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Editorial Board

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Olympiad Gold to Fields Medal: Congratulations to Jacob Tsimerman

On behalf of ***Crux***, we would like to extend our warmest congratulations to Jacob Tsimerman, Professor of Mathematics at the University of Toronto, who was awarded a 2026 Fields Medal.

Tsimerman's connection to the Canadian mathematical community goes back to his teenage years. As a student at University of Toronto Schools, he represented Canada at the International Mathematical Olympiad in 2003 and 2004, winning gold medals both times and achieving a perfect score in 2004. He subsequently gave back to the Canadian problem solving community, serving as an IMO team leader, chairing the CMS IMO Committee and of course contributing to ***Crux*** such as with article "Introduction to Inequalities." in Volume 41, No. 5.

On behalf of ***Crux***, we sincerely congratulate Jacob Tsimerman on receiving the 2026 Fields Medal and perhaps a special congratulations from the problem-solving community to one of its most distinguished alumni.

In Memoriam: Florian Luca

Florian specialized in number theory, focusing on Diophantine equations, linear recurrence sequences, and the distribution of arithmetic function values. He made remarkable contributions to proving that irrational automatic numbers are transcendental, as well as proving an Erdős conjecture regarding the intersection between Euler's totient function and the sum-of-divisors function.

Luca graduated from the Faculty of Mathematics at the "Alexandru Ioan Cuza" University of Iaşi in 1992 and defended his PhD in mathematics at the University of Alaska Fairbanks in 1996. Throughout his career, he held various academic positions at Syracuse University, Williams College, Bielefeld University, the Czech Academy of Sciences, the National Autonomous University of Mexico, and the University of the Witwatersrand. As of 2024, Florian was a distinguished professor at Stellenbosch University in South Africa. In early 2024, he received a new distinction, being named an Ambassador Fellow of the French National Centre for Scientific Research.

Florian Luca authored or co-authored nearly 800 scientific publications, some of them in top-tier journals, collaborating with 351 co-authors and holding an Erdős number of 1. On Google Scholar, his work has been cited over 8,000 times.

Florian never lost his foundational passion for elementary problem-solving, anchoring a three-decade relationship with ***Crux***. Florian was a master architect of problems that were deceptively simple in their statements but hidden within them were profound structural truths:

- Problem 2972 [Vol. 30(6), 2004]: A structural inequality exploring the boundary properties of composite numbers through Euler's totient and divisor functions.

- Problem 3175 [Vol. 32(7), 2006]: A deep Diophantine challenge exploring the factorization of $n!$ under the constraint of consecutive Fibonacci exponents.
- Problem 3452 [Vol. 35(5), 2009]: An advanced problem in recreational number theory proving the arithmetic impossibility of matching concatenated Fibonacci and Pell sequences.

Florian Luca treated mathematics not as a solitary pursuit of status, but as a shared language of joy, curiosity, and friendship. He used ***Crux*** to bridge pure research and competition math, sharing beautiful problems from Latin America and Africa to train and inspire generations of young minds globally. We bid farewell to a titan, a mentor, and a friend of the problem solving community.

Neculai Stanciu

MathemAttic

No. 77

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 15, 2026**.

MA381. *Proposed by Neculai Stanciu.*

Determine the five-digit numbers $abcde$ and $xyztu$ with distinct digits 1, 2, 3, 7 and 8 such that $4 \cdot abcde = xyztu$.

MA382. *Proposed by Nguyen Van An.*

There are 1001 boxes placed around a circle, labeled $B_1, B_2, \ldots, B_{1001}$ clockwise. Initially box B_i contains i balls ($i = 1, 2, \ldots, 1001$). Fix a positive integer k. A move consists of choosing k balls from k different nonempty boxes (one ball per chosen box) and, for each chosen ball that is in B_n, moving it into B_{n+1} (with the convention $B_{1002} \equiv B_1$). The game ends when all boxes contain the same number of balls.

(a) For $k = 1$, show how to finish the game.

(b) For $k = 1$, prove that the number of moves to finish the game is always divisible by 1001.

(c) For $2 \leq k \leq 1001$, is it always possible to finish the game? Explain and give a method when it is possible.

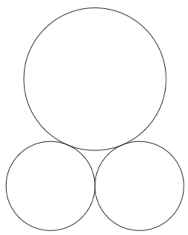
MA383. Two forgetful friends agree to meet in a coffee shop one afternoon but each has forgotten the agreed time. Each remembers that the agreed time was somewhere between 2 pm and 5 pm. Each decides to go to the coffee shop at a random time between 2 pm and 5 pm, wait for half an hour, and leave if the other doesn't arrive. Find the probability that they meet.

MA384. Starting from a random place, Hira decided to find the sum of 9 consecutive powers of 3. She was interested to know whether that could give her the same answer as adding up some number of consecutive integers starting at 1. Find a solution to Hira's problem. Show whether or not there are any other solutions.

MA385. Two circular discs of radius 5 cm and one circular disc of radius 8 cm are placed flat on a table with their edges touching.

(a) Determine the exact radius of the largest disc that can fit in the space between these three discs.

(b) Determine the exact radius of the smallest disc that can surround these three discs.



. . . .

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2026**.

MA381. *Soumis par Neculai Stanciu.*

Déterminez les nombres à cinq chiffres $abcde$ et $xyztu$, dont les chiffres distincts sont 1, 2, 3, 7 et 8, tels que $4 \cdot abcde = xyztu$.

MA382. *Soumis par Nguyen Van An.*

On dispose 1001 boîtes autour d'un cercle, étiquetées $B_1, B_2, \ldots, B_{1001}$ dans le sens horaire. Initialement, la boîte B_i contient i boules ($i = 1, 2, \ldots, 1001$). Fixons un entier positif k. Un coup consiste à choisir k boules provenant de k boîtes non vides distinctes (une boule par boîte choisie) et, pour chaque boule choisie se trouvant dans B_n, à la déplacer dans B_{n+1} (avec la convention $B_{1002} \equiv B_1$). Le jeu se termine lorsque toutes les boîtes contiennent le même nombre de boules.

(a) Pour $k = 1$, montrez comment terminer le jeu.

(b) Pour $k = 1$, montrez que le nombre de coups nécessaires pour terminer le jeu est toujours divisible par 1001.

(c) Pour $2 \leq k \leq 1001$, est-il toujours possible de terminer le jeu ? Justifiez votre réponse et donnez une méthode lorsque cela est possible.

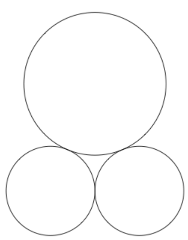
MA383. Deux amis distraits conviennent de se retrouver dans un café un après-midi, mais chacun a oublié l'heure convenue. Tous deux se souviennent que celle-ci se situait quelque part entre 14 h et 17 h. Chacun décide de se rendre au café à une heure choisie au hasard entre 14 h et 17 h, d'attendre une demi-heure, puis de partir si l'autre n'arrive pas. Déterminez la probabilité qu'ils se rencontrent.

MA384. En partant d'un endroit choisi au hasard, Hira a décidé de calculer la somme de 9 puissances consécutives de 3. Elle voulait savoir si elle pouvait obtenir le même résultat en additionnant un certain nombre d'entiers consécutifs à partir de 1. Trouvez une solution au problème de Hira. Montrez s'il existe ou non d'autres solutions.

MA385. Trois disques circulaires, deux de rayon 5 cm et un de rayon 8 cm, sont posés à plat sur une table de façon que leurs bords se touchent.

(a) Déterminez le rayon exact du plus grand disque pouvant tenir dans l'espace compris entre ces trois disques.

(b) Déterminez le rayon exact du plus petit disque pouvant contenir ces trois disques.



MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2026: 52(2), p. 63–64.

MA356. Let a, b, c, d be distinct positive numbers. What is the total number of different values for the expression $xy + zw$ if the variables x, y, z, w are assigned the values of a, b, c, d in all possible orders?

Originally from the World Federation of National Mathematics Competitions, First International Conference, Problem 100 (Revised Day 1 version).

We received 15 submissions, of which 11 were correct and complete. We present the solution by Parth Andhare.

Let a, b, c, d be distinct positive numbers. We first note that the arithmetic expression $xy + zw$ remains unchanged by swapping $x \leftrightarrow y$, $z \leftrightarrow w$, or $xy \leftrightarrow zw$, due to the commutative properties of addition and multiplication. Hence, the value of the expression depends only on the unordered partition of $\{a, b, c, d\}$ into two unordered pairs. There are exactly three ways to partition the set $\{a, b, c, d\}$ into two unordered pairs:

$$\mathcal{P}_1 = \{\{a, b\}, \{c, d\}\}, \quad \mathcal{P}_2 = \{\{a, c\}, \{b, d\}\}, \text{ and } \quad \mathcal{P}_3 = \{\{a, d\}, \{b, c\}\}.$$

The corresponding sums are:

$$S_1 = ab + cd, \quad S_2 = ac + bd, \text{ and } \quad S_3 = ad + bc.$$

It remains to show that all three sums are distinct. We note that:

$$\begin{aligned} S_1 - S_2 &= ab + cd - ac - bd = (a - d)(b - c), \\ S_1 - S_3 &= ab + cd - ad - bc = (a - c)(b - d), \text{ and} \\ S_2 - S_3 &= ac + bd - ad - bc = (a - b)(c - d). \end{aligned}$$

By assumption, a,b,c and d are all distinct. Therefore, all the factors $(a - b)$, $(a - c)$,$(a - d)$,$(b - c)$,$(b - d)$ and $(c - d)$ are nonzero, and the sums S_1, S_2, and S_3 are pairwise distinct. We conclude that the expression $xy + zw$ takes exactly 3 distinct values.

MA357. Forty-one rooks are placed on a 10×10 checkerboard. Prove that you can choose 5 of them that do not attack one another. (We say that one rook "attacks" another if they are in the same row or column of the checkerboard.)

Originally from the World Federation of National Mathematics Competitions, First International Conference, Problem 144 (Revised Day 1 version).

We received 7 submissions, of which 4 were correct and complete. We present the solution by José Luis Díaz-Barrero.

The key idea for this prove is to use the following lemma.

Lemma. For any configuration of rooks, the maximum number of pairwise non-attacking rooks equals the minimum number of rows and columns whose union contains all the rooks.

Proof. Any set of k non-attacking rooks occupies k distinct rows and k distinct columns, so any covering set of rows/columns must contain at least k lines. Thus

$$\text{max non-attacking rooks} \leq \text{min covering lines}.$$

Conversely, let m be the size of a minimal covering set of rows/columns. Each line in this cover contains a rook that lies in none of the other lines of the cover (otherwise the cover would not be minimal). Choosing one such rook from each line gives m rooks in distinct rows and columns. Hence

$$\text{max non-attacking rooks} \geq \text{min covering lines}.$$

Therefore, the two quantities are equal, and the lemma is proven. □

Now return to the 10×10 board with 41 rooks. Suppose, for contradiction, that no 5 rooks are pairwise non-attacking. Then by the lemma, all rooks can be covered by at most 4 rows and columns. Simple calculations show that any combination of four rows or columns will contain at most 40 rooks, so 4 lines cover at most 40 squares. But there are 41 rooks, each on a distinct square, so no 4 lines can cover all of them. This contradicts the assumption that no 5 non-attacking rooks exist. Therefore, there must exist 5 rooks no two of which attack each other.

MA358. *Proposed by Victor Manuel Mesa Solano.*

Show that if a number has exactly four proper factors (that is, four factors different from one or the number itself), then the sum of their reciprocals cannot be 1.

We received 16 submissions, of which 6 were correct and complete as most solvers forgot to consider one of the cases, most often that of p^5. We present the editor's solution.

For a number to have exactly four proper factors, it must be of the form p^5 or pq^2, where p, q are distinct primes.

