n+1}^2 - 3} = \frac{3\pi}{2}.$$

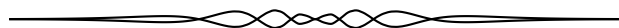
5180. *Proposed by Xicheng Peng.*

Let $ABCD$ be a quadrilateral and let E be the intersection of lines AB and CD . Let F and G be the midpoints of AB and CD , respectively. Let P be the intersection of the diagonals AC and BD and let H be the orthocenter of triangle ADP . Prove that $EH \perp FG$ if and only if the points A, B, C, D are concyclic.

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Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 décembre 2026**.



5171. *Soumis par Ziji Hu.*

Soit $\mathcal{P}$ un ensemble de points du plan euclidien, initialement vide. Soit m et n des entiers positifs fixés. Deux joueurs, A et B , jouent au jeu suivant. À chaque tour, les étapes suivantes sont effectuées:

- Le joueur A ajoute m nouveaux points à $\mathcal{P}$, de telle sorte qu'aucun triplet de points de $\mathcal{P}$ ne soit aligné. Le joueur A attribue ensuite à chacun de ces nouveaux points l'une des n couleurs disponibles.
- Le joueur B choisit deux points $u, v \in \mathcal{P}$ de même couleur, à condition que ni u ni v n'aient été choisis lors d'un tour précédent. Le joueur B trace ensuite le segment de droite uv . Aucun des segments tracés par le joueur B ne peut en couper un autre.

Montrez que le joueur B peut toujours jouer à chaque tour si et seulement si $m \geq n + 2$.

5172. *Soumis par Vasile Cîrtoaje.*

Déterminez la plus petite valeur positive de l'exposant k telle que

$$\left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \right)^k \geq \frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}$$

pour tout entier $n \geq 2$ et tous nombres réels non négatifs a_i tels qu'au plus un des a_i soit strictement inférieur à 1 et que $a_1 + a_2 + \cdots + a_n = n$.

5173. *Soumis par Michel Bataille.*

Soit ABC un triangle, non rectangle, inscrit dans un cercle Γ . Posons $BC = a$, $CA = b$, $AB = c$, $\angle BAC = \alpha$, $\angle CBA = \beta$ et $\angle ACB = \gamma$. Montrez que, pour tout point P du cercle Γ ,

$$\frac{a^2 - PA^2}{bc \cos \beta \cos \gamma} + \frac{b^2 - PB^2}{ca \cos \gamma \cos \alpha} + \frac{c^2 - PC^2}{ab \cos \alpha \cos \beta} = 4.$$

5174. *Soumis par Tatsunori Irie.*

Soient a , b et c les longueurs des côtés d'un triangle. Soient R le rayon du cercle circonscrit, r le rayon du cercle inscrit et S l'aire du triangle. Montrez que l'inégalité suivante est vérifiée :

$$\sqrt{3}Rr \geq \left(4\sqrt{3} - \sqrt{36 - \frac{a^2 + b^2 + c^2}{R^2}}\right) Rr \geq \frac{2}{3}S.$$

5175. *Soumis par Michael Friday.*

Soit ABC un triangle de centre du cercle circonscrit O . La tangente au cercle circonscrit à ABC en A coupe le côté BC au point P , tandis que les tangentes en B et en C se coupent au point Q . Montrez que le pied de la symédiane issue de A est l'orthocentre du triangle OPQ .

5176. *Soumis par Nguyen Viet Hung.*

Soit ABC un triangle de centre de gravité G , d'aire S et de rayon du cercle circonscrit R . On note respectivement par R_a , R_b et R_c les rayons des cercles circonscrits aux triangles BGC , CGA et AGB . Montrez que

$$3R \leq R_a + R_b + R_c \leq R + \frac{a^3 + b^3 + c^3}{6S}.$$

5177. *Soumis par Nguyen Van Huyen.*

Soit $n \geq 2$ et soient $a_1, a_2, \dots, a_n$ et $b_1, b_2, \dots, b_n$ des nombres réels non négatifs. Démontrez que

$$2^n \prod_{i=1}^n (a_i^2 - a_i b_i + b_i^2) \geq \prod_{i=1}^n (a_i^2 + b_i^2) + (2^n - 1) \prod_{i=1}^n (a_i - b_i)^2.$$

5178. *Soumis par Ion Patrascu.*

Soit ABC un triangle tel que $\angle BAC = 110^\circ$ et $\angle ABC = 40^\circ$. La bissectrice de l'angle ABC coupe le côté AC au point D . On considère un point E , situé à l'extérieur du triangle ABC , tel que $CE = AC$, $\angle BCE = 70^\circ$, et tel que les points E et A soient situés de part et d'autre de la droite BC . Les droites ED et BC se coupent au point F . Montrez que $BD = BF$.

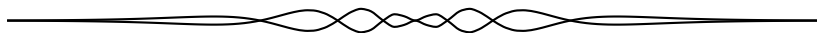
5179. *Soumis par Michel Bataille.*

Soit $(F_n)_{n \geq 0}$ la suite de Fibonacci définie par $F_0 = 0$, $F_1 = 1$ et, pour $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. Démontrez que

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{4F_{2n}}{F_{2n}^2 + 5} + \sum_{n=1}^{\infty} \tan^{-1} \frac{4F_{2n+1}}{F_{2n+1}^2 - 3} = \frac{3\pi}{2}.$$

5180. *Soumis par Xicheng Peng.*

Soit $ABCD$ un quadrilatère et soit E le point d'intersection des droites AB et CD . Soient F et G les milieux respectifs de AB et de CD . Soit P le point d'intersection des diagonales AC et BD , et soit H l'orthocentre du triangle ADP . Démontrez que $EH \perp FG$ si et seulement si les points A , B , C et D sont cocycliques.



