Crux Mathematicorum

Volume/tome 51, issue/numéro 7
September/septembre 2025

Canadian Mathematical Society
Société mathématique du Canada

Crux Mathematicorum is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

`https://publications.cms.math.ca/cruxbox/`

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n'est pas une revue scientifique. Soumission en ligne:

`https://publications.cms.math.ca/cruxbox/`

ISSN 1496-4309 (Online)

ISSN 1496-4309 (électronique)

Editorial Board

IN THIS ISSUE / DANS CE NUMÉRO

IN MEMORIAM

We are very saddened to announce the passing of George Apostolopoulos, a long-time *Crux* contributor from Greece. His contributions to *Crux* over the years have been substantial – he started as a regular solver in 2008 and as a problem proposer in 2011. Over the years, George submitted hundreds of original problems, which were always popular with solvers. His sustained contributions left an indelible mark on the *Crux* community and his presence will be greatly missed.

In memory of George Apostolopoulos, we share his first and last problem to appear in *Crux Mathematicorum*:

3603. *Proposed by George Apostolopoulos.*

Let ABC be a given triangle with $0 < \lambda < \frac{1}{2}$. Let D and E be points on AB such that $AD = BE = \lambda \cdot AB$ and F, G be points on AC such that $AF = CG = \lambda \cdot AC$. Let $BF \cap CE = H$ and $BG \cap CD = I$. Show that

1. $HI \parallel BC$;
2. $HI = \frac{1-2\lambda}{\lambda 2-\lambda+1} BC$.

5027. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with inradius r and circumradius R. Prove that

$$\cot^2 A + \cot^2 B + \cot^2 C \leq 8 \left(\frac{R}{2r}\right)^2 - 7.$$

W.J. Blundon Commemorative

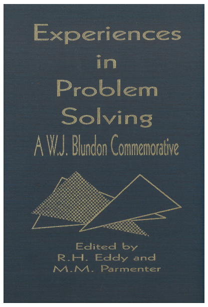
In the summer of 1973, having recently defended my PhD thesis at the University of British Columbia, I left the shores of the Pacific for the shores (and winds) of the Atlantic for what turned out to be a 40 year career at Memorial University. I knew only one person in the math department (maybe one in the province) at the time, Michael Parmenter, a classmate during my undergraduate years at the University of Toronto and now a life long friend. Professor John (Jack) Blundon — always Professor Blundon — was the Head of Department at the time and the only head the Mathematics Department at Memorial had ever had. I remember how very welcoming he was. All the new recruits were invited to his home that fall for what he called a "lunch", a term which means something quite different to a Newfoundlander. (Suffice to say I should not have eaten dinner that evening!) After his retirement and during my term as a co-editor of the *Canadian Mathematical Bulletin*, we would pass in the hall and he would always greet me with a smile and a "Good day, Mister Editor." He eventually suffered a stroke and died soon thereafter.

Whether as Head or a retired colleague, Professor Blundon spent a lot of time in the corridors of the Department poking into people's offices and discussing math problems. He spent a lot of time with Roland Eddy, who was also a keen problems guy. So was Mike Parmenter and, some time after Professor Blundon's passing, Roland and Mike had the idea of honouring Blundon with a book featuring all the problems he had solved and contributed to a variety of journals during his life.

For a period of time, I was a member of an advisory board of the National Research Council — something to do with publications — and in that context I met a lady — I'm sure her name was Sandra Woolfrey — who was the director of a small publishing house in Ontario and whom I contacted to ask if she might be willing to print the book. She agreed and the book was published. For some time, we advertised and sold copies, never recovering all the printing costs, but that wasn't the point. The book was used as a prize at APICS (now Science Atlantic) conferences for many years and given as a present to invited speakers.

It is a personal delight to know that the book continues to live on the internet and is freely available at `https://scienceatlantic.ca/blundon/`

Edgar Goodaire



MathemAttic

No. 67

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 15, 2025**.

MA331. *Proposed by Ivan Hadinata.*

Let M be the number of ordered pairs of natural numbers (a, b) satisfying the equation

$$a^b = (20!)^{24!}.$$

Find the last three digits of M.

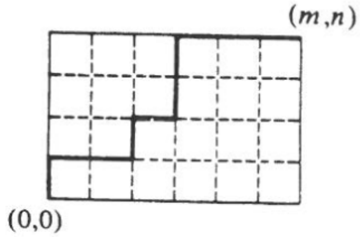
MA332. *Proposed by Michael Friday.*

In any triangle ABC, let H, O be the orthocenter and circumcenter, and let M_a, M_b, M_c be the midpoints of sides BC, CA, AB respectively. Prove that

$$OH^2 = (HM_a^2 - OM_a^2) + (HM_b^2 - OM_b^2) + (HM_c^2 - OM_c^2)$$

MA333.

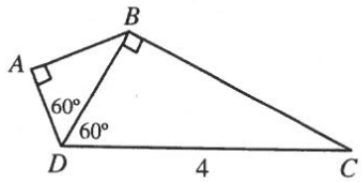
a) An $m \times n$ rectangle is divided into mn squares. A path is to be traced starting at $(0, 0)$ and concluding at (m, n) by moving only in a positive sense along the ruled lines.



Show that the number of distinct paths is $\binom{m+n}{n}$

b) An $n \times n \times n$ cube has each of faces ruled into n^2 squares. A path defined in part a), moving always in a positive sense on its faces, is to start at $(0, 0, 0)$ and reach the point (n, n, n). Determine the number of distinct paths.

MA334. In the quadrilateral $ABCD$, angles DBC and DAB are right angles. Also, angles ADB and BDC have measure of 60 degrees. If DC is 4 units, determine which one is greater $DA + AC$ or $DB + BC$.



MA335.

a) Find all geometric series such that the sum of the first two terms is 2 and the sum of the first three terms is 3.

b) For each of the sequences determined in part a), calculate the sum of all terms having value less than 1.

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 novembre 2025**.

MA331. *Soumis par Ivan Hadinata.*

Soit M le nombre de paires ordonnées de nombres naturels (a, b) satisfaisant l'équation

$$a^b = (20!)^{24!}.$$

Trouvez les trois derniers chiffres de M.

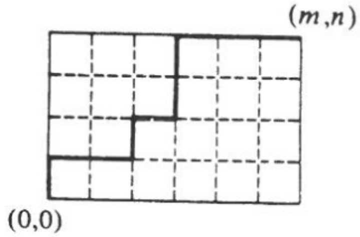
MA332. *Soumis par Michael Friday.*

Soit ABC un triangle quelconque et soit H et O respectivement l'orthocentre et le centre du cercle circonscrit au triangle. Soit encore M_a, M_b et M_c respectivement les milieux des côtés BC, CA et AB. Montrez que

$$OH^2 = (HM_a^2 - OM_a^2) + (HM_b^2 - OM_b^2) + (HM_c^2 - OM_c^2)$$

MA333.

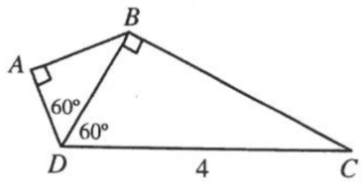
a) Un rectangle de dimensions $m \times n$ est divisé en mn carrés. Un chemin doit être tracé en partant de $(0, 0)$ et en terminant à (m, n) en se déplaçant uniquement dans le sens positif le long des lignes tracées.



Montrez que le nombre de chemins distincts est égal à $\binom{m+n}{n}$

b) Chacune des faces d'un cube $n \times n \times n$ est divisée en n^2 carrés. Un chemin défini à la partie a), se déplaçant toujours dans le sens positif sur les faces, doit partir de $(0, 0, 0)$ et atteindre le point (n, n, n). Déterminez le nombre de chemins distincts.

MA334. Dans le quadrilatère $ABCD$, les angles DBC et DAB sont des angles droits. De plus, les angles ADB et BDC mesurent 60 degrés. Si DC mesure 4 unités, déterminez lequel des deux est le plus grand parmi $DA + AC$ et $DB + BC$.



MA335.

a) Trouvez toutes les séries géométriques telles que la somme des deux premiers termes soit égale à 2 et la somme des trois premiers termes soit égale à 3.

b) Pour chacune des suites déterminées à la partie a), calculez la somme de tous les termes ayant une valeur inférieure à 1.

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(2), p. 57 – 60.

MA306. The power seven lottery awards prize money in powers of 7. For example, prize categories are $1 = 7^0$, $7 = 7^1$, $49 = 7^2$, $343 = 7^3$, etc. In this lottery there are several awards that amount to 777777 and there are no more than 6 awards for each prize category. How many awards are there?

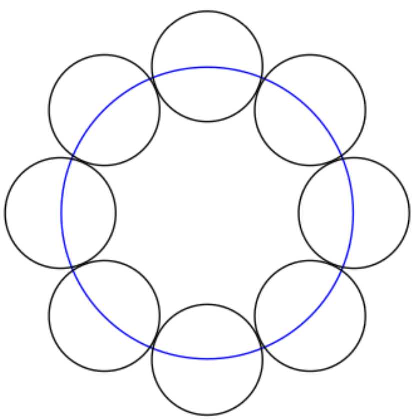
Originally from the 2018 Louisiana State University Math Contest, Team Session, problem 18.

We received 11 solutions for this problem, of which 9 were correct. We present the solution by Catherine Jian.

Since the prize amounts in each category are powers of 7, and there are no more than 6 awards per category, it's helpful to convert the total amount into base 7.

Note that the base 7 representation of 777777 is 6416400_7, therefore the number of awards is $6 + 4 + 1 + 6 + 4 = 21$.

MA307. Eight circles of radius 1 have centers on a larger common circle and adjacent circles are tangent. Find the area of the common circle. See the illustration below.



Originally from the 2018 Louisiana State University Math Contest, Team Session, problem 1.

We received 10 correct and 5 incorrect solutions. Most of the incorrect solutions involved an arithmetic error. We present 3 solutions due to multiple authors each.

Suppose that the centre of the large circle is O and its radius is r. Let A and B be the centres of two adjacent touching small circles and M their point of tangency, the midpoint of AB. Then $AO = OB = r$, the radius of the large circle, and $AM = MB = 1$. Then $\angle AOM = \theta$, $\angle AOB = 2\theta$ and $\angle OAB = 3\theta$, where $\theta = \pi/8$.

Solution 1.

By the Law of Cosines applied to triangle AOB,

$$4 = r^2 + r^2 - 2r^2 \cos 2\theta = (2 - \sqrt{2})r^2.$$

Hence $r^2 = 2(2 + \sqrt{2})$ and the area F of the large circle is $\pi r^2 = 2(2 + \sqrt{2})\pi$.

Solution 2.

From triangle AOM, $\sin\theta = 1/r$. Hence

$$r^2 = \frac{1}{\sin^2\theta} = \frac{2}{1 - \cos 2\theta} = \frac{2\sqrt{2}}{\sqrt{2} - 1} = 2\sqrt{2}(\sqrt{2} + 1),$$

from which $F = 2(2 + \sqrt{2})\pi$.

Solution 3.

In triangle AOB, let P be the foot of the altitude from A to OB. Then AOP is an isosceles right triangle with $AP = OP = r/\sqrt{2}$. Since triangle APB is right with hypotenuse $AB = 2$ and legs equal to $r(1 - 1/\sqrt{2})$ and $r/\sqrt{2}$, an application of Pythagoras' theorem leads to $r^2 = 2(2 + \sqrt{2})$ and $F = 2(2 + \sqrt{2})\pi$.

MA308. There are 46656 6-digit numbers that can be formed from the digits 1, 2, 3, 4, 5, and 6, with repetition of digits allowed. If these numbers are listed in order, what is the 2018th number in the list?

Originally from the 2018 Louisiana State University Math Contest, Team Session, problem 6.

We received 10 submissions for this question, 9 of which were correct. We present the solution by Catherine Jian.

If we replace digits $1, 2, 3, 4, 5,$ and 6 with $0, 1, 2, 3, 4,$ and 5 respectively, then the new 46656 6-digit numbers are just the base 6 representation of numbers from 0 ($(000000)_6$) to 46655 ($(555555)_6$). That means to find the 2018th number, we just need to find the base 6 representation of $2018 - 1 = 2017$. Since the base 6 representation of 2017 is $(013201)_6$, we know the 2018th number in the original list must be 124312.

MA309. Alice walks down to the bottom of an escalator that is moving up. Alice counts 150 steps. Alice's friend Bob walks up to the top of the escalator and counts 75 steps. Alice's speed of walking (number of steps per unit time) is 3 times Bob's walking speed. How many steps are visible on the escalator at a given time?

Originally from the W.J. Blundon Contest, 2023 Question 10.

We received 8 submissions, 7 of which were correct and complete. We present the solution by Catherine Jian.

Let x be the number of visible steps. We can assume that Bob's speed is 1 step per unit time and Alice's speed is 3 steps per unit time. Now let e be the speed of the escalator in steps per unit time.

Since Alice walks down and the escalator is moving up, her actual speed relative to the static world is $3-e$ steps per unit time. The time it takes her from top to the bottom of the escalator is then $\frac{x}{3-e}$ units of time. In the same time period, she walked 150 steps, so we have

$$\frac{x}{3-e} = \frac{150}{3} = 50.$$

Bob walks up, so his speed relative to the static world is $1+e$ steps per unit time. Similarly, we have

$$\frac{x}{1+e} = \frac{75}{1} = 75.$$

Solving the above equations yields $e = \frac{3}{5}$ and $x = 120$. Therefore there are 120 steps visible on the escalator.

MA310. In a town where any pair of people are either friends or strangers to each other, any two friends do not have common friends, while any two strangers have exactly two common friends. Prove that in this town everyone has exactly the same number of friends.

Originally from the W.J. Blundon Contest, 2024 Question 10.

We received 3 submissions, one of which was correct and complete. We present the solution by Catherine Jian.

We can represent this problem using graph theory. Let people in this town be vertices. If two people are friends, then there is an edge connecting them and if they are strangers, there is no edge connecting them.

Let A and B be friends. Suppose C is a friend of A other than B. Then C and B must be strangers since any two friends do not have common friends.

Since C and B are strangers with A being a common friend, there is D who is another common friend of B and C.

Now suppose E is another friend of A. By the same argument, there is F who is a friend of B and E.

We claim that D and F must be different people. Otherwise, A and D as two strangers will have three common friends, namely B,C and E.

Therefore for any new friend of A, there will be a unique corresponding friend of B. Therefore, we've shown that two friends always have the same number of friends.

