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IN THIS ISSUE / DANS CE NUMÉRO

In Memoriam: Andy Liu

This issue, we pay tribute to Andy Liu, a Canadian mathematician who passed away earlier this year. I searched my inbox to find out that our first email exchange, in which I introduced myself, happened in 2015. I never met Andy in person, but we have exchanged hundreds of emails. Andy started sending me papers, mostly by students, that he felt would be suitable for ***Crux*** and he was rarely wrong.

In 2016, we were discussing various international literature and sharing our frustrations at the unavailability of some well-known and well-loved resources in English. This is where the original idea for our book came from: we wanted to expose more of the Soviet tradition of mathematical circles, competitions and traditions to the English-speaking world. Andy had a lot of different ideas and materials to pick from and we decided to focus on the Leningrad Math Olympiads. I believe Andy had a vision for this project before I even fully grasped the vastness of the source materials. He had an incredible ability to transform a simple list of problems and solutions into usable and insightful learning resources. Four years later, in the midst of the pandemic, Springer published our *Grade Five Competition from the Leningrad Mathematical Olympiad*.

Our collaboration went well beyond this book itself. Andy had an immense amount of experience running math camps and was always generous in sharing the materials. At one point, he found a hard drive with math camp materials and mailed it to me, along with a cute mathematical puzzle that I still have. Every month or so, I would get an email from Andy, either with a student article or an idea for one or a note authored by him. One such note appears as this issue's *Teaching Problems* column.

Thanks Andy for all you have done for my personal mathematical enrichment and for the international mathematical community as a whole.

– Kseniya Garaschuk

Andy Liu Remembered

Andrew Chiang-Fung (Andy) Liu died on March 26, 2024, eleven days after his 77th birthday. He was a member of the mathematics faculty at the University of Alberta and a longtime contributor to **Crux Mathematicorum**. He served as Book Reviews editor of ***Crux*** (first in effect, later in name) from May 1989 until December 1998. His first contributed problem to ***Crux*** was number 404 way back in January 1979, and he continued to send problems to ***Crux*** for many years after.

Andy wrote or co-wrote a large number of books, mostly collections of problems from mathematics contests. These are listed on Andy's Wikipedia page, and include two books jointly published in 2009 by the CMS and the MAA: *The Alberta High School Mathematics Competitions, 1957 – 2006: A Canadian Problem Book*, and *Problems from Murray Klamkin*.

Andy was a key member of the Board for the Alberta High School Mathematics Competition for over 30 years. Every year on a Saturday in the early fall, it was the pleasant duty of my Calgary colleague Robert Woodrow and myself to make the trip up to Edmonton to help put together that year's contest. After a morning's work we would repair to Andy's favourite dim sum locale for lunch with the other Edmonton Board members, such as Murray Klamkin and Alvin Baragar. After a leisurely and substantial meal, spiced with Andy's always-irrepressible sense of humour, we would head back to the exam room to complete our task, then Rob and I would fly back to Calgary that evening. Inevitably, the online "Zoom" meetings of more recent years, although necessary and still enjoyable, cannot compare.

Andy was closely connected to other math competitions as well. His 35-odd years' involvement with the Edmonton Junior High Mathematics Contest ran the gamut from making up the entire contest, to counselling during the marking sessions, to donating book prizes to the contest winners. He introduced the Tournament of the Towns to Canada (first in Edmonton, then elsewhere), and later served as a vice-president for that competition. He was the Chair of the Problem Selection Committee for the 1995 International Mathematical Olympiad which was held in Canada. Later in 2000 and 2003 he served as the Leader of the Canadian Team to the IMO.

Andy was the recipient of many teaching honours, including: the David Hilbert International Award (1996); the Adrien Pouliot Award from the CMS (2002); and the PIMS Educational Prize (2010). In the February 1999 issue of ***Crux*** is a note congratulating Andy on yet another award, this time the 1998-1999 Canadian University Professor of the Year, as awarded by the Councils for Advancement and Support of Education of the USA and Canada. You can read about these and more of Andy's awards in [1999: 65-66], as well as in Andy's Wikipedia entry.

Unfortunately, the last few years of Andy's life were somewhat marred by serious criminal charges which were later shown to be without basis and entirely dropped. It is one of life's injustices that the dropping of these charges received no publicity whatsoever. In the end, Andy's legacy is assured through his books, which will continue to enrich the lives of their readers, and through the memories of the multitude of students to whom Andy helped introduce the beauties of mathematics.

Rest in peace, Andy.

– Bill Sands

MathemAttic

No. 56

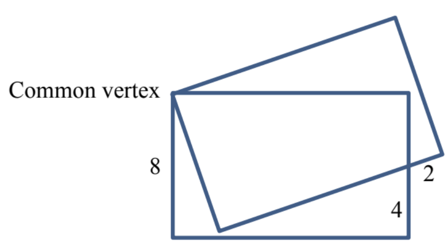
The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 30, 2024**.

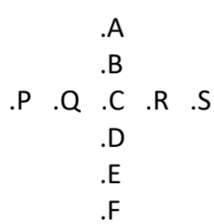
MA276. Table-tennis balls come in boxes of 8 and in boxes of 15. For example, an order for 38 balls may be filled without breaking open any box by sending 2 boxes of 15 and 1 box of 8 while an order for 37 balls cannot be filled without breaking open a box. What is the maximum number of balls that cannot be bought without breaking open a box?

MA277. Two congruent rectangles have a common vertex and overlap, as shown in the diagram.



Find the area of the drawing that is not common to both rectangles.

MA278. The points A, B, C, D, E, F are collinear and the points P, Q, C, R, S are collinear, as shown.



How many different triangles may be formed by using three of these given points as vertices?

MA279. A 4-digit number is a perfect square. When 1 is added to each of the digits, another 4-digit perfect square is formed. What number is the smaller of these perfect squares?

MA280. Prove that every odd integer n lies exactly halfway between a multiple of 3 and a multiple of 4 where $n \geq 5$.

. . . .

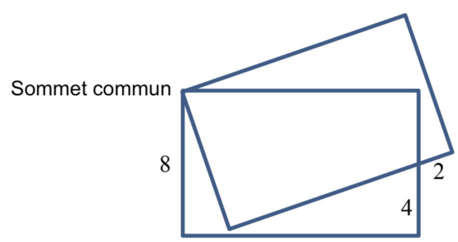
Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 août 2024**.

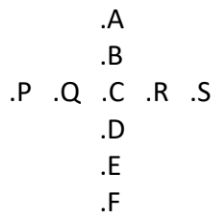
MA276. Les balles de tennis de table se présentent en boîtes de 8 et en boîtes de 15. Par exemple, une commande de 38 balles peut être exécutée sans ouvrir de boîte en envoyant 2 boîtes de 15 et 1 boîte de 8, alors qu'une commande de 37 balles ne peut pas être exécutée sans ouvrir de boîte. Quel est le nombre maximum de balles qui ne peuvent être achetées sans ouvrir une boîte ?

MA277. Deux rectangles congruents ont un sommet commun et se chevauchent, comme le montre le diagramme que voici.



Trouvez l'aire qui n'est pas commune aux deux rectangles.

MA278. Les points A, B, C, D, E et F sont colinéaires et les points P, Q, C, R et S sont colinéaires, tel qu'illustré.



Combien de triangles différents peuvent être formés en utilisant trois de ces points comme sommets ?

MA279. Un nombre à 4 chiffres donné est un carré parfait. Lorsque l'on ajoute 1 à chacun de ses chiffres, un autre carré parfait à 4 chiffres est formé. Quel est le plus petit de ces carrés parfaits ?

MA280. Montrez que tout entier impair n, où $n \geq 5$, se trouve exactement à mi-chemin entre un multiple de 3 et un multiple de 4.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2024: 50(4), p. 5–7.

In this issue of MathemAttic, we discovered some serious cases of plagiarism. Plagiarism of any sort will not be tolerated. The solvers must only submit their own original solutions. Copying solutions from online sources is unacceptable and will result in being banished from the publication.

MA251. Bob is practicing addition in base 2. Each time he adds two numbers in base 2, he counts the number of carries. For example, when summing the numbers 1001 and 1011 in base 2, there are three carries (shown on the top row).

1		1	1	
0	1	0	0	1
0	1	0	1	1
1	0	1	0	0

Suppose that Bob starts with the number 0 and adds 111 (i.e. 7 in base 2) to it one hundred times to obtain the number 1010111100 (i.e. 700 in base 2). How many carries occur (in total) in these one hundred calculations?

Originally from the 2023 University of Maryland High School Mathematics Competition, Question 24, Part I.

We received 4 submissions, of which 2 were plagiarized from the official source and 1 was not correct. We present the solution by Richard Hess.

To count the carries when adding 100 copies of 111 in base 2, we can imagine 111 piled in a stack of height 100.

The units column has 100 entries and will produce 50 carries to the 2's column.

The 2's column will have 150 1's to produce 75 carries to the 4's column.

The 4's column will have 175 1's to produce 87 carries to the 8's column.

From there on, the carries will be 43, 21, 10, 5, 2 and 1, where we have taken half the remaining carries and rounded down. The sum of these is 294 carries.

MA252. An Indian raga has two kinds of notes: a short note, which lasts for 1 beat, and a long note, which lasts for 2 beats. For example, there are 3 ragas which are 3 beats long: 3 short notes, a short note followed by a long note, and a long note followed by a short note. How many Indian ragas are 11 beats long? Justify your answer.

Originally from the 2023 University of Maryland High School Mathematics Competition, Question 1, Part II.

We received 6 submissions, 2 of which were plagiarized from the official source and 1 was incorrect. We present the solution by Henry Ricardo.

To find the number of ways to decompose a raga that is n beats long into short notes and long notes, take each possibility for $n-2$ beats and append a long note. Similarly, take each $n-1$ beat raga and append a short note. If r_n denotes the number of ragas that are n beats long, we have $r_n = r_{n-1} + r_{n-2}$, with $r_1 = 1, r_2 = 2$, and $r_3 = 3$. Therefore, $r_n = F_{n+1}$, the $(n+1)$st Fibonacci number, and $r_{11} = 144$.

MA253. Let $n \geq 2$ be an integer. There are n houses in a town. All distances between pairs of houses are different. Every house sends a visitor to the house closest to it. Find all possible values of n (with full justification) for which we can design a town with n houses where every house is visited.

Originally from the 2023 University of Maryland High School Mathematics Competition, Question 2, Part II.

We received 2 submissions, 1 of which was plagiarized from the official source and 1 was incomplete. We encourage you to look up the official solution and check it for completeness.

MA254. A sequence $a_1, a_2, \ldots$ satisfies $a_1 = \frac{5}{2}$ and $a_{n+1} = a_n^2 - 2$ for all $n \geq 1$. Let M be the integer closest to a_{2023}. Find the last digit of M.

Originally from the 2019 University of Maryland High School Mathematics Competition, Question 22, Part I.

We received 5 submissions, 1 of which was plagiarized from the official source and 1 was incorrect. We present the solution by the Missouri State University Problem Solving Group, adapted for completeness of the argument.

We claim that
$$a_n = \frac{2^{2^n} + 1}{2^{2^{n-1}}}.$$
We proceed by induction on n. The case $n = 1$ is immediate. For the inductive step,
$$\begin{aligned} a_{n+1} = a_n^2 - 2 &= \left(\frac{2^{2^n} + 1}{2^{2^{n-1}}}\right)^2 - 2 \\ &= \frac{2^{2^{n+1}} + 2 \cdot 2^{2^n} + 1 - 2 \cdot 2^{2^n}}{2^{2^n}} \\ &= \frac{2^{2^{n+1}} + 1}{2^{2^n}}, \end{aligned}$$
and we are done.

