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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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Call for submissions for an issue in honour of Robert Woodrow

We are deeply saddened by the news of Robert Woodrow's recent passing. Robert has been a constant presence and force in the Canadian mathematical community for decades. He played a key role in organizing and running numerous math competitions, championed ***Crux*** and supported generations of students through research, service and advocacy.

A dedicated member of the Canadian Mathematical Society for over 45 years, Robert served in many roles, most recently as Chair of the Canadian Open Mathematics Challenge (COMC). Robert has been heavily involved with ***Crux*** throughout the years: he served as the Editor of the Olympiad Corner from 1987 to 2011 and Editor-in-Chief alongside Bill Sands from 1992 to 1996; he also created and served as the Editor of the Skoliad from 1995 to 2001.

To honour Robert's memory, ***Crux*** will publish a special issue in Winter 2026. We invite submissions for this issue, including problem proposals, articles, expositions of Robert's contributions to mathematics and math education, tributes, reminiscences.

Please share this call widely, especially with those who knew Robert.

Submissions may be sent to `crux.eic@gmail.com` by November 15th.

MATHEMATTIC

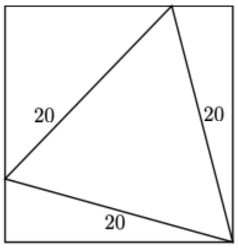
No. 66

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **September 30, 2025**.

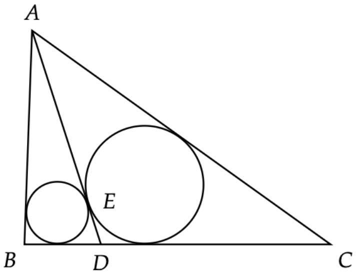
MA326. An equilateral triangle, 20 cm on a side, is inscribed in a square as shown in the diagram. Find the length of the side of the square.



MA327. Find all triples (p, q, r) where p, q, r are positive integers of which at least two are prime for which

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

MA328. In the diagram, triangle ABC has sides of length $AB = 7$, $AC = 12$, $BC = 10$. There is a point D on BC such that the circles inscribed in triangles ABD and ACD are both tangent to the line AD at a common point E. Find the length of the line segment BD.



MA329. Nine people attend a dinner where there are three choices for the type of meal. Three people order combo A, three order combo B and three order combo C. The server distributes the nine meals in random order. In how many different ways can exactly one person receive the correct meal?

MA330. The x-coordinates of the vertices of a square in the plane are 1, 3, 8 and 10. Determine the area of the square.

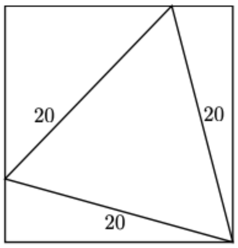
. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 septembre 2025**.

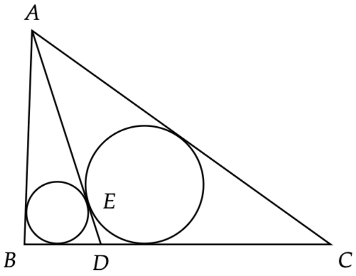
MA326. Un triangle équilatéral dont chacun des côtés mesure 20 cm est inscrit dans un carré comme indiqué sur le schéma que voici. Trouvez la longueur de chacun des côtés du carré.



MA327. Trouvez tous les triplets (p, q, r), où p, q, r sont des entiers positifs dont au moins deux sont premiers pour lesquels

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

MA328. Dans le diagramme, le triangle ABC a des côtés de longueur $AB = 7$, $AC = 12$ et $BC = 10$. Il existe un point D sur BC tel que les cercles inscrits dans les triangles ABD et ACD sont tous deux tangents à la droite AD en un point commun E. Trouvez la longueur du segment de droite BD.



MA329. Neuf personnes participent à un dîner au cours duquel trois choix de plats sont proposés. Trois personnes commandent le combo A, trois autres commandent le combo B et les trois autres commandent le combo C. Le serveur distribue les neuf repas dans un ordre aléatoire. De combien de façons différentes une personne peut-elle recevoir le bon repas ?

MA330. Les coordonnées sur l'axe des abscisses des sommets d'un carré inscrit dans le plan sont 1, 3, 8 et 10. Déterminez l'aire du carré.

◆

MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(1), p. 5–7.

MA301. Suppose n and m are positive integers. Mike and Mary each have a bag of $N = nm$ marbles labelled 1 to N. Mike randomly takes out a marble from his bag. Suppose a is the sum of the numbers on the remaining marbles in Mike's bag. Similarly, Mary randomly takes out a marble from her bag. Suppose b is the sum of the numbers on the remaining marbles in Mary's bag. Find the probability that $a - b$ is a multiple of n. Express your answer as a reduced rational function in terms of n and m.

Originally from the 2016 LSU Math Contest, Team Session, problem 7.

We received 10 submissions, all of which were correct and complete. We present the solution by Pranjal Agrawal.

Suppose Mike randomly removes a marble with number x. Then, the sum of the numbers on the remaining marbles in his bag is $a = S - x$, where

$$S = 1 + 2 + \cdots + nm = \frac{nm(nm+1)}{2}.$$

Similarly, if Mary randomly removes a marble with number y, the sum of the remaining marbles in her bag is

$$b = S - y.$$

Thus, we have

$$a - b = (S - x) - (S - y) = y - x.$$

We need to determine the probability that $a - b$ is a multiple of n; equivalently, we require that

$$n \mid (y - x) \quad \iff \quad y \equiv x \pmod{n}.$$

Notice that the numbers $1, 2, \ldots, nm$ are evenly distributed modulo n: each residue class modulo n appears exactly m times. Therefore, if Mike's chosen marble x is congruent to some residue $r \pmod{n}$, there are exactly m marbles (out of the total nm) in Mary's bag that are also congruent to r modulo n.

Thus, the probability that Mary picks a marble y such that $y \equiv x \pmod{n}$ is

$$\frac{m}{nm} = \frac{1}{n}.$$

Hence, the probability that $a - b$ is a multiple of n is $\frac{1}{n}$.

MA302. A $5 \times 5 \times 5$ wooden cube is painted on all 6 faces and then cut up into unit cubes. One unit is randomly selected and rolled. What is the probability that exactly one of the five visible faces (on the rolled die) is painted?

Originally from the 2016 LSU Math Contest, Team Session, problem 9.

There were 11 submissions, 9 of them complete and correct. We present the solution by Pranjal Agrawal.

A $5 \times 5 \times 5$ cube consists of 125 unit cubes. When the large cube is painted on all 6 faces, the number of painted faces on each unit cube depends on its location. We have:

- Corner cubes: 8 cubes, each painted on 3 faces.
- Edge cubes (not corners): 12 edges, each contributing $5 - 2 = 3$ cubes, so $12 \times 3 = 36$ cubes, each painted on 2 faces.
- Face-center cubes (not on edges): Each face has an inner 3×3 square of cubes, giving 9 cubes per face. With 6 faces, that is $6 \times 9 = 54$ cubes, each painted on 1 face.
- Interior cubes: The remaining $3 \times 3 \times 3 = 27$ cubes have 0 painted faces.

When a cube is rolled, one face (the bottom) is hidden and the other five faces are visible. We wish to have exactly one painted face among these five. We analyze each type.

Cubes with 0 painted faces (27 cubes). No matter how they are rolled, none of the visible faces is painted. So they contribute 0 favorable outcomes.

Cubes with 1 painted face (54 cubes). These cubes have a single painted face. In order to have exactly one painted face visible, the painted face must *not* be on the bottom. Since a cube has 6 faces and every face is equally likely to be on the bottom, the probability that the painted face is *not* on the bottom is $\frac{5}{6}$. Thus, the total number of favorable orientations for these cubes is:

$$54 \times \frac{5}{6} = 45.$$

Cubes with 2 painted faces (36 cubes). For an edge cube, the two painted faces are adjacent. To have exactly one painted face visible, one of the painted faces must be on the bottom (and the other, being adjacent, will be visible among the five). If neither painted face is on the bottom, then both will be visible; if both were somehow hidden (impossible, since only one face is hidden), that would not occur either. Thus, the favorable outcome is when the bottom face is one of the two painted faces. The probability for a cube with 2 painted faces is:

$$\frac{2}{6} = \frac{1}{3}.$$

So, the number of favorable outcomes for these cubes is:

$$36 \times \frac{1}{3} = 12.$$

Cubes with 3 painted faces (8 cubes). A corner cube has 3 painted faces. No matter which face is on the bottom, at least 2 painted faces will be visible (if the bottom is painted, the other two are visible; if the bottom is unpainted, then all three painted faces are visible). Thus, these cubes yield 0 favorable outcomes.

To calculate the total favorable outcomes, we add the contributions from each case to get:

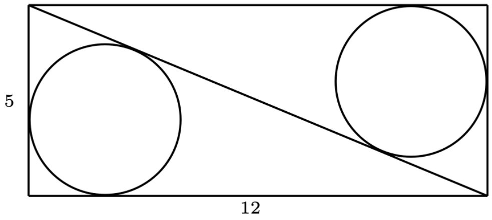
$$45 + 12 + 0 + 0 = 57.$$

Since there are 125 cubes and each cube is equally likely to be chosen and each has 6 equally likely orientations when rolled, the overall probability is $\frac{57}{125}$.

Editor's Comments. Honorable mention goes to Missouri State University Problem Solving Group for generalizing the problem into $n \times n \times n$ cubes.

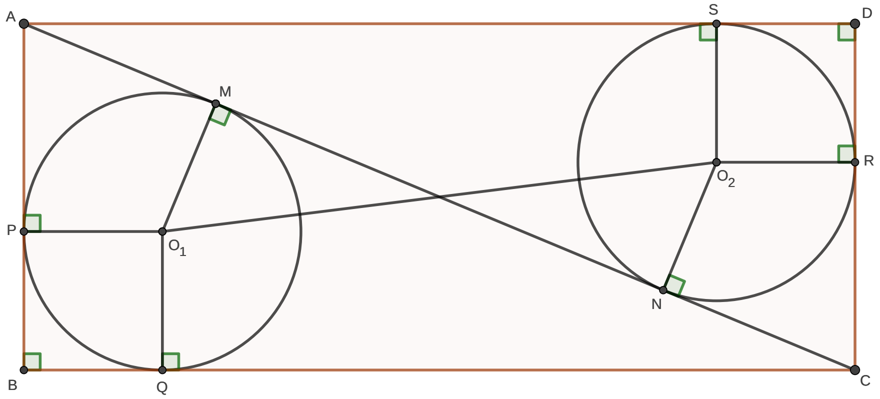
MA303. The rectangle below is 12×5. The diagonal forms two right triangles and a circle is inscribed in each triangle. Find the distance between the centers of the two circles.

Originally from the 2017 LSU Math Contest, Team Session, problem 2.



We received 17 submissions of which 16 were correct and complete. We present the solution by Soham Das.

