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MATHEMATTIC

No. 73

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 15, 2026**.

MA361. *Proposed by Nguyen Van An.*

There are 1001 boxes placed around a circle, labeled $B_1, B_2, \ldots, B_{1001}$ clockwise. Initially box B_i contains i balls ($i = 1, 2, \ldots, 1001$). Fix a positive integer k. A move consists of choosing k balls from k different nonempty boxes (one ball per chosen box) and, for each chosen ball that is in B_n, moving it into B_{n+1} (with the convention $B_{1002} \equiv B_1$). The game ends when all boxes contain the same number of balls.

(a) For $k = 1$, show how to finish the game.

(b) For $k = 1$, prove that the number of moves to finish the game is always divisible by 1001.

(c) For $2 \leq k \leq 1001$, is it always possible to finish the game? Explain and give a method when it is possible.

MA362. *Proposed by Neculai Stanciu.*

Determine the five-digit numbers $abcde$ and $xyztu$ with distinct digits 1, 2, 3, 7 and 8 such that $4 \cdot abcde = xyztu$.

MA363. The number $123456789101112\ldots484950$ has 91 digits. Wipe some of its digits off so that the largest possible number with the sum of its digits equal to 200 remains.

MA364. Consider a rectangular block with dimensions $m \times n \times r$, where m, n, r are positive integers and $m < n < r$. After the surfaces of the block are painted red, the block is cut into cubes with sides 1 unit. If the sum of the number of the cubes with no red surface and with two red surfaces minus the number of the cubes with one red surface equals 1991, find the values of m, n, and r.

MA365. $ABCD$ is a parallelogram in which $\angle ABC$ is acute and AD is a tangent to the circle ABC. Prove that

$$BD^2 = AB^2 + 2BC^2.$$

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2026**.

MA361. *Soumis par Nguyen Van An.*

On dispose 1001 boîtes autour d'un cercle, étiquetées $B_1, B_2, \ldots, B_{1001}$ dans le sens horaire. Au départ, la boîte B_i contient i boules ($i = 1, 2, \ldots, 1001$). Fixons un entier positif k. Un coup consiste à choisir k boules provenant de k boîtes non vides distinctes (une boule par boîte choisie) et, pour chaque boule choisie se trouvant dans B_n, à la déplacer dans B_{n+1} (avec la convention $B_{1002} \equiv B_1$). La partie se termine lorsque toutes les boîtes contiennent le même nombre de boules.

(a) Pour $k = 1$, montrez comment terminer la partie;

(b) Pour $k = 1$, prouvez que le nombre de coups nécessaires pour terminer la partie est toujours divisible par 1001;

(c) Pour $2 \leq k \leq 1001$, est-il toujours possible de terminer la partie ? Expliquez et donnez une méthode lorsque c'est possible.

MA362. *Soumis par Neculai Stanciu.*

Déterminez les nombres à cinq chiffres $abcde$ et $xyztu$, formés avec les chiffres distincts $1, 2, 3, 7$ et 8, tels que $4 \cdot abcde = xyztu$.

MA363. Le nombre $123456789101112\ldots484950$ a 91 chiffres. Effacez certains de ses chiffres de sorte qu'il reste le plus grand nombre possible dont la somme des chiffres soit égale à 200.

MA364. Considérons un pavé rectangulaire de dimensions $m \times n \times r$, où m, n et r sont des entiers positifs et $m < n < r$. Après avoir peint en rouge les faces du pavé, on le découpe en cubes de côté 1. Si la somme du nombre de cubes ne portant aucune face rouge et du nombre de cubes portant deux faces rouges, moins le nombre de cubes portant une seule face rouge, est égale à 1991, déterminez les valeurs de m, n et r.

MA365. $ABCD$ est un parallélogramme dans lequel $\angle ABC$ est aigu et AD est tangente au cercle passant par A, B et C. Montrez que

$$BD^2 = AB^2 + 2BC^2.$$

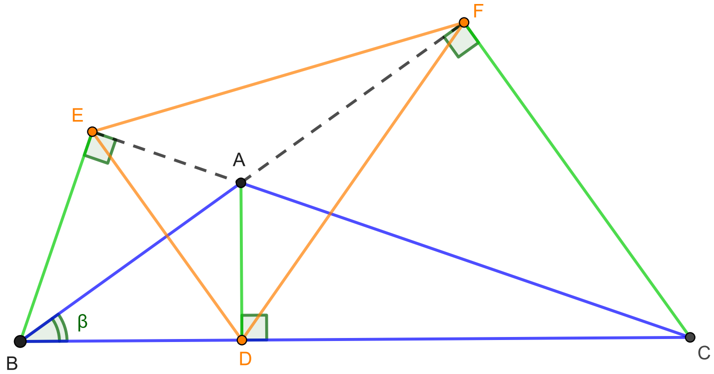
MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(8), p. 363–365.

MA336. *Proposed by Michael Friday.*

Suppose that in triangle ABC we have $\angle A - \angle B = 90^\circ$. Prove that the orthic triangle of ABC, which is the triangle formed by the feet of the altitudes of ABC, is isosceles.

We received 14 submissions of which 11 were correct and complete. We present 2 solutions.



Denote by D, E and F the feet of the altitudes from A, B and C respectively, as in the diagram above. Note that since $\angle A > 90^\circ$ we have that E and F are exterior to the line segments AC and AB. We will prove that $FE = FD$.

Let $\angle ABC = \beta$. We are given $\angle BAC = 90^\circ + \angle ABC = 90^\circ + \beta$.

Solution 1, by Lienhard Wimmer.

In $\triangle AEB$ we have

$$\angle EAB = 180^\circ - \angle BAC = 180^\circ - (90^\circ + \beta) = 90^\circ - \beta,$$

whence

$$\angle EBA = 90^\circ - (90^\circ - \beta) = \beta.$$

So $\angle ABE = \angle ABD = \beta$. Thus $\triangle ABE \cong \triangle ABD$ (as corresponding angles are congruent and they share the side AB), and it follows that $BE = BD$. The line l through B and A is the bisector of $\angle EBD$, and hence also the perpendicular bisector of ED. As the foot of the perpendicular from C onto l, F is also on l.

Since the perpendicular bisector of ED passes through the vertex F, we conclude that $\triangle FED$ is isosceles with $FE = FD$.

Solution 2, by Nicols Guasgua. We have

$$\angle FAC = 180^\circ - \angle BAC = 180^\circ - (90^\circ + \beta) = 90^\circ - \beta.$$

Since CF is the altitude from C, $\triangle ACF$ is right-angled; hence

$$\angle ACF = 90^\circ - \angle FAC = 90^\circ - (90^\circ - \beta) = \beta.$$

Quadrilateral $ADCF$ is cyclic ($\angle ADC + \angle AFC = 90^\circ + 90^\circ = 180^\circ$), so

$$\angle ADF = \angle ACF = \beta \text{ (they subtend the same arc).}$$

Since $\angle BEC = \angle BFC = 90^\circ$, quadrilateral $BCFE$ is also cyclic, which gives us

$$\begin{aligned} \angle EBF &= \angle ECF = \beta, \text{ and} \\ \angle FEC &= \angle FBC = \beta. \end{aligned}$$

Finally, $ADBE$ is cyclic because $\angle ADB + \angle AEB = 90^\circ + 90^\circ = 180^\circ$; this gives us

$$\begin{aligned} \angle EDA &= \angle EBA = \beta, \text{ and} \\ \angle DEA &= \angle DBA = \beta. \end{aligned}$$

Using the values calculated above,

$$\begin{aligned} \angle FDE &= \angle ADF + \angle EDA = 2\beta, \text{ and} \\ \angle DEF &= \angle DEA + \angle AEF = 2\beta, \end{aligned}$$

whence $\angle FDE = \angle DEF$ and $\triangle FDE$ is isosceles.

MA337. *Proposed by Alaric Pow.*

Prove or disprove that there cannot be a right-angled triangle whose sides are all of prime length.

We received 17 submissions of which 12 were correct and complete. We present an edited solution proposed by Peter McGregor.

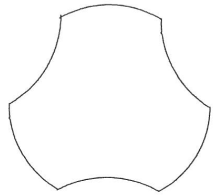
For the sake of contradiction, suppose that there is a Pythagorean triple of prime numbers of the form $a^2 + b^2 = c^2$. Since c is larger than a and b, c must be an odd prime number, as 2 is the only even prime number and it is the smallest prime number.

Therefore, c^2 is odd and so is $a^2 + b^2$. Exactly one of a^2 and b^2 is even. Without loss of generality, suppose that a^2 is even. Then $a = 2$ and we obtain

$$\begin{aligned} c^2 &= 4 + b^2 \\ (c+b)(c-b) &= 4. \end{aligned}$$

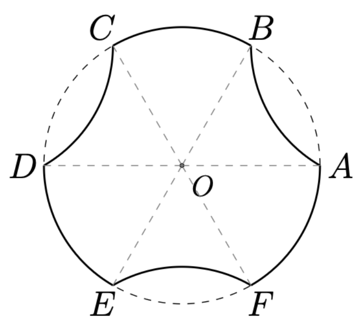
Since c and b are both odd primes, $c - b \geq 2$ and $c + b > 2$. Thus, $4 > 4$ and a contradiction is reached.

MA338. A *hexle* is constructed from a circle by reversing three non-intersecting arcs, each of which is 1/6 of the circumference. If the radius of the circle is 1, find the exact area of the hexle.



Originally from the 1995 Newfoundland and Labrador Senior Math League, Problem 5.

We received 16 submissions of which 14 were correct and complete. We present the solution by the Renert School Division 2 Problem Solving Group.



Let the circle have radius $r = 1$, so its area is $\pi r^2 = \pi$. Since arc BC is $\frac{1}{6}$ of the circumference, $\angle BOC = 60°$. Together with $BO = OC = 1$, this means $\triangle BOC$ is equilateral. Dropping the altitude from O to BC splits it into two 30-60-90 right triangles, giving an altitude of $\frac{\sqrt{3}}{2}$, and so the area of $\triangle BOC$ is

$$\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}.$$

The area of sector OBC is $\frac{1}{6}\pi$, so the circular segment between the chord and the arc has area $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{4}$. The hexle is obtained from the circle by removing six such congruent segments, so its area is

$$\pi - 6\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \pi - \pi + \frac{3\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

Editor's Comment. Observe that the irregular region can be rearranged into a regular hexagon. Each side of this hexagon has length 1, since it is a chord subtending a central angle of 60°. Hence its area is

$$6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}.$$

MA339. If p and q are positive integers such that

$$\frac{p}{q} = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \cdots + \frac{1}{478} + \frac{1}{479} - \frac{2}{480},$$

show that p is divisible by 641.

Originally from the 1987 Descartes Contest, Problem 10.

There were 9 submissions, all of them were complete and correct. We present the solution by Rutvaj Nehete.

Rearranging the sum, we get

$$\begin{aligned}\frac{p}{q} &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{480}\right) - 3\left(\frac{1}{3} + \frac{1}{6} + \cdots + \frac{1}{480}\right)\\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{480}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{160}\right)\\ &= \frac{1}{161} + \frac{1}{162} + \cdots + \frac{1}{480} = \sum_{k=161}^{320} \frac{641}{k(641-k)} = 641 \sum_{k=161}^{320} \frac{1}{k(641-k)}.\end{aligned}$$

None of the denominators in the latter sum are divisible by 641, a prime number, in the bounds for k, so we can let

$$\sum_{k=161}^{320} \frac{1}{k(641-k)} = \frac{P}{Q},$$

where $641 \nmid Q$, giving

$$\frac{p}{q} = \frac{641P}{Q},$$

showing that $641 \mid p$.

MA340. Suppose that m and n are positive integers. For what values of m and n can $m^4 + 4n^4$ be a prime number?

Originally from the 1982 Descartes Contest, Problem 8.

There were 16 submissions, 15 of them were complete and correct. We present the solution by Md. Rad Sarar Anando.

Let's seek positive integers $m, n \geq 1$ for which the expression $m^4 + 4n^4$ is prime. The main point is the algebraic identity

$$m^4 + 4n^4 = (m^2 + 2n^2 - 2mn)(m^2 + 2n^2 + 2mn).$$

It can be defined as

$$A = m^2 - 2mn + 2n^2, \quad B = m^2 + 2mn + 2n^2.$$

Therefore,

$$m^4 + 4n^4 = A \cdot B.$$

Let's rewrite A and B in a more suggestive form:

$$A = (m-n)^2 + n^2, \quad B = (m+n)^2 + n^2.$$

Since $m, n \geq 1$, both A and B are positive integers.

