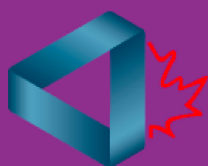


The top portion of the cover features a vibrant image of the Aurora Borealis (Northern Lights) in shades of green and blue, swirling across a dark night sky. Below the lights, the dark silhouettes of a dense forest of evergreen trees are visible against the lighter sky.

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Crux Mathematicorum with Mathematical Mayhem

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MATHEMATTIC

No. 68

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

*To facilitate their consideration, solutions should be received by **December 15, 2025**.*

MA336. *Proposed by Michael Friday.*

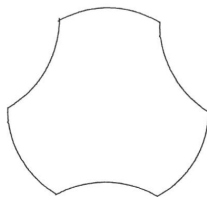
Suppose that in triangle ABC we have $\angle A - \angle B = 90^\circ$. Prove that the orthic triangle of ABC , which is the triangle formed by the feet of the altitudes of ABC , is isosceles.

MA337. *Proposed by Alaric Pow.*

Prove or disprove that there cannot be a right-angled triangle whose sides are all of prime length.

MA338.

A *hexle* is constructed from a circle by reversing three non-intersecting arcs, each of which is $1/6$ of the circumference. If the radius of the circle is 1, find the exact area of the hexle.



MA339.

If p and q are positive integers such that

$$\frac{p}{q} = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \cdots + \frac{1}{478} + \frac{1}{479} - \frac{2}{480},$$

show that p is divisible by 641.

MA340.

Suppose that m and n are positive integers. For what values of m and n can $m^4 + 4n^4$ be a prime number?

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 décembre 2025**.

MA336. *Soumis par Michael Friday.*

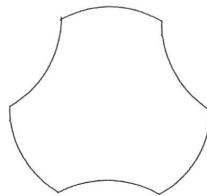
Supposons que dans le triangle ABC , nous ayons $\angle A - \angle B = 90^\circ$. Montrez que le triangle orthique, c'est-à-dire le triangle ayant pour sommets les pieds des hauteurs de ABC , est isocèle.

MA337. *Soumis par Alaric Pow.*

Prouvez ou réfutez l'affirmation voulant qu'il ne peut exister de triangle rectangle dont tous les côtés sont des nombres premiers.

MA338.

Un *hexle* est construit à partir d'un cercle en inversant trois arcs qui ne se rencontrent pas, chacun représentant $1/6$ de la circonférence. Si le rayon du cercle est égal à 1, calculez l'aire exacte de l'hexle.



MA339.

Si p et q sont des entiers positifs tels que

$$\frac{p}{q} = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \cdots + \frac{1}{478} + \frac{1}{479} - \frac{2}{480},$$

montrer que p est divisible par 641.

MA340.

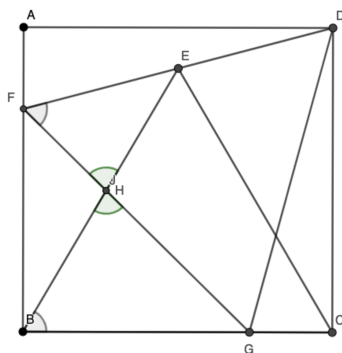
Supposons que m et n soient des nombres entiers positifs. Pour quelles valeurs de m et n le nombre $m^4 + 4n^4$ peut-il être premier ?

MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(3), p. 105–107.

MA311. *Proposed by Shafik Khalifa.*

Let $ABCD$ be a square with equilateral triangles BCE and DFG positioned inside it as shown. Prove that the ratio of the areas of EFH and HBG is 1:2.



We received 6 solutions, of which 5 were correct and complete. We present the solution by Miguel Amengual Covas.

In triangles EFH and HBG , we have

$$\angle EFH = \angle HBG, \quad \angle FHE = \angle GHB,$$

hence the two triangles are similar. Therefore,

$$\frac{\text{area } \triangle EFH}{\text{area } \triangle HBG} = \left(\frac{EF}{BG} \right)^2. \quad (1)$$

Since $\angle EFG = \angle EBG$, the quadrilateral $EFBG$ is cyclic. From this, we conclude that

$$\angle FGE = \angle FBE = 90^\circ - 60^\circ = 30^\circ.$$

Moreover, since $\angle FBG + \angle GEF = 180^\circ$, we know $\angle GEF = 90^\circ$. Then in right triangle GEF ,

$$EF = \frac{1}{2}FG. \quad (2)$$

Right-angled triangles DAF and CDG are congruent, with $AF = GC$. Thus,

$$FB = AB - AF = BC - GC = BG,$$

making $\triangle FBG$ isosceles right-angled, so that

$$BG = \frac{FG}{\sqrt{2}}. \quad (3)$$

By (2) and (3),

$$\frac{EF}{BG} = \frac{1}{\sqrt{2}}.$$

Substituting (1) yields

$$\frac{\text{area } \triangle EHF}{\text{area } \triangle HBG} = \frac{1}{2},$$

as desired.

MA312. *Proposed by Neculai Stanciu.*

Determine all pairs of integers (x, y) that satisfy $1 + x + x^2 + x^3 + x^4 = y^4$.

We received 10 submissions, of which 3 were correct and complete. The majority of solvers either missed a solution or had an incorrect justification. We present the solution by R. Achudhan and Srikanth Pai (a student-teacher collaboration).

The only solutions are $(0, 1), (0, -1), (-1, 1), (-1, -1)$. We actually prove a stronger claim, specifically that the equation

$$Y^2 = 1 + x + x^2 + x^3 + x^4$$

has integer solutions if and only if $x = 0, -1, 3$.

We prove this by showing that the left-hand side of the above equation is strictly sandwiched between two consecutive perfect squares for $|x| > 1$ unless $x = 3$. Specifically, we claim that for $|x| > 1, x \neq 3$:

$$\left(x^2 + \left\lfloor \frac{x}{2} \right\rfloor\right)^2 < 1 + x + x^2 + x^3 + x^4 < \left(x^2 + \left\lfloor \frac{x}{2} \right\rfloor + 1\right)^2.$$

To prove this, we consider two separate cases depending on the parity of x .

Case 1: x is even. We begin by estimating the right-hand side of the inequality. Observe that:

$$x^4 + \frac{x^2}{4} + 1 + x + 2x^2 + x^3 < 1 + x + x^2 + x^3 + x^4.$$

This simplifies to:

$$0 < \frac{5x^2}{4}$$

which is clearly true for all $x \neq 0$.

For the other side of the inequality, we deduce that:

$$x^4 + \frac{x^2}{4} + x^3 > x^4 + x^3 + x^2 + x + 1,$$

which gives a quadratic expression

$$\frac{3x^2}{4} + x + 1 > 0,$$

which is true for all real x since the discriminant of the quadratic is -2 .

Case 2: x is odd. We now turn to the case when x is odd and $|x| > 1$. We start with:

$$x^4 + x^3 - x^2 + \frac{(x-1)^2}{4} < 1 + x + x^2 + x^3 + x^4.$$

Rewriting and simplifying:

$$2x^2 - \frac{(x-1)^2}{4} + x + 1 > 0.$$

Expanding and collecting terms yields

$$8x^2 - (x^2 - 2x + 1) + 4x + 4 > 0 \iff 7x^2 + 6x + 3 > 0,$$

which is true since the discriminant is negative again.

Now we prove the right hand side of the inequality:

$$\begin{aligned} \left(x^2 + \frac{(x-1)}{2} + 1\right)^2 &> 1 + x + x^2 + x^3 + x^4 \\ x^4 + \frac{(x-1)^2}{4} + 1 + x^3 - x^2 + 2x^2 + (x-1) &> 1 + x + x^2 + x^3 + x^4 \end{aligned}$$

This reduces to showing $|x-1| > 2$ which is true unless $x = 3$. In this case $1 + x + x^2 + x^3 + x^4 = 11^2$ which is the only solution in this case.

Thus, in both even and odd cases, the original expression lies strictly between two consecutive squares, completing the proof.

MA313. The integers from 1 to 9 can be arranged into a 3×3 array so that the sum of the numbers in every row, column and diagonal is a multiple of 9.

A	B	C
D	E	F
G	H	I

- Prove that the number in the center of the array must be a multiple of 3.
- Give an example of such an array with 6 in the center.

Originally from the 1998–1999 USA Mathematical Talent Search, Round 3, Problem 3/3/10.

We received 9 solutions of which 4 were correct and complete. We present the solution by Srikanth Pai.

a) We know that the sum of the numbers in every row, column, and diagonal is a multiple of 9. Therefore, the following sums, and subsequently the sum of these sums, are all multiples of 9:

$$A + E + I, \quad B + E + H, \quad C + E + G, \quad F + E + D.$$

Moreover, since each integer from 1 to 9 appears exactly once in the square, then we can calculate the following sum:

$$A + B + C + D + E + F + G + H + I = \sum_{i=1}^9 i = 45.$$

It follows that for some $k \in \mathbb{Z}^+$, we have

$$\begin{aligned} S &= (A + E + I) + (B + E + H) + (C + E + G) + (F + E + D) \\ &= 3E + A + B + C + D + E + F + G + H + I \\ &= 3E + 45 \\ &= 9k. \end{aligned}$$

We can easily isolate E to get

$$E = \frac{9k - 45}{3} = 3k - 15 = 3(k - 5).$$

We conclude that E is a multiple of 3.

b) The following array satisfies the necessary conditions with $E = 6$.

$$\begin{bmatrix} 1 & 5 & 3 \\ 8 & 6 & 4 \\ 9 & 7 & 2 \end{bmatrix}$$

MA314. Determine the smallest five-digit positive integer N such that $2N$ is also a five-digit integer and all ten digits from 0 to 9 are found in N and $2N$.

Originally from the 2000–2001 USA Mathematical Talent Search, Round 1, Problem 1/1/12.

We received 4 submissions of which 2 were correct and complete. We present the solution by Sicheng Du.

For $N = 13485$ we have $2N = 26970$, so it satisfies the conditions. Assume that there exists $N \leq 13484$ also satisfying the condition.