If the number is p^5, then

$$\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} < 1.$$

Now suppose the number is pq^2. Its four proper factors are p, q, pq, q^2, whose reciprocals sum to

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{pq} + \frac{1}{q^2} = \frac{(q+1)(p+q)}{pq^2}.$$

Suppose this equals 1. If p and q are both odd, the numerator is even while the denominator is odd, a contradiction. If $q = 2$, then $3(p+2) = 4p$, giving $p = 6$. If $p = 2$, then

$$(q+1)(q+2) = 2q^2,$$

or $q^2 - 3q - 2 = 0$, which has no integer solutions.

Thus the sum of the reciprocals of the four proper factors cannot be 1.

MA359. *Proposed by Michael Friday.*

Let R be the circumradius of triangle ABC with $\angle A = 15^\circ$, $\angle B = 105^\circ$, and T the foot of the bisector of $\angle C$. Prove that $R^2 = BC \times CA$ and $CT^2 = AT \times BT$.

We received 19 solutions, all of which were correct and complete. We present the solution by Vasile Teodorovici.

Let O be the center of the circumcircle. We know

$$\begin{aligned}
\angle ACB = 60^\circ &\implies \angle ACT = \angle TCB = 30^\circ, \\
\angle ABC = 105^\circ &\implies \angle AOC = 150^\circ, \\
\angle BAC = 15^\circ &\implies \angle BOC = 30^\circ.
\end{aligned}$$

Applying the law of cosines in $\triangle AOC$,

$$AC^2 = R^2 + R^2 - 2R^2 \cos 150^\circ = 2R^2\left(1 + \frac{\sqrt{3}}{2}\right).$$

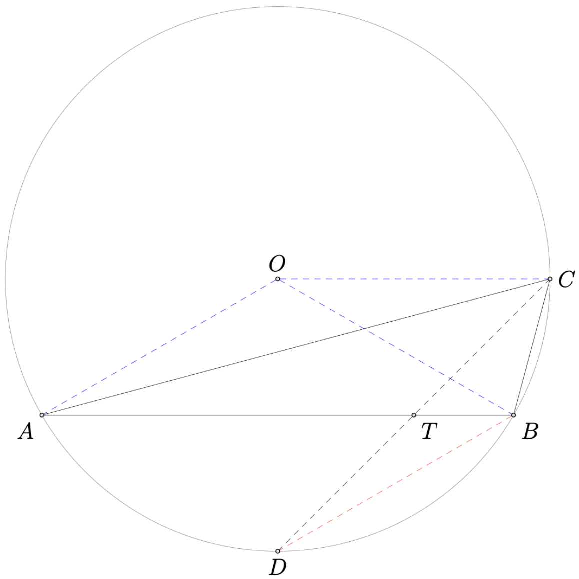
Applying the law of cosines in $\triangle BOC$,

$$BC^2 = R^2 + R^2 - 2R^2 \cos 30^\circ = 2R^2\left(1 - \frac{\sqrt{3}}{2}\right).$$

It follows that $AC \cdot BC = R^2$.

Next, extend the bisector CT so it intersects the circle at D. Using the Power of a Point Theorem at T gives

$$AT \cdot TB = DT \cdot TC.$$



We know $\angle TBD = \angle DCA = 30°$, and $\angle BDT = \angle BAC = 15°$. Then in $\triangle BDT$, $\angle BTD = \angle DBC = 135°$, so $\triangle BDT \sim \triangle CDB$, hence

$$\frac{BD}{DT} = \frac{CD}{DB}.$$

Applying the law of sines,

$$\frac{BD}{DT} = \frac{\sin 135°}{\sin 30°} = \sqrt{2}.$$

Thus,

$$CD = \frac{BD^2}{DT} = 2DT \implies DT = TC,$$

and so, $AT \cdot BT = TC^2$.

MA360. The geometry group 4G has a number (more than one) of committees, each of which contains 4 members of the club. Membership has been established to guarantee that for every two members of 4G there is exactly one committee to which both belong, and every two committees have at least one member in common.

a) Show that each person belongs to exactly 4 committees.

b) How many persons belong to 4G?

Originally from the Cariboo College High School Mathematics Contest, 1981 Senior Final Part B, Q5 (adapted).

We did not receive any submissions to this problem.

PROBLEM SOLVING VIGNETTES

No. 43

Shawn Godin

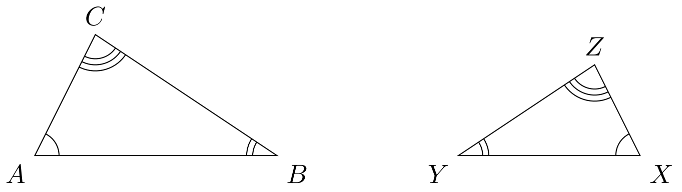
Similar Triangles

Over the years, the amount of geometry that is taught in schools has been on the decline. When I started teaching, I noticed the difference from when I was in school. By the time I retired, I had witnessed even more geometry being removed from the curriculum. This is a shame, not only because geometry is a beautiful field and a wonderful doorway to proof, but also because of the rise of dynamic geometry software, like *GeoGebra*™ (www.geogebra.org) that allows you to explore, discover, and understand geometry in a dynamic, hands on way. In my own schooling, geometric proofs using congruent or similar triangles were common. We explored congruent triangles in the very first number of this column [2019: 45(1), 13-16]. In this issue, we will look at similar triangles.

As we saw in the column on congruence, we defined two figures to be congruent if they have the same size and shape. For triangles, we would write $\Delta ABC \cong \Delta XYZ$ to state that triangle ABC and XYZ are congruent. This statement is equivalent to

$$\begin{array}{lll} AB = XY & BC = YZ & CA = ZX \\ \angle A = \angle X & \angle B = \angle Y & \angle C = \angle Z \end{array}$$

Two geometric figures in the plane are said to be *similar* if they have the same shape, but not necessarily the same size. This means that we can scale one of the figures up or down to make it congruent with the other.



For two triangles, we write $\Delta ABC \sim \Delta XYZ$ to state that triangle ABC and XYZ are similar. If two triangles are similar, their corresponding angles are equal, while their corresponding sides are in the same proportion. That is

$$\angle A = \angle X, \quad \angle B = \angle Y, \quad \angle C = \angle Z$$

and

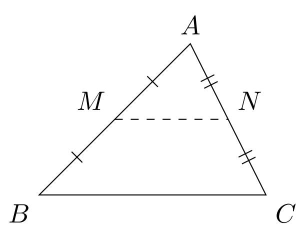
$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}.$$

With congruence, we had a number of shortcuts: side-side-side (SSS), side-angle-side (SAS), angle-side-angle (ASA), and hypotenuse-side (HS). Likewise, we have some shortcuts to similarity.

- **angle-angle (AA$\sim$)** - when two angles from one triangle are equal to two angles of another (e.g. $\angle A = \angle X$ and $\angle B = \angle Y$).
- **side-side-side (SSS$\sim$)** - when the sides from one triangle are in the same proportion with the sides from another triangle (e.g. $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$).
- **side-angle-side (SAS$\sim$)** - when two sides from one triangle are in the same proportion with two sides from another triangle and the angles between these two sides are equal (e.g. $\frac{AB}{XY} = \frac{BC}{YZ}$ and $\angle B = \angle Y$).
- **hypotenuse-side (HS$\sim$)** - when the hypotenuse and a leg from one right triangle are in the same proportion as the hypotenuse and a leg from another right triangle (e.g. $\frac{BC}{YZ} = \frac{CA}{ZX}$ and $\angle C = \angle Z = 90^\circ$).

Note that as some of the shortcuts are of the same form as those for congruence, we attach the similarity symbol, $\sim$, to differentiate them (that is, SSS$\sim$ verses SSS), although it should be apparent from the context. With similarity, since we are only after the shape being the same, if we have the angles of the triangles matching each other, it would imply similarity. However, since the three angles of *all* triangles in the plane sum to 180°, having *two* angles matching for the two triangles forces the third pair to be equal as well. As such, there is no ASA$\sim$, since two angles matching is enough for similarity (and we can't have a pair of sides "in the same proportion", we need something to compare it to).

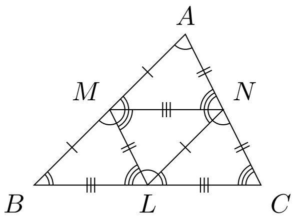
Let's take a look at similarity in action. Consider ΔABC, with M and N being the midpoints of sides AB and CA, respectively.



Joining M to N creates ΔAMN, such that $\angle MAN = \angle BAC$ and $\frac{AM}{AB} = \frac{NA}{CA} = \frac{1}{2}$, so, $\Delta AMN \sim \Delta ABC$, by SAS$\sim$. This tells us that $\frac{MN}{BC} = \frac{1}{2}$, $\angle AMN = \angle ABC$ and $\angle MNA = \angle BCA$. Since $\angle AMN$ and $\angle ABC$ are corresponding angles with respect to the transversal AB (as are $\angle MNA$ and $\angle BCA$), $\angle AMN = \angle ABC$ tells us that $MN \parallel BC$. Thus by joining the midpoints of two sides of a triangle, the new segment is half as long and parallel to the other side of the triangle. In doing so, we create a new triangle inside the other, similar to it and scaled down by a factor of $\frac{1}{2}$.

There was nothing special about choosing sides AB and CA. If we let L be the midpoint of BC, then, using a similar argument, we would get $\Delta MBL \sim \Delta ABC$,

$\Delta NLC \sim \Delta ABC$, $LM \parallel CA$, $LN \parallel AB$, and $\frac{LM}{CA} = \frac{1}{2} = \frac{LN}{AB}$. Since the segments joining the midpoints of ΔABC are half as long as the sides of the triangle, we can show that $\Delta LMN \sim \Delta ABC$ by SSS$\sim$. So, joining the midpoints of a triangle creates four smaller triangles, similar to the original (and one could show they are all congruent to each other).

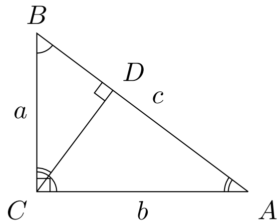


Note that each of our smaller triangles was the original scaled down by a factor of $\frac{1}{2}$, so the *area* of these new triangles is $\frac{1}{4} = \left(\frac{1}{2}\right)^2$ of the area of the original. This idea will hold in general, that is, if $\Delta ABC \sim \Delta XYZ$, and $\frac{AB}{XY} = r$, then

$$\frac{[\Delta ABC]}{[\Delta XYZ]} = r^2,$$

where $[\Delta ABC]$ denotes the area of ΔABC.

Let's look at another place where similarity shows up. Consider a right triangle ΔABC with $\angle C = 90^\circ$, and draw in the altitude CD, with D on the hypotenuse AB.



We can then, using right angles and shared angles, show that $\Delta ACD \sim \Delta ABC$ and $\Delta CBD \sim \Delta ABC$—and hence $\Delta ACD \sim \Delta CBD$—both by AA$\sim$. Thus, if we let the lengths of BC, CA, and AB be a, b, and c, respectively, we get, by similarity, that

$$\begin{aligned} &\Delta ACD \sim \Delta ABC \\ \Rightarrow\ &\frac{AD}{AC} = \frac{AC}{AB} \\ &\frac{AD}{b} = \frac{b}{c} \\ &AD = \frac{b^2}{c} \end{aligned} \qquad\qquad \begin{aligned} &\Delta CBD \sim \Delta ABC \\ \Rightarrow\ &\frac{BD}{BC} = \frac{CB}{AB} \\ &\frac{BD}{a} = \frac{a}{c} \\ &CD = \frac{a^2}{c}. \end{aligned}$$

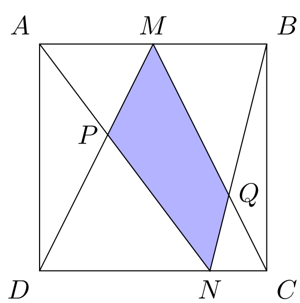
However, $AD + CD = AC = c$, hence

$$\begin{aligned}\frac{b^2}{c} + \frac{a^2}{c} &= c\\ b^2 + a^2 &= c^2.\end{aligned}$$

That is, we have demonstrated the Pythagorean theorem using similarity. This is like the proof attributed to the one Einstein reportedly came up with when he was a child.