Let m_n be the integer closest to a_n for $n \geq 2$. Then

$$a_n = \frac{2^{2^n}}{2^{2^{n-1}}} + \frac{1}{2^{2^{n-1}}} = 2^{2^{n-1}} + \frac{1}{2^{2^{n-1}}},$$

and so, since

$$\frac{1}{2^{2^{n-1}}} < \frac{1}{2}$$

for $n \geq 2$,

$$m_n = 2^{2^{n-1}} \quad \text{and hence} \quad m_{n+1} = m_n^2.$$

Since $m_3 = 16$ and $m_n = m_{n-1}^2$ for all $n \geq 4$, the last digit of m_n will be 6 for all $n \geq 3$. This follows inductively from the fact that the product of any two integers whose last digit is 6 will also have 6 as its last digit. Specifically, any such product can be written as $\pm(10k+6)(10l+6)$ for some positive integers k and l. This gives

$$(10k+6)(10l+6) = 100kl + 60(k+l) + 36 = 10(10kl + 6k + 6l + 3) + 6$$

i.e., the last digit of such product is 6. Therefore, since $M = m_{2023}$, its last digit is 6.

MA255. A $3\times3\times3$ cube of cheese is sliced into twenty-seven $1\times1\times1$ blocks. A mouse starts anywhere on the outside and eats one of the $1\times1\times1$ cubes. He then moves to an adjacent cube (in any direction), eats that cube, and continues until he has eaten all 27 cubes. (Two cubes are considered adjacent if they share a face.) Prove that no matter what strategy the mouse uses, he cannot eat the middle cube last. (Note: One should neglect gravity: intermediate configurations do not collapse.)

Originally from the 2017 University of Maryland High School Mathematics Competition, Question 4, Part II.

We received 2 solutions, 1 of which was plagiarized from an online source. We present the solution by Richard Hess.

The $3\times3\times3$ cube is divided into 27 blocks and a mouse attempts to consume all 27 blocks by starting on the surface of the cube and consuming adjacent blocks until he ends at the center block. The mouse's attempt will always fail owing to the following. Consider its path in reverse, which starts at the center block, and note that the remaining 26 blocks split into 12 edge blocks, 8 corner blocks and 6 face blocks. The mouse's first step from the center will always result in his being at a face block. From there on, each face or corner block must be followed by an edge block. Since there are only 12 edge blocks there are only 13 neighbors possible for the 14 edge and face blocks, a contradiction.

The mouse will be frustrated.

TEACHING PROBLEMS

No. 27

Andy Liu

A Factorization Problem

The core problems in this feature were provided by Andy Liu for potential consideration as an article. The Editors have decided to work with the material to make it into a submission suited for Teaching Problems.

. . . .

We start off with two problems that are structurally similar though different in that one can be solved mentally, whereas, the other involves large numbers that make any work with it to be seemingly impractical to consider. We then provide an organized approach shifting attention to a generalization that may serve solvers with value beyond the immediacy of solving the second problem posed in this piece.

Problem 1. For a group dinner, the diners agree to split the bill of 24 dollars equally, but two of them slip away without paying. As a result, each of the others has to pay one dollar more. How many diners end up footing the bill?

First Solution. We have $24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6$. The difference 2 must arise from $8 - 6 = 2$. Hence there are 8 diners each of whom is supposed to pay 3 dollars. Only 6 diners end up footing the bill, each paying 4 dollars, 1 dollar more than expected.

The key to solving this problem is knowing the factorizations of the value of the bill. When it is a small number like 24, there is no trouble. However, the next puzzle constitutes a considerable challenge.

Problem 2. For a group dinner, the diners agree to split the bill of 169828879009 dollars equally, but two of them slip away without paying. As a result, each of the others has to pay 824086 dollars more. How many diners end up footing the bill?

Theoretically, it is exactly the same as the preceding puzzle. However, there is a significant difference between the technical difficulties in factoring 24 and factoring 169828879009. This large number has no obvious factors.

Let us consider the puzzle in a more general form. We retain the setting but vary the actual question.

Problem 3. Let k and ℓ be positive integers. For a group dinner, the diners agree to split the bill equally, but ℓ of them slip away without paying. As a result, each of the others has to pay k dollars more. What are the possible values of the bill?

Solution. Suppose there are $n+\ell$ diners each paying m dollars in the agreement, but n diners each paying $m+k$ dollars in reality. Then the bill is

$$m(n+\ell) = (m+k)n.$$

It follows that $m\ell = kn$ so that $\frac{m}{n} = \frac{k}{\ell}$. Let $m = rk$ and $n = r\ell$. Then the bill is $r(r+1)k\ell$ for any positive rational number r such that $r(r+1)k\ell$ is integral.

Let us see how this can help us solve the preceding puzzles.

Second Solution to Problem 1. We have $k = 1$ and $\ell = 2$. Since the bill is

$$24 = r(r+1)k\ell,$$

we have $r(r+1) = 12$. It follows that $r = 3$, and the number of diners footing the bill is $r\ell = 6$.

Solution to Problem 2. We have $k = 824086$ and $\ell = 2$. Then the bill is

$$169828879009 = r(r+1)k\ell.$$

Note that we have

$$k\ell = 2 \times 824086 = 4 \times 412043.$$

Since

$$169828879009 \div 412043 = 412163,$$

we have $s(s+2) = 412163$, where $s = 2r$ is an odd number. Now $\sqrt{412163} \sim 642$. It follows that $s = 641$, $r = \frac{641}{2}$ and the number of diners footing the bill is $r\ell = 641$. The relevant factorizations are

$$169828879009 = 641 \times 264943649 = 643 \times 264119563$$

with

$$264943649 - 264119563 = 824086.$$

Editor's Remarks. The enormous dollar values and quantities in the second problem suggested something potentially unsolvable. The absence of any evident divisors to break down the high dollar amount reinforced the magnitude of the values. The idea of pursuing a generalization represents a worthy plan of attack, as a general result can apply with large or small values.

The presentation by the author offers a window into an avenue of problem solving that is not so commonly taken in my experience. The nature of the problem at hand perhaps motivates the approach in that it is not so easy to imagine the conditions for a simpler problem. A more familiar instinct given a challenge that may seem out of reach initially is to consider a simpler problem. Insight into a viable path of solution as well as enhanced understanding of the problem situation may facilitate solution of the more challenging version. It is the experience here of starting with a simpler problem that does not aid in solving the harder problem that makes the route of finding a generalization so valuable.

Gregarious and Reclusive Triples

Ed Barbeau

PART C: FINDING TRIPLES WHEN SOME ELEMENTS ARE KNOWN

This article is the third of three on this topic. For Part A, please refer to ***Crux***, 50(4), p. 190-192; for Part B, refer to ***Crux***, 50(5), p. 244-247.

Further application of Pell's equation

Once we start with a $k-$triple, (x, y, z), we can generate an infinite family of $k-$triples with the same values of x and y. We will suppose that xy is not a square and that $xy + k = c^2$. Recall that the diophantine Pell's equation $u^2 - (xy)v^2 = 1$ has infinitely many solutions (u, v) in integers.

Exercise 17. Let k, x, y, a and b be integers.

(a) Show that if there exists an integer z for which $yz + k = a^2$ and $xz + k = b^2$, then (a, b) must satisfy

$$xa^2 - yb^2 = (x - y)k.$$

(b) Suppose that (a, b) satisfies the equation in (a) and that (u, v) satisfies the diophantine equation $u^2 - (xy)v^2 = 1$. Verify that $xA^2 - yB^2 = (x - y)k$, when $(A, B) = (au + ybv, bu + xav)$.

(c) Suppose that $xy + k = c^2$ and that $xa^2 - yb^2 = (x - y)k$. Determine z so that (x, y, z) is a $k-$triple. Is this triple necessarily congenial or necessarily reclusive?

Exercise 18. Let c be an integer. Then the triple $(2, 4, 2c + 6)$ is a congenial $(c^2 - 8)-$triple. Use the method of Exercise 14 to construct other $k-$triples for which $(x, y) = (2, 4)$ and determine whether they are congenial or reclusive. Check for specific values of c.

Exercise 19. Determine a family of $(-1)-$triples for which $(x, y) = (1, 5)$. Look at the possible values of z and its relation to terms in the Fibonacci sequence. Make a conjecture and prove it directly. Which triples are congenial?

Constructing triples from the related squares

We can construct $k-$triples by starting with the squares involved. Let a, b, c be three arbitrary integers; we can factor the differences of their squares to construct a $k-$triple (x, y, z) for which $xy + k = c^2$, $xz + k = b^2$ and $yz + k = a^2$. For example, if $b^2 - c^2 = x(z - y)$, we can select different possibilities for the factors x and $z - y$.

Thus, $z-y$ will be among the divisors of b^2-c^2, $y-x$ among the divisors of a^2-b^2, and $z-x$ among the factors of a^2-c^2. However, the choice of divisors from the three differences of squares will be constrained by the fact that

$$z-x=(z-y)+(y-x).$$

From these choices for $z-x$, $z-y$, $y-x$, we can get x, y, z from the cofactors of the square differences and check that the values are consistent with their differences.

Exercise 20. Apply this approach to $(a,b,c)=(11,7,3)$ to obtain $k-$triples (x,y,z) for which $xy+k=9$, $xz+k=49$ and $yz+k=121$. What are the corresponding values of k?

Exercise 21. Determine $k-$triples (x,y,z) and associate squares (a,b,c) for which $x=b-c$, $y=a-c$, $z=a+b$. Are these congenial? superbly congenial?

Exercise 22. Investigate $k-$triples for which $x=a+b$, $y=a+c$, and $z=a+b$.

Exercise 23. Investigate $k-$triples for which $x=b-c$, $y=a-c$, $z=a-b$.

Exercise 24. Investigate $x=b-c$, $y=a+c$, $z=a+b$.

Exercise 25. Investigate the situation when $x=0$ or when $y=z$.

Exercise 26. What are the possible $k-$triples when $(a,b,c)=(5,5,4)$? Which ones are congenial? superbly congenial? reclusive?

Additional questions, some open

Question 1. For each nonzero integer k, what is the maximum number m of entries in a set S of integers for which the values of $xy+k$ for the $\binom{m}{2}$ pairs (x,y) of distinct elements of S are all squares, with no two equal?

Question 2. Must every congenial $k-$triple be superbly congenial?

Question 3. Can a triples (x,y,z) be a congenial $k-$triple for more than one integer k?

Question 4. For each integer k we form a graph whose vertices are equivalent classes of $k-$triples. Two $k-$triples are equivalent if the terms of one are the negative of the terms of the other, the terms of one are a permutation of those of the other, or a composite of these conditions. The vertices are the equivalence classes of $k-$triples and two vertices are connected by an edge if and only if a representative triple of one is an associate of a representative triple of the other. Is the graph formed by the equivalence classes of congenial $k-$triples connected?

Question 5. Are there any $k-$triples (x,y,z) for which none of x,y,z is equal to 0 or 1 and $xyz+k$ is also a square?

Question 6. Let k be an integer. Suppose that for some triple, the product $(xy+k)(yz+k)(zx+k)$ is square. Under what circumstances does this imply that each of the three factors is square?

Question 7. What are the possible values of the triple (k, m, d) for which there is a $k-$sequence with each term congruent to d modulo m?

For example, if m is a common divisor of r and s, then $(s^2, m, 0)$ is such a triple exemplified by the sequence

$$\ldots, r-s, 0, r+s, 4r+8s, 9r+21s, 25r+55s, \ldots.$$

Are there any examples for which $d \neq 0$?

Question 9. Which $k-$triples are arithmetic progressions? geometric progressions? harmonic progressions?

Question 10. Characterize triples (x, y, z) that are not $k-$triples for any value of k.