Finally we notice that this graph is indeed connected since any two strangers can always be connected by their common friends. By transitivity, we can now claim that in this town everyone has exactly the same number of friends.

PROBLEM SOLVING VIGNETTES

No. 38

Shawn Godin

Playing With Triangles

Recently, I was exploring a problem concerning cutting a square up into triangles of different types for my column in the *OAME Gazette*. Along the way, I was led to consider the following problem:

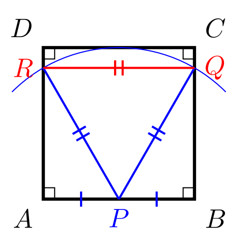
> *Construct an equilateral triangle PQR whose vertices lie on the sides of a given square $ABCD$ when P is also given.*

The investigation of this problem will be the focus of this column.

I first considered two special cases.

Case 1: When the vertex P of the inscribed equilateral triangle is at the midpoint of one of the sides of the square.

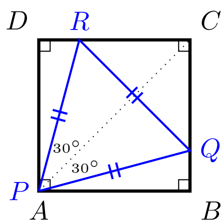
A good first step in solving a construction problem is to draw a reasonably accurate diagram of the completed figure. The diagram below shows the desired equilateral triangle which shares an axis of symmetry with the surrounding square. The diagram suggests that one side of the triangle is parallel to a side of the square, hence, the triangle's sides would necessarily have a length equal to a side of the square. For the construction we draw a circle with radius equal to AB and with center P at the midpoint of AB; it intersects the sides BC and AD at Q and R. From here we see that the right triangles formed are congruent by hypotenuse side. Hence the two points where the arc intersects the sides are equal lengths up the side of the square, so joining these points creates a segment that is indeed parallel to two of the sides of the square. Therefore, the segment is equal in length to the side of the square, so it is the third side of our equilateral triangle PQR.



It follows immediately from Case 1 that no two vertices of the inscribed equilateral triangle could lie on the same side of the square – otherwise its third vertex would lie inside the square. Consequently, at least one side of any inscribed equilateral triangle would necessarily have its end points on opposite sides of the square.

Case 2: When the vertex P of the inscribed equilateral triangle coincides with a vertex of the square.

Without loss of generality, taking vertex A of the square to be the location of P, a diagram indicates that the diagonal AC (PC) is an axis of symmetry for both the triangle and the square. Therefore, the sides PQ and PR would necessarily lie on the lines that make a 30° angle with PC. By symmetry, R is the image of Q in PC and hence $PQ = PR$, making PQR isosceles. However, since $\angle RPQ = 60^\circ$, this forces $\angle PQR = \angle QRP = 60^\circ$, making PQR equilateral.



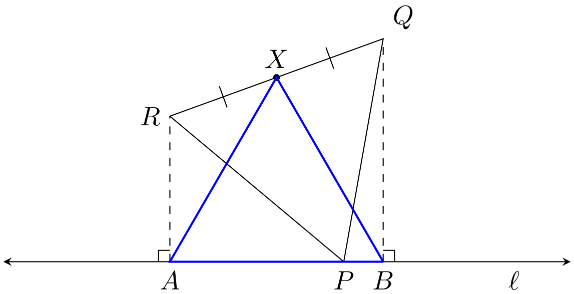
By symmetry, we can draw an equilateral triangle with a vertex coinciding with any vertex of the square. As such, we could have one with R at D, P on AB, and Q on BC such that $\angle PRB = \angle QRB = 30^\circ$. Hence, it is not hard to imagine starting with the diagram in Case 1 and moving P towards A, Q towards B, and R towards D in such a way that PQR remains equilateral (although the side length may change) right up until R arrives at D. If the process continues with P moving towards A, Q towards B and R towards C, we can continue to get equilateral triangle PQR until Q arrives at the midpoint of BC and we have an instance of Case 1. Hence, given any point, P, on the perimeter of a square, we should be able to find an equilateral triangle PQR, with Q and R also on the square.

At this point, I was reminded of a problem from my course with professor Honsberger that I presented in an earlier column [2019:45(6), 313-317].

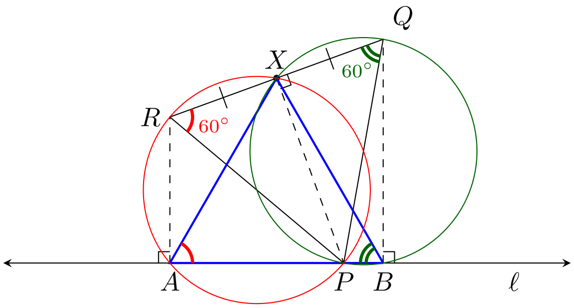
> **#20.** *Construct an equilateral triangle so that it has one vertex on each of three given parallel lines.*

I will return to this problem in a later column. However, while I was playing with this problem and its solution in GeoGebra, I made the following discovery.

> *Given a line ℓ and a point X not on ℓ. A point P is chosen on ℓ and the equilateral triangle PQR is constructed such that X is the midpoint of QR. Points A and B are the feet of the perpendiculars dropped from R and Q, respectively, to ℓ. Then ABX is an equilateral triangle.*



Proof: Since PQR is equilateral and X is the midpoint of QR, median PX lies on the perpendicular bisector of QR. Since $\angle PXR = 90^\circ = \angle RAP$, then $XRAP$ is cyclic and PR is a diameter of the circumcircle. Thus, $\angle XAP$ and $\angle XRP$ are inscribed angles subtended by the same arc, so $\angle XAP = \angle XRP = 60^\circ$.



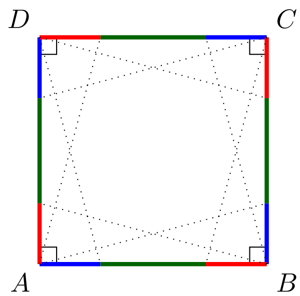
Similarly $QXPB$ is cyclic and $\angle PBX = \angle PQX = 60^\circ$. Therefore ABX is equilateral. □

What this means is as P moves along ℓ, the locus of points created by Q and R is two lines, perpendicular to ℓ at A and B, respectively, where ABX is equilateral.

What does that mean for our situation? Suppose we construct an equilateral triangle that shares one of its sides with the square and whose other vertex, X, is in the interior of the square. Then for *certain* points, P on the shared side, we can draw an equilateral triangle PQR such that X is the midpoint of QR, where Q and R are on the square. However, from above, some locations of P on the shared side would result in Q or R being outside the square.

If we go back to our original two special cases, we get a clue. In Case 1, P is at the midpoint of the shared side. As P moves along the shared side, one of Q and R moves towards the vertex of the square closest to it, while the other moves away from the vertex it is closest to. When the vertex of the square is reached we have Case 2. If we continue to move P in the same direction, the vertex of the triangle at the vertex of the square would move off of the square. As such, Case 2 provides limits for the movement of P on the side of the square.

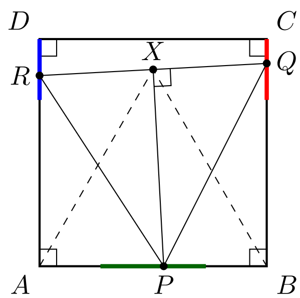
Draw two rays from each vertex, each at 30° to the diagonal through the vertex, and note the points of intersection break each side of the square into three pieces, as shown in the diagram below. If we pick point P in one of the green segments, then there will be an equilateral triangle PQR with Q and R on the sides perpendicular to the side that P is on. Also one of these vertices will be on a red segment and the other on a blue segment.



Knowing this gives us a way to construct our equilateral triangle given one point, P, on the perimeter of the square. There are two possible constructions:

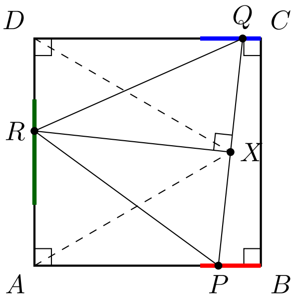
Construction 1: P is on a green segment.

Suppose, without loss of generality, that P is on side AB of square $ABCD$. Construct the equilateral triangle ABX where X is in the interior of the square. Draw a line through X perpendicular to PX, intersecting the sides of the square at Q and R. Then PQR is the desired equilateral triangle.



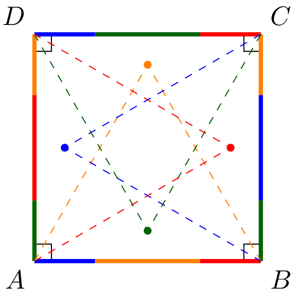
Construction 2: P is not on a green segment.

Suppose, without loss of generality, that P is on side AB of square $ABCD$, closer to B than A. Then DA is the side of the square $ABCD$ that is perpendicular to the side that P is on, AB, and is the furthest such side from P. Construct the equilateral triangle DAX where X is in the interior of the square. Draw the line PX, intersecting CD at Q. Construct the perpendicular bisector of PQ and let its point of intersection of DA be R. Then PQR is the desired equilateral triangle.



If one is unsure which construction to use, start with the first one and if the perpendicular to PX doesn't intersect two parallel sides, use the second construction. Note that when we have Case 2 (one vertex of the triangle coincides with a vertex of the square) then both constructions work.

It is interesting that if we construct the four equilateral triangles that share a side with the square and have the third vertex in the interior of the square, then (at least) one of these four points will be on any equilateral triangle with three vertices on the square. Also, as a bonus, if we colour the square as in the diagram below, the three vertices of an equilateral triangle will end up on sections of the same colour and the interior point of the same colour will be the midpoint of one of the sides.



. . . .

For your enjoyment, here are a few problems related to our exploration:

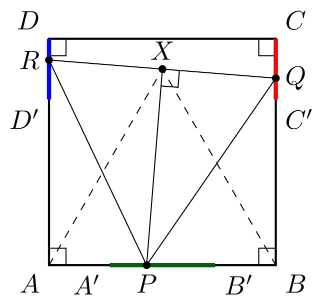
1. Prove that Case 1 produces the smallest inscribed equilateral triangle, Case 2, the largest.

2. Prove that the circumcentre of the equilateral triangle lies on a square, with sides parallel to the sides of $ABCD$, of side length $\left(\frac{2\sqrt{3}-3}{3}\right)s$, where s is the side length of $ABCD$.

3. For equilateral triangle PQR with vertices on the sides of square $ABCD$ given in the diagram below, prove

$$DR : RD' = A'P : PB' = C'Q : QC$$

where A', B', C' and D' are the end points of the coloured sections as described earlier.

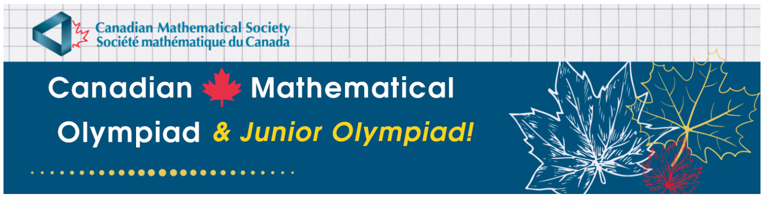


Once again, I would like to thank ***Crux*** editor Chris Fisher for his help and suggestions in the preparation of this column. Chris's insights resulted in a much better article.

Competition Highlights

Canadian Mathematical Olympiad and Junior Olympiad (CMO/CJMO)

Paweł Prałat and James Rickards



The Canadian Mathematical Olympiad (CMO) is an annual, invitational, proof-based competition for Canadian students. It is Canada's premier national advanced mathematics competition. Students attempt to solve 5 problems in three hours, with each problem graded on a scale from 0 to 7. In 2020, the CMS introduced the Canadian Junior Mathematical Olympiad (CJMO), also by invitation only, a variant specifically for students in grade at most 10. These 3-hour competitions are held each March at a selected time and date (by default, the second Thursday of March). All official participants write at the same time and are proctored by their local school faculty or staff. For more information visit `https://cms.math.ca/competitions/cmo/`.

The CMO is an important contest for students with international aspirations, as a good performance leads to the Canadian Team Selection Test, and then onto the International Mathematical Olympiad (IMO) itself. Qualification for the C(J)MO is primarily via the Canadian Open Mathematics Challenge (COMC), an open contest written in late October.

In total, the 2025 CMO was written by 95 students, with 93 official entrants. The CJMO was written by 17 students, all official entrants. Five Canadian provinces were represented, with the number of contestants as follows:

CMO: AB (8), BC (13), ON (47), QC (8), SK (1)
CJMO: AB (1), BC (1), ON (12), QC (1)

(note that Canadian citizens residing outside of Canada can also officially write the C(J)MO, accounting for the discrepancy in numbers).

Grading for both contests went relatively smoothly, with a team of 15 mathematicians, including professors, students, and former contestants, contributing their time. The top score on the CMO was 27 (out of 35), achieved by Warren Bei, and the mean score was 8.1. The Matthew Brennan Award for best solution went to Warren Bei for an excellent solution to problem 3. This was a difficult problem with most solutions being fairly technical. Warren's writeup was very short and clean. On the CJMO, a top score of 23 was achieved by Warren Maximilian Lin, and the mean score was 10.1. A full breakdown of the marks assigned problem by problem is in Table 1.

Score	P1	P2	P3	P4	P5
7	0	34	8	5	5
6	3	9	4	0	2
5	4	9	1	0	0
4	2	6	0	0	0
3	27	10	0	0	0
2	14	8	4	1	1
1	26	4	1	0	2
0	19	15	77	89	85
Avg	1.91	4.33	0.99	0.39	0.54

(a) CMO

Score	P1	P2	P3	P4	P5
7	0	2	0	1	0
6	15	0	0	0	1
5	1	0	1	0	0
4	0	0	0	2	0
3	1	0	3	4	0
2	0	0	3	1	0
1	0	2	2	1	0
0	0	13	8	8	16
Avg	5.76	0.94	1.29	1.76	0.35

(b) CJMO

Table 1 – C(J)MO score breakdown by problem.

An intrepid reader may note that CMO problem 1 (which was also CJMO problem 3) was significantly harder than usual, with no perfect 7's awarded. This was due to an unintentional clerical error. The question entailed a voting procedure among n hockey players, which took place in rounds. The question was intended to read "Prove that eventually, all players will unanimously vote for the same person." Instead, the final version asked to prove that "after n rounds, all players will unanimously vote for the same person." Despite several checks, this subtle change went unnoticed until after the contest.

Fortunately, the version with n rounds is still a correct problem, just more appropriately placed as the fourth or fifth problem on the contest. The increased difficulty had a large knock-on effect, with overall scoring averages being about a problem lower than last year's C(J)MO. It also offers a good lesson to future olympiad contestants: do not always trust the ordering of the problems! It is very common for test-setters to misestimate the relative difficulty of a contest, leading to misnumbered problems. Even a contest like the IMO occasionally has problems out of order (with the most famous example being 2011's "windmill problem").