Let's look at one more problem. This was problem 9 from Part I of the 2025-2026 Alberta High School Mathematics Competition.

> *In a square $ABCD$ with $AB = 1$, let M be the midpoint of side AB and N be a point on side DC such that $DN = 3NC$. The segment AN intersects DM at P, and the segment BN intersects CM at Q.*



> *The area of the quadrilateral $MPNQ$ is*
> *(A)* 1/5 *(B)* 7/30 *(C)* 11/45 *(D)* 1/4 *(E)* 4/15

Solution: Since $ABCD$ is a square, $AB \parallel CD$ and we get a number of equal alternate angles. Since $\angle PAM = \angle PND$ and $\angle PMA = \angle PDN$, we know that $\Delta PAM \sim \Delta PND$ by AA$\sim$. Since the triangles are similar, their sides are in the same proportion as well as the corresponding altitudes. From the information in the problem we will have $AM = \frac{1}{2}$ and $ND = \frac{3}{4}$. Therefore

$$\frac{h_{PAM}}{h_{PND}} = \frac{AM}{ND} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3} \Rightarrow h_{PAM} = \frac{2}{3}h_{PND}.$$

However, $h_{PAM} + h_{PND} = 1$ which leads to $h_{PAM} = \frac{2}{5}$, and $h_{PND} = \frac{3}{5}$.

Using a similar argument, $\Delta QMB \sim \Delta QCN$ by AA$\sim$ which leads to $h_{QMB} = \frac{2}{3}$, and $h_{QCN} = \frac{1}{3}$. We can now calculate the desired area

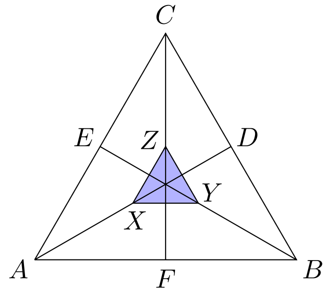
$$\begin{aligned}[MPNQ] &= [\Delta MCD] - [\Delta PND] - [\Delta QNC] \\ &= \frac{1}{2}(1)(1) - \frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{3}{5}\right) - \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) \\ &= \frac{1}{2} - \frac{9}{40} - \frac{1}{24} \\ &= \frac{60}{120} - \frac{27}{120} - \frac{5}{120} \\ &= \frac{28}{120} = \frac{7}{30}\end{aligned}$$

which gives us choice **(B)**.

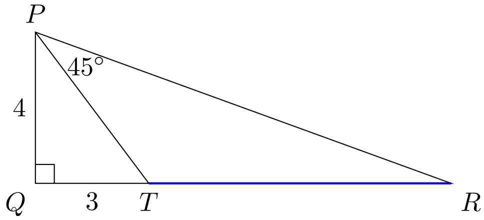
. . . .

Similarity pops up all the time in mathematics competitions, so keep your eyes open. I leave you with a few problems (including one to practice your French) from recent contests where similarity may prove useful.

1. Let ABC be an equilateral triangle with area 80. Let D, E, and F be the midpoints of BC, CA, and AB, respectively. Then, let X, Y, and Z be the midpoints of AD, BE, and CF, respectively. Determine the area of triangle XYZ. (***2025 Canadian Open Mathematics Challenge, problem A4.***)



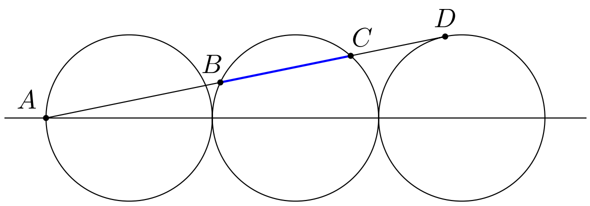
2. In the diagram, ΔPQR is right-angled at Q, $PQ = 4$, and T is on QR such that $QT = 3$ and $\angle TPR = 45°$. Determine the length of TR. (***2025 Canadian Senior Mathematics Contest, problem B2b.***)



3. AH and CP are heights of the acute triangle ABC. What is the value of the angle B if it is known that $AC = 2HP$? (***2026 British Columbia Secondary School Mathematics Contest, Senior, Final Round, Part B, problem 4.***)

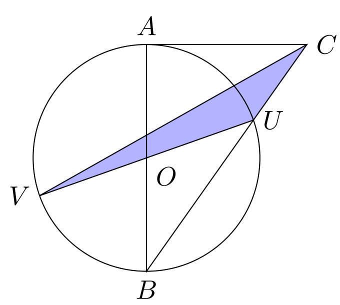
4. In a convex quadrilateral $MNPQ$, it is given that $MN = 4$, $PQ = 5$, $\angle MQN = \angle QPN$ and that $\angle QNM = \angle QNP + \angle QPN$. Find MQ. (***2025-2026 Alberta High School Mathematics Competition, Part II, problem 4.***)

5. Sur la figure, les trois cercles de rayon 12 cm sont tangents entre eux. La droite AD est tangente au troisième cercle en D et coupe le deuxième cercle en B et C. Déterminer la mesure de BC. (***Concours annuel de l'Association mathématique du Québec 2026, ordre secondaire, problème 4.***)



6. Let ABC be an isosceles triangle with $AB = AC$, whose area is 8 square units. Construct a square $PQRS$ so that P is on AB, Q is on AC, R is on BC, and S is on BC. Let x be the side length of the square. What is the maximum possible value of x? (***2025 Canada Lynx Mathematical Competition, problem C3.***)

7. In the diagram, the circle has centre O and radius 2. Points A. B, U, and V are on the circle so that AB and UV are diameters. Point C is outside the circle so that AC is tangent to the circle at A and so that BC intersects the circle at U. If $BU = 2UC$, determine the area of ΔCUV. (***2026 Euclid Contest, problem 8a.***)



OLYMPIAD CORNER

No. 445

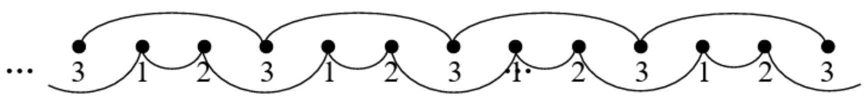
The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

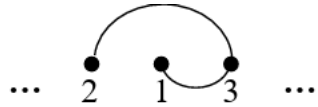
To facilitate their consideration, solutions should be received by **November 15, 2026**.

OC791. Let A, B and C be three-digit integers whose decimal representations together use exactly nine distinct digits. If $A < B < C$ and $A + B + C = 2026$, what is the smallest possible value of C?

OC792. On an infinite strip of adjacent squares (which can be associated with the integers), there is exactly one ball in each square. Written on these squares is a pattern of numbers that repeats periodically forever. The pattern is formed by a permutation of the numbers from 1 to n ($n \geq 3$) which we denote as $\overline{a_1 a_2 \ldots a_n}$. At a given signal, each ball jumps to the right over a number of squares equal to the value written in its starting square. The pattern is called ***collision-free*** if, after the jump, there is no square in which two or more balls land. For example, the patterns $\overline{123}$, $\overline{231}$, $\overline{312}$ are ***collision-free***, as shown below:



On the other hand, the pattern $\overline{213}$ is not ***collision-free***, as shown below:



Two patterns are considered identical if they can be obtained from one another through a cyclic shift. For example, $\overline{123}$ produces the same infinite strip as $\overline{231}$ or $\overline{312}$, that is $\overline{123} = \overline{231} = \overline{312}$. For $n = 3$, there is only one ***collision-free*** pattern.

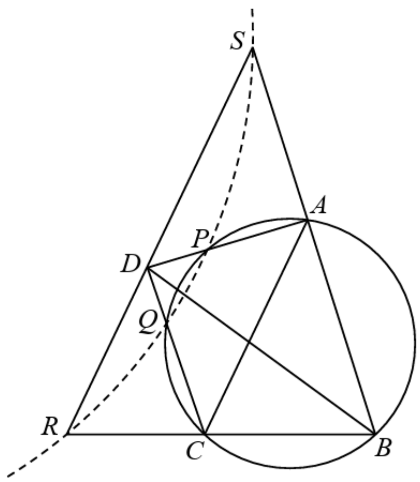
How many ***different collision-free*** patterns exist for $n = 5$? Explain.

OC793. Compute the integral

$$\int_0^{\pi} \ln(1 - 2a\cos x + a^2)\, dx$$

for $a > 1$.

OC794. As shown in the diagram below, $ABCD$ is a convex quadrilateral such that BD bisects $\angle ABC$, the circumcircle of triangle ABC intersects AD and CD at P and Q, respectively. A line drawn through D parallel to AC meets the extensions of BC and AB at R and S, respectively. Prove that P, Q, R and S are concyclic.



OC795. The real-valued infinitely differentiable function $f(x)$ is such that $f(0) = 1$, $f'(0) = 2$, and $f''(0) = 3$. Furthermore, f has the property that

$$f^{(n)}(x) + f^{(n+1)}(x) + f^{(n+2)}(x) + f^{(n+3)}(x) = 0$$

for all $n \geq 0$, where $f^{(n)}(x)$ denotes the nth derivative of f. Find $f(x)$.

. . . .

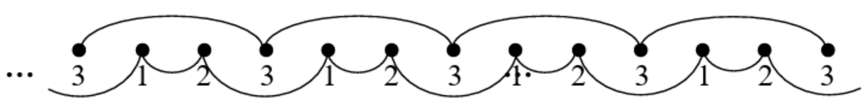
Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

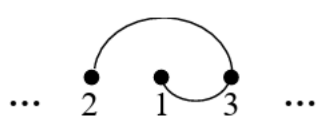
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 novembre 2026**.

OC791. Soient A, B et C des entiers à trois chiffres dont les représentations décimales utilisent ensemble exactement neuf chiffres distincts. Si $A < B < C$ et $A + B + C = 2026$, quelle est la plus petite valeur possible de C ?

OC792. Sur une bande infinie de cases adjacentes (que l'on peut associer aux entiers), il y a exactement une boule dans chaque case. Sur ces cases est inscrite une suite de nombres qui se répète périodiquement à l'infini. Le motif est formé d'une permutation des nombres de 1 à n ($n \geq 3$), que nous notons $\overline{a_1 a_2 \ldots a_n}$. À un signal donné, chaque boule saute vers la droite d'un nombre de cases égal au nombre inscrit dans sa case de départ. Le motif est dit ***sans collision*** si, après le saut, aucune case ne reçoit deux boules ou plus. Par exemple, les motifs $\overline{123}$, $\overline{231}$ et $\overline{312}$ sont ***sans collision***, comme illustré ci-dessous :



En revanche, le motif $\overline{213}$ n'est pas ***sans collision***, comme illustré ci-dessous :



Deux motifs sont considérés comme identiques s'ils peuvent être obtenus l'un de l'autre par un décalage cyclique. Par exemple, $\overline{123}$ produit la même bande infinie que $\overline{231}$ ou $\overline{312}$; ainsi, $\overline{123} = \overline{231} = \overline{312}$. Pour $n = 3$, il n'existe qu'un seul motif ***sans collision***.

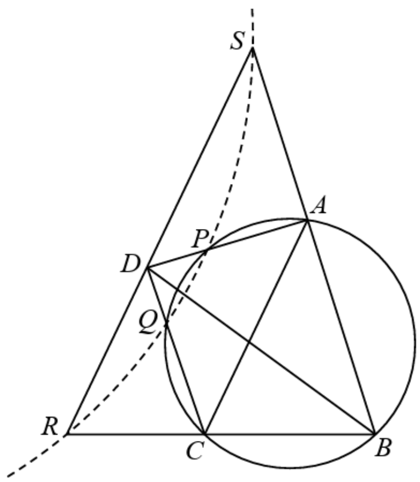
Combien existe-t-il de motifs ***sans collision distincts*** pour $n = 5$? Expliquez.

