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IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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MathemAttic

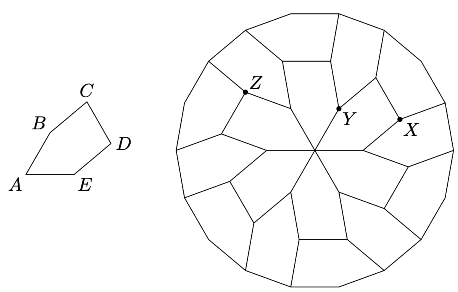
No. 64

The problems featured in this section are intended for students at the secondary school level.

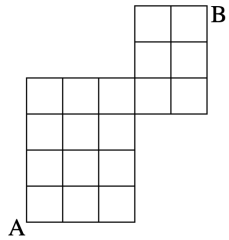
Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **June 15, 2025**.

MA316. A regular 18-gon is dissected into 18 pentagons, each of which is congruent to pentagon $ABCDE$, as shown. All sides of the pentagon have the same length. Show that points X, Y and Z are collinear.

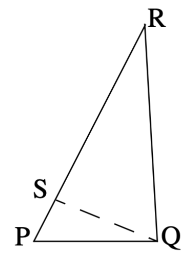


MA317. The figure below shows the map of Squareville, where each city block is of the same length. Two friends, Alexandra and Brianna, live at the corners marked by A and B, respectively. They start walking toward each other's house, leaving at the same time, walking with the same speed, and independently choosing a path to the other's house with uniform distribution out of all possible minimum-distance paths (that is, all minimum-distance paths are equally likely). What is the probability they will meet?



MA318. How many positive five-digit integers are there consisting of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, in which one digit appears once and two digits appear twice? For example, 41174 is one such number, while 75355 is not.

MA319. In triangle PQR, $PQ = 8$, $QR = 13$ and $RP = 15$. Prove that there is a point S on line segment PR, but not at its endpoints, such that PS and QS are also integers.



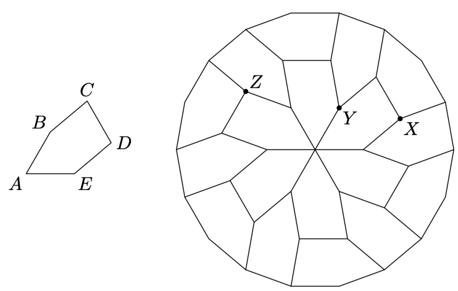
MA320. George has six ropes. He chooses two of the twelve loose ends at random (possibly from the same rope), and ties them together, leaving ten loose ends. He again chooses two loose ends at random and joins them, and so on, until there are no loose ends. Find, with proof, the expected value of the number of loops George ends up with.

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

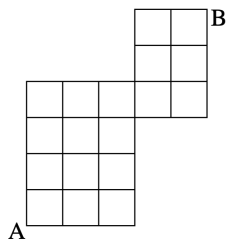
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2025**.

MA316. Un 18-gone régulier est divisé en 18 pentagones, dont chacun est congru au pentagone $ABCDE$, tel qu'illustré. Tous les côtés du pentagone ont la même longueur. Montrez que les points X, Y et Z sont colinéaires.

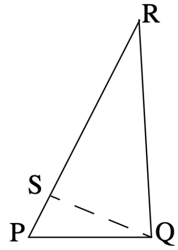


MA317. La figure ci-dessous montre la carte de Squareville, où chaque pâté de maisons a la même longueur. Deux amies, Alexandra et Brianna, vivent respectivement aux coins marqués par A et B. Elles commencent à marcher vers la maison de l'autre, en partant en même temps, en marchant à la même vitesse et en choisissant indépendamment un chemin vers la maison de l'autre avec une distribution uniforme parmi tous les chemins de distance minimale possibles (c'est-à-dire que tous les chemins de distance minimale sont également probables). Quelle est la probabilité qu'elles se rencontrent ?



MA318. Combien y a-t-il d'entiers positifs à cinq chiffres composés des chiffres 1, 2, 3, 4, 5, 6, 7, 8, 9, dans lesquels un chiffre apparaît une fois et deux chiffres apparaissent deux fois ? Par exemple, 41174 est un de ces nombres, mais pas 75355.

MA319. Dans le triangle PQR, $PQ = 8$, $QR = 13$ et $RP = 15$. Montrez qu'il existe un point S sur le segment de droite PR, mais pas à ses extrémités, tel que PS et QS soient aussi des entiers.



MA320. Georges dispose de six cordes. Il choisit au hasard deux des douze bouts libres (peut-être de la même corde) et les noue ensemble, ce qui laisse dix bouts libres. Il choisit à nouveau deux au hasard et les relie, et ainsi de suite jusqu'à ce qu'il n'y ait plus de bouts libres. Trouvez, avec preuve à l'appui, la valeur attendue du nombre de boucles que George finit par avoir.

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2024: 50(9), p. 433–434.

MA291. A jar has only red and blue jellybeans. The probability of drawing a red jellybean is $\frac{4}{11}$. After two red jellybeans and half of the blue jellybeans are removed from the jar, the probability of drawing a red jellybean is $\frac{1}{2}$. How many jellybeans were originally in the jar?

Originally problem T1 from the New York State Math League 2024 Final Questions.

We received 13 submissions, 12 of which were correct. We present the solution by Yeganeh Heydarizadeh.

Let $n(R)$ and $n(B)$ denote the number of red and blue jellybeans, respectively. According to the given information

$$\frac{n(R)}{n(R)+n(B)} = \frac{4}{11}$$

which implies $n(R) = \frac{4}{7}n(B)$. After removing two red jellybeans and half of the blue jellybeans, the probability of drawing a red jellybean becomes

$$\frac{n(R)-2}{\frac{1}{2}n(B)+n(R)-2} = \frac{1}{2}.$$

Thus, we deduce that

$$2n(R) - 4 = \frac{1}{2}n(B) + n(R) - 2$$

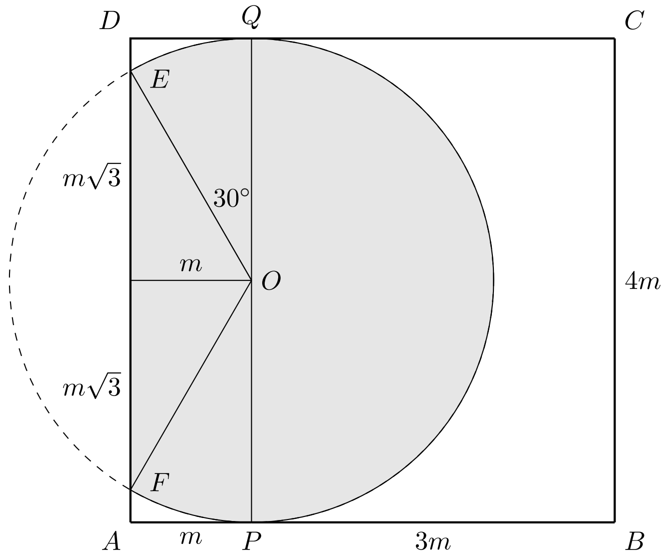
and so $n(R) - \frac{1}{2}n(B) = 2$. Substituting $n(R) = \frac{4}{7}n(B)$, we obtain

$$\frac{4}{7}n(B) - \frac{1}{2}n(B) = 2.$$

From this, we infer that $n(B) = 28$ and so $n(R) = \frac{4}{7}n(B) = 16$. Thus, the total number of jellybeans originally in the jar was 44.

MA292. Suppose that points P and Q lie on sides AB and CD of square $ABCD$, respectively, so that $AP : PB = DQ : QC = 1 : 3$. Given that a point X is chosen uniformly at random within the square, compute the probability that $\angle PXQ$ is obtuse.

Originally problem T4 from the New York State Math League 2024 Final Questions.



We received 4 submissions, all correct. We present the solution by Mihika Bansal, slightly abridged.

Note that $\angle PXQ$ is obtuse if and only if X lies inside the circle with diameter $\overline{PQ}$.

We have that

$$\text{probability} : \frac{\text{shaded area}}{\text{area of square}} = \frac{?}{16m^2}.$$

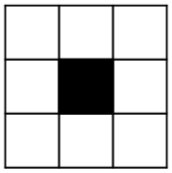
Because you have a right triangle where one of the legs is half of the hypotenuse, angle EOQ is 30°.

$$\begin{aligned}\text{shaded area} &= \text{area of } \triangle OEF + \text{area of sector } EOF \\ &= \frac{1}{2} \cdot 2m\sqrt{3} \cdot m + \frac{240^\circ}{360^\circ} \cdot \pi (2m)^2 \\ &= m^2\sqrt{3} + \frac{2}{3} \cdot \pi \cdot 4m^2 \\ &= m^2\left(\frac{8}{3}\pi + \sqrt{3}\right)\end{aligned}$$

Therefore, we get

$$\text{probability} = \frac{m^2\left(\frac{8}{3}\pi + \sqrt{3}\right)}{16m^2} = \frac{\frac{8}{3}\pi + \sqrt{3}}{16} = \frac{\pi}{6} + \frac{\sqrt{3}}{16} \approx 63.19\%.$$

MA293. Consider the following grid of squares, 8 of which are blank.



Compute the number of ways there are to place the numbers from 1 to 8 in the blank squares in the grid so that no two adjacent numbers share a common integer factor greater than 1.

Originally problem T5 from the New York State Math League 2024 Final Questions.

We received 6 submissions, of which 4 were correct and complete. We present the solution by Richard Hess.

We are asked to compute the number of ways the numbers 1 to 8 can be placed in the frames below so that each pair of adjacent numbers has a greatest common divisor equal to 1. The even numbers must be placed in either the four corners or the four middle spaces. Each set of four even numbers can be placed in 24 ways. Next, 3 can be placed in only two empty cells; 1 can be placed in any of three remaining cells; 5 can be placed in either of the two remaining cells; and 7 goes in the final cell. Thus, the even numbers can be placed in 48 ways and, for each of these ways, the odd numbers can be placed in 12 ways for a total of 576 ways.

MA294. The measure of an exterior angle of a regular polygon is $(2x + 6)^\circ$ for some integer x. The measure of an interior angle of the same regular polygon is $(ax + b)^\circ$, where a and b are positive integers whose sum is 29. Compute the number of diagonals of the regular polygon.

Originally problem T6 from the New York State Math League 2024 Final Questions.

We received eight solutions, and all are correct. We present the solution by Prithwijit De.

If n is the number of sides of the regular polygon then

$$n = \frac{360}{(2x+6)} = \frac{180}{x+3}.$$

Thus $x + 3$ is a divisor of 180. Also,

$$(ax + b) + (2x + 6) = 180$$

and $a + b = 29$ together imply

$$x = \frac{143}{a+2} + 1$$

which shows that $a+2$ is a divisor of 143. Since a and b are positive integers with $a+b=29$, we see that $1<a+2<31$, whence $a+2=11$ or $a+2=13$. That is $a=9$ or $a=11$.

If $a=9$ then $x=13$ but then $x+3=16$ is not a divisor of 180. Hence $a=11$ and $x=12$. Thus

$$n=\frac{180}{x+3}=12$$

and the number of diagonals of the given regular polygon is

$$\frac{n(n-3)}{2}=54.$$

MA295. Compute the number of ordered triples of positive integers (a,b,c) such that $a+b+c+\gcd(a,b,c)=18$.

Originally problem T9 from the New York State Math League 2024 Final Questions.

We received 7 solutions, 3 of which were correct. We present the solution and a more general problem by the Missouri State University Problem Solving Group.

We will use the well-known fact that the number of positive integer solutions to

$$a_1+a_2+\cdots+a_k=n$$

is the binomial coefficient $\binom{n-1}{k-1}$.

It must be the case that $\gcd(a,b,c)$ divides 18.

If $\gcd(a,b,c)=1$, then $a+b+c=17$. On the other hand, if a, b, and c are positive integers such that $a+b+c=17$, then $\gcd(a,b,c)=1$ or 17. In the latter case, $a+b+c\geq 51$, which is impossible. Therefore, the number of solutions in this case is just the number of positive solutions to $a+b+c=17$, which is $\binom{16}{2}=120$.

If $\gcd(a,b,c)=2$, then $a=2a'$, $b=2b'$, and $c=2c'$ with $\gcd(a',b',c')=1$. This gives $a'+b'+c'=8$. The solutions consist of three permutations of $(1,1,6)$ and of $(2,3,3)$ and six permutations of $(1,2,5)$ and of $1,3,4)$. This gives a total of 18 solutions.

If $\gcd(a,b,c)=3$, then $a=3a'$, $b=3b'$, and $c=3c'$ with $\gcd(a',b',c')=1$. This gives $a'+b'+c'=5$. As in the first case, any solution to this equation satisfies $\gcd(a',b',c')=1$, so there are $\binom{4}{2}=6$ solutions here.

This gives a total of $120+18+6=144$ solutions.

We wish to solve the more general problem of determining the number of solutions to

$$\gcd(a_1,\ldots,a_k)+\sum_{i=1}^{k}a_i=n,$$

where the a_i are positive integers.

Let $\psi_k(n)$ denote the number of solutions to

$$\sum_{i=1}^{k} x_i = n, \text{ where } x_i \in \mathbb{Z}^+ \text{ and } \gcd(x_1, \ldots, x_k) = 1. \tag{1}$$

Consider

$$\sum_{i=1}^{k} y_i = n, \text{ where } y_i \in \mathbb{Z}^+ \text{ and } \gcd(y_1, \ldots, y_k) = d. \tag{2}$$

Then $d|n$, $y_i = dy_i'$, and

$$\sum_{i=1}^{k} y_i' = \frac{n}{d}, \text{ where } y_i' \in \mathbb{Z}^+ \text{ and } \gcd(y_1', \ldots, y_k') = 1.$$

There are $\psi_k(n/d)$ solutions to this equation and hence to equation (2). The solutions to equation (1) can be partitioned by $\gcd(x_1, \ldots, x_k) = d$, and the total number of solutions to (1) is $\binom{n-1}{k-1}$, so

$$\binom{n-1}{k-1} = \sum_{d|n} \psi_k\left(\frac{n}{d}\right).$$

By the Möbius inversion formula,

$$\psi_k(n) = \sum_{d|n} \mu(d) \binom{n/d-1}{k-1},$$

where μ is the Möbius mu function for $n > 0$ and $\psi_k(0) = 0$.