It can be easily verified that N cannot begin with 10. Since each of the 10 digits of N and $2N$ is unique, N cannot begin with 11. If N begins with 12, then $2N$ begins with 2, which is also impossible.

So N must begin with 13, in which case $2N$ must begin with 2. Let

$$\begin{array}{r} N = \quad 1 \quad 3 \quad c \quad d \quad e \\ \times \quad \quad \quad \quad \quad 2 \\ \hline 2N = \quad 2 \quad v \quad w \quad x \quad y \end{array}$$

Since digits 1, 2, 3 have been used, then $c \geq 4$, so $c = 4$. Hence, $v = 6$ and $w = 8$ or 9. Looking at what remains, we get $e, y \in \{0, 5, 7, 8, 9\}$. Since $2e = y$ or $10 + y$, then $e = 5, y = 0$ or $e = 9, y = 8$. The latter is impossible because of the value of w , so $e = 5$ and $y = 0$. Now,

$$\begin{array}{r} N = \quad 1 \quad 3 \quad 4 \quad d \quad 5 \\ \times \quad \quad \quad \quad \quad 2 \\ \hline 2N = \quad 2 \quad 6 \quad w \quad x \quad 0 \end{array}$$

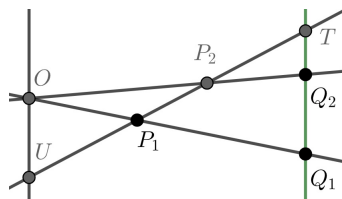
where $d \in \{7, 8, 9\}$. It is easy to check that only $d = 8$ works, so $N = 13485$, a contradiction. Therefore, $N = 13485$ is the desired smallest number.

Editor's Comment. So $N = 13485$ is the smallest possible number. How many solutions greater than $N = 13485$ are there?

MA315. *Proposed by Yagub N. Aliyev.*

A line passing through a point O and points Q_1 and Q_2 on a parallel line are drawn. A line intersects these lines at U and T , and the segments OQ_1 and OQ_2 at points P_1 and P_2 , respectively, so that $|P_1P_2| = |Q_1Q_2|$. Prove that $|OU| + |TQ_2| = |UP_1|$ and $|OU| + |TQ_1| = |UP_2|$.

We received 2 solutions for the problem. We present the solution by the proposer.



Since the lines OU and TQ_1 are parallel, $\triangle OUP_1 \sim Q_1TP_1$, $\triangle OUP_2 \sim Q_2TP_2$. Then

$$\frac{TQ_1}{UO} = \frac{TP_1}{UP_1} \quad \text{and} \quad \frac{TQ_2}{UO} = \frac{TP_2}{UP_2}.$$

So,

$$\frac{TQ_2 + Q_2Q_1}{UO} = \frac{TP_2 + P_2P_1}{UP_1} \quad \text{and} \quad \frac{TQ_2}{UO} = \frac{TP_2}{UP_1 + P_1P_2}.$$

By cross multiplying, we obtain

$$TQ_2 \cdot UP_1 + Q_2Q_1 \cdot UP_1 = TP_2 \cdot UO + P_2P_1 \cdot UO,$$

$$TQ_2 \cdot UP_1 + TQ_2 \cdot P_1P_2 = TP_2 \cdot UO.$$

Since $P_1P_2 = Q_1Q_2$, by subtracting, we obtain

$$(UP_1 - TQ_2) \cdot P_1P_2 = P_2P_1 \cdot UO.$$

Therefore,

$$UP_1 - TQ_2 = UO.$$

Consequently,

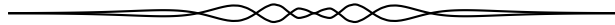
$$UP_1 = UO + TQ_2.$$

By adding the equality $P_1P_2 = Q_2Q_1$ to this equality we obtain also

$$UP_1 + P_1P_2 = UO + TQ_2 + Q_2Q_1,$$

or

$$UP_2 = UO + TQ_1.$$



Competition Highlights

The Canadian Mathematical Olympiad Qualifying Repêchage

Alex Zhuoqun Song

The Repêchage for the Canadian Math Olympiad was held in January 2025. The Repêchage is intended for students whose Canadian Open Mathematics Challenge (COMC) scores were just a bit below the cutoff score, for direct invitations to the Canadian Mathematical Olympiad (CMO). Students had a week to solve eight problems, with a blend of answer-based problems and proof-based problems. Because of the long time limit, students were required to write formal solutions for all eight problems, and were graded both on the quality and correctness of their writeup as well as their conclusions.

In 2025, we had eight problems, totalling 100 points. The median score was 55, and 21 students out of 83 participants qualified for the CMO, with 19 additional students qualifying for the Canadian Junior Mathematical Olympiad (CJMO).

Below, we will present a couple medium-level problems: a combinatorics problem involving expected value and a functional equation.

Problem: Initially, there are 2024 green balls and 1 red ball in a box. Every minute, Kate chooses a random ball from the box. If it is green, she paints it blue and puts it back into the box. If it is blue, she paints it green and puts it back into the box. Finally, if it is red, then she stops the process. What is the expected number of green balls at the end of her process?

Solution: Label the green balls as $G_1, \dots, G_{2024}$. Define the indicator random variable I_n to be 1 if the ball is green when the red ball is drawn, and 0 if the ball is blue when the red ball is drawn. The expected number of green balls at the end of the process is $E(I_1 + I_2 + \dots + I_{2024})$. From the linearity of expectation and the symmetry of all the green balls, we see that this expected value is equal to

$$2024E(I_1) = 2024 \cdot P(\text{ball 1 is green}).$$

In order to compute this probability, we consider the process of drawing balls. At each step, either ball G_1 is drawn; the red ball is drawn; or $G_2, \dots, G_{2024}$ is drawn. Note that in the last case, this does not affect the status of the ball G_1 at all, so we may discard this possibility. As such, we see that at each step, either G_1 is drawn or the red ball is drawn and they happen with equal probability. So we see that for the ball to be green in the end, the ball must be chosen exactly an even number of times before the red ball is chosen. So the probability is

$$P(\text{ball 1 is green}) = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots = \frac{2}{3}$$

As such, we see that the expected number of balls that are green at the end is $\frac{4048}{3}$.

Problem: Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $xy = f(x)f(y) - f(x+y)$ for all real numbers x and y .

Solution: We claim that the solutions are $f(x) = 1 \pm x$. It is easy to verify that these satisfy the functional equation.

To show that these are the only solutions, we first plug in $y = 0$ to the original equation to get $f(0)f(x) = f(x)$ for all x . It is also evident that $f(x)$ cannot be 0 for all x , so we get that $f(0) = 1$.

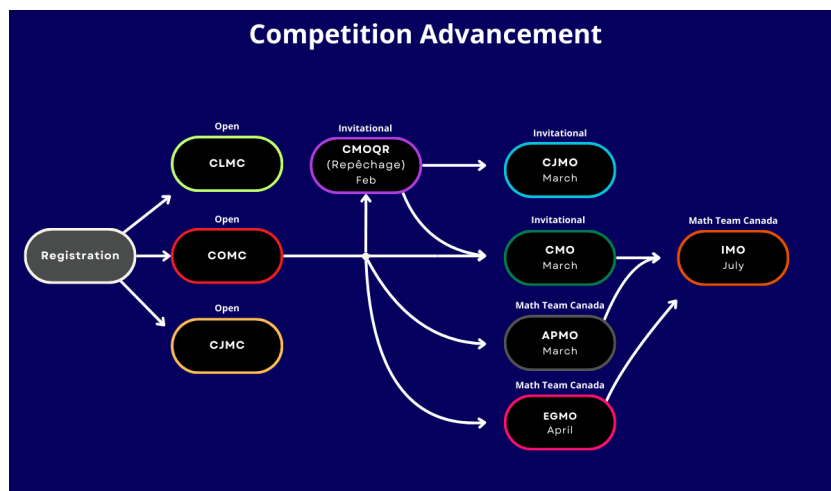
Next, plug in $x = 1$ and $y = -1$ to get that $f(1)f(-1) = 0$. We now split into two cases based on which of $f(1)$ and $f(-1)$ is zero.

Case 1: $f(1) = 0$. In this case, plug in $y = 1$ to get $f(x+1) = -x$, and by a change of variables, we get $f(x) = 1 - x$ for all x .

Case 2: $f(-1) = 0$. In this case, plug in $y = -1$ to get $f(x-1) = x$. Again, by a change of variables, we get $f(x) = 1 + x$ for all x .

.....

You can learn more about the various competitions run or supported by the CMS at <https://cms.math.ca/competitions/>. Competitions serve both as an enrichment activity as well as provide ways to advance and represent Team Canada at the International Mathematical Olympiad. Visit the CMS Competition website to learn more.



Trisecting an angle

Ed Barbeau

Section 0

After students learn about bisecting an angle using straightedge and compasses, the question of a similar construction for trisecting an angle may come up, and some students may attempt to find a method. Once I was contacted by a middle school teacher, one of whose students thought he had succeeded.

The proposed construction was pleasantly simple. Let POQ be the (acute) angle to be trisected. From any point A on OP , drop a perpendicular to meet OQ at B . Construct an equilateral triangle ABC with side AB with O and the vertex C on opposite sides of AB . Then it is claimed that $\angle COB$ is equal to one third of $\angle POQ$. If you check it out with a protractor the method is not bad at all, with numerical evidence suggesting that the error is within one or two degrees. In fact, it works for one acute angle; it is not hard to identify and check this angle.

However, there is a pedagogical difficulty here. One could don the mantle of authority and simply tell the student that it was rigorously proved long ago that no such method exists. It is more satisfactory to find an explanation that involves mathematics accessible to the student. I pose two problems for the reader and suggest solutions for them that I think can be improved upon.

- (1) Find an argument that the proposed trisection construction is faulty that involves facts of Euclidean geometry that the student might be expected to know. The more straightforward the argument the better.
- (2) Using standard high school mathematics, provide an analysis that identifies the situations for which the method delivers a trisection.

Section 1

We are asked to refute a construction that purports to produce a trisection for *every* acute angle. We employ a proof by contradiction: assume that the construction works for every angle and derive from this a false statement. All we have to do is to find *at least one* angle for which it does not work. The following argument will begin with the assumption that it works for both angles 30° and 60° and derive inconsistent conclusions. (In Section 3, you will see how the assumption that it works for $POQ = 60^\circ$ leads to a contradiction.)