OC793. Calculez l'intégrale

$$\int_0^\pi \ln(1 - 2a\cos x + a^2)\, dx$$

pour $a > 1$.

OC794. Comme l'illustre la figure ci-dessous, $ABCD$ est un quadrilatère convexe tel que BD bissecte l'angle $\angle ABC$, et le cercle circonscrit au triangle ABC coupe AD et CD en P et Q, respectivement. Une droite passant par D et parallèle à AC coupe les prolongements de BC et AB en R et S, respectivement. Montrez que P, Q, R et S sont cocycliques.



OC795. La fonction $f(x)$, à valeurs réelles et indéfiniment dérivable, est telle que $f(0) = 1$, $f'(0) = 2$ et $f''(0) = 3$. De plus, f possède la propriété que

$$f^{(n)}(x) + f^{(n+1)}(x) + f^{(n+2)}(x) + f^{(n+3)}(x) = 0$$

pour tout $n \geq 0$, où $f^{(n)}(x)$ désigne la n-ième dérivée de f. Déterminez $f(x)$.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2026: 52(2), p. 73–75.

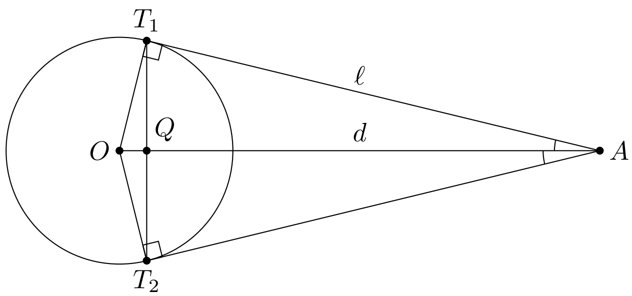
OC766. Suppose A is a point exterior to the unit sphere $\mathcal{S}$ in $\mathbb{R}^3$. The tangents from A to $\mathcal{S}$ form a right circular cone $\mathcal{K}$. (For our purposes this is a finite cone ending at the points of tangency. Note that the usual base of the cone is not included here.)

At what distance should A be from the centre of the sphere so that $\mathcal{S}$ and $\mathcal{K}$ have equal surface areas?

Originally from the 2025 Science Atlantic Math Problem Solving Contest, P15.

We received 6 submissions, 5 of which were correct and complete. We present the solution by Henry Diaz Bordon.

Let d be the distance between A and the center of $\mathcal{S}$, call it O. Because a line tangent to a circle forms a right angle with its radius, taking any plane containing A and O yields the following scenario:



Since T_1 is constructed as such a tangency point, by Pythagoras' theorem we have:

$$\ell^2 + 1 = d^2 \implies \ell = \sqrt{d^2 - 1};$$

by symmetry, it is trivial that $T_1A = T_2A$.

Let Q be the intersection of $\overline{T_1T_2}$ and $\overline{OA}$. Note that $\angle T_1QO = \angle OQT_2 = 90^\circ$ because their sum must equal 180° and since $\triangle AT_1Q \cong AT_2Q$ by the side-side-side criterion. Therefore, letting $r = T_1Q$, we get

$$\sin(\angle T_1AQ) = \frac{1}{d} = \frac{r}{\ell} \implies r = \frac{\ell}{d}.$$

Thus, the radius and slant height (r and d, respectively) of $\mathcal{K}$ are known in terms of d, so by computing the surface of the cone (this formula can be derived by unfolding this lateral face into a circular sector, whose area is straightforward to compute), setting it equal to 4π, and solving for d, an exact value for the desired magnitude may be finally obtained:

$$\pi r\ell = \pi\frac{d^2-1}{d} = 4\pi \implies d^2-4d-1=0 \implies d = 2+\sqrt{5}.$$

OC767. Sam goes into the gym and puts a marble of radius 1cm in the corner, touching two walls and the floor. (Assume all surfaces are flat and the angles are right.) A second student comes along and puts a ball in the corner so that it just touches both walls, the floor, and the marble. Several more students do the same, each ball touching the walls, the floor, and the previous ball.

What is the first ball radius that is greater than or equal to 1m? Give your answer in the form

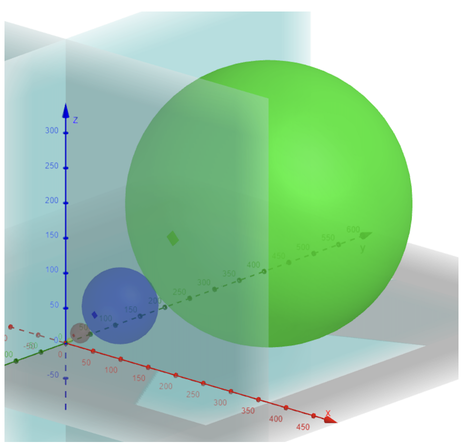
$$\frac{a+b\sqrt{c}}{d},$$

where a, b, c, d are integers.

Originally from the 2025 Science Atlantic Math Problem Solving Contest, P12.

We received 9 submissions, all correct though one significantly lacking detail. We present the common approach taken by several solvers.

Fix the coordinate system so that the origin is located in the corner of the gym, the coordinate planes coincide with two walls and the floor with the balls (and the marble) lying in the first octant. Note that the centers of the balls are collinear as they are equidistant from the three coordinates planes due to tangency.



Let O_0 be the center of the marble with $r_0 = 1$. For $n \geq 1$, let O_n and r_n denote the center and radius, respectively, of the nth ball. We have that for every $n \geq 0$

$$(r_n + r_{n+1})^2 = |O_nO_{n+1}|^2 = 3(r_{n+1}-r_n)^2$$

Hence,

$$0 = 3(r_{n+1} - r_n)^2 - (r_n + r_{n+1})^2 = 2\left(r_{n+1} - \left(2 - \sqrt{3}\right) r_n\right)\left(r_{n+1} - \left(2 + \sqrt{3}\right) r_n\right).$$

Since $r_{n+1} > r_n$, we deduce $r_{n+1} = \left(2 + \sqrt{3}\right) r_n$. Therefore, $r_n = \left(2 + \sqrt{3}\right)^n$.

Finally, since $1.8 > \sqrt{3} > 1.7$, by simple estimations, so

$$r_5 = (2 + \sqrt{3})^4 > 3.7^4 > 100 \qquad \text{and} \qquad r_4 = (2 + \sqrt{3})^3 < 4^3 = 64 < 100,$$

so the first radius that is greater or equal to 1m or 100 cm is

$$\left(2 + \sqrt{3}\right)^4 = 97 + 56\sqrt{3}.$$

Editor's Comments. Thanks to Henry Diaz Bordon and Brian Beasley for providing the graphics, with the latter also available in Desmos for exploration: `https://www.desmos.com/3d/uqw1ajynxa`.

OC768. Late in the year 1 CE, the villagers of Marzipan invented a winter solstice tradition: each household sent a fruitcake to one other household. None of these fruitcakes were ever eaten: instead, each year afterward, each household passed their fruitcake on to the same household that they had given one to in the previous year. This tradition has lasted over the centuries, and last year (2025 CE), for the first time, every household got its original fruitcake back.

What is the smallest possible number of households in the village of Marzipan?

Originally from the 2025 Science Atlantic Math Problem Solving Contest, P3.

We received 6 submissions, all correct though some were missing minor details in the justification. The presented approach was the one taken by most solvers.

The fruitcake-passing rule defines a permutation of the households as each household passes its fruitcake to the same household every year.

Since the first time every household received its original fruitcake back was after 2025 years, the fruitcake-passing permutation needs to have order exactly 2025. Recall that a permutation can be decomposed into disjoint cycles and its order is the least common multiple (LCM) of the cycle lengths. Therefore, we are looking for cycle lengths with smallest possible sum, whose LCM is 2025. Note that

$$2025 = 3^4 \cdot 5^2 = 81 \cdot 25,$$

with 81 and 25 are co-prime and the LCM of 81 and 25 is 2025. So taking one cycle of length 3^4 and one cycle of length 5^2 will produce the sum of $81 + 25 = 106$ households. Introducing any additional cycles would only increase the sum without changing the LCM. Therefore, 106 is the smallest possible number of households.

OC769. For each natural number n, define an $n \times n$ matrix $P(n)$ with $(P(n))_{ij} = \binom{n+i-2}{j-1}$ for $i, j = 1, 2, \ldots, n$. Find, with proof, $\det(P(n))$.

Originally from the 2025 Science Atlantic Math Problem Solving Contest, P7.

We received 7 submissions, all correct. We present the solution by José Luis Díaz-Barrero.

For $n = 1, 2$, we have

$$P(1) = \begin{pmatrix} 1 \end{pmatrix}, \qquad P(2) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

In both cases the determinant is 1, so we conjecture that for all $n \in \mathbb{N}$ it holds that

$$\det(P(n)) = 1.$$

Indeed, we argue by mathematical induction. For $n = 1$, we have $P(1) = (1)$, so $\det(P(1)) = 1$. Now assume $\det(P(n-1)) = 1$ for some $n \geq 2$. We will show that $\det(P(n)) = 1$. To do that, we perform the row operations

$$R_i \longleftarrow R_i - R_{i-1}, \qquad i = 2, \ldots, n.$$

These operations do not change the determinant. Let $P'(n)$ be the resulting matrix. Its entries for $i \geq 2$ satisfy

$$P'(n)_{i,j} = \binom{n+i-2}{j-1} - \binom{n+i-3}{j-1} = \binom{n+i-3}{j-2},$$

using Pascal's identity. Thus

$$P'(n)_{1,j} = \binom{n-1}{j-1}, \qquad P'(n)_{i,j} = \binom{n+i-3}{j-2} \ (i \geq 2).$$

For the first column ($j = 1$), we have

$$P'(n)_{1,1} = \binom{n-1}{0} = 1, \qquad P'(n)_{i,1} = \binom{n+i-3}{-1} = 0 \ (i \geq 2).$$

Hence the first column of $P'(n)$ is $(1, 0, \ldots, 0)^T$.

Now consider the submatrix formed by rows $2, \ldots, n$ and columns $2, \ldots, n$. Let $i' = i - 1$ and $j' = j - 1$. Then for $i \geq 2$, $j \geq 2$,

$$P'(n)_{i,j} = \binom{n+i-3}{j-2} = \binom{(n-1)+i'-2}{j'-1} = (P(n-1))_{i',j'}.$$

Thus the lower-right $(n-1) \times (n-1)$ block of $P'(n)$ is exactly $P(n-1)$.

Therefore,

$$P'(n) = \begin{pmatrix} 1 & * \\ 0 & P(n-1) \end{pmatrix},$$

where $*$ is some row vector we do not need explicitly. Hence

$$\det(P(n)) = \det(P'(n)) = 1 \cdot \det(P(n-1)) = \det(P(n-1)).$$

By the induction hypothesis, $\det(P(n-1)) = 1$, and therefore $\det(P(n)) = 1$, as claimed.

OC770. You are given positive integers p, q, r, s satisfying

$$qr - ps = 1.$$

Suppose x, y are positive integers with

$$\frac{p}{q} < \frac{x}{y} < \frac{r}{s}.$$

Find the smallest value of y satisfying this last condition and determine all x that go with it.

Originally from the 2025 Science Atlantic Math Problem Solving Contest, P22.

We received 7 submissions (4 by the same author), all correct, but some redundant. We present the solution by Theo Koupelis.

From the second of the given conditions we get

$$qx - py > 0 \Longrightarrow qx - py \geq 1 \Longrightarrow s(qx - py) \geq s,$$

because q, x, p, y are positive integers; similarly, we get

$$ry - sx > 0 \Longrightarrow ry - sx \geq 1 \Longrightarrow q(ry - sx) \geq q.$$

Adding the above two inequalities we get

$$y(qr - sp) \geq q + s \Longrightarrow y \geq q + s,$$

because $qr - ps = 1$. Thus, $y_{\min} = q + s$. Substituting $y = y_{\min}$ into the second of the given equations and using $qr - ps = 1$ we get

$$(q+s) \cdot \frac{p}{q} < x < (q+s) \cdot \frac{r}{s} \Longrightarrow p + r - \frac{1}{q} < x < p + r + \frac{1}{s}.$$

Therefore, the value of x that goes with $y = y_{\min}$ is $p + r$.