We now determine the number of solutions to

$$\gcd(a_1, \ldots, a_k) + \sum_{i=1}^{k} a_i = n, \text{ with } a_i \in \mathbb{Z}^+. \tag{3}$$

Let $d = \gcd(a_1, \ldots, a_k)$. Then $d|n$ and $a_i = a_i'd$ with $\gcd(a_1', \ldots, a_k') = 1$. Hence

$$\sum_{i=1}^{k} a_i' = \frac{n}{d} - 1, \text{ with } a_i' \in \mathbb{Z}^+ \text{ and } \gcd(a_1', \ldots, a_k') = 1.$$

There are $\psi_k(n/d-1)$ solutions to this equation. Therefore the number of solutions to (3) is

$$\sum_{d|n} \psi_k\left(\frac{n}{d} - 1\right).$$

For example, if $k = 3$ and $n = 18$, the number of solutions is

$$\psi_3(17) + \psi_3(8) + \psi_3(5) + \psi_3(2) + \psi_3(1) + \psi_3(0).$$

Now

$$\begin{aligned}
\psi_3(17) &= \mu(1)\binom{16}{2} + \mu(17)\binom{0}{2} = 120 - 0 = 120,\\
\psi_3(8) &= \mu(1)\binom{7}{2} + \mu(2)\binom{3}{2} + \mu(4)\binom{1}{2} + \mu(8)\binom{0}{2} = 21 - 3 + 0 + 0 = 18,\\
\psi_3(5) &= \mu(1)\binom{4}{2} + \mu(5)\binom{0}{2} = 6 - 0 = 6,
\end{aligned}$$

and the rest of the ψ_3 are 0. This gives 144 solutions (in agreement with the result above).

Competition Highlights

The Canada Lynx Mathematical Competition

by Richard Hoshino

The second annual Canada Lynx Mathematical Competition (CLMC) took place on October 3 and 4, 2024. This new contest was created by the Canadian Mathematical Society (CMS), and is aimed at students in grades 7 to 12. The CMS created this competition to foster an interest in mathematics among students regardless of their skill level, to increase student confidence in their math abilities, and to present mathematics as a fun and playful subject to students. The contest is 90 minutes long, consisting of 15 multiple-choice questions.

The first five questions are worth 4 points each, the next five questions are worth 5 points each, and the final five questions are worth 7 points each. Thus, the maximum possible score is 80 points.

The 2024 CLMC attracted 3,058 participants from Canada, the United States, China, Hong Kong, Australia, New Zealand, India, Thailand, Singapore, England, Ireland, Azerbaijan and Rwanda. The median score was 39 out of 80, with 10 students achieving perfect scores, including 7 perfect scores from Canada.

The CLMC is a fun contest taking place at the start of the year, giving students a chance to assess their skills in preparation for other contests later in the Fall semester, most notably the Canadian Open Mathematics Challenge (COMC). The COMC, in turn, is the primary gateway to the elite invitation-only Canadian Mathematical Olympiad (CMO) contest.

We end this column by presenting Question #14 of the 2024 CLMC.

Question #14. For each positive integer n, define $S(n)$ to be the sum of the positive divisors of n, and define $P(n)$ to be the number of prime divisors of n.

For example, $S(20) = 1 + 2 + 4 + 5 + 10 + 20 = 42$, and $P(20) = 2$ because the only prime divisors of 20 are 2 and 5.

If n is a positive integer for which $S(n) > 4n$, what is the minimum possible value of $P(n)$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5 (f) 6

Solution.

Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n, where each p_i is prime, and each a_i is a positive integer.

Every divisor of n must only contain the primes in the set $\{p_1, p_2, \ldots, p_k\}$, and be of the form $p_1^{b_1} \cdot p_2^{b_2} \cdots p_k^{b_k}$, where $0 \le b_i \le a_i$ for each $1 \le i \le k$.

Since each b_i can range from 0 to a_i, there are $a_i + 1$ options for each term $p_i^{b_i}$. And this is true for each index i from 1 to k. Therefore, we observe that $S(n)$, the sum of all of the positive divisors of n, must equal

$$S(n) = (p_1^0 + p_1^1 + \cdots + p_1^{a_1}) \cdot (p_2^0 + p_2^1 + \cdots + p_2^{a_2}) \cdots (p_k^0 + p_k^1 + \cdots + p_k^{a_k}).$$

Consider the first term $(p_1^0 + p_1^1 + \cdots + p_1^{a_1})$, which is the sum of the first $a_1 + 1$ terms of a geometric sequence with first term p_1^0 and common ratio p_1. Letting T be this total, we have the following two equations:

$$\begin{aligned} T &= p_1^0 + p_1^1 + p_1^2 + p_1^3 + \cdots + p_1^{a_1}, \\ p_1 T &= p_1^1 + p_1^2 + p_1^3 + p_1^4 + \cdots + p_1^{a_1+1}. \end{aligned}$$

The second equation is formed by multiplying the first equation by p_1. If we subtract the first equation from the second, we get

$$p_1T - T = (p_1^1 + p_1^2 + p_1^3 + p_1^4 + \cdots + p_1^{a_1+1}) - (p_1^0 + p_1^1 + p_1^2 + p_1^3 + \cdots + p_1^{a_1}) = p_1^{a_1+1} - p_1^0.$$

Therefore, $p_1T - T = p_1^{a_1+1} - p_1^0$, which is equivalent to $T(p_1 - 1) = p_1^{a_1+1} - 1$, or

$$T = \frac{p_1^{a_1+1} - 1}{p_1 - 1}.$$

We have shown that the first term of our product $S(n)$ is equal to $\frac{p_1^{a_1+1}-1}{p_1-1}$, and we get a similar formula for all of the other terms. Therefore,

$$S(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1}.$$

Since $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, we have

$$\frac{S(n)}{n} = \frac{S(n)}{p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}} = \frac{p_1^{a_1+1} - 1}{p_1^{a_1}(p_1 - 1)} \cdot \frac{p_2^{a_2+1} - 1}{p_2^{a_2}(p_2 - 1)} \cdots \frac{p_k^{a_k+1} - 1}{p_k^{a_k}(p_k - 1)}.$$

And this simplifies to

$$\frac{S(n)}{n} = \frac{p_1 - \frac{1}{p_1^{a_1}}}{p_1 - 1} \cdot \frac{p_2 - \frac{1}{p_2^{a_2}}}{p_2 - 1} \cdots \frac{p_k - \frac{1}{p_k^{a_k}}}{p_k - 1} < \frac{p_1 - 0}{p_1 - 1} \cdot \frac{p_2 - 0}{p_2 - 1} \cdots \frac{p_k - 0}{p_k - 1}.$$

In the question, we are given that $S(n) > 4n$, i.e., $\frac{S(n)}{n} > 4$. Our goal is to find the minimum possible value of $P(n)$, which equals k, the number of prime divisors of n.

We will show that if $k = 3$, then $\frac{S(n)}{n}$ must be less than 4. Observe that each fraction $\frac{p_i-0}{p_i-1}$ is *maximized* when we make the prime p_i as *small* as possible.

If n has only 2, 3, and 5 as its prime divisors, then the above analysis shows that

$$\frac{S(n)}{n} < \frac{p_1-0}{p_1-1} \cdot \frac{p_2-0}{p_2-1} \cdot \frac{p_3-0}{p_3-1} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{4} = 3.75 < 4.$$

Thus, if $k = 3$, then $\frac{S(n)}{n} < 3.75$, which implies that we cannot have $S(n) > 4n$. A similar argument shows that $S(n) < 4n$ if $k = 2$ or $k = 1$.

We now show that if $n = 2^{100} \times 3^{100} \times 5^{100} \times 7^{100}$, then $S(n) > 4n$. This will prove that $k = 4$ is the desired answer.

When a_i is large, the fraction $\frac{p_1 - \frac{1}{p_1^{a_1}}}{p_1-1}$ is essentially equal to $\frac{p_1-0}{p_1-1}$.

And so, for this value of n, we have

$$\frac{S(n)}{n} \sim \frac{p_1-0}{p_1-1} \cdot \frac{p_2-0}{p_2-1} \cdot \frac{p_3-0}{p_3-1} \cdot \frac{p_4-0}{p_4-1} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} = \frac{105}{24} = 4.375 > 4.$$

We have found a positive integer n with $k = 4$ prime divisors for which $S(n) > 4n$, and proven that if n has at most three prime divisors, then we must have $S(n) < 4n$. Thus, $k = 4$ is the minimum possible value of $P(n)$.

The answer is (d).

Note #1: If $n = 2^5 \times 3^3 \times 5^1 \times 7^1 = 30240$, we can show that $S(n) = 4n$. And so any multiple of this number, say $n = 2^6 \times 3^3 \times 5^1 \times 7^1 = 60480$, is guaranteed to satisfy $S(n) > 4n$.

Note #2: There is one integer $n < 30240$ for which $S(n) > 4n$. Can you find it?

OLYMPIAD CORNER

No. 432

The problems in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **June 15, 2025**.

OC726. A sequence $\{u_n\}$ is defined by

$$u_0 = 2,\ u_1 = 5/2,\ u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \quad \text{for} \quad n = 1, 2, \dots$$

Prove that for positive integers n, $[u_n] = 2^{[2^n-(-1)^n]/3}$, where $[x]$ denotes the greatest integer less than or equal to x.

OC727. Let C, M and S be positive integers that satisfy the equation

$$C^2 + M^2 + 2^{2024} = 25 \times 2^S.$$

Given that $C \leq M$, what is the solution (C, M, S) of the equation?

OC728. Anne and Bob will play a number game, where they use one piece of paper with only the number 2024 written on it. Starting with Anne, each player takes turns writing down a positive integer that is at least one-third of the previous number but less than the previous number. For the first step, Anne should write a number less than 2024 but no less than $\frac{2024}{3}$. The player who writes down the number 1 wins the game. Determine, with complete proof, which player has a winning strategy. What is the strategy?

OC729. Let $\mathbb{R}^+ = (0, \infty)$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and polynomials $P(x)$ with non-negative real coefficients such that $P(0) = 0$ which satisfy the following equality for all reals $x > y > 0$:

$$f(f(x) + P(y)) = f(x - y) + 2y.$$

OC730. Let ABC be an acute-angled triangle with $AC > AB$ and let D be the foot of the A-angle bisector on BC. The reflections of lines AB and AC in line BC meet AC and AB at points E and F respectively. A line through D meets AC and AB at G and H respectively such that G lies strictly between A and C while H lies strictly between B and F. Prove that the circumcircles of $\triangle EDG$ and $\triangle FDH$ are tangent to each other.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2025**.

OC726. Une suite $\{u_n\}$ est définie ainsi :

$$u_0 = 2,\ u_1 = 5/2,\ u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \quad \text{pour} \quad n = 1, 2, \ldots$$

Montrez que pour tous les entiers positifs n, $[u_n] = 2^{[2^n - (-1)^n]/3}$ où $[x]$ désigne le plus grand entier $\leq x$.

OC727. Soient C, M et S des entiers positifs qui satisfont l'équation

$$C^2 + M^2 + 2^{2024} = 25 \times 2^S.$$

Étant donné que $C \leq M$, quelle est la solution (C, M, S) de l'équation?

OC728. Anne et Bob vont jouer à un jeu de chiffres en utilisant une feuille de papier sur laquelle est inscrit le nombre 2024. En commençant par Anne, chaque joueur écrit à tour de rôle un nombre entier positif qui représente au moins un tiers du nombre précédent, mais qui est inférieur à ce dernier. Pour la première étape, Anne doit écrire un nombre inférieur à 2024 mais pas inférieur à $\frac{2024}{3}$. Le joueur qui écrit le nombre 1 gagne la partie. Déterminez, avec une preuve complète, quel joueur a une stratégie gagnante. Quelle est la stratégie ?

OC729. Soit $\mathbb{R}^+ = (0, \infty)$ l'ensemble des nombres réels positifs. Trouvez toutes les fonctions $f : \mathbb{R}^+ \to \mathbb{R}^+$ et les polynômes $P(x)$ à coefficients réels non négatifs tels que $P(0) = 0$ qui satisfont l'égalité

$$f(f(x) + P(y)) = f(x - y) + 2y$$

pour tous les nombres réels $x > y > 0$.

OC730. Soit ABC un triangle acutangle avec $AC > AB$ et soit D le pied de la bissectrice de l'angle A sur BC. Les réflexions des droites AB et AC sur la droite BC rencontrent AC et AB aux points E et F respectivement. Une droite passant par D rencontre AC et AB respectivement aux points G et H de sorte que G est strictement compris entre A et C tandis que H est strictement compris entre B et F. Montrez que les circonférences de $\triangle EDG$ et $\triangle FDH$ sont tangentes l'une à l'autre.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2024: 50(9), p. 451–453.

OC701. We will say that we have shortened a number if we have erased its last digit. A natural number greater than a million is such that if we shorten it, we get the square of a natural number, if we shorten this square, we get the cube of a natural number, by shortening this cube, we get the fourth power of a natural number, and by shortening this fourth power, we get the fifth power of a natural number. Prove that if we shorten this fifth power, we get the sixth power of a natural number.

Originally Problem 9, Day 2 of the Leonhard Euler Competition 2023.

We received one correct solution. We present the solution by C. R. Pranesachar.