In the diagram below, $\angle POQ = 60^\circ$ and $\angle DOB = 30^\circ$. We will suppose that $AB = 3$, from which we find that $BD = 1$, $AD = 2$ and $OB = \sqrt{3}$. Triangles

ABC and DBE are equilateral, and $\angle CBQ = \angle EBQ = 30^\circ$. Since $AC \parallel DE$, $CE = AD = 2$.

Assuming the method is valid, $\angle EOB = 10^\circ$, so

$$\angle OEB = \angle EBQ - \angle EOB = 20^\circ.$$

Also, by hypothesis, $\angle COQ = 20^\circ$, so $\angle COE = 10^\circ$. Since

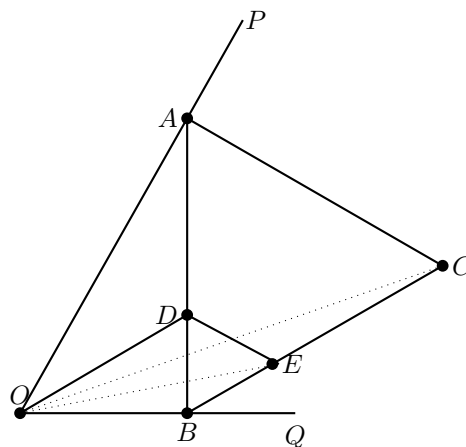
$$\angle OCE = \angle OEB - \angle COE = 10^\circ,$$

triangle COE is isosceles with $OE = CE = 2$.

Consider triangle OBE , On the one hand, $\angle OBE$ is obtuse. On the other,

$$OB^2 + BE^2 = 3 + 1 = 4 = OE^2,$$

which can occur only if $\angle OBE = 90^\circ$. Since these two statements are incompatible, the method fails for at least one of 30° and 60° .



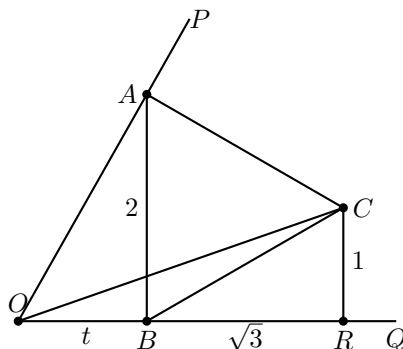
Section 2

Now look at the general situation of a proper acute angle. In the diagram below, assume that $AB = 2$ and $OB = t$ with $t > 0$. From the diagram, we see that

$$\tan \angle COQ = \frac{1}{t + \sqrt{3}}$$

and

$$\tan \angle POQ = \frac{2}{t}.$$



It can be verified that

$$\tan 3\angle COQ = \frac{3t^2 + 6t\sqrt{3} + 8}{t^3 + 3t^2\sqrt{3} + 6t}.$$

Therefore

$$\begin{aligned} & (t^3 + 3t^2\sqrt{3} + 6t) \cdot [\tan 3\angle COQ - \tan \angle POQ] \\ &= (3t^2 + 6t\sqrt{3} + 8) - 2(t^2 + 3t\sqrt{3} + 6) \\ &= t^2 - 4 \\ &= (t - 2)(t + 2). \end{aligned}$$

This vanishes if and only if $t = 2$ and $\angle POQ = 45^\circ$. (There is a degenerate situation when $\angle POQ = 90^\circ$. Here $Q = B$ and $\angle COQ = 30^\circ$.)

Section 3

A different assignment of lengths suggested by J. Chris Fisher gives a more transparent relationship between the tangents of the angles POQ and COQ . Let $AB = \sqrt{3}$ and $OB = t$; then $CR = \sqrt{3}/2$, $BR = 3/2$ and $\tan \angle COQ = \sqrt{3}/(2t+3)$. Then

$$\tan 3\angle COQ = \frac{3\sqrt{3}(t^2 + 3t + 2)}{t(2t^2 + 9t + 9)} = \tan \angle POQ \left(\frac{3(t^2 + 3t + 2)}{2t^2 + 9t + 9} \right).$$

When $\angle POQ = 60^\circ$, then $t = 1$ and

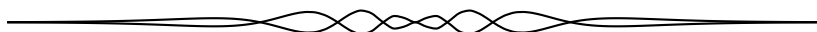
$$\tan 3\angle COQ = (9/10)\sqrt{3} = (9/10) \tan 60^\circ.$$

In this case, the trisection method produces an angle of about 19.1° .

Let

$$f(t) = [3(t^2 + 3t + 2)]/[2t^2 + 9t + 9].$$

Suppose that $\angle POQ = \theta$. If t is the value of OB that corresponds to θ , then $3/t$ is the value of OB that corresponds to the complement, $90^\circ - \theta$. Then $f(3/t) \cdot f(t) = 1$. The function $f(t)$ increases from $2/3$ when $t = 0$ (and $\theta = 90^\circ$) to $3/2$ when $t = \infty$ (and $\theta = 0$). However, the effect of the values of $f(t)$ in the accuracy of the trisection when θ is further from 45° and $f(t)$ is further from 1 is offset by the fact that when θ is close to 90° a large change in its tangent corresponds to a small change in the angle, and when θ is small, the error from a true trisection will also be small. For what angle is the deviation from a proper trisection maximum?



OLYMPIAD CORNER

No. 436

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

*To facilitate their consideration, solutions should be received by **December 15, 2025**.*

OC746. Two different integers u and v are written on a board. We perform a sequence of steps. At each step, we do one of the following two operations:

- (i) If a and b are different integers on the board, then we can write $a + b$ on the board, if it is not already there.
- (ii) If a , b and c are three different integers on the board, and if an integer x satisfies $ax^2 + bx + c = 0$, then we can write x on the board, if it is not already there.

Determine all pairs of starting numbers (u, v) from which any integer can eventually be written on the board after a finite sequence of steps.

OC747. Find all positive integers d for which there exists a degree d polynomial P with real coefficients such that there are at most d different values among $P(0), P(1), P(2), \dots, P(d^2 - d)$.

OC748. The complex $n \times n$ matrices A, B satisfy the relation $A^2B + BA^2 = 2ABA$. Check that $X = AB - BA$ commutes with A , and either using this or in any other way prove that there exists $k \in \{1, \dots, n\}$ such that $X^k = 0$.

OC749. If H is a set containing a given number $n > 1$ of (arbitrary) positive integers, how many elements can be in $\{xy + z | x, y, z \in H\}$ at most and at least?

OC750. Find

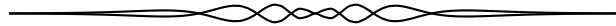
$$\lim_{n \rightarrow \infty} \sum_{k=n}^{2n} \binom{k-1}{n-1} 2^{-k}.$$

.....

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 décembre 2025**.



OC746. Deux entiers différents u et v sont écrits sur un tableau. On effectue une séquence d'étapes. À chaque étape, on réalise l'une des deux opérations suivantes :

- (i) Si a et b sont des entiers différents sur le tableau, alors on peut écrire $a + b$ sur le tableau, s'il n'y est pas déjà.
- (ii) Si a, b et c sont trois entiers différents sur le tableau, et si un entier x satisfait $ax^2 + bx + c = 0$, alors on peut écrire x sur le tableau, s'il n'y est pas déjà.

Déterminez toutes les paires de nombres de départ (u, v) à partir desquelles un nombre entier quelconque peut éventuellement être écrit sur le tableau après une séquence finie d'étapes.

OC747. Trouvez tous les entiers positifs d pour lesquels il existe un polynôme P de degré d à coefficients réels tel qu'il y ait au plus d valeurs différentes parmi $P(0), P(1), P(2), \dots, P(d^2 - d)$.

OC748. Les matrices $n \times n$ complexes A et B satisfont la relation $A^2B + BA^2 = 2ABA$. Vérifiez que $X = AB - BA$ commute avec A , et soit en utilisant ce fait, soit d'une autre manière, montrez qu'il existe $k \in \{1, \dots, n\}$ tel que $X^k = 0$.

OC749. Si H est un ensemble contenant un nombre donné $n > 1$ d'entiers positifs (arbitraires), au plus et au moins combien d'éléments peuvent se trouver dans $\{xy + z | x, y, z \in H\}$?

OC750. Trouvez

$$\lim_{n \rightarrow \infty} \sum_{k=n}^{2n} \binom{k-1}{n-1} 2^{-k}.$$



OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(3), p. 121–122.

OC721. Petya and Vasya know only natural numbers that do not exceed $10^9 - 4000$. Petya considers good numbers that can be represented in the form $abc + ab + bc + ca$, where a, b, c are natural numbers not less than 100. Vasya considers good numbers that can be represented in the form $xyz - x - y - z$, where x, y, z are natural numbers greater than 100. For which of them are there more good numbers?

Originally from the All Russian Mathematical Olympiad 2024 - Final Round, Grade 9, Problem 1.

We received 5 correct solutions. We present the solution by Theo Koupelis.

Using the identity

$$abc + ab + bc + ca = (a + 1)(b + 1)(c + 1) - (a + 1) - (b + 1) - (c + 1) + 2,$$

we see that if a number $N = xyz - x - y - z$ is considered good by Vasya, where $(x, y, z) = (a + 1, b + 1, c + 1)$, then the number $N + 2$ is considered good by Petya. Also, for $a, b, c \geq 100$ we get $x, y, z \geq 101$. The number $N_{\max} = 10^9 - 4000$ is achieved by Vasya for $(x, y, z) = (10^3 - 1, 10^3, 10^3 + 1)$ or permutations, but the number $N_{\max} + 2$ is outside the allowed range, and thus not achievable by Petya. Thus, there are more good numbers for Vasya.

OC722. Let p and q be distinct prime numbers. Given an infinite decreasing arithmetic sequence in which each of the numbers p^{23} , p^{24} , q^{23} and q^{24} occurs, prove that the numbers p and q are sure to occur in this sequence.

Originally from the All Russian Mathematical Olympiad 2024 - Final Round, Grade 10, Problem 1.

We received 9 correct solutions. We present the solution by Oliver Geupel.