Moreover,

$$A = (m-n)^2 + n^2 \geq n^2 \geq 1,$$

and

$$B = (m+n)^2 + n^2 \geq (1+1)^2 + 1^2 = 4 + 1 = 5.$$

The equality holds in the bound for B when $m = n = 1$.

For the product $A \cdot B$ to be prime, and since $A, B > 0$, one of the factors must be 1 and the other a prime number. As $B \geq 5 > 1$, the only possibility is

$$A = 1$$

with B prime.

By solving the equation,

$$(m-n)^2 + n^2 = 1.$$

The left side is the sum of two squares of integers, each nonnegative. Since $n \geq 1$, we have $n^2 \geq 1$, so the only way the sum equals 1 is if

$$n^2 = 1 \quad \text{and} \quad (m-n)^2 = 0.$$

Thus $n = 1$ (as $n > 0$) and $m - n = 0$, so $m = 1$.

For $m = n = 1$,

$$A = 1^2 - 2 \cdot 1 \cdot 1 + 2 \cdot 1^2 = 1 - 2 + 2 = 1,$$
$$B = 1^2 + 2 \cdot 1 \cdot 1 + 2 \cdot 1^2 = 1 + 2 + 2 = 5,$$

and

$$m^4 + 4n^4 = 1 + 4 = 5,$$

which is prime.

In order to confirm uniqueness, let's assume that there exists another pair $m', n' \geq 1$ with $A = 1$. Then $(m'-n')^2 + (n')^2 = 1$. The only integer solution to $x^2 + y^2 = 1$

with $y \geq 1$ is $x = 0$, $y = 1$, forcing $n' = 1$ and $m' = 1$. Hence, the only solution is $m = n = 1$.

For completeness, if $m > n \geq 1$, then $m - n \geq 1$, so

$$A \geq 1 + n^2 \geq 2 > 1.$$

If $m < n$, then $m - n \leq -1$, so $(m - n)^2 \geq 1$ and

$$A \geq 1 + n^2 \geq 2 > 1.$$

In all cases except $m = n = 1$, it shows $A > 1$ and $B > 1$, so the product is composite.

Therefore, the only positive integers m, n that make $m^4 + 4n^4$ prime are $m = 1$, $n = 1$.

Editor's Comment. Special mention to Konstantine Zelator for looking into a generalisation of the problem.

PROBLEM SOLVING VIGNETTES

No. 41

Shawn Godin

An Open Problem

Long-time ***Crux*** editor Ed Barbeau is well-known in mathematics education circles. He has been involved in teacher preparation, contest construction, and mathematical outreach and enrichment for decades. His website contains numerous resources for teachers and students alike. In this issue, we will focus on a problem from the document *A collection of problems and investigations* found under the heading *Talks, articles and compilations.*

> **Problem 33.** *The Erdős-Straus conjecture dates back to 1948. It states that, for any integer $n \geq 3$, the fraction $\frac{4}{n}$ can be written as the sum of three distinct reciprocals of integers. For example:*
>
> $$\frac{4}{7} = \frac{1}{2} + \frac{1}{15} + \frac{1}{210}$$

Let's see what we can do with this problem. First, we can get a little help by looking at the difference between the reciprocals of two consecutive positive integers, n and $n+1$. Subtracting yields

$$\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$$

which can be rearranged to obtain

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} \tag{1}$$

This tells us we can write the reciprocal of any positive integer as the sum of two reciprocals of positive integers, which we can use to our advantage.

For example, consider when $n > 3$ is even. Then we can write $n = 2k$, for some $k > 1$. and thus

$$\begin{aligned}\frac{4}{n} &= \frac{4}{2k}\\ &= \frac{2}{k}\\ &= \frac{1}{k} + \frac{1}{k}\\ &= \frac{1}{k} + \frac{1}{k+1} + \frac{1}{k(k+1)} \qquad (2)\end{aligned}$$

We can demonstrate (2) with a couple of cases. Checking $n = 6$ gives

$$\frac{4}{6} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$$

while $n = 20$ gives

$$\frac{4}{20} = \frac{1}{10} + \frac{1}{11} + \frac{1}{110}$$

The interested reader can verify that these identities hold.

The results given in (2) are not necessarily unique. Note that if n is divisible by 4, we could write $n = 4k$, $k \geq 1$ and then

$$\begin{aligned}\frac{4}{n} &= \frac{4}{4k} \\ &= \frac{1}{k} \\ &= \frac{1}{k+1} + \frac{1}{k(k+1)} \\ &= \frac{1}{k+2} + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \end{aligned} \tag{3}$$

Thus, we could also write

$$\frac{4}{20} = \frac{1}{7} + \frac{1}{30} + \frac{1}{42}$$

giving us a second example for $\frac{4}{20}$.

What if n is odd? Equation (1) gives us a hint that if $n = 4k - 1$, for $k \geq 1$, then

$$\begin{aligned}\frac{4}{4k-1} &= 4\left(\frac{1}{4k} + \frac{1}{4k(4k-1)}\right) \\ &= \frac{1}{k} + \frac{1}{k(4k-1)}\end{aligned}$$

from which we could use (1) again to obtain

$$\frac{4}{4k-1} = \frac{1}{k+1} + \frac{1}{k(k+1)} + \frac{1}{k(4k-1)} \tag{4}$$

or

$$\begin{aligned}\frac{4}{4k-1} &= \frac{1}{k} + \frac{1}{k(4k-1)+1} + \frac{1}{k(4k-1)(k(4k-1)+1)} \\ &= \frac{1}{k} + \frac{1}{4k^2-k+1} + \frac{1}{k(4k-1)(4k^2-k+1)}\end{aligned} \tag{5}$$

A little algebraic sleight of hand suggests that divisibility of n or $n+1$ by 3 can also be handled, as

$$\frac{4}{n} = \frac{3}{n} + \frac{1}{n}$$

In the case where $n = 3k$, we get

$$\begin{aligned}\frac{4}{n} = \frac{4}{3k} &= \frac{3}{3k} + \frac{1}{3k}\\ &= \frac{1}{k} + \frac{1}{3k}\end{aligned}$$

which, with the help from (1), yields

$$\frac{4}{3k} = \frac{1}{k+1} + \frac{1}{k(k+1)} + \frac{1}{3k} \tag{6}$$

$$\frac{4}{3k} = \frac{1}{k} + \frac{1}{3k+1} + \frac{1}{3k(3k+1)} \tag{7}$$

In the case where $n = 3k - 1$, we get

$$\begin{aligned}\frac{4}{n} = \frac{4}{3k-1} &= \frac{3}{3k-1} + \frac{1}{3k-1}\\ &= 3\left(\frac{1}{3k} + \frac{1}{3k(3k-1)}\right) + \frac{1}{3k-1}\\ &= \frac{1}{k} + \frac{1}{3k-1} + \frac{1}{k(3k-1)} \qquad (8)\end{aligned}$$

Thus we have solved for all $n \equiv 0, 2, 3 \pmod 4$ and $n \equiv 0, 2 \pmod 3$, which means that we have solved the problem for all cases *except* when $n \equiv 1 \pmod 4$ **and** $n \equiv 1 \pmod 3$, that is, $n \equiv 1 \pmod{12}$.

If we let $n = 12k + 1$, for $k > 1$, then we can take a hint from continued fractions and write

$$\begin{aligned}\frac{4}{12k+1} &= \frac{1}{\frac{12k+1}{4}}\\ &= \frac{1}{3k + \frac{1}{4}}\end{aligned}$$

from which we can deduce

$$\frac{1}{3k+1} < \frac{4}{12k+1} < \frac{1}{3k}$$

Thus $\frac{1}{3k+1}$ is close to, yet smaller than $\frac{4}{12k+1}$, the difference between them being

$$\begin{aligned}\frac{4}{12k+1} - \frac{1}{3k+1} &= \frac{4(3k+1) - (12k+1)}{(3k+1)(12k+1)}\\ &= \frac{3}{(3k+1)(12k+1)}\\ &= \frac{1}{(3k+1)(12k+1)} + \frac{2}{(3k+1)(12k+1)} \qquad (9)\end{aligned}$$

If k is odd, then we can write $k = 2\ell + 1$ for some $\ell > 1$ and (9) becomes

$$\begin{aligned}\frac{4}{12k+1} - \frac{1}{3k+1} &= \frac{1}{(3k+1)(12k+1)} + \frac{2}{(3(2\ell+1)+1)(12(2\ell+1)+1)}\\ &= \frac{1}{(3k+1)(12k+1)} + \frac{2}{(6\ell+4)(24\ell+13)}\\ &= \frac{1}{2(3\ell+2)(24\ell+13)} + \frac{1}{(3\ell+2)(24\ell+13)}\end{aligned}$$

from which we can deduce

$$\frac{4}{24\ell+13} = \frac{1}{6\ell+4} + \frac{1}{2(3\ell+2)(24\ell+13)} + \frac{1}{(3\ell+2)(24\ell+13)} \qquad (10)$$

Equation (10) takes care of the case where $n \equiv 13 \pmod{24}$ leaving only the case of $n \equiv 1 \pmod{24}$ unaccounted for.

Unfortunately, our trick above — or any other trick I could think of — will yield any results for $n \equiv 1 \pmod{24}$. Does that mean the identity is not true for all numbers of this form? A little playing yields

$$\begin{aligned}\frac{4}{25} &= \frac{1}{10} + \frac{1}{25} + \frac{1}{50}\\ \frac{4}{49} &= \frac{1}{15} + \frac{1}{98} + \frac{1}{210}\end{aligned}$$

showing there is some hope. Both of these came from examining

$$\frac{4}{n} - \frac{1}{2n} = \frac{7}{2n}$$

which leads to

$$\frac{4}{n} = \frac{1}{2n} + \frac{1}{m}$$

when $7 \mid n$, and $m = 2n \div 7$. From here we can use (1) to come up with a couple identities of the desired form. If n isn't a multiple of 7 we can write

$$\frac{4}{n} = \frac{1}{2n} + \frac{2}{2n} + \frac{5}{2n} = \frac{1}{2n} + \frac{1}{n} + \frac{1}{m'} \qquad (11)$$

when $5 \mid n$ and $m' = 2n \div 5$.

We could continue in this way, finding further restrictions on our remaining case of $n \equiv 1 \pmod{24}$ or we could back up a bit and reexamine the original problem. Suppose, for some n we find positive integers a, b, and c such that

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

then for *any* positive integer $m > 1$, multiplying the equation above by $\frac{1}{m}$ yields

$$\frac{4}{mn} = \frac{1}{ma} + \frac{1}{mb} + \frac{1}{mc}$$

Thus, to find an identity of the form for any n, we need only find an identity of the desired form for *any* prime p that divides n.

For example, if we look at our recently discovered identities for $n \equiv 1 \pmod{24}$ when $5 \mid n$, the candidate after 25 is 145. The identity that we discovered, (11), yields

$$\frac{4}{145} = \frac{1}{290} + \frac{1}{145} + \frac{1}{58}$$

However, we can factor $145 = 5 \times 29$. Since $5 = 2 \times 3 - 1$ and $29 = 10 \times 3 - 1$, then equation (8) will give us

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$$
$$\frac{4}{29} = \frac{1}{10} + \frac{1}{29} + \frac{1}{290}$$

from which we can obtain

$$\frac{4}{145} = \frac{1}{29} \times \frac{4}{5} = \frac{1}{58} + \frac{1}{145} + \frac{1}{290}$$
$$\frac{4}{145} = \frac{1}{5} \times \frac{4}{29} = \frac{1}{50} + \frac{1}{145} + \frac{1}{1450}$$

We have reduced the problem to showing that it works for primes, $p \equiv 1 \pmod{24}$. For $n \leq 1000$, that only leaves 14 candidates — 73, 97, 193, 241, 313, 337, 409, 433, 457, 577, 601, 673, 769, and 937 — to check (they all work). For arbitrary n, this remains an open problem in number theory. Will you be the one to finally solve it? Good luck!

OLYMPIAD CORNER

No. 441

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **May 15, 2026**.

OC771. Determine the smallest constant c such that the inequality

$$(x+y)^2(x^2+y^2+c)+(1-xy)^2 \geq 0$$

holds for all real numbers x and y. For which values of x and y does equality hold for this smallest constant c?

OC772. Let ABC be an acute triangle with $AB > AC$. Let D, E and F denote the feet of its altitudes on BC, AC and AB, respectively. Let S denote the intersection of lines EF and BC. Prove that the circumcircles k_1 and k_2 of the two triangles AEF and DES touch in E.