Let M be number, which satisfies the given conditions. Since M is greater than one million, we infer that M has at least seven digits. Let M end in the four digits $x_3x_2x_1x$, that is,

$$M = \cdots x_3x_2x_1x.$$

By the given conditions, we have

$$\left\lfloor \frac{M}{10} \right\rfloor = d^2, \quad \left\lfloor \frac{d^2}{10} \right\rfloor = c^3, \quad \left\lfloor \frac{c^3}{10} \right\rfloor = b^4, \quad \left\lfloor \frac{b^4}{10} \right\rfloor = a^5,$$

for positive integers a, b, c, d.

From

$$d^2 = 10c^3 + x_1 \quad \text{and} \quad c^3 = 10b^4 + x_2,$$

we get

$$d^2 = 100b^4 + 10x_2 + x_1.$$

Also, after deletion of the last three digits x, x_1, x_2 in M we get that M has at least 4 digits, so b^4 has at least 4 digits, that is, $b^4 \geq 1000$ and so $b > 5$.

Since $0 \leq x_1, x_2 \leq 9$ and $b > 5$ we have

$$(10b^2)^2 = 100b^4 \leq d^2 < 100b^4 + 100 < (10b^2 + 1)^2 = 100b^4 + 20b^2 + 1.$$

Hence $d = 10b^2$, forcing $x_1 = x_2 = 0$. Thus

$$c^3 = 10b^4 \quad \text{and} \quad d^2 = 10c^3.$$

Both show that b, c, d are multiples of 10. In the same way, from $b^4 = 10a^5 + x_3$, we see that b^4 is a multiple of 10. Hence $x_3 = 0$ and $b^4 = 10a^5$. Therefore a, b, c, d are all multiples of 10.

Combining

$$b^4 = 10a^5, \quad c^3 = 10b^4, \quad d^2 = 10c^3$$

implies

$$(10a)^5 = (10b)^4 = (10c)^3 = (10d)^2 = T,$$

for some T. Thus T is a number which is simultaneously a square, a cube, a fourth power, and a fifth power. Its exponent must be a multiple of $\operatorname{lcm}(2,3,4,5) = 60$. Since each is also a multiple of 10, we get $T = 10^{60}m^{60}$ for some positive integer m.

From $d^2 = \dfrac{T}{10}$ and $M = 10d^2 + x$, it follows

$$M = 10\left(\frac{T}{10}\right) + x = 10^{59}m^{60} + x.$$

Removing digits from the end of M indeed yields successively

$$\left\lfloor \frac{M}{10} \right\rfloor = (m^{30}\,10^{29})^2, \quad \left\lfloor \frac{M}{10^2} \right\rfloor = (m^{20}\,10^{19})^3,$$

$$\left\lfloor \frac{M}{10^3} \right\rfloor = (m^{15}\,10^{14})^4, \quad \left\lfloor \frac{M}{10^4} \right\rfloor = (m^{12}\,10^{11})^5.$$

Finally, shortening once more gives

$$\left\lfloor \frac{M}{10^5} \right\rfloor = m^{60}\,10^{54} = \left(m^{10}\,10^9\right)^6,$$

which is a perfect sixth power. This completes the proof.

OC702. There are two piles of stones on the table, in which there are respectively 100 and 101 stones. Two people play a game making their own move at each turn. During a turn, it is allowed to take a pile, remove a certain number of stones from it (at least one) and split the remaining stones in this pile into two non-empty piles. The one who cannot make a move loses. Who has a winning strategy: the one who makes the first move, or his opponent?

Originally Problem 10, Day 2 of the Leonhard Euler Competition 2023.

We received one solution. We present the solution by Roy Barbara.

The first player (the one who makes the first move) has a winning strategy. More generally, let the game starts with two piles of n stones and m stones respectively, where $m \geq n$ are positive integers. We have the result:

(a) If $m = n$, then, the second player has a winning strategy.

(b) If $m \geq n + 2$, then, the first player has a winning strategy.

(c) If $m = n + 1$, then, the first player has a winning strategy if n is even and the second player has a winning strategy if n is odd.

Let A denote the first player and B the second player. W.S. stands for "winning strategy" and r-pile stands for "pile of r stones".

At each moment, the state S of the game can be represented by a pair $[M, M']$ of multisets of positive integers, where an integer r in M or in M' represents an r-pile. Example: $S = [M, M'] = [\{1, 1, 1, 2, 3, 3, 5, 8\}, \{2, 2, 19, 31\}]$ is the state S where M represents the collection of three 1-piles, one 2-pile, two 3-piles, one 5-pile and one 8-pile and M' represents the collection of two 2-piles, one 19-pile and one 31-pile. Note that the 1-piles and the 2-piles being unusable can be omitted. Thus, the state above can be written as $S = [\{3, 3, 5, 8\}, \{19, 31\}]$.

Definition 1: The state $S = [M, M']$ is symmetric if the multisets M and M' are equal.

Definition 2: Let $S = [M, M']$ be a symmetric state. Suppose that player X selects an r-pile ($r \geq 3$) from M (resp. M'), removes r_1 stones from this pile and leaves a r_2-pile and a r_3-pile ($r_1 \geq 1$ and $r_1 + r_2 + r_3 = r$). Since $M = M'$, there is at least one r-pile in M' (resp. M). We say that Y makes the "symmetric move" if Y selects an r-pile from M' (resp. M), removes from it r_1 stones and leaves a r_2-pile and a r_3-pile. It should be clear that after one move of X and the symmetric move of Y, the new state of the game remains symmetric.

Proof of (a): The initial state is $S_0 = [M, M'] = [\{n\}, \{n\}]$ (symmetric). If $n = 1$ or 2, A loses. Let $n \geq 3$: if A makes a move on M (resp. M'), then, B makes the symmetric move on M' (resp. M). It results in a new symmetric state. Again, for each move of A, B makes the symmetric move, and so on. Cleary, the first player unable to move is A. B wins.

Proof of (b): The initial state is $S_0 = [M, M'] = [\{n\}, \{m\}]$, $m \geq n + 2$. A removes $m - n - 1$ (≥ 1) stones from the m-pile and leaves an n-pile and a 1-pile. The new state is $S_1 = [\{n\}, \{n, 1\}]$ or $S_1 = [\{n\}, \{n\}]$. Now B must play. By (a), A has a W.S.

Proof of (c): The initial state is $S_0 = [M, M'] = [\{n\}, \{n+1\}]$. We use induction on n: The result is obvious if $n = 1$ or 2. Let $n \geq 3$. We prove the result for n, given that it holds for $n' < n$.

If n is even, A removes one stone from the $(n + 1)$-pile and leaves a $(n - 1)$-pile and a 1-pile. We get the state $S_1 = [\{n\}, \{n - 1, 1\}]$ or $S_1 = [\{n - 1\}, \{n\}]$. Now, B must play. Since $n - 1$ is odd and $n - 1 < n$, then, by the induction hypothesis, A has a W.S.

From now on, we assume n odd and we consider 3 cases:

Case 1: A removes n_1 stones from the n-pile and leaves an n_2-pile and an n_3-pile ($n_i \geq 1$ and $n_1 + n_2 + n_3 = n$). We get the state $S_1 = [\{n_2, n_3\}, \{n + 1\}]$. Then, B removes $n_1 + 1$ stones from the $(n + 1)$-pile and leaves an n_2-pile and an n_3-pile. We get the symmetric state $S_2 = [\{n_2, n_3\}, \{n_2, n_3\}]$. Now, A must play. By (a), B has a W.S.

Case 2: A removes $n_1 \geq 2$ stones from the $(n + 1)$-pile and leaves an n_2-pile and an n_3-pile ($n_i \geq 1$ and $n_1+n_2+n_3 = n+1$). We get the state $S_1 = [\{n\}, \{n_2, n_3\}]$. Then, B removes $n_1 - 1$ (≥ 1) stones from the n-pile and leaves an n_2-pile and an n_3-pile. We get the symmetric state $S_2 = [\{n_2, n_3\}, \{n_2, n_3\}]$. Now, A must play. By (a), B has a W.S.

Case 3: A removes *exactly one* stone from the $(n+1)$-pile and leaves an n_2-pile and an n_3-pile with $n_2, n_3 \geq 1$ and $1 + n_2 + n_3 = n + 1$. We get the state $S_1 = [\{n\}, \{n_2, n_3\}]$. Since $n_2 + n_3 = n$ is *odd*, one of n_2, n_3, say n_2, is *even* ($n_2 \geq 2$). Then, B removes *one* stone from the n-pile and leaves a $(n_2 - 1)$-pile and an n_3-pile. We get the state $S_2 = [\{n_2 - 1, n_3\}, \{n_2, n_3\}]$.

At this stage, view the game as two *parallel* games G and G' with respective initial states

$$S_0 = [\{n_2 - 1\}, \{n_2\}] \quad \text{and} \quad S_0' = [\{n_3\}, \{n_3\}],$$

and where A must start. Since $n_2 - 1$ is *odd* and $n_2 - 1 < n$, then, by the induction hypothesis, B has a W.S. for G. But by (a), B has a W.S. for G'. It clearly follows that B has a W.S. for the parallel games G and G'.

OC703. In the Far Far Away Kingdom there are 100 cities, and every two cities are connected by at most one road. One day the king ordered that one-way traffic be introduced on each road, and that each road be painted white or black. The Minister of Transport proudly announced that after the order was carried out, one could travel from any city to any other by roads, alternating their colors, and so that the first road on the way would be white. What is the smallest number of roads that could be in this country? When traveling from city to city, one can pass through intermediate cities any number of times.

Originally Problem 7, Day 2 of the Leonhard Euler Competition 2023.

We received one solution. We present the solution by Oliver Geupel.

We prove that the answer is 150. Let $(V; B, W)$ be an oriented graph with an even number $|V|$ of vertices and such that the set of directed edges is partitioned into two classes B and W. We call $(V; B, W)$ *nice* if each vertex is reachable from every other vertex by a B/W-alternating path, starting with a W-edge. We call a nice graph $(V; B, W)$ *minimal* if for every nice graph $(V; B', W')$ it holds

$$|B \cup W| \leq |B' \cup W'|.$$

We prove that for every $n \geq 3$ there is a minimal nice graph with $|V| = |W| = 2n$ and $|B| = n$.

First, we construct a nice graph $\mathcal{G}_n = (V; B, W)$ for any $n \geq 3$ where $|V| = 2n$, $|B| = n$, and $|W| = 2n$. Let $V = \{1, 2, \dots, 2n\}$, $B = \{(2k, 2k+1) : 1 \leq k \leq n\}$ and $W = \{(2k-1, 2k), (2k, 2k+2) : 1 \leq k \leq n\}$, where vertices are taken modulo $2n$. An alternating path from vertex $2k - 1$ to a vertex m, beginning with a W-edge, is

$$(2k-1, 2k), (2k, 2k+1), (2k+1, 2k+2), \dots, (m-1, m).$$

Similarly, an alternating path from vertex $2k$ to vertex m, starting with a W-edge, is

$$(2k, 2k+2), (2k+2, 2k+3), (2k+3, 2k+4), \ldots, (m-1, m).$$

Hence, $\mathcal{G}_n$ is nice. (We will see that it is also minimal.)

Note that in every nice graph $(V; B, W)$ it holds $|W| \geq |V|$ by definition.

We now show that every minimal nice graph $(V; B, W)$ with $|V| = 2n \geq 4$ satisfies $|B| \geq n$ (thus $|B \cup W| = 3n$ by the example). The proof is by induction on n.

For the base case $n = 2$, a nice graph with four vertices has at least $4 \cdot 3 = 12$ paths between two distinct vertices but not more than $\binom{4}{2} = 6$ edges; whence there must be a shortest path of length at least two. Thus, $B \neq \varnothing$. If $(k, m) \in B$, any path from vertex m to vertex k has a length greater than one, since $(m, k) \notin B \cup W$; thus $|B| \geq 2$, which completes the base case.

We move on to the induction step and assume that for some fixed $n \geq 2$ every minimal nice graph $(V'; B', N')$ with $|V'| = 2n$ satisfies $|B'| \geq n$; therefore

$$|B'| + |W'| \geq n + |V'| = 3n.$$

Let $(V; B, W)$ be a minimal nice graph such that $|V| = 2(n+1)$. Because the graph $\mathcal{G}_{n+1}$ from the example above is nice, we have

$$|B \cup W| \leq 3(n+1).$$

We have to show that $|B| \geq n+1$. Assume to the contrary that $|B| \leq n$. Then there are two distinct vertices that are not neighbours and not adjacent to a B-edge. Let u and v be such nodes. Each of u and v is adjacent to at least one inbound and one outbound edge. The induced subgraph $(V \smallsetminus \{u, v\}; B', W')$ is again nice. Thus

$$3n \leq |B' \cup W'| \leq |B \cup W| - 4 \leq 3(n+1) - 4,$$

which is impossible. The induction is complete.

Consequently, the example $\mathcal{G}_n$ above is a minimal nice graph for every $n \geq 3$.

OC704. Fix positive integers n and k such that $2 \leq k \leq n$ and a set M consisting of n fruits. A *permutation* is a sequence $x = (x_1, x_2, \ldots, x_n)$ such that $\{x_1, \ldots, x_n\} = M$. Ivan *prefers* some (at least one) of these permutations. He notes that for every preferred permutation x, there exist k indices $i_1 < \ldots < i_k$ with the following property: for every $1 \leq j < k$, if he swaps x_{i_j} and $x_{i_{j+1}}$ he obtains another preferred permutation.

Prove that he prefers at least $k!$ permutations.

Originally Problem 5 of Day 1 of the International Mathematics Competition 2023.

We received no submissions to this problem.

OC705. Ivan writes the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ on the board. Then he performs the following operation on the matrix several times:

- he chooses a row or a column of the matrix, and
- he multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ after finitely many steps?

Originally Problem 6 of Day 2 of the International Mathematics Competition 2023.