Denote the rectangle by $ABCD$, the incenters by O_1 and O_2 and the points of tangency as shown in the diagram on the next page:



Apply the Phythagorean Theorem in $\triangle ABC$ to get that

$$AC = \sqrt{5^2 + 12^2} = 13.$$

The radius of the circle inscribed in $\triangle ABC$ is

$$r = \frac{\text{Area of } \triangle ABC}{\text{semiperimeter}} = \frac{\frac{1}{2} \cdot 12 \cdot 5}{\frac{1}{2} \cdot (12 + 5 + 13)} = 2.$$

The side of square PO_1QB is thus 2, and it follows that

$$AP = AB - PB = 5 - 2 = 3.$$

Since $AM = AP$ we get $AM = 3$ as well. By symmetry, the radius of the circle inscribed in $\triangle ACD$ is 2 and $NC = 3$. Thus

$$MN = AC - (AM + NC) = 13 - (3 + 3) = 7.$$

Finally, the length of the common transverse tangent of two circles is given by $\sqrt{d^2 - (r_1 + r_2)^2}$, where d is the distance between the centers and r_1, r_2 are the radii. In this case, the common transverse tangent MN has length 7 and $r_1 = r_2 = 2$, so we can solve for d:

$$7 = \sqrt{d^2 - (2 + 2)^2} \Rightarrow 49 + 16 = d^2 \Rightarrow d = \sqrt{65}.$$

Therefore the distance O_1O_2 between the centers of the two circles is $\sqrt{65}$.

MA304. At a picnic, there are c children, m mothers, and f fathers, with $2 \leq f < m < c$. Every person shakes hands with every other person. The sum of the number of handshakes amongst the children, amongst the mothers, and amongst the fathers is 80. How many persons attended the picnic?

Originally from the 2016 LSU Math Contest, Team Session, problem 10.

We received 7 submissions, all of which were correct and complete. We present the solution by Meryem Bourget.

There are $\binom{f}{2}$ handshakes among the fathers, $\binom{m}{2}$ handshakes among the mothers, and $\binom{c}{2}$ handshakes among the children. By assumption, we have

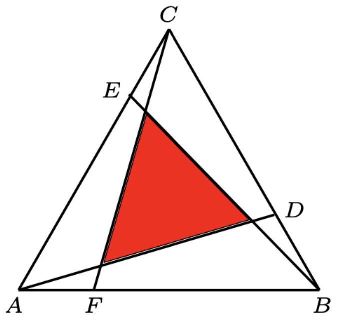
$$\binom{f}{2} + \binom{m}{2} + \binom{c}{2} = \frac{f(f-1)}{2} + \frac{m(m-1)}{2} + \frac{c(c-1)}{2} = 80 \qquad (1)$$

To solve equation (1), we must find three distinct numbers, each of the form $\frac{n(n-1)}{2}$, whose sum is 80. In the table below, we list all the possible values $\frac{n(n-1)}{2}$ for $n = 2$ to $n = 13$.

n	$\frac{n(n-1)}{2}$	n	$\frac{n(n-1)}{2}$
2	1	3	3
4	6	**5**	**10**
6	**15**	7	21
8	28	9	36
10	45	**11**	**55**
12	66	13	78

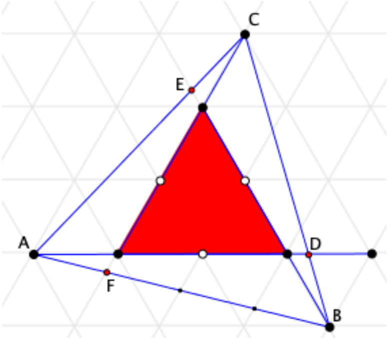
The only possible way to pick three distinct numbers from columns 2 and 4 whose sum is 80 is by choosing 10, 15 and 55. These correspond to $n = 5$, $n = 6$ and $n = 11$ respectively. Since $f < m < c$, it follows that $f = 5$, $m = 6$ and $c = 11$, so the total number of people that came to the picnic is $f + m + c = 5 + 6 + 11 = 22$.

MA305. Choose points D, E, and F on the sides of the equilateral triangle ABC so that $|AF| = |EC| = |DB| = 1$ and $|FB| = |DC| = |AE| = 3$. The line segments EB, AD, and CF enclose a triangle that is shaded in the diagram. Find the ratio of the area of the shaded region and the area of triangle ABC.



Originally from the 2017 LSU Math Contest, Team Session, problem 10.

We received 14 solutions, of which 11 were correct and complete. We present the solution by J. Chris Fisher.



This problem is an example of the special case of Routh's theorem where the points D, E, F divide the sides of the outer triangle ABC in the same ratio r, namely $r = \frac{1}{3}$. H.S.M. Coxeter tells the story in Section 13.5 of his *Introduction to Geometry*. The theorem states that the ratio of areas is

$$\frac{(r-1)^3}{r^3-1} = \frac{\left(\frac{2}{3}\right)^3}{\frac{26}{3^3}} = \frac{4}{13}.$$

Coxeter adds that this special case of Routh's theorem is an immediate consequence of Pick's theorem. The latter was a favorite of Andy Liu, a friend of *Crux Mathematicorum* from its earliest days until his death last year. One of his essays on the topic can be found in this journal, Volume 4, no. 9 (November, 1978), p. 242-244. Pick's theorem states that the area of any simple polygon whose vertices are lattice points is given by the formula

$$\frac{1}{2}b + c - 1,$$

where b is the number of lattice points on the boundary while c is the number of lattice points inside. In the accompanying figure, the lattice points of each cevian of ΔABC are projected (by a family of parallel lines) onto the nearest side of the triangle; for example, AD is projected onto AB with the point F one-fourth of the distance from A to B. Since the central triangle has $b = 6, c = 0$ (giving an area of 2), while ABC has $b = 3, c = 6$ (for an area of $\frac{13}{2}$) we again find the desired ratio to be $\frac{2}{(13/2)} = \frac{4}{13}$.

* * *

TEACHING PROBLEMS

No. 30

John Grant McLoughlin

Consecutive Integers

The product of two consecutive integers is 30. What are the integers? How about if the product is 156? Now how about 1722? Smaller products can be used to bring forth the idea that the two consecutive integers will be slightly less than and greater than the square root of the product. The square root of 1722 is approximately 41.5 and hence, the consecutive integers would be 41 and 42. This example is not challenging for the readership here, but rather an illustration of how consecutive integers can be employed to develop mathematical thinking.

Let us extend the discussion of products of consecutive integers by way of two multiple choice questions.

> Which of these numbers is the product of two consecutive integers?
>
> A) 5402 B) 5404 C) 5405 D) 5408 E) 5409

Here we will use an approach that focuses on the final digit of the product. In fact, we are considering the products of numbers modulo 10. The principles of modular arithmetic overlay neatly upon the ideas around consecutive integers. Whether it is formally called modular arithmetic is not so important. Rather the idea of seeking patterns around the units (ones) digit of numbers serves as a valuable tool in many problem situations. How does this work here?

The final digit of a product is determined by the final digits being multiplied together. There are ten such possibilities, namely, 0×1, 1×2, 2×3, and so on through to 9×0. The result is that only certain digits become possibilities for the units digit of the product, namely, 0, 2, and 6. From our list of choices it becomes apparent that the answer is A) 5402.

A second question is offered here:

> Which of these numbers is not the product of three consecutive integers?
>
> A) 7980 B) 13800 C) 32736 D) 48168 E) 148824

Considering products of $0 \times 1 \times 2$, $1 \times 2 \times 3$, $2 \times 3 \times 4$, and up through $9 \times 0 \times 1$, we find that the only possibilities for the units digit of the product are 0, 4 and 6. Hence, it is D) 48168 that is impossible to represent as a product of three consecutive integers.

The recent *40th New Brunswick Mathematics Competition* hosted on May 8, 2025 by University of New Brunswick and Université de Moncton featured three separate contest papers for Grades 7, 8, and 9 students respectively. Two questions addressed the subject of consecutive integers with the latter being the most challenging problem on the Grade 9 paper. These questions are discussed here beginning with Question 11 from the Grade 8 paper.

> Which of these values could be the sum of four consecutive integers?
>
> A) 20 B) 36 C) 60 D) 72 E) 86

Earlier we discussed the idea of modulo 10 in terms of considering the units digit. Curiously modulo arithmetic can be applied neatly here. While it is likely that many students used a trial and error approach seeking sums, it is nice to consider a more elegant approach that transfers to larger values for which the seeking of actual sums would become more difficult.

Note that with any four consecutive integers, there must be an integer corresponding to all possible remainders upon division by 4. That is a number in the set of integers will be divisible by 4 while others will leave remainders of 1, 2, and 3 respectively. Hence, upon addition of these four numbers, there will be a remainder of 2 upon division by 4. (Check that makes sense as there will 6 left over which corresponds to 2 being left over when grouped in fours.) The only number with that property in the set of choices is E) 86.

Before moving on from this problem, let's consider what would happen with two consecutive integers. The sum would have a remainder of 1 when divided by 2. What about with three consecutive integers? Do you agree that the sum would be divisible by 3? What about with five consecutive integers? Six? Seven? Some nice mathematics falls out here in terms of considering the sums of n consecutive integers with respect to the remainder upon division by n.

Now for a big leap in level of difficulty... Question 26, Grade 9.

> The product of n consecutive two-digit numbers is divisible by 2025. What is the smallest possible value of n?
>
> A) 3 B) 4 C) 5 D) 6 E) 7

How might we begin here? First, it would be helpful to factor 2025. You may use the fact that $2025 = 45 \times 45$ if you know that, or perhaps seeing it as 81×25 is another option for breaking it down into prime factors.

We find that $2025 = 3 \times 3 \times 3 \times 3 \times 5 \times 5$. Hence, the product must include four 3's and two 5's. If we arbitrarily take any consecutive multiples of 5, it will often require at least six numbers in the product. For example, to include both 15 and 20 would require us to use $15 \times 16 \times 17 \times 18 \times 19 \times 20$. It follows that finding a product of 6 consecutive two-digit numbers divisible by 2025 is plausible. Can we do better?

Improving upon this result will require us to not need two separate multiples of 5. That is, the inclusion of 25, 50 or 75 in the product will take care of both factors of 5. Is there a number near any of these values that gives us plenty of 3's? The value of 27 jumps out. Considering the product of $25 \times 26 \times 27$, we find that there are two 5's and three 3's, thus, making it one factor of 3 short of producing divisibility by 2025. Extending the product to include 24 takes care of this. The smallest possible value of n is given by B) 4. It is noteworthy that $54 = 2 \times 27$ and it turns out that $50 \times 51 \times 52 \times 53 \times 54$ gives a set of five consecutive integers that is divisible by 2025. As suggested above, sets of six consecutive integers could readily be found to satisfy the required divisibility by 2025.