OC773. Initially, the numbers $1, 2, \ldots, 2024$ are written on a blackboard. Trixi and Nana play a game, taking alternate turns. Trixi plays first. The player whose turn it is chooses two numbers a and b, erases both, and writes their (possibly negative) difference $a-b$ on the blackboard. This is repeated until only one number remains on the blackboard after 2023 moves. Trixi wins if this number is divisible by 3, otherwise Nana wins. Which of the two has a winning strategy?

OC774. Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u, v of vertices, let $d(u,v)$ denote the distance between u and v, that is, the number of edges in the shortest path in T that connects u with v.

Consider the sums

$$W(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} d(u,v) \qquad \text{and} \qquad H(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} \frac{1}{d(u,v)}.$$

Prove that

$$W(T)\cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

OC775. Let V be the set of all continuous functions $f : [0,1] \to \mathbb{R}$, differentiable on $(0,1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0,1)$ such that

$$f(\xi) + \alpha = f'(\xi).$$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2026**.

OC771. Déterminez la plus petite constante c telle que l'inégalité

$$(x+y)^2(x^2+y^2+c) + (1-xy)^2 \geq 0$$

soit vérifiée pour tous les nombres réels x et y. Pour quelles valeurs de x et y y a-t-il égalité pour cette plus petite constante c ?

OC772. Soit ABC un triangle acutangle tel que $AB > AC$. Soient D, E et F les pieds des hauteurs issues de A, B et C sur BC, AC et AB, respectivement.

Soit S le point d'intersection des droites EF et BC. Montrez que les cercles circonscrits k_1 et k_2 des triangles AEF et DES sont tangents en E.

OC773. Initialement, les nombres $1, 2, \ldots, 2024$ sont écrits sur un tableau. Trixi et Nana jouent à un jeu en alternant les tours, Trixi jouant en premier. La joueuse dont c'est le tour choisit deux nombres a et b, les efface tous les deux, puis écrit leur différence (possiblement négative) $a-b$ sur le tableau. On répète ce procédé jusqu'à ce qu'il ne reste plus qu'un seul nombre sur le tableau après 2023 coups. Trixi gagne si ce nombre est divisible par 3 ; sinon, c'est Nana qui gagne. Laquelle des deux joueuses possède une stratégie gagnante ?

OC774. Soit T un arbre à n sommets ; c'est-à-dire un graphe simple connexe à n sommets ne contenant aucun cycle. Pour toute paire u, v de sommets, on note $d(u,v)$ la distance entre u et v, c'est-à-dire le nombre d'arêtes dans un plus court chemin de T reliant u à v.

On considère les sommes

$$W(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} d(u,v) \qquad \text{et} \qquad H(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} \frac{1}{d(u,v)}.$$

Montrez que

$$W(T)\cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

OC775. Soit V l'ensemble des fonctions continues $f:[0,1]\to\mathbb{R}$, dérivables sur $(0,1)$, vérifiant $f(0)=0$ et $f(1)=1$. Déterminez tous les $\alpha\in\mathbb{R}$ tels que, pour toute fonction $f\in V$, il existe $\xi\in(0,1)$ tel que

$$f(\xi)+\alpha = f'(\xi).$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2026: 51(8), p. 377–378.

Note to all readers. We received several **plagiarized** solutions to the problems in this section. Indeed, the official solutions to Olympiad problems are often available online. However, in ***Crux***, we feature original submissions and have **zero** tolerance for plagiarism. Those who decide to partake in academic dishonesty will be **banned** from the publication.

OC746. Two different integers u and v are written on a board. We perform a sequence of steps. At each step, we do one of the following two operations:

(i) If a and b are different integers on the board, then we can write $a+b$ on the board, if it is not already there.

(ii) If a, b and c are three different integers on the board, and if an integer x satisfies $ax^2+bx+c=0$, then we can write x on the board, if it is not already there.

Determine all pairs of starting numbers (u,v) from which any integer can eventually be written on the board after a finite sequence of steps.

Originally from the 2024 EGMO 2024, Day 1, Problem 1.

We received 5 solutions, 4 of which were complete. We present the solution by Oliver Geupel.

We claim that every integer can eventually be written if and only if

$$\max\{u,v\}\geq 1 \qquad \text{and} \qquad uv\neq 0 \qquad \text{and} \qquad \{u,v\}\neq\{-1,1\}.$$

The conditions are necessary. In fact, if neither u nor v is positive, then no positive integer can ever be written: Any outcome of iterated operations (i) is negative, and for distinct nonpositive a, b, c and positive x, it holds

$$ax^2+bx+c<0,$$

which also rules out the operation (ii).

If u or v is zero, no further integer can be written. If $\{u,v\}=\{-1,1\}$, then the process stops after having written 0 on the board. It remains to prove that the conditions are sufficient. Suppose u and v satisfy the conditions where $u\geq 1$. We start writing $u+v$ which is different from both u and v. Next, operation (ii) gives -1 because

$$u\cdot(-1)^2+(u+v)\cdot(-1)+v=0.$$

Iterated addition of -1 to u yields $u-1,u-2,\ldots,1$. By successive addition of ± 1 we can now eventually reach every integer.

OC747. Find all positive integers d for which there exists a degree d polynomial P with real coefficients such that there are at most d different values among $P(0), P(1), P(2), \dots, P(d^2 - d)$.

Originally from the 2024 EGMO 2024, Day 2, Problem 6.

We received 1 incomplete submission to this problem. Polynomials as described in the problem exist if and only if $d \leq 3$. *The complete proof of this statement can be found at* `https://www.egmo.org/egmos/egmo13/solutions.pdf`.

OC748. The complex $n \times n$ matrices A, B satisfy the relation $A^2B + BA^2 = 2ABA$. Check that $X = AB - BA$ commutes with A, and either using this or in any other way prove that there exists $k \in \{1, \dots n\}$ such that $X^k = 0$.

Originally from the 2019 BME Mathematics Contest, Problem 6.

We received 3 submissions, all complete and correct. We feature a common solution.

We have

$$XA - AX = (AB - BA)A - A(AB - BA) = 2ABA - (A^2B + BA^2) = 0,$$

so X commutes with A. Therefore, for each positive integer i, X^i commutes with A. It follows that for each positive integer i, we have

$$\begin{aligned} X^i &= X^{i-1}(AB - BA) = (X^{i-1}A)B - X^{i-1}BA \\ &= (AX^{i-1})B - X^{i-1}BA = A(X^{i-1}B) - (X^{i-1}B)A, \end{aligned}$$

which in particular implies that $\operatorname{tr}(X^i) = 0$.

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of X. Then we have $\operatorname{tr}(X^i) = \sum_{j=1}^{n} \lambda_j^i = 0$ for all positive integers i. It follows that all the eigenvalues of X are zero. Thus X is nilpotent, and there exists $k \in \{1, \dots, n\}$ such that $X^k = 0$.

OC749. If H is a set containing a given number $n > 1$ of (arbitrary) positive integers, how many elements can be in $\{xy + z | x, y, z \in H\}$ at most and at least?

Originally from the 2019 BME Mathematics Contest, Problem 3.

We received 3 submissions. One solver misinterpreted the problem and two other submissions were only partially complete. We leave it up to the reader to look up official solutions or try to create a complete one of their own.

OC750. Find
$$\lim_{n\to\infty}\sum_{k=n}^{2n}\binom{k-1}{n-1}2^{-k}.$$

Originally from the 2019 BME Mathematics Contest, Problem 2.

We received 5 submissions, 4 of which were correct. We present the solution by Michel Bataille.

We first prove the following lemma: if n is a nonnegative integer, then
$$\sum_{k=0}^{n}2^{-k}\binom{n+k}{k}=2^n. \tag{1}$$

The proof is by induction: let X_n denote the sum on the left-hand side. Clearly, $X_0=1=2^0$. Assume that $X_{n-1}=2^{n-1}$ for some positive integer n. Then,
$$\begin{aligned}X_n&=\sum_{k=0}^{n}\binom{n+k-1}{k}2^{-k}+\sum_{k=1}^{n}\binom{n+k-1}{k-1}2^{-k}\\&=X_{n-1}+2^{-n}\binom{2n-1}{n}+\frac{1}{2}\sum_{j=0}^{n-1}\binom{n+j}{j}2^{-j}\\&=X_{n-1}+\frac{1}{2}X_n+\frac{1}{2^n}\left[\binom{2n-1}{n}-\frac{1}{2}\binom{2n}{n}\right]\\&=X_{n-1}+\frac{1}{2}X_n+0,\end{aligned}$$
hence $X_n=2X_{n-1}$ and so $X_n=2^n$, completing the induction step.

Now, let
$$S_n=\sum_{k=n}^{2n}\binom{k-1}{n-1}2^{-k}.$$

We have
$$\begin{aligned}S_n&=\sum_{k=n}^{2n}\binom{k-1}{k-n}2^{-k}\\&=\sum_{j=0}^{n}\binom{n-1+j}{j}2^{-n-j}\\&=\frac{1}{2^n}\left(\binom{2n-1}{n}2^{-n}+\sum_{j=0}^{n-1}\binom{n-1+j}{j}2^{-j}\right),\end{aligned}$$
hence, using (1),
$$S_n=\frac{1}{2}+\frac{1}{4^n}\binom{2n-1}{n}.$$

From Stirling's formula, we deduce that

$$\binom{2n-1}{n} = \frac{1}{2} \cdot \frac{(2n)!}{(n!)^2} \sim \frac{1}{2} \cdot \frac{\sqrt{4\pi n} e^{-2n} 2^{2n} n^{2n}}{2\pi n e^{-2n} n^{2n}} = \frac{4^n}{2\sqrt{\pi n}}$$

as $n \to \infty$.

Thus,

$$\lim_{n\to\infty} \frac{1}{4^n} \binom{2n-1}{n} = 0, \quad \text{and therefore} \quad \lim_{n\to\infty} S_n = \frac{1}{2}.$$

La formule de Descartes

Robert Guilloteau

1 La formule de Descartes

La relation de Descartes, aujourd'hui bien connue sous ce nom, relie élégamment les courbures de quatre cercles tangents deux à deux. Descartes l'avait trouvée à travers l'analyse d'un cas particulier du dixième problème d'Apollonius. Rappelons qu'Apollonius avait posé et résolu, dans un traité malheureusement perdu, un problème plus général où il demandait à trouver les cercles passant par ou tangents à trois objets pris parmi des points, des droites et des cercles. Le dixième problème, réputé le plus difficile, consiste à trouver les cercles tangents à trois cercles donnés.

Cette relation apparaît pour la première fois sous la plume de Descartes dans un échange épistolaire avec la princesse Elisabeth de Bohême, en 1643, alors qu'ils cherchaient chacun de leur côté à établir un théorème qui pourrait relier les rayons des cercles tangents à trois cercles donnés, dans le cas général du dixième problème d'Apollonius. Le cas de cercles donnés tangents apparaît comme une simplification du problème, abordable par la princesse, conduisant potentiellement à un tel théorème. Dans cet échange, René Descartes ne décrit pas les calculs qui aboutissent à la formule qu'il a découverte.

Il faudra attendre 1936 et Frederick Soddy (1877-1956) (chimiste britannique, prix Nobel de chimie en 1921 pour sa découverte des isotopes) pour redécouvrir cette relation, qui, pour un temps, va porter le nom du chimiste. Il avait su si bien en faire la promotion à travers un poème intitulé "The Kiss Precise", paru dans la revue *Nature* en 1936 [3], poème dont le dernier vers se terminait par ces mots

> The sum of the squares of all four bends
> Is half the square of their sum.

Il semble que ce soit Dan Pedoe [4] qui lui restituera le nom de Descartes près avoir pris connaissance d'un article d'Alfred Aeppli paru en 1960 et intitulé *Une généralisation d'une formule de Descartes* [5]. Dans cet article Alfred Aeppli fait directement référence à la lettre de Descartes donnant cette relation.