Let A be the given sequence and let d denote its common difference. Note that d divides the number

$$p^{24} - p^{23} = (p - 1)p^{23}.$$

But d is not a multiple of p , because d divides the number $p^{23} - q^{23}$, which is not divisible by p . Hence, d is a divisor of $p - 1$. Thus, d divides the number

$$(p - 1) \sum_{k=1}^{22} p^k = p^{23} - p.$$

It follows that p occurs in A . Similarly, d is a divisor of $q - 1$ and therefore also divides $q^{23} - q$. Consequently, q occurs in A . This completes the proof.

OC723. Oleg has a set of 2024 different checkered rectangles of sizes 1×1 , 1×2 , 1×3 , $\dots$, 1×2024 (one rectangle of each size). Can he, by choosing some of them, make up a checkered square with an area greater than 1?

Originally from the All Russian Mathematical Olympiad 2024 - Regional Round, Grade 9, Problem 1.

We received 5 solutions. We present the solution by Theo Koupelis.

Let the side size of the square be n , where n is a positive integer in some appropriate unit. Let m , $1 \leq m \leq 2024$, be the number of the checkered rectangles used to create the square; the width of each of these rectangles is 1 and their lengths are $k_1, k_2, \dots, k_m$ units in increasing order, where $1 \leq k_i \leq 2024$ are positive integers and $i \in [1, 2, \dots, m]$. Clearly $n \geq k_m$. The areas of the used checkered rectangles, in increasing order, are $k_1 \times 1, k_2 \times 1, \dots, k_m \times 1$ squared units, and thus

$$n^2 = (k_1 + k_2 + \dots + k_m) \times 1. \quad (1)$$

But

$$k_1 + k_2 + \dots + k_m \leq 1 + 2 + \dots + k_m = \frac{k_m(k_m + 1)}{2}. \quad (2)$$

From (1) and (2) we get

$$k_m^2 \leq n^2 \leq \frac{k_m(k_m + 1)}{2} \implies k_m \leq 1.$$

Thus, Oleg cannot create a square of area greater than 1 by using some of the checkered rectangles.

OC724. Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \rightarrow \mathbb{R}$ be a twice differentiable function on I such that $f(x) \cdot f''(x) = 0$ for all $x \in I$. Prove that $f''(x) = 0$ for all $x \in I$.

Originally from the Romanian Mathematical Olympiad 2024 - Final Round, Grade 11, Problem 1.

We received 10 solutions. We present the solution by Michel Bataille.

Let $Z = \{x \in I : f(x) = 0\}$. We set $I = (\omega, \Omega)$ (possibly $\omega = -\infty$, possibly $\Omega = \infty$).

If $Z = \emptyset$, the desired result directly follows from the hypothesis.

Suppose that $Z = \{a\}$ for some a in I . Then, $f''(x) = 0$ for any $x \in (\omega, a)$, hence there exist real numbers α, β such that $f(x) = \alpha x + \beta$ for all $x \in (\omega, a)$. Since f' is continuous at a , we have $\lim_{x \rightarrow a^-} f'(x) = f'(a)$, hence $\alpha = f'(a)$. Similarly, the continuity of f at a gives

$$0 = f(a) = af'(a) + \beta,$$

hence $\beta = -af'(a)$ and

$$f(x) = xf'(a) - af'(a)$$

for all $x \in (\omega, a)$. In the same way, considering f on the interval (a, Ω) , we obtain $f(x) = xf'(a) - af'(a)$ for all $x \in (a, \Omega)$. Finally,

$$f(x) = xf'(a) - af'(a)$$

for all $x \in I$ and $f''(x) = 0$ for all $x \in I$ follows.

Suppose that Z contains at least two elements, say b, c with $b < c$. We show the stronger result: $f(x) = 0$ for all $x \in I$.

We will use the following lemma: if $u, v \in Z$ with $u < v$, then $f(x) = 0$ for all $x \in [u, v]$.

Proof. Since

$$[f(x)f'(x)]' = (f'(x))^2 + f(x)f''(x) = (f'(x))^2$$

for all $x \in I$, we have

$$\int_u^v (f'(x))^2 dx = f(v)f'(v) - f(u)f'(u) = 0,$$

hence (since f' is continuous) $f'(x) = 0$ for all $x \in [u, v]$. Therefore f is constant on $[u, v]$ and $f(x) = 0$ for all $x \in [u, v]$.

Now, let $a_0 = \inf Z$. We show that $a_0 = \omega$. Clearly, $a_0 \neq \Omega$. Assume that $a_0 \in I$. Since a_0 is the limit of a sequence of elements of Z , we have $f(a_0) = 0$ (by continuity of f) and we have $f(x) \neq 0$ whenever $x \in (\omega, a_0)$. Then $f''(x) = 0$ in (ω, a_0) and, as above, we deduce that

$$f(x) = xf'(a_0) - a_0f'(a_0)$$

for all $x \in (\omega, a_0]$. From the lemma we have $f(x) = 0$ for $x \in [a_0, c]$, hence $f'(a_0) = 0$, contradicting $f(x) \neq 0$ for $x < a_0$. Thus, we must have $a_0 = \omega$. Similarly, we obtain that $\sup Z = \Omega$.

To conclude, take any $x_0 \in I$. Since $\inf Z < x_0 < \sup Z$, there exist $a_1, a_2 \in Z$ such that $a_1 < x_0 < a_2$. From the lemma, we have $f(x) = 0$ for all $x \in [a_1, a_2]$. In particular, $f(x_0) = 0$. This being true for any $x_0 \in I$, we conclude that $f(x) = 0$ for all $x \in I$.

OC725. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function so that $f(x) + \sin(f(x)) \geq x$ for all $x \in \mathbb{R}$. Prove that

$$\int_0^\pi f(x) dx \geq \frac{\pi^2}{2} - 2.$$

Originally from the Romanian Mathematical Olympiad 2024 - Final Round, Grade 12, Problem 1.

We received 7 solutions. We present the solution by Oliver Geupel.

The definite integral is well-defined since the function f is continuous. The function

$$g : [0, \pi] \rightarrow [0, \pi] : x \mapsto x + \sin x$$

is strictly increasing and surjective. Hence, its inverse function $g^{-1} : [0, \pi] \rightarrow [0, \pi]$ exists. For every $x \in [0, \pi]$, it holds

$$g(f(x)) = f(x) + \sin(f(x)) \geq x = g(g^{-1}(x)).$$

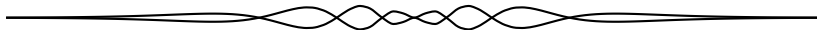
Thus,

$$f(x) \geq g^{-1}(x)$$

by monotonicity of g . We conclude

$$\int_0^\pi f(x) \, dx \geq \int_0^\pi g^{-1}(x) \, dx = \pi^2 - \int_0^\pi g(x) \, dx = \pi^2 - \left[\frac{x^2}{2} - \cos x \right]_0^\pi = \frac{\pi^2}{2} - 2$$

as desired.



PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **December 15, 2025**.



5071. *Proposed by Michel Bataille.*

Let triangle ABC be inscribed in a circle Γ with center O and let its incircle γ , with center I , touch BC at D . Let M be the midpoint of the arc BC of Γ containing A . If the line MD intersects γ at $U \neq D$ and Γ at $V \neq M$, prove that IU and OV are parallel.

5072. *Proposed by Seán M. Stewart.*

Consider the sequence of polynomials $\{P_n(x)\}_{n \geq 0}$ defined by the exponential generating function

$$\frac{1}{(1-x)e^t + x} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}.$$

Show that

$$\sum_{k=0}^n \binom{n}{k} \int_0^1 P_k(x) P_{n-k}(x) dx = (-1)^n.$$

5073. *Proposed by Yun Zhang.*

Let A_1, A_2, A_3, A_4 be the vertices of a tetrahedron with volume V , and let P be an arbitrary point in its interior. For each $k = 1, 2, 3, 4$, let M_k denote the midpoint of the segment $A_k P$. For each k , construct a plane that passes through M_k that is parallel to the face opposite A_k and intersects three edges of the tetrahedron. This divides the original tetrahedron into four internal tetrahedra of volumes V_k . Show that

$$V_1 + V_2 + V_3 + V_4 \geq \frac{27}{128} V,$$

with equality if and only if the point P is the centroid of the tetrahedron.

5074. *Proposed by Vasile Cîrtoaje.*

Let a, b, c, d be nonnegative real numbers such that at most one of them is larger than 1 and $ab + bc + cd + da = 4$. Prove that

$$\frac{1}{ab+2} + \frac{1}{ac+2} + \frac{1}{ad+2} + \frac{1}{bc+2} + \frac{1}{bd+2} + \frac{1}{cd+2} \geq 2.$$

5075. *Proposed by Matt Olechnowicz.*

Let m and k be positive integers. Show that

$$\binom{(m+1)k}{m} \leq (m+1) \binom{mk}{m}.$$

5076. *Proposed by Michael Friday.*

Let O and H be the circumcenter and orthocenter of triangle ABC satisfying $\angle B - \angle A = 90^\circ$. Prove that the circumcenters of triangles BCO , CAO , BCH , CAH , and vertex C , all lie on the same circle.

5077. *Proposed by Nguyen Viet Hung.*

Given a triangle ABC with the centroid G , let M be an arbitrary point inside the triangle. The lines that pass through M parallel to the median lines intersect the sides BC, CA, AB at X, Y, Z respectively. Prove that M, E, G are collinear, where E is the centroid of triangle XYZ .

5078. *Proposed by Tatsunori Irie.*

Let n be an integer such that $n \geq 2$ and let x be a positive integer. Show that the following holds:

$$\left(1 + \frac{x}{n}\right)^n \geq x \left(\frac{1}{n-1} + 1\right)^{n-1}.$$

5079. *Proposed by Torabi Dashti.*

For a square $ABCD$, let P be a point on AB and consider a triangle PQR , where Q is the point of intersection between DP and the diagonal AC , and R is on BC such that $\angle ADP = \angle BPR$. Prove that $PQ = PR$ if and only if $AP : PB = \frac{\sqrt{5}+1}{2}$.