Note that $\gcd(p, q) = \gcd(r, s) = 1$ because $qr - ps = 1$, and the fraction $\frac{x}{y} = \frac{p+r}{q+s}$ is the mediant of the fractions $\frac{p}{q}$ and $\frac{r}{s}$.

On p-adic valuations of sums

CJ Quines

1 Valuations

Let p be a prime. The ***p*-adic valuation** of a positive integer n, written $\nu_p(n)$, is the largest integer e such that $p^e \mid n$. We extend this to nonzero rational numbers via $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$. For nonzero rational numbers $a_1, a_2, \ldots, a_n$, two key properties of ν_p are:

- $\nu_p(a_1 a_2 \cdots a_n) = \nu_p(a_1) + \nu_p(a_2) + \cdots + \nu_p(a_n)$.
- $\nu_p(a_1 + a_2 + \cdots + a_n) \geq \min\{\nu_p(a_1), \nu_p(a_2), \ldots \nu_p(a_n)\}$. If the minimum is unique, then equality holds.

We'll call these the *product rule* and the *sum rule*, respectively. While the most famous application of p-adic valuation to mathematical olympiads is the *lifting-the-exponent lemma* (see [1]), we can solve several problems using only these two properties.

In this note, we'll focus on the *equality case* of the sum rule, which is more useful than the inequality. All solutions have the same shape: translate the problem to a p-adic condition, then use the sum rule, handling the equality case as needed.

2 Examples

Example 1. *Let $n \geq 2$ be an integer. Show that $H_n = 1/1 + 1/2 + \cdots + 1/n$ is not an integer.*

Proof. We will show that $\nu_2(H_n)$ is negative: this will complete the proof as all integers have non-negative p-adic valuations. By the sum rule,

$$\begin{aligned}\nu_2(H_n) &\geq \min\{\nu_2(1/1), \nu_2(1/2), \ldots, \nu_2(1/n)\}\\ &= \min\{-\nu_2(1), -\nu_2(2), \ldots, -\nu_2(n)\}\\ &= -\max\{\nu_2(1), \nu_2(2), \ldots, \nu_2(n)\}.\end{aligned}$$

Let 2^k be the largest power of 2 that's at most n. We claim 2^k is the only positive integer $a \leq n$ such that $\nu_p(a) = k$. If there was another such a, we'd have $a = a'2^k$ for some positive integer a'. As a is different from 2^k, we have $a' \geq 2$. Then $n \geq a \geq 2 \cdot 2^k = 2^{k+1}$, contradicting our choice of 2^k as the largest power of 2 that's at most n.

Thus, the minimum in $\{\nu_2(1/1), \nu_2(1/2), \ldots, \nu_2(1/n)\}$ is $\nu_2(1/2^k)$, which is unique. This means we have equality in the sum rule, so $\nu_2(H_n) = -k$. As $n \geq 2$, we have $k \geq 1$, so $\nu_2(H_n)$ is negative, as desired. □

Example 2 (Bay Area MO 2018 P4b)**.** *Show that if a, b, c are nonzero integers such that $a/b + b/c + c/a$ is an integer, then abc is a perfect cube.*

Proof. Note that abc is a perfect cube if and only if $3 \mid \nu_p(abc)$ for all primes p. Let p be a prime, and let x, y, and z be $\nu_p(a)$, $\nu_p(b)$, and $\nu_p(c)$, respectively. We will show that $3 \mid x + y + z$.

Because $a/b + b/c + c/a$ is an integer, we have $\nu_p(a/b + b/c + c/a) \geq 0$. By the sum rule, we have

$$\begin{aligned}\nu_p(a/b + b/c + c/a) &\geq \min\{\nu_p(a/b), \nu_p(b/c), \nu_p(c/a)\}\\ &= \min\{x - y, y - z, z - x\}.\end{aligned}$$

We can't combine these two inequalities, so we reach for the equality case by reasoning about uniqueness.

- **Case 1.** The three integers $x - y$, $y - z$, and $z - x$ are all distinct. Then the minimum must be unique, and we have the equality case of the sum rule, and thus $\nu_p(a/b + b/c + c/a) = \min\{x - y, y - z, z - x\}$. These are three distinct integers that sum to zero, so one of them must be negative. But this contradicts $\nu_p(a/b + b/c + c/a) \geq 0$, so this case is impossible.
- **Case 2.** Two of these integers are equal. Suppose we had $x - y = y - z$. Then $x + y + z = 3y$, which is divisible by 3, as desired. The cases $y - z = z - x$ and $z - x = x - y$ are similar.

□

Example 3 (Polish MO 2019, 2nd Round, P3)**.** *Let $f(t) = t^3 + t$. Decide if there exist rational numbers x, y and positive integers m, n such that $xy = 3$ and $f^m(x) = f^n(y)$. (Here, $f^m(x) = \underbrace{f(f(\cdots f}_{m \text{ times}}(x)\cdots))$.)*

Proof. There do not. We proceed by contradiction, examining $\nu_3(f^m(x))$ and $\nu_3(f^n(y))$. By the sum rule,

$$\nu_3(f(x)) \geq \min\{\nu_3(x^3), \nu_3(x)\} = \min\{3\nu_3(x), \nu_3(x)\}.$$

To get the equality case for the sum rule, we do casework on the sign of $\nu_3(x)$. We have three cases.

- **Case 1.** $\nu_3(x) > 0$. Then the minimum in $\{3\nu_3(x), \nu_3(x)\}$ is $\nu_3(x)$, which is unique. We have the equality case of the sum rule, so $\nu_3(f(x)) = \nu_3(x) > 0$. By induction, $\nu_3(f^m(x)) > 0$.
- **Case 2.** $\nu_3(x) < 0$. This is similar to the first case. The minimum is $3\nu_3(x)$, which is unique, so we have the equality case, and $\nu_3(f(x)) = 3\nu_3(x) < 0$. By induction, $\nu_3(f^m(x)) < 0$.

- **Case 3.** $\nu_3(x) = 0$. The sum rule doesn't help in this case, so we can examine $f(x)$ directly. As $\nu_3(x) = 0$, we must have $x \not\equiv 0 \pmod 3$. By Fermat's little theorem,

  $$f(x) \equiv x^3 + x \equiv x + x \equiv 2x \not\equiv 0 \pmod 3\,,$$

  so $f(x)$ and 3 are again relatively prime, and $\nu_3(f(x)) = 0$. By induction, $\nu_3(f^m(x)) = 0$.

Across the cases, we see that $\nu_3(x)$ and $\nu_3(f^m(x))$ have the same sign. So if, for the sake of contradiction, $\nu_3(f^m(x)) = \nu_3(f^n(y))$, then $\nu_3(x)$ and $\nu_3(y)$ must also have the same sign.

As $xy = 3$, we have $\nu_3(x) + \nu_3(y) = 1$. As the two addends are integers, they cannot have the same sign, contradiction. $\square$

3 Problems

Problem 1 (Iranian MO 2017, 3rd Round, 1st Number Theory, P1)**.** *Let n be a positive integer and P a set of prime numbers. Let A be the set of all positive integers less than n not divisible by a prime in P. If $|A| > 1$, show that the sum of the reciprocals of the elements of A is not an integer.*

Problem 2 (Austrian MO 2016, Part 2, P6)**.** *Let a, b, and c be nonzero integers such that $ab/c + bc/a + ca/b$ is an integer. Prove that each of the three numbers ab/c, bc/a, and ca/b are integers.*

Problem 3 (IMO Shortlist 2007 N2)**.** *Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.*

Problem 4 (APMO 2017 P4)**.** *Call a rational number r powerful if r can be expressed in the form p^k/q for some relatively prime positive integers p, q and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all powerful.*

References

[1] Parvardi, Amir H. (2011). "Lifting the Exponent Lemma (LTE)". `https://s3.amazonaws.com/aops-cdn.artofproblemsolving.com/resources/articles/lifting-the-exponent.pdf`

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 15, 2026**.

5161. *Proposed by Surajit Dutta.*

Let P be a simple polygon with vertices in $\mathbb{Z}^2$. Suppose that each side of P contains exactly m lattice points, including endpoints. What restrictions do these conditions impose on the area of P?

5162. *Proposed by Nazar Kirgizbaev.*

Let x, y, z be positive real numbers such that $x^2+y^2+z^2=3$. Prove the following inequality:

$$\sum_{\text{cyc}} \left(\frac{2xy}{xy+z}\right)^2 \geq 1+\frac{2}{3}(xy+yz+zx).$$

5163. *Proposed by Michel Bataille.*

Let ABC be a triangle with centroid G and circumcircle Γ. Prove that the reflection of G in the line BC is on Γ if and only if BC is tangent to the circumcircles of ΔGAB and ΔGAC.

5164. *Proposed by Paul Bracken.*

Let n be a non-negative integer and let $H_n = \sum_{k=1}^n \frac{1}{k}$ be the standard definition of harmonic numbers. Prove the following finite sum formulas:

a) $\sum_{k=1}^n (-1)^{k-1}\binom{n}{k} H_k = \frac{1}{n}$

b) $\sum_{k=1}^n (-1)^{k-1}\binom{n}{k} \frac{H_{k+1}}{k+1} = \frac{1}{(n+1)^2}$

5165. *Proposed by Nguyen Viet Hung.*

Determine all values of k such that the inequality

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}+\frac{k(ab+bc+ca)}{a+b+c} \geq \left(\frac{1}{2}+\frac{k}{3}\right)(a+b+c)$$

holds for all non-negative real numbers a, b, c no two of which are zero.

5166. *Proposed by Quang Hung Tran.*

Let $ABCD$ be a fixed tetrahedron inscribed in the sphere ω. Let P be a variable point on ω. Let G be the centroid of tetrahedron $PBCD$. Prove that the plane through G perpendicular to PA always passes through a fixed point as P moves.

5167. *Proposed by Vasile Cîrtoaje.*

Find the largest positive value of the constant k such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq a^k+b^k+c^k$$

for any positive numbers a, b, c with $ab+bc+ca=3$ and at most one of them less than 1.

5168. *Proposed by Mihaela Berindeanu.*

Determine the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $\frac{1}{2}f(2xy-1)+f(x)f(y)=4xy-1$.

5169. *Proposed by Tatsunori Irie.*

Let $n \geq 3$ be an integer and let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $0 \leq x_i \leq 1$ for all $i = 1, \ldots, n$. Find the maximum possible value of

$$\frac{x_1+x_2+\cdots+x_n}{n}+\sqrt{x_1(1-x_1)+x_2(1-x_2)+\cdots+x_n(1-x_n)}.$$

5170. *Proposed by Michael Friday.*

In triangle ABC, the perpendicular bisector of BC meets CA at A_1 and the perpendicular bisector of CA meets BC at B_1. Prove that the intersection point $AB_1 \cap BA_1$ is on the Euler line of ABC.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 novembre 2026**.

5161. *Soumis par Surajit Dutta.*

Soit P un polygone simple dont les sommets appartiennent à $\mathbb{Z}^2$. Supposons que chaque côté de P contienne exactement m points du réseau, y compris ses extrémités. Quelles restrictions ces conditions imposent-elles à l'aire de P ?

5162. *Soumis par Nazar Kirgizbaev.*

Soient x, y et z des nombres réels positifs tels que $x^2+y^2+z^2=3$. Montrez que l'inégalité suivante est vérifiée :

$$\sum_{\text{cyc}}\left(\frac{2xy}{xy+z}\right)^2 \geq 1+\frac{2}{3}(xy+yz+zx).$$

5163. *Soumis par Michel Bataille.*

Soit ABC un triangle de centre de gravité G et de cercle circonscrit Γ. Montrez que le symétrique de G par rapport à la droite BC appartient à Γ si et seulement si la droite BC est tangente aux cercles circonscrits des triangles GAB et GAC.