Notes

Exercise 18. In this case, we are led to the equation $a^2 - 2b^2 = -(c^2 - 8)$, where $(a, b) = (c+4, c+2)$ is the starting solution. The corresponding $(c^2-8)-$triple is $(2, 4, 12c^2 + 7c + 102)$. Here are some examples for specific values of c; they are all k triples, but some are gregarious for another value of k. The value of k for which the triple is gregarious is appended to the triple, thus $(x, y, z; k)$.

c	k	$(x, y, z; k))$
-3	1	(2, 4, 0; 1) (2, 4, 12; 1) (2, 4, 420; 42841) (2, 4, 14280; 50936761)
-2	-4	(2, 4, 2; -4) (2, 4, 10; -4) (2, 4, 290; 20156)
-1	-7	(2, 4, 4; -7) (2, 4, 8; -7) (2, 4, 44; 353) (2, 4, 184; 7913) (2, 4, 1408; 491391)
0	-8	(2, 4, 6; -8) (2, 4, 102; 2296) (2, 4, 3366; 2822392)

Exercise 19. We obtain the (-1)-triples $(1, 5, 10)$, $(1, 5, 65)$, $(1, 5, 442)$ and $(1, 5, 3026)$. These are all of the form $(1, 5, f_{2n}^2 + 1)$ with the associated squares of $(f_{2n-1} + f_{2n+1}, f_{2n}, 2)$. For the product of the second and third entries, we have

$$\begin{aligned}
&5(f_{2n}^2 + 1) - 1 - (f_{2n-1} + f_{2n+1})^2 \\
&= 5f_{2n-1}f_{2n+1} - 1 - (f_{2n-1} + f_{2n+1})^2 \\
&= f_{2n-1}(f_{2n+1} - f_{2n-1}) + (f_{2n-1} - f - 2n + 1)f_{2n+1} + f_{2n-1}f_{2n+1} - 1 \\
&= -f_{2n}(f_{2n+1} - f_{2n-1}) + f_{2n-1}f_{2n+1} - 1 \\
&= -f_{2n^2} + f_{2n-1}f_{2n+1} - 1 \\
&= 0.
\end{aligned}$$

Exercise 20. We obtain the (-23)-triple $(4, 8, 18)$ and the $(-131)-$triple $(10, 14, 18)$, both congenial.

Exercise 26. $(a, b, c) = (5, 5, 4)$ gives rise to the congenial $(-65)-$sequence $\{\ldots, 61, 26, 9, 9, 10, 29, 69, \ldots\}$ and congenial $15-$sequence

$\{\ldots, 61, 21, 10, 1, 1, -6, -11, -35, \ldots\}$. However, the factorization $b^2 - c^2 = 3 \times 3$ yields the reclusive $7-$triple $(3, 3, 6)$. Notice that this is also a congenial $(-9)-$triple.

Question 6. This has been investigated for $k = 1$ in the paper by Kiran S. Kedlaya, When is $(xy+1)(yz+1)(zx+1)$ a square? *Mathematics Magazine*, 71:1 (February, 1998), 61-63.

The problems in this article undoubtedly have been well studied over a long period. The best historical reference I have come across is the book by Andrej Dujella, *Diophantine $m-$tuples and elliptic curves.* Springer, 2024. This has recently been published and the author has provided a summary on the webpage as well as a list of open problems.

What we call a $k-$triple, Dujella calls a $D(k)-$triple, with analogous terminology for $m-$tuples; a superb gregarious triple is, in his terms, *regular* (in my opinion, an overworked word in definitions). Dujella dates interest in this problem to the discovery by Diophantus that $(\frac{1}{16}, \frac{33}{16}, \frac{17}{4}.\frac{105}{16})$ is a $1-$quadruple with rational entries. This is equivalent to $(1, 33, 68, 105)$ being a $256-$quadruple. We note that the triple $(1, 33, 68)$ is congenial with left associate $(0, 1, 33)$ and right associate $(33, 68, 201)$, However, $(33, 68, 105)$ is reclusive, since neither its left or right associates, $(97, 33, 68)$ and $(68, 105, 313)$ are $256-$triples. Diophantus also discovered other examples of $k-$quadruples with $k \neq 1$.

Fermat is credited with finding the first $1-$quadruple $(1, 3, 8, 120)$; in 1969, Baker and Davenport showed that 120 is the only value of d that makes $(1, 3, 8, d)$ a $1-$triple. Euler made significant progress, initiating over 200 years of intermittent and increasingly deep progress.

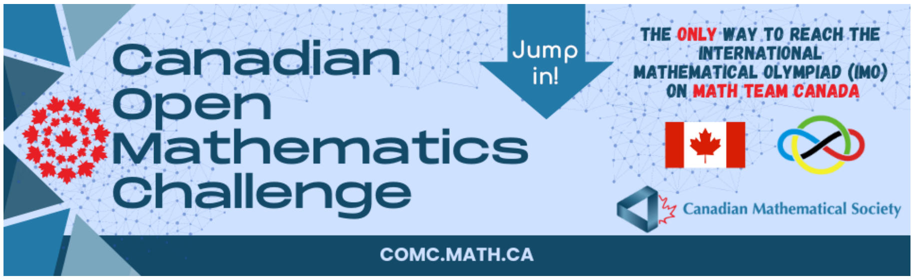
To construct a $k-$quadruple (x, y, z, w) by extending a $k-$triple, we have to determine w so that $v^2 = (xw+k)(yw+k)(zw+k)$, the equation of an elliptic curve in the wv-plane. Accordingly, the bulk of Dujella's book is the development of the theory of elliptic curves to support research in this area.

For the cases $k = \pm 1$, a few results and additional references are given on pages 153-155, 157-159 of the book Edward J. Barbeau, *Power play.* The Mathematical Association of America, 1997 ISBN 0-88385-523-2

Competition Highlights

Canadian Open Mathematics Challenge

by Margo Kondratieva and Robert Woodrow



The Canadian Open Mathematics Challenge (COMC) is an annual, individual competition designed for secondary school students and available to any interested participant who enjoys problem solving and wishes to enhance their mathematical knowledge and ability.

At the same time, for the most capable and ambitious Canadian students it is the main starting point if they aim to represent Canada at the International Math Olympiad (IMO). Indeed, students with the top COMC scores can earn invitations directly to the Canadian Math Olympiad (CMO) or its qualifying Repêchage (CMOQR). The winners of CMO are potential candidates for the Math Team Canada competing at the IMO.

The European Girls' Mathematics Competition (EGMO) is another invitational competition to which the best COMC students could be invited.

Students performing well at COMC can earn invitations to other higher-level mathematics competitions. The best scores receive monetary prizes. In addition, the best performers have an advantage in obtaining scholarships and admissions to prestigious universities.

Given the double role of the COMC, to engage all and to select a few, the contest is composed of both easy and challenging problems organized in three parts: A, B, and C.

Part A consists of four relatively easy problems, each of which is worth 4 points. Each of the four problems in part B is worth 6 points. The full score in parts A and B is given to every student who provides the correct answer, with or without an explanation. However, explaining one's reasoning in parts A and B could be helpful for receiving partial credit in the case when an answer is incorrect but the solution contains some valid ideas.

Each of the four problems in part C is worth 10 points and consists of sub-problems, typically (a), (b) and (c). Parts (a) and (b) aim to help the solver to mingle with the problem and to understand it better via consideration of a special case or a specific example that is generalized in part (c) of the same problem. Part (c) is therefore the most challenging. Oftentimes it requires writing a proof or making a conclusion addressing the situation in greater generality. In contrast with parts A and B of the contest, to receive a credit in Part C requires writing an explanation of the answer in any of the sub-problems.

The contest covers traditional topics such as arithmetic, algebra, geometry, combinatorics, probability, some elements of number theory, and functions.

We will illustrate the challenge presented in parts A, B and C using three geometrical problems offered in the most recent contest COMC 2023.

The problem A4 from part A asks to find the area of the quadrilateral $XCYZ$, which is a part of the 10×10 square $ABCD$, given that points W, X, Y, and Z are the midpoints of the segments AB, BC, CD and WX, respectively. Refer to Figure 1.

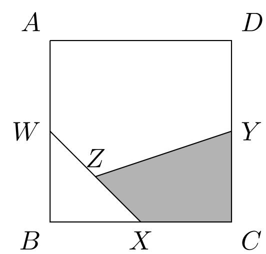


Figure 1: An illustration of the problem from part A.

Let us draw the auxiliary lines XY and WY (Figure 2). Then one can argue that the two blue triangles WXB and XYC are congruent, so they have the same area. The two pink triangles XYZ and WYZ also have the same area as having the same base $WZ = ZX$ and the same height from vertex Y. In conclusion, the required area is $[XCYZ] = \frac{1}{2}[WBCY] = \frac{1}{4}[ABCD] = 25\,\text{cm}^2$.

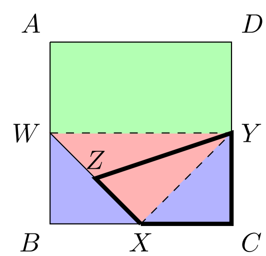


Figure 2: An illustration of a solution of the problem from part A.

Alternatively, the areas of all these triangles can be explicitly found, leading to the same answer. Among 6196 submitted papers, 4353 received the full score for this problem, while 796 papers received a partial credit. The average mark for this problem was 75%.

The problem B4 from part B asked to find the angle CPE shown in Figure 3, given that $\angle BAC = 24°$ and $\angle ACB = 28°$. Here point D is constructed such that AB is parallel to CD, $AD = BC$, and AD and BC are not parallel. Similarly, point E is constructed such that AE is parallel to BC, $AB = CE$, and AB and CE are not parallel. The first challenge of this problem was that the figure was not given, only the words describing this construction.

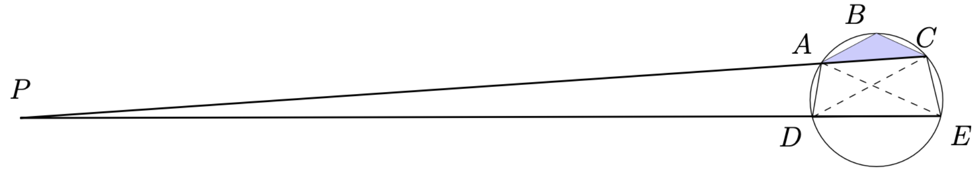


Figure 3: An illustration of the solution of the problem from part B.

From the description we realize that $ABCD$ and $ABCE$ are isosceles trapezoids and the points A, B, C, D, E are all concyclic. From triangle ABC, $\angle ABC = 128°$. Then trapezoid $ABCE$ has angles $128°$ and $52°$. Now we have:

$$\angle ACE = \angle BCE - \angle ACB = 128° - 28° = 100°,$$
$$\angle DEA = \angle DCA = \angle BAC = 24°,$$
$$\angle DEC = \angle DEA + \angle AEC = 24° + 52° = 76°.$$

Thus, from triangle CPE, $\angle CPE = 180° - 100° - 76° = 4°$.

For this problem only 953 students received the full score, while 1613 students received a partial credit. The average mark for B4 was 22%.

The part C geometry problem, C2, consisted of the sub-problems:

a) Find the distance from the point $(1, 0)$ to the line connecting the origin and the point $(0, 1)$.

b) Find the distance from the point $(1, 0)$ to the line connecting the origin and the point $(1, 1)$.

c) Find the distance from the point $(1, 0, 0)$ to the line connecting the origin and the point $(1, 1, 1)$.

In (a) we observe that the line connecting the origin and the point $(0, 1)$ is the y-axis. The closest point to $(1, 0)$ on the y-axis is the origin, so the distance from $(1, 0)$ to the y-axis is 1. Refer to Figure 4 (top left).