5080. *Proposed by Nguyen Van Huyen.*

Let a, b, c, d be positive real numbers. Prove that

$$\frac{a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2}{abcd} + 10 \geq (a+b+c+d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right).$$

When does equality hold?

.....

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 décembre 2025**.



5071. *Soumis par Michel Bataille.*

Soit le triangle ABC inscrit dans un cercle Γ de centre O et soit son cercle inscrit γ , de centre I , tangent à BC en D . Soit M le milieu de l'arc BC de Γ contenant A . Si la droite MD coupe γ en $U \neq D$ et Γ en $V \neq M$, montrez que IU et OV sont parallèles.

5072. *Soumis par Seán M. Stewart.*

Considérons la suite de polynômes $\{P_n(x)\}_{n \geq 0}$ définie par la fonction génératrice exponentielle

$$\frac{1}{(1-x)e^t + x} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}.$$

Montrez que

$$\sum_{k=0}^n \binom{n}{k} \int_0^1 P_k(x) P_{n-k}(x) dx = (-1)^n.$$

5073. *Soumis par Yun Zhang.*

Soient A_1, A_2, A_3, A_4 les sommets d'un tétraèdre de volume V , et soit P un point arbitraire à l'intérieur de celui-ci. Pour chaque $k = 1, 2, 3, 4$, soit M_k le milieu du segment $A_k P$. Pour chaque k , construisez un plan passant par M_k qui soit parallèle à la face opposée à A_k et qui coupe trois arêtes du tétraèdre. Cela divise le tétraèdre d'origine en quatre tétraèdres internes de volumes V_k . Montrez que

$$V_1 + V_2 + V_3 + V_4 \geq \frac{27}{128} V,$$

avec égalité si et seulement si le point P est le centre de gravité du tétraèdre.

5074. *Soumis par Vasile Cîrtoaje.*

Soient a, b, c, d des nombres réels non négatifs tels que tout au plus un d'entre eux soit supérieur à 1 et que $ab + bc + cd + da = 4$. Montrez que

$$\frac{1}{ab+2} + \frac{1}{ac+2} + \frac{1}{ad+2} + \frac{1}{bc+2} + \frac{1}{bd+2} + \frac{1}{cd+2} \geq 2.$$

5075. *Soumis par Matt Olechnowicz.*

Soient m et k des entiers positifs. Montrez que

$$\binom{(m+1)k}{m} \leq (m+1) \binom{mk}{m}.$$

5076. *Soumis par Michael Friday.*

Soient O et H respectivement le centre du cercle circonscrit et le centre du cercle inscrit au triangle ABC tel que $\angle B - \angle A = 90^\circ$. Montrez que les centres des cercles circonscrits aux triangles BCO , CAO , BCH et CAH et au sommet C sont tous situés sur le même cercle.

5077. *Soumis par Nguyen Viet Hung.*

Soit un triangle ABC dont le centre de gravité est G . Soit M un point arbitraire à l'intérieur du triangle. Les droites passant par M parallèles aux médianes coupent les côtés BC , CA et AB respectivement en X , Y et Z . Montrez que M , E et G sont colinéaires, où E est le centre de gravité du triangle XYZ .

5078. *Soumis par Tatsunori Irie.*

Soit n un entier tel que $n \geq 2$ et soit x un entier positif. Montrez que la relation suivante est vérifiée :

$$\left(1 + \frac{x}{n}\right)^n \geq x \left(\frac{1}{n-1} + 1\right)^{n-1}.$$

5079. *Soumis par Torabi Dashti.*

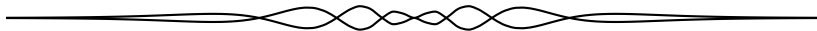
Étant donné un carré $ABCD$, soit P un point sur AB et considérons un triangle PQR , où Q est le point d'intersection de DP et de la diagonale AC et R est sur BC de telle sorte que $\angle ADP = \angle BPR$. Montrez que $PQ = PR$ si et seulement si $AP : PB = \frac{\sqrt{5}+1}{2}$.

5080. *Soumis par Nguyen Van Huyen.*

Soient a , b , c , d des nombres réels positifs. Montrez que

$$\frac{a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2}{abcd} + 10 \geq (a+b+c+d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right).$$

Quand a-t-on égalité ?



SOLUTIONS

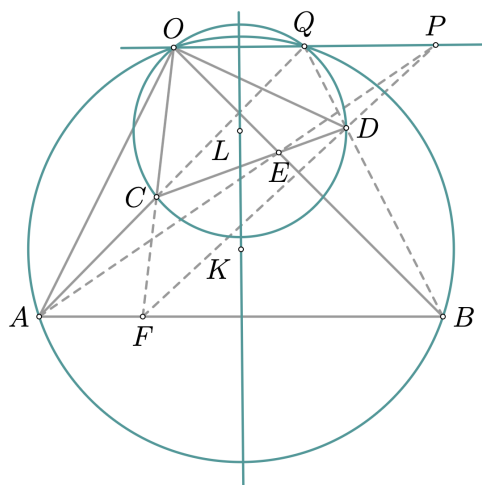
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2025: 51(3), p. 136–140.

5021. Proposed by Tran Quang Hung.

Let OAB and OCD be two equilateral triangles oriented in the same direction, with centers K and L respectively. The lines AB and CD intersect the lines OC and OB at F and E , respectively. Let P be the intersection point of lines AE and DF . Prove that OP is perpendicular to KL .

We received 12 solutions, all correct. The majority of solutions involve heavy algebraic manipulations through the introduction of coordinates. We present the solution by Chikara Tsugawa (slightly edited), which adopts an elegant geometric approach. The same solution was independently proposed by J. Chris Fisher.



We prove the following more general statement.

Suppose that $\triangle OAB \sim \triangle OCD$ have the same orientation and let K and L be their circumcenters. The lines AB and CD intersect the lines OC and OB at F and E respectively. Let P be the intersection of AE and DF . Then OP is perpendicular to KL .

Let Q be the intersection of AC and BD .

Applying Pappus's theorem to the two triples (A, B, F) and (D, C, E) shows that the points O, P, Q are collinear. Furthermore, since $\triangle OAB \sim \triangle OCD$ with the same orientation, it follows that the spiral similarity with center O that carries

A to B also sends C to D . This implies that $\triangle OCA \sim \triangle ODB$ with the same orientation. Hence, working with directed angles,

$$\begin{aligned}\angle OAQ &= \angle OAC = \angle OBD = \angle OBQ, \\ \angle OCQ &= \angle OCA = \angle ODB = \angle ODQ.\end{aligned}$$

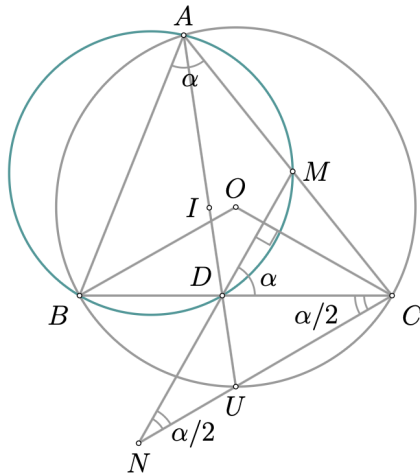
Consequently, the points O, A, B, Q are concyclic, as are the points O, C, D , and Q . Hence OP is the radical axis of these two circles and is therefore perpendicular to the line through their centers, namely KL .

5022. *Proposed by Michel Bataille.*

Let ABC be a triangle inscribed in a circle Γ with center O and let I be its incenter. The line AI intersects BC at D and Γ again at U . The perpendicular to OC through D intersects the lines AC and CU at M and N , respectively. Prove that M, A, B, D are concyclic and that $DN = DC$.

We received 12 solutions, all correct. We present the solution by Chikara Tsugawa (slightly edited). Many readers proposed similar angle chasing solutions.

Let $\angle CAB = \alpha$.



By the inscribed angle theorem, $\angle BOC = 2\angle BAC = 2\alpha$, and since $OB = OC$, we have $\angle OCB = \frac{\pi}{2} - \alpha$. Consequently,

$$\angle MDC = \frac{\pi}{2} - \angle OCB = \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right) = \alpha = \angle MAB.$$

It follows that the points M, A, B, D are concyclic. Since AI bisects $\angle CAB$,

$$\angle DCN = \angle BAD = \frac{\alpha}{2}.$$

Moreover,

$$\angle DNC = \angle MDC - \angle DCN = \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.$$

Therefore, we have $DN = DC$.

Editor's Comments. Chikara Tsugawa also observed that for the first part of the problem the assumption that AD bisects $\angle BAC$ can be dropped.

5023. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Let $f(x) = \frac{\ln x}{x^2 + n^2}$. Find $\lim_{n \rightarrow \infty} \frac{\int_1^{n^2} f(x) dx}{f(n)}$. †

We received 20 solutions, of which 14 were correct and complete. We present the solution by Didier Pinchon.

For $f(x) = \frac{\ln x}{x^2 + n^2}$, $n > 1$, let I_n denote the function

$$I_n = \frac{1}{nf(n)} \int_1^{n^2} f(x) dx = \frac{2n}{\ln n} \int_1^{n^2} \frac{\ln x}{x^2 + n^2} dx.$$

I_n may be written $I_n = J_n + K_n$ with

$$J_n = \frac{2n}{\ln n} \int_1^n \frac{\ln x}{x^2 + n^2} dx \quad \text{and} \quad K_n = \frac{2n}{\ln n} \int_n^{n^2} \frac{\ln x}{x^2 + n^2} dx.$$

Using the change of variables $x = nu$ in J_n ,

$$J_n = \frac{2}{\ln n} \int_{1/n}^1 \frac{\ln(nu)}{1 + u^2} du = 2 \int_{1/n}^1 \frac{du}{1 + u^2} + \frac{2}{\ln n} \int_{1/n}^1 \frac{\ln u}{1 + u^2} du,$$

and using the change of variables $x = n/u$ in K_n ,

$$K_n = \frac{2}{\ln n} \int_{1/n}^1 \frac{\ln(n/u)}{1 + u^2} du = 2 \int_{1/n}^1 \frac{du}{1 + u^2} - \frac{2}{\ln n} \int_{1/n}^1 \frac{\ln u}{1 + u^2} du.$$

Therefore

$$I_n = J_n + K_n = 4 \int_{1/n}^1 \frac{du}{1 + u^2} = 4 \left[\arctan(1) - \arctan\left(\frac{1}{n}\right) \right] = \pi - 4 \arctan\left(\frac{1}{n}\right),$$

so

$$\lim_{n \rightarrow +\infty} I_n = \pi \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\int_1^{n^2} f(x) dx}{f(n)} = \lim_{n \rightarrow \infty} nI_n = \infty.$$

†Please note that while computer-generated solutions are welcome, they will not be counted in the total tally when the problem can be solved without such aids. At **CruX**, we would like to encourage the pursuit of the solutions, not just the arrival at the final destination.