2 La correspondance avec Elisabeth de Bohême

Parmi les nombreux correspondants de René Descartes figure une princesse amateur de philosophie et de mathématiques, Elisabeth de Bohême (1618-1680) qui restera en liaison épistolaire de 1640 à la mort de Descartes en 1650. Descartes fut principalement son professeur de philosophie mais il lui enseigna son approche analytique des problèmes géométriques jusqu'à lui suggérer, à l'automne 1643, de résoudre par ces méthodes la configuration du dixième problème d'Apollonius. Précisons qu'il réduisait implicitement ce problème au seul cas du cercle tangent extérieurement à trois cercles donnés disjoints. Le 21 octobre 1643 [1, lettre CC-CXX, pp. 25-27] il s'inquiétait auprès de son ami Alphonse Pollot, pensant avoir transmis un problème trop difficile sur lequel la princesse allait perdre beaucoup de temps. Dans une nouvelle lettre [1, lettre CCCXXVI, p. 43] adressée au même Pollot en novembre 1643 , toujours inquiet vis à vis de la princesse, et d'autant plus qu'il pensait que son approche du problème, la princesse ayant choisi de n'utiliser qu'une inconnue (le rayon du cercle solution), serait probablement vouée à l'échec, il joignait à son courrier une lettre destinée à cette dernière [1, lettre CCCXXV, pp. 37-42] où il proposait sa propre approche et tentait de la dissuader de poursuivre, laissant libre Pollot de la lui transmettre ou non selon l'avancement de son travail. Pollot lui transmit.

2.1 La lettre CCCXXV

Dans un premier temps Descartes tente de convaincre la princesse qu'une approche avec plusieurs inconnues était préférable à celle n'utilisant qu'une inconnue. Il le justifie par le fait que cela permet, en général, de n'utiliser ensuite que les propriétés des triangles semblables ou le théorème de Pythagore et réduit le plus souvent, affirme-t-il, le volume de calcul. Puis il donne une description de sa propre approche.

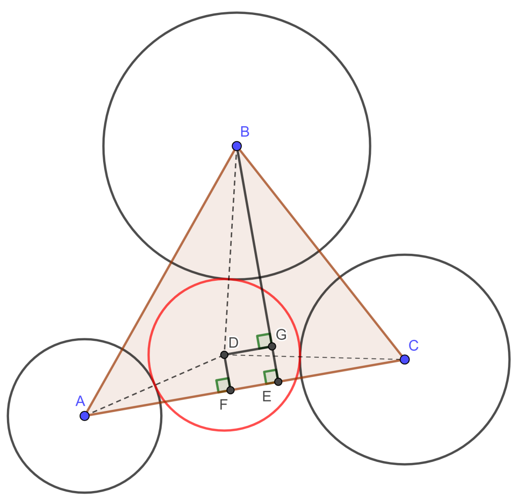


Figure 1: Descartes et le problème CCC.

Soient trois cercles donnés disjoints de centres respectifs A, B et C et de rayons respectifs a, b, c. Notons D le centre du cercle cherché, tangent extérieurement à ces trois cercles. De B menons sur AC une perpendiculaire coupant ce côté en E. Faisons de même avec D pour placer le point F. De D menons une autre perpendiculaire à BE coupant cette droite en G.

Le triangle $\triangle ABC$ étant connu, les grandeurs AE, BE et EC sont connues.

Descartes ne les calcule pas mais les utilise par la suite sous les notations respectives d, e et f. Il choisit trois inconnues : x le rayon du cercle cherché , $y = DF = GE$, $z = DG = FE$.

Le théorème de Pythagore appliqué aux trois triangles rectangles $\triangle AFD$, $\triangle DGB$ et $\triangle FCD$ donne respectivement

$$(a+x)^2 = (d-z)^2 + y^2, \quad (b+x)^2 = (e-y)^2 + z^2, \quad (c+x)^2 = (f+z)^2 + y^2,$$

soit encore

$$\begin{aligned} a^2 + 2ax + x^2 &= d^2 - 2dz + z^2 + y^2, \\ b^2 + 2bx + x^2 &= e^2 - 2ey + y^2 + z^2, \\ c^2 + 2cx + x^2 &= f^2 + 2fz + z^2 + y^2. \end{aligned}$$

Il fait remarquer qu'en soustrayant membre à membre la première égalité des deux suivantes il se débarrasse des termes inconnus au carré. Il obtient

$$b^2 - a^2 + 2(b-a)x = e^2 - d^2 - 2ey + 2dz,$$

$$c^2 - a^2 + 2(c-a)x = f^2 - d^2 + 2(d+f)z,$$

De ces deux équations il peut tirer des expressions de y et z en fonction de x:

$$\begin{aligned} z &= \frac{1}{2}(d-f) + \frac{c^2 - a^2 + 2cx - 2ax}{2(d+f)}, \\ y &= \frac{e}{2} - \frac{b^2}{2e} - \frac{bx}{e} - \frac{df}{2e} + \frac{c^2 d - a^2 d + 2cdx - 2adx}{2e(d+f)}. \end{aligned}$$

En substituant ces valeurs dans une des trois premières égalités il obtient une équation du second degré en x qui fournit la solution. Descartes ne développe pas plus les calculs. Compte tenu de la forme de l'équation il en conclut que le problème est *plan* et que le cercle tangent peut être construit à la règle et au compas. En 2003 Henk Bos [2, p. 205] nous délivre, avec l'aide du logiciel *Mathematica*, l'équation du second degré complète. Elle ne comprend pas moins de 87 termes dont chacun est de degré 6 en données et inconnues (par exemple $-2b^2e^2d^2$, $4a^2cdfx$ ou $-8abdfx^2$).

Elisabeth de Bohême lui enverra en retour l'état de ses travaux. Si ce texte a été perdu nous conservons la lettre d'accompagnement [1, lettre CCCXXVII, pp. 44-45] où la princesse exprime sa reconnaissance. Descartes lui répondra quelques jours plus tard [1, lettre CCCXXVIII, pp. 45-50] pour commenter cette solution et lui suggérer une autre configuration plus simple, celle où les trois cercles donnés sont tangents deux à deux.

2.2 La lettre CCCXXVIII

Dans cette lettre Descartes commente la solution de la princesse. Henk Bos [2, pp. 206-207] a tenté de reconstruire cette dernière à partir de ces commentaires.

La princesse n'utilise qu'une inconnue, x, le rayon du cercle cherché. Elle s'appuie sur les points E, F et G proposés par Descartes mais change les notations. Les rayons des cercles de centres A, B et C deviennent d, e, f et les côtés du triangle $\triangle ABC$ sont notés, $a = AB$, $b = BC$ et $c = AC$. Elisabeth s'appuyait ensuite sur les deux triangles $\triangle ABC$ et $\triangle ADC$.

Dans le premier, la loi du cosinus appliquée au côté BC s'écrit

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos \angle CAB,$$

mais $AB \cos \angle CAB = AE$. Cela permet d'exprimer $AE = \dfrac{a^2 - b^2 + c^2}{2c}$, puis EC

$$EC = AC - AE = c - \frac{a^2 - b^2 + c^2}{2c} = \frac{-a^2 + b^2 + c^2}{2c}.$$

BE peut se calculer à partir de la formule de Héron donnant la surface S du triangle en fonction de ses côtés

$$16S^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c).$$

Comme S est aussi égale au demi-produit de la base par la hauteur, en élevant au carré

$$S^2 = \frac{1}{4} AC^2 \cdot BE^2.$$

De ces deux égalités nous obtenons

$$BE^2 = \frac{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}{4c^2}..$$

De la même façon dans le deuxième triangle nous pouvons exprimer les segments AF, FC et DF. Il suffit dans les formules précédentes de remplacer a par $d + x$ et b par $f + x$.

$$\begin{aligned}
AF &= \frac{(d + x)^2 - (f + x)^2 + c^2}{2c} = \frac{c^2 + d^2 - f^2 + 2(d - f)x}{2c}, \\
FC &= AC - AF = \frac{c^2 - d^2 + f^2 - 2(d - f)x}{2c}, \\
DF^2 &= \frac{(c + d + f + 2x)(c - d + f)(c + d - f)(-c + d + f + 2x)}{4c^2}.
\end{aligned}$$

De ces expressions nous pouvons tirer

$$DG = EF = AE - AF \quad \text{et} \quad BG = BE - GE = BE - DF.$$

Ces dernières grandeurs s'exprimeront en fonction des grandeurs a, b, c, d, e, f, x.

Dans le triangle $\triangle BGD$ le théorème de Pythagore donne $BD^2 = DG^2 + BG^2$ avec $BD = x + e$. Nous obtenons ainsi une équation du second degré en x. Henk Bos a pu la reconstituer avec l'aide de *Mathematica*, elle possède 78 termes, neuf de moins que l'équation à laquelle serait arrivé Descartes. Il est possible que la princesse ait abouti à cette équation, mais sa complexité n'a pas pu lui permettre d'en déduire le théorème simple qu'elle imaginait pouvoir découvrir dans la relation d'un cercle tangent à trois autres.

Après ces commentaires, Descartes suggère une simplification du problème en considérant que les trois cercles donnés sont tangents deux à deux. Descartes ayant changé les notations, nous conserverons, pour simplifier, celles de la lettre CCCXXVIII avec pour inconnue le seul rayon x du cercle solution.

La tangence des cercles donnés se traduit par

$$a = d + e, \quad b = e + f, \quad c = d + f.$$

Il lui donne aussi le résultat intermédiaire ci-dessous au cas où la princesse tenterait de résoudre ce nouveau problème. Ces résultats intermédiaires ne sont autres que les valeurs de EC et FC que Henk Bos avait reconstituées et dans lesquelles les valeurs de a, b et c sont remplacées par les sommes de rayons ci-dessus. Soit

$$EC = \frac{f^2 + df - ed + ef}{d + f}, \quad FC = \frac{f^2 + df - dx + fx}{d + f}.$$

Descartes aboutit finalement à la formule symétrique suivante

$$\begin{aligned} &d^2e^2f^2 + d^2e^2x^2 + d^2f^2x^2 + e^2f^2x^2 \\ &= 2(def^2x^2 + de^2f^2x + de^2fx^2 + d^2ef^2x + d^2efx^2 + d^2e^2fx) \end{aligned}$$

qu'il énonce sous forme de théorème, qui portera plus tard le nom de théorème de Descartes sur les cercles

> Les quatre sommes qui se produisent en multipliant ensemble les carrés de trois de ces rayons, sont le double de six, qui se produisent en multipliant deux de ces rayons l'un par l'autre, et par les carrés des deux autres.

Descartes ne décrit pas comment il aboutit à cette formule.

2.3 La forme habituelle de la formule de Descartes

Il semble très étonnant que Descartes n'ait pas eu l'idée de simplifier la relation trouvée en utilisant les courbures. Il aurait alors pu donner un énoncé plus simple à la manière de Frédéric Soddy.

Voyons comment retrouver cette forme habituelle à partir de la relation énoncée par Descartes. La courbure d'un cercle est définie comme l'inverse du rayon. Si nous notons d', e', f' et x' les courbures des cercles de rayons d, e, f et x, la formule donnée par Descartes devient

$$\begin{aligned} &\frac{1}{d'^2}\frac{1}{e'^2}\frac{1}{f'^2} + \frac{1}{d'^2}\frac{1}{e'^2}\frac{1}{x'^2} + \frac{1}{d'^2}\frac{1}{f'^2}\frac{1}{x'^2} + \frac{1}{e'^2}\frac{1}{f'^2}\frac{1}{x'^2} \\ &= 2\left(\frac{1}{d'}\frac{1}{e'}\frac{1}{f'^2}\frac{1}{x'^2} + \frac{1}{d'}\frac{1}{e'^2}\frac{1}{f'^2}\frac{1}{x'} + \frac{1}{d'}\frac{1}{e'^2}\frac{1}{f'}\frac{1}{x'^2} + \frac{1}{d'^2}\frac{1}{e'}\frac{1}{f'^2}\frac{1}{x'} + \frac{1}{d'^2}\frac{1}{e'}\frac{1}{f'}\frac{1}{x'^2} + \frac{1}{d'^2}\frac{1}{e'^2}\frac{1}{f'}\frac{1}{x'}\right). \end{aligned}$$

Multiplions les deux membres par $d'^2e'^2f'^2x'^2$ et simplifions, l'égalité devient

$$(d' + e' + f' + x')^2 = 2(d'^2 + e'^2 + f'^2 + x'^2).$$

C'est sous cette dernière forme que la formule de Descartes, reliant les courbures de quatre cercles deux à deux tangents extérieurement est généralement présentée. Le théorème s'énonce alors plus simplement

> Le carré de la somme des courbures de quatre cercles tangents deux à deux est égal au double de la somme des carrés de leurs courbures.

Si d, e et f sont les rayons de trois cercles donnés tangents deux à deux, la relation de Descartes, du second degré en x nous donnera le rayon x du cercle tangent extérieurement aux trois cercles tangents donnés. La construction du cercle peut ensuite se faire facilement. Le centre du cercle cherché sera un des deux points d'intersection de deux cercles dérivés de deux des cercles donnés en conservant leurs centres et en augmentant leur rayon de la valeur donnée par la relation de Descartes pour le cercle cherché. Son rayon sera évidemment cette même valeur.