We received 3 solutions. We present the solution by the Missouri State University Problem Solving Group.

The answer is no. This is equivalent to performing elementary row and column operations on the matrix

$$\begin{pmatrix} \ln 2 & \ln 3 \\ \ln 2 & \ln 4 \end{pmatrix}$$

to obtain the matrix

$$\begin{pmatrix} \ln 2 & \ln 4 \\ \ln 2 & \ln 3 \end{pmatrix}.$$

However these elementary row operations do not change the determinant of a matrix, but

$$\det \begin{pmatrix} \ln 2 & \ln 3 \\ \ln 2 & \ln 4 \end{pmatrix} = \ln 2(\ln 4 - \ln 3), \text{ while } \det \begin{pmatrix} \ln 2 & \ln 4 \\ \ln 2 & \ln 3 \end{pmatrix} = \ln 2(\ln 3 - \ln 4).$$

Since these are unequal, we obtain a contradiction.

Gregarious and Reclusive Triples

Ed Barbeau

PART D: FURTHER CONSIDERATIONS

This article is the fourth on this topic. For Part A, please refer to ***Crux***, 50(4), p. 190-192; for Part B, refer to ***Crux***, 50(5), p. 244-247; for part C, refer to 50(6), p. 290-293.

A communication from Steve Hzindar led me to reflect further on the existence of integer triples (x, y, z) for which $xy + k$, $yz + k$ and $zx + k$ are all perfect squares for some integer value of k. Such $k-$**triples** are **gregarious** if their left and right **associates** $(2(x + y) - z, x, y)$ and $(y, z, 2(y + z) - x)$ are also $k-$triples. They are **superbly** gregarious if, in addition, $z - (x + y) = 2c$ and $xy + k = c^2$ for some integer c. Otherwise, the $k-$triple is **reclusive**. Previous articles in *Crux* (50:4, 190-192; 50:5, 244-247; 50:6, 290-293) described a number of ways in which such triples can be discovered, their properties and some open questions. The purpose of this note is to carry the investigations further.

Exercise 27. $(1, 3, 8)$ and $(1, 3, 120)$ are $1-$triples. Generalize these to find families of $k-$triples $(1, y, z)$. Are these triples gregarious?

Exercise 28. Determine $1-$triples (x, y, z), where x, y, z are in arithmetic progression.

Exercise 29. If $(x, y, z) = (v - u, v, v + u)$ is a $1-$triple in arithmetic progression and $b^2 = (v - u)(v + u) + 1$, then $v^2 + 1 = b^2 + u^2$. Investigate situations in which $v^2 + 1$ has an alternative representation as a sum of squares which lead to a $1-$triple. Are any of them reclusive?

Exercise 30. Find $k-$triples (x, y, z) in arithmetic progression for other values of k.

Exercise 31. Determine $k-$triples (x, y, z) where x, y, z are in geometric progression.

Exercise 32. (a) Determine $k-$triples (x, y, z) where x, y, z are in harmonic progression.

(b) Verify that $(x, y, z) = (r^2 - 2s, r^2 - s^2, r^2 + rs)$ is a triple in harmonic progression. Prove that, if $yz + k = a^2$, $zx + k = b^2$ and $xy + k = c^2$ for some integers k, a, b, c, then $a^2 + c^2 = 2b^2$ or $u^2 + v^2 = b^2$, where $a = u + v$, $b = u - v$. Use this to construct families of $k-$triples. Can you find any reclusive triples?

Euler discovered that $1-$quadruples are in the form

$$(x, y, z, w) = (x, y, x + y + 2c, 4c(x + c)(y + c)).$$

When you test this as a $k-$quadruple for $k \neq 1$, something rather interesting happens: $k = 1$ is special.

Exercise 33. For each consecutive triple (x, y, z) in the following subsequences of Fibonacci numbers, determine the fourth member (x, y, z, w) to make an Euler quadruple, and check out the differences between nearby squares and the six products of pairs of them. What do you notice?

$$0, 1, 3, 8, 21, 55, 144, \ldots$$

$$1, 2, 5, 13, 34, 89, 233, \ldots$$

Exercise 34. Investigate the Euler quadruples (x, y, z, w) formed from $(x, y; c) = (1, y; 1), (1, y; 2)$ as described. Check quadruples arising from $k-$triples (x, y, z) when k is small. Make a conjecture.

Exercise 35. Since two of the entries of a fundamental Pythagorean triple are odd and the other even, such a triple must be a $k-$triple for some value of k. Check some Pythagorean triples (x, y, z) and determine k so that they are $k-$triples. Suppose $(yz + k, zx + k, xy + k) = (a^2, b^2, c^2)$. Find a connection between (x, y, z) and (a, b, c).

Exercise 36. In Exercise 22, triples $(x, y, z) = (b+c, c+a, a+b)$ were investigated. Determine conditions on (a, b, c) such that (x, y, z) is a Pythagorean triple.

In the next exercise, we focus on $k = -1$.

Exercise 37. (a) Determine triples of integers r, s, t for which

$$(r^2 + 1)(s^2 + 1) = t^2 + 1.$$

(b) Let m be a positive integer. Determine a finite sequence $(x_1, x_2, \cdots, x_m)$ for which $x_i x_j - 1 = c_{ij}^2$ for $1 \leq i < j \leq m$ and integers c_{ij}.

A more general investigation

Let $p(t)$ be a monic polynomial with integer coefficients. We can generalize the investigation to ask for triples (x, y, z) for which xy, xz and yz are all of the form $p(t)$ for integers t. When the linear coefficient of the polynomial is even, by completing the square, we can write it in the form $q(t)^2 + k$, so we get essentially the problem just discussed.

Suppose that $p(t) = t^2 + (2c+1)t + d = (t + c)(t + c + 1) + (d - c(c + 1))$, so that $p(t)$ is up to a constant summand the product of two consecutive integers. This indicates that we should be looking at oblong $k-$ triples (x, y, z) for which $xy + k$,

$yz+k$ and $zx+k$ are *oblong numbers*, *i.e.*, the product of two consecutive integers. For example, $(2,5,6)$ is a $(-10)-$triple since $2\times 5-10=0\times 1$, $2\times 6-10=1\times 2$ and $5\times 6-10=4\times 5$. Quite a few results for $k-$triples involving squares have their analogues for this involving oblong numbers.

Exercise 38. (a) Show that $(x,y,x+y+2c+1)$ is an oblong $k-$triple with $k=c(c+1)-xy$.

(b) Show that the triple $(2,5,6)$ is of the form in (a). Embed this in a bilateral sequence $\{x_n\}$ that satisfies the recursion $x_{n+3}+x_n=2(x_{n+1}+x_{n+2})$. Do any three consecutive terms of the sequence constitute an oblong $(-10)-$triple? Answer the same question more generally for the triple in (a).

(c) Consider the quadruple

$$(x+y-(2c+1),x,y,x+y+2(c+1)).$$

Show that this is an oblong $k-$quadruple, with $k=c(c+1)-xy$ if and only if

$$x^2+xy+y^2=(3c^2+3c+1)+d(d+1)=3c(c+1)+d(d+1)+1$$

for some integer d. Find some examples.

Exercise 39. Prove that if (x,y,z) is an oblong $k-$triple, then $(2x,2y,2z)$ is an ordinary $k'-$triple for a suitable value of k. Investigate the relationship between the two types of $k-$triples.

Notes on the exercises

Exercise 27. If $(x,y,z)=(1,n^2-k,(n+1)^2-k)$, then

$$yz+k=[n(n+1)-k]^2,$$

and the triple is superbly gregarious.

We can consider $1-$triples of the form $(1,3,w^2-1)$, where $3(w^2-1)+1=3w^2-2$ is a square. Determining values of w involves a Pell's equation $v^2-3w^2=-2$. Some solutions are $(v,w)=(1,1),(5,3),(19,11),(71,41)$, giving rise to the $1-$triples $(1,3,0)$, $(1,3,8)$, $(1,3,120)$, $(1,3,1680)$.

Related to this is Problem 10238 in the *American Mathematical Monthly* 99:7 (August-September, 1992), 674, which asks for a sequence of values of a_n for which a_n+1 and $3a_n+1$ are squares and $a_na_{n+1}+1$ is also square.

Exercise 29. This analysis is due to Steve Hzindar. Suppose that the triple $(x,y,z)=(v-u,v,v+u)$ is a $1-$triple, with $a^2=yz+1$, $b^2=zx+1$ and $c^2=xy+1$. One way to get a possible triple is to ensure that $z-(x+y)=2u-v$ is even, and get a superb congenial triple. So we may assume that $v=2w$. We are led to

$$a^2 = 4w^2 + 2uw + 1;$$
$$b^2 = 4w^2 - u^2 + 1;$$
$$c^2 = 4w^2 - 2uw + 1.$$

One possibility is to let $u = 2w$, which leads us to $(x, y, z) = (0, 2w, 4w)$ where $8w^2 + 1$ is a square. Alternatively, we can create a superbly congenial triple by making $a^2 = (u - w)^2$. Since

$$a^2 = (u - w)^2 - (u^2 - 3w^2 - 1);$$
$$b^2 = w^2 - (u^2 - 3w^2 - 1);$$
$$c^2 = (u + w)^2 - (u^2 - 3w^2 - 1),$$

we can achieve this when the Pell equation $u^2 - 3w^2 = 1$ is satisfied. The solutions are given by $(u_0, w_0) = (1, 0)$, $(u_1, w_1) = (2, 1)$ and $(u_{n+1}, w_{n+1}) = (4u_n - u_{n-1}, 4w_n - w_{n-1})$ for $n \geq 0$. Alternatively

$$(u_{n+1}, w_{n+1}) = (2u_n + 3w_n, u_n + 2w_n).$$

It can be checked directly that $(x, y, z) = (w_{n-1}, 2w_n, w_{n+1})$ is a superbly congenial 1-triple.

This was the content of problem 10622, which appeared in the *American Mathematical Monthly* 104:9 (November, 1997), 870 and 106:9 (November, 199), 867-868.

Exercise 30. If $(x, y, z) = (2w - u, 2w, 2w + u)$, we are led to

$$a^2 = (w + u)^2 - (u^2 - 3w^2 - k);$$
$$b^2 = w^2 - (u^2 - 3w^2 - k);$$
$$c^2 = (w - u)^2 - (u^2 - 3w^2 - k).$$

Thus we can find triples whenever $u^2 - 3w^2 = k$ is solvable. For example, when $k = 13$, we are led to the solutions $(u, w) = (4, 1), (5, 2), (11, 6), (16, 9), \ldots$.

Exercise 31. Let $(x, y, z) = (s, sr, sr^2)$. For a superbly congenial $k-$triple, we need s to be even. For example, if $s = 2$, then $(x, y, z) = (2, 2r, 2r^2)$ is a $k-$triple with $k = (r^2 + r + 1)(r^2 - 3r + 1)$ and

$$(a, b, c) = (r^2 + r - 1, r^2 - r + 1, r^2 - r - 1).$$

Exercise 33. For the quadruple

$$(x, y, z, w) = (x, y, x + y + 2c, 4c(x + c)(y + c)),$$

let the first three entries constitute a $k-$triple with $k = c^2 - xy$. Then

$$xy + k = c^2; \qquad yz + k = (y + c)^2; \qquad zx + k = (x + c)^2.$$

However, when $k \neq 1$, we find that $xw + k$, $yw + k$, $zw + k$ are not generally squares. Rather $k^2 = c^4 - 2c^2xy + x^2y^2$,

$$xw + k^2 = (c^2 + 2cx + xy)^2;$$
$$yw + k^2 = (c^2 + 2cy + xy)^2;$$
$$zw + k^2 = (3c^2 + 2(x + y)c + xy)^2.$$

Exercise 35. We have the examples $(x, y, z) = (3, 4, 5)$ and $(x, y, z) = (8, 15, 17)$. Respectively, these are $(-11)-$triples and $(-111)-$triples with associated squares $(3, 2, 1)$ and $(12, 5, 3)$. In both cases, we find that $(x, y, z) = (b + c, c + a, a + b)$. In general, we have:

(x, y, z)	k	(a, b, c)
$(2r + 1, 2r^2 + 2r, 2r^2 + 2r + 1)$	$-r(4r^2 + 5r + 2)$	$(r(2r + 1), r + 1.r)$
$(2r, r^2 - 1, r^2 + 1)$	$-(r - 1)(2r^2 + r + 1)$	$(r(r - 1), r + 1, r - 1)$
$(m^2 - n^2, 2mn, m^2 + n^2)$	$-n(2m^3 - m^2n - n^3)$	$(n(m + n), m(m - n), n(m - n))$

Exercise 36. $(b + c, c + a, a + b)$ is a congenial $k-$triple with $k = -(ab + bc + ca)$. It is a Pythagorean triple if and only if $ab = (a + b + c)c$, or

$$c^2 + (a + b)c - ab = 0.$$

Since the equation $t^2 + (a + b)t + ab = 0$ has integer roots $-a$ and $-b$, this leads us to study the conditions under which there are integer coefficients p and q for which both equations $t^2 + pt \pm q = 0$ have integer roots. This will happen when the discriminants $p^2 - 4q = r^2$ and $p^2 + 4q = s^2$ for some integers r and s. This leads us to the equations $r^2 + s^2 = 2p^2$ and $u^2 + v^2 = p^2$ where $r = u - v$ and $s = u + v$. For example

$$(u, v; r, s : p, q) = (4, 3; 1, 7; 5, 6)$$

leads us to the equations $x^2 + 5x \pm 6 = 0$ where

$$x^2 + 5x + 6 = (x + 2)(x + 3); \qquad x^2 + 5x - 6 = (x + 6)(x - 1).$$

In our problem, this corresponds to $(a, b) = (2, 3)$ and c equal to either 1 or -6. Both lead to Pythagorean triples for (x, y, z).