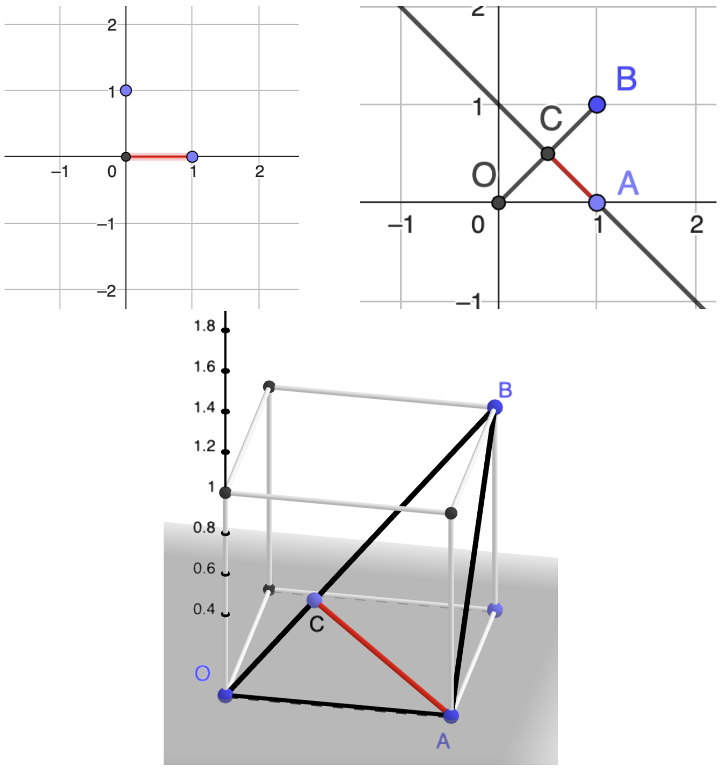


Figure 4: An illustration of the solution of the part C problem, (a) (b) and (c).

In (b) we note that the three given points $O(0,0)$, $A(1,0)$ and $B(1,1)$ are vertices of a unit square and so they form a right triangle with legs $AO = AB = 1$ and the hypotenuse $BO = \sqrt{2}$ (Figure 4, top right). The required distance is the height from point A to the line BO, hitting BO at C. We can write the area of the triangle ABO in two different ways:

$$\frac{1}{2} = \frac{AO \cdot AB}{2} = \frac{BO \cdot AC}{2} = \frac{\sqrt{2}}{2}AC.$$

Therefore, $AC = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

In (c) we need a 3D image which is more challenging to visualize. However, we can employ the same idea as in our solution for (b). Denote: $O(0,0,0)$ and $B(1,1,1)$. The required distance is the height from point $A(1,0,0)$ to the line BO, hitting BO at C. Refer to Figure 4 (bottom). Note that ABO is a right triangle with $AB = \sqrt{2}, BO = \sqrt{3}$, and $AO = 1$. Thus, we can write the area of the triangle ABO in two different ways:

$$\frac{\sqrt{2}}{2} = \frac{AO \cdot AB}{2} = \frac{BO \cdot AC}{2} = \frac{\sqrt{3}}{2}AC, \quad \text{so} \quad AC = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}.$$

Students' solutions of this problem were awarded 751 full marks, while 3959 papers received a partial credit. The average mark for C2 was 40.4% . This means that many students pretty successfully attempted sub-problems (a) and (b) even though they missed (c).

Restricting ourselves to the topic of geometry, we see that the ideas required in any of the considered solutions are a part of the secondary school curriculum. This includes the area of a triangle and a quadrilateral, Cartesian coordinates, measurement of angles and the length of a segment using the Pythagorean theorem. However, sometimes the standard ideas need creative use and students' ability to read the problem carefully and to make an appropriate sketch. Extracurricular knowledge is typically not required but may be beneficial in some cases to produce a shorter solution. This could be decisive for the result in the situation of time constraint.

Looking at the total COMC picture for the year 2023, we observe that the four average scores of the problems in part A ranged between 63% and 92%, the four average scores of problems in part B ranged between 22% and 73%, and the four average scores of problems in part C ranged between 3% and 52%. We note that while there were 8 perfect papers and a number of very insightful solutions, the average score for the contest was below 50%.

This is pretty typical for COMC because one of the goals of the competition is to select a small group of about 80 students to invite for the CMO.

Therefore, all participants of COMC should keep in mind that they already benefited from challenging themselves with interesting mathematical problems, which they may continue to think about after the exam.

Students, who successfully solved selected problems should be proud of their current achievement.

Students who want to go further have to prepare more and work systematically by studying previous years' solutions and other resources.

As an additional training, students are advised to participate in the Canada Lynx Mathematical Competition (CLMC) prior to COMC.

It would be also helpful to develop optimal strategies for time management during the competition.

We welcome all interested participants to the future rounds of the COMC, usually in the last week of October. For more information visit `https://cms.math.ca/Competitions/COMC/`

OLYMPIAD CORNER

No. 424

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **August 30, 2024**.

OC686. A sequence of Xs and Os is given, such that no three consecutive characters in the sequence are all the same, and let N be the number of characters in this sequence. Maia may swap two consecutive characters in the sequence. After each swap, any consecutive block of three or more of the same character will be erased (if there are multiple consecutive blocks of three or more characters after a swap, then they will be erased at the same time), until there are no more consecutive blocks of three or more of the same character. For example, if the original sequence were $XXOOXOXO$ and Maia swaps the fifth and sixth character, the end result will be $XXOOOXXO \to XXXXO \to O$. Find the maximum value N for which Maia can't necessarily erase all the characters after a series of swaps.

OC687. For certain real constants p, q, r, we are given a system of equations

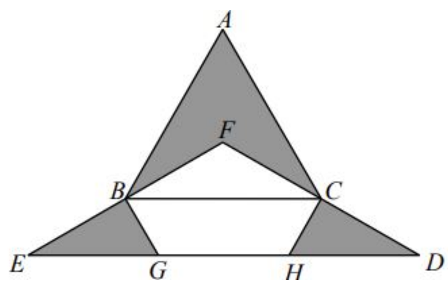
$$\begin{cases} a^2 + b + c = p \\ a + b^2 + c = q \\ a + b + c^2 = r \end{cases}$$

What is the maximum number of solutions of real triplets (a, b, c) across all possible p, q, r? Give an example of the p, q, r that achieves this maximum.

OC688. Fix integers a and b greater than 1. For any positive integer n, let r_n be the (non-negative) remainder that b^n leaves upon division by a^n. Assume there exists a positive integer N such that $r_n < 2^n/n$ for all integers $n \geq N$. Prove that a divides b.

OC689. The product of five consecutive positive integers is 120 times greater than $\overline{ABABAB}$, where A and B are non-zero digits. What is the largest of those five numbers?

OC690. Let ABC be an equilateral triangle whose area is 36 cm^2, and EFD be an isosceles triangle with $EF = FD$. The point F is the centre of triangle ABC, and the points B and C are the midpoints of EF and FD, respectively, as shown in the diagram below. If $BG \perp EF$ and $CH \perp DF$, what is the area, in cm^2, of the shaded region?



. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 aout 2024**.

OC686. Soit une suite de X et de O dans laquelle on ne retrouve trois caractères consécutifs identiques. Soit N le nombre de caractères dans cette suite. Maia peut échanger deux caractères consécutifs dans la suite. Après chaque permutation, tout bloc consécutif de trois caractères ou plus du même caractère sera effacé (s'il y a plusieurs blocs consécutifs de trois caractères ou plus après une permutation, ils seront effacés en même temps), jusqu'à ce qu'il n'y ait plus de blocs consécutifs de trois caractères ou plus du même caractère. Par exemple, si la suite originale était $XXOOXOXO$ et que Maia intervertit le cinquième et le sixième caractère, alors le résultat final sera $XXOOOXXO \to XXXXO \to O$. Trouvez la valeur maximale N pour laquelle Maia ne peut pas nécessairement effacer tous les caractères après une série de permutations.

OC687. Pour certaines constantes réelles p, q et r, on nous donne un système d'équations

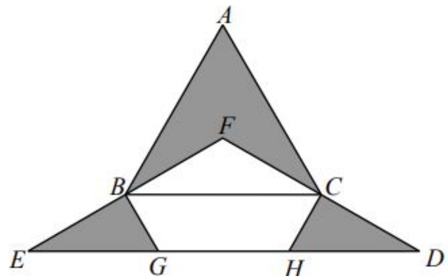
$$\begin{cases} a^2 + b + c = p \\ a + b^2 + c = q \\ a + b + c^2 = r \end{cases}$$

Quel est le nombre maximum de solutions de triplets réels (a, b, c) pour toutes les valeurs possibles de p, q et r ? Donnez un exemple de p, q et r pour lesquels ce maximum est atteint.

OC688. Fixons des entiers a et b supérieurs à 1. Pour tout entier positif n, notons r_n le reste (non négatif) que b^n laisse après division par a^n. Supposons qu'il existe un entier positif N tel que $r_n < 2^n/n$ pour tous les entiers $n \geq N$. Montrez que a divise b.

OC689. Le produit de cinq entiers positifs consécutifs est 120 fois supérieur à $\overline{ABABAB}$, où A et B sont des chiffres non nuls. Quel est le plus grand de ces cinq nombres ?

OC690. Soit ABC un triangle équilatéral dont l'aire est 36 cm^2 et EFD un triangle isocèle avec $EF = FD$. Le point F est le centre du triangle ABC et les points B et C sont les points milieux de EF et FD, respectivement, tel qu'illustré sur le schéma ci-dessous. Si $BG \perp EF$ et $CH \perp DF$, quelle est l'aire, en cm^2, de la région ombrée ?



OLYMPIAD CORNER SOLUTIONS

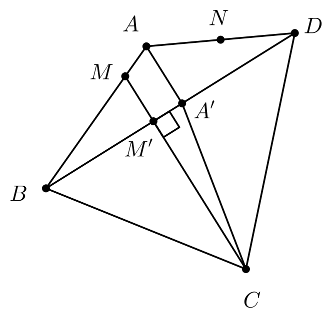
Statements of the problems in this section originally appear in 2024: 50(1), p.16–17.

OC661. Point N is the midpoint of side AD of a convex quadrilateral $ABCD$, and point M on side AB is such that $CM \perp BD$. Prove that if $BM > MA$, then $2BC + AD > 2CN$.

Originally 2023 Leonhard Euler Competition, 3rd Round, Day 1, Problem 2.

We received 6 submissions, all of which were correct and complete. We present two solutions.

Solution 1, by UC Lan Cyprus Problem Solving Group.



Let M', A' be the projections of M, A on BD. Since $BM > MA$, we have that $BM' > M'A'$ and so $BC > CA'$. We also have $NA' = NA = ND$. Therefore

$$2BC + AD = 2(BC + AN) > 2(CA' + NA') \geqslant 2CN$$

as required.

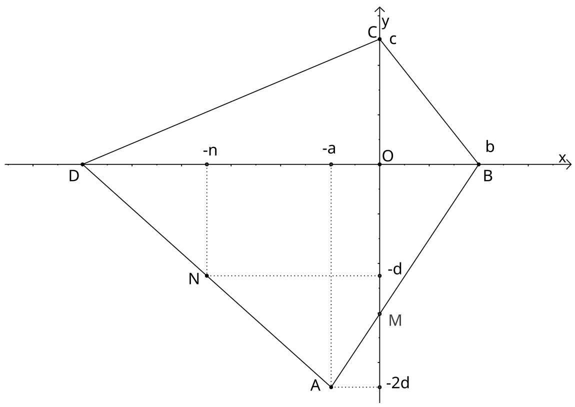
Solution 2, by Oliver Geupel.