3 Cas particulier de trois cercles de même rayon

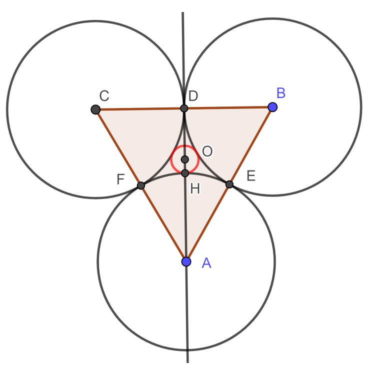


Figure 2: Cas particulier : trois cercles donnés de même rayon.

Vérifions la formule sur un cas particulier, le cas où les trois cercles donnés sont de même rayon r, inverse de la courbure commune r'. La courbure x' du cercle tangent cherché doit vérifier

$$(x' + 3r')^2 = 2(x'^2 + 3r'^2).$$

Équation du second degré dont les racines sont $x'_{1,2} = (3 \pm 2\sqrt{3})r'$. L'unique racine positive nous donne le rayon cherché. Il vaut $x' = (3 + 2\sqrt{3})r'$. Les rayons seront dans le rapport inverse des courbures, soit

$$x = \frac{r}{3 + 2\sqrt{3}}.$$

Le cercle solution peut alors être tracé comme décrit ci-dessus.

Confirmons le par une approche différente. Les centres A, B et C des trois cercles donnés forment un triangle équilatéral de côté $2r$. Dans un tel triangle les médiatrices des côtés sont simultanément hauteurs, bissectrices et médianes. Leur intersection O, centre du cercle circonscrit, est donc aussi le centre de gravité, situé au tiers de la médiane à partir de la base. Le rayon OA du cercle circonscrit vaut donc les deux tiers de la hauteur soit

$$OA = \frac{2}{3} \cdot \frac{2r\sqrt{3}}{2} = \frac{2}{\sqrt{3}}r.$$

Soit H le point d'intersection de la droite OA avec le cercle de centre A, H étant le point d'intersection le plus proche de O. Le cercle de centre O et de rayon OH sera tangent au cercle de centre A et pour des raisons de symétrie tangent aussi aux deux autres cercles. Son rayon vaut

$$OH = OA - HA = \frac{2}{\sqrt{3}}r - r = \frac{2-\sqrt{3}}{\sqrt{3}}r.$$

Multiplions dénominateur et numérateur par $2+\sqrt{3}$, on obtient $OH = \dfrac{1}{3+2\sqrt{3}}r$, qui est bien la valeur donnée par la formule de Descartes.

4 Reconstitution du calcul de Descartes

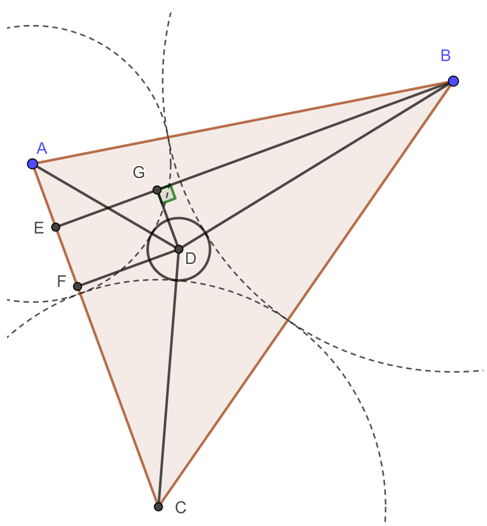


Figure 3: Configuration de Descartes.

Reprenons les notations de la reconstitution par Henk Bos de l'approche de la princesse. Les cercles deux à deux tangents centrés respectivement sur A, B, C, D ont pour rayons respectifs d, e, f et x.

Nous avons vu que BE avait été évalué en utilisant le théorème de Héron.

$$BE^2 = \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4c^2}$$

où $AB = a$, $BC = b$ et $AC = c$. Avec nos nouvelles hypothèses liées aux cercles tangents

$$a = d + e, \quad b = e + f, \quad c = d + f.$$

BE^2 prend alors la forme suivante

$$BE^2 = \frac{4def(d+e+f)}{(d+f)^2},$$

DF^2 se déduit de l'égalité précédente en remplaçant e par x.

$$DF^2 = \frac{4dfx(d+f+x)}{(d+f)^2}.$$

Cela ouvre la voie aux deux relations suivantes rappelées par Descartes avant qu'il ne donne sa célèbre formule.

$$EC = \frac{f^2 + df - ed + ef}{d+f} \quad \text{et} \quad FC = \frac{f^2 + df - dx + fx}{d+f}.$$

D'où

$$EF = EC - FC = \frac{(f-d)(e-x)}{d+f}$$

Soit G le pied de la perpendiculaire abaissée de D sur BE. Appliquons le théorème de Pythagore dans le triangle rectangle $\triangle BDG$ avec $BD = e + x$:

$$\begin{aligned} BG &= BE - GE = BE - DF = \frac{2\sqrt{def(d+e+f)} - 2\sqrt{dfx(d+f+x)}}{d+f}, \\ DG &= EF = \frac{(f-d)(e-x)}{d+f}. \end{aligned}$$

Qui nous conduit à la relation

$$(e+x)^2(d+f)^2 = (f-d)^2(e-x)^2 + \left(2\sqrt{def(d+e+f)} - 2\sqrt{dfx(d+f+x)}\right)^2.$$

Après quelques simplifications, nous obtenons

$$\begin{aligned} &d^2e^2f^2 + d^2e^2x^2 + d^2f^2x^2 + e^2f^2x^2 \\ &= 2(def^2x^2 + de^2f^2x + de^2fx^2 + d^2ef^2x + d^2efx^2 + d^2e^2fx) \end{aligned}$$

la relation que Descartes révèle dans sa deuxième lettre à la princesse Elisabeth de Bohême.

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PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 15, 2026**.

5121. *Proposed by Michel Bataille.*

Let ABC be a triangle with no right angle and let the circle Γ centered at the midpoint M of BC with radius MB intersect again AC at E and AB at F. If the circumcircles of ΔAME and ΔAMF intersect Γ again at P and Q, respectively, prove that $BPCQ$ is a rectangle.

5122. *Proposed by Nguyen Viet Hung.*

Let a, b, c be pairwise distinct real numbers such that

$$\frac{ab}{(a-b)^2} + \frac{bc}{(b-c)^2} + \frac{ca}{(c-a)^2} = -\frac{1}{4}.$$

Evaluate

$$\frac{a}{a-b} + \frac{b}{b-c} + \frac{c}{c-a}.$$

5123. *Proposed by Andrei Marinca.*

Let $ABCD$ be a convex quadrilateral in which we denote by O_1, O_2, O_3, O_4 the circumcenters of triangles ABD, ABC, BCD and CDA, respectively. Moreover, let R_1, R_2, R_3 and R_4 be the lengths of the circumradii of these respective triangles. Prove that the following inequality holds with square brackets denoting the area:

$$\frac{[BO_1DO_3]}{R_1R_3} + \frac{[AO_2CO_4]}{R_2R_4} \leq 4.$$

5124. *Proposed by Ziji Hu.*

Consider N points in the Euclidean plane, no three collinear, with the property that for any triple P, Q, R of the points, the measure in degrees of $\angle PQR$ is an integer.

(a) What is the maximum size of N?

(b) What is the maximum size of N if, in addition, all the given points have rational coordinates; that is, all points have rational x and y coordinates in the Cartesian plane?

5125. *Proposed by Tatsunori Irie, modified by the Editorial Board.*

Given a positive integer n such that $n+1$ is not a perfect power, show that the value of $\dfrac{\mathrm{rad}(n)}{\mathrm{rad}(n+1)}$ can be arbitrarily large, where $rad(n)$ is the product of distinct prime divisors of n.

5126. *Proposed by Phuc Nguyen.*

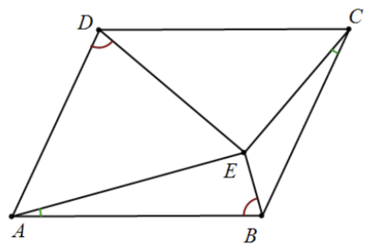
Let a, b, c, d be positive real numbers with $abcd = 1$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 2\sqrt{a^2 + b^2 + c^2 + d^2}.$$

5127. *Proposed by Xicheng Peng.*

With square brackets denoting area, prove that for any point E inside the parallelogram $ABCD$,

$$[AED](\cot\angle EBA - \cot\angle ADE) = [BCE](\cot\angle BAE - \cot\angle ECB).$$



5128. *Proposed by Vasile Cîrtoaje.*

Let a, b, c be positive real numbers such that $a \geq b \geq c$ and $ab^2c^3 \geq 1$. Prove that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \geq \frac{3}{\sqrt[3]{abc}+1}.$$

5129. *Proposed by José Rodolfo Das López.*

Given $n \in \mathbb{N}$, we consider the product of factorials

$$(4n)! \cdot (4n-1)! \cdot (4n-2)! \dots 4! \cdot 3! \cdot 2! \cdot 1!$$

Prove that for each n we can remove one factorial, $k!$, with $1 \leq k \leq 4n$, such that the product of the remaining factorials is a perfect square.

5130. *Proposed by Mihaela Berindeanu.*

Let a, b, c be distinct complex numbers with the property $|a| = |b| = |c| = 1$. Show that

$$|2a - b - c| \geq \frac{|a-b|^2 + |a-c|^2}{2}.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2026**.

5121. *Soumis par Michel Bataille.*

Soit ABC un triangle sans angle droit et soit le cercle Γ, de centre M (milieu de BC) et de rayon MB. Ce cercle coupe à nouveau AC en E et AB en F. Si les cercles circonscrits aux triangles AME et AMF recoupent Γ en P et Q, respectivement, montrez que $BPCQ$ est un rectangle.

5122. *Soumis par Nguyen Viet Hung.*

Soient a, b et c des nombres réels deux à deux distincts tels que

$$\frac{ab}{(a-b)^2} + \frac{bc}{(b-c)^2} + \frac{ca}{(c-a)^2} = -\frac{1}{4}.$$

Évaluez

$$\frac{a}{a-b} + \frac{b}{b-c} + \frac{c}{c-a}.$$

5123. *Soumis par Andrei Marinca.*

Soit $ABCD$ un quadrilatère convexe. On note O_1, O_2, O_3 et O_4 les centres des cercles circonscrits aux triangles ABD, ABC, BCD et CDA, respectivement. De plus, soient R_1, R_2, R_3 et R_4 les rayons des cercles circonscrits de ces triangles correspondants. Montrez que l'inégalité suivante est vérifiée :

$$\frac{[BO_1DO_3]}{R_1R_3} + \frac{[AO_2CO_4]}{R_2R_4} \leq 4.$$

5124. *Soumis par Ziji Hu.*

Considérons N points du plan euclidien dont aucun triplet n'est colinéaire et ayant la propriété suivante : pour tout triplet P, Q, R de ces points, la mesure en degrés de l'angle $\angle PQR$ est un entier.

(a) Quelle est la valeur maximale de N ?

(b) Quelle est la valeur maximale de N si, de plus, tous les points donnés ont des coordonnées rationnelles, c'est-à-dire que leurs coordonnées x et y dans le plan cartésien sont rationnelles.

5125. *Soumis par Tatsunori Irie, modifié par le comité de rédaction.*

Montrez que, pour les entiers positifs n tels que $n+1$ ne soit pas une puissance parfaite, la valeur de $\dfrac{\mathrm{rad}(n)}{\mathrm{rad}(n+1)}$ peut être arbitrairement grande, où $\mathrm{rad}(n)$ désigne le produit des diviseurs premiers distincts de n.

5126. *Soumis par Phuc Nguyen.*

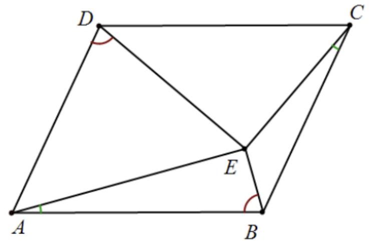
Soient a, b, c et d des nombres réels positifs tels que $abcd = 1$. Montrez que

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 2\sqrt{a^2 + b^2 + c^2 + d^2}.$$

5127. *Soumis par Xicheng Peng.*

En notant par des crochets l'aire d'une figure, montrez que, pour tout point E situé à l'intérieur du parallélogramme $ABCD$,

$$[AED](\cot\angle EBA - \cot\angle ADE) = [BCE](\cot\angle BAE - \cot\angle ECB).$$



5128. *Soumis par Vasile Cîrtoaje.*

Soient a, b et c des nombres réels positifs tels que $a \geq b \geq c$ et $ab^2c^3 \geq 1$. Montrez que

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \geq \frac{3}{\sqrt[3]{abc}+1}.$$

5129. *Soumis par José Rodolfo Das López.*

Soit $n \in \mathbb{N}$. On considère le produit de factorielles

$$(4n)! \cdot (4n-1)! \cdot (4n-2)! \dots 4! \cdot 3! \cdot 2! \cdot 1!$$

Montrez que, pour chaque n, on peut retirer une factorielle $k!$, avec $1 \leq k \leq 4n$, de sorte que le produit des factorielles restantes soit un carré parfait.