Concluding remarks

Consecutive integers represent an entry point into discussing more advanced ways of thinking about mathematics. Mention of modular arithmetic is likely to draw blank faces from most middle school teachers. However, ideas related to remainders or patterns associated with the units digit are accessible and understandable. Though not directly discussed here, the ideas of consecutive integers can be related to measures of central tendency, arithmetic sequences and other topics. I am reminded of the infamous puzzle in which unmarked 3 litre and 5 litre jugs are to be used to measure 1 litre. Mathematically this is equivalent to finding integer solutions to the equation $3x + 5y = 1$. Mathematical thinking can be developed through accessing relatively simple situations and extending at a later time to the level of formality required to express those thoughts with enhanced levels of elegance.

In closing, let us return to the divisibility of a product of n consecutive two-digit numbers by 2025 with a twist.

> What would be the largest value of n for which the product of n consecutive two-digit numbers is not divisible by 2025?

Competition Highlights

The Alberta High School Mathematics Competition

by Nicolae Strungaru

Running since 1957, the Alberta High School Mathematics Competition is the oldest province-wide competition in Canada, with the 69th edition having taken place last fall and earlier this year. It is open to all students from Alberta, and students from Northwest Territories can also write the contest. Since 1981 it is a qualifying round for the CMO.

The contest consists of two parts. Part I usually runs in November; the students have 80 minutes to answer 16 multiple-choice questions. The top students and schools in overall standing, as well as the top students and schools in each of the four zones (Edmonton, Calgary, rural north Alberta and rural south Alberta) receive certificates and book prizes. The top 50 students province-wide are invited to write Part II of the contest, in February.

In Part II, the students have 3 hours to answer five long answer questions. The top students receive prizes, with the third-place award sponsored by the Canadian Mathematical Society. The top 3 students are invited to write the CMO.

In the 69th AHSMC, 526 students from 41 schools wrote Part I of the contest. Among the students, 182 were girls, and 2 were gender X.

While the contest is aimed at high-school students, we had kids as young as grade 4 writing it. Of the top 50 students, four were in grades 7 and 8, and three grade 9 students were among the awards winners.

We complete this column by presenting two questions from this year's contest, one from each part, with their solutions. We start by presenting a multiple-choice question.

Problem 15 from Part I.

m, n are positive integers with $6 \leq n \leq 99$. Subject to this condition, $\frac{m}{n}$ is the reduced fraction that is the closest possible to $\frac{2}{5}$. What is the value of $m+n$?

(A) 46 (B) 68 (C) 137 (D) 138 (E) 139

Solution. We have

$$\left|\frac{2}{5}-\frac{m}{n}\right|=\frac{|5m-2n|}{5n}\,.$$

We want $\frac{|5m-2n|}{5n}$ to be the smallest possible.

We split the solution in three cases:

Case 1: $|5m-2n|=0$. Then $5m=2n$ and hence

$$\frac{m}{n}=\frac{2}{5}\,.$$

Since $\frac{m}{n}$ is reduced, we get $n=5$, a contradiction. Therefore, there is no fraction in this case.

Case 2: $|5m-2n|=1$. Then

$$\left|\frac{2}{5}-\frac{m}{n}\right|=\frac{1}{5n}\,.$$

This is the smallest when n is the largest possible.

Note here that

$$\begin{aligned}|5m-2n|=1 &\iff\\ 5m-2n=\pm 1 &\implies\\ 2n\equiv \pm 1 \pmod 5 &\implies\\ n\equiv \pm 3 \pmod 5\end{aligned}$$

The largest n which satisfies this condition is $n=98$. Note that in this case $2n=196$ and hence $m=39$ is the only value for $n=98$ which satisfies

$$|5m-2n|=1\,.$$

Therefore, this case $m=39, n=98$ leads to the smallest error of

$$\left|\frac{2}{5}-\frac{m}{n}\right|=\frac{1}{5\cdot 98}\,.$$

Case 3: $|5m-2n|\geq 2$. Then

$$\left|\frac{2}{5}-\frac{m}{n}\right|\geq\frac{2}{5n}\geq\frac{2}{5\cdot 99}>\frac{1}{5\cdot 98}\,.$$

In this case the error is larger than the one for $m=39, n=98$.

We conclude that the closest fraction to $\frac{2}{5}$ is $\frac{39}{98}$.

Answer: (C)

Next, we present Problem 2 from Part II with its solution.

Problem 2 from Part II.

Anna writes on a piece of paper all the digits of 4^{2025}. Then she writes all the digits of 25^{2025}.

How many digits did she write altogether?

Solution. Let m, n be the number of digits of these two numbers.

Then

$$10^{m-1} < 4^{2025} < 10^{m}$$
$$10^{n-1} < 25^{2025} < 10^{n}$$

Multiplying we get

$$10^{m+n-2} < 100^{2025} < 10^{m+n}\,,$$

that is

$$10^{m+n-2} < 10^{4050} < 10^{m+n}\,.$$

This implies that $m + n = 4051$.

* * *

OLYMPIAD CORNER

No. 434

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **September 30, 2025**.

OC736. Solve in $\mathbb{R}$ the equation $[\log_2 x] = \sqrt{x} - 2$, where $[x]$ denotes the integer part of x.

OC737. Find all real solutions of the equation

$$7^{\log_5\left(x^2+\frac{4}{x^2}\right)} + 2\left(x + \frac{2}{x}\right)^2 = 25$$

OC738. Prove that for each $z \in \mathbb{C}$ the following inequality holds

$$|z^2 + 2z + 2| + |z - 1| + |z^2 + z| \geq 3.$$

When does the equality hold?

OC739. In triangle ABC with $AB = AC$ let I denote the incenter of the triangle. Line BI meets the circumcircle a second time in point D. Find the measures of the angles of the triangle if $BC = ID$.

OC740. Find the 73rd digit from the end of $111\ldots1^2$, where the number of ones is 2012.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 septembre 2025**.

* * *

OC736. Résolvez dans $\mathbb{R}$ l'équation $[\log_2 x] = \sqrt{x} - 2$, où $[x]$ représente la partie entière de x.

OC737. Trouvez toutes les solutions réelles de l'équation

$$7^{\log_5\left(x^2+\frac{4}{x^2}\right)} + 2\left(x + \frac{2}{x}\right)^2 = 25.$$

OC738. Montrez que pour chaque $z \in \mathbb{C}$ l'inégalité suivante est vérifiée

$$|z^2 + 2z + 2| + |z - 1| + |z^2 + z| \geq 3.$$

Quand a-t-on égalité ?

OC739. Dans le triangle ABC avec $AB = AC$, on désigne par I le centre du cercle inscrit au triangle. La droite BI rencontre le centre du cercle circonscrit au triangle au point D. Trouvez les mesures des angles du triangle si $BC = ID$.

OC740. Trouvez le 73-ième chiffre à partir de la fin de $111\ldots1^2$, où le nombre de uns est 2012.

* * *

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2025: 51(1), p. 18–19.

OC711. Place different positive integers, not greater than 25, in the cells of a 3×3 square so that in any pair of adjacent cells one number is divisible by the other.

Originally from the Moscow Mathematical Olympiad 2024 - Grade 7, Problem 1.

We received 8 solutions. We present the solution by Theo Koupelis.

2	4	16
12	24	8
6	3	1

Another example:

6	12	4
18	3	24
2	1	8

Another example:

3	6	12
1	2	24
4	16	8

OC712. The teacher dictated to Vovochka the slopes and y-intercepts of three different linear functions whose graphs are all parallel. Inattentive Vovochka, when writing down each of the functions, swapped the slope and the y-intercept of each function and plotted the graphs of the resulting lines. How many points could there be through which at least two graphs pass?

Originally from the Moscow Mathematical Olympiad 2024 - Grade 8, Problem 1.

We received 7 submissions, 6 of which were correct and complete. We present the solution by Oliver Geupel.

We show that the answer is 1. The lines given by the teacher have all the same slope, but distinct y-interepts, because they are parallel and different. Vovochka, by swapping slope and y-intercept, draws three lines that have all the same y-intercept, but distinct slopes. These are three distinct straight lines with a common point of intersection on the y-axis.

OC713. Let a, b, c, d be real numbers such that

$$\frac{a}{b}+\frac{b}{a}=\frac{c}{d}+\frac{d}{c}.$$

Prove that the product of two numbers from a, b, c, d is equal to the product of the other two.

Originally from the Moscow Mathematical Olympiad 2024 - Grade 9, Problem 1.

We received 11 solutions.

We present the solution by the Missouri State University Problem Solving Group.

Let $x = a/b$ and $y = c/d$. Then $x + 1/x = y + 1/y$. Clearing denominators and rewriting gives

$$x^2y + y - xy^2 - x = 0, \text{ so } (x - y)(xy - 1) = 0.$$

We have two cases. Either $x = y$, so $a/b = c/d$ and $ad = bc$ or $y = 1/x$, so $a/b = d/c$ and $ac = bd$.

Editor's Comment. The statement of the problem has been corrected. Thanks to Roy Barbara, Richard Hess, Theo Koupelis, and C.R. Pranesachar for pointing out that the original statement was incorrect.

OC714. Petya and Vasya play on the segment $[0, 1]$ in which the points 0 and 1 are marked. The players take turns, Petya starts. Each move, the player marks a previously unmarked point on the segment. If, after the next player's move, three consecutive segments are found between adjacent marked points from which a triangle can be formed, then the player who made such a move is declared the winner, and the game ends. Will Petya be guaranteed to win?

Originally from the Moscow Mathematical Olympiad 2024 - Grade 10, Problem 1.

We received 5 solutions. We present the solution by Theo Koupelis.

Let O, A be the points marked as $0, 1$, respectively, on the line containing the segment $[0, 1]$. Petya is guaranteed to win by first marking the midpoint M at 0.5.

Without loss of generality, let Vasya mark a point N on the segment $(0.5, 1)$. Let $MN = y$, where $0 < y < 0.5$. The three segments OM, MN, NA cannot form a non-degenerate triangle because $OM = MN + NA$. To win, Petya chooses any point K on the segment OM such that $OK = x$ where

$$0 < \frac{1-2y}{4} < x < \frac{1+2y}{4} < \frac{1}{2},$$

which is clearly possible. In this case, the consecutive segments OK, KM, MN can form a triangle because

$$\begin{aligned}
&OK + KM > MN \iff \frac{1}{2} > y,\\
&OK + MN > KM \iff x + y > \frac{1}{2} - x \iff x > \frac{1-2y}{4},\\
&KM + MN > OK \iff \frac{1}{2} - x + y > x \iff x < \frac{1+2y}{4}.
\end{aligned}$$

Note that the above analysis shows that the result is the same even if the points 0 and 1 were not marked. If Vasya were to mark an end point, say point A in the first move, then Petya marks the point O (or vice-versa). The analysis above guarantees that Petya wins independent of what point Vasya chooses next.

OC715. A mathematician has 19 different weights, the masses of which in kilograms are equal to $\ln 2, \ln 3, \ln 4, \ldots, \ln 20$, and an absolutely precise two-pan scale. He puts several weights on the scale so that equilibrium is established. What is the greatest number of weights that could be on the scale?