SOLUTIONS

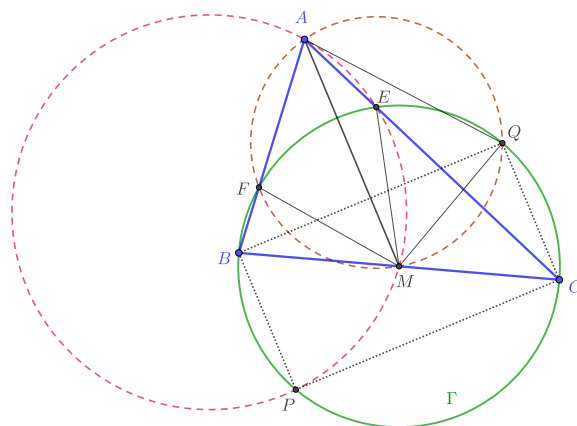
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2026: 52(3), p. 140–144.

5121. *Proposed by Michel Bataille.*

Let ABC be a triangle with no right angle and let the circle Γ centered at the midpoint M of BC with radius MB intersect again AC at E and AB at F . If the circumcircles of $\triangle AME$ and $\triangle AMF$ intersect Γ again at P and Q , respectively, prove that $BPCQ$ is a rectangle.

We received 13 solutions, of which 6 were complete and correct. Submissions which omitted complex calculations were not credited. We present the solution by Theo Koupelis, slightly edited.



Note that if $AB = AC$ then Q and C will coincide, as will B and P , so that $BPCQ$ is a degenerate rectangle.

Assume now without loss of generality that $AB < AC$. For simplicity, we will only show the case when $\triangle ABC$ is acute, as in the diagram below. The case when $\triangle ABC$ is obtuse is solved similarly; a diagram illustrating that case is included at the end of the proof.

The segment BC is a diameter of Γ , and thus $\angle BQC = \angle BPC = 90^\circ$.

By construction, $AFMQ$ is a cyclic quadrilateral (all the vertices are on the circumcircle of $\triangle AMF$), so $\angle AQM = \angle BFM$. BM and FM are radii of Γ , so

$\angle BFM = \angle FBM = \angle ABM$. Combining these two statements we get that

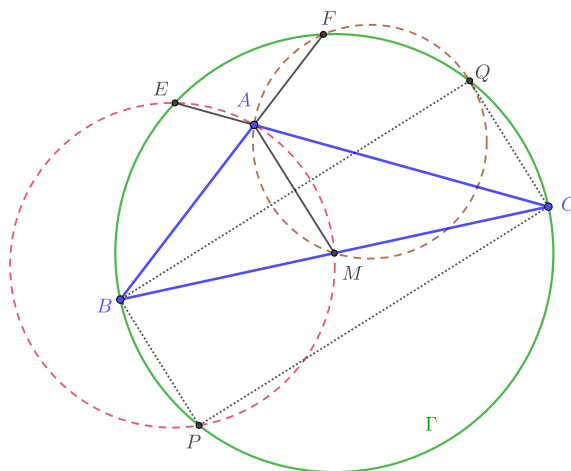
$$\angle AQM = \angle ABM.$$

MB and MQ are radii of Γ , so $\triangle MBQ$ is isosceles with $\angle MBQ = \angle MQB$. Hence

$$\angle ABQ = \angle ABM - \angle MBQ = \angle AQM - \angle MQB = \angle AQB.$$

So $\triangle ABQ$ is also isosceles, and $ABMQ$ is a kite. Therefore AM is the perpendicular bisector of BQ . Mutatis mutandis AM is also the perpendicular bisector of PC , whence $BQ \parallel PC$. Since $\angle BQC = \angle BPC = 90^\circ$, we conclude that $BPCQ$ is a rectangle.

The diagram below illustrates the case when $\triangle ABC$ is obtuse, and E and F are exterior to the triangle:



5122. *Proposed by Nguyen Viet Hung.*

Let a, b, c be pairwise distinct real numbers such that

$$\frac{ab}{(a-b)^2} + \frac{bc}{(b-c)^2} + \frac{ca}{(c-a)^2} = -\frac{1}{4}.$$

Evaluate

$$\frac{a}{a-b} + \frac{b}{b-c} + \frac{c}{c-a}.$$

We received 27 solutions, of which 26 were correct and complete. We present one common approach, sampling particularly from the submission of Ansh Agarwal.

Set $x = \frac{a+b}{a-b}$, $y = \frac{b+c}{b-c}$, $z = \frac{c+a}{c-a}$. We have from the hypothesis that

$$\begin{aligned} -1 &= \frac{4ab}{(a-b)^2} + \frac{4bc}{(b-c)^2} + \frac{4ca}{(c-a)^2} \\ &= \frac{(a+b)^2 - (a-b)^2}{(a-b)^2} + \frac{(b+c)^2 - (b-c)^2}{(b-c)^2} + \frac{(c+a)^2 - (c-a)^2}{(c-a)^2} \\ &= \left(\frac{a+b}{a-b}\right)^2 + \left(\frac{b+c}{b-c}\right)^2 + \left(\frac{c+a}{c-a}\right)^2 - 3, \end{aligned}$$

so

$$x^2 + y^2 + z^2 = 2.$$

Also, note that

$$\frac{x-1}{x+1} = \frac{\frac{a+b}{a-b} - \frac{a-b}{a-b}}{\frac{a+b}{a-b} + \frac{a-b}{a-b}} = \frac{\frac{2b}{a-b}}{\frac{2a}{a-b}} = \frac{b}{a}.$$