We use Cartesian coordinates such that $B = (b, 0)$ and $C = (0, c)$ with positive b and c and the intersection of BD and CM is the origin. Let $A = (-a, -2d)$. By convexity of $ABCD$, the numbers a and d are positive. By the hypothesis $BM > MA$, it holds $a < b$. Since D lies on the x-axis and N is the midpoint of AD, we have $N = (-n, -d)$ with a positive number n. Then,

$$2BC + AD = 2(BC + AN) = 2\left(\sqrt{b^2 + c^2} + \sqrt{|n - a|^2 + d^2}\right).$$

Note that $b + |n - a| \geq b + n - a > n$. It follows by Minkowski's inequality that

$$\sqrt{b^2 + c^2} + \sqrt{|n - a|^2 + d^2} \geq \sqrt{(b + |n - a|)^2 + (c + d)^2} > \sqrt{n^2 + (c + d)^2} = CN.$$



This proves the required result.

OC662. Let $a_1, \ldots, a_k$ be distinct positive integers such that the difference between the largest and smallest of them is less than 1000. What is the largest k for which it is possible that all quadratic equations $a_i x^2 + 2a_{i+1}x + a_{i+2} = 0$, where $1 \le i \le k-2$, have no real roots?

Originally 2023 Leonhard Euler Competition, 3rd Round, Day 1, Problem 3.

We received 4 submissions, 2 of which were correct and complete. We present the solution by Sicheng Du.

Let $x_i = a_{i+1} - a_i$, where $1 \le i \le k-1$. Since the quadratic equation,

$$a_i x^2 + 2a_{i+1}x + a_{i+2} = 0,$$

has no real roots $a_{i+2}a_i > a_{i+1}^2$. By AM-GM inequality,

$$a_{i+2} + a_i \ge 2\sqrt{a_{i+2}a_i} > 2a_{i+1}.$$

Hence

$$a_{i+2} - a_{i+1} > a_{i+1} - a_i,$$

or $x_{i+1} > x_i$, or $x_{i+1} - x_i \ge 1$ for any i, $1 \le i \le k-2$.

Let t, $1 \le t \le k-1$, such that

$$x_1 < x_2 < \cdots < x_{t-1} < 0 < x_t < \cdots < x_{k-1}.$$

If $x_{t-1} = -1$ and $x_t = 1$, then $a_{t+1} - a_t = 1$ and $a_t - a_{t-1} = -1$, and so $a_{t+1} = a_{t-1}$, a contradiction. So either $x_{t-1} \le -2$ or $x_t \ge 2$.

If $x_t \geq 2$, then

$$a_k - a_t = \sum_{i=t}^{k-1}(a_{i+1} - a_i) = \sum_{i=t}^{k-1} x_i = \sum_{i=t}^{k-1}\left[x_t + \sum_{j=t}^{i-1}(x_{j+1} - x_j)\right]$$
$$\geq \sum_{i=t}^{k-1}(2 + i - t) = \frac{(k-t+3)(k-t)}{2},$$

while $a_k - a_t \leq \max\{a_i\} - \min\{a_i\} \leq 1000$. So we deduce $k - t \leq 43$. Similarly,

$$1000 \geq a_1 - a_t = \sum_{i=1}^{t-1}(-x_i) \geq \sum_{i=1}^{t-1}(t-i) = \frac{t(t-1)}{2},$$

equivalently $t \leq 45$. Hence $k = (k-t) + t \leq 45 + 43 = 88$. If $x_{t-1} \leq -2$, the same method gives $k - t \leq 45$ and $t \leq 43$, hence $k \leq 88$ as well.

On the other hand, let

$$x_i = \begin{cases} i - 45, & \text{if } 1 \leq i \leq 44, \\ i - 43, & \text{if } 45 \leq i \leq 87. \end{cases}$$

We can check that $\max\{a_i\} = a_1$ and $\min\{a_i\} = a_{45}$, with $a_1 - a_{45} = 990$. We also need

$$a_{i+1}a_{i-1} > a_i^2 \iff (a_i - x_{i-1})(a_i + x_i) > a_i^2$$
$$\iff a_i > \frac{x_i x_{i-1}}{x_i - x_{i-1}} \text{ for all } 1 \leq i \leq 86.$$

If we increase all a_i ($1 \leq i \leq 88$) by the same positive integer N, then none of the neither of x_i ($1 \leq i \leq 87$) changes. For sufficiently large N, we can satisfy the inequality above.

We conclude that the largest possible k is 88.

OC663. There are 100 cities in the Far Far Away Kingdom, and every two cities are connected by no more than one road. One day the king ordered the introduction of one-way traffic on every road, and at the same time every road was painted white or black. The Minister of Transport proudly announced that after carrying out the order, one can get from any city to any other along roads alternating their colors, and so that the first road along the way will be white. What is the smallest number of roads there could be in this country? When getting from city to city, you can pass through intermediate cities any number of times.

Originally 2023 Leonhard Euler Competition, 3rd Round, Day 2, Problem 6.

We received 2 submissions, both of which were correct and complete. We present the solution by UCLan Cyprus Problem Solving Group.

We will show that the country must have 150 roads but no less.

To do it with this many roads, label the cities as $x_1, x_2, \ldots, x_{100}$ and consider a Hamilton cycle $x_1 x_2 x_3 \cdots x_{100} x_1$ with edges alternating between white and black, where $x_1 x_2$ is white. This fact enables us to start from any city with an odd index and reach any city we wish.

Now we also add the white edges $x_2 \to x_4, x_4 \to x_6, \ldots, x_{98} \to x_{100}, x_{100} \to x_2$. Together with the previous edges, these enable us to reach any city we wish, starting from any city with an even index. Starting from x_{2n}, we move via a white road to x_{2n+2} and then to x_{2n+3} with a black road. Afterward, we can reach any city we wish, starting from there.

To show that we need at least 150 roads, let A be the set of all cities from which a black road starts. Let B be the set of all cities for which no black road passes through it, either towards them or outwards from them.

We claim that from every city x, be it in the set A or not, there must be a white road ending in a city $y \in A$. Alternatively, from x we can reach cities connected via a white road starting at x. In particular, we cannot reach any city in A. Note that A must be non-empty.

We claim that for every city $x \in B$, there must be a white road ending in B. Indeed there is no black road ending in B so we can also reach B via a white road.

From the above two paragraphs, we deduce that we have at least $100 + |B|$ white roads. We must also have at least $|A|$ black roads. By $|A|$ we denote the number of elements of the set A.

Now observe that there are at most $|A|$ cities in which a black road ends. Some of those might be cities from the set A. Therefore $|B| \geqslant 100 - 2|A|$. In particular, we deduce that

$$|A| + |B| \geqslant \frac{2|A| + |B|}{2} \geqslant 50\,.$$

Thus we have at least $(100 + |B|) + |A| \geqslant 150$ roads.

OC664. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$.

Originally from the 2023 IMC 2023, Day 1, Problem 1.

We received 10 submissions, all correct and complete. We present the solution by Piyush Kumar Jha.

Differentiate twice the relationship $f(7x+1) = 49f(x)$ to get $f''(7x+1) = f''(x)$. Here replace x by $(x-1)/7$ to get $f''(x) = f''\left(\frac{x-1}{7}\right)$. Repeat to get inductively that

$$f''(x) = f''\left(\frac{6x - 7^n + 1}{7^n \cdot 6}\right).$$

Take the limit with respect to n by keeping x fixed. Use the fact that f'' is continuous:

$$f''(x) = \lim_{n\to\infty} f''\left(\frac{6x-7^n+1}{7^n\cdot 6}\right) = f''\left(\lim_{n\to\infty}\frac{6x-7^n+1}{7^n\cdot 6}\right) = f''\left(\frac{-1}{6}\right).$$

Since x was arbitrarily selected, f'' is constant. It follows that f is a second-order polynomial. Assume $f(x) = ax^2 + bx + c$. This implies

$$a(7x+1)^2 + b(7x+1) + c = 49ax^2 + 49bx + 49c.$$

By coefficient comparison we get $k = a = 3b = 36c$, so $f(x) = k\left(x+\frac{1}{6}\right)^2$ for any arbitrary $k \in \mathbb{R}$.

OC665. Let A, B and C be $n\times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \qquad \text{and} \qquad B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

Originally from the 2023 IMC 2023 Day 1, Problem 2.

We received 10 submissions, all of which were correct and complete. We present the solution by Michel Bataille.

From

$$A^2B = B^2B = B^3 = ABC + 2I,$$

we deduce that $A(AB - BC) = 2I$, which implies that A is invertible with inverse $A^{-1} = \frac{1}{2}(AB - BC)$. Similarly, since

$$BC^2 = B^3 = ABC + 2I,$$

we have $(BC - AB)C = 2I$, hence C is also invertible with inverse $C^{-1} = \frac{1}{2}(BC - AB)$. We remark that $A^{-1} = -C^{-1}$ and deduce that

$$A = A^{-1}A^2 = -C^{-1}A^2 = -C^{-1}C^2 = -C.$$

Thus, A, B satisfy $A^2 = B^2$ and $B^3 = -ABA + 2I$. Observing that

$$(ABA)^2 = ABA\cdot ABA = AB\cdot B^2\cdot BA = AB^4A = AA^4A = A^6,$$

we obtain

$$B^6 = (2I - ABA)^2 = 4I - 4ABA + (ABA)^2 = 4I - 4ABA + A^6.$$

Since $A^6 = (A^2)^3 = (B^2)^3 = B^6$, it follows that $ABA = I$ and therefore $B^3 = I$. Finally, we have $A^6 = B^6 = I$.

Proving a Class of Symmetric Inequalities by Using Partition Method II

Yanhui Xu

Introduction

In the first part of this article (***Crux*** 50(4), p. 193-198), we gave three different strategies and examples of using the partition method to solve a class of symmetric inequalities. In this part, we give two other strategies and provide more examples of using the partition method.

Construct homogeneous inequalities

The symmetric inequality suitable for this technique is a homogeneous inequality or it can be transformed into a homogeneous inequality. We will first use the partition method to transform the proof of a sum of inequalities into the proof of a single inequality. Then we will construct a homogeneous inequality using the AM-GM inequality or by other methods. Finally, using symmetry and summation, we will obtain the desired result.

Example 9. Let a, b be positive real numbers. Prove that $\sqrt{\frac{a}{a+3b}}+\sqrt{\frac{b}{b+3a}}\geq 1$.

Proof. By using the AM-GM inequality, we can conclude that

$$\begin{aligned} & b^{\frac{3}{2}}+a^{\frac{3}{4}}b^{\frac{3}{4}}+a^{\frac{3}{4}}b^{\frac{3}{4}}\geq 3a^{\frac{1}{2}}b \\ \Leftrightarrow\ & a^{\frac{3}{2}}+b^{\frac{3}{2}}+2a^{\frac{3}{4}}b^{\frac{3}{4}}\geq a^{\frac{3}{2}}+3a^{\frac{1}{2}}b=a^{\frac{1}{2}}(a+3b) \\ \Leftrightarrow\ & a^{\frac{3}{4}}+b^{\frac{3}{4}}\geq a^{\frac{1}{4}}(a+3b)^{\frac{1}{2}}, \end{aligned}$$

that is

$$\sqrt{\frac{a}{a+3b}}\geq\frac{a^{\frac{3}{4}}}{a^{\frac{3}{4}}+b^{\frac{3}{4}}}.$$

Similarly, we can obtain

$$\sqrt{\frac{b}{b+3a}}\geq\frac{b^{\frac{3}{4}}}{a^{\frac{3}{4}}+b^{\frac{3}{4}}}.$$

Therefore $\sqrt{\frac{a}{a+3b}}+\sqrt{\frac{b}{b+3a}}\geq 1$.