Problem 5 on the CMO was a neat problem about an ant traveling around a rectangle.

Problem 5. *A rectangle R is divided into a set S of finitely many smaller rectangles with sides parallel to the sides of R such that no three rectangles in S share a common corner. An ant is initially located at the bottom-left corner of R. In one operation, we can choose a rectangle $r \in S$ such that the ant is currently located at one of the corners of r, say c, and move the ant to one of the two corners of r adjacent to c. Suppose that after a finite number of operations, the ant ends up at the top-right corner of R. Prove that some rectangle $r \in S$ was chosen in at least two operations.*

A natural approach to solving the problem is to prove the contrapositive: assume that the ant does not choose a rectangle twice, and prove that it cannot travel from the bottom left to the top right.

A good next step is to draw a few sample dissections S, and attempt to walk the ant to the top right following the moves. For example, see Figure 1.

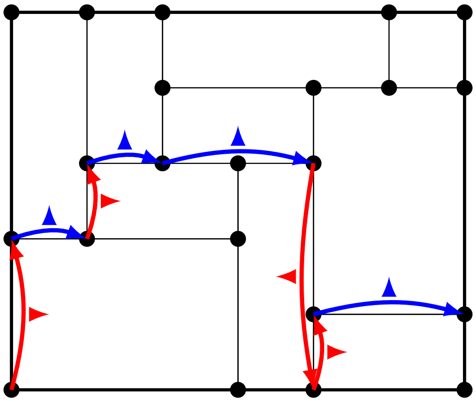


Figure 1 – A possible path by the ant. Arrows indicate direction of travel, and point to the chosen rectangles.

One will find that the ant can travel pretty far, in fact, it seems like it can visit most vertices in S, and it can get very close to the top right corner. Furthermore, if we drop the repeated rectangle rule, it is very easy to walk to the top right. This signals that there must be some sort of *invariant* that is blocking the ant.

Such an invariant is clearly broken when choosing the same rectangle twice. Let us examine the given conditions a bit more closely: there is the clause that "no three rectangles in S share a common corner." This is clearly very important, as the problem would be false without it! See Figure 2 for a simple path that reaches the top-right corner of R, and does not choose a rectangle twice.

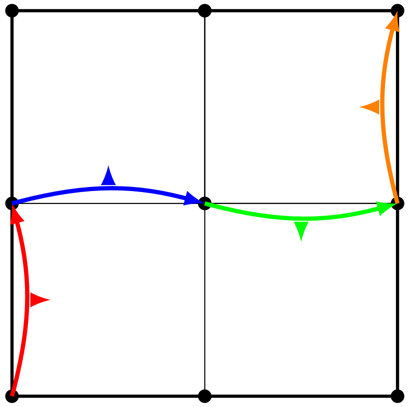


Figure 2 – How the ant can win, if three corners may intersect. (Note that three corners intersecting implies that a fourth must also intersect.)

What is special about forbidding a three corner intersection? This implies that all corner intersections are "T-intersections": they form a pattern like $\vdash$, $\dashv$, $\perp$, or $\top$.

In particular, to reach an intersection, there are exactly two possible rectangles that can be chosen. One rectangle must be chosen to reach the intersection point, and the other is therefore chosen to leave it, since we cannot repeat rectangles.

In fact, this is exactly where the non-repeating condition gets used. It also implies a strengthening of the problem: in order for the ant to get to the top right, not only did the same rectangle have to be used twice, but it had to be used twice *consecutively*!

Heading back to the problem, it seems like we are close. We have identified a key property of the setup, which limits the possible moves, and need to form this into an invariant that is somehow preserved. There is very little to work with here, other than the location of the ant's path relative to the chosen rectangle.

In particular, we can track if the ant is traveling vertically or horizontally, and if the chosen rectangle is to the left or to the right of the ant as they travel. This makes four combinations, and we can keep track of them as we walk (an analogous invariant would be tracking the cardinal directions, e.g. SW corner to NW corner). Colour the ant's moves as follows:

Red: vertical, with chosen rectangle on the right (with respect to ant's travel);

Orange: vertical, with chosen rectangle on the left;

Green: horizontal, with chosen rectangle on the right;

Blue: horizontal, with chosen rectangle on the left.

In Figure 2, we made it to the top-right corner of R, and used all four colours. On the other hand, in Figure 1, there are no intersections of three rectangles, we followed the no rectangle repeats rule, and only red and blue were used!

At this point it is clear that this observation must lead to a solution. Indeed, note that the first move (from the bottom left) must choose the bottom left rectangle, and either be vertical and right (red), or horizontal and left (blue). By analyzing the different T-intersection possibilities, we find that if the ant walks a blue or a red path, their next move is again blue or red, hence all future moves are as well. See Figure 3 for a demonstration of the possibilities after a move north or east (180 degree rotation gives the south and west cases).

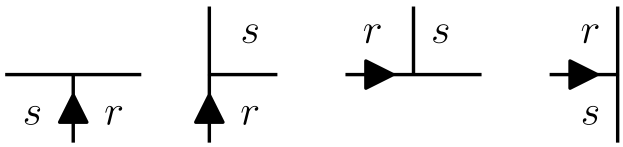


Figure 3 – Possible ant moves going north or east. The ant enters on rectangle r, and leaves on rectangle s. There are two possible choices, and each one remains blue or red.

Finally, we must go back to the original problem. Why can the ant not reach the top right corner? Well, the final move must be to choose the upper-right rectangle, which is left of the vertical move to make it there, and right of the horizontal move. These moves are coloured green and orange, which cannot occur after a blue or red move. This provides the contradiction we require!

A takeaway of this problem is that even very difficult questions can sometimes be solved by a series of small observations. Nowhere in this solution did we need to come up with a difficult and clever idea: it all fell apart naturally from a slow and methodical investigation into how the ant can travel.

OLYMPIAD CORNER

No. 435

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **November 15, 2025**.

OC741. A triple of positive real numbers (a, b, c) is called *mysterious* if

$$\sqrt{a^2+\frac{1}{a^2c^2}+2ab}+\sqrt{b^2+\frac{1}{b^2a^2}+2bc}+\sqrt{c^2+\frac{1}{c^2b^2}+2ca}=2(a+b+c).$$

Prove that if the triple (a, b, c) is mysterious, then the triple (c, b, a) is also mysterious.

OC742. Let $A \in \mathcal{M}_n(\mathbb{R})$ be an invertible matrix.

(a) Show that the matrix AA^T has real and positive eigenvalues.

(b) Suppose that there exist distinct positive integers p and q such that $(AA^T)^p = (A^TA)^q$. Prove that $A^T = A^{-1}$.

OC743. Let $(K, +, \cdot)$ be a division ring such that $x^2y = yx^2$ for all $x, y \in K$. Prove that $(K, +, \cdot)$ is a field.

OC744. Given a rectangle $ABCD$ and a point X lying inside it. The bisectors of angles DAX and CBX intersect at point P. Point Q satisfies the equalities $\angle QAP = \angle QBP = 90^\circ$. Prove that $PX = QX$.

OC745. Let n be a positive integer. Bolek draws $2n$ points on the plane, no two of which define a vertical or horizontal line. Then for each of these $2n$ points, Lolek draws two rays starting at that point, one of which is vertical and the other horizontal. Lolek wants to do this in such a way that the rays drawn divide the plane into as many areas as possible. Determine the largest integer k such that Lolek can obtain at least k areas regardless of the position of the points chosen by Bolek.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 novembre 2025**.

OC741. Un triplet de nombres réels positifs (a, b, c) est qualifié de *mystérieux* si

$$\sqrt{a^2 + \frac{1}{a^2c^2} + 2ab} + \sqrt{b^2 + \frac{1}{b^2a^2} + 2bc} + \sqrt{c^2 + \frac{1}{c^2b^2} + 2ca} = 2(a + b + c).$$

Prouvez que si le triplet (a, b, c) est mystérieux, alors le triplet (c, b, a) l'est aussi.

OC742. Soit $A \in \mathcal{M}_n(\mathbb{R})$ une matrice inversible.

(a) Montrez que la matrice AA^T a des valeurs propres réelles et positives.

(b) Supposons qu'il existe des entiers positifs distincts p et q tels que $(AA^T)^p = (A^TA)^q$. Montrez que $A^T = A^{-1}$.

OC743. Soit $(K, +, \cdot)$ un anneau à division tel que $x^2y = yx^2$ pour tout $x, y \in K$. Montrez que $(K, +, \cdot)$ est un corps.

OC744. Soit un rectangle $ABCD$ et un point X situé à l'intérieur de celui-ci. Les bissectrices des angles DAX et CBX se coupent au point P. Le point Q satisfait aux égalités $\angle QAP = \angle QBP = 90°$. Montrez que $PX = QX$.

OC745. Soit un entier positif n. Bolek trace $2n$ points sur le plan, dont aucun ne définit une ligne verticale ou horizontale. Ensuite, pour chacun de ces $2n$ points, Lolek trace deux rayons à partir de point, dont l'un est vertical et l'autre horizontal. Lolek veut faire cela de façon à ce que les rayons tracés divisent le plan en autant de zones que possible. Déterminez le plus grand entier k tel que Lolek puisse obtenir au moins k aires quelle que soit la position des points choisis par Bolek.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(2), p. 74–75.

OC716. Show that for every integer $n \geq 6$, there exists a convex hexagon which can be dissected into exactly n congruent triangles.

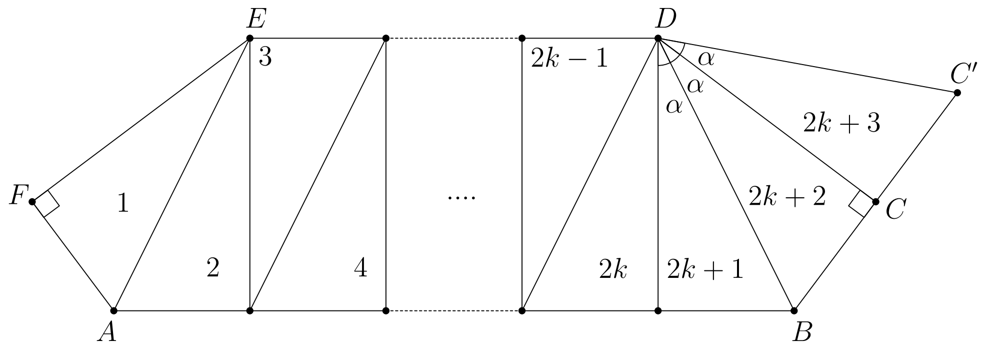
Originally from the 1990 Asian Pacific Mathematical Olympiad.

We received 4 submissions, all of which were correct and complete. We present the solution by Oliver Geupel.

We use n copies of a right triangle with legs of lengths 1 and 2. If α is the angle opposite to the leg of length 1, it holds

$$\tan\alpha = \frac{1}{2} < \frac{1}{\sqrt{3}} = \tan 30^\circ.$$

Hence $\alpha < 30^\circ$. Our construction is as follows.



For $n = 2k+2$, where $k \geq 2$, $ABCDEF$ is a hexagon with the desired property. For $n = 2k+3$, where $k \geq 2$, $ABC'DEF$ is a hexagon with the desired property. The convexity follows from $\alpha < 30^\circ$.

OC717. Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of the products of $a_1, a_2, \ldots, a_n$ taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \ldots a_n,$$

for $k = 1, 2, \ldots, n-1$.

Originally from the 1990 Asian Pacific Mathematical Olympiad.

We received 10 submissions, all of which were correct and complete. We present a typical solution.

The inequality is established as a consequence of the Cauchy-Schwarz inequality. Indeed, if $p_1, \ldots, p_m$ are positive real numbers, we have by the Cauchy-Schwarz inequality that

$$\begin{aligned}(p_1 + \cdots + p_m)&\left(\frac{1}{p_1} + \cdots + \frac{1}{p_m}\right)\\ &= \left((\sqrt{p_1})^2 + \cdots + (\sqrt{p_m})^2\right)\left(\left(\frac{1}{\sqrt{p_1}}\right)^2 + \cdots + \left(\frac{1}{\sqrt{p_m}}\right)^2\right)\\ &\geq \left(\sqrt{p_1} \times \frac{1}{\sqrt{p_1}} + \cdots + \left(\sqrt{p_m} \times \frac{1}{\sqrt{p_m}}\right)^2 = m^2.\end{aligned}$$

Let $m = \binom{n}{k}$, and let $p_1, \ldots, p_m$ be the products of $a_1, \ldots, a_n$, taken k at a time. Then,

$$S_k = p_1 + \cdots + p_m, \qquad S_{n-k} = a_1 \cdots a_n \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m}\right).$$

Consequently,

$$\begin{aligned}S_k S_{n-k} &= (p_1 + \cdots + p_m)\left(\frac{1}{p_1} + \cdots + \frac{1}{p_m}\right) a_1 \cdots a_n\\ &\geq m^2 a_1 \cdots a_n = \binom{n}{k}^2 a_1 \cdots a_n.\end{aligned}$$

OC718. In a triangle ABC, the median and the angle bisector at A meet the side BC at M and N respectively. The perpendicular at N to NA meets MA in Q and BA in P, and the perpendicular at P to BA meets AN produced in O. Prove that QO is perpendicular to BC.

Originally from the 2000 Asian Pacific Mathematical Olympiad.

We received 5 submissions, all of which were correct and complete. We present two solutions.

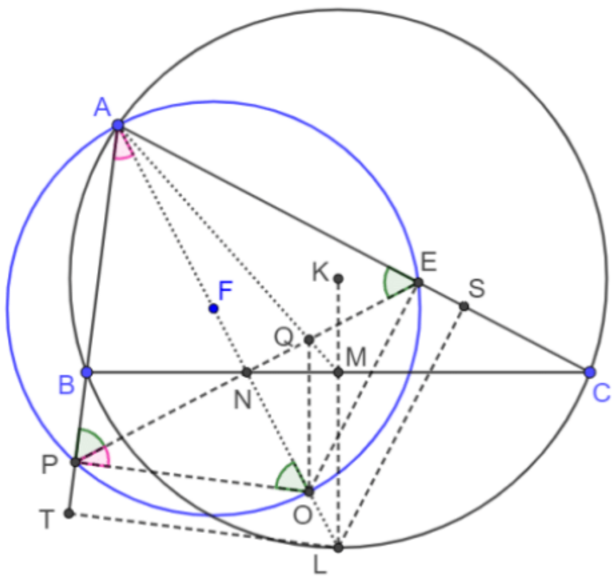
Solution 1, by Theo Koupelis.