Similarly $\frac{y-1}{y+1} = \frac{c}{b}$ and $\frac{z-1}{z+1} = \frac{a}{c}$, so

$$\left(\frac{x-1}{x+1}\right)\left(\frac{y-1}{y+1}\right)\left(\frac{z-1}{z+1}\right) = \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c} = 1,$$

and ergo

$$(x-1)(y-1)(z-1) = (x+1)(y+1)(z+1).$$

Rearranging this equation gives that

$$xy + yz + zx = -1.$$

Therefore

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = 2 + 2(-1) = 0,$$

and so $x + y + z = 0$. That is,

$$\frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a} = 0.$$

We can now evaluate the value of the desired sum as

$$\begin{aligned} &\frac{a}{a-b} + \frac{b}{b-c} + \frac{c}{c-a} \\ &= \frac{1}{2} \left(\frac{(a+b) + (a-b)}{a-b} + \frac{(b+c) + (b-c)}{b-c} + \frac{(c+a) + (c-a)}{c-a} \right) \\ &= \frac{1}{2} \left[\left(\frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a} \right) + \frac{a-b}{a-b} + \frac{b-c}{b-c} + \frac{c-a}{c-a} \right] \\ &= \frac{1}{2} (0 + 1 + 1 + 1) \\ &= \frac{3}{2}. \end{aligned}$$

5123. *Proposed by Andrei Marinca.*

Let $ABCD$ be a convex quadrilateral in which we denote by O_1, O_2, O_3, O_4 the circumcenters of triangles ABD, ABC, BCD and CDA , respectively. Moreover, let R_1, R_2, R_3 and R_4 be the lengths of the circumradii of these respective triangles. Prove that the following inequality holds with square brackets denoting the area:

$$\frac{[BO_1DO_3]}{R_1R_3} + \frac{[AO_2CO_4]}{R_2R_4} \leq 4.$$

We received 8 correct solutions. We present the solution by Michal Adamaszek.

The upper bound given in the problem is too generous; it can be reduced to 2, but no further.

Since O_1 and O_3 are equidistant from B and D , O_1O_3 right bisects BD and the triangles O_1BO_3 and O_1DO_3 are congruent with $O_1B = O_1D = R_1$ and $O_3B = O_3D = R_3$. Using the fact that the area of any triangle does not exceed half the product of two of its sides, we have

$$[BO_1DO_3] = [O_1BO_3] + [O_1DO_3] \leq 2\left(\frac{1}{2}(R_1R_3)\right) = R_1R_3.$$

A similar argument applies to AO_2CO_4 and we find that each term of the left of the inequality does not exceed 1, so their sum does not exceed 2.

Comments by the editor. When the quadrilateral $ABCD$ is concyclic, all four circumcentres coincide. In particular, when it is a rectangle, the left side vanishes. Note that the maximum angle of any quadrilateral is at least 90° so that the centres O_1 and O_3 , say, can appear in either order and on either side of BD .

Is it possible for the left side to equal 2? Note that $2[BO_1DO_3] = BD \cdot O_1O_3$ and derive expressions for each factor. Let $\alpha, \beta, \gamma, \delta$ be the respective angles at A, B, C, D . Then $BD = 2R_1 \sin \alpha = 2R_3 \sin \gamma$. Let $M = BD \cap O_1O_3$. Depending on whether M lies between O_1 and O_3 , O_1O_3 is either the sum or difference of O_1M and O_3M . In any case, it turns out that

$$O_1M + O_3M = |R_1 \cos \alpha| + |R_3 \cos \gamma| = |R_1 \cos \alpha + R_3 \cos \gamma|.$$

Hence

$$[BO_1DO_3] = R_1R_3 |\sin \gamma \cos \alpha + \sin \alpha \cos \gamma| = R_1R_3 (\sin(\alpha + \gamma)).$$

Similarly,

$$[AO_2CO_4] = R_2R_4 (\sin(\beta + \delta)).$$

Since $(\alpha + \gamma) + (\beta + \delta) = 360^\circ$, the left side of the inequality is equal to

$$2 \sin(\alpha + \gamma) = 2 \sin(\beta + \gamma).$$

We conclude that one pair of opposite angles of the quadrilateral add up to 90° and the other pair to 270° .

An alternative approach is to note that, for equality,

$$\angle O_1BO_3 = \angle O_1DO_3 = \angle O_2AO_4 = \angle O_2CO_4 = 90^\circ,$$

and do a little angle-chasing. An example, identified by three solvers, is a parallelogram with angles equal to 45° and 135° .

5124. *Proposed by Ziji Hu.*

Consider N points in the Euclidean plane, no three collinear, with the property that for any triple P, Q, R of the points, the measure in degrees of $\angle PQR$ is an integer.

- What is the maximum size of N ?
- What is the maximum size of N if, in addition, all the given points have rational coordinates; that is, all points have rational x and y coordinates in the Cartesian plane?

We received 2 submissions of which both were correct and complete. We present an edited solution proposed by Agarwal Ansh.

We first show that $N = 180$ is maximal. Let 180 points $A_1, A_2, \dots, A_N$ be spaced equally on the circumference of the unit circle. Then, each arc A_iA_j on the unit circle is a multiple of 2° . As any angle $\angle A_iA_kA_j$ is half the measure of arc A_iA_j (a multiple of 2°), then $\angle A_iA_kA_j$ is an integer, showing $N = 180$ is achievable.