Originally from the Moscow Mathematical Olympiad 2024 - Grade 11, Problem 1. We received 7 solutions and we present 2 of them.

Solution 1, by Theo Koupelis.

Let $a_i = i$, where $i = 2, 3, \ldots, 20$. For the scale to be balanced, the product of some of the numbers a_i whose corresponding weights $\ln a_i$ are on one of the two pans must equal to the product of some different numbers a_i whose corresponding weights are on the second pan, because $\ln x + \ln y = \ln(xy)$. Thus, the weights corresponding to the primes $11, 13, 17$, and 19 cannot be used because $i \in [2, 20]$. On the other hand, using the prime factorization of the numbers a_i other than $11, 13, 17, 19$, we get that there are eighteen 2s, eight 3s, four 5s, and two 7s. It is possible to balance the corresponding weights as follows: on one pan we have the weights corresponding to $a_2, a_3, a_7, a_8, a_9, a_{10}, a_{12}, a_{20}$, and on the second pan we have the weights corresponding to $a_4, a_5, a_6, a_{14}, a_{15}, a_{16}, a_{18}$. The total weight on each of the pans is

$$\ln(7257600) = \ln(2 \cdot 3 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 12 \cdot 20) = \ln(4 \cdot 5 \cdot 6 \cdot 14 \cdot 15 \cdot 16 \cdot 18).$$

Thus, the greatest number of weights that can be on the scale is 15.

Solution 2, by Missouri State University Problem Solving Group.

The problem is equivalent to finding two selections from $2, 3, 4, \ldots, 20$ whose products are equal. Since 11, 13, 17, and 19 only appear once in the factorizations of the integers from 2 to 20, they cannot be part of the selection. This leaves 15 as the greatest possible number of weights. Since

$$2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdot 9 \cdot 18 \cdot 20 = 3 \cdot 6 \cdot 10 \cdot 12 \cdot 14 \cdot 15 \cdot 16,$$

15 is, in fact, the greatest number of weights.

From the lecture notes of . . .

Chris Sangwin

In this feature of ***Crux****, we share some of our favourite problems from first and second year undergraduate courses. These problems tend to be non-standard, elegant or unexpected. If you have a problem you would like to share (and it fits on one or so page), please send it along with its solution and a description of the course/audience it is intended for to* `crux.eic@gmail.com`*.*

This month's column is brought to you by Chris Sangwin. Chris joined the University of Edinburgh in 2015 as Professor of Technology Enhanced Science Education. His learning and teaching interests include digital educational technology and automatic assessment of mathematics using computer algebra. Chris has written a number of books, including "Computer Aided Assessment of Mathematics" and the popular science book "How Round is Your Circle", which illustrates and investigates many links between mathematics and engineering using physical models. He's also co-authored textbooks for the International Baccalaureate published by Haese Mathematics.

. . . .

Audience and rationale

This set of problems was written for participants of the *Institute for Mathematical Pedagogy* (IMP) which took place at Ripon College, Cuddesdon, in 2023. `http://www.pmtheta.com/imp.html` Each Institute is a week long meeting of mathematics teachers, advisers, academics, etc. who meet and work together on mathematical tasks. The point is not to do the tasks, but rather to notice what is involved in working on the tasks, making sense, making progress, getting stuck, working with others, and the pedagogical implications. The theme in 2023 was *Engineering the unexpected*, i.e. creating surprise in the classroom. This sequence of problems provoked genuine surprise in the participants, and has also provoked surprise when used subsequently with undergraduate mathematics students at all levels (undergraduate years 1–4).

It is important to work through the problems in order. Indeed, sequences of problems like this can form a mathematical journey.

The problem sheet

Mathematical introduction and notation

Decimals are an important way of representing real numbers.

1. $0.001001001\cdots$ is periodic with period 3.
   Without calculating, guess the period of $(0.001001001\cdots)^2$.
2. You would like to multiply two decimals together, and get the first p decimal digits correct. How many places of each decimal do you need to take to *guarantee* the first p decimal digits of the product are correct? For example, do you need to take only $2p$ digits, or p^2 digits or something else?

We take the following theorem for granted.

Theorem. A real number is rational if and only if the decimal terminates or is eventually periodic.

Some examples

1. $\frac{3}{10}$ terminates: $\frac{3}{10} = 0.3$.
2. $\frac{1}{3}$ is periodic: $\frac{1}{3} = 0.333\cdots$.
3. $\frac{1}{12}$ is eventually periodic: $\frac{1}{12} = 0.08333\cdots$.
4. π is irrational so that $\pi = 3.1415\cdots$ is never periodic.
5. $0.1010010001000010000001\cdots = 0.1\overbrace{0}^{1}1\underbrace{00}_{2}1\overbrace{000}^{3}1\underbrace{0000}_{4}1\overbrace{00000}^{\cdots}1\cdots$ is not periodic and so must be irrational.

The period of the decimal expansion of a rational number varies; for example

1. $\frac{1}{11}$ is periodic with period 2: $\frac{1}{11} = 0.090909\cdots$
2. $\frac{1}{7}$ is periodic with period 6: $\frac{1}{7} = 0.142857142857142857\cdots$

These exercises investigate the period of some simple decimals.

- Three repetitions of a pattern followed by $\cdots$ indicate the pattern recurs indefinitely.
- Please perform long multiplication and look for patterns in the repeated digits.
- Please do not reformulate as fractions!

Exercises

1. Using long multiplication find each of the following terminating decimals

$$(0.11)^2, \quad (0.111)^2, \quad (0.1111)^2, \quad (0.11111)^2$$

2. What pattern occurs when we square longer finite sequences $0.111\cdots 1$?
3. Using long multiplication find the finite decimal multiplied by $0.111\cdots$:

$$\begin{gathered} 0.1 \times 0.111\cdots \\ 0.11 \times 0.111\cdots \\ 0.111 \times 0.111\cdots \end{gathered}$$

4. Using long multiplication find the value and period of $(0.111\cdots)^2$.
5. Rather than single-digit columns, create "double-digit columns" and write 0.0101 as $0.(01)(01)$. Find $(0.0101)^2$. That is, calculate

$$\begin{array}{rcccc} & 0 & . & (01) & (01) \\ \times & 0 & . & (01) & (01) \\ \hline & 0 & . & \cdots & \end{array}$$

using an adapted long multiplication method.

6. Find each of the following finite decimals $(0.010101)^2$, $(0.01010101)^2$.
7. Using long multiplication find the value and period of $(0.010101\cdots)^2$.
8. By grouping in blocks of three digits, or otherwise (!), and using adapted long multiplication find the value and period of $(0.001001001\cdots)^2$.

Background

This sequence of problems was inspired by [2, 3].

> This confidence that we can use decimal fractions to give arithmetical sense to expressions like $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ is what we still teach, and it still forms the basis of our understanding of much of mathematics. I would not change it, and even regret that this approach may be being diluted. But, underlying it all, is what I think is the confidence trick. [2, p. 21]

The mathematics of floating point numbers is essential in all modern computing, and the mathematics of contemporary floating point algorithms remains important, see [1].

Underlying this exploration is the fact that if you write $\frac{1}{999^2} = \frac{1}{998001}$ as a decimal you get all three digit numbers from 000 to 999 except 998,

$$\frac{1}{998001} = 0.000, 001, 002, 003, 004, 005, 006, 007, \cdots 997, 999, \cdots$$

$$\begin{aligned}
\frac{1}{998001} = \frac{1}{999^2} = \frac{1}{(1000-1)^2} &= \frac{1}{1000^2}\frac{1}{\left(1-\frac{1}{1000}\right)^2} \\
&= \frac{1}{1000^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right)\Bigg|_{x=\frac{1}{1000}} \\
&= \frac{1}{1000^2}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n\Bigg|_{x=10^{-3}} \\
&= 10^{-6}\sum_{n=1}^{\infty} n x^{n-1}\Bigg|_{x=10^{-3}} \\
&= 10^{-6}\sum_{n=1}^{\infty} n 10^{-3n+3} \\
&= \sum_{n=1}^{\infty} n 10^{-3(n+1)} \\
&= 1\times 10^{-6} + 2\times 10^{-9} + 3\times 10^{-12}\cdots \\
&= 0.000001 + 0.000000002 + 0.000000000003 + \cdots \\
&= 0.000, 001, 002, 003, 004, \cdots
\end{aligned}$$

Note, the pattern continues until

$$\cdots + 998\times 10^{-2997} + 999\times 10^{-3000} + 1000\times 10^{-3003}\cdots$$

Then

$$\begin{aligned}
& 998\times 10^{-2997} + 999\times 10^{-3000} + 1000\times 10^{-3003} \\
= \quad & 10^{-2997}\,(998 + 0.999 + 0.001) \\
= \quad & 10^{-2997}\,(998 + 1) \\
= \quad & 999\times 10^{-2997}
\end{aligned}$$

This carry effectively removes the 998 term from the sum.

References

[1] Goldberg, D., What Every Computer Scientist Should Know About Floating-Point Arithmetic, *Computing Surveys*, 1991, 23(1), p. 5–48, DOI: 10.1145/103162.103163.

[2] Fowler, D., 400 Years of decimal fractions, *Mathematics Teaching*, 1985 (110), p. 20–21.

[3] Fowler, D., 400.25 Years of decimal fractions, *Mathematics Teaching*, 1985 (111), p. 30–31.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **September 30, 2025**.

5051. *Proposed by Giuseppe Fera.*

A climber starts at altitude 0 at time 0. Until he reaches the mountain top, every second he tosses a biased coin that gives heads with probability p such that $0 < p < 1/2$ and tails with probability $1 - p$. If the coin shows heads, the climber moves up one meter; otherwise, he either moves down one meter or he remains at altitude 0. The mountain top is at an altitude N meters, where N is a positive integer. Find the average climbing time to the top.

5052. *Proposed by Tatsunori Irie.*

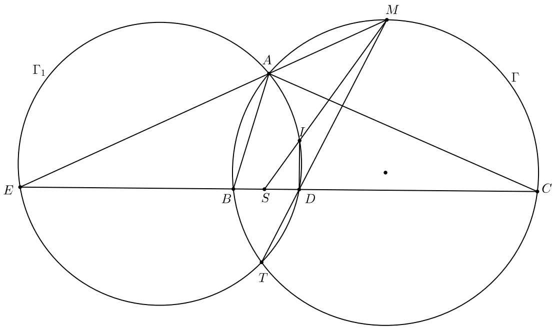
Let p be a prime number with $p \geq 3$ and let m be a natural number that is relatively prime to p and that is not congruent to 1 modulo p. Also, let n be an integer with $n \geq 2$. Define

$$N = \frac{(1+p)^{p^{m-1}} - 1}{p^{m-1}} + p(m-1).$$

Determine whether it is possible for N, when expressed in base n, to be a p-digit number consisting solely of the digit 1.