Let the line NP intersect the line AC at E, and let the angle bisector at A intersect the circumcircle of the triangle at $L \not\equiv A$. If K is the circumcenter, then points K, M, L are collinear and $KL \perp BC$. Let T, S be the projections of L on the lines AB, AC, respectively. The points T, M, S are collinear because the line TS is the Simson line of point L. The points O, P, A, E are concyclic because $\angle OPE = \angle PAO = \angle OAE$, and thus $\angle OEA = 90°$. From the similar right triangles APO, ATL, and AEO, ASL we get

$$\frac{AP}{AT} = \frac{AO}{AL} = \frac{AE}{AS}. \quad (1)$$

Thus, $PE \parallel TS$, and therefore, from the similar triangles APQ, ATM we get

$$\frac{AP}{AT} = \frac{AQ}{AM}. \quad (2)$$



From (1) and (2) we get that the triangles AOQ, ALM are similar, and thus $OQ \parallel LM$. Therefore, $QO \perp BC$.

Solution 2, by Prithwijit De.

Place the triangle ABC in the coordinate plane in such a way that A coincides with the origin and the angle bisector AN falls along the x-axis. Let $N \equiv (t, 0)$ and $\angle BAN = \angle CAN = \theta$. Then the equation of line AB is

$$y = -x \tan\theta$$

and that of AC is

$$y = x \tan\theta.$$

Suppose $B \equiv (b, -b\tan\theta)$ and $C \equiv (c, c\tan\theta)$. The perpendicular at N to NA lies along the y-axis and so $Q \equiv (t, q)$ where q is obtained by solving

$$x = t$$

and

$$y = \left(\frac{(c-b)\tan\theta}{c+b}\right)x.$$

Thus $q = \left(\frac{t(c-b)}{c+b}\right)\tan\theta$. We also have $P \equiv (t, -t\tan\theta)$ and therefore the equation of the line ℓ through P and perpendicular to BA is

$$y + t\tan\theta = \cot\theta(x - t).$$

Since $O \equiv (k, 0)$ is the point of intersection of ℓ and NA we have

$$t \tan\theta = \cot\theta(k - t)$$

whence $k = t\sec^2\theta$. Hence $O \equiv (t\sec^2\theta, 0)$ and $Q \equiv \left(t, \dfrac{t(c-b)}{c+b}\tan\theta\right)$, and the slope of OQ is

$$m_{OQ} = \left(\frac{b-c}{b+c}\right)\cot\theta.$$

But the slope of BC is

$$m_{BC} = \left(\frac{c+b}{c-b}\right)\tan\theta.$$

Therefore

$$m_{OQ} \times m_{BC} = -1$$

and we conclude that QO is perpendicular to BC.

OC719. Let p be a prime number and $n_1, n_2, \ldots, n_p \in \mathbb{N}^*$, at least two of which are distinct. Denote by d the greatest common divisor of the numbers $n_1, n_2, \ldots, n_p$. Prove that the polynomial

$$f = \frac{X^{n_1} + X^{n_2} + \cdots + X^{n_p} - p}{X^d - 1}$$

is irreducible in $\mathbb{Q}[X]$.

Originally from the 2010 Selection test for Romanian team at IMO and BMO.

We received 3 submissions, 2 of which were correct and complete. We present the solution by Oliver Geupel.

We have

$$f(X) = \sum_{k=1}^{p} f_k(X)$$

where

$$f_k(X) = \frac{X^{n_k} - 1}{X^d - 1} \in \mathbb{Z}[X].$$

Hence

$$f(X) \in \mathbb{Z}[X].$$

Let

$$g(X) = X^{n_1} + X^{n_2} + \cdots + X^{n_p} - p.$$

For complex numbers z such that $|z| < 1$, we have

$$|g(z)| = |p - z^{n_1} - z^{n_2} - \cdots - z^{n_p}| > p - |z^{n_1}| - |z^{n_2}| - \cdots - |z^{n_p}| > 0.$$

Thus, the modulus of any root of $g(X)$, and of any root of $f(X)$ is at least 1.

Let ζ be a root of $g(X)$ such that $|\zeta| = 1$. Then

$$\zeta^{n_1} + \zeta^{n_2} + \cdots + \zeta^{n_p} = p,$$

which can be satisfied only if

$$\zeta^{n_1} = \zeta^{n_2} = \cdots = \zeta^{n_p} = 1.$$

It follows that ζ has the form $\zeta = e^{2k\pi/n}$ with appropriate coprime integers k and n, where n is a common divisor of $n_1, n_2, \ldots, n_p$. Hence, ζ is a root of the polynomial $X^d - 1$. Let $g'(X)$ denote the derivative of $g(X)$. Since

$$g'(\zeta) = \frac{n_1 + n_2 + \cdots + n_p}{\zeta} \neq 0,$$

ζ is a simple root of $g(X)$ and, therefore, is not a root of $f(X)$. We deduce that every root of $f(X)$ is of modulus greater than 1.

For the sake of obtaining a contradiction, suppose that $f(X)$ is reducible in $\mathbb{Q}[X]$. By Gauss's Lemma, $f(X)$ is then reducible in $\mathbb{Z}[X]$. Since the constant term of $f(X)$ is a prime, we find that $f(X)$ is divisible by a nonconstant integer polynomial with constant term 1, all of whose roots being of modulus greater than 1. But this contradicts Vieta's formula. Hence the result.

OC720. A cube is sub-divided into 27 rectangular prisms by planes parallel to its faces. If exactly two of these prisms are cubes, prove that the two have equal sides.

Originally from the 2009 Romanian Mathematical Olympiad, Final Round.

We received 3 submissions, 2 of which were correct and complete. We present the solution by Theo Koupelis.

Let a cube of side length a have a vertex at the origin of a Cartesian coordinate system, and let the three edges that are concurrent at this vertex be along the x, y, z axes. If n planes, parallel to a face of the cube, subdivide it, then the subdivision will result into $n+1$ rectangular prisms. Thus, n, m, k planes, parallel to the three faces of a cube, subdivide it in $(n+1)(m+1)(k+1)$ rectangular prisms. But

$$27 = 1 \times 1 \times 27, \quad \text{or} \quad 27 = 1 \times 3 \times 9, \quad \text{or} \quad 27 = 3 \times 3 \times 3.$$

The first case does not allow the presence of a cube among the 27 rectangular prisms because two of the sides of the prisms have lengths equal to a but the third side is less than a. Similarly, for the second case, one of the sides of the prisms is equal to a but the other two are smaller than a. Thus, we have $n = m = k = 2$.

Let ℓ_1, ℓ_2, ℓ_3 be the lengths of the rectangular prisms formed by the two planes perpendicular to the x-axis, where

$$\ell_1 + \ell_2 + \ell_3 = a;$$

let w_1, w_2, w_3 be the widths of the rectangular prisms formed by the two planes perpendicular to the y-axis, where

$$w_1 + w_2 + w_3 = a;$$

and let h_1, h_2, h_3 be the heights of the rectangular prisms formed by the two planes perpendicular to the z-axis, where

$$h_1 + h_2 + h_3 = a.$$

Without loss of generality, let one of the two rectangular prisms that are cubes have dimensions $\ell_1 = w_1 = h_1$. If the second cube belongs in the group of 19 rectangular prisms that have one of their dimensions equal to ℓ_1 or w_1 or h_1, then clearly the two cubes will have the same size. Let the second cube belong to the group of 8 rectangular prisms that have no dimension equal to ℓ_1 or w_1 or h_1; let its dimensions be $\ell_2 = w_2 = h_2$. Then there is a rectangular prism whose dimensions are

$$a - \ell_1 - \ell_2, \quad a - w_1 - w_2, \quad a - h_1 - h_2,$$

all of which are equal; thus, there is a third cube, which contradicts the statement of the problem. Therefore, if exactly two of the prisms are cubes, they have equal sides.

How To Write A *Crux* Article 2025 Update!

Almost ten years ago (November 2015, to be precise), I wrote a note for ***Crux*** detailing what a ***Crux*** article should be – and what it should not be. Some time later, we became an electronic-only journal. There are still some readers who miss the old purple paper problem poser, but there are advantages to the new regime. Not only is the new e-***Crux*** free to all the day that it comes out, but we have no printing costs. While that doesn't mean that we can publish everything people send us, it does give us a little more flexibility; so in 2019, I wrote an update to the note. Since then, some things have changed (notably the advancement of AI to the point where it's actually of use for some purposes) and some people have forgotten the 2019 note.

My first point in 2015 was that ***Crux*** is accessible. "***Crux*** is read by university professors and graduate students. It's also read by undergraduates, school teachers, school students, and amateurs whose day jobs have nothing to do with mathematics. We ask prospective writers to write for a very clever high school student. Assume high intelligence but not a lot of specialized knowledge." Still true! We stop about where the Putnam does – if your article needs much more than standard sophomore math, it's not for us.

Also in the interest of accessibility: the International Math Olympiad (IMO) community has developed a certain amount of private jargon of its own over the years, not understood by the wider mathematical community, often identifying a technique by the name of a trainer who popularized it. Not all our readers are IMO contestants either: please explain or (better) avoid such usages. (A note along the lines of "this technique was popularized in the IMO community by X. Wisehead around 2005" would be quite appropriate, however.)

My second point was that ***Crux*** is for problem solvers. "The sort of thing that might appear on a regional or national math contest." That's still the case, and we still want articles about tricks for problem solving. "Assume that most ***Crux*** readers know the standard tricks of the trade. Don't stop and explain mathematical induction or double counting unless you're explaining something new or unusual about those topics." A typical ***Crux*** article will be about one problem-solving technique. It will introduce it and explain it, including how the reader should recognize a problem where it might be useful. You should then give two or three worked demonstration problems with explanations. The first should be the easiest problem that the technique works on, even if it's familiar. The second should be a typical example, and you might add a third showing some virtuosity in application. Finish up with three or four unsolved problems given with at most short hints, for the reader to try.

Not everything to do with problem solving makes a good ***Crux*** article.

A new problem, with or without solution: send it to ***Crux*** as a problem (not an article).

A solution to a ***Crux*** problem: submit it to ***Crux*** as a solution (not an article).

One problem, many solutions: unlikely to be of much interest.

One technique that works for multiple problems: that's a ***Crux*** article.

I wrote that "***Crux*** is short. Our articles don't usually run more than five or six pages, and we're more likely to run it if it's three or four. That's the right length for the sort of thing we publish. We do like well-done illustrations, nice examples, and interesting asides – but please keep it all brief." This is still true – we have a bit more flexibility but our readers' free time is still the same.

Crux is still not a research journal. If your research paper is high level, send it to the *CMS Bulletin* or one of the many other great math journals out there. If it's elementary, try the *College Math Journal*, *Mathematics Magazine*, or the *American Mathematical Monthly*. But don't send it to us, that's not what we do. And in particular, we don't want "research" that other journals won't print. "You cannot trisect the angle, square the circle, or duplicate the cube using classical tools. If you understand Wantzel's classic 1837 proof, you won't try. If you don't understand it, you have not done your preparatory work and you have no business trying."

However, with our newfound freedom we have the luxury of running a few general-interest articles. I'm thinking about light and entertaining survey articles, the sort of thing that Martin Gardner used to write. The level will have to be just right – I anticipate rejecting some articles for being too difficult, others for being too elementary, more for being too specialized and still others for being common knowledge. I suspect that the successful writer will know a lot about the subject, and write much less than they know.

This raises the question of originality. We don't expect original research – indeed, like Wikipedia, we feel it's probably not right for us. But we don't want articles that just rephrase material already available from one source, with or without attribution. Your article should combine material from several sources, and say something that none of them says in quite the same way. (Young writers: please run it past a teacher or professor, explaining all your sources, before submitting.)

We absolutely do not want AI-written articles, even though AI software can now sometimes generate solutions to certain sorts of contest problem. In case you don't yet know how chatbots generate text, they (in effect) look over their notes from scanning millions of other sources and think "now, what would one of those clever human writers have put here?" without understanding why. We want articles direct from the clever human writers that will help readers understand problem solving techniques. If you really understand the technique you want to write about, you have some chance of writing such an article. If you have the professional confidence to believe that you're the right person to do it, you won't want a chatbot that doesn't understand the technique messing with your ideas. If you don't meet these strict criteria, that's okay: wait until you do have a good ***Crux*** idea. We'll still be here.

A note on format. Normally, we'd rather that you use LaTeX. Make sure that you know how to use things like theorem environments, `\label`, `\cite` and `\ref`. If you're a mathematician you already know LaTeX; if you're going to be one, you'll have to learn some day, so might as well do it now. If your article has very few equations in it you may submit it as a .doc or .docx file, or even flat text. (PDF is okay but be prepared to submit something editable on request.) Please don't use a theorem-proof-lemma form. It's not our style.

We love diagrams and pictures. Please make them neat; you can create them with Maple, Cinderella, Geogebra, or other software. Now we're paperless, colour and video are both fine, where appropriate.

Okay? Now you know what we're looking for. Write it – get somebody to check it over to make sure the math and English/French is correct – and send it to us.

Robert Dawson

Le Cerf-Volant

Robert Guilloteau

1 Enoncé

Emporte-moi mon cerf-volant !

Emporte-moi haut dans le vent !

Ces vers de Maurice Carême révèlent la fascination que ces oiseaux de toile exercent sur nos âmes d'enfant. Sublimes par leurs couleurs et leurs vols hésitant de papillon, ils le sont aussi par leurs formes géométriques. Des formes souvent très simples comme pour le cerf-volant qui nous intéresse ici.

Notre cerf-volant est découpé dans un triangle équilatéral $\triangle ABC$. Il a pour sommets le point A et les trois milieux M_a, M_b, M_c des côtés respectifs BC, CA, AB. Les côtés M_aM_b et M_aM_c sont des arcs de cercle de centres respectifs C et B. Sur le cerf-volant sont positionnés le cercle inscrit dans le cerf-volant et le centre O du triangle équilatéral. Il semble que le cercle inscrit passe par O. Il semble seulement car ce n'est pas le cas. Montrez le.