More generally, we can take $(a, b) = (r + 1, r(2r + 1))$ and find that

$$t^2 + (2r^2 + 2r + 1)t - r(r + 1)(2r + 1) = [t + (r + 1)(2r + 1)][t - r]$$

and arrive at the congenial triple $(2r + 1, 2r^2 + 2r, 2r^2 + 2r + 1)$. Alternatively, we have the pair of factorizations:

$$t^2 + (r^2 + 1)t + (r - 1)r(r + 1) = [t + (r + 1)][t + r(r - 1)];$$
$$t^2 + (r^2 + 1)t - (r - 1)r(r + 1) = [t - (r - 1)][t + r(r + 1)];$$

to arrive at the triple $(2r, r^2 - 1, r^2 + 1)$.

Another approach is to suppose that

$$t^2 + (a+b)t - ab = [t + (ab)/r][t - r].$$

Then $(ab/r) - r = a + b$, from which it follows that

$$(a - r)(b - r) = 2r^2.$$

Then, we can find a possibility for each way of factoring $2r^2$ as a product of two integers. For example, if $a - r = 2$ and $b - r = r^2$, then $(a, b) = (r + 2, r(r + 1))$. We find that

$$c^2 + (r^2 + 2r + 2)c - r(r + 1)(r + 2) = (c + (r + 1)(r + 2))(c - r).$$

A final approach is to note that the discriminant of $c^2 + (a + b)c - ab$ is equal to $(a + 3b)^2 - 8b^2$. Determining when this is square leads us to the Pell equation

$$\alpha^2 - 8\beta^2 = \gamma^2.$$

When $\gamma = 1$, this has solutions $(1, 0), (3, 1), (17, 6), (99, 35), \ldots$. For general γ, there are the obvious solutions

$$(\alpha, \beta) = (\gamma, 0), (3\gamma, \gamma), (17\gamma, 6\gamma), \ldots.$$

For many values of γ, that is all that there is. However, $\alpha^2 - 8\beta^2 = 49$ has two other fundamental solutions $(\alpha, \beta) = (9, 2), (11, 3)$. Since $b = \beta$ and $a + 3b = \alpha$, we can backtrack to get $(a, b) = (3, 2), (2, 3)$.

Likewise, since $\alpha^2 - 8\beta^2 = 17^2$ is satisfied by $(\alpha, \beta) = (19, 3), (33, 10)$, we are led to $(a, b) = (10, 3), (3, 10)$.

Exercise 37. (a) Such triples are given by $(r, s, t) = (r, r + 1, r^2 + r + 1)$ and triples (r, s, t) that satisfy a Pell's equation of the form $t^2 - (r^2 + 1)s^2 = r^2$. For example, $(r, s, t) = (1, 12, 17), (2, 21, 47)$.

(b) Let all the entries be 1 except for two of the form $r^2 + 1$, $s^2 + 1$ where r and s are as in (a), or except for one of the form $r^2 + 1$ or with no exceptions.

I am indebted to Andrej Dujella for the reference to the paper "There is no Diophantine $D(-1)-$quadruple" by N.C. Bonciocat, M. Cipu and M. Mignotte in the *Journal of the London Mathematical Society* 105 (2022), 63-99. There are no quadruples with all integers distinct. He also provided a link He also provided a link https://web.math.pmf.unizg.hr/˜duje/dn.html indicating values of k for which $k-$quadruples exist.

Exercise 38. This is open territory. Note that

$$(d - c)^2 + (d - c)(2c + 1) + (2c + 1)^2 = 3c(c + 1) + d(d + 1) + 1.$$

We have the example of the oblong $k-$quadruple $(d - c, d - c, 2c + 1, 3c + d + 2)$ with $k = 3c^2 + 2c - 2cd - d$.

Exercise 39. If $xy + k = a(a + 1)$, then $(2x)(2y) + (4k + 1) = (2a + 1)^2$.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **June 15, 2025**.

5031. *Proposed by Tran Quang Hung.*

Given a triangle ABC with incircle (I) touching the sides BC, CA, AB at D, E, F, respectively. The line through A parallel to EF intersects DE at P. Prove that the line through C and perpendicular to IP bisects the segment EF.

5032. *Proposed by Michael Friday, modified by the Editorial Board.*

Let ABC be a triangle with side-lengths a, b, c.

(a) Prove that the associated Euler line is parallel to side BC if and only if

$$(b^2 - c^2)^2 = 2a^4 - a^2(b^2 + c^2).$$

(b) Show that there are non-isosceles triangles satisfying this condition.

(c)* Are there triangles with integer sides satisfying this condition?

5033. *Proposed by Decebal S. Getan.*

Find all real numbers a and b such that $2(2a + 2b - 3ab)^2 = (a + b)(a^2 - ab + b^2)^2$.

5034. *Proposed by Nguyen Minh Ha.*

Let $ABCD$ be a tetrahedron and M be a variable point inside the tetrahedron. Let N, P and Q be the intersection points of BM, CM and DM with the planes (ACD), (ADB) and (ABC), respectively. Find the point M in the tetrahedron $ABCD$ such that the volume of tetrahedron $MNPQ$ is maximized.

5035. *Proposed by Vasile Cîrtoaje.*

Find the largest positive value of the constant k such that the inequality

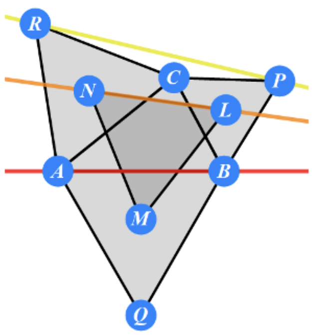
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^k + b^k + c^k$$

holds for any $a \geq b \geq 1 \geq c > 0$ with $a^2 + b^2 + c^2 = 3$.

5036. *Proposed by Xicheng Peng.*

As illustrated in the figure below, equilateral triangles CBP, ACR and BAQ are constructed outward on the sides of triangle ABC with centers L, N and M, respectively. Prove that

1. the lines AB, NL and PR are concurrent;
2. the lines AL, BN and CM are concurrent;
3. the lines AP, BR and CQ are concurrent;
4. the lines PL, QM and RN are concurrent.



5037. *Proposed by Nguyen Tuan Anh, modified by the Editorial Board.*

Let n be a positive integer. Let A be the set of all n-digit numbers in which all digits are 1's, 2's, 3's and 4's, and so that the numbers of 1's and 2's are equal. Show that $|A| = \binom{2n}{n}$.

5038. *Proposed by Michel Bataille.*

Let x be a positive real number and n be a positive integer. Prove that

$$e^x > 1 + x\sqrt{1 + \frac{(n-1)x}{n}\sqrt{1 + \frac{(n-2)x}{n}\sqrt{\cdots\cdots\sqrt{1 + \frac{2x}{n}\sqrt{1 + \frac{x}{n}}}}}}.$$

5039. *Proposed by Tran Ngoc Khuong Trang.*

Find the minimal value of the expression

$$\sqrt{\frac{a}{8ab+1}} + \sqrt{\frac{b}{8bc+1}} + \sqrt{\frac{c}{8ca+1}}$$

for all non-negative real numbers a, b, c satisfying $ab + bc + ca = 1$.

5040. *Proposed by Bowen Zhou.*

Ming starts at $O(0,0,0)$ in 3-dimensional space and moves 1 unit at every step. He chooses every step in the direction of positive or negative $x, y, z-$ axes, which means that every step is parallel to a vector $(1,0,0)$,$(-1,0,0)$, $(0,1,0)$, $(0,-1,0)$, $(0,0,1)$ or $(0,0,-1)$. He returns to O after 2024 steps. Denote the number of paths by n. Prove that

$$n > \binom{2024}{1012} \cdot 3^{2013}.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2025**.

5031. *Soumis par Tran Quang Hung.*

Étant donné un triangle ABC dont le cercle inscrit (I) touche les côtés BC, CA et AB respectivement en D, E et F. La droite passant par A et parallèle à EF coupe DE en P. Montrez que la droite passant par C et perpendiculaire à IP coupe en deux le segment EF.

5032. *Soumis par Michael Friday, modifié par le comité de rédaction.*

Soit ABC un triangle dont les côtés sont a, b et c.

(a) Montrez que la droite d'Euler associée est parallèle au côté BC si et seulement si

$$(b^2 - c^2)^2 = 2a^4 - a^2(b^2 + c^2).$$

(b) Montrez qu'il existe des triangles non isocèles satisfaisant à cette condition.

(c)* Existe-t-il des triangles à côtés entiers satisfaisant à cette condition ?

5033. *Soumis par Decebal S. Getan.*

Trouvez tous les nombres réels a et b tels que $2(2a+2b-3ab)^2 = (a+b)(a^2-ab+b^2)^2$.

5034. *Soumis par Nguyen Minh Ha.*

Soit $ABCD$ un tétraèdre et M un point variable à l'intérieur du tétraèdre. Soit N, P et Q les points d'intersection de BM, CM et DM avec les plans (ACD), (ADB) et (ABC), respectivement. Trouvez le point M du tétraèdre $ABCD$ tel que le volume du tétraèdre $MNPQ$ soit maximal.

5035. *Soumis par Vasile Cîrtoaje.*

Trouvez la plus grande valeur positive de la constante k telle que l'inégalité

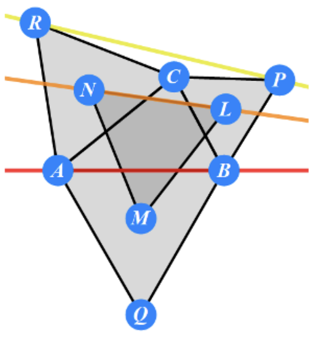
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq a^k+b^k+c^k$$

est vérifiée pour tout $a \geq b \geq 1 \geq c > 0$ avec $a^2+b^2+c^2=3$.

5036. *Soumis par Xicheng Peng.*

Comme le montre la figure ci-dessous, les triangles équilatéraux CBP, ACR et BAQ sont construits vers l'extérieur sur les côtés du triangle ABC dont les centres sont respectivement L, N et M. Montrez que

1. les droites AB, NL et PR sont concourantes;
2. les droites AL, BN et CM sont concourantes;
3. les droites AP, BR et CQ sont concourantes;
4. les droites PL, QM et RN sont concourantes.



5037. *Soumis par Nguyen Tuan Anh, modifié par le comité de rédaction.*

Soit n un entier positif. Soit A l'ensemble de tous les nombres à n chiffres dont tous les chiffres sont des 1, des 2, des 3 et des 4, et dont les nombres de 1 et de 2 sont égaux. Montrez que $|A| = \binom{2n}{n}$.

5038. *Soumis par Michel Bataille.*

Soit x un nombre réel positif et n un entier positif. Montrez que

$$e^x > 1 + x\sqrt{1 + \frac{(n-1)x}{n}\sqrt{1 + \frac{(n-2)x}{n}\sqrt{\cdots\cdots\sqrt{1 + \frac{2x}{n}\sqrt{1 + \frac{x}{n}}}}}}.$$

5039. *Soumis par Tran Ngoc Khuong Trang.*

Trouvez la valeur minimale de l'expression

$$\sqrt{\frac{a}{8ab+1}} + \sqrt{\frac{b}{8bc+1}} + \sqrt{\frac{c}{8ca+1}}$$

pour tous les nombres réels non négatifs a, b et c vérifiant $ab + bc + ca = 1$.

5040. *Soumis par Bowen Zhou.*

Ming débute à $O(0,0,0)$ dans un espace à 3 dimensions et se déplace d'une unité à chaque pas. Il choisit chaque pas dans l'une ou l'autre des directions x, y et z positives ou négatives, ce qui signifie que chaque pas est parallèle à un vecteur $(1,0,0)$,$(-1,0,0)$, $(0,1,0)$, $(0,-1,0)$, $(0,0,1)$ ou $(0,0,-1)$. Il revient à O après 2024 étapes. On note n le nombre de chemins. Montrez que

$$n > \binom{2024}{1012} \cdot 3^{2013}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2024: 50(9), p. 465–469.

4981. *Proposed by Ted Barbeau.*

(a) Let n be a positive integer. Determine all vectors $(w_0, w_1, w_2, \ldots, w_n)$ of nonnegative integers for which w_i is the number of times that the entry i appears in the vector for $0 \leq i \leq n$.

(b) Let W_n be the set of all vectors with $n+1$ entries, all nonnegative integers between 0 and n inclusive, not all the same. Define a map $\mathbf{w} \to \mathbf{w}'$, where the ith entry of $\mathbf{w}'$ is the number of times i appears as an entry in the vector $\mathbf{w}$. Prove that the trajectory of any vector under this map leads to a fixed vector or a periodic orbit, and find them all.

We did not receive any submission. We present the solution by the proposer, slightly modified by the editor.

(a) The number of terms in the vector can be counted in three ways. It is clearly $n+1$. It is also $w_0 + w_1 + \cdots + w_n$. Finally, since there are w_0 zeros, w_1 1s, w_2 2s, and so on, the sum of the terms can be written as $w_1 + 2w_2 + 3w_3 + \cdots + nw_n$. Equating these two expressions leads to

$$w_0 = w_2 + 2w_3 + \cdots + (n-1)w_n. \tag{1}$$

We divide our discussion according to the value of w_0:

1. $w_0 = 0$. This is impossible.
2. $w_0 = 1$. Equation (1) gives $w_2 = 1$ and $w_k = 0$ for $k \geq 3$. This leads to the possibility $(1, 2, 1, 0)$.
3. $w_0 = 2$. Then $w_2 \geq 1$ and equation (1) forces $w_2 = 2$ and $w_k = 0$ for $k \geq 3$. This leads to two possibilities $(2, 0, 2, 0)$ and $(2, 1, 2, 0, 0)$.
4. $w_0 = r \geq 3$. Then $w_r \geq 1$. Since $2(r-1) > r$, equation (1) forces $w_r = 1$, $w_2 = 1$, and $w_k = 0$ for $k \geq 3$ and $k \neq r$. It follows $w_1 = 2$. Therefore $n+1 = w_0 + w_1 + w_2 + w_r = r + 2 + 1 + 1 = r + 4$, and thus $r = n - 3$. Since $r \geq 3$ and $n \geq 6$, we obtain the sequence $(n-3, 2, 1, 0_{n-6}, 1, 0, 0, 0)$, where 0_{n-6} represents a string of $n-6$ zeros.