If a and b are replaced with a^2 and b^2 respectively, we can obtain a new inequality given in Example 10.

Example 10. Let a, b be positive real numbers. Prove that $\frac{a}{\sqrt{a^2+3b^2}}+\frac{b}{\sqrt{b^2+3a^2}}\geq 1$.

Example 11. Let a, b be positive real numbers with $abc=1$. Prove that

$$\frac{1}{1+2a}+\frac{1}{1+2b}+\frac{1}{1+2c}\geq 1.$$

Proof. By using the AM-GM inequality, we get

$$b^{-\frac{2}{3}} + c^{-\frac{2}{3}} \geq 2(bc)^{-\frac{1}{3}} = 2a^{\frac{1}{3}}.$$

Then

$$\frac{1}{1+2a} \geq \frac{a^{-\frac{2}{3}}}{a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}}}.$$

Similarly, we can obtain

$$\frac{1}{1+2b} \geq \frac{b^{-\frac{2}{3}}}{a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}}} \quad \text{and} \quad \frac{1}{1+2c} \geq \frac{c^{-\frac{2}{3}}}{a^{-\frac{2}{3}} + b^{-\frac{2}{3}} + c^{-\frac{2}{3}}}.$$

Adding everything together, we get the desired result.

Example 12. Let a, b be positive real numbers. Prove the following:

a) $\sqrt{\dfrac{a}{b+c}} + \sqrt{\dfrac{b}{a+c}} + \sqrt{\dfrac{c}{a+b}} > 2,$

b) $\sqrt[3]{\dfrac{a}{b+c}} + \sqrt[3]{\dfrac{b}{a+c}} + \sqrt[3]{\dfrac{c}{a+b}} > 2.$

Proof. a) Let's start by proving that $\sqrt{\dfrac{a}{b+c}} \geq \dfrac{2a}{a+b+c}$. This is equivalent to proving

$$\frac{a}{b+c} \geq \frac{4a^2}{(a+b+c)^2} \iff (a+b+c)^2 \geq 4a(b+c) \iff (a-(b+c))^2 \geq 0.$$

Similarly, we can obtain that

$$\sqrt{\frac{b}{a+c}} \geq \frac{2b}{a+b+c} \quad \text{and} \quad \sqrt{\frac{c}{a+b}} \geq \frac{2c}{a+b+c}.$$

Clearly, the three equality signs do not hold simultaneously, so we have proved the original inequality.

b) We start by proving that $\dfrac{a^{1/3}}{(b+c)^{1/3}} \geq \dfrac{2a^{2/3}}{a^{2/3}+b^{2/3}+c^{2/3}}$, which is equivalent to proving that

$$a^{2/3} + b^{2/3} + c^{2/3} \geq 2a^{1/3}(b+c)^{1/3}.$$

By the AM-GM inequality, we get

$$a^{2/3} + (b+c)^{2/3} \geq 2a^{1/3}(b+c)^{1/3}.$$

What is left to prove is that $b^{2/3} + c^{2/3} \geq (b+c)^{2/3}$, which is equivalent to $(b^{2/3} + c^{2/3})^3 \geq (b+c)^2$, which is obvious.

Similarly, we can obtain

$$\frac{b^{1/3}}{(a+c)^{1/3}} \geq \frac{2b^{2/3}}{a^{2/3}+b^{2/3}+c^{2/3}} \quad \text{and} \quad \frac{c^{1/3}}{(a+b)^{1/3}} \geq \frac{2c^{2/3}}{a^{2/3}+b^{2/3}+c^{2/3}}.$$

Adding the three inequalities, we get the desired result.

Note that the inequalities in Example 12 hold even when we generalize the expressions to roots of degree n. In that case, we would need to prove

$$a^{2/n} + b^{2/n} + c^{2/n} \geq 2a^{1/n}(b+c)^{1/n}.$$

Applying the AM-GM inequality, we get $a^{2/n} + (b+c)^{2/n} \geq 2a^{1/n}(b+c)^{1/n}$ or $b^{2/n} + c^{2/n} \geq (b+c)^{2/n}$. Letting $x = b^{1/n}$ and $y = c^{1/n}$, this inequality becomes $(x^2+y^2)^n \geq (x^n+y^n)^2$, which is easily seen by applying the binomial theorem to both sides.

Transform and piece out

It is often difficult to prove the symmetric inequality by directly using the partition method. Instead, we will use a technique we call "transform and piece out". The key is figuring out how to use flexible transformations to prove a single inequality and then use symmetry to get the desired result. The technique requires the problem-solver to both have a certain amount of experience in transformations and to also have a good mathematical intuition.

Example 13. Let $a, b, c, d > 0$ such that $abcd = 1$. Prove that

$$\frac{1}{b+c+d+1} + \frac{1}{c+d+a+1} + \frac{1}{d+a+b+1} + \frac{1}{a+b+c+1} \leq 1.$$

Proof. Let $x, y, z, w > 0$ and $x^4 = a, y^4 = b, z^4 = c, w^4 = d$ then $xyzw = 1$. By using Chebyshev's inequality and the AM-GM inequality, we get:

$$\begin{aligned} b+c+d+1 \geq y^4+z^4+w^4+xyzw &\geq \frac{1}{3}(y^3+z^3+w^3)(y+z+w)+xyzw \\ &\geq yzw(y+z+w)+xyzw \\ &= yzw(x+y+z+w) \\ &= \frac{x+y+z+w}{x}, \end{aligned}$$

that is

$$\frac{1}{b+c+d+1} \leq \frac{x}{x+y+z+w}.$$

Similarly, we can obtain the other three symmetric inequalities. Adding everything gives the desired result.

Example 14. Let $x, y, z \geq 0$ such that $x^2+y^2+z^2 = 3$. Prove that

$$\frac{x}{\sqrt{x^2+y+z}} + \frac{y}{\sqrt{y^2+z+x}} + \frac{z}{\sqrt{z^2+x+y}} \leq \sqrt{3}.$$

Proof. Using Cauchy's inequality, we get that $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$, that is, $x+y+z \leq x^2+y^2+z^2$. Also by Cauchy's inequality, we get

$$\sqrt{(x^2+y+z)(1+y+z)} \geq x+y+z.$$

Therefore

$$\frac{x}{\sqrt{x^2+y+z}} \le \frac{x\sqrt{1+y+z}}{x+y+z}.$$

Similarly, we can obtain the other there symmetric inequalities and adding everything together, we get:

$$\frac{x}{\sqrt{x^2+y+z}}+\frac{y}{\sqrt{y^2+z+x}}+\frac{z}{\sqrt{z^2+x+y}} \le \frac{x\sqrt{1+y+z}+y\sqrt{1+z+x}+z\sqrt{1+x+y}}{x+y+z}.$$

The Cauchy inequality gives:

$$\begin{aligned}
&(x\sqrt{1+y+z}+y\sqrt{1+z+x}+z\sqrt{1+x+y})^2\\
&\le \left(\sqrt{x}\cdot\sqrt{x+xy+xz}+\sqrt{y}\cdot\sqrt{y+yz+xy}+\sqrt{z}\cdot\sqrt{z+zx+zy}\right)^2\\
&\le (x+y+z)\left[2(xy+yz+zx)+x+y+z\right]\\
&\le (x+y+z)\left(2xy+2yz+2zx+x^2+y^2+z^2\right)\\
&= (x+y+z)^3.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{x\sqrt{1+y+z}+y\sqrt{1+z+x}+z\sqrt{1+x+y}}{x+y+z} &\le \frac{(x+y+z)^{3/2}}{x+y+z}\\
&= \sqrt{x+y+z}\\
&\le \sqrt{x^2+y^2+z^2}\\
&= \sqrt{3}.
\end{aligned}$$

The result follows. The equal sign holds if and only if $x=y=z=1$.

Example 15. Let a, b, c be positive real numbers. Prove that

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \le \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}}.$$

Proof. We have that

$$a+\sqrt{ab}+\sqrt[3]{abc} \le a+\sqrt[3]{ab\cdot\frac{a+b}{2}}+\sqrt[3]{abc}.$$

Now we will prove that

$$a+\sqrt[3]{ab\cdot\frac{a+b}{2}}+\sqrt[3]{abc} \le \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}}.$$

By the AM-GM inequality, we get

$$\sqrt[3]{1\cdot\frac{2a}{a+b}\cdot\frac{3a}{a+b+c}} \le \frac{1+\frac{2a}{a+b}+\frac{3a}{a+b+c}}{3}$$

and its two symmetric equivalents for the cyclic rotation of the variables. Adding everything together gives the desired result. The equal sign holds if and only if $a = b = c$.

In short, the characteristic of using the partition method to prove symmetric inequalities is to first use the partition method to transform the proof of sum inequality into the proof of a single inequality. When proving the single inequality, based on the known conditions of the problem, we summarized five different strategies. When using these five common strategies to prove symmetric inequalities, the key is for the problem solver to analyze the hidden features of the problem based on their own experience and intuition.

Exercises.

1. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{1+8a} + \frac{1}{1+8b} + \frac{1}{1+8c} \geq \frac{1}{3}.$$

2. Let a, b, c be non-negative real numbers such that $a^4 + b^4 + c^4 = 1$. Prove that

$$\frac{1}{4-ab} + \frac{1}{4-bc} + \frac{1}{4-ac} \leq 1.$$

3. Prove that for any $a, b, c \in (1, 2)$, we have

$$\frac{b\sqrt{a}}{4b\sqrt{c}-c\sqrt{a}} + \frac{c\sqrt{b}}{4c\sqrt{a}-a\sqrt{b}} + \frac{a\sqrt{c}}{4a\sqrt{b}-b\sqrt{c}} \geq 1.$$

4. Let a, b, c be positive real numbers such that $abc = 1$. Show that

$$\frac{2}{(a+1)^2+b^2+1} + \frac{2}{(b+1)^2+c^2+1} + \frac{2}{(c+1)^2+a^2+1} \leq 1.$$

5. Consider real numbers $x_1, x_2, \ldots, x_n$ such that $x_1 x_2 \cdots x_n = 1$. Prove that

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

Hints.

1. First prove that

$$\frac{3}{1+8a} \geq \frac{a^{-8/9}}{a^{-8/9}+b^{-8/9}+c^{-8/9}}.$$

2. Start by proving that

$$\frac{1}{4-ab} \leq \frac{1}{2}\left(\frac{1}{4-a^2} + \frac{1}{4-b^2}\right),$$

then show that

$$\frac{1}{4-a^2} - \frac{1}{3} \leq \frac{1}{18}(a^4 - 1).$$

Do the same for the symmetric equivalents and add everything up.

3. Prove that each summand is at least $\frac{x}{a+b+c}$, where x is one of a, b, c.

4. Begin by proving that

$$\frac{2}{(a+1)^2 + b^2 + 1} \leq \frac{1}{ab + a + 1}.$$

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 30, 2024**.

4951. *Proposed by Michel Bataille.*

Let n be a positive integer. Prove that the sums

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k}^{-1} \quad \text{and} \quad \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k-1}$$

are equal and find their common value.

4952. *Proposed by Mihaela Berindeanu.*

Let ABC be a non-isosceles triangle with the circumcircle Γ, the centroid G, $AG \cap \Gamma = \{X\}$, $BG \cap \Gamma = \{Y\}$ and $CG \cap \Gamma = \{Z\}$. Prove that the triangle XYZ is isosceles with $XY = XZ$ if and only if $BXCG$ is a parallelogram.

4953. *Proposed by Michel Bataille.*

Let $a \in \mathbb{R} - \{0, 1\}$. Find all a and all functions $f : \mathbb{R} \to \mathbb{R}$ that are not identically zero such that

$$(x + y)f(x + y) - xf(x) - yf(y) = a(yf(x) + xf(y))$$

for all x, y.