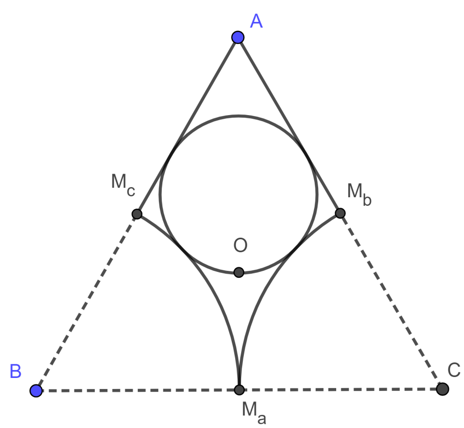


Figure 4 – Le Cerf-Volant.

Informé de cette "imperfection", le fabriquant entreprit de la corriger. Pour cela il décida de jouer sur le rayon des cercles de découpe centrés en B et C. Quelle est la valeur de ce rayon pour que le cercle inscrit dans le cerf-volant passe exactement par le centre du triangle ?

Il s'avère alors que la longueur du cerf-volant, mesurée de la tête à la queue, est réduite par rapport à la version initiale. Quelle est en pourcentage de la longueur initiale la réduction de taille ? Quelle est en pourcentage la réduction de voilure ?

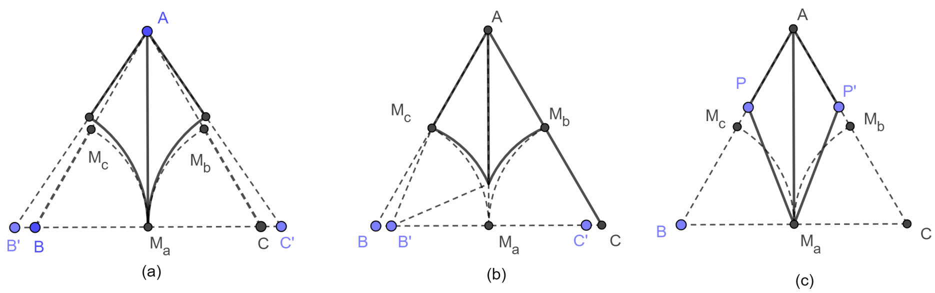


Figure 5 – Trois autres stratégies.

Il y aurait d'autres stratégies pour "remonter" le cercle inscrit de façon à le faire passer exactement par O. Nous vous en proposons ici trois et vous laissons libres de les étudier à votre guise.

La première (Figure 5 (a)) consisterait à élargir la base du triangle équilatéral. Certes, ce n'est plus alors qu'un triangle isocèle, mais l'opération fera remonter le cercle inscrit qui pourra alors passer par le centre O du triangle équilatéral initial. Si $B'C'$ est la base de ce triangle quelle serait la valeur de BB' ?

La deuxième (Figure 5 (b)) serait de déplacer les centres des cercles de découpe. Situés sur le segment BC, ces nouveaux centres B' et C' passent par les milieux M_b et M_c des côtés CA et AB. Quelle serait la valeur de BB' ?

La troisième (Figure 5 (c)) abandonnerait les cercles de découpe pour des droites. Soit P un point de AB. Le nouveau cerf-volant est représenté par le quadrilatère APM_aP' où P' est le symétrique de P par rapport à l'axe principal du cerf-volant. Une telle figure, formée par deux triangles isocèles rattachés par leur base commune, est d'ailleurs en géométrie nommée...cerf-volant. Quelle est la valeur de AP pour que le cercle inscrit dans ce quadrilatère passe par le centre du triangle équilatéral ?

Dans les trois cas, la variation de la longueur de l'axe principal et celle de la surface de la voilure pourront, le cas échéant, être recherchées.

Attention, certaines de ces configurations peuvent mener à des calculs passablement rébarbatifs.

2 Solution.

2.1 Le cercle inscrit ne passe pas par O.

Soit a le côté du triangle équilatéral.

Abaissons la hauteur AM_a. Elle passe par O et D, centre du cercle inscrit. La perpendiculaire menée de D sur AB coupe ce côté en T.

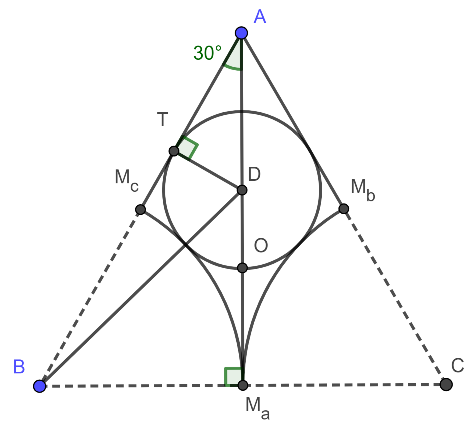


Figure 6 – Le Cerf-Volant - Solution.

Posons

$$x = DT \quad \text{et} \quad y = AD.$$

Dans le triangle rectangle $\triangle DTA$, le côté DT vaut

$$DT = DA \sin \angle TAD.$$

L'angle $\angle TAD$ vaut 30°. L'égalité précédente devient

$$x = y \sin 30^\circ = \frac{y}{2}.$$

Appliquons le théorème de Pythagore dans le triangle rectangle $\triangle DM_aB$.

$$DM_a^2 + BM_a^2 = BD^2.$$

Avec

$$DM_a = AM_a - AD,$$

$$AM_a = \frac{\sqrt{3}}{2}a,$$

$$BM_a = \frac{a}{2}$$

et DB somme des rayons des cercles de centres B et D

$$BD = \frac{a}{2} + x.$$

La relation de Pythagore devient

$$\left(\frac{\sqrt{3}}{2}a - y\right)^2 + \frac{a^2}{4} = \left(\frac{a}{2} + x\right)^2.$$

En remplaçant y par $2x$, en développant et en réarrangeant, nous obtenons

$$12x^2 - 4(2\sqrt{3}+1)ax + 3a^2 = 0,$$

dont les racines sont

$$x = \frac{2\sqrt{3}+1 \pm 2\sqrt{1+\sqrt{3}}}{6}a.$$

Une seule convient, car x doit être inférieur à a. Soit

$$x = \frac{2\sqrt{3}+1 - 2\sqrt{1+\sqrt{3}}}{6}a \approx 0.1931a$$

qui est le rayon du cercle cherché. Son centre D se trouve sur la hauteur AM_a à une distance y de A avec

$$y = 2x = \frac{2\sqrt{3}+1 - 2\sqrt{1+\sqrt{3}}}{3}a.$$

Si le cercle inscrit dans le cerf-volant passait par O nous aurions

$$DO = x.$$

Mais O, centre de gravité du triangle est aux deux-tiers de la hauteur AM_a de A et

$$AO = \frac{2}{3}\cdot\frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{3}a,$$

d'où

$$DO = AO - AD = \frac{\sqrt{3}}{3}a - y = \frac{2\sqrt{1+\sqrt{3}} - \sqrt{3} - 1}{3}a \approx 0,1912a.$$

DO est différent de x et le cercle inscrit dans le cerf-volant ne passe pas par O. Mais il s'en faut de vraiment peu, le rapport de DO à x valant approximativement

$$\frac{DO}{x} \approx 0.99.$$

2.2 Nouveau rayon de découpe.

Référons nous à la figure 6 et conservons les notations précédentes.

$$DT = x \quad \text{et} \quad AD = y.$$

Le centre O du triangle équilatéral étant légèrement à l'intérieur du cercle inscrit, il va falloir "remonter" ce dernier et pour cela augmenter légèrement la valeur du rayon de découpe. Nous continuons à noter D le centre de ce cercle même s'il est modifié par rapport à la situation initiale.

Le nouveau rayon de découpe des cercles centrés en B et C sera désigné par z.

Comme précédemment, le triangle de base étant toujours équilatéral, nous avons

$$y = 2x.$$

Le cercle inscrit passe maintenant par le centre du triangle, nous pouvons écrire

$$AO = AD + DO = y + x.$$

Mais AO est connu. Il vaut les deux tiers de la hauteur du triangle. D'où

$$x + y = a\frac{\sqrt{3}}{3}.$$

Comme y vaut le double de x nous obtenons immédiatement la position du centre du cercle inscrit ainsi que son rayon.

$$x = a\frac{\sqrt{3}}{9},$$

$$y = a\frac{2\sqrt{3}}{9}.$$

Pour obtenir la valeur de z appliquons le théorème de Pythagore au triangle $\triangle DM_aB$.

$$DM_a^2 + BM_a^2 = BD^2.$$

Avec

$$DM_a = DO + OM_a = x + a\frac{\sqrt{3}}{6},$$

$$BM_a = \frac{a}{2},$$

$$BD = z + x.$$

Soit

$$(x + z)^2 = \left(x + a\frac{\sqrt{3}}{6}\right)^2 + \frac{a^2}{4}.$$

Soit, en remplaçant x par sa valeur et après simplification

$$\left(a\frac{\sqrt{3}}{9} + z\right)^2 = \frac{13}{27}a^2.$$

Et finalement

$$z = \frac{(\sqrt{13} - 1)\sqrt{3}}{9}a \approx 0,5014a.$$

Avec ce rayon de découpe, légèrement supérieur au rayon précédent, le cercle inscrit dans le cerf-volant passe exactement par le centre du triangle.

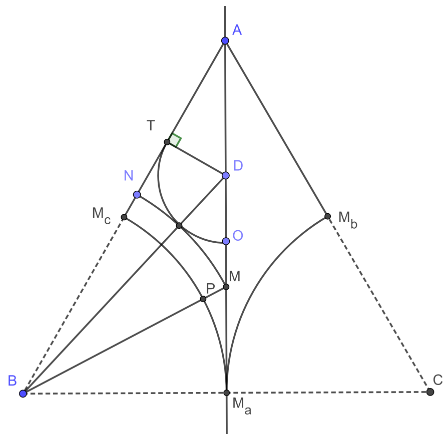


Figure 7 – Nouvelle découpe.

2.3 Réduction de longueur du cerf-volant.

La figure ci-dessus est volontairement déformée pour éviter que des points soient confondus.

Les deux cercles de découpe de rayon z, supérieur à la moitié du côté, vont ce couper en deux points sur la hauteur $ADOM_a$. Appelons M le point le plus proche de A et appliquons le théorème de Pythagore au triangle rectangle $\triangle MM_aB$:

$$MM_a^2 = BM^2 - BM_a^2 = z^2 - \frac{a^2}{4}.$$

Remplaçons z par sa valeur

$$MM_a^2 = \frac{29 - 8\sqrt{13}}{108}a^2$$

et

$$MM_a = \frac{\sqrt{29 - 8\sqrt{13}}}{6\sqrt{3}}a.$$

Le taux de réduction de la longueur du cerf-volant vaut alors

$$\frac{MM_a}{AM_a} = \frac{\sqrt{29 - 8\sqrt{13}}}{9} \approx 4,38\%.$$

2.4 Modification de la surface de la voilure.

La réduction de surface du cerf-volant, pour la moitié gauche de celui-ci, est égale à la surface du quadrilatère curviligne M_cM_aMN où N est l'intersection du nouveau cercle de découpe avec le côté AB. Les deux côtés MN et M_cM_a sont des arcs de cercles centrés en B. L'arc $\widehat{M_cM_a}$ est sous-tendu par un angle au centre de 60°, l'arc $\widehat{MN}$ par un angle au centre égal à $60° - \theta$ où θ a pour tangente

$$\tan\theta = \frac{MM_a}{BM_a} = \frac{\sqrt{29 - 8\sqrt{13}}}{3\sqrt{3}} \quad \Leftrightarrow \quad \tan\theta \approx 0,0759 \quad \Leftrightarrow \quad \theta \approx 0,0758 \text{ radian.}$$

La surface de M_cM_aMN peut s'écrire

$$S(M_cM_aMN) = S(BMN) + S(BMM_a) - S(BM_aM_c).$$

BMN est un secteur de cercle de rayon z et d'angle égal à $\frac{\pi}{3} - \theta$ et

$$S(BMN) = \pi z^2 \frac{\pi - 3\theta}{6\pi}.$$

BMM_a est un triangle

$$S(BMM_a) = \frac{1}{2} MM_a \cdot BM_a = \frac{a}{4}\sqrt{z^2 - \frac{a^2}{4}}.$$

BM_aM_c est un secteur de cercle de rayon $\frac{a}{2}$ et d'angle égal à $\frac{\pi}{3}$

$$S(BM_aM_c) = \pi \frac{a^2}{4}\frac{1}{6}.$$

La réduction totale de voilure vaut donc

$$\Delta S = 2S(M_cM_aMN) = 2\left(z^2 \frac{\pi - 3\theta}{6} + \frac{a}{4}\sqrt{z^2 - \frac{a^2}{4}} - \frac{\pi a^2}{24}\right).$$

Nous connaissons la valeur de z

$$z \approx 0,5014a = k \cdot a$$

et

$$\Delta S = \frac{a^2}{12}(4k^2(\pi - 3\theta) + 3\sqrt{4k^2 - 1} - \pi),$$

$$\Delta S \approx 0,0011a^2.$$

La surface initiale du cerf-volant, égale à la surface du triangle équilatéral diminuée de deux fois la surface d'un secteur de cercle de rayon $\frac{a}{2}$ et d'angle 60° valait

$$S = \left(\frac{\sqrt{3}}{4} - \frac{\pi}{12}\right) a^2,$$

$$S \approx 0,1712a^2.$$

La réduction de voilure est donc égale à

$$\frac{\Delta S}{S} \approx 0,0064.$$

Soit une réduction de surface de $0,64\%$.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 15, 2025**.

5061. *Proposed by Nguyen Van Huyen.*

Consider the polynomial $f(x) = x^4 - ax^3 + 6x^2 - bx + c$. Suppose that $f(x)$ has four distinct real roots. Prove that

$$a^3b + 4b^2 + 256 \leqslant 12a(a + 2b).$$

5062. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with $AC > BC$. The midpoint of AB is M, the orthocenter of ΔABC is H, and the feet of the altitudes from A, B, C are D, E, F, respectively. Let X be the point of intersection of AB and ED. If O is the circumcenter of ΔCMX, then prove that

$$\overrightarrow{OH} = \frac{2(\overrightarrow{OC} + \overrightarrow{OX}) + \overrightarrow{OA} + \overrightarrow{OB}}{2}.$$

5063. *Proposed by Bing Jian.*

Given a circle ω with center O and a line ℓ not tangent to ω, let m be the line passing through O and perpendicular to ℓ, and denote by A one of the points where m intersects the circle. For points P and Q on ℓ, let P^* and Q^* be their respective reflections in the mirror m, and let P' and Q' be the second points of intersection of the lines P^*A and Q^*A with ω. Prove that the cross-joins PQ' and QP' intersect on ω.