5130. *Soumis par Mihaela Berindeanu.*

Soient a, b et c des nombres complexes distincts tels que $|a| = |b| = |c| = 1$. Montrez que

$$|2a-b-c| \geq \frac{|a-b|^2 + |a-c|^2}{2}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2025: 51(4), p. 184–188 and 51(8), p. 383–386.

5032. *Proposed by Michael Friday, modified by the Editorial Board.*

Let ABC be a triangle with side-lengths a, b, c.

(a) Prove that the associated Euler line is parallel to side BC if and only if

$$(b^2 - c^2)^2 = 2a^4 - a^2(b^2 + c^2).$$

(b) Show that there are non-isosceles triangles satisfying this condition.

(c)* Are there triangles with integer sides satisfying this condition?

The solution to parts a) and b) of the problem appeared in November 2025 issue, p. 434. For part c) here, we feature a sketch of a solution based on the submission by Javier Badesa, modified by the editor.

We show that all such triangles are equilateral. Clearly all equilateral triangles satisfy the condition. Suppose otherwise that there is such a triangle with $b \neq c$. Then we have

$$(b^2 - c^2)^2 = a^2(2a^2 - b^2 - c^2).$$

Let $S = b^2 + c^2$ and $D = b^2 - c^2$. Then

$$D^2 = a^2(2a^2 - S)$$

and thus $a \mid D$. Write $D = at$. Then we have

$$a^2t^2 = 2a^2(a^2 - S) \implies S = 2a^2 - t^2.$$

It follows that

$$\begin{aligned}(2bc)^2 &= (b^2 + c^2)^2 - (b^2 - c^2)^2 \\ &= S^2 - D^2 \\ &= (2a^2 - t^2)^2 - (at)^2 \\ &= (2a^2 - at - t^2)(2a^2 + at - t^2) \\ &= (a^2 - t^2)(4a^2 - t^2)\end{aligned}$$

and thus

$$(2bc/t)^2 = \big((a/t)^2 - 1\big)\big(4(a/t)^2 - 1\big).$$

Consider the curve

$$C : y^2 = (x^2 - 1)(4x^2 - 1).$$

The above argument shows that $(a/t, 2bc/t)$ is a rational point of C. Observe that C has the following rational points:

$$(0, \pm 1), (\pm 1, 0), (\pm 1/2, 0).$$

To derive a contradiction, it suffices to show that C has no other rational points. This requires tools from the theory of elliptic curves. Here we sketch the key steps:

1. Convert C to a Weierstrass model of an elliptic curve E using a birational map. For example,

$$E: \quad V^2 = (U-5)(U^2-16) = U^3 - 5U^2 - 16U + 80$$

   is a Weierstrass model of an elliptic curve over $\mathbb{Q}$.

2. Show that E has rank 0 and has torsion $\mathbb{Z}/2 \times \mathbb{Z}/4$.

3. Conclude that C has no other rational points.

5071. *Proposed by Michel Bataille.*

Let triangle ABC be inscribed in a circle Γ with center O and let its incircle γ, with center I, touch BC at D. Let M be the midpoint of the arc BC of Γ containing A. If the line MD intersects γ at $U \neq D$ and Γ at $V \neq M$, prove that IU and OV are parallel.

We received 15 solutions, of which 13 were correct and complete. The remaining two involved the use of computer software that was inappropriate for this straightforward problem. The first solution below represents that of 9 solvers. The second, due to the proposer, seemed particularly interesting.

Solution 1.

Since triangle OVM is isosceles, $\angle OVM = \angle OMV$. Since triangle IUD is isosceles, $\angle IUD = \angle IDU$. MO and ID are both perpendicular to BC so that $MO \| ID$.

Therefore

$$\angle OVM = \angle OMV = \angle IDU = \angle IUD$$

and $OV \| IU$.

Solution 2, by the proposer.

Let $\mathbf{I}$ denote the inversion in the circle with centre M and radius $MB = MC$. Since $\mathbf{I}$ fixes B and C, Γ is the inverse of the line BC. It follows that $\gamma' = \mathbf{I}(\gamma)$ is a circle tangent to Γ at $V = \mathbf{I}(D)$. We know that γ' is also the image of γ under a homothety $\mathbf{H}$ with centre M. Since $V = \mathbf{I}(D)$, $V = \mathbf{H}(U)$. Now $\mathbf{H}(I) = J$, where J is the centre of γ' so that $IU \| JV$. Since γ' and Γ are tangent at V, then J, V, O are collinear and $IU \| OV$.

5072. *Proposed by Seán M. Stewart.*

Consider the sequence of polynomials $\{P_n(x)\}_{n\geqslant 0}$ defined by the exponential generating function

$$\frac{1}{(1-x)e^t+x}=\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{n!}.$$

Show that

$$\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k(x)P_{n-k}(x)\,dx=(-1)^n.$$

We received 13 solutions, of which 11 were correct and complete. We present a solution by multiple independent solvers.

We know that

$$P_n(x)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\frac{1}{(1-x)e^t+x}\bigg|_{t=0}.$$

From the general Leibniz rule, for two n-times differentiable functions g and h,

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(g(t)h(t)\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[g(t)\right]\frac{\mathrm{d}^{n-k}}{\mathrm{d}t^{n-k}}\left[h(t)\right].$$

Thus,

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k(x)P_{n-k}(x)\mathrm{d}x\\
=&\int_0^1\left[\sum_{k=0}^{n}\binom{n}{k}P_k(x)P_{n-k}(x)\right]\mathrm{d}x\\
=&\int_0^1\left[\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{1}{(1-x)e^t+x}\bigg|_{t=0}\cdot\frac{\mathrm{d}^{n-k}}{\mathrm{d}t^{n-k}}\frac{1}{(1-x)e^t+x}\bigg|_{t=0}\right]\mathrm{d}x\\
=&\int_0^1\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{1}{(1-x)e^t+x}\right)^2\bigg|_{t=0}\mathrm{d}x\\
=&\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\int_0^1\frac{1}{\left((1-x)e^t+x\right)^2}\mathrm{d}x\right]\bigg|_{t=0}\\
=&\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[-\frac{1}{(1-e^t)\left((1-x)e^t+x\right)}\bigg|_0^1\right]\bigg|_{t=0}\\
=&\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[e^{-t}\right]\bigg|_{t=0}\\
=&(-1)^n.
\end{aligned}$$

5073. *Proposed by Yun Zhang.*

Let A_1, A_2, A_3, A_4 be the vertices of a tetrahedron with volume V, and let P be an arbitrary point in its interior. For each $k = 1, 2, 3, 4$, let M_k denote the midpoint of the segment A_kP. For each k, construct a plane that passes through M_k that is parallel to the face opposite A_k and intersects three edges of the tetrahedron. This divides the original tetrahedron into four internal tetrahedra of volumes V_k. Show that

$$V_1 + V_2 + V_3 + V_4 \geq \frac{27}{128}V,$$

with equality if and only if the point P is the centroid of the tetrahedron.

All 9 submissions were correct; we feature the solution by Theo Koupelis.

Let A_1 be the apex of the tetrahedron and let A_1' be the intersection point of the line A_1P with the base triangle $A_2A_3A_4$. Let $V_{\scriptscriptstyle P}$ and $V_{\scriptscriptstyle M_1}$ be the volumes of the tetrahedra of apex A_1 whose bases are the triangles formed by the planes parallel to the base triangle $A_2A_3A_4$ that pass through P and M_1, respectively. When the barycentric coordinates of P with respect to points A_1, A_2, A_3, A_4 are x, y, z, t such that $x + y + z + t = 1$, we get

$$\frac{A_1P}{A_1A_1'} = \frac{y+z+t}{x+y+z+t} = y + z + t = 1 - x.$$

Thus,

$$\frac{V_1}{V} := \frac{V_{\scriptscriptstyle M_1}}{V} = \frac{1}{8} \cdot \frac{V_{\scriptscriptstyle P}}{V} = \frac{1}{8} \cdot \left(\frac{A_1P}{A_1A_1'}\right)^3 = \frac{(1-x)^3}{8},$$

with similar expressions for $V_2/V, V_3/V$, and V_4/V. Therefore, it suffices to show that

$$(1-x)^3 + (1-y)^3 + (1-z)^3 + (1-t)^3 \geq \frac{27}{16}.$$

This is an immediate consequence of the power mean inequality which says that

$$\left(\frac{(1-x)^3 + (1-y)^3 + (1-z)^3 + (1-t)^3}{4}\right)^{1/3} \geq \frac{(1-x) + (1-y) + (1-z) + (1-t)}{4} = \frac{3}{4}.$$

Equality occurs if and only if $x = y = z = t = 1/4$; that is, if and only if P is the centroid of the tetrahedron.

Editor's comment. Walther Janous observed that by using the same argument, we obtain the analogous result for simplices in all dimensions, namely,

$$V_1 + V_2 + \cdots + V_{n+1} \geq (n+1)\left(\frac{n}{2(n+1)}\right)^n V.$$

In particular, for triangles (that is, when $n = 2$ with areas F in place of V) we get

$$F_1 + F_2 + F_3 \geq \frac{1}{3}F.$$

5074. *Proposed by Vasile Cîrtoaje.*

Let a, b, c, d be nonnegative real numbers such that at most one of them is larger than 1 and $ab + bc + cd + da = 4$. Prove that

$$\frac{1}{ab+2} + \frac{1}{ac+2} + \frac{1}{ad+2} + \frac{1}{bc+2} + \frac{1}{bd+2} + \frac{1}{cd+2} \geq 2.$$

We received 6 submissions, of which 3 were correct and complete. We present 2 solutions.

Solution 1, by Michal Adamaszek.

We have that $(a+c)(b+d) = 4$. Since at least one of the variables has to be at least 1 to satisfy the condition $(a+c)(b+d) = 4$, and at most one variable is greater than one, we may assume without loss of generality that $b, c, d \leq 1 \leq a$.

Let $a + c = p$, and so $b + d = \frac{4}{p}$. Since $(a-1)(c-1) \leq 0$, then

$$0 \leq ac \leq a + c - 1 = p - 1.$$

We know that for any positive $x, y \in \mathbb{R}$, $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{x+y}$. We derive the following inequalities:

$$\frac{1}{ab+2} + \frac{1}{ad+2} \geq \frac{4}{a(b+d)+4} = \frac{4}{\frac{4a}{p}+4} = \frac{p}{a+p}, \tag{1}$$

$$\frac{1}{bc+2} + \frac{1}{cd+2} \geq \frac{4}{c(b+d)+4} = \frac{4}{\frac{4c}{p}+4} = \frac{p}{c+p}. \tag{2}$$

Since

$$(b-d)^2 = b^2 + d^2 - 2bd \geq 0,$$

then

$$b^2 + d^2 + 2bd \geq 4bd,$$

we have that

$$bd \leq \frac{(b+d)^2}{4} = \frac{4}{p^2}.$$

Now, we derive the following third inequality:

$$\frac{1}{bd+2} \geq \frac{1}{\frac{4}{p^2}+2} = \frac{p^2}{4+2p^2}. \tag{3}$$

We combine the three inequalities (1)-(3) and add the remaining term $\frac{1}{ac+2}$, and conclude that we have the following bound on the six-term sum we want to estimate:

$$\begin{aligned}\frac{p}{a+p}+\frac{p}{c+p}+\frac{1}{ac+2}+\frac{p^2}{4+2p^2} &= \frac{p(a+c)+2p^2}{ac+(a+c)p+p^2}+\frac{1}{ac+2}+\frac{p^2}{4+2p^2}\\ &= \frac{3p^2}{ac+2p^2}+\frac{1}{ac+2}+\frac{p^2}{4+2p^2}\\ &\geq \frac{3p^2}{2p^2+p-1}+\frac{1}{p+1}+\frac{p^2}{4+2p^2}\\ &= 2+\frac{(p-2)^2}{(2p-1)(4+2p^2)}.\end{aligned}$$

The last term is nonnegative for $p = a + c \geq 1$, which is always true by our assumptions. This completes our proof.