5164. *Soumis par Paul Bracken.*

Soit n un entier non négatif, et soit $H_n=\sum_{k=1}^{n}\frac{1}{k}$ la définition usuelle des nombres harmoniques. Montrez les formules de somme finie suivantes :

a) $\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k=\frac{1}{n}$

b) $\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{H_{k+1}}{k+1}=\frac{1}{(n+1)^2}$

5165. *Soumis par Nguyen Viet Hung.*

Déterminez toutes les valeurs de k telles que l'inégalité

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}+\frac{k(ab+bc+ca)}{a+b+c}\geq\left(\frac{1}{2}+\frac{k}{3}\right)(a+b+c)$$

soit vérifiée pour tous nombres réels non négatifs a, b et c, dont au plus un est nul.

5166. *Soumis par Quang Hung Tran.*

Soit $ABCD$ un tétraèdre fixe inscrit dans la sphère ω. Soit P un point variable de ω. Soit G le centre de gravité du tétraèdre $PBCD$. Montrez que le plan passant par G et perpendiculaire à PA passe toujours par un point fixe lorsque P parcourt ω.

5167. *Soumis par Vasile Cîrtoaje.*

Déterminez la plus grande valeur positive de la constante k telle que

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq a^k+b^k+c^k$$

pour tous nombres réels positifs a, b et c tels qu'au plus l'un d'entre eux soit strictement inférieur à 1 et que $ab+bc+ca=3$.

5168. *Soumis par Mihaela Berindeanu.*

Déterminez les fonctions $f:\mathbb{R}\to\mathbb{R}$ vérifiant $\frac{1}{2}f(2xy-1)+f(x)f(y)=4xy-1$.

5169. *Soumis par Tatsunori Irie.*

Soit $n\geq 3$ un entier, et soient $x_1,x_2,\ldots,x_n$ des nombres réels vérifiant $0\leq x_i\leq 1$ pour tout $i=1,\ldots,n$. Déterminez la plus grande valeur possible de

$$\frac{x_1+x_2+\cdots+x_n}{n}+\sqrt{x_1(1-x_1)+x_2(1-x_2)+\cdots+x_n(1-x_n)}.$$

5170. *Soumis par Michael Friday.*

Soit ABC un triangle. La médiatrice de BC coupe CA en A_1, et la médiatrice de CA coupe BC en B_1. Montrez que le point d'intersection des droites AB_1 et BA_1 appartient à la droite d'Euler du triangle ABC.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2026: 52(2), p. 88–91.

5111. *Proposed by Michel Bataille.*

Let ABC be an acute-angled triangle, Γ its circumcircle, B' and C' the feet of its altitudes from B and C. Let P and Q be distinct points on the arc BC of Γ not containing A and let AP and AQ intersect BB' and CC' at M and N, respectively. Prove that M, N, P, Q are concyclic if and only if MN is parallel to $B'C'$.

We received 13 solutions, of which 11 were correct. There was one incomplete solution and one that relied heavily on Maple. *We present 3 solutions.*

Solution 1, by Michal Adamaszek.

We first establish the following result.

Lemma. Suppose that XYZ is a triangle for which UXV is the tangent to its circumcircle at X and K and L are respectively two points on the sides XY and XZ. Then $KLZY$ is concyclic if and only if KL is parallel to UXV.

To see this, note that $\angle XZY = \angle UXY$. The quadrilateral $KLZY$ is concyclic if and only if $\angle XKL = \angle XZY = \angle UXY$ if and only if $KL \| UXV$.

Apply this to the problem. Triangle ABC and APQ have a common circumcircle Γ. Since BC subtends right angles at C' and B', $B'C'BC$ is concyclic and so $B'C'$ is parallel to the tangent of Γ at A.

Using the lemma for triangle APQ and points M and N, we find that $MNQP$ is concyclic if and only if MN is parallel to the tangent of Γ at A, *i.e.* when $MN \| B'C'$.

Solution 2, by Oliver Geupel.

Let AQ and $C'B'$ intersect at N'. Then

$$\angle N'AB' = \angle QAC = \angle QPC = \angle QPA - \angle CPA = \angle QPM - \angle CBA.$$

Since $BCB'C'$ is concyclic,

$$\angle AB'N' = \angle AB'C' = \angle CBC' = \angle CBA.$$

Hence

$$\begin{aligned}\angle C'N'Q &= \angle B'N'A \\ &= 180^\circ - \angle N'AB' - \angle AB'N' \\ &= 180^\circ - \angle QPC - \angle CBA = 180^\circ - (\angle QPA - \angle CPA) - \angle CBA \\ &= 180^\circ - (\angle QPM - \angle CBA) - \angle CBA = 180^\circ - \angle QPM.\end{aligned}$$

The quadrilateral $MNQP$ is concyclic if and only if

$$\angle MNQ = 180^\circ - \angle QPM = \angle C'N'Q,$$

which is satisfied if and only if MN is parallel to the line $C'N' = B'C'$.

Solution 3, by the proposer.

Let U be the point common to the perpendicular to BB' through M and the perpendicular to CC' through N. The centres of the circumcircles γ_m and γ_n of triangles BMP and CNQ lie on BC. (In the case of γ_n, let $R = MU \cap BC$ so that $MR \| AC$. Then

$$\angle BPM = \angle BPA = \angle BCA = \angle BRM,$$

so that R lies on γ_m and BR is its diameter.)

Suppose first that $MN \| C'B'$. Since also $MU \| AB'$ anad $NU \| AC'$, we can appeal to Desargues' theorm to conclude that triangles UMN and $AB'C'$ are prespective; thus AU, MB' and NC' meet at a common point H and U is on the altitude AA' of triangle ABC.

Let $\mathbf{I}$ be the inversion with centre H that interchanges A' and U. This inversion transforms the line BC to a circle with diameter HU (since $HA' \perp BC$, HU is perpendicular to this circle) which contains the points M and N (HU subtends right angles at M and N). Hence $\mathbf{I}(M) = B$ and $\mathbf{I}(N) = C$, so that

$$HB \cdot HM = HC \cdot HN$$

and H is on the radical axis of γ_m and γ_n. Since AH is perpendicular to the line of centres of γ_m and γ_n, A is on the radical axis and $AM \cdot AP = AN \cdot AQ$, from which it follows that $MNQP$ is concyclic.

Conversely, suppose that $MNQP$ is concyclic. Then $AM \cdot AP = AN \cdot AQ$ and so the radical axis of γ_m and γ_n is AA'. Therefore $HB \cdot HM = HC \cdot HN$ and the inversion $\mathbf{J}$ with centre H which exchanges B and M also exchanges N and C. The line MU is inverted into a circle with diameter HB and the line NU into a circle with diameter HC. The point A' lies on both these circles and is inverted to the intersection point U on both these lines; thus U is on the altitude AA'. Therefore triangles UMN and $AB'C'$ are perspective from H, so by Desargues' theorem and because $AB' \| MU$ and $AC' \| NU$, it follows that $MN \| B'C'$.

5112. *Proposed by Vasile Cîrtoaje.*

What is the smallest positive value of k such that

$$\frac{1}{a^2+b^2+k}+\frac{1}{b^2+c^2+k}+\frac{1}{c^2+a^2+k}\leq\frac{3}{2+k}$$

for any triangle with side lengths satisfying $a+b+c=3$?

There were 5 solutions submitted, of which 3 were correct. The following solution draws on the correct solutions submitted.

Without loss of generality, suppose that $a\geq b\geq c>0$. We first establish that it suffices to prove the inequality for isosceles triangles whose smaller sides are equal.

Let $f(a,b,c)$ be the quantity on the left side of the inequality and let $x=\frac{1}{2}(b+c)$, so that

$$2(b^2+c^2-2x^2)=(b-c)^2,\quad 2(b-x)=2(x-c)=b-c.$$

Then

$$\begin{aligned}
&f(a,x,x)-f(a,b,c)\\
&=\frac{b^2+c^2-2x^2}{(b^2+c^2+k)(2x^2+k)}+\frac{b^2-x^2}{(a^2+b^2+k)(a^2+x^2+k)}+\frac{c^2-x^2}{(a^2+c^2+k)(a^2+x^2+k)}\\
&=\frac{(b-c)}{2(a^2+x^2+k)}\left[\frac{(b-c)(a^2+x^2+k)}{(b^2+c^2+k)(2x^2+k)}+\frac{b+x}{a^2+b^2+k}-\frac{c+x}{a^2+c^2+k}\right]\\
&=\frac{(b-c)^2}{2(a^2+x^2+k)}\left[\frac{a^2+x^2+k}{(b^2+c^2+k)(2x^2+k)}+\frac{a^2-bc-2x^2+k}{(a^2+b^2+k)(a^2+c^2+k)}\right].
\end{aligned}$$

To show that $f(a,x,x)\geq f(a,b,c)$, it is necessary to show that

$$(a^2+x^2+k)(a^2+b^2+k)(a^2+c^2+k)+(2x^2+k)(b^2+c^2+k)(a^2-bc-2x^2+k)\geq 0.$$

Since $a^2+b^2\geq 2x^2$ and $a^2+c^2\geq b^2+c^2$, it suffices to establish that

$$(a^2+x^2+k)+(a^2-bc-2x^2+k)\geq 0.$$

This is straighforward, since the left side is equal to

$$(a^2-x^2)+(a^2-bc)+2k,$$

all of whose three terms are nonnegative.

Noting that $a+2x=3$ and that $x<a=3-2x<2x$, we have $3/4<x<1$. The inequality

$$\frac{3}{2+k}\geq\frac{1}{2x^2+k}+\frac{2}{a^2+x^2+k}$$

is equivalent to

$$\begin{aligned}
0&\leq 3(9-12x+5x^2+k)(2x^2+k)-(2+k)(9-12x+9x^2+3k)\\
&=(30x^4-72x^3+36x^2+24x-18)+(12x^2-24x+12)k\\
&=6(x-1)^2[(5x^2-2x-1)+2k].
\end{aligned}$$

On the interval $(3/4,1)$, the function $5x^2-2x-3$ increases from $-27/16$ to 0. Hence, for the inequality to be satisfied for all $x\in(3/4,1)$ it is necessary and sufficient that $k\ge 27/32$.

Hence, the minimum value of k yielding the inequality for all triangles of perimeter 3 is $27/32$.

5113. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Let u,v be complex numbers such that $|u-v|\ge\max(|u|,|v|)$ and let w be another complex number. Show that

$$|u+v+w|\le|u-v+w|+|v-u+w|.$$

We received 7 solutions, 6 of which were correct and complete. We present the solution by Michal Adamaszek.

We have

$$(u+v+w)(u-v)=(u-v+w)\cdot u+(v-u+w)\cdot(-v).$$

Therefore,

$$\begin{aligned}|u+v+w|\cdot|u-v|&=|(u-v+w)\cdot u+(v-u+w)\cdot(-v)|\\&\le|u-v+w|\cdot|u|+|v-u+w|\cdot|v|\\&\le|u-v+w|\cdot|u-v|+|v-u+w|\cdot|u-v|.\end{aligned}$$

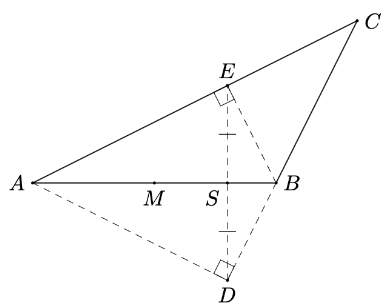
If $u=v$, then $u=v=0$ and the target inequality clearly holds. Otherwise, we obtain the target inequality by dividing the above by $|u-v|$.

Note that if ABC is a triangle whose longest side is AB, then for any point P we have $PA+PB\ge PC$. This is equivalent to the problem we have here.

5114. *Proposed by Michael Friday, modified by the Editorial Board.*

Let ABC be a triangle in which $B-A=90^\circ$. If D and E are the feet of the altitudes from A and B, prove that the midpoint of DE is the foot of the symmedian from vertex C to the side AB.