5064. *Proposed by Michel Bataille.*

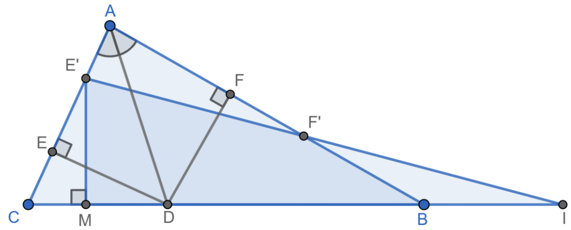
Let the sequence $(a_n)_{n\geq 1}$ be defined by $a_1 = 0$ and $a_{n+1} = a_n + \ln(2^n e^{a_n} - 1)$ for all $n \geq 1$. Evaluate

$$\ell = \lim_{n\to\infty} \frac{a_n}{2^n} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{a_n - 2^n\ell}{n}.$$

5065. *Proposed by Yagub Aliyev.*

Let ABC be a triangle with acute angles at the vertices B and C, such that $\angle B < \angle C$. The angle bisector AD of the triangle ABC is drawn. Let DE and

DF be perpendiculars to the sides AC and AB, respectively. Let E' and F' be points on the sides AC and AB, respectively, such that $AE = CE'$ and $AF = BF'$. Let $E'M$ be perpendicular to the side BC. Prove that $E'M + E'I > AB + AC$.



5066. *Proposed by Tatsunori Irie.*

Let n be a positive integer. Initially, n stones – each coloured either white or black – are arranged in a single row. The game is played by repeatedly performing the following operation:

- Randomly select two white stones that are not adjacent (i.e. if two stones appear consecutively, they cannot be selected as a pair).
- Reverse the colour (i.e. switch from white to black or black to white) of every stone located between the two selected white stones.
- Finally, change the colours of the two chosen white stones to black.

The game terminates when no pair of white stones satisfying the above condition (that is, non-adjacent) can be selected.

Prove that, regardless of the initial configuration of the stones and irrespective of the order and combination in which the valid pairs of white stones are chosen, the game always terminates.

5067. *Proposed by Paul Bracken.*

Prove that

$$\sum_{n=1}^{\infty} \frac{4^n}{(2n-1)^2(4n+1)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{16}{9} \cdot \sqrt{2} - \frac{20}{9}.$$

5068. *Proposed by Nguyen Viet Hung.*

Prove that in any triangle ABC,

$$\frac{4R}{r} \geq \left(\frac{1}{r_a} + \frac{1}{r_b}\right) (\sqrt{r_a} + \sqrt{r_b})^2.$$

When does the equality happen?

5069. *Proposed by Michael Friday.*

Let ABC be a triangle in which $B - A = 90°$. Let M and S be the feet of the median and symmedian, respectively, from vertex C. Prove that triangles ABC and CMS have the same orthocenter, and the circumcircle of CMS is internally tangent to the circumcircle of ABC at C.

5070. *Proposed by Vasile Cîrtoaje.*

Prove that 2 is the largest positive value of the constant k such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - 4 \geq k(a + b + c + d - 4)$$

for any positive real numbers a, b, c, d with at most one of them less than 1 and $ab + bc + cd + da = 4$.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 novembre 2025**.

5061. *Soumis par Nguyen Van Huyen.*

Soit le polynôme $f(x) = x^4 - ax^3 + 6x^2 - bx + c$. Supposons que $f(x)$ ait quatre racines réelles distinctes. Montrez que

$$a^3b + 4b^2 + 256 \leqslant 12a(a + 2b).$$

5062. *Soumis par Mihaela Berindeanu.*

Soit ABC un triangle aigu avec $AC > BC$. Le milieu de AB est M, l'orthocentre de ΔABC est H, et les pieds des hauteurs depuis A, B et C sont respectivement D, E et F. Soit X le point d'intersection de AB et ED. Si O est le centre du cercle circonscrit à ΔCMX, montrez que

$$\overrightarrow{OH} = \frac{2(\overrightarrow{OC} + \overrightarrow{OX}) + \overrightarrow{OA} + \overrightarrow{OB}}{2}.$$

5063. *Soumis par Bing Jian.*

Soit un cercle ω de centre O et une droite ℓ non tangente à ω. Soit m la droite passant par O et perpendiculaire à ℓ, et soit A l'un des points d'intersection de m avec le cercle. Étant donné des points P et Q sur ℓ, soit P^* et Q^* leurs reflets respectifs dans le miroir m, et soit P' et Q' les seconds points d'intersection des droites P^*A et Q^*A avec ω. Montrez que les droites PQ' et QP' se coupent sur ω.

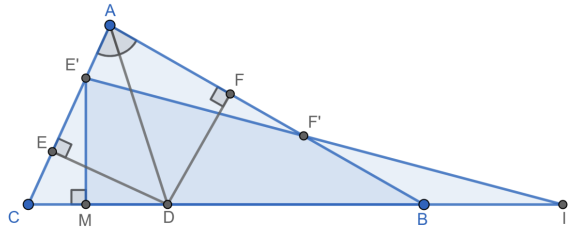
5064. *Soumis par Michel Bataille.*

Soit $(a_n)_{n\geq 1}$ la suite définie par $a_1 = 0$ et $a_{n+1} = a_n + \ln(2^n e^{a_n} - 1)$ pour tout $n \geq 1$. Évaluez l'expression suivante

$$\ell = \lim_{n\to\infty} \frac{a_n}{2^n} \qquad \text{et} \qquad \lim_{n\to\infty} \frac{a_n - 2^n\ell}{n}.$$

5065. *Soumis par Yagub Aliyev.*

Soit ABC un triangle avec des angles aigus aux sommets B et C, tels que $\angle B < \angle C$. La bissectrice AD du triangle ABC est tracée. Soient DE et DF les perpendiculaires aux côtés AC et AB, respectivement. Soient E' et F' des points sur les côtés AC et AB, respectivement, tels que $AE = CE'$ et $AF = BF'$. Soit $E'M$ la perpendiculaire au côté BC. Montrez que $E'M + E'I > AB + AC$.



5066. *Soumis par Tatsunori Irie.*

Soit n un nombre entier positif. Au départ, n pierres, chacune de couleur blanche ou noire, sont disposées en une seule rangée. Le jeu consiste à répéter l'opération suivante :

- Sélectionnez au hasard deux pierres blanches qui ne sont pas adjacentes (c'est-à-dire que si deux pierres apparaissent consécutivement, elles ne peuvent pas être sélectionnées comme paire).
- Inversez les couleurs (c'est-à-dire faites passer du blanc au noir ou du noir au blanc) de toutes les pierres situées entre les deux pierres blanches sélectionnées.
- Enfin, changez la couleur des deux pierres blanches choisies en noir.

Le jeu se termine lorsqu'il n'est plus possible de sélectionner une paire de pierres blanches satisfaisant à la condition ci-dessus (c'est-à-dire non adjacentes).

Montrez que, quelle que soit la configuration initiale des pierres et indépendamment de l'ordre et de la combinaison dans lesquels les paires valides de pierres blanches sont choisies, le jeu se termine toujours.

5067. *Soumis par Paul Bracken.*

Montrez que

$$\sum_{n=1}^{\infty} \frac{4^n}{(2n-1)^2(4n+1)} \frac{\binom{2n}{n}}{\binom{4n}{2n}} = \frac{16}{9} \cdot \sqrt{2} - \frac{20}{9}.$$

5068. *Soumis par Nguyen Viet Hung.*

Montrez que dans tout triangle ABC,

$$\frac{4R}{r} \geq \left(\frac{1}{r_a} + \frac{1}{r_b}\right) (\sqrt{r_a} + \sqrt{r_b})^2.$$

Quand a-t-on égalité ?

5069. *Soumis par Michael Friday.*

Soit ABC un triangle dans lequel $B - A = 90°$. Soient M et S respectivement les pieds de la médiane et de la symédiane issues du sommet C. Montrez que les triangles ABC et CMS ont le même orthocentre, et que le cercle circonscrit à CMS est tangent intérieurement au cercle circonscrit à ABC en C.

5070. *Soumis par Vasile Cîrtoaje.*

Montrez que 2 est la plus grande valeur positive de la constante k telle que

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - 4 \geq k(a + b + c + d - 4)$$

pour tout ensemble de nombres réels positifs a, b, c et d dont au plus un d'entre eux est inférieur à 1 et $ab + bc + cd + da = 4$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2025: 51(2), p. 81–84.

5011. *Proposed by Fedor Petrov and Max A. Alekseyev.*

Let a, b, c be the zeros of a cubic polynomial, and let α, β be the zeroes of its derivative. Prove that

$$\frac{(a-b)^4+(a-c)^4+(b-c)^4}{(\alpha-\beta)^4}$$

is a constant that does not depend on the polynomial.

We received 37 submissions of which 35 were correct and complete. The incorrect solutions contained mistakes in the calculations. We present the solution by the Missouri State University Problem Solving Group.

The polynomial

$$f(x)=x^3-(a+b+c)x^3+(ab+bc+ca)x-abc$$

has a, b, and c as its zeros. We have

$$f'(x)=3x^2-2(a+b+c)x+(ab+bc+ca).$$

Its zeros are

$$\alpha,\beta=\frac{a+b+c\pm\sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}.$$

Therefore,

$$\alpha-\beta=\frac{2\sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}.$$

We have

$$(\alpha-\beta)^4=\frac{16}{81}\left(a^4+b^4+c^4-2(a^3b+ab^3+a^3c+ac^3+b^3c+bc^3)+3(a^2b^2+a^2c^2+b^2c^2)\right)$$

and

$$\begin{aligned}&(a-b)^4+(a-c)^4+(b-c)^4\\&=2(a^4+b^4+c^4)-4(a^3b+ab^3+a^3c+ac^3+b^3c+bc^3)+6(a^2b^2+a^2c^2+b^2c^2).\end{aligned}$$

Hence,

$$\frac{(a-b)^4+(a-c)^4+(b-c)^4}{(\alpha-\beta)^4}=\frac{81}{8}.$$

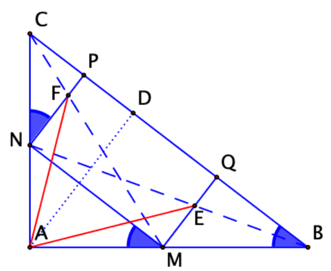
5012. *Proposed by Tran Quang Hung and Nguyen Minh Ha.*

Given a right triangle ABC with $\angle A$ being the right angle. Construct a rectangle $MNPQ$ such that M and N lie on AB and AC, respectively, while P and Q lie on BC. Let E be the intersection of MQ and BN, and F be the intersection of NP and CM. Prove that $AE = AF$ if and only if $AB = AC$ or $MNPQ$ is a square.

All but one of the 11 submissions were correct; we feature the solution by Corneliu Manescu-Avram.

Choose a cartesian system of coordinates with A at the origin, AB of unit length along the positive x-axis, and AC along the positive y-axis. We have

$$A = (0, 0), \quad B = (1, 0), \quad C = (0, c), c > 0.$$



By definition,

$$AB = AC \text{ if and only if } c = 1. \tag{1}$$

We begin by proving

$$\text{The rectangle } MNPQ \text{ is a square if and only if } \quad m = \frac{c}{c^2 + c + 1}. \tag{2}$$

Indeed, the triangles AMN and ABC are similar (since $MN||BC$), hence

$$\frac{MN}{BC} = \frac{AM}{AB} = m,$$

so that

$$MN = am,$$

where $a = BC = \sqrt{c^2 + 1}$. The triangles QBM and ABC are also similar so that

$$\frac{MQ}{AC} = \frac{MB}{BC},$$

whence

$$MQ = \frac{c(1 - m)}{a}.$$

Finally, the rectangle $MNPQ$ is a square if and only if $MN = MQ$; that is,

$$am = \frac{c(1 - m)}{a}$$

if and only if

$$m = \frac{c}{a^2 + c} = \frac{c}{c^2 + c + 1},$$

which proves (2).

Lines MQ and BN satisfy the equations $y = \frac{1}{c} \cdot (x - m)$ and $x + \frac{y}{cm} = 1$; their intersection is therefore

$$E = \left(\frac{m(c^2 + 1)}{c^2 m + 1}, \frac{cm(1 - m)}{c^2 m + 1} \right).$$

Lines NP and CM satisfy the equations $y - cm = \frac{1}{c} \cdot x$ and $\frac{x}{m} + \frac{y}{c} = 1$; their intersection is therefore

$$F = \left(\frac{c^2 m(1 - m))}{c^2 + m}, \frac{cm(c^2 + 1)}{c^2 + m} \right).$$

The following statements are equivalent:

$$\begin{aligned}
&AE = AF \\
&\iff x_E^2 + y_E^2 = x_F^2 + y_F^2 \\
&\iff \left(\frac{m}{c^2 m + 1} \right)^2 [(c^2 + 1)^2 + c^2(1 - m)^2] = \left(\frac{cm}{c^2 + m} \right)^2 [c^2(1 - m)^2 + (c^2 + 1)^2] \\
&\iff \frac{m}{c^2 m + 1} = \frac{cm}{c^2 + m} \\
&\iff (c - 1) \cdot [(c^2 + c + 1)m - c] = 0;
\end{aligned}$$

consequently (1) or (2) holds, as claimed.

Editor's comments. One correspondent based his argument on the rather surprising observation that $\angle MAE = \angle FAN$. This follows easily from our featured solution because the tangents of these angles are just the ratios of the x- and y-coordinates of the points E and F. Unfortunately, the correspondent's argument was flawed. Perhaps the reader sees how to use the observation to complete an alternative proof of the main result.

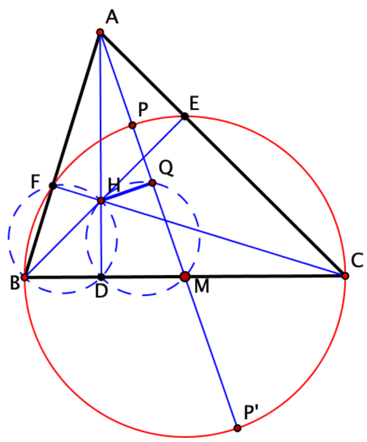
5013. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Let AM be the median from the vertex A of a triangle ABC to the midpoint M of BC, and let P be the point of that median (extended, if necessary beyond A) for which $PM = BM = CM$, while Q is the projection of the orthocenter of the triangle on that median. Prove that

$$\frac{AP}{PM} = \frac{PQ}{QM}.$$

All but one of the 15 submissions were correct; we feature the solution by Michal Adamaszek, with some of his notation modified by the editor.