For the sake of contradiction, suppose that $N > 180$ is possible. Suppose there are k points $A_1, A_2, \dots, A_k$ on the convex hull of the N points. From the properties of polygons,

$$\angle A_1A_2A_3 + \angle A_2A_3A_4 + \dots + \angle A_kA_1A_2 = 180(k-2).$$

Hence, there exists some i s.t.

$$\angle A_{i-1}A_iA_{i+1} \leq \frac{180(k-2)}{k},$$

where indices are treated mod k . Wlog, assume that $\angle A_1A_2A_3 \leq 180(k-2)/k$. Now, when taking a ray from A_2 pointed to A_1 , and rotating it towards A_3 , this ray hits all points, as all points lie in the convex hull. Suppose it hits the points in the order $B_1, B_2, \dots, B_{N-1}$, where $B_1 = A_1$ and $B_{N-1} = A_3$. By definition,

$$\sum_{i=1}^{N-2} \angle B_iA_2B_{i+1} = \angle A_1A_2A_3.$$

Hence, there is some i such that

$$\angle B_iA_2B_{i+1} \leq \frac{\angle A_1A_2A_3}{N-2} = \frac{180(k-2)}{k(N-2)} \leq \frac{180}{N},$$

where the last inequality holds as $\frac{k-2}{k} = 1 - \frac{2}{k}$ is an increasing function for $k > 0$, and $k \leq N$. As B_i, A_2, B_{i+1} are distinct points, $\angle B_i A_2 B_{i+1}$ is a positive integer, hence at least 1. But, $\frac{180}{N} < 1$, a contradiction.

For part b), we show $N = 4$ is maximal. Observe that it is achieved by $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. To show the bound, we prove the following two claims.

- (1) Suppose distinct points A, B, C have rational coordinates in the Cartesian plane. Then, $\cot(\angle BAC)$ is rational or undefined.
- (2) *Niven's Theorem*. If a is a rational number s.t. $\cot(a\pi)$ is rational (and not undefined), then $\cot(a\pi) = 0, \pm 1$.

Proof of 1. Let $\vec{AB} = (a_1, a_2, 0)$ and $\vec{AC} = (b_1, b_2, 0)$ in 3-D. Let $\angle BAC = \theta$. If $\theta = 0, 180^\circ$ we're done. Otherwise, we get

$$|\cos(\theta)| = \left| \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|} \right| = \left| \frac{a_1 b_1 + a_2 b_2}{|\vec{AB}||\vec{AC}|} \right|.$$

Similarly,

$$|\sin(\theta)| = \left| \frac{\vec{AB} \times \vec{AC}}{|\vec{AB}||\vec{AC}|} \right| = \left| \frac{a_1 b_2 - a_2 b_1}{|\vec{AB}||\vec{AC}|} \right|.$$

Hence,

$$|\cot(\theta)| = \left| \frac{a_1 b_1 + a_2 b_2}{a_1 b_2 - a_2 b_1} \right|.$$

As A, B, C are rational points, so are a_1, a_2, b_1, b_2 , hence $\cot(\theta)$ is as well. An elementary proof can be done by letting D be a point to the right of A , then noting that $\tan(\angle BAD), \tan(\angle CAD)$ are the slopes of BA, CA suffices as they are both rational. Hence, by the tangent addition formula, so is $\tan(\angle BAC)$. $\square$

Proof of 2. This is Niven's Theorem, but an elementary proof is provided for the reader's convenience. For the sake of contradiction, assume that $\cot(a\pi) \neq 0, \pm 1$. Let $\cot(a \cdot 2^k \pi) = \frac{p_k}{q_k}$, for integers p_k, q_k with $\gcd(p_k, q_k) = 1$, and $q_k > 0$. We first check when this is well defined. Note that

$$\cot(a \cdot 2^{k+1} \pi) = \cot(2 \cdot a \cdot 2^k \pi) = \frac{\frac{p_k^2}{q_k^2} - 1}{\frac{2p_k}{q_k}} = \frac{p_k^2 - q_k^2}{2p_k q_k}.$$

Hence, $\cot(a \cdot 2^{k+1} \pi)$ will be rational as well, unless $\cot(a \cdot 2^k \pi)$ is 0. For the sake of contradiction, assume this does happen. Take $k \geq -1$ minimal such that $\cot(a \cdot 2^{k+1} \pi) = 0$. By definition, $k \neq -1$. So, $\cot(a \cdot 2^k \pi) \neq 0$ and is rational, but $\cot(a \cdot 2^{k+1} \pi) = 0$. This implies $a \cdot 2^{k+1} \pi = (2r+1)\pi/2$. So

$$\cot(a \cdot 2^k \pi) = \cot((2r+1)\pi/4) = \pm 1,$$

which implies $k \neq 0$. Hence

$$\cot(a \cdot 2^{k-1}\pi) = \cot((2r+1)\pi/8),$$

which is known to always be irrational, contradiction.