5053. *Proposed by Michel Bataille.*

Let triangle ABC with $AB \neq AC$ be inscribed in circle Γ and let D be the projection of its incenter I onto BC. Let M be the midpoint of the arc BC of Γ containing A and let the line MI intersect BC at S. Prove that the line AS, the line MD, Γ and the circumcircle of ΔAID have a common point.



5054. *Proposed by Eugen J. Ionascu, modified by the Editorial Board.*

Find the smallest possible number of 1's in the binary representation of a positive integer which is a multiple of 2025.

5055. *Proposed by Bing Jian.*

In triangle ABC, let AD be the altitude from vertex A to side BC. Let $DE \perp AB$ and $DF \perp AC$, with points E and F lying on AB and AC, respectively. Let points P and Q lie on the line AB and the line AC, respectively, such that $DP \parallel AC$ and $DQ \parallel AB$. Prove that the lines PQ, EF, and BC are concurrent or parallel.

5056. *Proposed by Mihaela Berindeanu.*

If $a_n = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{n^3}$ $\forall n \in \mathbb{N}^*$, show that $\frac{1}{a_1^2} + \frac{1}{8a_2^2} + \frac{1}{27a_3^2} + \cdots + \frac{1}{n^3 a_n^2} < \frac{6}{5}$.

5057. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Evaluate

$$\lim_{n\to\infty} n(-1)^n \sum_{k=1}^{n} \frac{(-1)^k}{k(n-k)!}.$$

5058. *Proposed by Vasile Cîrtoaje.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that at most one of them is less than 1 and $a_1^3 + a_2^3 + \cdots + a_n^3 = n$. Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq a_1 + a_2 + \cdots + a_n.$$

5059. *Proposed by Tatsunori Irie, modified by the Editorial Board.*

Let T be the Fermat-Torricelli point of triangle ABC with angles less than 120 degrees, that is T is the point such that the sum of the three distances from each of the three vertices of the triangle to the point is the smallest possible. Prove that Area(ABC) is less than the area of an equilateral triangle with sides equal to $TA + TB + TC$.

5060. *Proposed by Nguyen Van Huyen.*

Find the smallest constant k such that the inequality

$$2(a+b+c+abc) + k[(ab-1)^2 + (bc-1)^2 + (ca-1)^2] \geqslant (a+1)(b+1)(c+1),$$

holds for all non-negative real numbers $a,\ b,\ c.$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 september 2025**.

5051. *Soumis par Giuseppe Fera.*

Un alpiniste commence à l'altitude 0 au temps 0. Jusqu'à ce qu'il atteigne le sommet de la montagne, il lance chaque seconde une pièce biaisée qui donne *face* avec une probabilité p telle que $0 < p < 1/2$ et *pile* avec une probabilité $1 - p$. Si la pièce tombe sur *face*, l'alpiniste monte d'un mètre; sinon, il descend d'un mètre ou reste à l'altitude 0. Le sommet de la montagne se trouve à une altitude de N mètres, où N est un nombre entier positif. Calculez le temps moyen nécessaire pour atteindre le sommet.

5052. *Soumis par Tatsunori Irie.*

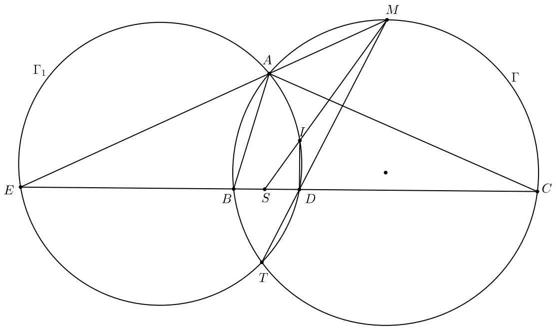
Soit p un nombre premier tel que $p \geq 3$ et soit m un nombre naturel qui est premier avec p et qui n'est pas congruent à 1 modulo p. Soit également n un entier tel que $n \geq 2$. Définissons

$$N = \frac{(1+p)^{p^{m-1}} - 1}{p^{m-1}} + p(m-1).$$

Déterminez s'il est possible que N, lorsqu'il est exprimé en base n, soit un nombre à p chiffres composé uniquement du chiffre 1.

5053. *Soumis par Michel Bataille.*

Soit le triangle ABC avec $AB \neq AC$ inscrit dans le cercle Γ et soit D la projection sur BC du centre I de son cercle inscrit. Soit M le milieu de l'arc BC de Γ contenant A et soit S le point d'intersection de la droite MI et de BC. Montrez que la droite AS, la droite MD, Γ et le cercle circonscrit à ΔAID ont un point commun.



5054. *Soumis par Eugen J. Ionascu, modifié par le comité de rédaction.*

Trouvez le plus petit nombre possible de 1 dans la représentation binaire d'un nombre entier positif qui est un multiple de 2025.

5055. *Soumis par Bing Jian.*

Dans un triangle ABC, soit AD la hauteur issue du sommet A et abaissée sur BC. Soient $DE \perp AB$ et $DF \perp AC$, où les points E et F appartiennent à AB et AC, respectivement. Soient P et Q deux points situés respectivement sur la droite AB et la droite AC, tels que$DP \parallel AC$ et $DQ \parallel AB$. Montrez que les droites PQ, EF et BC sont concourantes ou parallèles.

5056. *Soumis par Mihaela Berindeanu.*

Si $a_n = \frac{1}{1^3}+\frac{1}{2^3}+...+\frac{1}{n^3}$ $\forall n \in \mathbb{N}^*$, montrez que $\frac{1}{a_1^2}+\frac{1}{8a_2^2}+\frac{1}{27a_3^2}+...+\frac{1}{n^3a_n^2} < \frac{6}{5}$.

5057. *Soumis par Ovidiu Furdui and et Sîntămărian.*

Évaluez l'expression suivante

$$\lim_{n\to\infty} n(-1)^n \sum_{k=1}^{n} \frac{(-1)^k}{k(n-k)!}.$$

5058. *Soumis par Vasile Cîrtoaje.*

Soient $a_1, a_2, \ldots, a_n$ des nombres réels positifs tels que tout au plus un d'entre eux soit inférieur à 1 et que $a_1^3 + a_2^3 + \cdots + a_n^3 = n$. Montrez que

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq a_1 + a_2 + \cdots + a_n.$$

5059. *Soumis par Tatsunori Irie, modifié par le comité de rédaction.*

Soit T un point de Fermat-Torricelli du triangle ABC dont les angles sont inférieurs à 120 degrés, c'est-à-dire que T est un point tel que la somme des trois distances entre chacun des trois sommets du triangle et ce point est la plus petite possible. Montrez que l'aire (ABC) est inférieure à l'aire d'un triangle équilatéral dont les côtés sont égaux à $TA + TB + TC$.

5060. *Soumis par Nguyen Van Huyen.*

Trouvez la plus petite constante k telle que l'inégalité

$$2(a+b+c+abc) + k[(ab-1)^2 + (bc-1)^2 + (ca-1)^2] \geqslant (a+1)(b+1)(c+1),$$

est vérifiée pour tous les nombres réels non négatifs $a,\ b,\ c$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2025: 51(1), p. 32–35.

5001. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with the incircle $\mathit{\Gamma}$, the incenter I and A', B', C' the points of tangency of circle $\mathit{\Gamma}$ with the sides BC, AC and AB. If $A'I \cap B'C' = A_1$, $B'I \cap A'C' = B_1$, $C'I \cap A'B' = C_1$, $AA_1 \cap BC = A_2$, $BB_1 \cap AC = B_2$ and $CC_1 \cap AB = C_2$, show that $\overrightarrow{AA_2} + \overrightarrow{BB_2} + \overrightarrow{CC_2} = \overrightarrow{0}$.

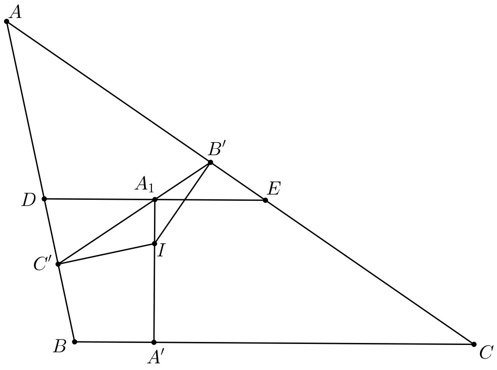
All but one of the 11 submissions were correct and complete; we feature the solution by Michel Bataille.

As will be shown by the lemma below, A_2, B_2, C_2 are the midpoints of BC, CA, AB, respectively. If G is the centroid of ABC, we thus have

$$\overrightarrow{AA_2} + \overrightarrow{BB_2} + \overrightarrow{CC_2} = \frac{3}{2}\left(\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG}\right) = \overrightarrow{0},$$

as desired.

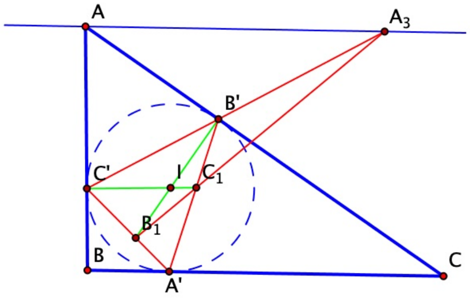
Lemma. Let the parallel to BC through A_1 intersect AB at D and AC at E. Then A_1 is the midpoint of DE and, therefore, AA_1 intersects BC at its midpoint.
Proof.



Since $\angle IA_1D = \angle IC'D = 90°$, the points A_1 and C' are on the circle with diameter ID. Also, the angles $\angle IC'B'$ and $\angle IDA_1$ are acute (because $\Delta B'IC'$ is isosceles while ΔIA_1D is right-angled) and are subtended by the arc IA_1 of this circle. It follows that $\angle IC'B' {=} \angle IDA_1$. Similarly, we obtain $\angle IEA_1 = \angle IB'C'$

and, therefore, ΔDIE is isosceles with $ID = IE$. Since IA_1 is its altitude from I, A_1 is the midpoint of DE. Segment BC being the image of DE of a homothety with center A, AA_1 intersects BC at its midpoint.

Comment by Chetan Balwe. Using the fact that AA_2 is a median of ΔABC, one can show (by chasing cross-ratios) that if A_3 is the point of intersection of $B'C'$ and B_1C_1, then AA_3 is parallel to BC. This is rather interesting and I wonder if there is an elegant ("non-computational") way to see this.



5002. *Proposed by Max A. Alekseyev.*

Let m and n be positive integers such that n does not divide m. Prove that

$$\cot\frac{\pi m}{n} = \frac{1}{n}\sum_{j=0}^{n-1}(2n-1-2j)\sin\frac{2\pi mj}{n}.$$

We received 15 submissions and they were all complete and correct. We present the following solution by the majority of solvers.