Let AD, BE, CF be the altitudes and H the orthocenter. The circle with diameter BC and center M passes through B, C, E, F. It intersects the line AM in two points, call them P and P'. Note that the proposer intended for P to lie on the same side of the line BC as A, but we shall see that it makes no difference which of the two points is labelled P — the desired result holds when P' appears in place of P.



We shall prove that, equivantly, $AP \cdot QM = PM \cdot PQ$. To avoid the need for considering special cases, we shall make use of directed distances.

Notation. For any two points X and Y, the pair XY represents directed distance; that is, XY is the distance directed from X to Y, while $YX = -XY$ is the distance directed from Y to X. Thus, for any three points X, Y, Z on a line we have

$$XY + YX = 0, \quad XY + YZ = XZ, \quad YZ = XZ - XY.$$

The quadrilaterals $DHQM$, $BDHF$ have opposite right angles, hence they have a circumcircle. By comparing the powers of point A with respect to circles $DHQM$, $BDHF$ and $BPFP'$ we arrive at:

$$AQ \cdot AM = AH \cdot AD = AF \cdot AB = AP \cdot AP' = AP \cdot (AM + MP') = AP \cdot (AM + PM).$$

From this,

$$AP \cdot PM = AQ \cdot AM - AP \cdot AM = AM \cdot (AQ - AP) = AM \cdot (AQ + PA) = AM \cdot PQ.$$

Since $PM = PQ + QM$ and $AM = AP + PM$, we have

$$AP \cdot (PQ + QM) = (AP + PM) \cdot PQ,$$

and so,

$$AP \cdot QM = PM \cdot PQ,$$

as required.

Editor's comments. Note that our result is a theorem of inversive geometry (where here, one point of the inversive plane has been designated as the point at infinity of the Euclidean plane). Other submitted solutions made use of properties of the Euclidean model such as the length of the median AM, the Law of Cosines, and various properties of the 9-point circle $EFDM$, but none of those extraneous results lead to an easier proof.

5014. *Proposed by Michel Bataille.*

Let n be an integer with $n \geq 2$ and $A \in \mathcal{M}_n(\mathbb{C})$ of rank $r \geq 1$.

a) Show that $A = XY$ for some pair $(X, Y) \in \mathcal{M}_{n,r}(\mathbb{C}) \times \mathcal{M}_{r,n}(\mathbb{C})$ such that $\text{rank}(X) = \text{rank}(Y) = r$.

b) If $A^2 = A$, prove that the product YX is independent of the pair (X, Y).

We received 3 solutions. We present the solution by Pranjal Agrawal.

(a) Since $\text{rank}(A) = r$, the column space $\mathcal{C}(A)$ is an r-dimensional subspace of $\mathbb{C}^n$. Choose any ordered basis for $\mathcal{C}(A)$ and form the $n \times r$ matrix X whose columns are these basis vectors. Then the columns of A are contained in $\mathcal{C}(A)$, so there exists a (unique) linear mapping

$$Y : \mathbb{C}^n \to \mathbb{C}^r$$

such that

$$A = X\,Y.$$

In other words, we may write

$$A = XY, \quad X \in M_{n,r}(\mathbb{C}), \quad Y \in M_{r,n}(\mathbb{C}).$$

Since the columns of X are linearly independent, $\text{rank}(X) = r$. Moreover, since $A = XY$ has rank r, we must have $\text{rank}(Y) = r$ as well.

(b) Now assume that A is idempotent, i.e. $A^2 = A$. Suppose that

$$A = XY$$

where

$$X \in M_{n,r}(\mathbb{C}), \quad Y \in M_{r,n}(\mathbb{C}), \quad \text{rank}(X) = \text{rank}(Y) = r.$$

We show that

$$YX = I_r.$$

Let $v \in \mathbb{C}^r$ be arbitrary. Then $Xv \in \mathcal{C}(A)$ and, since A is idempotent,

$$A(Xv) = Xv.$$

But $A = XY$, so

$$XY(Xv) = Xv.$$

Since X has full column rank, it is injective; hence,

$$YXv = v \quad \text{for every } v \in \mathbb{C}^r.$$

This proves that

$$YX = I_r.$$

In particular, the product YX does not depend on the particular factorization $A = XY$.

5015. *Proposed by Mohammad Bakkar.*

Let $\mathcal{C}, \mathcal{P}$ be the sets of circles and points in the plane, respectively. Find all functions $f : \mathcal{C} \to \mathcal{P}$ such that for all nonconcentric circles ω_1, ω_2 from $\mathcal{C}$, we have $f(\omega_1), f(\omega_2), M_{\omega_1,\omega_2}$ are collinear, where M_{ω_1,ω_2} is the intersection of the radical axis of the two circles with the line that passes through their centers.

We received two submissions for this unusual, but interesting problem. Only the proposer's was correct and complete, so we present his solution (modified by the editor).

We shall see that either f takes every circle to its center, or its range consists of a single point.

The function that takes every circle to its center satisfies the definition of f. Let us therefore assume that there exists a circle ω with center O whose image under f is a point Ω that is different from O.

Notation. Circles will be denoted by lowercase letters; their image under f will be denoted by the corresponding capital letter.

Let γ be a circle with center Ω that is tangent to ω at a point B. Let $\Gamma = f(\gamma)$. It is easy to see that

$$\Gamma \in O\Omega$$

as follows: Since the radical axis of the two circles is their common tangent, we must have $M_{\omega,\gamma} = B \in O\Omega$; by the definition of f, the image Γ of γ must also lie on $B\Gamma$, which is just another name for the line $O\Omega$. We want to show that, in fact, $\Gamma = \Omega$. Assume to the contrary that $\Gamma \neq \Omega$.

Let ℓ be the line perpendicular to $O\Omega$ at Ω, and O_1 and O_2 be the points where ℓ intersects γ (as in the Figure 1). Consider the circles c_1 with center O_1, radius $O_1\Omega$, and its mirror image $c_2(O_2, O_2\Omega)$. Then $M_1 := M_{c_1,\gamma}$ lies on ℓ (in fact, it is the midpoint of $O_1\Omega$), while $M_1' := M_{c_1,\omega}$ lies on O_1O; in particular, it does not lie on ℓ. Consequently,

$$C_1 = f(c_1) = \Gamma M_1 \cap \Omega M_1'$$

is a well-defined point that lies on neither ℓ nor $O\Omega$. Observe that $C_2 = f(c_2)$ must be the reflection of C_1 in the line $O\Omega$, so that the line C_1C_2 is parallel to and distinct from ℓ. On the other hand, $M_{c_1,c_2} = \Omega$, which (by the definition of the function f) forces C_1C_2 to contain Ω, which contradicts our assumption that

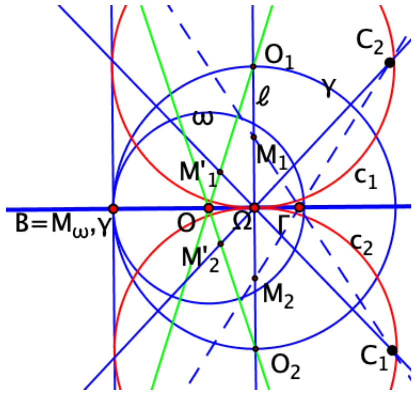


Figure 1 – The configuration under the assumption that $\Gamma \neq \Omega$.

$\Gamma \neq \Omega$. In other words, we must have $\Gamma = \Omega$. In summary, under the assumption that there exists a circle ω whose center is O such that $\Omega = f(\omega) \neq O$, we have constructed a second circle γ for which $f(\gamma) = \Omega$. We will now show in four steps that for every circle c, $f(c) = \Omega$. As before, ℓ is the line that is perpendicular to $O\Omega$ at Ω.

Step 1. Every circle c with center $A \neq \Omega$ on ℓ is mapped by f to Ω.

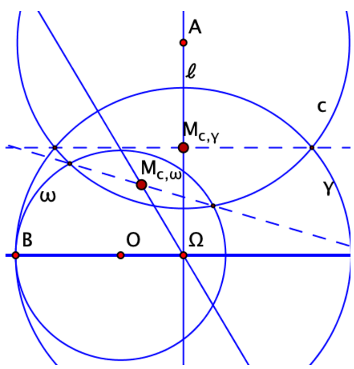


Figure 2 – Step 1.

Because $M_{c,\gamma} \in \ell$, we have $f(c) \in \Omega M_{c,\gamma} = \ell$; however, we also have $f(c) \in \Omega M_{c,\omega}$. But, $M_{c,\omega} \notin \ell$, whence we conclude that $f(c) = \Omega M_{c,\gamma} \cap \Omega M_{c,\omega} = \Omega$, as claimed.

Step 2. Every circle c with center Ω that is different from γ is taken by f to Ω.

Note that $f(c)$ must lie on the line joining $f(\omega) = \Omega$ to $M_{\omega,c}$, namely $O\Omega$. Furthermore, if we take any circle c' with center on ℓ different from Ω (as in Step 1), we find that $f(c) \in \Omega M_{c,c'} = \ell$. We conclude that $f(c) = O\Omega \cap \ell = \Omega$, as claimed.

Step 3. Every circle c with center A on line $O\Omega$ is mapped by f to Ω.

Since Step 2 has already dealt with $A = \Omega$, assume here that $A \neq \Omega$. We know that $f(c)$ is on the line $M_{c,\gamma}f(\gamma) = O\Omega$. Define the circle c' to be tangent to c and to have its center different from Ω on ℓ. Then $M_{c,c'}$ is the point of tangency. By Step 1 we know that $f(c') = \Omega$; consequently, $f(c) \in M_{c,c'}\Omega$ and, therefore, $f(c) = M_{c,c'}\Omega \cap O\Omega = \Omega$, as claimed.

Step 4. Every circle c whose center lies on neither ℓ nor $O\Omega$ is taken by f to Ω.

Define the circles c_1 and c_2 with centers on $O\Omega$ and ℓ, respectively, to be tangent to c at M_{c,c_1} and M_{c,c_2}, respectively, chosen in such a way that the line joining the two tangency points does not contain Ω. From the previous steps we know that $f(c)$ is on both lines

$$M_{c,c_1}f(c_1) = M_{c,c_1}\Omega \quad \text{and} \quad M_{c,c_2}f(c_2) = M_{c,c_2}\Omega;$$

therefore, $f(c) = M_{c,c_1}\Omega \cap M_{c,c_2}\Omega = \Omega$, as claimed.

Thus, we have proved that if there exists just one circle, ω, that is taken by f to a point Ω that is different from its center, then f takes every circle to Ω.

5016. *Proposed by Yagub Aliyev.*

Let $n > 2$ and polynomial $p_n(t)$ be defined by

$$(t-1)^2 p_n(t) = ((n-1)-(n-2)t)(1-nt^{n-1}+(n-1)t^n).$$

Prove that $p_n(t)$ has exactly one maximum point in interval $[0, +\infty)$.

We received 11 solutions, all of which were correct and complete. We present two of them, one of which does not use Descartes' rule of signs and the other one does.

Solution 1, by Michal Adamaszek.

Starting from

$$\sum_{i=0}^{n-1} t^i = \frac{t^n - 1}{t-1}$$

and differentiating we get

$$\sum_{i=0}^{n-2}(i+1)t^i = \frac{nt^{n-1}(t-1)-(t^n-1)}{(t-1)^2} = \frac{(n-1)t^n - nt^{n-1}+1}{(t-1)^2} = \frac{p_n(t)}{(n-1)-(n-2)t}$$

hence

$$p_n(t) = -(n-2)(n-1)t^{n-1} + \sum_{i=0}^{n-2}(i+n-1)t^i$$

and in particular the derivative $p_n'(t)$ has the form

$$p_n'(t) = -a_{n-2}t^{n-2} + \sum_{i=0}^{n-3} a_i t^i = t^{n-2}\left(\sum_{i=1}^{n-2}\frac{a_{n-2-i}}{t^i} - a_{n-2}\right)$$

with all $a_i > 0$. The expression in the bracket is strictly decreasing from $+\infty$ when $t \to 0$ down to $-a_{n-2} < 0$ when $t \to +\infty$. Therefore $p'_n(t)$ has a unique zero, is positive to the left of it, and negative to the right, which ends the proof.

Solution 2, by Didier Pinchon.

For an integer $n > 2$, let $q_n(t)$ be the polynomial defined by

$$(t-1)^2 q_n(t) = 1 - nt^{n-1} + (n-1)t^n.$$

For $n = 3$, $(t-1)^2 q_3(t) = 1 - 3t^2 + 2t^3 = (t-1)^2(1+2t)$, and thus $q_3(t) = 1 + 2t$. As, for $n > 3$,

$$q_n(t) - q_{n-1}(t) = \frac{(n-1)t^n - nt^{n-1} - (n-2)t^{n-1} + (n-1)t^{n-2}}{(t-1)^2} = (n-1)t^{n-2},$$

it follows by recurrence that

$$q_n(t) = \sum_{k=0}^{n-2}(k+1)t^k \text{ and } p_n(t) = ((n-1)-(n-2)t)\sum_{k=0}^{n-2}(k+1)t^k,\ n > 2.$$

Therefore

$$\begin{aligned} p'_n(t) &= (n-1)\sum_{k=1}^{n-2} k(k+1)t^{k-1} - (n-2)\sum_{k=0}^{n-2}(k+1)^2 t^k - (n-2)\sum_{k=1}^{n-2} k(k+1)t^k \\ &= \sum_{k=0}^{n-3}(k+1)(k+n)t^k - (n-2)(n-1)^2 t^{n-2}. \end{aligned}$$

The polynomial $p'_n(t)$, of degree $n-2$ in t, has only one change of sign in its ordered coefficients. Therefore by Descartes's rule of signs, it has only one root t_0 in $(0, +\infty)$.

Because $p'_n(0) = n > 0$, $p_n(t)$ is increasing in an interval $[0, \epsilon)$ with $\epsilon > 0$, and $p(t) < 0$ for $t > (n-1)/(n-2)$, it follows that $0 < t_0 < (n-1)/(n-2)$ and $p_n(t_0)$ is the unique maximum of $p_n(t)$ in $(0, +\infty)$.