Therefore the definition of p_k, q_k is valid. Now, note that $\gcd(p_k^2 - q_k^2, p_k q_k) = 1$. Indeed, if a prime $r \mid \gcd(p_k^2 - q_k^2, p_k q_k)$, then $r \mid p_k q_k$. WLOG let $r \mid q_k$, so $r \mid p_k^2 - q_k^2 + q_k^2 = p_k^2$, so $r \mid p_k$, a contradiction to $\gcd(p_k, q_k) = 1$. Hence, $\gcd(p_k^2 - q_k^2, 2p_k q_k) \leq 2$, and so

$$q_{k+1} = \left\lfloor \frac{2p_k q_k}{\gcd(p_k^2 - q_k^2, 2p_k q_k)} \right\rfloor \geq |p_k q_k|.$$

As the p_i are never 0, this already implies that $q_{k+1} \geq q_k$. If $q_{k+1} = q_k$, then we must have $\gcd(p_k^2 - q_k^2, 2p_k q_k) = 2$. Hence, $2 \mid p_k^2 - q_k^2$, which implies $2 \mid p_k - q_k, p_k + q_k$. This implies that both p_k, q_k are odd, and $4 \mid p_k^2 - q_k^2$. In particular,

$$|p_{k+1}| = \left\lfloor \frac{p_k^2 - q_k^2}{\gcd(p_k^2 - q_k^2, 2p_k q_k)} \right\rfloor$$

is even. So, this implies $q_{k+2} > q_{k+1}$, and so we always have $q_{k+2} > q_k$. In particular, the denominators are always unbounded. But, as a is rational, $a \cdot 2^k \pi$ is periodic mod 2π , hence the q_k are periodic as well, a contradiction, proving the claim. $\square$

From here, we are almost done. For the sake of contradiction, suppose $N > 4$. By similar logic to part a), we may find distinct points A, B, C such that

$$\angle ABC \leq \frac{180}{N} < 45^\circ.$$

So, $\cot(\angle ABC) > 1$. By claim 1, $\cot(\angle ABC)$ is rational, and as $\angle ABC$ is a rational multiple of π , this is in contradiction to claim 2. We are done.

5125. *Proposed by Tatsunori Irie, modified by the Editorial Board.*

Given a positive integer n such that $n+1$ is not a perfect power, show that the value of $\frac{\text{rad}(n)}{\text{rad}(n+1)}$ can be arbitrarily large, where $\text{rad}(n)$ is the product of distinct prime divisors of n .

We received 5 solutions, of which 3 were correct and complete. We present all 3 solutions.

Solution 1, by Ansh Agarwal.

Let p be an arbitrarily large odd prime and $n = p^2(4k+2) - 1$ for a positive integer k . Then $n+1 = 2p^2(2k+1)$ is divisible by 2, but not by 2^2 , so cannot be

a perfect power. Also $p(\text{rad}(n+1)) \leq n+1$. If n happens to be squarefree, then $\text{rad}(n) = n$ and

$$\frac{\text{rad}(n)}{\text{rad}(n+1)} \geq \frac{np}{n+1} \geq \frac{p}{2}.$$

So we need to find a value of k to make n squarefree.

Let N be a positive integer and $1 \leq k \leq N$. Suppose q is a prime such that q^2 divides $n = 2p^2(2k+1) - 1$. Then $q < 3p\sqrt{k} \leq 3p\sqrt{N}$ and

$$k \equiv \frac{1}{4p^2} - \frac{1}{2} \pmod{q^2}.$$

Thus, q^2 divides at most $(N/q^2) + 1$ of the N values of n .

Hence the number of values of k for which there exists a prime q whose square divides n is at most

$$\sum_{q=3}^{3p\sqrt{N}} \left(\frac{N}{q^2} + 1 \right) < \sum_{q=3}^{\infty} \frac{N}{q^2} + 3p\sqrt{N} < N \left(\frac{\pi^2}{6} - 1.25 \right) + 3p\sqrt{N} < 0.5N + 3p\sqrt{N}.$$

For sufficiently large N , this is less than N and so there is some $k \in [1, N]$ for which there is no prime q with q^2 dividing n . Thus, for each prime p a squarefree value of n can be found.

Solution 2, by Walther Janous.

For each positive integer k , let $n_k = 3 \cdot 2^k - 1$, so that $n_k + 1 = 3 \cdot 2^k$ and $\text{rad}(n_k + 1) = 6$. Let $a_1 = n_1 = 5$. We define an increasing subsequence $\{a_i\}$ of $\{n_k\}$ inductively whose entries are pairwise coprime. This will ensure that for any integer m , some a_i will be divisible by a prime exceeding m and so $\text{rad}(a_i)$ will exceed m as well. This yields the result, since $\text{rad}(a_i)/\text{rad}(a_i + 1) > m/6$.

Suppose that $a_1 < a_2 < \dots < a_i$ have already been selected. Let $b = a_1 a_2 \dots a_i$. By the pigeonhole principle, we can select 2^u and 2^v with $0 \leq u < v \leq b$ that are congruent modulo b . Hence $2^w \equiv 1 \pmod{b}$ where $w = v - u$.

Then $2^w - 1 = cb$ for some positive integer c , so that

$$3 \cdot 2^w - 1 = 3cb + 2.$$

This number is coprime with b , and thus to each of its factors. Let $a_{i+1} = n_w$. Also

$$a_{i+1} > 3b > b > a_i.$$

Solution 3, by Michal Adamaszek.

Let $a_k = 2 \cdot 3^{p_k-1} - 1$, where p_k is the k th prime, beginning with 5. If the sequence $\{\text{rad}(a_k)\}$ is unbounded, then so is $\{\text{rad}(a_k)/\text{rad}(a_k + 1)\}$.

On the other hand, if $\text{rad}(a_k) \leq M$ for all k , then

$$\text{rad}(a_k - 1) = \text{rad}(2 \cdot (3^{p_k-1} - 1)) \geq 2p_k,$$

since $3^{p_k-1} \equiv 1$ modulo p_k , so the ratio of the rads of $a_k - 1$ and a_k can be arbitrarily large. We need to show that a_k is not a perfect power.