4954. *Proposed by Sicheng Du.*

In triangle ABC, let I be the incenter and D, E, F be the points of tangency of the incircle on BC, CA, AB, respectively. If the circumcircles of triangles ABD and AIC intersect at A and G, show that the lines AG, DE, and FI are concurrent.

4955. *Proposed by Vasile Cîrtoaje.*

For given $n \geq 6$, find the largest positive constant k_n such that

$$\frac{1}{a_1 + k_n} + \frac{1}{a_2 + k_n} + \cdots + \frac{1}{a_n + k_n} \geq \frac{n}{1 + k_n}$$

whenever $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n \geq 0$ satisfying $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$.

4956. *Proposed by Toyesh Sharma.*

If $a, b, c, x, y, z > 0$, show that

$$1 \leq \log_{abc}(a^x b^y c^z) \cdot \log_{abc}(a^{1/x} b^{1/y} c^{1/z}).$$

4957. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with I as its incenter. Let R and r be the circumradius and inradius of ABC, respectively. Prove that

$$3\sqrt{3} \leq \frac{AB}{IC} + \frac{BC}{IA} + \frac{CA}{IB} \leq \frac{3R}{2r^2}\sqrt{R^2 - r^2}.$$

4958. *Proposed by Nguyen Viet Hung.*

Let a, b, c be non-negative real numbers, no two of which are zero, satisfying $a^2 + b^2 + c^2 = 2$. Prove that

$$\frac{a(2+bc)^2}{b+c} + \frac{b(2+ca)^2}{c+a} + \frac{c(2+ab)^2}{a+b} \geq 8 + \frac{(3abc)^2}{2(ab+bc+ca)}.$$

4959. *Proposed by Jeromin Rocklage.*

For $\alpha > 0$, evaluate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^k \left(\frac{k}{2n}\right)^{\alpha}.$$

4960. *Proposed by Yagub N. Aliyev.*

a) Suppose that a, b and c are real numbers. Prove that if $b^2 = a^2 + ac$ and $c^2 = b^2 + ab$, then $abc = (a+b+c)(-a^2 - b^2 + c^2)$.

b) Show that there are no positive integers a, b and c such that $b^2 = a^2 + ac$ and $c^2 = b^2 + ab$.

c) Find positive integers a, b and c such that $abc = (a+b+c)(-a^2 - b^2 + c^2)$.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 août 2024**.

4951. *Soumis par Michel Bataille.*

Soit n un entier positif. Montrez que les sommes

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k}^{-1} \quad \text{et} \quad \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k-1}$$

sont égales et trouvez leur valeur commune.

4952. *Soumis par Mihaela Berindeanu.*

Soit ABC un triangle non isocèle de cercle circonscrit Γ, de centre de masse G, $AG \cap \Gamma = \{X\}$, $BG \cap \Gamma = \{Y\}$ et $CG \cap \Gamma = \{Z\}$. Prouvez que le triangle XYZ est isocèle avec $XY = XZ$ si et seulement si $BXCG$ est un parallélogramme.

4953. *Soumis par Michel Bataille.*

Soit $a \in \mathbb{R} - \{0, 1\}$. Trouvez tous les a et toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ qui ne sont pas identiquement nulles tels que

$$(x + y)f(x + y) - xf(x) - yf(y) = a(yf(x) + xf(y))$$

pour tout x et y.

4954. *Soumis par Sicheng Du.*

Dans le triangle ABC, notons I le centre de son cercle inscrit et notons D, E et F les points de tangence du cercle inscrit sur BC, CA et AB, respectivement. Si les cercles circonscrits aux triangles ABD et AIC se coupent en A et G, montrez que les droites AG, DE et FI sont concourantes.

4955. *Soumis par Vasile Cîrtoaje.*

Pour $n \geq 6$ donné, trouvez la plus grande constante positive k_n telle que

$$\frac{1}{a_1 + k_n} + \frac{1}{a_2 + k_n} + \cdots + \frac{1}{a_n + k_n} \geq \frac{n}{1 + k_n}$$

pour tout $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n \geq 0$ satisfaisant $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$.

4956. *Soumis par Toyesh Sharma.*

Si $a, b, c, x, y, z > 0$, montrez que

$$1 \leq \log_{abc}(a^x b^y c^z) \cdot \log_{abc}(a^{1/x} b^{1/y} c^{1/z}).$$

4957. *Soumis par George Apostolopoulos.*

Soit ABC un triangle et soit I le centre de son cercle inscrit. Soient R et r les rayons des cercles circonscrits et inscrits á ABC, respectivement. Montrez que

$$3\sqrt{3} \leq \frac{AB}{IC} + \frac{BC}{IA} + \frac{CA}{IB} \leq \frac{3R}{2r^2}\sqrt{R^2 - r^2}.$$

4958. *Soumis par Nguyen Viet Hung.*

Soient a, b et c des nombres réels non négatifs, dont au moins deux d'entre eux sont non nuls, et satisfaisant $a^2 + b^2 + c^2 = 2$. Montrez que

$$\frac{a(2+bc)^2}{b+c} + \frac{b(2+ca)^2}{c+a} + \frac{c(2+ab)^2}{a+b} \geq 8 + \frac{(3abc)^2}{2(ab+bc+ca)}.$$

4959. *Soumis par Jeromin Rocklage.*

Pour $\alpha > 0$, évaluez la limite

$$\lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^k \left(\frac{k}{2n}\right)^{\alpha}.$$

4960. *Soumis par Yagub N. Aliyev.*

a) Supposons que a, b et c sont des nombres réels. Montrez que si $b^2 = a^2 + ac$ et $c^2 = b^2 + ab$, alors $abc = (a+b+c)(-a^2 - b^2 + c^2)$.

b) Montrez qu'il n'existe pas d'entiers positifs a, b et c tels que $b^2 = a^2 + ac$ et $c^2 = b^2 + ab$.

c) Trouvez des entiers positifs a, b et c tels que $abc = (a+b+c)(-a^2 - b^2 + c^2)$.

SOLUTIONS

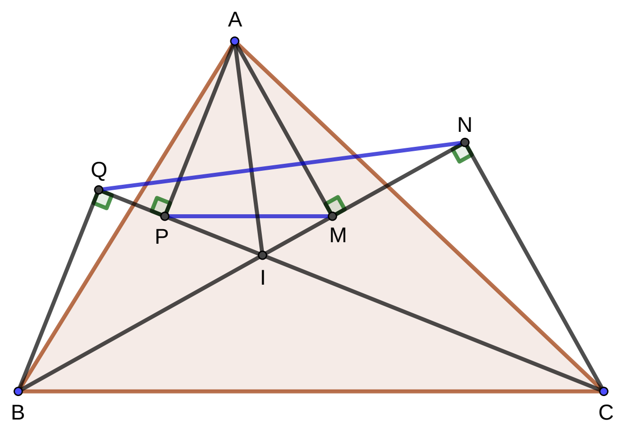
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2024: 50(1), p. 36–40.

4901. *Proposed by Michel Bataille.*

Let ABC be a triangle and I its incenter. Let M, N on the line BI and P, Q on the line CI be such that AM, CN (resp. AP, BQ) are perpendicular to BI (resp. CI). Prove that M, N, P, Q are concyclic and that MP is parallel to BC.

We received 21 solutions. We present the solution by Sicheng Du.



Since $\angle API = \angle AMI = \frac{\pi}{2}$, the points A, P, I and M are concyclic. Thus

$$\begin{aligned}\angle IPM = \angle IAM &= \frac{\pi}{2} - \angle AIM \\ &= \frac{\pi}{2} - (\angle ABI + \angle BAI) \\ &= \frac{\pi}{2} - \frac{\angle ABC + \angle BAC}{2} \\ &= \frac{\angle ACB}{2} \\ &= \angle BCI,\end{aligned}$$

concluding the proof that PM is parallel to BC.

Since

$$\angle BQC = \angle BNC = \frac{\pi}{2},$$

the points B, Q, N and C are concyclic. So

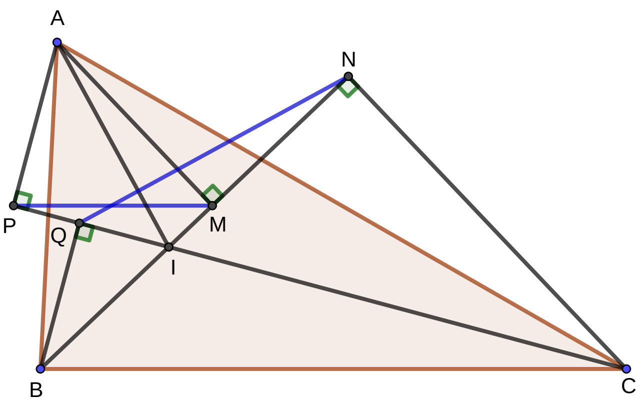
$$\angle BNQ = \angle BCQ = \frac{\angle ACB}{2}.$$

Combining this with the previous calculation

$$\angle IPM = \angle BNQ.$$

Thus we can conclude that P, Q, N and M are concyclic.

Editor's Comments. Note that the proof still works without any changes if, for example, $\triangle ABC$ is such that the order of Q and P are switched, as in the diagram below:



Note that in this case PM and QN are the diagonals of quadrilateral $PQMN$ instead of being two sides of quadrilateral $PMNQ$.

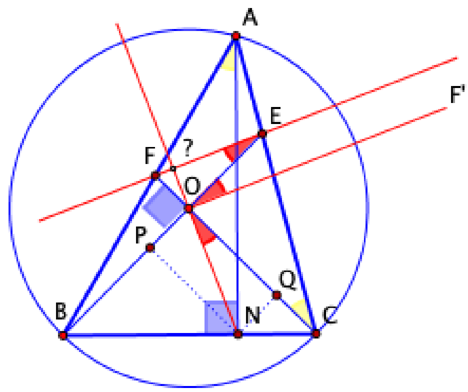
The configuration of Problem 4901 has appeared in earlier *Crux* problems. A summary of previous occurrences can be found in the solution to Problem 4171 [Vol. 43(8), October 2017, pages 361-363]. Among other interesting properties, the line PM contains the midpoints of AB and AC, and the feet of the perpendiculars from A to the external bisectors of B and C. See also the solution to Problem 3 of the Hungarian Mathematical Olympiad 2002-2003 First Round [Vol. 33(3), April 2007, pages 160-161].

4902. *Proposed by Titu Zvonaru.*

Let O be the circumcenter of triangle ABC. Let AN be the altitude from A. Lines BO and CO intersect the lines AC and AB at E and F, respectively. Prove that if $\angle BAC = 45°$, then the lines NO and EF are perpendicular.

We received 18 submissions, all correct, and sample two of the various approaches.

Solution 1, by Sicheng Du.



Since O is the circumcenter of $\triangle ABC$, $\angle COA$ is twice $\angle CBA$ so that

$$\angle ACO = \frac{\pi}{2} - \angle CBA = \angle BAN.$$

Also,

$$\angle BOC = 2\angle BAC = \frac{\pi}{2},$$

so that $\angle COE = \frac{\pi}{2} = \angle BNA$. Consequently,

$$\triangle EOC \sim \triangle BNA. \text{ Similarly, } \triangle CNA \sim \triangle FOB.$$

Therefore,

$$\frac{NC}{OF} = \frac{NA}{OB} = \frac{NA}{OC} = \frac{NB}{OE},$$

and so

$$\frac{NC}{NB} = \frac{OF}{OE}.$$

Let the feet of the perpendiculars from N to OB and OC be P and Q respectively. Note that $POQN$ is a rectangle and BPN, NQC are isosceles right triangles. Therefore,

$$\tan\angle NOQ = \frac{QN}{OQ} = \frac{\sqrt{2}QN}{\sqrt{2}PN} = \frac{NC}{NB} = \frac{OF}{OE} = \tan\angle FEO,$$

whence $\angle NOQ = \angle FEO =: \alpha$. Define OF' to be the half-line parallel to FE with initial point O. The rotation through α takes OF' to OE and ON to OC. Because we know that $OE \perp OC$, it follows that $FE \perp ON$.