5024. *Proposed by Tatsunori Irie.*

Consider the Fibonacci sequence f_n for $n = 1, 2, \dots$ defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$. For any odd prime p , find p different terms among $f_1, \dots, f_{2p}$ whose sum is divisible by p .

We received 11 solutions, out of which we present the one by the Eagle Problem Solvers, lightly edited.

If $p = 5$, then

$$f_1 + f_3 + f_5 + f_7 + f_9 = 1 + 2 + 5 + 13 + 34 = 55,$$

which is divisible by $p = 5$. The remaining odd primes are either congruent to ± 1 modulo 10 or congruent to ± 3 modulo 10. Let $\pi(p)$ denote the period of the Fibonacci sequence modulo p , the *Pisano* period. It is known that $\pi(p)$ divides $p - 1$, if $p \equiv \pm 1 \pmod{10}$, whereas $\pi(p)$ divides $2p + 2$, if $p \equiv \pm 3 \pmod{10}$.

Note that the sum of the first p even Fibonacci numbers is

$$\sum_{k=1}^p f_{2k} = \sum_{k=1}^p (f_{2k+1} - f_{2k-1}) = f_{2p+1} - f_1 = f_{2p+1} - 1. \quad (1)$$

Let $p \equiv \pm 1 \pmod{10}$. Then

$$f_{2p+1} \equiv f_3 \equiv 2 \pmod{p}.$$

Replacing $f_4 = 3$ with $f_3 = 2$ in (1) we obtain

$$f_2 + f_3 + f_6 + f_8 + \dots + f_{2p} = f_{2p+1} - 2 \equiv f_3 - 2 \equiv 0 \pmod{p}.$$

On the other hand, if $p \equiv \pm 3 \pmod{10}$, then

$$\begin{aligned} f_{2p+2} &\equiv f_0 \equiv 0 \pmod{p}, \quad \text{and} \\ f_{2p+3} &\equiv f_1 \equiv 1 \pmod{p}, \end{aligned}$$

which means that

$$f_{2p+1} = f_{2p+3} - f_{2p+2} \equiv f_1 - f_0 \equiv 1 - 0 \equiv 1 \pmod{p}.$$

Thus, from (1),

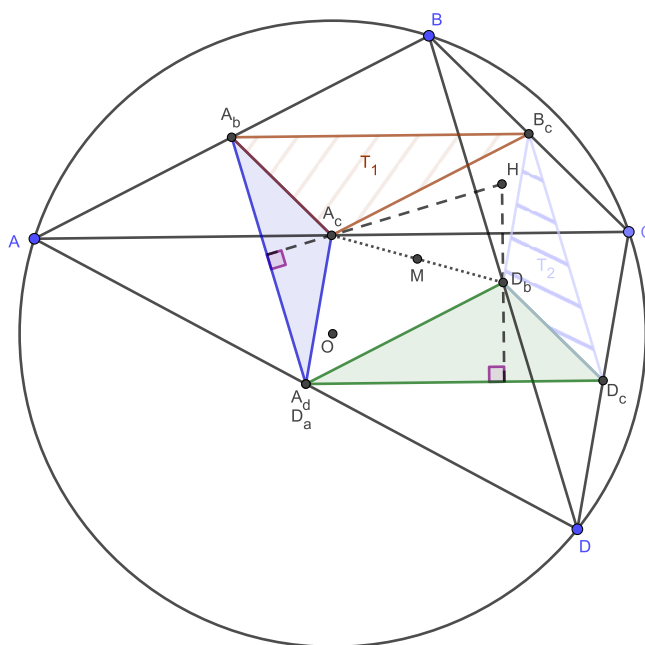
$$f_2 + f_4 + \dots + f_{2p} = f_{2p+1} - 1 \equiv 1 - 1 \equiv 0 \pmod{p}.$$

Editor's Comments. As was pointed out by our reader C.R. Pranesachar, this result also follows directly from the Erdős-Ginzburg-Ziv theorem that states that for any positive integer n , any multiset of $\mathbb{Z}/n\mathbb{Z}$ of size $2n-1$ contains a submultiset of size n whose sum is congruent to 0 modulo n .

5025. *Proposed by Michael Friday, modified by the Editorial Board.*

Let $ABCD$ be a convex cyclic quadrilateral. Let $D_aD_bD_c$ be the triangle obtained by joining the midpoints of DA , DB , DC ; similarly define $A_bA_cA_d$. Prove that the triangles $D_aD_bD_c$ and $A_bA_cA_d$ have the same orthocenter.

We received 13 solutions of which 11 were complete and correct. We present the solution by M. Bello, M. Benito, Ó. Ciaurri and E. Fernández.



Let O be the center of the circumscribed circle of quadrilateral $ABCD$, B_c the midpoint of BC and M the midpoint of A_cD_b .

First show that $\triangle A_bB_cA_c$ and $\triangle B_cD_cD_b$ have the same orthocenter. Note that $\triangle A_bB_cA_c$ (T_1 in the diagram) has as its vertices the midpoints of the sides of $\triangle ABC$, whence the orthocenter of $\triangle A_bB_cA_c$ is the circumcenter of $\triangle ABC$, namely O . Similarly, the orthocenter of $\triangle B_cD_cD_b$ (T_2 in the diagram) is the circumcenter of $\triangle BCD$, which is likewise O .

Next, consider $\triangle D_aD_bD_c$. Since D_a and D_b are the midpoints of DA and DB respectively, we have $D_aD_b \parallel AB$ and $D_aD_b = \frac{1}{2} AB$. Continuing in this manner for all the edges of $\triangle D_aD_bD_c$ we conclude that $\triangle D_aD_bD_c$ has all its edges parallel and congruent to those of $\triangle B_cA_cA_b$, so that a 180° rotation about M (the midpoint of A_cD_b) maps $\triangle B_cA_cA_b$ to $\triangle D_aD_bD_c$. Moreover, a similar argument shows that

the same 180° rotation about M maps $\triangle D_c D_b B_c$ to $\triangle A_b A_c A_d$. We can thus conclude that $\triangle A_b A_c A_d$ and $\triangle D_a D_b D_c$ have a common orthocenter, namely the image of O under a 180° rotation about M (this common orthocenter is denoted H in the diagram).

Editor's Comments. It follows quickly that the point H lies on the line through the midpoint of a side of the cyclic quadrilateral that is perpendicular to the opposite side; see Nathan Altshiller Court's *College Geometry*, Theorem 258, p. 131. The Wikipedia article on cyclic quadrilaterals calls these lines *multitudes* (which is short for midpoint altitudes), and their common point H , the *anticenter of the cyclic quadrilateral*. The proposer suggests R.A. Johnson's *Advanced Euclidean Geometry*, pages 251-253 for a discussion of further properties of the configuration.

5026. *Proposed by Eugen J. Ionaşcu.*

Let $n \geq 3$ be an integer. We denote by $[x]$ the greatest integer part of $x = \frac{n}{\sqrt{3}}$ and let $\{x\} = x - [x]$ be its fractional part.

(a) If $C \in (0, \frac{1}{3}]$, prove that the following equivalence holds:

$$\{x\} < \frac{1}{3} \quad (1)$$

if and only if

$$\{x\} < \sqrt{[x]^2 + \frac{2[x]}{3}} + C - [x], \quad (2)$$

(b) If $C > \frac{1}{3}$, prove that for every $M > 0$, there exists $n > M$ such that (2) is true but (1) is not.

(c) If $C \leq 0$, prove that for every $M > 0$, there exists $n > M$ such that (1) is true but (2) is not.

The problem is inspired by problem 1285 from The College Mathematics Journal.

We received 4 submissions and 3 of them were complete and correct. We present the following solution by the majority of solvers.

Observe that (1) is equivalent to $x < [x] + \frac{1}{3}$, which is equivalent to $9x^2 < (3[x] + 1)^2$, that is, $3n^2 < (3[x] + 1)^2$. Observe that (2) is equivalent to

$$\begin{aligned} x^2 &< [x]^2 + \frac{2[x]}{3} + C, \\ \iff 9x^2 &< 9[x]^2 + 6[x] + 9C, \\ \iff 3n^2 &< (3[x] + 1)^2 + (9C - 1). \end{aligned}$$

(a) Assume that $C \in (0, \frac{1}{3}]$. Note that $3n^2 \equiv 0 \pmod{3}$ and $(3[x] + 1)^2 \equiv 1 \pmod{3}$. If (1) holds, then

$$3n^2 < (3[x] + 1)^2;$$

by considering modulo 3, we also have

$$3n^2 \leq (3[x] + 1)^2 - 1$$

and thus (2) holds. If (2) holds, that is,

$$3n^2 < (3[x] + 1)^2 + 2;$$

by considering modulo 3, we also have

$$3n^2 < (3[x] + 1)^2$$

and thus (1) holds.

(b) In view of the proof of part (a), it suffices to construct sufficiently large n with

$$3n^2 = (3[x] + 1)^2 + 2.$$

To this end, we consider the positive integer solutions of the generalized Pell equation $y^2 - 3z^2 = -2$ with $y \equiv 1 \pmod{3}$. The fundamental solution of the associated Pell equation $y^2 - 3z^2 = 1$ is $(y, z) = (2, 1)$, and the smallest solution to $y^2 - 3z^2 = -2$ is $(y, z) = (1, 1)$. Thus, the general solution to $y^2 - 3z^2 = -2$ is

$$y_m + \sqrt{3}z_m = (1 + \sqrt{3})(2 + \sqrt{3})^m.$$

Note that $(y_0, z_0) = (1, 1)$, $(y_1, z_1) = (5, 3)$, $(y_2, z_2) = (19, 11)$. Using induction, it is easy to show that $y_m \equiv 1 \pmod{3}$ if and only if m is even.