Let $\omega = \exp(2\pi i/n)$. Then

$$\sin\frac{2\pi mj}{n} = \mathfrak{Im}\exp(2\pi mj/n) = \mathfrak{Im}\omega^{mj}$$

for each $0 \le j \le n-1$. It follows that

$$\frac{1}{n}\sum_{j=0}^{n-1}(2n-1-2j)\sin\frac{2\pi mj}{n} = \frac{1}{n}\mathfrak{Im}\sum_{j=0}^{n-1}(2n-1-2j)\omega^{mj}.$$

On the other hand, $n \nmid m$ implies that $\omega^m \neq 1$. Note that

$$(1-\omega^m)\sum_{j=0}^{n-1}(2n-1-2j)\omega^{mj} = \sum_{j=0}^{n-1}(2n-1-2j)\omega^{mj} - \sum_{j=0}^{n-1}(2n-1-2j)\omega^{m(j+1)}$$
$$= 2n - 2\sum_{j=0}^{n-1}\omega^{mj} = 2n.$$

Therefore,

$$\frac{1}{n}\mathfrak{Im}\sum_{j=0}^{n-1}(2n-1-2j)\omega^{mj} = \mathfrak{Im}\frac{2}{1-\omega^m} = \frac{2\mathfrak{Im}(1-\omega^m)}{|1-\omega^m|^2}$$
$$= \frac{2\sin\frac{2\pi m}{n}}{(1-\cos\frac{2\pi m}{n})^2+\sin^2\frac{2\pi m}{n}}$$
$$= \frac{\sin\frac{2\pi m}{n}}{1-\cos\frac{2\pi m}{n}} = \cot\frac{\pi m}{n}.$$

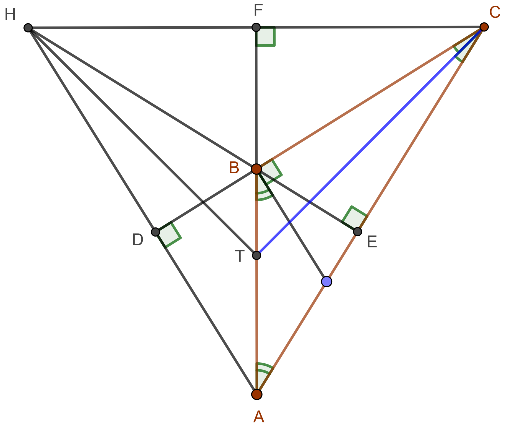
Editor's Comment. Walther Janous pointed out that a similar computation leads to the identity

$$1 = \frac{1}{n}\sum_{j=0}^{n-1}(2n-1-2j)\cos\frac{2\pi mj}{n}.$$

5003. *Proposed by Michael Friday, modified by the Editorial Board.*

Let H be the orthocenter of a triangle ABC for which $B = A + 90°$. Denote by D, E, F the feet of the altitudes from A, B, C respectively, and by T the foot of the bisector of $\angle C$. Prove that the circle with center F and radius FH contains the points C, D, E, and T.

We received 17 submissions of which 14 were correct and complete. We present the solution by Michal Adamaszek, lightly edited.



Denote by Γ the circle with center F and radius FH. Let $\angle FAC = \alpha$. By assumption $\angle ABC = 90^\circ + \alpha$, so

$$\angle BCA = 180^\circ - (\angle BAC + \angle ABC) = 180^\circ - (\alpha + (90^\circ + \alpha)) = 90^\circ - 2\alpha.$$

An an angle exterior to $\triangle ABC$,

$$\angle ABD = \angle BAC + \angle BCA = \alpha + (90^\circ - 2\alpha) = 90^\circ - \alpha,$$

whence

$$\angle FAH = \angle BAD = 90^\circ - \angle ABD = 90^\circ - (90^\circ - \alpha) = \alpha = \angle FAC.$$

Hence the altitude AF of $\triangle CAH$ is also the bisector of $\angle CAH$, so $\triangle CAH$ is isosceles and F is the midpoint of CH. Hence CH is a diameter of Γ.

As an exterior angle to $\triangle ATC$,

$$\angle FTC = \angle TAC + \angle TCA = \angle BAC + \frac{1}{2}\angle BCA = \alpha + \frac{1}{2}(90^\circ - 2\alpha) = 45^\circ.$$

By symmetry, $\angle HTF = 45^\circ$ as well, so $\angle CTH = 90^\circ$. As all the points E, T and D span 90° angles with respect to the diameter CH of Γ we conclude that E, T and D lie on Γ.

5004. *Proposed by Rajesh Sharma and Vijay Kumar.*

Let z_1, z_2, z_3, z_4 denote four distinct complex numbers. Prove that following conditions are equivalent:

(i) The points z_1, z_2, z_3, z_4 form the vertices of a rectangle in the plane.

(ii) $m_3 = 0$ and $\operatorname{Re} \overline{m_2}\sqrt{m_4 - m_2^2} = 0$, where $m_r = \frac{1}{4}\sum_{j=1}^4 (z_j - \widetilde{z})^r$ and $\widetilde{z} = \frac{1}{4}\sum_{j=1}^4 z_j$.

All seven of the submissions were correct, although a few failed to explicitly state the necessary assumption that the rectangle must be labeled so that z_1 is the vertex opposite z_3. We feature the solution by Chetan Balwe.

For $i = 1, \ldots 4$, let $w_i = z_i - \tilde{z}$. Thus

$$m_r = \frac{1}{4}\sum_{i=1}^{4} w_i^r.$$

Under the assumption that the vertex z_2 is opposite z_4, we have $w_1 = -w_3$ and $w_2 = -w_4$, whence

$$\begin{aligned} m_4 - m_2^2 &= (2w_1^4 + 2w_2^4)/4 - (2w_1^2 + 2w_2^2)^2/4^2 \\ &= (w_1^4 + w_2^4)/2 - (w_1^2 + w_2^2)^2/4 \\ &= (w_1^2 - w_2^2)^2/4. \end{aligned}$$

Consequently,

$$\sqrt{m_4 - m_2^2} = \pm(w_1^2 - w_2^2)/2. \tag{1}$$

Proof of (i) → (ii): Suppose that $z_1, \ldots, z_4$ are vertices of a rectangle in $\mathbb{C}$, with z_1 opposite z_3.

Clearly,

$$m_3 = (w_1^3 + w_2^3 + w_3^3 + w_4^3)/4 = (w_1^3 + w_2^3 - w_1^3 - w_2^3)/4 = 0.$$

Note that $|w_1| = |w_3| =: s$, and let $w_1 = se^{i\alpha}$ and $w_2 = se^{i\beta}$. Then from (1) we have

$$\begin{aligned}\overline{m}_2\sqrt{m_4 - m_2^2} &= \pm(1/4)(w_1^2 - w_2^2)(\overline{w_1^2 + w_2^2}) \\ &= \pm(1/4)(s^2e^{2i\alpha} - s^2e^{2i\beta})(s^2e^{-2i\alpha} + s^2e^{-2i\beta}) \\ &= \pm(1/4)s^4(e^{2i(\alpha-\beta)} - e^{2i(\beta-\alpha)}) \\ &= \pm(1/4)s^4(2i\sin(2(\alpha-\beta))).\end{aligned}$$

Thus, we see that $\overline{m}_2\sqrt{m_4 - m_2^2}$ is pure imaginary. Therefore (ii) holds, completing the proof that (i) implies (ii).

Proof of (ii) → (i): Let us assume that (ii) holds. Then, as $m_3 = 0$, we have $\sum_{i=1}^4 w_i^3 = 0$. Since $w_4 = -(\sum_{i=1}^3 w_i)$, we obtain the equality

$$(w_1 + w_2 + w_3)^3 - w_1^3 - w_2^3 - w_3^3 = 0.$$

The left-hand side of the above equation equals

$$3(w_1 + w_2)(w_2 + w_3)(w_3 + w_1).$$

It follows that one of the three numbers w_1+w_2, w_2+w_3 and w_3+w_1 is equal to 0. Without loss of generality, let us assume that $w_1 + w_3 = 0$. Then, as $\sum_{i=1}^4 w_i = 0$, we see that $w_2 + w_4 = 0$ as well, and equation (1) again holds. In that equation set $w_1 = r_1e^{i\alpha}$ and $w_2 = r_2e^{i\beta}$, where r_1, r_2 are non-negative real numbers. Then,

$$\begin{aligned}\overline{m}_2\sqrt{m_4 - m_2^2} &= \pm(1/4)(w_1^2 - w_2^2)(w_1^2 + w_2^2) \\ &= \pm(1/4)(r_1^2e^{2i\alpha} - r_2^2e^{2i\beta})(r_1^2e^{-2i\alpha} + r_2^2e^{-2i\beta}) \\ &= \pm(1/4)\left[(r_1^4 - r_2^4) + r_1^2r_2^2(e^{2i(\alpha-\beta)} - e^{2i(\beta-\alpha)})\right] \\ &= \pm(1/4)\left[(r_1^4 - r_2^4) + r_1^2r_2^2(2i\sin(2(\alpha-\beta)))\right].\end{aligned}$$

This number is strictly imaginary if and only if $r_1^4 = r_2^4$; that is, $r_1 = r_2$. This shows that w_1, w_2, w_3, w_4 are the vertices of a rectangle centered at 0. Hence z_1, z_2, z_3, z_4 are the vertices of a rectangle as well, proving that (ii) implies (i).

5005. *Proposed by Daniel Sitaru.*

Let $x, y, z > 0$ such that $xyz = 1$. Show that

$$\left(\frac{x}{1+x+xy}+\frac{y}{1+y+yz}+\frac{z}{1+z+zx}\right)^3 \leq \frac{x^3}{1+x+xy}+\frac{y^3}{1+y+yz}+\frac{z^3}{1+z+zx}$$

There were 25 correct solutions from 23 contributors. We present 4 solutions.

We make some preliminary observations. Let

$$(a,b,c)=\left(\frac{1}{1+x+xy},\frac{1}{1+y+yz},\frac{1}{1+z+zx}\right)$$

and $(u,v,w)=(xa,yb,zc)$. Then $a=z(z+zx+xyz)^{-1}=z(1+z+zx)^{-1}=w$, $b=u$ and $c=v$. Then

$$\begin{aligned} u+v+w=a+b+c &= \frac{1}{1+x+xy}+\frac{x}{x+xy+xyz}+\frac{xy}{xy+xyz+xyzx} \\ &= \frac{1+x+xy}{1+x+xy}=1. \end{aligned}$$

Solution 1, by Henry Ricardo and the proposer (independently).

Since the function $f(t)=t^3$ is convex, we have

$$(ax+by+cz)^3 \leq ax^3+by^3+cz^3,$$

which is the desired result.

Solution 2, by Didier Pinchon.

By the weighted power mean inequality,

$$(ax^3+by^3+cz^3)^{1/3} \geq ax+by+cz=u+v+w,$$

which establishes the result.

Solution 3, by Michel Bataille and Oliver Geupel, independently.

By Hölder's inequality (applied with $(p,q)=(3/2,3)$), we have

$$\begin{aligned} (u+v+w)=(ax+by+cz) &= a^{2/3}a^{1/3}x+b^{2/3}b^{1/3}y+c^{2/3}c^{1/3}z \\ &\leq (a+b+c)^{2/3}(ax^3+by^3+cz^3)^{1/3}. \end{aligned}$$

Cubing this inequality gives the result.