We received 21 solutions, of which 16 were correct and complete. We present the solution by Chikara Tsugawa.



Let M and S be the midpoints of AB and DE, respectively.

Since $\angle BEA = \angle ADB = 90^\circ$, we know A, D, B, and E are concyclic. Hence, $\angle EAB = \angle EDB$ and so $\triangle CAB \sim \triangle CDE$. Then $\angle ACM = \angle DCS$ meaning S is on the symmedian from C.

Let $\angle CAB = \alpha$. Then, $\angle ABC = 90^\circ + \alpha$, so $\angle BAD = 90^\circ - \angle DBA = \alpha$. Thus, $\angle BAE = \angle BAD = \alpha$, and $\triangle AEB \cong \triangle ADB$. Therefore, AB is the perpendicular bisector of DE, and S lies on AB.

Note that the relation $\angle ACM = \angle DCS$ holds for an arbitrary triangle ABC. The assumption $B - A = 90^\circ$ is required only for $S \in AB$.

5115. *Proposed by Tran Nhat Quang.*

Let n be a positive integer. Find all pairs of functions $f, g : (0, +\infty) \to (0, +\infty)$ such that:

$$f\big(g(x) + f(x)f(y)\big) = x^n + yf(x)$$

for al $x, y > 0$.

We received 1 submission but it is incorrect. We present the solution by the proposer, slightly modified by the editor.

It is easy to check that

$$f(x) = x, \qquad g(x) = x^n$$

satisfy the equation. We show that these are the only solutions.

Let $P(x, y)$ denote the given equation. First, f is injective: if $f(u) = f(v)$, then comparing $P(x, u)$ and $P(x, v)$ gives $uf(x) = vf(x)$, and hence $u = v$.

Next, f is surjective. Indeed, given $c > 0$, choose

$$x = \left(\frac{c}{2}\right)^{1/n}, \qquad y = \frac{c}{2f(x)}.$$

Then $P(x, y)$ gives

$$f\big(g(x) + f(x)f(y)\big) = c.$$

In particular, there exists $a > 0$ such that $f(a) = 1$. From $P(a, y)$,

$$f\big(g(a) + f(y)\big) = a^n + y. \tag{1}$$

Set $k = g(a) + a^n > 0$. Replacing y by $g(a) + f(y)$ in equation (1) and using it again gives

$$f(y + k) = f(y) + k, \qquad \forall y > 0. \tag{2}$$

Now apply $P\left(x, y + \frac{k}{f(x)}\right)$. Using $P(x, y)$ and equation (2), we obtain

$$\begin{aligned} f\left(g(x) + f(x)f\left(y + \frac{k}{f(x)}\right)\right) &= x^n + yf(x) + k \\ &= f\big(g(x) + f(x)f(y) + k\big). \end{aligned}$$

Since f is injective,
$$f\left(y+\frac{k}{f(x)}\right)=f(y)+\frac{k}{f(x)}.$$
As f is surjective onto $(0,\infty)$, the quantity $k/f(x)$ ranges over all positive real numbers. Therefore

$$f(y+z)=f(y)+z \qquad \forall y,z>0.$$

Interchanging y and z gives

$$f(y)-y=f(z)-z,$$

so $f(y)=y+C$ for some constant C. Since $f(y)>0$ for all $y>0$, we have $C\geq 0$. Substituting $f(t)=t+C$ into the original equation yields

$$g(x)=x^n-Cx-C(C+1).$$

If $C>0$, then $g(x)<0$ for all sufficiently small $x>0$, a contradiction. Hence $C=0$, and consequently $f(x)=x$ and $g(x)=x^n$.

5116. *Proposed by Wanlong Han.*

For $x_0>0$, we define
$$x_{n+1}=\sqrt{\frac{\sum_{k=0}^{n}(\arctan x_k)^2}{n+1}}.$$
for any nonnegative integer n. Find $\lim_{n\to\infty}x_n\sqrt{\ln n}$.

We received 7 submissions, of which 2 were correct and complete. While not fundamentally flawed, the incomplete solutions lacked rigour or glossed over too many details. We present the solution by Zhuang Shilong modified by the editor, with concluding calculation by the proposer.

We claim that

$$0<\sqrt{\frac{n}{n+1}}x_n<x_{n+1}<x_n<\min\left(\frac{\pi}{2},x_0\right). \tag{1}$$

Indeed, from the definition,

$$(n+1)x_{n+1}^2=\sum_{k=0}^{n}\arctan^2(x_k), \qquad nx_n^2=\sum_{k=0}^{n-1}\arctan^2(x_k). \tag{2}$$

It follows that
$$(n+1)x_{n+1}^2-nx_n^2=\arctan^2(x_n). \tag{3}$$

Since $x_0>0$, $x_1=\arctan x_0>0$ and for all $n\in\mathbb{N}$, $x_n\geq\sqrt{\frac{\arctan^2(x_0)}{n}}>0$. From (3), $(n+1)x_{n+1}^2-nx_n^2>0$ and therefore $\sqrt{\frac{n}{n+1}}x_n<x_{n+1}$. Moreover, using the well known fact that for any $y>0$, $\arctan(y)<y$, we obtain the other

desired inequality, $x_{n+1} < x_n$. We conclude, as arctan is upper bounded by $\frac{\pi}{2}$, that $0 < x_1 < \min\{\frac{\pi}{2}, x_0\}$.

Therefore, x_n is monotone and bounded, which implies that it has a limit that we will denote by a. From the continuity of arctan, we observe that $\lim_{n\to\infty} \arctan(x_n) = \arctan(a)$. From,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n+1} \arctan^2 x_k - \sum_{k=0}^{n} \arctan^2 x_k}{n+1-n} = \lim_{n\to\infty} \arctan^2 x_{n+1}. \tag{4}$$

Using the Stolz-Cesàro theorem for real-valued sequence ratios where the denominator's sequence goes to $+\infty$ yields

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \arctan^2 x_k}{n+1} = \lim_{n\to\infty} \arctan^2 x_{n+1}. \tag{5}$$

The LHS term corresponds to $\lim_{n\to\infty} x_{n+1}^2$, resulting in

$$a^2 = \arctan^2(a) \implies a = 0. \tag{6}$$

Additionally, from (3),

$$1 > \frac{x_{n+1}^2}{x_n^2} = \frac{n}{n+1} + \frac{\arctan^2(x_n)}{nx_n^2} > \frac{n}{n+1}. \tag{7}$$

Applying the squeeze theorem results in

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1. \tag{8}$$

We conclude with a similar application of the Stolz-Cesàro theorem.

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{(x_n\sqrt{\ln(n)})^2} &= \lim_{n\to\infty} \frac{1}{x_n^2 \ln(n)} &(9)\\
&= \lim_{n\to\infty} \frac{1/x_n^2}{\ln n} &(10)\\
&= \lim_{n\to\infty} \frac{1/x_{n+1}^2 - 1/x_n^2}{\ln(n+1) - \ln n} &(11)\\
&= \lim_{n\to\infty} \frac{n}{x_n^2} \left[\left(1 + \frac{x_n - x_{n+1}}{x_{n+1}}\right)^2 - 1\right] &(12)\\
&= 2\lim_{n\to\infty} \frac{n(x_n - x_{n+1})}{x_n^3}. &(13)
\end{aligned}$$

Using the Taylor expansion $\arctan x = x - x^3/3 + o(x^3)$, $\sqrt{1+x} = 1 + x/2 + o(x^2)$ and the observation that $x_{n+1}^2 = \frac{nx_n^2 + \arctan^2(x_n)}{n+1}$

$$\begin{aligned}
x_{n+1} = \sqrt{\frac{nx_n^2 + \left(x_n - \frac{x_n^3}{3} + o(x_n^3)\right)^2}{n+1}} &= \sqrt{\frac{(n+1)x_n^2 - \frac{2}{3(n+1)}x_n^4 + o(x_n^4)}{n+1}} &(14)\\
&= x_n\sqrt{1 - \frac{2}{3(n+1)}x_n^2 + o(x_n^2)} &(15)
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{n(x_n - x_{n+1})}{x_n^3} = \lim_{n\to\infty} \frac{nx_n}{x_n^3}\left(1 - \sqrt{1 - \frac{2}{3(n+1)}x_n^2 + o(x_n^2)}\right) \tag{16}$$
$$= \lim_{n\to\infty} \frac{nx_n}{x_n^3}\left(\frac{2}{6(n+1)}x_n^2 + o(x_n^2)\right) \tag{17}$$
$$= \frac{1}{3}. \tag{18}$$

We conclude,

$$\lim_{n\to\infty} x_n\sqrt{\ln n} = \sqrt{\frac{3}{2}}. \tag{19}$$

5117. *Proposed by Nguyen Minh Ha.*

Let $OXYZ$ be a tetrahedron and let k be a given positive constant. Points M, N, P lie respectively on the edges YZ, ZX, XY. Points A, B, C move respectively on the segments OX, OY, OZ such that

$$V[AXNP] + V[BYPM] + V[CZMN] = k,$$

where V represents volume. Prove that the center of the circumscribed sphere of the tetrahedron $OABC$ moves on a fixed plane.

We received five submissions; all were correct and complete although two of them relied on a computer. We feature the solution by Michal Adamaszek.

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be fixed linearly independent vectors, and $x_1, x_2, x_3 > 0$ be scalars that are allowed to vary. Our problem deals with the family of nondegenerate tetrahedra with vertices $\mathbf{0}$ (the origin), $x_1\mathbf{v}_1$, $x_2\mathbf{v}_2$, $x_3\mathbf{v}_3$, and circumcenter $\mathbf{s}$ (considered as a column vector $\mathbf{s}(x_1, x_2, x_3)$ dependent on the variables x_i). These circumcenters are determined by the three equations

$$\|\mathbf{s}\|^2 = \|\mathbf{s} - x_i\mathbf{v}_i\|^2,\ i = 1, 2, 3.$$

These are equivalent to

$$2x_i\langle \mathbf{s}, \mathbf{v}_i\rangle = x_i^2\|\mathbf{v}_i\|^2,$$

and subsequently

$$2\langle \mathbf{s}, \mathbf{v}_i\rangle = x_i\|\mathbf{v}_i\|^2.$$

The three equations can be written in matrix form as

$$2\mathbf{s}^T\left[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\right] = \left[x_1\|\mathbf{v}_1\|^2, x_2\|\mathbf{v}_2\|^2, x_3\|\mathbf{v}_3\|^2\right],$$

and from here we have the explicit formula for $\mathbf{s}$:

$$\mathbf{s}^T = \frac{1}{2}\cdot\left[x_1\|\mathbf{v}_1\|^2, x_2\|\mathbf{v}_2\|^2, x_3\|\mathbf{v}_3\|^2\right]\cdot\left[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\right]^{-1}.$$

In particular *the coordinates of* $\mathbf{s}$ *are linear functions of* x_1, x_2, x_3, or in other words the map s which takes (x_1, x_2, x_3) to the circumcenter $\mathbf{s}(x_1, x_2, x_3)$ is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$.

Now we proceed to solve the problem. Let O be the origin, $\mathbf{v}_1 = \overrightarrow{OX}$, $\mathbf{v}_2 = \overrightarrow{OY}$, $\mathbf{v}_3 = \overrightarrow{OZ}$, and let

$$k_1 = V[OXNP],\ k_2 = V[OYPM],\ k_3 = V[OZMN].$$

All the values defined until now are constant. Moreover, let

$$\overrightarrow{OA} = x_1\mathbf{v}_1,\ \overrightarrow{OB} = x_2\mathbf{v}_2,\ \overrightarrow{OC} = x_3\mathbf{v}_3$$

for variable $0 < x_1, x_2, x_3 < 1$. The condition from the problem statement is equivalent to

$$(1 - x_1)k_1 + (1 - x_2)k_2 + (1 - x_3)k_3 = k. \tag{1}$$

Consequently, the triples (x_1, x_2, x_3) vary in the fixed affine plane π defined by (1); therefore, the circumcenters of $OABC$, namely $\mathbf{s}(x_1, x_2, x_3)$, lie in the image $\mathbf{s}(\pi)$. As established in the previous paragraph s is linear, therefore $\mathbf{s}(\pi)$ is also a (fixed) affine plane.