Let m be a sufficiently large even number so that $n = z_m > M$. Note that

$$y_m - \sqrt{3}n = y_m - \sqrt{3}z_m = \frac{-2}{y_m + \sqrt{3}z_m} = \frac{-2}{(2 + \sqrt{3})^m(1 + \sqrt{3})} \in (-1, 0).$$

It follows that

$$\frac{y_m}{3} < \frac{n}{\sqrt{3}} < \frac{y_m + 1}{3}.$$

Since $y_m \equiv 1 \pmod{3}$, it follows that $[x] = [n/\sqrt{3}] = \frac{y_m - 1}{3}$ and therefore we have

$$3n^2 = (3[x] + 1)^2 + 2,$$

as required.

(c) In view of the proof of part (a), it suffices to construct sufficiently large n with

$$3n^2 = (3[x] + 1)^2 - 1.$$

Similar to (b), we consider solutions of the Pell equation $y^2 - 3z^2 = 1$ with $y \equiv 1 \pmod{3}$. The general solution to the equation is

$$y_m + \sqrt{3}z_m = (2 + \sqrt{3})^m.$$

By a similar argument as in the proof of (b), if we take $n = z_m$, where m is a sufficiently large even number so that $n = z_m > M$, then $[x] = [n/\sqrt{3}] = \frac{y_m-1}{3}$ and $3n^2 = (3[x] + 1)^2 - 1$.

5027. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with inradius r and circumradius R . Prove that

$$\cot^2 A + \cot^2 B + \cot^2 C \leq 8 \left(\frac{R}{2r} \right)^2 - 7.$$

We received 18 submissions of which 14 were correct and complete. Heron's formula together with Euler's inequality is sufficient to solve the problem. We present two solutions.

Solution 1, by the proposer (In Memoriam) and Kevin Soto Palacios independently, slightly altered by the editor.

Let a, b, c be the lengths of the sides BC, CA, AB , respectively. Then

$$\begin{aligned} \cot^2 A + \cot^2 B + \cot^2 C &= \frac{\cos^2 A}{\sin^2 A} + \frac{\cos^2 B}{\sin^2 B} + \frac{\cos^2 C}{\sin^2 C} \\ &= \frac{1 - \sin^2 A}{\sin^2 A} + \frac{1 - \sin^2 B}{\sin^2 B} + \frac{1 - \sin^2 C}{\sin^2 C} \\ &= \frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C} - 3 \\ &= \frac{1}{\left(\frac{a}{2R}\right)^2} + \frac{1}{\left(\frac{b}{2R}\right)^2} + \frac{1}{\left(\frac{c}{2R}\right)^2} - 3 \\ &= 4R^2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) - 3. \end{aligned} \quad (1)$$

On the right hand side, we have

$$8 \left(\frac{R}{2r} \right)^2 - 7 = 2 \left(\frac{R}{r} \right)^2 - 7 \geq \left(\frac{R}{r} \right)^2 + 4 - 7 = 4R^2 \cdot \frac{1}{4r^2} - 3, \quad (2)$$

where we used Euler's inequality $R \geq 2r$.

Thus, to complete the proof, it suffices to show the following lemma.

Lemma. $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}.$

Proof of Lemma. We have $(b-c)^2 \geq 0$, which implies $a^2 - (b-c)^2 \leq a^2$ and hence

$$\frac{1}{a^2} \leq \frac{1}{a^2 - (b-c)^2} = \frac{1}{(a+b-c)(a-b+c)}.$$

Let $2s = a + b + c$. Then $a + b - c = 2(s - c)$, $a - b + c = 2(s - b)$, so

$$\frac{1}{a^2} \leq \frac{1}{4(s-b)(s-c)}. \text{ Similarly, } \frac{1}{b^2} \leq \frac{1}{4(s-c)(s-a)}, \quad \frac{1}{c^2} \leq \frac{1}{4(s-a)(s-b)}.$$

Thus we have

$$\begin{aligned}
 \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} &\leq \frac{1}{4} \left(\frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} + \frac{1}{(s-a)(s-b)} \right) \\
 &\leq \frac{3s - (a+b+c)}{4(s-a)(s-b)(s-c)} \\
 &= \frac{s}{4r^2s} \quad (\text{Heron: } rs = \sqrt{s(s-a)(s-b)(s-c)}) \\
 &= \frac{1}{4r^2}.
 \end{aligned}$$

Using the lemma to connect equations (1) and (2), we conclude that

$$\cot^2 A + \cot^2 B + \cot^2 C \leq 8 \left(\frac{R}{2r} \right)^2 - 7,$$

with equality if and only if the triangle is equilateral.

Solution 2, by Theo Koupelis.

Let a, b, c, E be the side lengths and area of the triangle, respectively.

From $E = \frac{1}{2}bc \sin A$, we deduce

$$\cot^2 A = \frac{1 - \sin^2 A}{\sin^2 A} = \frac{b^2 c^2}{4E^2} - 1; \quad \text{similarly } \cot^2 B = \frac{c^2 a^2}{4E^2} - 1, \quad \cot^2 C = \frac{a^2 b^2}{4E^2} - 1.$$

Then, by substituting $4R = abc/E$ and $r = 2E/(a+b+c)$ into the desired inequality, we obtain an equivalent expression

$$(16E^2)^2 + 16E^2(a^2b^2 + b^2c^2 + c^2a^2) - 2a^2b^2c^2(a+b+c)^2 \leq 0. \quad (*)$$

Let $(a, b, c) = (x+y, y+z, z+x)$, where x, y, z are the lengths of the external tangents from the vertices B, C, A , respectively, to the incircle of the triangle. For convenience, let $p := x+y+z$, $q := xy+yz+zx$, and $r := xyz$. Recall Heron's formula $E^2 = xyz(x+y+z)$. With these changes of variables (*) can be rewritten as

$$32pr^2 + 2r(p^4 + q^2 - 2p^2q + 4pr) - p(pq - r)^2 \leq 0.$$

Simplifying the expression, it becomes

$$q^2(p^3 - 2r) + 2p^2rq - 2p^4r - 39pr^2 \geq 0.$$

By the AM–GM inequality we have $pq \geq 9r$, and $q^2 \geq 3pr$. Thus it suffices to show that

$$3pr(p^3 - 2r) + 2pr \cdot 9r - 2p^4r - 39pr^2 \geq 0 \iff pr(p^3 - 27r) \geq 0,$$

which is true since $p^3 \geq 27r$ by AM–GM again. Equality holds for equilateral triangles.

Editor's Comments. Another common approach was to first cite a theorem expressing the cotangents directly in terms of R , r , and s , and then proceed via Gerretsen inequality. Some solvers also employed additional tools, such as the general Muirhead inequalities. Among all the creative submissions, that of the late proposer stands out as especially elementary and elegant.

5028. *Proposed by Traian M. Ulpui.*

Find all differentiable functions $f : (0, \infty) \rightarrow \mathbb{R}$ with $f(1) = 1$ that satisfy the equation $f'(x) = f(1/x)$ for all $x > 0$.

We received 17 submissions of which 14 were correct and complete. We present the solution by Brian Bradie.

From $f'(x) = f(1/x)$ for $x > 0$ it follows in particular that $f'(1) = f(1) = 1$. Because f is differentiable on $(0, \infty)$ and $1/x$ is differentiable on $(0, \infty)$, it follows that $f(1/x)$ is differentiable on $(0, \infty)$; hence, $f'(x) = f(1/x)$ is differentiable on $(0, \infty)$. Moreover,

$$f''(x) = (f'(x))' = (f(1/x))' = -\frac{1}{x^2}f'(1/x) = -\frac{1}{x^2}f(x);$$

that is,

$$x^2 f''(x) + f(x) = 0,$$

which is an Euler-Cauchy equation for f . Using the test function $f(x) = x^r$ for some constant r yields the characteristic equation

$$r(r-1) + 1 = 0$$

whose roots are

$$r = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

The general solution to the Euler-Cauchy equation for f is thus

$$f(x) = c_1 \sqrt{x} \cos\left(\frac{\sqrt{3}}{2} \ln x\right) + c_2 \sqrt{x} \sin\left(\frac{\sqrt{3}}{2} \ln x\right).$$

The initial condition $f(1) = 1$ leads to $c_1 = 1$. Next,

$$f'(x) = \left(\frac{c_1}{2\sqrt{x}} + \frac{c_2\sqrt{3}}{2\sqrt{x}}\right) \cos\left(\frac{\sqrt{3}}{2} \ln x\right) + \left(\frac{c_2}{2\sqrt{x}} - \frac{c_1\sqrt{3}}{2\sqrt{x}}\right) \sin\left(\frac{\sqrt{3}}{2} \ln x\right),$$

so the condition $f'(1) = 1$ leads to $c_2 = 1/\sqrt{3}$. Therefore the real-valued function satisfying the given conditions is

$$f(x) = \sqrt{x} \cos\left(\frac{\sqrt{3}}{2} \ln x\right) + \sqrt{\frac{x}{3}} \sin\left(\frac{\sqrt{3}}{2} \ln x\right).$$

It is easy to verify that for this function we indeed have $f(1) = f'(1) = 1$, as well as

$$f'(x) = \frac{1}{\sqrt{x}} \cos\left(\frac{\sqrt{3}}{2} \ln x\right) - \frac{1}{\sqrt{3}x} \sin\left(\frac{\sqrt{3}}{2} \ln x\right) = f(1/x).$$

5029. *Proposed by Vasile Cirtoaje and Vo Quoc Ba Can.*

Let n be a positive integer with $n \geq 3$. Prove that $n - 1$ is the largest positive value of the constant k such that the inequality

$$\frac{1}{a_1 + k} + \frac{1}{a_2 + k} + \cdots + \frac{1}{a_n + k} \geq \frac{n}{1 + k}.$$

holds for any nonnegative real numbers $a_1, a_2, \dots, a_n$ with

$$\sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}.$$

We received 3 submissions and 2 of them were complete and correct. We feature the following two solutions, slightly modified by the editor.