Solution 4, by Boris Čolaković.

Let $(x, y, z) = (\alpha/\beta, \beta/\gamma, \gamma/\alpha)$ and $\delta = \beta\gamma + \gamma\alpha + \alpha\beta$. The left side of the proposed inequality is equal to 1, while the right side ρ is equal to

$$\rho = \frac{1}{\delta}\left(\frac{\alpha^3\gamma}{\beta^2} + \frac{\beta^3\alpha}{\gamma^2} + \frac{\gamma^3\beta}{\alpha^2}\right) = \frac{1}{\delta}\left(\frac{\alpha^3\gamma^3}{\beta^2\gamma^2} + \frac{\beta^3\alpha^3}{\gamma^2\alpha^2} + \frac{\gamma^3\beta^3}{\alpha^2\beta^2}\right).$$

By the Radon inequality,

$$\frac{\lambda^3 + \mu^3 + \nu^3}{\lambda^2 + \mu^2 + \nu^2} \geq \frac{(\lambda + \mu + \nu)^3}{(\lambda + \mu + \nu)^2}$$

applied to $(\lambda, \mu, \nu) = (\beta\gamma, \gamma\alpha, \alpha\beta)$, it follows that

$$\rho \geq \frac{1}{\delta}\frac{(\beta\gamma + \beta\alpha + \alpha\beta)^3}{(\beta\gamma + \beta\alpha + \alpha\beta)^2} = \frac{\delta^3}{\delta \cdot \delta^2} = 1.$$

The desired result follows.

5006. *Proposed by Pericles Papadopoulos.*

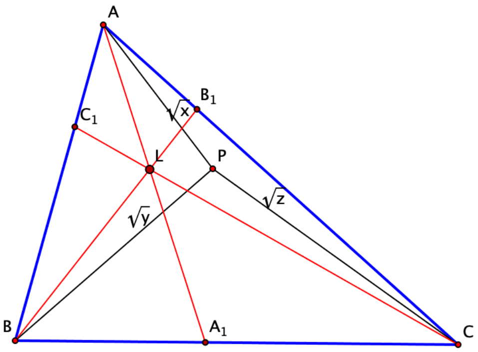
In the plane of a triangle ABC, consider a point P not lying on a side of the triangle. The P-symmedians of the triangles PBC, PAC and PAB meet the sides BC, AC and AB at A_1, B_1 and C_1 respectively. Prove that AA_1, BB_1 and CC_1 concur at a point L satisfying

$$\frac{AL}{LA_1} + \frac{BL}{LB_1} + \frac{CL}{LC_1} = \frac{(PA^2 + PB^2)(PA^2 + PC^2)(PB^2 + PC^2)}{PA^2PB^2PC^2} - 2.$$

All but one of the 12 submissions were correct; our featured solution by C.R. Pranesachar is typical of the majority of the submissions.

As indicated in the accompanying figure, we let

$$PA^2 = x,\ PB^2 = y,\ PC^2 = z.$$



It is known that the symmedian through a vertex of a triangle divides the opposite side in the ratio of the squares of adjacent sides. This means that

$$\frac{BA_1}{A_1C}=\frac{y}{z},\quad \frac{CB_1}{B_1A}=\frac{z}{x},\quad \text{and}\quad \frac{AC_1}{C_1B}=\frac{x}{y},$$

whence

$$\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}\cdot\frac{AC_1}{C_1B}=\frac{y}{z}\cdot\frac{z}{x}\cdot\frac{x}{y}=1.$$

Therefore, by Ceva's theorem, $AA_1,\ BB_1,\ CC_1$ are concurrent, say at L. By Van Aubel's sum theorem (see, for example, *Crux Mathematicorum* **39**(4) (April 2013), pages 182-183),

$$\frac{AL}{LA_1}=\frac{AC_1}{C_1B}+\frac{AB_1}{B_1C}=\frac{x}{y}+\frac{x}{z}.$$

Similarly

$$\frac{BL}{LB_1}=\frac{y}{z}+\frac{y}{x}\quad \text{and}\quad \frac{CL}{LC_1}=\frac{z}{x}+\frac{z}{y}.$$

Hence,

$$\begin{aligned}\frac{AL}{LA_1}+\frac{BL}{LB_1}+\frac{CL}{LC_1} &= \frac{x}{y}+\frac{x}{z}+\frac{y}{z}+\frac{y}{x}+\frac{z}{x}+\frac{z}{y}\\ &= \frac{x^2z+x^2y+y^2x+y^2z+z^2y+z^2x}{xyz}\\ &= \frac{(x+y)(y+z)(z+x)-2xyz}{xyz}\\ &= \frac{(x+y)(y+z)(z+x)}{xyz}-2\\ &= \frac{(PA^2+PB^2)(PA^2+PC^2)(PB^2+PC^2)}{PA^2\,PB^2\,PC^2}-2,\end{aligned}$$

as desired. This completes the proof.

5007. *Proposed by Michel Bataille.*

Let m and n be non-negative integers. Evaluate in closed form:

$$\sum_{k=0}^{n}\binom{m+k}{k}2^{n-k}+\sum_{k=0}^{m}\binom{n+k}{k}2^{m-k}.$$

We received 14 solutions, of which 12 were correct. We present 2 solutions.

Solution 1, by Ulrich Abel and Vitaliy Kushnirevych.

We show the following generalization:

$$(1-x)^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}x^k+x^{n+1}\sum_{k=0}^{m}\binom{n+k}{k}(1-x)^k=1$$

for all integers $m, n \geq 0$ and all real (or complex) numbers x.

Toss a coin, whose probability for heads is p, until reaching $m+1$ tails or $n+1$ tails. The sum of the probabilities of all the possible final outcomes equates to the probability 1, i.e.,

$$\sum_{k=0}^{n} \binom{m+k}{k} p^k \left(1-p\right)^{m+1} + \sum_{k=0}^{m} \binom{n+k}{k} p^{n+1} \left(1-p\right)^k = 1.$$

Since this polynomial identity is valid for $0 < p < 1$, it holds for all complex numbers p. Choosing the special value $p = 1/2$ and multiplying with 2^{n+m+1} yields

$$\sum_{k=0}^{n} \binom{m+k}{k} 2^{n-k} + \sum_{k=0}^{m} \binom{n+k}{k} 2^{m-k} = 2^{n+m+1}.$$

Solution 2, by Didier Pinchon. For m and n nonnegative integers and x a nonzero real number, let $g(m,n,x)$ be the function defined by

$$g(m,n,x) = \sum_{k=0}^{n} \binom{m+k}{k} x^{n-k} = \sum_{k=0}^{n} \binom{m+n-k}{m} x^k. \tag{1}$$

The problem statement is to evaluate in closed form the function

$$f(m,n) := g(m,n,2) + g(n,m,2).$$

Of course, $f(m,n) = f(n,m)$ for $m, n \geq 0$.

For $n = 0$, (1) gives $g(m,0,x) = 1$, while for $m = 0$,

$$g(0,m,x) = \sum_{k=0}^{m} \binom{k}{k} x^{m-k} = \frac{x^{m+1}-1}{x-1}\cdot \tag{2}$$

It follows that

$$f(m,0) = f(0,m) = 2^{m+1},\ m \geq 0. \tag{3}$$

Using Pascal's rule,

$$\binom{m+n-k}{m} = \binom{m+n-k-1}{m} + \binom{m+n-k-1}{m-1},\ m \geq 1,\ n \geq 1,$$

it comes

$$g(m,n,x) = g(m,n-1,x) + g(m-1,n,x),\ m \geq 1,\ n \geq 1, \tag{4}$$

and exchanging m and n,

$$g(n,m,x) = g(n,m-1,x) + g(n-1,m,x),\ m \geq 1,\ n \geq 1. \tag{5}$$

Adding (4) and (5) for $x = 2$ gives

$$f(m,n) = f(m-1,n) + f(m,n-1),\ m \geq 1,\ n \geq 1. \tag{6}$$

For p a nonnegative integer, let C_p be the set

$$C_p = \{(m,n),\ m \geq 0,\ n \geq 0,\ m+n = p\}\,.$$

As $C_0 = \{(0,0)\}$ and $C_1 = \{(1,0),(0,1)\}$, it is already proven by (3) that $f(m,n) = 2^{p+1},\ (m,n) \in C_p$ for $p = 0$ and $p = 1$.

Let us now suppose that, for some $p \geq 2$, $f(m,n) = 2^p$ for $(m,n) \in C_{p-1}$.

Using (3), $f(p,0) = f(0,p) = 2^{p+1}$, and for $m > 0,\ n > 0,\ m+n = p$, (6) proves that $f(m,n) = 2^p + 2^p = 2^{p+1}$.

Therefore $f(m,n) = 2^{p+1}$ for $(m,n) \in C_p,\ p \geq 0$, which means

$$f(m,n) = 2^{m+n+1},\ m,n \geq 0.$$

Editor's Comments. We clarify that if a sequence of coin tosses reaches $m+1$ tails before reaching $n+1$ heads, then the number k of heads is at most m; and in that sequence of total length $m+k+1$, the locations of the heads define a size-k subset of the first $m+k$ tosses—whence the factor $\binom{m+k}{k}$. An equivalent probabilistic interpretation uses random walks on a grid, as in the solutions by José Luis Arregui and W. Janous, the latter of whom described this problem as "one of the evergreens since the days of Stefan Banach". A few solvers used generating functions but most solutions were elementary and *ad hoc.* J. Santmyer simply appealed to the formula

$$\begin{aligned}(a-b)^{m+n+1} \quad &= \quad (z-b)^{n+1}\sum_{k=0}^{m}(-1)^k\binom{n+k}{k}(z-a)^k(a-b)^{m-k} \\ &\quad + (-1)^{m+1}(z-a)^{m+1}\sum_{k=0}^{n}\binom{m+k}{k}(z-b)^k(a-b)^{n-k}\end{aligned}$$

valid for any $a,b,z \in \mathbb{C}$ with $a \neq b$ and $m,n \in \mathbb{N}$, derived from Problem 1268 of the *College Mathematics Journal* (Volume 55, Issue 1, January 2024).

5008. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with angles denoted by A, B, C. Prove that

$$\sum_{k=A,B,C}\left(\sec\frac{k}{2} - \frac{3}{2}\tan\frac{k}{2}\right) \leq \frac{\sqrt{3}}{2}.$$

There were 5 correct solutions, 1 incomplete solution and 1 incorrect solution. We present the solution by the proposer.