Solution2, by Paul Bracken.

Recall the inequality,

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n} \geq \frac{n^2}{x_1+\ldots+x_n}.$$

Equality holds when $x_1 = \ldots = x_n$. We set $x_1 = ab+2$, $x_2 = ac+2$, $x_3 = ad+2$, $x_4 = bc+2$, $x_5 = bd+2$ and $x_6 = cd+2$. We get the following inequality:

$$\frac{1}{ab+2}+\frac{1}{ac+2}+\frac{1}{ad+2}+\frac{1}{bc+2}+\frac{1}{bd+2}+\frac{1}{cd+2} \geq \frac{36}{16+ac+bd},$$

where equality holds when $a = b = c = d$. We want to show that

$$\frac{36}{16+ac+bd} \geq 2,$$

which is equivalent to showing that $ac + bd \leq 2$.

Let $f(a, b, c, d) = ac + bd$ and we introduce a Lagrange multiplier λ to define the Lagrange function

$$L = ac + bd + \lambda(ab + bc + cd + da - 4).$$

We differentiate L with respect to the variables a, b, c, d and λ and obtain the following homogeneous system of equations

$$\begin{aligned} L_a &= c + \lambda(b + d) = 0\\ L_b &= d + \lambda(a + c) = 0\\ L_c &= a + \lambda(b + d) = 0\\ L_d &= b + \lambda(c + a) = 0\\ L_\lambda &= ab + bc + cd + da - 4 = 0.\end{aligned}$$

We solve this system and find that there exists one critical point that satisfies our conditions on the variables, and that is for $a = b = c = d = 1$, $\lambda = -\frac{1}{2}$. This critical point gives us $f(1,1,1,1) = 2$, and thus, is a maximum. Therefore, $ac+bd \leq 2$ holds for all nonnegative real values of a, b, c, and d, and this completes our proof.

5075. *Proposed by Matt Olechnowicz.*

Let m and k be positive integers. Show that

$$\binom{(m+1)k}{m} \leq (m+1)\binom{mk}{m}.$$

We received 13 solutions, 12 of which were correct. We present 2 solutions.

Solution 1, byt Ulrich Abel and Vitaliy Kushnirevych.

We have

$$\begin{aligned}\binom{(m+1)k}{m}\binom{mk}{m}^{-1} &= \prod_{j=0}^{m-1} \frac{(m+1)k - j}{mk - j} \leq \prod_{j=0}^{m-1} \frac{(m+1) - j}{m - j} \\ &= \binom{m+1}{m}\binom{m}{m}^{-1} = m+1,\end{aligned}$$

since, for each fixed $j = 0, \ldots, m-1$,

$$\frac{(m+1)x - j}{mx - j} = 1 + \frac{x}{mx - j} = 1 + \frac{1}{m - j/x}$$

is non-increasing for $x \in [1, \infty)$.

Solution 2, by Michał Adamaszek.

Consider an office building with $m+1$ floors and k offices on each floor. A company wants to refurbish one floor into an open space for joint work on recreational math problems, and also to choose m offices for its employees. How many ways are there to achieve this?

First we choose a floor for the open space ($m+1$ ways) and then choose m offices from the remaining mk options. It shows that the number of ways is exactly $(m+1)\binom{mk}{m}$.

Alternatively we can first choose m offices from the whole building in $\binom{(m+1)k}{m}$ ways. Since there are $m+1$ floors, there is at least one floor without a chosen office, and we pick any of them for the refurbishment (for example the highest). It shows that the number of ways is at least $\binom{(m+1)k}{m}$.

That proves the inequality.

5076. *Proposed by Michael Friday.*

Let O and H be the circumcenter and orthocenter of triangle ABC satisfying $\angle B - \angle A = 90^\circ$. Prove that the circumcenters of triangles BCO, CAO, BCH, CAH, and vertex C, all lie on the same circle.

There were 8 correct solutions submitted by 7 solvers. Two additional solutions involved brute force applied by computer software. We present 3 solutions.

Solution 1, by Theo Koupelis.

Before we begin, let us establish some useful properties. Suppose that D, E, F are the respective feet of the altitudes from A, B, C on their opposite sides BC, CA, AB. Let $\alpha = \angle BAC$, so that $\angle ABC = 90^\circ + \alpha$ and $\angle ACB = 90^\circ - 2\alpha$. Since

$$\angle BAC + \angle ABC < 180^\circ,$$

$\alpha < 45^\circ$ and $180^\circ - 2\alpha > 90^\circ$. We have that

$$\angle BCF = 90^\circ - (180^\circ - \angle ABC) = 90^\circ - (90^\circ - \alpha) = \alpha.$$

Therefore FC is tangent to the circumcircle of ABC, so that $OC \perp CF$. Hence $OC \| AB$.

Also, since the right triangles ACF and HCE are similar,

$$\angle BHC = 90^\circ - \angle ACF = \angle CAF = \alpha,$$

so that $BC = CH$ and $CF = FH$.

Note that

$$\begin{aligned}
&\angle OCA = \angle OAC = \angle FAH = \angle BHC = \alpha\\
&\angle BOC = \angle OBA = \angle OAB = 2\alpha\\
&\angle BCO = \angle CBF = \angle FBH = 90^\circ - \alpha\\
&\angle AOC = 180^\circ - 2\alpha
\end{aligned}$$

Let K be the circumcentre of triangle BCO. Then

$$\angle KCB = \angle KBC = 90^\circ - \tfrac{1}{2}\angle BKC = 90^\circ - \angle BOC = 90^\circ - 2\alpha = \angle ACB,$$

so that K lies on AC as well as the right bisectors of OC and BC.

Let L be the circumcentre of triangle CAO. Since this triangle is obtuse, L and O lie on opposite sides of AC. Then

$$\angle LCA = \angle LAC = 90^\circ - \tfrac{1}{2}\angle ALC = 90^\circ - (180^\circ - \angle AOC) = 90^\circ - 2\alpha = \angle ACB,$$

so that L lies on BC produced as well as on the right bisectors of AC and OC. Because $LK \perp OC$, we have that

$$\angle CKL = 90^\circ + \angle OCK = 90^\circ + \alpha = \angle ABC$$

and

$$\angle CLK = 90^\circ - \angle OCB = 90^\circ - (90^\circ - \alpha) = \alpha.$$

Let M be the circumcentre of the isosceles triangle BCH. M lies on AB (the right bisector of CH) and on the right bisector of BC. Thus O, K, M are collinear. (Note that $\angle CBH$ is obtuse and so F lies between B and M.) Since

$$\angle CMK = \angle BMK = \tfrac{1}{2}\angle BMC = \angle BHC = \alpha = \angle CLK,$$

the points K, L, M, C lie on a circle γ.

Let N be the circumcentre of the isosceles triangle CAH. N lies on AB and on the right bisector of AC. The points A, B, M, N are collinear as are the points O, L, N. Then

$$\angle NCK = \angle NCA = \angle NAC = \angle BAC = \alpha = \angle BMK = \angle NMK,$$

so that N must lie on γ. The result follows.

Moreover, let P be the centre of γ. Then

$$\angle KCP = \angle PKC = 90^\circ - \angle KMC = 90^\circ - \alpha = \angle ACH = \angle KCH.$$

Then P lies on CH.

Solution 2, by Corneliu Manescu-Avram.

In the Cartesian plane, assign coordinates $A(-1, 0)$, $B(1, 0)$, $C(a, b)$ with $a, b, > 0$. Let F be the point $(a, 0)$. Since

$$\angle BCF = 90^\circ - \angle CBF = \angle CBA - 90^\circ = \angle CAB,$$

triangles ACF and CBF are similar, so that

$$(a + 1) : b = AF : CF = CF : BF = b : (a - 1).$$

Therefore $a^2 = b^2 + 1$ and the circumcenter O of triangle ABC is at $(0, b)$ since $OC = OA$.

The orthocenter H is at (a, d) for some $d < 0$ since $CH \perp AB$. Since $AH \perp BC$,

$$\frac{d}{a+1} \times \frac{b}{a-1} = 1,$$

whence $d = b$.

By a straightforward computation, it can be verified that the respective circumcentres of triangle BCO, CAO, BCH and CAH are the points

$$\left(\frac{a}{2}, \ \frac{b^2 + a - 1}{2b}\right), \ \left(\frac{a}{2}, \frac{b^2 - a - 1}{2b}\right), \ (a + 1, 0), \ (a - 1, 0).$$

The circle through $(a+1,0)$, $(a-1,0)$ and $C(a,b)$ has centre

$$\left(a, \frac{b^2-1}{2b}\right)$$

and radius $(b^2+1)/2b$. It is straightforward to check that the remaining circumcentres lie on this circle.

Solution 3, by Corneliu Manescu-Avram.

We begin by showing that $\angle B - \angle A = 90^\circ$ if and only if $AB \| CO$. Let J be the diametrically opposite point to C on the circumcircle of triangle ABC. Then

$$AB\|CO \Leftrightarrow AB\|CJ \Leftrightarrow \angle A = \angle CAB = \angle ACJ = \angle BJC = \angle ABJ = \angle B - 90^\circ.$$

Let the circumcircle of triangle ABC be the unit circle in the complex plane and, wolog, choose affixes $A(a)$, $B(b)$, $C(1)$ with $a, b \in \mathbb{C}$ and $|a| = |b| = 1$. Since $AB\|CO$,

$$(a, b) = (e^{i\alpha}, e^{i(\pi-\alpha)})$$

for some real α and $ab = -1$. Thus, $b = -1/a$.

Also, using the Euler relation $HO = 3GO$ (where $G = \frac{1}{3}(a+b+1)$ is the centroid), we have H at $(a+b+1) = (a - \frac{1}{a} + 1)$. We show that the points

$$K: \left(\frac{-1}{a-1}\right) = \left(1 - \frac{a}{a-1}\right),$$
$$L: \left(\frac{a}{a+1}\right) = \left(1 - \frac{1}{a+1}\right),$$
$$M: \left(\frac{a-1}{a}\right) = \left(1 - \frac{1}{a}\right),$$

and $N : (a+1)$ are the respective circumcentres of triangles BCO, CAO, BCH, CAH. Indeed

$$[K] \qquad \left|-\frac{1}{a} + \frac{1}{a-1}\right| = \left|1 + \frac{1}{a-1}\right| = \left|0 + \frac{1}{a-1}\right| = \frac{1}{|a-1|};$$

$$[L] \qquad \left|1 - \frac{a}{a+1}\right| = \left|a - \frac{a}{a+1}\right| = \left|0 - \frac{a}{a+1}\right| = \frac{1}{|a+1|};$$

$$[M] \qquad \left|-\frac{1}{a} - \frac{a-1}{a}\right| = \left|1 - \frac{a-1}{a}\right| = \left|a - \frac{1}{a} + 1 - \frac{a-1}{a}\right| = 1;$$

$$[N] \qquad |1-a-1| = |a-a-1| = \left|a - \frac{1}{a} + 1 - a - 1\right| = 1.$$

It remains to prove that the points K, L, M, N are concyclic. Let $1+c$ be the centre of the circle with radius R through C, N and L. Then, noting that $\bar{a} = 1/a$ and $\overline{a+1} = (a+1)/a$,

$$R^2 = c\bar{c} = (c-a)\left(\bar{c} - \frac{1}{a}\right) = \left(c + \frac{1}{1+a}\right)\left(\bar{c} + \frac{a}{1+a}\right).$$

Therefore

$$\bar{c}a + \frac{c}{a} = 1 \qquad \text{and} \qquad \bar{c} + ca + \frac{a}{1+a} = 0.$$

Eliminating $\bar{c}$ leads to

$$c\left(\frac{a^3-1}{a^2}\right) = -\left(\frac{1}{a} + \frac{a}{1+a}\right) = -\frac{a^2+a+1}{a(1+a)}.$$

Thus $c = -a(a^2-1)^{-1}$ and $R = |a^2-1|^{-1}$. It is straightforward to check that K and M also lie on the circle with centre $1+c = (a^2-a-1)(a^2-1)^{-1}$ and radius $|a^2-1|^{-1}$.

We note in passing that c is real. This can be seen directly from the fact that $\bar{a} = 1/a$ or from the equation

$$(c-a)(\bar{c} - \tfrac{1}{a}) = (c + \tfrac{1}{a})(\bar{c} + a)$$

derived from using the points L and N to find c.