(b) Suppose we start with a vector $\mathbf{w}$. Since the number of possible vectors in W_n is finite, repeated application of $\to$ will eventually lead to a vector that was encountered earlier, and this will initiate a periodic orbit. After the first application of $\to$, the sum of the entries of every vector in the trajectory is $n+1$, so we will assume that this is the case.

A little experimentation reveals that the tendency under the operation is for the number of positive entries in $\mathbf{w}$ to decrease. However, since the sequence of images is eventually periodic, then, at some point, $\mathbf{w}'$ must have at least as many positive entries as $\mathbf{w}$. Let us see when this might occur. Let n be a positive integer exceeding 2. We will use the notation $\langle v_0, v_1, \ldots, v_n\rangle$ to refer to any vector of nonnegative integers v_i with $\sum_i v_i = n+1$ and $0 \le v_i \le n$, where the order of the entries is not specified. (The parentheses will signify that the integers are in a particular order.)

Let $\mathbf{w}$ be a vector in which there are k distinct positive entries a_i $(1 \le i \le k)$ where a_i appears with positive frequency b_i, along with r zeros. Suppose b_1 is the maximum frequency. Then

$$n+1 = \sum_{i=1}^{k} b_i a_i = \sum_{i=1}^{k} b_i + r.$$

Then $\mathbf{w}'$ has at most $k+1$ entries consisting of the b_i's and possibly r. If $\mathbf{w}'$ has at least as many positive entries as $\mathbf{w}$, then $b_1 + b_2 + \cdots + b_k \le k+1$. Clearly, there are two possibilities. In the first instance, $b_1 = b_2 = \cdots = b_k = 1$; in the second instance, $b_1 = 2$ and $b_2 = b_3 = \cdots = b_k = 1$. Let $\mathbf{w}''$ be the image of $\mathbf{w}'$. In both cases, $\mathbf{w}''$ has up to four nonzero entries. Next, we divide our discussion into a few cases, according to the nonzero entries of $\mathbf{w}''$.

Case 1. Suppose $\mathbf{w}'' = \langle a, b, c, 0_{n-2}\rangle$ with three nonzero distinct entries a, b, c. Since $a+b+c = n+1$, $n \ge 5$. When $n = 5$, we have

$$\langle 3,2,1,0,0,0\rangle \to (3,1,1,1,0,0) \leftrightarrow (2,3,0,1,0,0),$$

a cycle of period 2. When $n = 6$, we are led to

$$\begin{aligned}\langle 4,2,1,0_4\rangle &\to (4,1,1,0,1,0,0) \to (3,3,0,0,1,0,0) \to (4,1,0,2,0,0,0)\\ &\to (4,1,1,0,1,0,0),\end{aligned}$$

a cycle of period 3. When $n \ge 7$, we have

$$\begin{aligned}\langle a,b,c,0_{n-2}\rangle &\to \langle n-2,1,1,1,0_{n-3}\rangle \to (n-3,3,0_{n-4},1,0,0)\\ &\leftrightarrow (n-2,1,0,1,0_{n-7},1,0,0,0),\end{aligned}$$

a cycle of period 2.

Case 2. Suppose $\mathbf{w}'' = \langle a, a, b, 0_{n-2}\rangle$ with $ab \ne 0$, $a \ne b$ and $2a+b = n+1$. Then $n \ge 3$. When $n = 3$, we have $\langle 1,1,2,0\rangle \to (1,2,1,0)$, ending in a fixed vector. When $n = 4$, we have

$$\langle 1,1,3,0,0\rangle \to (2,2,0,1,0) \to (2,1,2,0,0),$$

ending in a fixed vector; and $\langle 1,2,2,0,0\rangle \to (2,1,2,0,0)$, ending in the same fixed vector. When $n \ge 5$, we have $\langle a,a,b,0_{n-2}\rangle \to \langle n-2,2,1,0_{n-2}\rangle$ and we are in Case 1 again.

Case 3. Suppose $\mathbf{w}'' = \langle a, a, a, 0_{n-2}\rangle$ with $a \neq 0$. Then we have $3a = n + 1$ and $n \geq 5$. When $n = 5$, we have

$$\langle 2, 2, 2, 0, 0, 0\rangle \to (3, 0, 3, 0, 0, 0) \to (4, 0, 0, 2, 0, 0) \to (4, 0, 1, 0, 1, 0)$$

and we are in Case 2 again. When $n > 5$, we have

$$\langle a, a, a, 0_{n-2}\rangle \to (n - 2, 3, 0_{n-1}) \to (n - 1, 1, 1, 0_{n-2})$$

and we are in Case 2 again.

Case 4. Suppose $\mathbf{w}'' = \langle a, b, 0_{n-1}\rangle$ with $ab \neq 0$ and $a \neq b$. Then $\langle a, b, 0_{n-1}\rangle \to \langle n - 1, 1, 1, 0_{n-2}\rangle$, and we are in one of Case 1, Case 2, or Case 3 again.

Case 5. Suppose $\mathbf{w}'' = \langle a, a, 0_{n-1}\rangle$ with $a \neq 0$. When $n = 3$, we have $\langle 2, 2, 0, 0\rangle \to (2, 0, 2, 0)$, ending in a fixed vector. When $n \geq 4$, then $\langle a, a, 0_{n-1}\rangle \to \langle n - 1, 2, 0_{n-1}\rangle$ and we are in Case 4 again.

Case 6. Suppose $\mathbf{w}'' = \langle a, 0_n\rangle$ with $a \neq 0$. It follows that $a = n + 1$, which is impossible.

Case 7. Suppose $\mathbf{w}'' = \langle a, b, c, d, 0_{n-3}\rangle$ with $abcd \neq 0$. Note that the number of nonzero entries in the image of $\mathbf{w}''$ is one more than the number of distinct elements in a, b, c, d. If the number of distinct elements in a, b, c, d is at most 2, then we are in one of the previous cases again. If a, b, c, d are distinct, then we have $n + 1 = a + b + c + d \geq 10$ and thus $n \geq 9$; in this case, $\langle a, b, c, d, 0_{n-3}\rangle \to \langle n - 3, 1, 1, 1, 1, 0_{n-4}\rangle \to \langle n - 4, 4, 1, 0_{n-2}\rangle$ and we are again in Case 1. Finally, if there are exactly three distinct elements in a, b, c, d, then $a + b + c + d \geq 7$ and $n \geq 6$; in this case, we have

$$\langle a, a, b, c, 0_{n-3}\rangle \to \langle n - 3, 2, 1, 1, 0_{n-3}\rangle \to (n - 3, 2, 1, 0_{n-6}, 1, 0, 0, 0),$$

ending in a fixed vector.

To summarize, we have the fixed vectors

$$(1, 2, 1, 0), (2, 0, 2, 0), (2, 1, 2, 0, 0)$$

and

$$(n - 3, 2, 1, 0_{n-6}, 1, 0, 0, 0)$$

for $n \geq 6$. We have the following period-2 orbits:

$$(3, 1, 1, 1, 0, 0) \leftrightarrow (2, 3, 0, 1, 0, 0)$$

and

$$(n - 2, 1, 0, 1, 0_{n-7}, 1, 0, 0, 0) \leftrightarrow (n - 3, 3, 0_{n-4}, 1, 0, 0)$$

for $n \geq 7$.

Finally, there is a period-3 orbit:

$$(4, 1, 1, 0, 1, 0, 0) \to (3, 3, 0, 0, 1, 0, 0) \to (4, 1, 0, 2, 0, 0, 0) \to \dots.$$

4982. *Proposed by Michel Bataille.*

Let ABC be a triangle inscribed in a circle Γ and let $BC = a, CA = b, AB = c$. For any point P of the arc BC of Γ not containing A, prove that

$$\frac{PA \cdot PB}{ab} - \frac{PB \cdot PC}{bc} + \frac{PC \cdot PA}{ca} = 1.$$

We received 25 submissions, all of which were correct. Our featured solution by Michal Adamaszek is an example of the majority of the submissions.

Let R be the radius of Γ and let $|\cdot|$ denote the area of a polygon. We have

$$|PAB| + |PAC| = |PCAB| = |PBC| + |ABC|,$$

and this can be written using the formula for triangle area involving its three sides and circumradius as

$$\frac{PA \cdot PB \cdot c}{4R} + \frac{PC \cdot PA \cdot b}{4R} = \frac{PB \cdot PC \cdot a}{4R} + \frac{abc}{4R}.$$

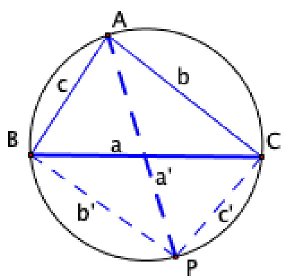
Dividing by $abc/4R$, we obtain

$$\frac{PA \cdot PB}{ab} + \frac{PC \cdot PA}{ca} = \frac{PB \cdot PC}{bc} + 1,$$

as required.

Editor's comments. C.R. Pranesachar observed that Problem 4982 is a known property of cyclic quadrilaterals, a sort of quotient relative of Ptolemy's theorem:

> *In a cyclic quadrilateral, the ratio of the diagonals equals the ratio of the sums of the products of the sides that share the diagonal's endpoints.*



According to Pranesachar, the theorem is attributed to Brahmagupta, but this editor has never seen any attribution. The result is perhaps most easily understood using appropriate notation. With A, B, C, P defined as in our problem, let the sides of ΔABC be a, b, c, and let the segments joining P to the respective vertices of the triangle be a', b', c' (as in the accompanying figure). With this notation, the known theorem becomes

$$\frac{a}{a'} = \frac{bc' + b'c}{bc + b'c'},$$

while the formula to demonstrate in Bataille's problem is

$$\frac{a'b'}{ab} - \frac{b'c'}{bc} + \frac{c'a'}{ca} = 1.$$

To transform the first equation to the second, cross multiply and divide both sides of the resulting equation by abc.

4983. *Proposed by Mihaela Berindeanu.*

Calculate $\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{1}{n}\ln\left(1+\frac{k}{n}\right)\right)$.

Originally, the problem was published with a typo that was soon corrected. We received 10 submissions to the revised problem of which 8 were correct and complete. We present the solution submitted independently by Henry Ricardo, Borche Joshevski, and the proposer, slightly altered by the editor.

Let $f(x) = \ln(1+x)$. Then for $x \in [0,1]$, $|f(x)| \leq \ln(2)$ and for $x > 0$, we have $x - \frac{x^2}{2} \leq \ln(1+x) \leq x$.

Setting $x = (1/n)\ln(1+k/n)$ and adding the resulting inequalities yields

$$\sum_{k=1}^{n} \frac{1}{n}\ln\left(1+\frac{k}{n}\right) - \sum_{k=1}^{n} \frac{1}{2n^2}\ln^2\left(1+\frac{k}{n}\right) \leq \sum_{k=1}^{n} \ln\left(1+\frac{1}{n}\ln\left(1+\frac{k}{n}\right)\right) \leq \sum_{k=1}^{n} \frac{1}{n}\ln\left(1+\frac{k}{n}\right).$$

Taking limits as $n \to \infty$, we get

$$\int_0^1 \ln(1+x)dx - \lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{2n^2}\ln^2\left(1+\frac{k}{n}\right) \leq \lim_{n\to\infty}\sum_{k=1}^{n} \ln\left(1+\frac{1}{n}\ln\left(1+\frac{k}{n}\right)\right) \leq \int_0^1 \ln(1+x)dx.$$

But

$$0 \leq \lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{2n^2}\ln^2\left(1+\frac{k}{n}\right) \leq \lim_{n\to\infty}\frac{\ln(2)^2}{2n} = 0,$$

and the squeeze principle implies the existence of the limit and that

$$\lim_{n\to\infty}\sum_{k=1}^{n} \ln\left(1+\frac{1}{n}\ln\left(1+\frac{k}{n}\right)\right) = \int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1 \approx 0.38629.$$

Editor's Comments. When this problem was originally published it contained a typo (missing a summation). The following suggested alterations to the original statement were received before the correction of the typo was made:

Ulrich Abel and Brian Bradie argued that for any real number $k > -1$,

$$\lim_{n\to\infty} \ln\left(1 + n\ln\left(1+\frac{k}{n}\right)\right) = \ln(1+k).$$

Richard Hess noted that

$$\lim_{n\to\infty} n\ \ln\left(1 + \ln\left(1+\frac{k}{n}\right)\right) = k$$

and

$$\lim_{n\to\infty} n^2\ \ln\left(1 + \frac{1}{n}\ln\left(1+\frac{k}{n}\right)\right) = k.$$

4984. *Proposed by Vasile Cîrtoaje.*

Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 \le a_2 \le \cdots \le a_n$ and $a_1 + a_2 + \cdots + a_{n-1} = n - 1$. Prove that

$$\sum_{1\le i<j\le n} a_i a_j \ge \frac{n(n-1)}{2}.$$

There were 12 solvers, 7 of whom presented essentially the following solution.

We have

$$\begin{aligned}
2\sum_{1\le i<j\le n} a_i a_j &= 2\sum_{1\le i<j\le n-1} a_i a_j + 2a_n\sum_{i=1}^{n-1} a_i \\
&= \Big(\sum_{i=1}^{n-1} a_i\Big)^2 - \sum_{i=1}^{n-1} a_i^2 + 2a_n\sum_{i=1}^{n-1} a_i \\
&\ge (n-1)^2 - a_n\sum_{i=1}^{n-1} a_i + 2a_n\sum_{i=1}^{n-1} a_i = (n-1)^2 + a_n(n-1) \\
&\ge (n-1)^2 + \sum_{i=1}^{n-1} a_i = (n-1)^2 + (n-1) = n(n-1),
\end{aligned}$$

as desired. Equality occurs if and only if $a_i = 1$ for all i.