We need two preliminary results. First, for $t \in \mathbb{R}$

$$\begin{aligned}\tan\left(\frac{\pi}{4} - t\right) &= \frac{1 - \tan t}{1 + \tan t} = \frac{\cos t - \sin t}{\cos t + \sin t} = \frac{1 - 2\cos t \sin t}{\cos^2 t - \sin^2 t} \\ &= \frac{1 - \sin 2t}{\cos 2t} = \sec 2t - \tan 2t.\end{aligned}$$

Secondly, the Popoviciu inequality provides that

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \ge 2\left[f\left(\frac{y+z}{2}\right) + f\left(\frac{x+x}{2}\right) + f\left(\frac{x+y}{2}\right)\right]$$

whenever $f(t)$ is a convex function and $x, y, z \in \mathbb{R}$.

Apply this inequality to the convex function $f(t) = \tan t$ and use the fact that $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ to get

$$\begin{aligned}\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \sqrt{3} &\ge 2\left[\tan\left(\frac{B}{4} + \frac{C}{4}\right) + \tan\left(\frac{C}{4} + \frac{A}{4}\right) + \tan\left(\frac{A}{4} + \frac{B}{4}\right)\right] \\ &= 2\left[\tan\left(\frac{\pi}{4} - \frac{A}{4}\right) + \tan\left(\frac{\pi}{4} - \frac{B}{4}\right) + \tan\left(\frac{\pi}{4} - \frac{C}{4}\right)\right] \\ &= 2\left(\sec\frac{A}{2} + \sec\frac{B}{2} + \sec\frac{C}{2}\right) - 2(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}).\end{aligned}$$

Rearranging the terms gives the desired inequality.

Comments by the editor. This problem still awaits a straightforward solution that does not rely on an obscure inequality. A natural approach is to check the convexity properties of the function $g(x) = \sec x - \frac{3}{2}\tan x$ for $0 \le x \le \frac{\pi}{2}$; but this takes us only so far. We have that

$$g'(x) = \sec x \tan x - \tfrac{3}{2}\sec^2 x = \sec^2 x(\sin x - \tfrac{3}{2})$$

and

$$g''(x) = \sec^3 x(1 + \sin^2 - 3\sin x) = \sec^3 x[(\sin x - \tfrac{3}{2})^2 - \tfrac{5}{4}].$$

Then $g'(x) < 0$. Also, $g''(x) \ge 0$ when $0 \le \sin x \le \frac{3-\sqrt{5}}{2}$, and $g''(x) \le 0$ when $\frac{3-\sqrt{5}}{2} \le \sin x \le 1$. Let $0 < \theta < \pi/2$ and $\sin\theta = \frac{3-\sqrt{5}}{2}$. Note that $\theta < \pi/6$. Suppose that angle A, B and C all exceed 2θ. Since $g(x)$ is concave on $[\theta, \pi/2]$ and $A + B + C = \pi$,

$$g(\tfrac{A}{2}) + \tfrac{B}{2} + \tfrac{C}{2}) \ge 3g(\tfrac{\pi}{6}) = 3/(2\sqrt{3}) = \sqrt{3}/2,$$

giving the result in this case.

The remaining possibilities present a much thornier situation. Four solvers took an analytic approach to identify a maximum of a multivariate function on a bounded region that led to formidable computations; one resorted to *Maple* to sort things out. One broke the analysis into three cases according as one, two or three of the half-angles exceeded θ.

5009. *Proposed by Vasile Cirtoaje.*

Let a, b, c, d be nonnegative real numbers such that $ab + bc + cd = 7$. Prove that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} \geq \frac{3}{2}.$$

We received 5 solutions, of which 2 were correct and complete. We present the solution by the proposer.

By the Cauchy-Schwarz inequality, we have

$$((a+1)b + (d+1)c)\left(\frac{1}{a+1} + \frac{1}{d+1}\right) \geq (\sqrt{b} + \sqrt{c})^2,$$

$$\frac{1}{a+1} + \frac{1}{d+1} \geq \frac{b + c + 2\sqrt{bc}}{b + c - bc + 7}.$$

So, it suffices to show that

$$\frac{b + c + 2\sqrt{bc}}{b + c - bc + 7} + \frac{1}{b+1} + \frac{1}{c+1} \geq \frac{3}{2}.$$

Let

$$s = \frac{b+c}{2}, \qquad p = \sqrt{bc}, \qquad s \geq p.$$

We need to show that

$$\frac{2s + 2p}{2s - p^2 + 7} + \frac{2s + 2}{2s + p^2 + 1} \geq \frac{3}{2},$$

which is equivalent to $F \geq 0$, where

$$\begin{aligned} F &= 4s^2 + 8(p-1)s + 3p^4 + 4p^3 - 22p^2 + 4p + 7 \\ &= 4(s + p - 1)^2 + 3p^4 + 4p^3 - 26p^2 + 12p + 3. \end{aligned}$$

For $p \leq \frac{1}{2}$, we have

$$F \geq 3p^4 + 4p^3 - 26p^2 + 12p + 3 > -28p^2 + 12p + 1 = (1 - 2p)(1 + 14p) \geq 0.$$

For $p \geq \frac{1}{2}$, we have $s + p - 1 \geq 2p - 1 \geq 0$, therefore

$$\begin{aligned} F &\geq 4(2p-1)^2 + 3p^4 + 4p^3 - 26p^2 + 12p + 3 \\ &= 3p^4 + 4p^3 - 10p^2 - 4p + 7 \\ &= (p-1)^2(p+1)(3p+7) \\ &\geq 0. \end{aligned}$$

The equality occurs for $a = d = 3$ and $b = c = 1$.

5010. *Proposed by Nguyen Viet Hung, modified by the Editorial Board.*

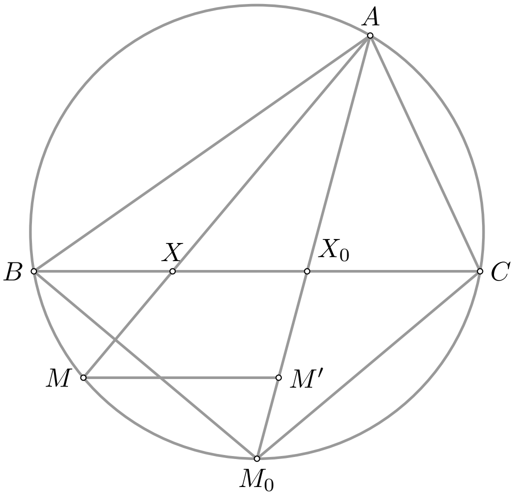
Let M be on the arc BC opposite A of the circumcircle of triangle ABC. Prove that

$$\frac{b+c}{a} \leq \sqrt{\frac{[ABMC]}{[MCB]}}\,,$$

where $a = BC, b = CA, c = AB$, and square brackets represent areas. When does equality hold?

We received 19 solutions for this problem. The majority of the solutions used more or less trigonometry and the formulae for the area of a triangle like Heron's. The first one of the following two is the most geometric solution. The second one connects this problem to Crux problem 4982.

Solution 1, by Chikara Tsugawa. Let M_0 be the midpoint of arc BC and let $X = AM \cap BC$, and $X_0 = AM_0 \cap BC$. Let M' be the intersection of AM_0 and the line through M parallel to BC.



Then,

$$\frac{[ABMC]}{[MCB]} = \frac{AM}{MX} = \frac{AM'}{M'X_0} = 1 + \frac{AX_0}{M'X_0} \geq 1 + \frac{AX_0}{M_0X_0} = \frac{AM_0}{M_0X_0}.$$

Equality holds if and only if $M = M_0$. Note that we have

$$BX_0 = \frac{ac}{b+c}$$

$$CX_0 = \frac{ab}{b+c}$$

Let $\angle M_0AB = \angle M_0AC = \theta$. Then, $\angle M_0BC = M_0CB = \theta$. It follows that

$$BM_0 = CM_0 = \frac{a}{2\cos\theta}$$

Since $\triangle ABM_0 \sim \triangle BX_0M_0$,

$$\begin{aligned} AB : AM_0 &= BX_0 : BM_0 \\ c : AM_0 &= \frac{ac}{b+c} : \frac{a}{2\cos\theta} \\ AM_0 &= \frac{b+c}{2\cos\theta} \end{aligned}$$

Furthermore,

$$\begin{aligned} AB : BM_0 &= BX_0 : X_0M_0 \\ c : \frac{a}{2\cos\theta} &= \frac{ac}{b+c} : X_0M_0 \\ X_0M_0 &= \frac{a^2}{2\,(b+c)\cos\theta} \end{aligned}$$

Therefore,

$$\frac{AM_0}{M_0X_0} = \frac{b+c}{2\cos\theta} \cdot \frac{2(b+c)\cos\theta}{a^2} = \left(\frac{b+c}{a}\right)^2$$

Thus, we have

$$\sqrt{\frac{[ABMC]}{[MCB]}} \geq \sqrt{\frac{AM_0}{M_0X_0}} = \frac{b+c}{a}$$

Equality holds when M is the midpoint of the arc BC.

Note: Chikara Tsugawa mentioned that the above solution could have been done purely geometrically without using theta because it cancels out.

Solution 2, by Kenji Tonozuka. By the result of problem 4982 in Volume 50, Issue 9, November 2024,

$$\frac{MA \cdot MB}{ab} - \frac{MB \cdot BC}{bc} + \frac{MC \cdot MA}{ca} = 1.$$

Multiplying by bc,

$$\frac{c}{a}MA \cdot MB - MB \cdot BC + \frac{b}{a}MC \cdot MA = bc. \tag{1}$$

Let $\theta = \angle BMC$, as $\angle BAC = \pi - \theta$ and $\sin(\pi - \theta) = \sin\theta$, then

$$\begin{aligned} [ABMC] &= \frac{1}{2}bc \cdot \sin\angle BAC + \frac{1}{2}MB \cdot MC \cdot \sin\angle BMC \\ &= \frac{1}{2}(bc + MB \cdot MC)\sin\theta \\ &= \frac{1}{2}\left(\frac{c}{a}MA \cdot MB + \frac{b}{a}MC \cdot MA\right)\sin\theta. \quad \text{(by (1))}. \end{aligned}$$

Since $[\triangle MCB] = \frac{1}{2}MB \cdot MC \cdot \sin\theta$,

$$\frac{[ABMC]}{[\triangle MCB]} = \frac{\frac{c}{a}MA \cdot MB + \frac{b}{a}MC \cdot MA}{MB \cdot MC} = \left(\frac{b \cdot MB + c \cdot MC}{a}\right)\left(\frac{b}{a} \cdot \frac{1}{MB} + \frac{c}{a} \cdot \frac{1}{MC}\right).$$

By Ptolemy's theorem,

$$\begin{aligned} a \cdot MA &= b \cdot MB + c \cdot MC \\ &= \frac{bc}{a^2}\left(\frac{MB}{MC} + \frac{MC}{MB}\right) + \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2 \\ &\geq \frac{2bc}{a^2} + \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2 = \left(\frac{b+c}{a}\right)^2 \end{aligned}$$

and

$$\frac{b+c}{a} \leq \sqrt{\frac{[ABMC]}{[\triangle MCB]}}.$$

The equality holds when $MB = MC$.

* * *