Editor's Comments. Triangles with one angle exceeding another by 90° have a number of interesting properties. The proposer recently highlighted one such property in Problem 5069 (statement in 51:7, p. 344, 346, solution in 52:1, p. 107). He pointed out other appearances on the pages of *Crux*. Problem 5003 (51:1, p. 32, 34 and 51:6, p. 290-291) establishes a similar property, to wit, that if D, E, F are the respective pedal points of the altitudes from A, B, C, then the circle with centre F and radius FH passes through C, D, E and the foot of the bisector from angle C.

J. Chris Fisher, in an article *Triangles whose angles satisfy* $B = 90^\circ + C$, appearing in *Crux* 38:10, lists a number of properties and discusses problems 2525, 2765 and 2867.

Giuseppe Fera solved the problem using barycentric coordinates based on the triangle ABC with side lengths a, b, c. Assume only that the triangle is not right-angled. The general equation of a circle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)(fx+gy+hz) = 0.$$

The circumcentre of ABC is at

$$[a^2(b^2+c^2-a^2) : b^2(a^2+c^2-b^2) : c^2(a^2+b^2-c^2)]$$

and the orthocentre is at

$$[(a^2+b^2-c^2)(a^2+c^2-b^2) : (a^2+b^2-c^2)(b^2+c^2-a^2) : (a^2+c^2-b^2)(b^2+c^2-a^2)].$$

The values of the parameters for the various circumcircles are

$$
\begin{array}{ll}
ABC: & f = g = h = 0; \\
BCO: & f = \dfrac{b^2c^2}{b^2+c^2-a^2}, \quad g = h = 0; \\
CAO: & g = \dfrac{c^2a^2}{a^2+c^2-b^2}, \quad f = h = 0; \\
BCH: & f = b^2+c^2-a^2, \quad g = h = 0; \\
CAH: & g = a^2+c^2-b^2, \quad f = h = 0.
\end{array}
$$

Up until now, the computations are remarkably benign. A more formidable calculation identifies the various circumcenters and the circle passing through them. This circle passes through C if and only if $c^2(a^2+b^2) = (a^2-b^2)^2$. Finally, some trigonometry demonstrates that this in turn is equivalent to $\angle B = \angle A + 90^\circ$.

5077. *Proposed by Nguyen Viet Hung.*

Given a triangle ABC with the centroid G, let M be an arbitrary point inside the triangle. The lines that pass through M parallel to the median lines intersect the sides BC, CA, AB at X, Y, Z respectively. Prove that M, E, G are collinear, where E is the centroid of triangle XYZ.

We will sample 2 of the 10 submissions, all of which were correct.

Solution 1. There were 8 solvers who used coordinates; we feature a composite of the 5 solutions that used cartesian coordinates.

We shall prove that if M is any point in the plane of ΔABC (not necessarily inside the triangle), then E will be the midpoint of GM (and, therefore, M, E, G are collinear). Because the problem is concerned with the affine properties of a triangle, we can set (without loss of generality) $A = (0,2), B = (0,0)$, and $C = (2,0)$. Then the centroid

$$G = \frac{1}{3}(A+B+C) = (\frac{2}{3}, \frac{2}{3}).$$

Let M have coordinates (p,q) for arbitrary real numbers p,q. Consequently, the midpoint of GM is

$$\frac{1}{2}\left(\frac{2}{3}+p,\ \frac{2}{3}+q\right) = \left(\frac{p}{2}+\frac{1}{3},\ \frac{q}{2}+\frac{1}{3}\right).$$

Since the slope of AG is -2, the line through M parallel to AG satisfies the equation $y = -2x + (2p+q)$ and meets the line $BC: y = 0$ at

$$X = \left(p + \frac{q}{2}, 0\right).$$

Since the slope of BG is 1, the line through M parallel to BG satisfies the equation $y = x + (q-p)$ and meets the line $CA: x + y = 2$ at

$$Y = \left(\frac{p-q+2}{2}, \frac{q-p+2}{2}\right).$$

Since the slope of CG is $-\frac{1}{2}$, the line through M parallel to CG satisfies the equation $y = -\frac{1}{2}x + (\frac{p}{2} + q)$, and it meets the line $AB : x = 0$ at

$$Z = \left(0,\ \frac{p}{2} + q\right).$$

Finally, the centroid of ΔXYZ has coordinates $\frac{1}{3}(X + Y + Z) = \left(\frac{p}{2} + \frac{1}{3},\ \frac{q}{2} + \frac{1}{3}\right)$, which is the midpoint of GM, as claimed.

Solution 2, by Chikara Tsugawa.

We prove that E is the midpoint of MG. Moreover, the argument works for any point M in the plane of ΔABC, allowing X, Y, Z to lie on the extensions of the sides. Let O be an arbitrary point *not* lying in the plane of triangle ABC. Let $\overrightarrow{OA} = \overrightarrow{a}$, $\overrightarrow{OB} = \overrightarrow{b}$, and $\overrightarrow{OC} = \overrightarrow{c}$. Then,

$$\overrightarrow{OM} = k\overrightarrow{a} + l\,\overrightarrow{b} + m\overrightarrow{c},$$

where the vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ are linearly independent and $k + l + m = 1$. Thus, the centroid G of ΔABC must satisfy

$$\overrightarrow{OG} = \frac{1}{3}\left(\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}\right).$$

Since $MX \parallel AG$, there exists a real number t such that

$$\overrightarrow{OX} = \overrightarrow{OM} + t\overrightarrow{AG} = k\overrightarrow{a} + l\,\overrightarrow{b} + m\overrightarrow{c} + t\left(\frac{1}{3}\left(\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}\right) - \overrightarrow{a}\right).$$

Since X is on BC, the coefficient of $\overrightarrow{a}$ must be 0: $k - \frac{2}{3}t = 0$; hence, $t = \frac{3}{2}k$. Therefore,

$$\overrightarrow{OX} = \left(l + \frac{k}{2}\right)\overrightarrow{b} + \left(m + \frac{k}{2}\right)\overrightarrow{c}.$$

Similarly,

$$\overrightarrow{OY} = \left(m + \frac{l}{2}\right)\overrightarrow{c} + \left(k + \frac{l}{2}\right)\overrightarrow{a},$$

$$\overrightarrow{OZ} = \left(k + \frac{m}{2}\right)\overrightarrow{a} + \left(l + \frac{m}{2}\right)\overrightarrow{b}.$$

Therefore,

$$\begin{aligned}
\overrightarrow{OE} &= \frac{1}{3}\left(\overrightarrow{OX} + \overrightarrow{OY} + \overrightarrow{OZ}\right) \\
&= \frac{1}{3}\left(\left(\frac{3k}{2} + \frac{1}{2}\right)\overrightarrow{a} + \left(\frac{3l}{2} + \frac{1}{2}\right)\overrightarrow{b} + \left(\frac{3m}{2} + \frac{1}{2}\right)\overrightarrow{c}\right) \\
&= \frac{1}{2}\left(k\overrightarrow{a} + l\,\overrightarrow{b} + m\overrightarrow{c}\right) + \frac{1}{6}\left(\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}\right) \\
&= \frac{1}{2}\overrightarrow{OM} + \frac{1}{2}\overrightarrow{OG}.
\end{aligned}$$

It follows that E is the midpoint of MG, as desired.

Editor's comments. Tran Quang Hung observed that our Problem 5077 is the planar case of his n-dimensional version that appeared three years ago as Problem 12369 in *The American Mathematical Monthly*, 130:1, p. 87, under the title, "One Simplex Begets Another." The published solution (131:9, pp. 818-9) made use of normalized barycentric coordinates, much like the solutions submitted by three of our readers.

5078. *Proposed by Tatsunori Irie.*

Let n be an integer such that $n \geq 2$ and let x be a positive integer. Show that the following holds:

$$\left(1+\frac{x}{n}\right)^n \geq x\left(\frac{1}{n-1}+1\right)^{n-1}.$$

We received 24 solutions, of which 20 were correct and complete.

We present two solutions that showcase different approaches to the problem.

Solution 1, proposed both by Kee-Wai Lau and by M. Bello, M. Benito, Ó. Ciaurri and E. Fernández, adapted by the edtior.

By the AM–GM inequality,

$$\sqrt[n]{x\left(\frac{1}{n-1}+1\right)^{n-1}} \leq \frac{1}{n}\left(x+(n-1)\left(\frac{1}{n-1}+1\right)\right) = 1+\frac{x}{n}$$

for any $x \geq 0$. The inequality of the problem follows by taking the nth power.

Solution 2, proposed by Adam J. Frederickson, adapted by the editor.

Let

$$f(x) = \left(1+\frac{x}{n}\right)^n - x\left(1+\frac{1}{n-1}\right)^{n-1}.$$

If the inequality $f(x) \geq 0$ holds for all $x > 0$, then the desired inequality holds. To find the critical points of f, we see that

$$f'(x) = \left(1+\frac{x}{n}\right)^{n-1} - \left(1+\frac{1}{n-1}\right)^{n-1}.$$

The critical points occur when $f'(x) = 0$, so we solve

$$1+\frac{x}{n} = 1+\frac{1}{n-1} \iff x = \frac{n}{n-1}.$$

f is also convex on $(0, \infty)$, since

$$f''(x) = \frac{n-1}{n}\left(1+\frac{x}{n}\right)^{n-2} > 0$$

for all $x > 0$, so all critical points are minima. If we evaluate the function at the critical point $x = n/(n-1)$, we get

$$f\left(\frac{n}{n-1}\right) = \left(1 + \frac{1}{n-1}\right)^n - \left(1 + \frac{1}{n-1}\right)\left(1 + \frac{1}{n-1}\right)^{n-1} = 0.$$

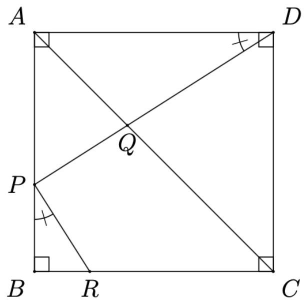
Thus, $f(x) \geq 0$ for all $x > 0$, and the inequality holds.

5079. *Proposed by Torabi Dashti.*

For a square $ABCD$, let P be a point on AB and consider a triangle PQR, where Q is the point of intersection between DP and the diagonal AC, and R is on BC such that $\angle ADP = \angle BPR$. Prove that $PQ = PR$ if and only if $AP : PB = \frac{\sqrt{5}+1}{2}$.

We received 24 solutions, all of which were correct and complete.

We present the solution by Chikara Tsugawa.



Let $PB = 1$, $AP = x$. Then, $AD = AB = x + 1$. Since $\angle ADP = \angle BPR$, the triangles $\triangle ADP$ and $\triangle BPR$ are similar. Hence,

$$PR = PD \cdot \frac{BP}{AD} = PD \cdot \frac{1}{x+1}.$$

Since the triangles $\triangle APQ$ and $\triangle CDQ$ are also similar,

$$PQ = PD \cdot \frac{AP}{AP + CD} = PD \cdot \frac{x}{2x+1}.$$

Therefore, $PR = PQ$ if and only if

$$\frac{1}{x+1} = \frac{x}{2x+1}, \iff x^2 - x - 1 = 0.$$

Since $x > 0$, it follows that

$$\frac{AP}{PB} = x = \frac{\sqrt{5}+1}{2}.$$

5080. *Proposed by Nguyen Van Huyen.*

Let a, b, c, d be positive real numbers. Prove that

$$\frac{a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2}{abcd}+10 \geq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

When does equality hold?

We received 13 submissions and they were all complete and correct. We present the following solution by the majority of solvers.

Observe that

$$a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2+6abcd=(ab+cd)^2+(ac+bd)^2+(ad+bc)^2.$$

Note that we have the general inequality $x^2+y^2+z^2 \geq xy+yz+zx$, with equality holding if and only if $x=y=z$. Thus, we have

$$(ab+cd)^2+(ac+bd)^2+(ad+bc)^2 \geq (ab+cd)(ac+bd)+(ac+bd)(ad+bc)+(ad+bc)(ab+cd),$$

with equality holding if and only if

$$ab+cd=ac+bd=ad+bc \iff (a-d)(b-c)=0=(a-b)(c-d),$$

that is, at least 3 of the numbers a,b,c,d are equal.

We conclude that

$$\begin{aligned}&\frac{a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2}{abcd}+10\\ =&\frac{a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2+6abcd}{abcd}+4\\ \geq&\frac{(ab+cd)(ac+bd)+(ac+bd)(ad+bc)+(ad+bc)(ab+cd)+4abcd}{abcd}\\ =&(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right),\end{aligned}$$

with equality holding if and only if at least 3 of the numbers a,b,c,d are equal.