Comments by the editor. Where $s = a_1 + \cdots + a_{n-1}$, Michal Adamaszek and Michel Bataille proved more generally that

$$\sum_{1\le i<j\le n} a_i a_j \ge \frac{ns^2}{2(n-1)},$$

Adamaszek arguing as in the solution and Bataille using induction. Two further induction arguments were presented, using the substitution $b_i = (n-1)a_i/s$ in the induction step.

4985. *Proposed by Tran Ngoc Khuong Trang.*

Prove that the following inequality

$$\sqrt{a^2+6ab}+\sqrt{b^2+6bc}+\sqrt{c^2+6ca}\geq 5\sqrt{\frac{ab+bc+ca}{2}}$$

holds for all non-negative real numbers a, b, c. When does equality occur?

We received 5 submissions 3 of which were correct and complete. We present the solution submitted by Titu Zvonaru, slightly altered by the editor.

We will prove the stronger inequality:

$$\sqrt{a^2+6bc}+\sqrt{b^2+6ca}+\sqrt{c^2+6ab}\geq 4\sqrt{ab+bc+ca}.$$

We may suppose that $c=\min\{a,b,c\}$, and first establish that

$$\sqrt{a^2+6bc}+\sqrt{b^2+6ca}\geq\sqrt{(a+b)^2+12c(a+b)}. \qquad (1)$$

In fact, inequality (1) is equivalent to each of the following inequalities:

$$a^2+b^2+6bc+6ca+2\sqrt{\left(a^2+6bc\right)\left(b^2+6ca\right)}\geq a^2+b^2+2ab+12ac+12bc,$$
$$\sqrt{\left(a^2+6bc\right)\left(b^2+6ca\right)}\geq ab+3ac+3bc,$$
$$2a^3c+2b^3c+6abc^2\geq 3a^2c^2+3b^2c^2+2a^2bc+2ab^2c,$$
$$2a^2c(a-b)-2b^2c(a-b)-3c^2(a-b)^2\geq 0,$$
$$(a-b)^2\left(2ac+2bc-3c^2\right)\geq 0.$$

Since $c=\min\{a,b,c\}$, inequality (1) is established, and so, because $(a+b)^2\geq 4ab$ and $c^2+6ab\geq 6ab$, it suffices to prove that

$$\sqrt{4ab+12c(a+b)}+\sqrt{6ab}\geq 4\sqrt{ab+bc+ca}. \qquad (2)$$

Inequality (2) is equivalent to each of the following inequalities:

$$4ab+12c(a+b)+6ab+2\sqrt{6ab(4ab+12c(a+b))}\geq 16(ab+bc+ca),$$
$$\sqrt{6ab(4ab+12c(a+b))}\geq 3ab+2bc+2ac,$$
$$\frac{7}{6}\sqrt{6ab\left(\frac{144}{49}ab+\frac{432}{49}c(a+b)\right)}\geq 3ab+2bc+2ac,$$
$$\sqrt{6ab\left(\frac{144}{49}ab+\frac{432}{49}c(a+b)\right)}\geq\frac{18ab}{7}+\frac{12bc}{7}+\frac{12ac}{7}. \qquad (3)$$

Since

$$6ab=\frac{18ab}{7}+\frac{12ab}{7}+\frac{12ab}{7}\geq\frac{18ab}{7}+\frac{12bc}{7}+\frac{12ac}{7}$$

and

$$\frac{144ab}{49} + \frac{432c(a+b)}{49} \geq \frac{18ab}{7} + \frac{12bc}{7} + \frac{12ac}{7},$$

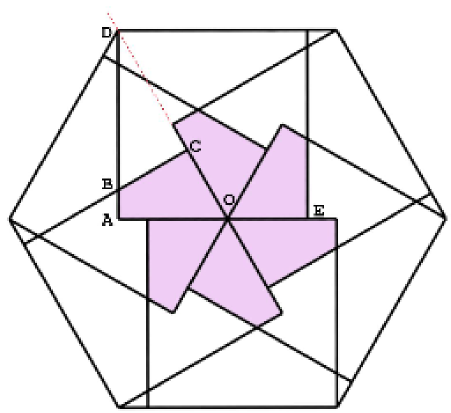
it follows that the inequality (3) is true.

From (1) and (2), we deduce that for equality it is necessary that $a = b, c = 0$. Thus, equality occurs if and only if $a = b = c = 0$.

Editor's Comments. Theo Koupelis noted that equality holds in the inequality as originally posed when $(a, b, c) = (2t, t, 0)$, and for cyclic permutations thereof, where t is an arbitrary non-negative real number.

4986. *Proposed by Arsalan Wares.*

There are six congruent squares inside a regular hexagon. Each square shares a vertex with the hexagon, as shown. One side of each square passes through the center of the hexagon and is parallel to a side of the hexagon, as shown. The squares split the hexagon into 24 disjoint regions; six of those regions are shaded. Exactly what fraction of the hexagon is shaded?



All but one of the 21 submissions were correct. We will sample three of them.

Solution 1 is an example of the approach used for the majority of the submissions; we feature a composite of the nearly identical (except for notation), almost wordless solutions by Catherine Jian and by Ricard Peiró i Estruch, with a few words added by the editor.

We first introduced the labelling of the vertices as in the diagram above. By symmetry, the six shaded regions are congruent; consequently,

- $\angle COA = 60°$ (where O is the center of the regular hexagon); and
- O, C, D are on same line. (A line through the center of a regular hexagon that is parallel to a side must be a diagonal.)

Let the side length of the regular hexagon be 2; then we have the following:

1. $|OA| = 1$
2. The side length of each square is $a = \sqrt{3} \quad (|AD| = |OA| \cdot \tan 60^\circ)$;
3. $|OC| = |OE| = a - |OA| = \sqrt{3} - 1$
4. $|BC| = \frac{\sqrt{3}}{3}|DC| = \frac{\sqrt{3}}{3}(2 - |OC|) = \sqrt{3} - 1 = |OC|$
5. $|AB| = a - 2|BC| = 2 - \sqrt{3}$

Using square brackets to denote area, we have

$$[OABC] = [OAB] + [BCO] = \frac{1}{2}(|OA||AB| + |OC||BC|) = \frac{6 - 3\sqrt{3}}{2}.$$

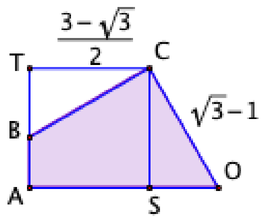
The shaded area is, therefore,

$$6[OABC] = 6 \cdot \left(3 - \frac{3\sqrt{3}}{2}\right),$$

while the hexagon is composed of six equilateral triangles of area $\sqrt{3}$; consequently, the desired fraction is

$$\frac{3 - \frac{3\sqrt{3}}{2}}{\sqrt{3}} = \sqrt{3} - \frac{3}{2}.$$

Solution 2 by Chikara Tsugawa provided an alternative treatment of the final paragraph of Solution 1. Here is that paragraph (using the notation of Solution 1).



Denote by S the projection of C on AO, and by T its projection on AB. From step 4 above we know that $|BC| = |OC| = \sqrt{3} - 1$. Thus the right triangles CSO and CTB are congruent; moreover,

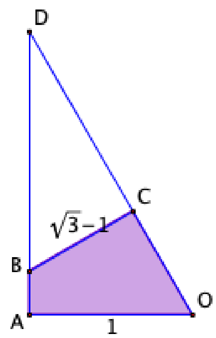
$$|CT| = |CS| = \frac{\sqrt{3}|OC|}{2} = \frac{3 - \sqrt{3}}{2}.$$

Therefore,

$$[OABC] = [ATCS] = \left(\frac{3 - \sqrt{3}}{2}\right)^2 = \frac{12 - 6\sqrt{3}}{4} = \frac{6 - 3\sqrt{3}}{2},$$

as we saw in Solution 1.

Solution 3 by Katsuhiro Shimada is an example of an approach used by a half dozen submissions. Here is a summary of Shimada's solution.



Continuing with the notation of Solution 1 and the lengths obtained there, we see that ΔDBC is similar to ΔDOA; moreover, since $|BC| = \sqrt{3} - 1$ while $|OA| = 1$, the sides of the smaller triangle are $\sqrt{3} - 1$ times the sides of the larger. Denoting $[DOA]$ by Δ, we have

$$[OABC] = \Delta - (\sqrt{3} - 1)^2\Delta = (2\sqrt{3} - 3)\Delta.$$

Because ΔDOA is half of an equilateral triangle that makes up one-sixth the area of the given regular hexagon, the ratio of the shaded area to the area of the hexagon must be

$$\frac{[OABC]}{2\Delta} = \frac{(2\sqrt{3} - 3)\Delta}{2\Delta} = \sqrt{3} - \frac{3}{2}.$$

4987. *Proposed by Jakson Hong.*

In the triangle ABC, let K be a point inside ABC, let KD, KE and KF be perpendicular to BC, AC and AB, respectively, and intersect BC, AC and AB at points D, E, F. Suppose Ω is the circumcircle of ABC and AK intersects Ω at M. Suppose MN is perpendicular to BC and meets Ω at N. Finally, let J be a point on EF such that JD is parallel to AK and let T be the point on KD such that AT is parallel to BC. Prove that the triangle TJD is similar to the triangle NKM.

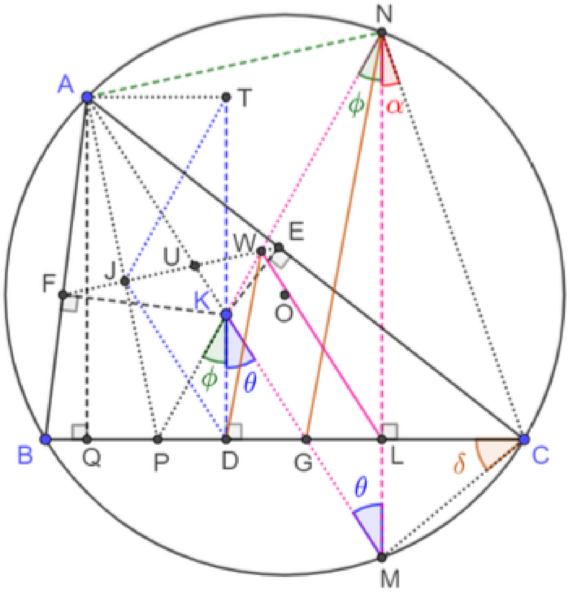
All four submissions were correct; we feature the solution by Theo Koupelis.

In triangles TJD and NKM we are given that $JD \parallel KM$ and $DT \parallel MN$, so that $\angle JDT = \angle KMN$. Therefore, to show that the two triangles are similar it suffices to prove that $DJ/DT = MK/MN$.

Let U, G be the intersection points of the line AK with the lines FE, BC, respectively; let L be the intersection point of the line MN with the line BC; and let W, P be the intersection points of the line NK with the lines FE, BC respectively.

Also, let

$$\begin{aligned} \alpha &:= \angle MNC = \angle MAC, \quad \delta := \angle BCM = \angle BAM, \\ \phi &:= \angle KNM = \angle PKD, \quad \theta := \angle KMN = \angle DKG. \end{aligned}$$



We first prove that $FE \parallel AN$. The quadrilateral $BANM$ is cyclic by construction, and the quadrilateral $KFAE$ is cyclic because $\angle KFA = \angle KEA = 90^\circ$. Thus,

$$\begin{aligned} \angle EFA &= \angle EKA = 90^\circ - \angle KAE = 90^\circ - \alpha = 90^\circ - \angle MBC \\ &= \angle BML = \angle BMN = 180^\circ - \angle BAN = 180^\circ - \angle FAN. \end{aligned}$$

Therefore, $FE \parallel AN$.

Next step is to show that $WL \parallel KM$. Again using the circles $BANM$ and $KFAE$, we have

$$\delta = \angle BCM = \angle BAM = \angle BAK = \angle FEK; \quad \text{also,} \quad \alpha = \angle KAE.$$

From the law of sines in triangles UEK and UEA we get

$$\frac{UK}{\sin\delta} = \frac{UE}{\cos\alpha}, \quad \text{and} \quad \frac{AU}{\cos\delta} = \frac{UE}{\sin\alpha} \quad \Longrightarrow \quad \frac{AU}{UK} = \frac{1}{\tan\alpha \cdot \tan\delta}.$$

But $\tan\alpha = LC/NL$ and $\tan\delta = LM/LC$. Thus, $AU/UK = NL/LM$. But $AN \parallel UW$, and thus $AU/UK = NW/WK$. Therefore, $NW/WK = NL/LM$, and $WL \parallel KM$.

Next, $PW/WN = PD/DG$ and $DW \parallel GN$. In triangle PWL, because GK is parallel to the base LW, it cuts the sides PW, PL proportionally; in particular,

$PW/PL = KW/GL$. Similarly, with triangle NKM and transversal WL we get $KW/ML = WN/LN$. From the similar right triangles KDG and MLG we get $DG/LG = KD/ML$; finally, from the similar right triangles PKD and PNL we get $KD/NL = PD/PL$. Multiplying the eight fractions and simplifying, we get $PW/WN = PD/DG$ and, thus, $DW \parallel GN$.

Next, we show that $J \in AP$. Let J' be the intersection point of the line PA with the line FE. Using $FE \parallel AN$ we get $PJ'/J'A = PW/WN = PD/DG$. Thus, $DJ' \parallel AG$, and therefore $J' \equiv J$.