Suppose that $2 \cdot 3^m - 1 = x^n$ for some $m, x, n \geq 2$. Then $x = 2s - 1$ and $n = 2k + 1$ for positive integers s and k (squares not congruent to -1 modulo 3), and

$$2 \cdot 3^m = (x + 1)(x^{2k} - x^{2k-1} + \cdots - x + 1) = 2s(x^{2k} - x^{2k-1} + \cdots - x + 1).$$

Hence $s = 3^a$ and $x^{2k} - x^{2k-1} + \cdots - x + 1 = 3^b$ for positive integers a and b with $a + b = m$. Therefore $x \equiv -1 \pmod{3}$ and

$$0 \equiv 3^b = x^{2k} - x^{2k-1} + \cdots - x + 1 \equiv 2k + 1 = n \pmod{3}.$$

Thus n is divisible by 3. Replacing x^n by $(x^{n/3})^3$, we see that it suffices to prove that there are no solutions for the case $n = 3$.

For $n = 3$, $x = 2 \cdot 3^a - 1$ and $x^2 - x + 1 = 3^b$, from which

$$3^a(4 \cdot 3^{a-1} - 2) + 1 = 3^{b-1},$$

which is not possible.

The argument for the second case may be moot, since it is likely that $\{\text{rad}(a_k)\}$ is unbounded.

Comment by the editor. As you would expect, the sequence $\{a_i\}$ in Solution 2 grows very rapidly: $a_1 = 5$, $a_2 = 47$ and

$$a_3 = 11 \times 13 \times 83 \times 5279089 \times 237087000454357307,$$

according to AI. If we start with $a_1 = n_1 = 5$ and $a_2 = n_2 = 11$, then $a_3 = 13 \times 241979$. If we look at the sequence $\{n_k\}$ itself, seven of the first twelve entries are prime and the remaining five are the product of two distinct primes, so all are equal to their rads. The smallest two entries in the sequence that are not square free are $n_{13} = 24575 = 5^2 \times 983$ and $n_{22} = 12582911 = 11^2 \times 103991$.

5126. *Proposed by Phuc Nguyen.*

Let a, b, c, d be positive real numbers with $abcd = 1$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 2\sqrt{a^2 + b^2 + c^2 + d^2}.$$

The above inequality does not hold. We received 8 submissions, of which 4 were counterexamples. We present the one by Michal Adamaszek.

We show that the inequality is incorrect by presenting a counterexample. Consider the positive real numbers $a = 1, b = \frac{1}{2}, c = \frac{1}{2}, d = 4$. The left-hand side gives us

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = \frac{1}{\frac{1}{2}} + \frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{1}{2}}{4} + \frac{4}{1} = 2 + 1 + \frac{1}{8} + 4 = 7 + \frac{1}{8},$$

and the right-hand side gives us

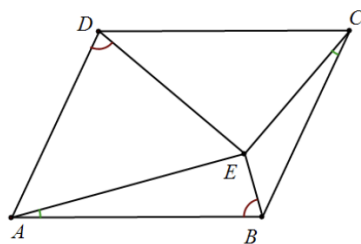
$$a^2 + b^2 + c^2 + d^2 = 1^2 + \frac{1}{2^2} + \frac{1}{2^2} + 4^2 = 1 + \frac{2}{4} + 16 = 17 + \frac{1}{2}.$$

Since $2\sqrt{17 + \frac{1}{2}} > 2\sqrt{16} = 8 > 7 + \frac{1}{8}$, then we may conclude that the inequality does not hold for all positive real numbers a, b, c , and d .

5127. *Proposed by Xicheng Peng.*

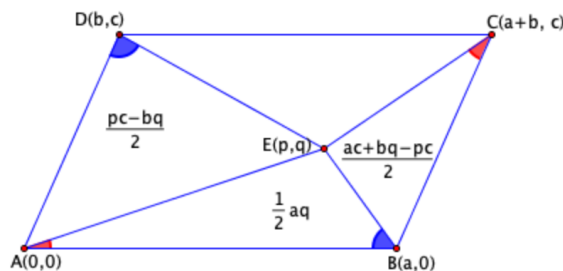
With square brackets denoting area, prove that for any point E inside the parallelogram $ABCD$,

$$[AED](\cot \angle EBA - \cot \angle ADE) = [BCE](\cot \angle BAE - \cot \angle ECB).$$



We received 14 submissions, all of which were correct. Most of the solutions just verified that equality held. We will sample one such verification, then present a second solution that was unique.

Solution 1, by C.R. Pranesachar.



We shall use coordinate geometry. For $a, b, c, p, q > 0$ let

$$A = (0, 0), B = (a, 0), C = (a + b, c), D = (b, c), \text{ and } E = (p, q).$$

We have

$$[ADE] = \frac{1}{2}(pc - bq), \quad [BCE] = \frac{1}{2}(ac + bq - pc), \quad [ABE] = \frac{1}{2}aq.$$

Observe that

$$\cos \angle ADE = \frac{AD^2 + DE^2 - AE^2}{2 \cdot AD \cdot DE}, \quad \text{while} \quad \sin \angle ADE = \frac{2[ADE]}{AD \cdot DE}, \quad \text{etc.,}$$

The desired equation may be rewritten as

$$[AED] \cot \angle ADE - [BCE] \cot \angle ECB = [AED] \cot \angle EBA - [BCE] \cot \angle BAE.$$