Editor's Comments. The polynomial $p_n(t)$ plays an important role for the determination of the best constant for the inequality

$$\frac{A_n^n - G_n^n}{H_n A_n^{n-1} - G_n^n} \geq C$$

for $n > 2$, where H_n, G_n, and A_n are respectively harmonic, geometric, and arithmetic means (see Aliyev, Y.N. "The best constant for inequality involving the sum of the reciprocals and product of positive numbers with unit sum". *J Inequal Appl* 2024, 29 (2024). `https://doi.org/10.1186/s13660-024-03107-3`):

$$H_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}},\ G_n = \sqrt[n]{\prod_{i=1}^{n} x_i},\ A_n = \frac{\sum_{i=1}^{n} x_i}{n}.$$

5017. *Proposed by Michel Bataille.*

Let x, y, and z be non-negative real numbers such that $x+y+z=1$. Prove that

$$(1-x)(1-y)(1-z) \geq \frac{8\sqrt{3}}{9}\sqrt{xyz}.$$

We received 28 solutions, of which 25 were correct and complete. Common methods used Muirhead's inequality or the $n-1$ equal value principle. We present an elementary solution by Hoang Duc Hung and Yuhan Huang (done independently).

Using the AM-GM inequality, we have

$$x+y+z \geq 3\sqrt[3]{xyz} \Rightarrow xyz \leq \frac{1}{27} \Rightarrow \sqrt{xyz} \leq \frac{\sqrt{3}}{9},$$

Furthermore, we observe

$$\begin{aligned}(xy+yz+xz)^2 &= x^2y^2+y^2z^2+z^2x^2+2xyz(x+y+z)\\ &= \frac{x^2y^2+x^2z^2}{2}+\frac{y^2x^2+y^2z^2}{2}+\frac{x^2z^2+y^2z^2}{2}+2xyz(x+y+z)\\ &\geq xyz(x+y+z)+2xyz(x+y+z)\\ &= 3xyz(x+y+z).\end{aligned}$$

Using the above two facts, we have

$$\begin{aligned}(1-x)(1-y)(1-z) &= 1-(x+y+z)+xy+yz+zx-xyz\\ &= xy+yz+zx-xyz\\ &= \sqrt{(xy+yz+zx)^2}-\sqrt{xyz}\cdot\sqrt{xyz}\\ &\geq \sqrt{3xyz(x+y+z)}-\frac{\sqrt{3}}{9}\cdot\sqrt{xyz}\\ &= \frac{8\sqrt{3}}{9}\sqrt{xyz}.\end{aligned}$$

Equality holds if and only if $x=y=z=\frac{1}{3}$; $x=1, y=z=0$; $x=z=0, y=1$; $x=y=0; z=1$.

Editor's Comments. Honourable mentions go to Walther Janous and Clemens Heuberger, who submitted solutions generalizing the problem for when $x+y+z=s$ for some positive $s \leq 1$, showing that

$$(1-x)(1-y)(1-z) \geq \left(\sqrt{\frac{3}{s}}-\sqrt{\frac{s}{3}}\right)^3\sqrt{xyz}.$$

5018. *Proposed by Nguyen Viet Hung.*

Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2b}{2a+b}+\frac{b^2c}{2b+c}+\frac{c^2a}{2c+a}+\frac{2}{9}M\geq 1,$$

where $M=\max\{(a-b)^2,(b-c)^2,(c-a)^2\}$.

We received 11 submissions, 8 of which were correct and complete. We present two solutions.

Solution 1 is a compilation of independent solutions submitted by Sicheng Du and the proposer.

$$\begin{aligned}1-\sum_{\text{cyc}}\frac{a^2b}{2a+b}&=\frac{(a+b+c)^2}{9}-\sum_{\text{cyc}}\frac{a^2b}{2a+b}=\frac{\sum\limits_{\text{cyc}}a^2+2\sum\limits_{\text{cyc}}ab}{9}-\sum_{\text{cyc}}\frac{a^2b}{2a+b}\\&=\sum_{\text{cyc}}\left(\frac{a^2+2ab}{9}-\frac{a^2b}{2a+b}\right)=\sum_{\text{cyc}}\frac{2}{9}\cdot\frac{a}{2a+b}(a-b)^2\\&\leq\frac{2}{9}\sum_{\text{cyc}}\frac{a}{2a+b}M=\frac{2}{9}M\sum_{\text{cyc}}\frac{a}{2a+b}.\end{aligned}\tag{1}$$

By Titu's inequality,

$$\begin{aligned}\sum_{\text{cyc}}\frac{a}{2a+b}&=\sum_{\text{cyc}}\frac{1}{2}\left(1-\frac{b}{2a+b}\right)\\&=\frac{3}{2}-\sum_{\text{cyc}}\frac{b^2}{2(2ab+b^2)}\\&\leq\frac{3}{2}-\frac{(a+b+c)^2}{2(2ab+b^2+2bc+c^2+2ca+a^2)}\\&=1.\end{aligned}$$

So (1) implies that

$$1-\sum_{\text{cyc}}\frac{a^2b}{2a+b}\leq\frac{2M}{9}$$

completing the proof.

Solution 2, by Theo Koupelis, slightly altered by the editor.

Without loss of generality, let

$$(a-b)^2\geq(b-c)^2\iff(a-c)(2b-a-c)\leq 0,$$

and

$$(a-b)^2 \geq (c-a)^2 \Longleftrightarrow (b-c)(2a-b-c) \leq 0.$$

Thus, $a \leq c \leq b$ or $b \leq c \leq a$. By Titu's inequality we have

$$(a+b+c)^2 \left[\frac{a^2b^2}{b(2a+b)} + \frac{b^2c^2}{c(2b+c)} + \frac{c^2a^2}{a(2c+a)}\right] \geq (ab+bc+ca)^2.$$

Thus, it suffices to show that

$$(ab+bc+ca)^2 + 2(a-b)^2 \geq 9, \quad (*)$$

where $a+b+c=3$ and c is between a and b. Without loss of generality, let

$$b = x,\ c = x+t,\ \text{and } a = x+t+s,$$

where $x = (3-2t-s)/3$, where $t, s \geq 0$, and $2t+s < 3$. Substituting into $(*)$ we get

$$[3x^2 + x(4t+2s) + t^2 + ts]^2 + 2(t+s)^2 - 9 \geq 0,$$

which is equivalent to each of

$$\left[3 \cdot \frac{(3-2t-s)^2}{9} + (4t+2s) \cdot \frac{3-2t-s}{3} + t^2 + ts\right]^2 + 2(t+s)^2 - 9 \geq 0,$$

$$[9 - (t^2+ts+s^2)]^2 + 18(t+s)^2 - 81 \geq 0,$$

and

$$t^4 + 2t^3s + 3t^2s^2 + 2ts^3 + 18ts + s^4 \geq 0,$$

which is clearly true. Equality occurs when $t = s = 0$, or $a = b = c = 1$.

5019. *Proposed by Vasile Cirtoaje.*

Let $a_1 \geq a_2 \geq \cdots \geq a_9 \geq 0$ such that $a_1a_2 + a_2a_3 + \cdots + a_9a_1 = 9$. Prove that

$$a_1 + a_2 + \cdots + a_6 \geq 6.$$

We received 9 submissions, 6 of which were correct and complete. We present two solutions.

Solution 1, by Michal Adamaszek.

We have:

$$\begin{aligned}
&(a_1+a_3+a_5)(a_2+a_4+a_6)\\
&= (a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6) + (a_1a_4 + a_1a_6 + a_3a_6 + a_2a_5)\\
&= (9 - a_6a_7 - a_7a_8 - a_8a_9 - a_9a_1) + (a_1a_4 + a_1a_6 + a_3a_6 + a_2a_5)\\
&= 9 + (a_1a_4 - a_6a_7) + (a_1a_6 - a_1a_9) + (a_3a_6 - a_7a_8) + (a_2a_5 - a_8a_9)\\
&\geq 9.
\end{aligned}$$

Hence

$$a_1 + \cdots + a_6 \geq 2\sqrt{(a_1 + a_3 + a_5)(a_2 + a_4 + a_6)} \geq 2\sqrt{9} = 6.$$

Solution 2, by the proposer, slightly altered by the editor.

Write the desired inequality in the homogeneous form

$$(a_1 + a_2 + a_3 + 3S)^2 \geq 4(a_1a_2 + a_2a_3 + \cdots + a_9a_1),$$

where $S = \dfrac{a_4 + a_5 + a_6}{3}$. Since

$$a_3a_4 + a_4a_5 + a_5a_6 \leq a_3(a_4 + a_5 + a_6) = 3a_3S, \quad a_6a_7 + a_7a_8 + a_8a_9 \leq 3S^2, \quad a_9a_1 \leq a_1S,$$

it suffices to prove that $E \geq 0$, where

$$E = (a_1 + a_2 + a_3 + 3S)^2 - 4(a_1a_2 + a_2a_3 + 3a_3S + 3S^2 + a_1S).$$

For fixed a_1, a_2 and a_3, the expression E is a concave quadratic function of S. Since $0 \leq S \leq a_3$, it suffices to prove that $E \geq 0$ for $S = 0$ and $S = a_3$.

For $S = 0$, we have

$$E = (a_1 + a_2 + a_3)^2 - 4(a_1a_2 + a_2a_3) = (a_1 - a_2 + a_3)^2 \geq 0.$$

For $S = a_3$, we have

$$\begin{aligned} E &= (a_1 + a_2 + 4a_3)^2 - 4(a_1a_2 + a_2a_3 + a_3a_1 + 6a_3^2) \\ &= (a_1 + a_2 + 4a_3)^2 - 36a_3^2 - 4(a_1a_2 + a_2a_3 + a_3a_1 - 3a_3^2) \\ &= (a_1 + a_2 - 2a_3)(a_1 + a_2 + 10a_3) - 4(a_1a_2 + a_2a_3 + a_3a_1 - 3a_3^2) \\ &\geq (a_1 + a_2 - 2a_3)(a_1 + a_2 + 6a_3) - 4(a_1a_2 + a_2a_3 + a_3a_1 - 3a_3^2) \\ &= (a_1 - a_2)^2 \geq 0. \end{aligned}$$

The proof is completed. The equality occurs when $a_1 = a_2 = \cdots = a_9 = 1$, and also when $a_1 = a_2 = 3$ and $a_3 = \cdots = a_9 = 0$.

5020. *Proposed by Mihaela Berindeanu.*

Let a, b, c be three numbers greater than zero with the propriety $abc = 1$. Show that

$$\frac{\ln a}{1 + a + ab} + \frac{\ln b}{1 + b + bc} + \frac{\ln c}{1 + c + ca} \leq 0.$$

We received 26 submissions, of which 22 were correct and complete. The incomplete solutions omitted too many intermediate steps and calculations, while incorrect solutions failed to verify one of the two conditions for applying Jensen's inequality. We present two different solutions:

Solution 1, by Michal Adamaszek and Tsugawa Chikara (done independently).

Since $a, b, c \in \mathbb{R}_{>0}$ and $abc = 1$, we know there exist three positive real numbers x, y, z such that

$$a = \frac{x}{y}, \quad b = \frac{y}{z}, \quad c = \frac{z}{x}$$

This allows us to rewrite

$$\begin{aligned} 1 + a + ab &= 1 + \frac{x}{y} + \frac{x}{y} \cdot \frac{y}{z} = \frac{xy + yz + zx}{yz} \\ 1 + b + bc &= 1 + \frac{y}{z} + \frac{y}{z} \cdot \frac{z}{x} = \frac{xy + yz + zx}{zx} \\ 1 + c + ac &= 1 + \frac{z}{x} + \frac{x}{y} \cdot \frac{z}{x} = \frac{xy + yz + zx}{xy} \end{aligned}$$

Thus, the original inequality becomes

$$\frac{yz}{xy + yz + zx} \ln \frac{x}{y} + \frac{zx}{xy + yz + zx} \ln \frac{y}{z} + \frac{xy}{xy + yz + zx} \ln \frac{z}{x} \leq 0$$

or

$$\left(\frac{x}{y}\right)^{\frac{yz}{xy+yz+zx}} \cdot \left(\frac{y}{z}\right)^{\frac{zx}{xy+yz+zx}} \cdot \left(\frac{z}{x}\right)^{\frac{xy}{xy+yz+zx}} \leq 1.$$

This expression is a weighted geometric mean of the numbers in the bases, with weights in the exponents. Note that the weights add up to one as

$$\frac{yz}{xy + yz + zx} + \frac{xz}{xy + yz + zx} + \frac{xy}{xy + yz + zx} = \frac{xy + yz + zx}{xy + yz + zx} = 1$$

so, we can use the weighted AM-GM inequality. Notice that the corresponding weighted arithmetic mean is

$$\frac{x}{y} \cdot \frac{yz}{xy + yz + zx} + \frac{y}{z} \cdot \frac{zx}{xy + yz + zx} + \frac{z}{x} \cdot \frac{xy}{xy + yz + zx} = 1$$

and we can conclude that

$$\left(\frac{x}{y}\right)^{\frac{yz}{xy+yz+zx}} \cdot \left(\frac{y}{z}\right)^{\frac{zx}{xy+yz+zx}} \cdot \left(\frac{z}{x}\right)^{\frac{xy}{xy+yz+zx}} \leq 1$$

proving the given inequality.

Solution 2, by Didier Pinchon.

Let $\lambda_1, \lambda_2, \lambda_3$ be the positive real numbers defined by

$$\lambda_1 = \frac{1}{1 + a + ab}, \quad \lambda_2 = \frac{1}{1 + b + bc}, \quad \lambda_3 = \frac{1}{1 + c + ca}.$$

Substituting $c = \frac{1}{ab}$, we compute:

$$\lambda_2 = \frac{1}{1+b+\frac{1}{a}} = \frac{a}{1+a+ab} = a\lambda_1,$$
$$\lambda_3 = \frac{1}{1+\frac{1}{ab}+\frac{1}{b}} = \frac{ab}{1+a+ab} = ab\lambda_1.$$

Therefore, we observe that $b\lambda_2 = ab\lambda_1$ and $c\lambda_3 = \lambda_1$. It follows that:

$$\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + a\lambda_1 + ab\lambda_1 = (1+a+ab)\lambda_1 = 1,$$
$$a\lambda_1 + b\lambda_2 + c\lambda_3 = a\lambda_1 + ab\lambda_1 + \lambda_1 = (1+a+ab)\lambda_1 = 1.$$

Since $\lambda_1, \lambda_2, \lambda_3 > 0$ and their sum is one, we may apply Jensen's inequality to the strictly concave function $f(x) = \ln x$ on $(0, \infty)$. This gives:

$$\lambda_1 \ln a + \lambda_2 \ln b + \lambda_3 \ln c \le \ln(\lambda_1 a + \lambda_2 b + \lambda_3 c) = \ln 1 = 0,$$

with equality if and only if $a = b = c = 1$.