5118. *Proposed by Tatsunori Irie.*

Let a, b, c, d be positive real numbers. Prove that

$$\left(\frac{a+b+c+d}{4}\right)^{\frac{a+b+c+d}{4}} \geq (a^b b^c c^d d^a)^{\frac{1}{4}}.$$

We received 19 submissions and they were all complete and correct. We present the following solution by the majority of solvers.

Taking logarithms on both sides of the desired inequality yields

$$b\log a + c\log b + d\log c + a\log d \leq (a+b+c+d)\log\left(\frac{a+b+c+d}{4}\right).$$

Now, since the logarithm function is concave on $(0, \infty)$, Jensen's inequality implies that

$$\frac{b\log a + c\log b + d\log c + a\log d}{b+c+d+a} \leq \log\left(\frac{ab+bc+cd+da}{b+c+d+a}\right).$$

Thus, it suffices to prove that

$$\frac{ab+bc+cd+da}{b+c+d+a} \leq \frac{a+b+c+d}{4},$$

that is, $4(a+c)(b+d) \leq (a+b+c+d)^2$. We are done since the latter follows from the AM-GM inequality.

5119. *Proposed by Nazar Kirgizbaev, modified by the Editorial Board.*

For a triangle ABC with circumcenter O and a point X on the circumcircle, let D, E, and F be the points where the cevians AX, BX, and CX meet the opposite sides (or their extensions). Prove that if the circles with centers D, E, F and radii DX, EX, FX have a common tangent line, then that line passes through O.

We received only two submissions, both of which required considerable perseverance. One verified the result using an intricate argument based on barycentric coordinates. We shall, however, feature the proposer's solution.

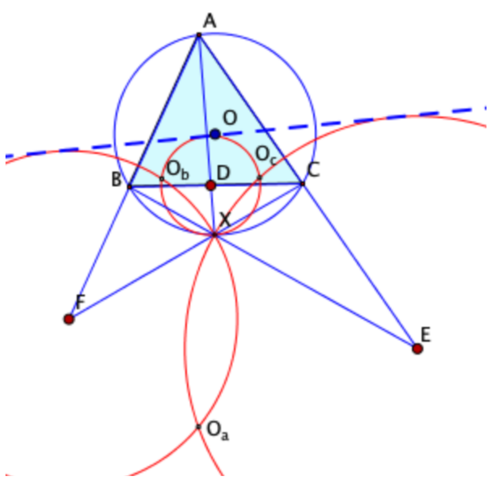


Figure 1

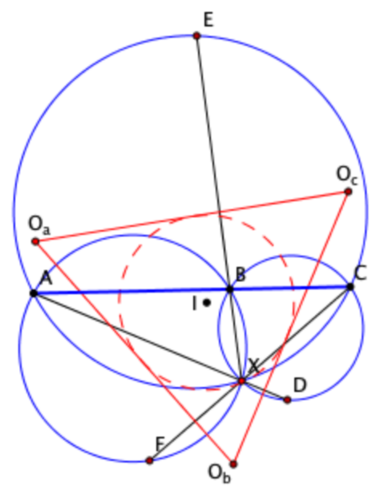


Figure 2

Our argument is based on two known results.

Lemma 1. *Given three collinear points A, B, C and a point X not on that line (as in Figure 2), let D be the second intersection of the circumcircle of ΔBCX with the line AX, and F the second intersection of the circumcircle of ΔABX and the line CX. If O_b is the circumcenter of FXD, then $O_bB \perp AC$.*

Proof. Let P be the intersection of the lines CD and AF. We first note that P lies on the circle (FDX): This is just Miquel's theorem applied to the triangle ACP with points B on AC, D on CP, and F on PA (which states that the three circles $(ABF), (CBD)$, and (PFD) meet at the Miquel point, which must be X, the point where the first two circles meet). Next, define $Q = FD \cap PX$ and consider the quadrangle $FPDX$ (which is inscribed in the circle with center O_b) with its diagonal points A, C, and Q. Brocard's theorem states that the diagonal triangle ΔACQ of the complete cyclic quadrangle is self-polar (that is, each vertex is the pole of the opposite side), and its orthocenter is the quadrangle's circumcenter, namely O_b; in particular, $O_bQ \perp AC$ at B, as claimed.

Recall that the incircle and the nine-point circle of a triangle are tangent at a point called the *Feuerbach point* of the triangle. For any point P on the circumcircle of a triangle, its reflections in the sides of the triangle lie on a line, namely the *Steiner line* of P with respect to the triangle, and that line passes through the orthocenter of the triangle. These two notions are related by the following result:

Lemma 2. *The Steiner line of a triangle's Feuerbach point with respect to its midpoint triangle passes through the incenter of the original triangle.*

A complete proof can be found in the article *Some Properties of the Feuerbach Point* by Kyaw Shin Thant.

Returning to the original problem, we apply an inversion centered at X to the given configuration (shown in Figure 1), and our problem is transformed into a problem concerning the configuration of Figure 2:

> Let A, B, C be points on a line ℓ, and let X be a point not on ℓ. Let D, E, F be the second intersections of the circumcircles of ΔBCX, ΔCAX, ΔABX with lines AX, BX, CX, respectively. Denote by O_a, O_b, O_c the points of intersection of the perpendicular bisectors of the line segments XD, XE, XF, labeled as in Figure 2 (with O_a the intersection of the bisectors of XE and XF, etc.). If the incircle of $\Delta O_aO_bO_c$ passes through X, prove that the incenter I of this triangle lies on the line ℓ.

Note that the labels of points in the original statement have been retained for their images in the restatement except for the center of inversion X, which was exchanged with the point at infinity by the inversion. Lines that contain X are fixed by the inversion, while lines not containing X (namely, the sides of ΔABC in Figure 1) are are taken to circles through X (such as ABX in Figure 2). Finally, the circles through X (namely the circumcircle of the given triangle ΔABC and the circles centered at D, E, F) are taken to lines not through X (namely the line ℓ and the sides of $\Delta O_aO_bO_c$). Because a circle through X together with a pair of points that are exchanged by an inversion in that circle (such as the circle with center F and radius FX together with the points F and infinity) is sent by our inversion into a line not through X and a pair of points related by reflection (namely the line O_aO_b with points F and X), that image line must be the perpendicular bisector of the two image points. Because orthogonality is preserved by inversion, the common tangent line to the circles in the original statement of the problem contains the circumcenter O if and only if the image ℓ of the given circumcircle is the diameter of the circle that is tangent to the sides of the image triangle $\Delta O_aO_bO_c$. Note that that tangent circle must be the incircle (and not an excircle) because inversions preserve betweenness (so that pairs of vertices separate the tangency points from the point at infinity). We are now ready for the proof of the restatement.

Claim 1. The midpoint M_a of the line segment O_bO_c is the circumcenter of ΔBCX.

Proof. The line O_bO_c is, by definition, the perpendicular bisector of XD; thus, any point on this line, including its midpoint M_a, is equidistant from D and X, whence $M_aD = M_aX$. To prove that M_a is the circumcenter of ΔBCX, it suffices to show that $M_aB = M_aC$. From Lemma 1 we deduce that O_bB and O_cC are both perpendicular to ℓ; therefore, the perpendicular bisector of the segment BC lies halfway between the parallel lines O_bB and O_cC; in particular, it contains the midpoint M_a of O_bO_c, so that $M_aB = M_aC$, and M_a is the circumcenter of ΔBCX, as claimed.

Similarly, the midpoints M_b of O_cO_a and M_c of O_aO_b are the circumcenters of ΔCAX and ΔABX.

Claim 2. The point X lies on the nine-point circle of $\Delta O_aO_bO_c$.

Proof. The reflections of the point X in the lines M_aM_b, M_bM_c, M_cM_a (joining pairs of circumcenters) are the points C, B, A, respectively (the other ends of the common chords XC, XB, XA). Since these points are assumed to be collinear on the line ℓ, a theorem of Steiner implies two things:

1. The point X must lie on the circumcircle of $\Delta M_aM_bM_c$.
2. The line ℓ is the Steiner line of X with respect to $\Delta M_aM_bM_c$.

The circumcircle of the midpoint triangle $\Delta M_aM_bM_c$ is the nine-point circle of $\Delta O_aO_bO_c$. Consequently X lies on the nine-point circle, as claimed.

We can now conclude the proof. We are given that X lies on the incircle of $\Delta O_aO_bO_c$. From Claim 2 we know that X lies also on the nine-point circle of $\Delta O_aO_bO_c$. But, a point that lies on both the incircle and the nine-point circle of a triangle is, by definition, its Feuerbach point. Thus X is the Feuerbach point of $\Delta O_aO_bO_c$. From the proof of Claim 2 we established that ℓ is the Steiner line of X with respect to the medial triangle $\Delta M_aM_bM_c$. By Lemma 2, the Steiner line of the Feuerbach point with respect to the midpoint triangle must pass through the incenter of the triangle. Therefore, the incenter I of $\Delta O_aO_bO_c$ must lie on the line ℓ, which is exactly what we needed to prove.

5120. *Proposed by Michel Bataille.*

Let m, n be integers such that $0 \leq m < n$. Prove that

$$\left(\sum_{k=0}^{m} \binom{m}{k} \frac{(-2)^k}{n-k}\right) \left(\sum_{k=0}^{m} \binom{n}{k} (-2)^k\right) = \left(\sum_{k=0}^{m} \binom{m}{k} \frac{1}{n-k}\right) \left(\sum_{k=0}^{m} \binom{n}{k}\right).$$

We received 4 solutions, all correct and complete. We present the solution by the problem proposer Michel Bataille (a similar approach was employed by Michał Adamaszek and Walther Janous).

As a lemma, we first prove the following identity

$$\sum_{k=0}^{m} \binom{n}{k} (-1)^k (x+1)^k = \frac{(-1)^m n(n-1)\cdots(n-m)}{m!} \sum_{k=0}^{m} \binom{m}{k} \frac{x^k}{n-k}. \qquad (1)$$

Proof. The binomial theorem and well-known formulas

$$\binom{r}{s}\binom{s}{t}=\binom{r}{t}\binom{r-t}{s-t} \quad \text{and} \quad \sum_{k=0}^{p}(-1)^k\binom{m}{k}=(-1)^p\binom{m-1}{p},$$

show that the left-hand side is equal to

$$\begin{aligned}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}(-1)^k x^j &= \sum_{j=0}^{m}\sum_{k=j}^{m}\binom{n}{j}\binom{n-j}{k-j}(-1)^k x^j\\ &= \sum_{j=0}^{m}\binom{n}{j}x^j(-1)^j\sum_{i=0}^{m-j}(-1)^i\binom{n-j}{i}\\ &= (-1)^m\sum_{j=0}^{m}\binom{n}{j}\binom{n-j-1}{m-j}x^j,\end{aligned}$$

and (1) is obtained by remarking that

$$\binom{n}{j}\binom{n-j-1}{m-j}=\frac{n(n-1)\cdots(n-m)}{m!}\cdot\frac{1}{n-j}\cdot\binom{m}{j}.$$

Taking successively $x=-2$ and $x=1$ in (1), we obtain

$$\sum_{k=0}^{m}\binom{n}{k}=\frac{(-1)^m n(n-1)\cdots(n-m)}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{(-2)^k}{n-k}$$

and

$$\sum_{k=0}^{m}\binom{n}{k}(-2)^k=\frac{(-1)^m n(n-1)\cdots(n-m)}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{1}{n-k}.$$

The result follows.

Note. More generally, if we set

$$P(x)=\left(\sum_{k=0}^{m}\binom{m}{k}\frac{x^k}{n-k}\right)\left(\sum_{k=0}^{m}\binom{n}{k}x^k\right),$$

then the above argument shows that $P(-1-x)=P(x)$.