Consequently,

$$\begin{aligned} \text{LHS} &= [AED] \cdot \frac{AD^2 + DE^2 - AE^2}{4[AED]} - [BCE] \cdot \frac{EC^2 + CB^2 - EB^2}{4[BCE]} \\ &= \frac{1}{4} ((b^2 + c^2) + ((b-p)^2 + (c-q)^2) - (p^2 + q^2)) \\ &\quad - \frac{1}{4} (((a+b-p)^2 + (c-q)^2) + (b^2 + c^2) - ((a-p)^2 + q^2)) \\ &= \frac{1}{2}(b^2 + c^2 - bp - cq) - \frac{1}{2}(b^2 + c^2 - bp - cq + ab) \\ &= -\frac{1}{2}ab. \end{aligned}$$

Similarly,

$$\begin{aligned} \text{RHS} &= \frac{1}{2}(pc - bq) \cdot \frac{EB^2 + BA^2 - EA^2}{4[ABE]} - \frac{1}{2}(ac + bq - pc) \cdot \frac{BA^2 + AE^2 - BE^2}{4[ABE]} \\ &= \frac{1}{2}(pc - bq) \frac{(((a-p)^2 + q^2) + a^2 - (p^2 + q^2))}{4 \cdot \frac{1}{2}aq} \\ &\quad - \frac{1}{2}(ac + bq - pc) \frac{(a^2 + (p^2 + q^2) - ((a-p)^2 + q^2))}{4 \cdot \frac{1}{2}aq} \\ &= \frac{1}{4}(pc - bq) \cdot \frac{(2a^2 - 2ap)}{aq} - \frac{1}{4}(ac + bq - pc) \cdot \frac{2ap}{aq} \\ &= (pc - bq) \cdot \frac{(a-p)}{2q} - (ac + bq - pc) \cdot \frac{p}{2q} \\ &= -\frac{1}{2}ab. \end{aligned}$$

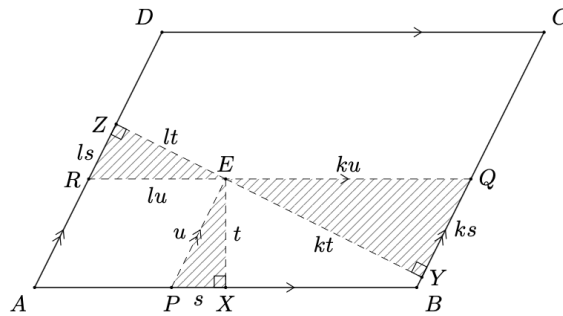
The left-hand side equals the right-hand side, which completes the proof.

Solution 2 by Chikara Tsugawa.

Let X, Y and Z be the feet of the perpendiculars from E to the lines AB, BC and DA , respectively. Let the line through E parallel to AD meet AB at P , and let the

line through E parallel to AB meet BC and DA at Q and R , respectively. Finally, let $PX = s$, $XE = t$, $EP = u$. Since the right triangles $\triangle EPX, \triangle EQY, \triangle ERZ$ are similar, there exist positive numbers k and l such that

$$QY = ks, YE = kt, EQ = ku, \text{ and } RZ = ls, ZE = lt, ER = lu.$$



Hence,

$$\begin{aligned} AB \cdot \frac{s}{t} &= RQ \cdot \frac{s}{t} = (ku + lu) \cdot \frac{s}{t} \\ &= (ks + ls) \cdot \frac{u}{t} \\ &= ((CY - CQ) + (DR - DZ)) \cdot \frac{u}{t} \\ &= (CY - DZ) \cdot \frac{u}{t} \\ &= CY \cdot \frac{EQ}{EY} - DZ \cdot \frac{ER}{EZ} \\ &= ku \cdot \frac{CY}{EY} - lu \cdot \frac{DZ}{EZ} \\ &= ku \cot \angle ECB - lu \cot \angle ADE. \end{aligned}$$

On the other hand,

$$\begin{aligned} AB \cdot \frac{s}{t} &= AB \cdot \frac{PX}{EX} \\ &= \frac{AB(ku - XB)}{EX} \\ &= \frac{AB \cdot ku - (lu + ku)XB}{EX} \\ &= \frac{-lu \cdot XB + ku(AB - XB)}{EX} \\ &= \frac{-lu \cdot XB + ku \cdot AX}{EX} \\ &= -lu \cot \angle EBA + ku \cot \angle BAE. \end{aligned}$$

Therefore,

$$\begin{aligned} ku \cot \angle ECB - lu \cot \angle ADE &= -lu \cot \angle EBA + ku \cot \angle BAE \\ l(\cot \angle EBA - \cot \angle ADE) &= k(\cot \angle BAE - \cot \angle ECB). \end{aligned}$$

Since

$$\frac{[AED]}{[BCE]} = \frac{[APD]}{[BCP]} = \frac{AP}{PB} = \frac{l}{k},$$

we have

$$[AED](\cot \angle EBA - \cot \angle ADE) = [BCE](\cot \angle BAE - \cot \angle ECB),$$

as desired.

5128. *Proposed by Vasile Cîrtoaje.*

Let a, b, c be positive real numbers such that $a \geq b \geq c$ and $ab^2c^3 \geq 1$. Prove that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \geq \frac{3}{\sqrt[3]{abc} + 1}.$$

We received 5 solutions, all correct and complete. We present the solution by Ansh Agarwal.

Let $f(x) = \frac{1}{x+1}$. The main claim is as follows.