Finally, we prove that $DJ/DT = MK/MN$. Let Q be the foot of the perpendicular from A onto BC. By construction, $DT = AQ$. From the similar right triangles AQG and MLG, and from the similar triangles PJD and PAG, we get

$$\frac{AG}{MG} = \frac{AQ}{ML} = \frac{DT}{ML} \quad \text{and} \quad \frac{JD}{AG} = \frac{PD}{PG} \implies \frac{DJ}{DT} = \frac{PD}{PG} \cdot \frac{MG}{ML}. \tag{1}$$

Using the law of sines in triangles PKG and MKN we get

$$\frac{PG}{\sin(\phi+\theta)} = \frac{PK}{\cos\theta} = \frac{PK \cdot MG}{ML} \quad \text{and} \quad \frac{MN}{\sin(\phi+\theta)} = \frac{MK}{\sin\phi} = \frac{MK \cdot PK}{PD}.$$

Dividing $\dfrac{PG}{\sin(\phi+\theta)}$ by $\dfrac{MN}{\sin(\phi+\theta)}$ and rearranging we get

$$\frac{PD}{PG} \cdot \frac{MG}{ML} = \frac{MK}{MN}. \tag{2}$$

Equations (1) and (2) yield $\frac{DJ}{DT} = \frac{MK}{MN}$, which completes the proof.

Editor's comments. Two of the three other submissions verified the result using coordinates with the help of computer calculations (or, perhaps, with considerable perseverance). The proposer's solution chased angles much like the featured solution.

4988. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=1}^{\infty} \left[(2n-1) \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \right) - 2 \right].$$

We received 17 solutions, all correct. We present two solutions.

Solution 1, by Michel Bataille.

For $n \in \mathbb{N}$, let $R_n = \sum\limits_{k=n}^{\infty} \frac{1}{k^2}$ and $u_n = (2n-1)R_n - 2$. We show that $\sum\limits_{n=1}^{\infty} u_n = -\frac{1}{2}$.

Let $a_{k,n} = \dfrac{1}{k(k+1)\cdots(k+n-1)} = \dfrac{(k-1)!}{(k+n-1)!}$, $(k=1,2,\ldots)$. We first prove:

$$(1) \quad \sum_{k=1}^{\infty} a_{k,n+1} = \frac{1}{n(n!)} \qquad\qquad (2) \quad \sum_{k=1}^{\infty} \frac{a_{k,n}}{k} = \frac{R_n}{(n-1)!}.$$

(1) directly follows from $a_{k,n+1} = \frac{1}{n}\left(a_{k,n} - a_{k+1,n}\right)$. We prove (2) by induction.

The equality holds if $n = 1$. Assume that the equality holds for some positive integer n. Observing that $\frac{a_{k+1,n}}{k} = a_{k,n+1}$ and using (1), we then obtain

$$\sum_{k=1}^{\infty} \frac{a_{k,n+1}}{k} = \frac{1}{n}\sum_{k=1}^{\infty} \frac{a_{k,n} - a_{k+1,n}}{k} = \frac{1}{n}\cdot\frac{R_n}{(n-1)!} - \frac{1}{n}\sum_{k=1}^{\infty} a_{k,n+1} = \frac{1}{n!}\left(R_n - \frac{1}{n^2}\right) = \frac{R_{n+1}}{n!}.$$

This completes the induction step and the proof of (2).

We deduce that

$$\begin{aligned} u_n &= (2n-1)\cdot[(n-1)!]\sum_{k=1}^{\infty}\frac{a_{k,n}}{k} - 2 \\ &= (2(n!) - (n-1)!)\left(\frac{1}{n!} + \frac{1}{2(n+1)!} + b_n\right) - 2 \\ &= -\frac{3}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) + 2(n!b_n) - (n-1)!b_n \end{aligned}$$

where

$$b_n = \sum_{k=3}^{\infty} \frac{a_{k,n}}{k} = \sum_{j=1}^{\infty} \frac{a_{j+2,n}}{j+2}.$$

We calculate

$$\begin{aligned} \sum_{n=1}^{\infty} n!b_n &= \sum_{n=1}^{\infty} n! \sum_{j=1}^{\infty} \frac{a_{j+2,n}}{j+2} \\ &= \sum_{j=1}^{\infty} \frac{(j+1)!}{j+2} \sum_{n=1}^{\infty} \frac{n!}{(j+n+1)!} \\ &= \sum_{j=1}^{\infty} \frac{(j+1)!}{j+2} \sum_{m=2}^{\infty} a_{m,j+1} \\ &= \sum_{j=1}^{\infty} \frac{(j+1)!}{j+2} \left(\frac{1}{j(j!)} - \frac{1}{(j+1)!}\right) \\ &= \frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{1}{j} - \frac{1}{j+2}\right) = \frac{3}{4} \end{aligned}$$

and similarly

$$\sum_{n=1}^{\infty}(n-1)!b_n = \sum_{j=1}^{\infty}\frac{(j+1)!}{j+2}\sum_{n=1}^{\infty}a_{n,j+2} = \sum_{j=1}^{\infty}\frac{(j+1)!}{j+2}\cdot\frac{1}{(j+1)(j+1)!}$$
$$= \sum_{j=1}^{\infty}\left(\frac{1}{j+1}-\frac{1}{j+2}\right) = \frac{1}{2}.$$

Thus, $\displaystyle\sum_{n=1}^{\infty}u_n = -\frac{3}{2}+2\cdot\frac{3}{4}-\frac{1}{2} = -\frac{1}{2}.$

Solution 2, by the Missouri State University Problem Solving Group.

We notice that for any $a > 0$,

$$\frac{1}{a^2} = \int_0^{\infty} te^{-at}dt = -\int_0^1 x^{a-1}\log x dx,$$

where we used the transformation $x = e^{-t}$ to get the second integral expression. Hence

$$\sum_{k=0}^{\infty}\frac{1}{(n+k)^2} = -\sum_{k=0}^{\infty}\int_0^1 x^{n+k-1}\log x dx = -\int_0^{\infty} x^{n-1}\log x\sum_{k=0}^{\infty}x^k dx = -\int_0^1\frac{x^{n-1}}{1-x}\log x dx.$$

Denote the summand in the problem as a_n, which could be written as

$$\begin{aligned} a_n &= (2n-1)\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots\right)-2 \\ &= -2n\int_0^1\frac{x^{n-1}}{1-x}\log x dx + \int_0^1\frac{x^{n-1}}{1-x}\log x dx - 2 \\ &= -\left.\frac{2x^n\log x}{1-x}\right|_0^1 + 2\int_0^1\frac{x^{n-1}-x^n+x^n\log x}{(1-x)^2}dx + \int_0^1\frac{x^{n-1}}{1-x}\log x dx - 2 \\ &= \int_0^1\frac{x^{n-1}(2-2x+\log x+x\log x)}{(1-x)^2}dx. \end{aligned}$$

Denote the desired summation by S, hence

$$\begin{aligned} S &= \sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\int_0^1\frac{x^{n-1}(2-2x+\log x+x\log x)}{(1-x)^2}dx \\ &= \int_0^1\frac{2-2x+\log x+x\log x}{(1-x)^3}dx \\ &= \left.\frac{1-x+x\log x}{(1-x)^2}\right|_0^1 = \lim_{x\to 1}\frac{1-x+x\log x}{(1-x)^2}-1 \\ &= \lim_{x\to 1}\frac{\log x}{-2(1-x)}-1 = \lim_{x\to 1}\frac{\frac{1}{x}}{2}-1 = -\frac{1}{2}. \end{aligned}$$

4989. *Proposed by Yagub N. Aliyev.*

Suppose that $a > b > c > 0$. Prove that:

a) if $b \geq 2c$ and $2b^2 + 2c^2 - ab - ac - 2bc \geq 0$, then $\frac{1}{a} + \frac{1}{c} > \frac{3}{b}$;

b) if $2b^2 + 2c^2 - ab - ac - 2bc \geq 0$, then $a + c < 2b$.

We received 12 correct solutions for this problem. The following is the solution by Manuel Sanchez-Guijarro.

a) One may see that

$$\frac{1}{a} + \frac{1}{c} > \frac{3}{b} \iff \frac{bc + ab - 3ac}{abc} > 0 \iff bc + ab - 3ac > 0,$$

so we will prove the last inequality instead.

The hypothesis $2b^2 + 2c^2 - ab - ac - 2bc \geq 0$ may be rewritten as

$$-ac \geq -2b^2 - 2c^2 + ab + 2bc.$$

Using the latter yields

$$bc + ab - 2ac - ac \geq bc + ab - 2ac - 2b^2 - 2c^2 + ab + 2bc.$$

Reordering the last expression and applying $b \geq 2c$ gives

$$bc + 2bc + 2ab - 2ac - 2b^2 - 2c^2 \geq 2c^2 + 2bc + 2ab - 2ac - 2b^2 - 2c^2.$$

Simplifying the $2c^2$ terms, since $2bc+2ab-2ac-2b^2 > 0 \iff bc+ab-ac-b^2 > 0$, we may just check the latter, which is true because

$$bc + ab - ac - b^2 = b(c - b) + a(b - c) > b(c - b) + b(b - c) = 0,$$

where we used $a > b > c > 0$.

b) We rewrite the hypothesis $2b^2 + 2c^2 - ab - ac - 2bc \geq 0$ as

$$ab + bc - 2b^2 \leq 2c^2 - ac - bc.$$

Now, noting that $a + c < 2b \iff a + c - 2b < 0 \iff ab + bc - 2b^2 < 0$, along with the above, gives

$$ab + bc - 2b^2 \leq 2c^2 - ac - bc = c(2c - a - b) < 0 \iff 2c - a - b < 0.$$

And the last inequality is true because $2c = c + c < a + b$ holds as a result of $a > b > c > 0$.

Editor's note. The proved inequalities play an important role in the classification of the intersections of the ellipsoid $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$ and its caustics (see for example p. 412-413 of Aliyev, Y. N., Apollonius Problem and Caustics of an Ellipsoid, *International Electronic Journal of Geometry*, 17 (2), 2024.).

4990. *Proposed by Tuan anh Nguyen.*

Prove that

$$\sum_{k=0}^{n-1}\binom{n+k}{2k+1}\frac{(-1)^k+1}{2}-\frac{F_{2n}}{2}=\frac{(-1)^{\left\lfloor\frac{n-1}{3}\right\rfloor}+(-1)^{\left\lfloor\frac{n}{3}\right\rfloor}}{4},\quad \forall n\in\mathbb{N}^*,$$

where F_i is the Fibonacci numbers with $F_1=F_2=1$, $F_{i+1}=F_i+F_{i-1}$ and $\mathbb{N}^*=\{1,2,\ldots\}$ denotes the set of positive integers.

We received 9 solutions, all correct and complete. Below we present the solution by Missouri State University Problem Solving Group.

Let

$$f_n(x)=\sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^k \text{ and } g_n(x)=\sum_{k=0}^{n}\binom{n+k}{2k}x^k.$$

We have

$$\begin{aligned}
f_n(x)&=\sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^k\\
&=\sum_{k=0}^{n-1}\left(\binom{n+k-1}{2k+1}+\binom{n+k-1}{2k}\right)x^k\\
&=f_{n-1}(x)+g_{n-1}(x)
\end{aligned}$$

and

$$\begin{aligned}
g_n(x)&=\sum_{k=0}^{n}\binom{n+k}{2k}x^k\\
&=\sum_{k=0}^{n-1}\left(\binom{n+k-1}{2k-1}+\binom{n+k-1}{2k}\right)x^k\\
&=xf_n(x)+g_{n-1}(x)\\
&=x\left(f_{n-1}(x)+g_{n-1}(x)\right)+g_{n-1}(x)\\
&=xf_{n-1}(x)+(x+1)g_{n-1}(x).
\end{aligned}$$

Therefore,

$$\begin{pmatrix}f_n(x)\\g_n(x)\end{pmatrix}=\begin{pmatrix}1&1\\x&x+1\end{pmatrix}\begin{pmatrix}f_{n-1}(x)\\g_{n-1}(x)\end{pmatrix}.$$

Letting

$$M=\begin{pmatrix}1&1\\x&x+1\end{pmatrix},$$

we have

$$\begin{pmatrix}f_n(x)\\g_n(x)\end{pmatrix}=M\begin{pmatrix}f_{n-1}(x)\\g_{n-1}(x)\end{pmatrix}=M^2\begin{pmatrix}f_{n-2}(x)\\g_{n-2}(x)\end{pmatrix}=\cdots=M^n\begin{pmatrix}f_0(x)\\g_0(x)\end{pmatrix}.$$

Now

$$M^2 - (x+2)M + I = 0, \text{ so } M^n - (x+2)M^{n-1} + M^{n-2} = 0$$

and hence, $f_n(x) = (x+2)f_{n-1}(x) - f_{n-2}(x)$.

Letting $x = 1$, we have $f_n(1) = 3f_{n-1}(1) - f_{n-2}(1)$ with $f_0(1) = 0$ and $f_1(1) = 1$. These are precisely the initial conditions and recurrence relation for F_{2n}, so $f_n(1) = F_{2n}$.

Letting $x = -1$, we have $f_n(-1) = f_{n-1}(-1) - f_{n-2}(-1)$ with $f_0(-1) = 0$ and $f_1(-1) = 1$. The sequence

$$a_n = \frac{(-1)^{\lfloor \frac{n-1}{3} \rfloor} + (-1)^{\lfloor \frac{n}{3} \rfloor}}{2}$$

yields the periodic sequence $0, 1, 1, 0, -1, -1, 0, 1, 1, \ldots$ with period 6 (starting with $n = 0$). This sequence satisfies the same initial conditions and recurrence relation as $f_n(-1)$, so $f_n(-1) = a_n$.

Finally

$$\sum_{k=0}^{n-1} \binom{n+k}{2k+1} \frac{(-1)^k + 1}{2} = \frac{f_n(-1) + f_n(1)}{2}$$

and the result follows.