Suppose k is such that the conclusion holds. Let $x > 0$ and consider the sequence $(a_1, a_2, \dots, a_n) = (x, \frac{n(n-1)}{2x}, 0, \dots, 0)$; we get

$$\frac{1}{x + k} + \frac{1}{\frac{n(n-1)}{2x} + k} + \frac{n-2}{k} \geq \frac{n}{1 + k}.$$

Letting $x \rightarrow \infty$, we conclude that

$$\frac{1}{k} + \frac{n-2}{k} \geq \frac{n}{1 + k},$$

that is, $k \leq n - 1$.

It remains to show that the given inequality holds for $k = n - 1$, that is,

$$\frac{1}{a_1 + (n-1)} + \cdots + \frac{1}{a_n + (n-1)} \geq 1. \quad (1)$$

Next, we present two different proofs of inequality (1).

Solution 1, by Michal Adamaszek.

For each $0 \leq i \leq n$, let σ_i be the i -th elementary symmetric polynomial of a_i . Since $\sigma_2 = \binom{n}{2}$, by Maclaurin's inequality, $\sigma_i \leq \binom{n}{i}$ for each $i \geq 2$.

By clearing the denominators, observe that inequality (1) is equivalent to

$$\sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} (a_{i_1} + (n-1)) \cdots (a_{i_{n-1}} + (n-1)) \geq (a_1 + (n-1)) \cdots (a_n + (n-1)).$$

After expansion, it suffices to show that

$$\sum_{i=0}^{n-1} (n-1)^{n-i-1} (n-i) \sigma_i \leq \sum_{i=0}^n (n-1)^{n-i} \sigma_i,$$

equivalently,

$$(n-1)^{n-1} \geq \sum_{i=2}^n (n-1)^{n-i-1} (i-1) \sigma_i.$$

Since $\sigma_i \leq \binom{n}{i}$ for each $i \geq 2$, by the binomial theorem, we have

$$\begin{aligned} & \sum_{i=2}^n (n-1)^{n-i-1} (i-1) \sigma_i \\ & \leq \sum_{i=2}^n (n-1)^{n-i-1} (i-1) \binom{n}{i} \\ & = \frac{n}{n-1} \sum_{i=2}^n (n-1)^{n-1-(i-1)} \binom{n-1}{i-1} - \frac{1}{n-1} \sum_{i=2}^n (n-1)^{n-i} \binom{n}{i} \\ & = \frac{n}{n-1} (n^{n-1} - (n-1)^{n-1}) - \frac{1}{n-1} (n^n - (n-1)^n - n(n-1)^{n-1}) \\ & = (n-1)^{n-1}, \end{aligned}$$

as required.

Solution 2, by the proposer.

For each $1 \leq i \leq n$, set

$$b_i = \frac{a_i}{a_i + n - 1} \in [0, 1).$$

Note that

$$\sum_{i=1}^n b_i = \sum_{i=1}^n \frac{a_i}{a_i + n - 1} = n - (n-1) \sum_{i=1}^n \frac{1}{a_i + n - 1}.$$

Thus, to show inequality (1), it suffices to show that $\sum_{i=1}^n b_i \leq 1$. Since

$$a_i = \frac{(n-1)b_i}{1-b_i}$$

for each i , we have

$$\sum_{1 \leq i < j \leq n} \frac{b_i b_j}{(1-b_i)(1-b_j)} = \frac{n}{2(n-1)}.$$

By the Cauchy-Schwarz inequality, we have

$$\left[\sum_{1 \leq i < j \leq n} b_i b_j (1-b_i)(1-b_j) \right] \left[\sum_{1 \leq i < j \leq n} \frac{b_i b_j}{(1-b_i)(1-b_j)} \right] \geq \left(\sum_{1 \leq i < j \leq n} b_i b_j \right)^2.$$

It follows that

$$\frac{4(n-1)}{n} \left(\sum_{1 \leq i < j \leq n} b_i b_j \right)^2 \leq 2 \sum_{1 \leq i < j \leq n} b_i b_j (1 - b_i)(1 - b_j). \quad (2)$$

Next we give an upper bound on the right-hand side of (2). Set $x_i = b_i(1 - b_i)$ for $1 \leq i \leq n$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & 2 \sum_{1 \leq i < j \leq n} b_i b_j (1 - b_i)(1 - b_j) \\ &= 2 \sum_{1 \leq i < j \leq n} x_i x_j = \left(\sum_{i=1}^n x_i \right)^2 - \sum_{i=1}^n x_i^2 \leq \left(\sum_{i=1}^n x_i \right)^2 - \frac{1}{n} \left(\sum_{i=1}^n x_i \right)^2 \\ &= \frac{n-1}{n} \left(\sum_{i=1}^n x_i \right)^2 = \frac{n-1}{n} \left(\sum_{i=1}^n b_i - \sum_{i=1}^n b_i^2 \right)^2. \end{aligned} \quad (3)$$

Thus, comparing inequalities (2) and (3), we get

$$\left(\sum_{i=1}^n b_i \right)^2 - \sum_{i=1}^n b_i^2 = 2 \sum_{1 \leq i < j \leq n} b_i b_j \leq \sum_{i=1}^n b_i - \sum_{i=1}^n b_i^2,$$

that is, $\sum_{i=1}^n b_i \leq 1$, as required.

5030. *Proposed by Daniel Sitaru.*

Let $0 < a \leq b$ and $c > 0$ be real numbers. Show that

$$\int_a^b \int_a^b \frac{x+y}{\sqrt{xy+c}} dx dy \leq (b^2 - a^2) \ln \left(\frac{b + \sqrt{b^2 + c}}{a + \sqrt{a^2 + c}} \right)$$

We received 8 submissions, of which 3 were correct and complete. We present the solution by Michal Adamaszek, modified by the editor.

Let $x, y, c \in \mathbb{R}^+$. We start by proving the following inequality:

$$\frac{x+y}{\sqrt{xy+c}} \leq \frac{x}{\sqrt{y^2+c}} + \frac{y}{\sqrt{x^2+c}}.$$

The proof of this inequality is obtained by the following calculation:

$$\begin{aligned}
& \frac{x+y}{\sqrt{xy+c}} - \frac{x}{\sqrt{y^2+c}} - \frac{y}{\sqrt{x^2+c}} \\
&= x \left(\frac{1}{\sqrt{xy+c}} - \frac{1}{\sqrt{y^2+c}} \right) + y \left(\frac{1}{\sqrt{xy+c}} - \frac{1}{\sqrt{x^2+c}} \right) \\
&= x \frac{\sqrt{y^2+c} - \sqrt{xy+c}}{\sqrt{xy+c}\sqrt{y^2+c}} + y \frac{\sqrt{x^2+c} - \sqrt{xy+c}}{\sqrt{xy+c}\sqrt{x^2+c}} \\
&= x \frac{y^2 - xy}{\sqrt{xy+c}\sqrt{y^2+c}(\sqrt{xy+c} + \sqrt{y^2+c})} + y \frac{x^2 - xy}{\sqrt{xy+c}\sqrt{x^2+c}(\sqrt{xy+c} + \sqrt{x^2+c})} \\
&= \frac{xy(y-x)}{\sqrt{xy+c}} \left(\frac{1}{\sqrt{y^2+c}(\sqrt{xy+c} + \sqrt{y^2+c})} - \frac{1}{\sqrt{x^2+c}(\sqrt{xy+c} + \sqrt{x^2+c})} \right) \\
&= \frac{xy(y-x)}{\sqrt{xy+c}} \left(\frac{\sqrt{x^2+c}(\sqrt{xy+c} + \sqrt{x^2+c}) - \sqrt{y^2+c}(\sqrt{xy+c} + \sqrt{y^2+c})}{\sqrt{y^2+c}\sqrt{x^2+c}(\sqrt{xy+c} + \sqrt{y^2+c})(\sqrt{xy+c} + \sqrt{x^2+c})} \right) \\
&= \frac{xy(y-x)}{\sqrt{xy+c}} \frac{\sqrt{xy+c}(\sqrt{x^2+c} - \sqrt{y^2+c}) + x^2 - y^2}{\sqrt{y^2+c}\sqrt{x^2+c}(\sqrt{xy+c} + \sqrt{y^2+c})(\sqrt{xy+c} + \sqrt{x^2+c})}
\end{aligned}$$

Note that the denominator is positive. We focus on the numerator.

$$\begin{aligned}
& xy(y-x) \left(\sqrt{xy+c}(\sqrt{x^2+c} - \sqrt{y^2+c}) + x^2 - y^2 \right) \\
&= xy(y-x) \left(\frac{\sqrt{xy+c}}{\sqrt{x^2+c} + \sqrt{y^2+c}} (x^2 - y^2) + x^2 - y^2 \right) \\
&= -xy(y-x)^2(x+y) \left(\frac{\sqrt{xy+c}}{\sqrt{x^2+c} + \sqrt{y^2+c}} + 1 \right) \leq 0.
\end{aligned}$$

This completes the proof of the inequality. As for the problem, we now have:

$$\begin{aligned}
\int_a^b \int_a^b \frac{x+y}{\sqrt{xy+c}} dx dy &\leq \int_a^b \int_a^b \frac{x}{\sqrt{y^2+c}} dx dy + \int_a^b \int_a^b \frac{y}{\sqrt{x^2+c}} dx dy \\
&= 2 \int_a^b \int_a^b \frac{x}{\sqrt{y^2+c}} dx dy \\
&= 2 \int_a^b x dx \cdot \int_a^b \frac{1}{\sqrt{y^2+c}} dy \\
&= 2 \cdot \left(\frac{b^2 - a^2}{2} \right) \cdot \left(\frac{\ln(b + \sqrt{b^2+c})}{\ln(a + \sqrt{a^2+c})} \right),
\end{aligned}$$

as desired.