Claim. Let x, y be positive real numbers such that $xy \geq 1$. Then,

$$f(x) + f(y) \geq 2f(\sqrt{xy}).$$

Proof. Let $m = \sqrt{x}$ and $n = \sqrt{y}$, so we wish to prove

$$\frac{1}{m^2+1} + \frac{1}{n^2+1} \geq \frac{2}{mn+1}.$$

But we have

$$\frac{1}{m^2+1} + \frac{1}{n^2+1} - \frac{2}{mn+1} = \frac{(mn-1)(m-n)^2}{(m^2+1)(n^2+1)(mn+1)} \geq 0.$$

The last inequality holds as $mn = \sqrt{xy} \geq 1$, and all other terms are trivially non-negative (as m, n are positive). This proves the claim.

Now, applying the Claim we have

$$\begin{aligned} f(a) + f(b) + f(c) + f(\sqrt[3]{abc}) &\geq 2f(\sqrt{ab}) + f(c) + f(\sqrt[3]{abc}) \\ &\geq 2f(\sqrt[4]{abc^2}) + f(\sqrt{ab}) + f(\sqrt[3]{abc}) \\ &\geq 2f(\sqrt[4]{abc^2}) + 2f(\sqrt[12]{a^5b^5c^2}) \\ &\geq 4f(\sqrt[3]{abc}). \end{aligned}$$

The first inequality holds because $a^3b^3 \geq ab^2c^3 \geq 1$, hence $\sqrt{ab} \geq 1$. The second inequality holds because $a^{3/2}b^{3/2}c^3 \geq ab^2c^3$, hence $\sqrt[4]{abc^2} \geq 1$. The third inequality holds because $a^{5/2}b^{5/2}c \geq ab^2c^3$, hence $\sqrt[12]{a^5b^5c^2} \geq 1$. The last inequality holds because $a^2b^2c^2 \geq ab^2c^3$, hence $\sqrt[3]{abc} \geq 1$. Overall, this implies

$$f(a) + f(b) + f(c) + f(\sqrt[3]{abc}) \geq 4f(\sqrt[3]{abc}) \text{ or } f(a) + f(b) + f(c) \geq 3f(\sqrt[3]{abc})$$

and we are done.

5129. *Proposed by José Rodolfo Das López.*

Given $n \in \mathbb{N}$, we consider the product of factorials

$$(4n)! \cdot (4n-1)! \cdot (4n-2)! \cdots 4! \cdot 3! \cdot 2! \cdot 1!$$

Prove that for each n we can remove one factorial, $k!$, with $1 \leq k \leq 4n$, such that the product of the remaining factorials is a perfect square.

There were 21 solutions submitted, all correct except for one that was incomplete. The following 3 solutions cover all the solution methods presented.

Solution 1.

Let P_n be the given product. Then

$$\begin{aligned} P_n &= \prod_{k=1}^{2n} (2k)!(2k-1)! = \prod_{k=1}^{2n} (2k)[(2k-1)!]^2 \\ &= \prod_{k=1}^{2n} (2k) \cdot \left[\prod_{k=1}^{2n} (2k-1)! \right]^2 \\ &= 2^{2n} \cdot (2n)! \cdot \left[\prod_{k=1}^{2n} (2k-1)! \right]^2, \end{aligned}$$

from which it follows that P_n divided by $(2n)!$ is a square.

Solution 2.

$P_1 = 2! \times (12)^2$. Assume that $P_n = (2n)!S_n^2$ for some $n \geq 1$ and integer S_n . Then

$$\begin{aligned} P_{n+1} &= (4n+4)!(4n+3)!(4n+2)!(4n+1)!P_n \\ &= (4n+4)(4n+2)(2n)![(4n+3)!(4n+1)!S_n]^2 \\ &= (2n+2)(2n+1)(2n)!S_{n+1}^2 = (2n+2)!S_{n+1}^2, \end{aligned}$$

where

$$S_{n+1} = 2(4n+3)!(4n+1)!S_n.$$

It follows by induction that P_n divided by $(2n)!$ is square.

Solution 3.

Each number k with $1 \leq k \leq 4n$ appears only in the $4n - k + 1$ factorials $k!, \dots, (4n)!$. Thus, any odd number k divides P_n to an even power and any even number k divides P_n to an odd power. Thus P_n is equal to $2 \cdot 4 \cdot 6 \cdots (4n) = 2^{2n}(2n)!$ times a square. Therefore, dividing P_n by $(2n)!$ yields a square.

5130. *Proposed by Mihaela Berindeanu.*

Let a, b, c be distinct complex numbers with the property $|a| = |b| = |c| = 1$. Show that

$$|2a - b - c| \geq \frac{|a - b|^2 + |a - c|^2}{2}.$$

We received 24 solutions, all of which were correct and complete. We present the solution by Prithwijit De.

Without loss of generality, let $a = 1$ (else, divide both sides of the equation by a and relabel $\frac{b}{a}$ and $\frac{c}{a}$ to b and c respectively). Let $u = 1 - b$ and $v = 1 - c$. Then $|1 - u| = |b| = 1$ and $|1 - v| = |c| = 1$, and we are to show that

$$|u + v| \geq \frac{|u|^2 + |v|^2}{2}.$$

We have

$$1 = |1 - u|^2 = 1 - u - \bar{u} + |u|^2 = 1 - 2 \operatorname{Re} u + |u|^2$$

so $\operatorname{Re} u = \frac{|u|^2}{2}$, and likewise $1 = |1 - v|^2 = 1 - 2 \operatorname{Re} v + |v|^2$ so $\operatorname{Re} v = \frac{|v|^2}{2}$. Hence,

$$|u + v| \geq \operatorname{Re}(u + v) = \frac{|u|^2 + |v|^2}{2}.$$