Solution 2 is a composite of the similar solutions submitted independently by Prithwijit De, Marie-Nicole Gras, Corneliu Manescu-Avram, and Didier Pinchon.

If $\angle BAC = 45°$, the central angle $\angle BOC = 90°$, and a Euclidean system of coordinates may be chosen such that the circumcircle of the triangle ABC is the unit circle, and B, C, A have coordinates $B = (-1, 0)$, $C = (0, -1)$, $A = (a, b)$, with $a^2 + b^2 = 1$ and a, b real numbers that are not both negative. (If we allow both a and b to be negative, then A is on the arc of the unit circle in the third quadrant, and we get a more general result: If $\angle BAC$ equals either 135° or 45°, then $NO \perp EF$. Compare the editor's comments below.)

The point N, located on the line BC of equation $x + y = -1$ has coordinates $N = (x_N, -1 - x_N)$ and the vector $\overrightarrow{AN} = (a - x_N, b + 1 + x_N)^T$ is orthogonal to vector $\overrightarrow{BC} = (1, -1)^T$.

So $\overrightarrow{AN} \cdot \overrightarrow{BC} = 0$, that is $a - x_N - (b + 1 + x_N) = 0$. It follows that

$$x_N = \frac{a - b - 1}{2}$$

and

$$\overrightarrow{ON} = \frac{1}{2}\begin{pmatrix} a - b - 1 \\ b - a - 1 \end{pmatrix}.$$

The line BO is the x-axis, and the point $E = (x_E, 0)$ is such that the vector $\overrightarrow{CE} = (x_E, 1)^T$ is parallel to $\overrightarrow{CA} = (a, 1 + b)^T$. Therefore $x_E = a/(1 + b)$.

In the same way, the line CO is the y-axis, and the point $F = (0, y_F)$ is such that the vector $\overrightarrow{BF} = (1, y_F)^T$ is parallel to $\overrightarrow{BA} = (1 + a, b)^T$. Therefore we have that

$$y_F = \frac{b}{1 + a}.$$

It follows that

$$\overrightarrow{EF} = \begin{pmatrix} -\dfrac{a}{1 + b} \\ \dfrac{b}{1 + a} \end{pmatrix}.$$

Recalling that $a^2 + b^2 = 1$, we have

$$\begin{aligned} 2(1 + b)(1 + a)\,\overrightarrow{ON} \cdot \overrightarrow{EF} &= \begin{pmatrix} a - b - 1 \\ b - a - 1 \end{pmatrix} \cdot \begin{pmatrix} -(1 + a)a \\ (1 + b)b \end{pmatrix} \\ &= -(a - 1 - b)(a + 1)a + (b - 1 - a)(b + 1)b \\ &= [-a(a^2 - 1) + a^2 b + ab] + [b(b^2 - 1) - ab^2 - ab] \\ &= ab^2 + a^2 b + ab - a^2 b - ab^2 - ab \\ &= 0. \end{aligned}$$

Therefore ON and EF are perpendicular.

Editor's comments. Walther Janous devised a converse: If a triangle ABC satisfies $NO \perp EF$, then either $\angle BAC$ equals 45° or 135°, or ΔABC is isosceles with $AB = AC$. His proof uses coordinates, but the algebra requires a computer and has not been independently verified.

4903. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{2n-1}-\frac{1}{2n+1}+\frac{1}{2n+3}-\cdots\right)-\frac{1}{4n}\right].$$

We received 20 submissions, of which 15 were correct and complete. We present the solution by UCLan Cyprus Problem Solving Group.

We have

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2n+2k-1}=\sum_{k=0}^{\infty}\int_0^1(-1)^k x^{2n+2k-2}\,\mathrm{d}x=\int_0^1\sum_{k=0}^{\infty}(-1)^k x^{2n+2k-2}\,\mathrm{d}x=\int_0^1\frac{x^{2n-2}}{1+x^2}\,\mathrm{d}x\,.$$

Here, the interchange between the integral and the summation is allowed by the Dominated Convergence Theorem since

$$\sum_{n=1}^{N}(-1)^k x^{2n+2k-2}=\frac{x^{2n-2}(1-(-x^2)^{N+1})}{1+x^2}$$

is dominated on $[0,1]$ by 1.

We now have that

$$\begin{aligned}\sum_{n=1}^{2N}\sum_{k=0}^{\infty}\frac{(-1)^k}{2n+2k-1}&=\sum_{n=1}^{2N}\int_0^1\frac{x^{2n-2}}{1+x^2}\,\mathrm{d}x\\&=\int_0^1\frac{1-x^{4N}}{1-x^4}\,\mathrm{d}x\\&=\int_0^1\left(\frac{1-x^3}{1-x^4}-\frac{x^{4N}(1-x^3)}{1-x^4}+\frac{x^3(1-x^{4N})}{1-x^4}\right)\mathrm{d}x\end{aligned}$$

Note that

$$\frac{1-x^3}{1-x^4}=\frac{1+x+x^2}{(1+x)(1+x^2)}=\frac{1}{2}\left[\frac{x+1}{x^2+1}+\frac{1}{x+1}\right]=\frac{2x}{4(x^2+1)}+\frac{1}{2(x^2+1)}+\frac{1}{2(x+1)}$$

therefore

$$\int_0^1\frac{1-x^3}{1-x^4}\,\mathrm{d}x=\left[\frac{\ln(x^2+1)}{4}+\frac{\arctan(x)}{2}+\frac{\ln(x+1)}{2}\right]_0^1=\frac{\pi}{8}+\frac{3\ln 2}{4}\,.$$

Since $\frac{x^{4N}(1-x^3)}{1-x^4}$ is dominated by the integrable function $\frac{(1-x^3)}{1-x^4}$ on $(0,1)$, then

$$\lim_{N\to\infty}\int_0^1\frac{x^{4N}(1-x^3)}{1-x^4}\,\mathrm{d}x=0\,.$$

Finally,

$$\int_0^1\frac{x^3(1-x^{4N})}{1-x^4}\,\mathrm{d}x=\int_0^1\left(x^3+x^7+\cdots+x^{4N-1}\right)\mathrm{d}x=\frac{1}{4}\sum_{k=1}^{N}\frac{1}{k}=\frac{H_N}{4}.$$

Therefore

$$\lim_{N\to\infty}\sum_{n=1}^{2N}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{2n+2k-1}-\frac{1}{4n}\right)=\frac{\pi}{8}+\frac{3\ln 2}{4}-\lim_{N\to\infty}\frac{H_{2N}-H_N}{4}=\frac{\pi}{8}+\frac{\ln 2}{2}$$

where we used the fact that $H_N-\log N\to\gamma$.

Letting

$$a_n=\sum_{k=0}^{\infty}\frac{(-1)^k}{2n+2k-1}=\int_0^1\frac{x^{2n-2}}{1+x^2}\,\mathrm{d}x$$

we have $a_n\geqslant a_{n+1}$ (as $x^{2n-2}\geqslant x^{2n}$) and

$$a_n+a_{n+1}=\int_0^1 x^{2n-2}\,\mathrm{d}x=\frac{1}{2n-1}\,.$$

Thus $a_n\geqslant\frac{1}{4n-2}$. We also have $a_{n+1}\geqslant\frac{1}{4n+2}$ and so $a_n\leqslant\frac{1}{2n-1}-\frac{1}{4n+2}$. Thus

$$\frac{1}{4n-2}-\frac{1}{4n}\leqslant a_n-\frac{1}{4n}\leqslant\frac{1}{2n-1}-\frac{1}{4n+2}-\frac{1}{4n}\,.$$

From this it follows that $n(a_n-\frac{1}{4n})\to 0$ and so $a_n-\frac{1}{4n}\to 0$.

This implies that

$$\lim_{N\to\infty}\sum_{n=1}^{2N+1}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{2n+2k-1}-\frac{1}{4n}\right)=\frac{\pi}{8}+\frac{\ln 2}{2}$$

and therefore

$$\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{2n+2k-1}-\frac{1}{4n}\right)=\frac{\pi}{8}+\frac{\ln 2}{2}\,.$$

4904. *Proposed by Ivan Hadinata.*

Find all pairs (x,y) of prime numbers x and y such that $x\geq y$, $x+y$ is prime and x^x+y^y is divisible by $x+y$.

We received 24 submissions and they were all correct. We present the following solution by the majority of solvers.

If $y\geq 3$, then $x+y$ is even and it cannot be a prime. Thus, $y=2$ and the problem reduces to determine all possible prime x such that $x+2$ is a prime and $x^x+4\equiv 0\pmod{x+2}$. Let x be such a prime. Since $x+2$ is a prime, by Fermat's little theorem, $x^{x+1}\equiv 1\pmod{x+2}$. It follows that

$$0\equiv x(x^x+4)=x^{x+1}+4x\equiv 4x+1\pmod{x+2}\,,$$

and thus $x=5$. Finally, one can easily verify that $(x,y)=(5,2)$ is indeed a solution.

4905. *Proposed by Aravind Mahadevan.*

In a right-angled triangle, the acute angles x and y satisfy the following equation:

$$\tan x + \tan y + \tan^2 x + \tan^2 y + \tan^3 x + \tan^3 y = 70.$$

Find x and y.

We received 37 submissions, all of which were correct; we feature a composite of the nearly identical solutions submitted independently by Aryan Desai, Marie-Nicole Gras, and Yunyong Zhang.

We shall see that $\{x, y\} = \{15^\circ, 75^\circ\}$ or, if you prefer, $\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}$. Since the triangle is right angled we have $x + y = 90^{\mathrm{o}}$, so that $\tan y = \dfrac{1}{\tan x} > 0$ and the given equation may be rewritten as

$$\left(\tan x + \frac{1}{\tan x}\right) + \left[\left(\tan x + \frac{1}{\tan x}\right)^2 - 2\right] + \left[\left(\tan x + \frac{1}{\tan x}\right)^3 - 3\left(\tan x + \frac{1}{\tan x}\right)\right] = 70. \tag{1}$$

Put

$$t = \tan x + \frac{1}{\tan x}$$

so that equation (1) becomes

$$t^3 + t^2 - 2t - 72 = 0. \tag{2}$$

We observe that $t = 4$ is a root of (2), and obtain

$$t^3 + t^2 - 2t - 72 = (t - 4)(t^2 + 5t + 18) = 0.$$

Because $t > 0$, $t^2 + 5t + 18 \neq 0$, and we deduce that $t = 4$ is the single real solution of (2). We compute

$$4 = t = \tan x + \frac{1}{\tan x} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin(2x)}.$$

Hence, the single solution of (1) is such that $\sin(2x) = \frac{1}{2}$, whence $2x = 30^\circ, 150^\circ$. Thus,

$$x = 15^{\mathrm{o}} \text{ and } y = 75^{\mathrm{o}},$$

or vice versa.

Editor's comments. Almost all submissions were similar to the featured solution except that they solved $4 = \tan x + \dfrac{1}{\tan x}$ for $\tan x$ and found that

$$\tan x = 2 \pm \sqrt{3}.$$

A century ago every kid in high school knew that $\arctan(2 - \sqrt{3}) = \dfrac{\pi}{12}$ while $\arctan(2 + \sqrt{3}) = \dfrac{5\pi}{12}$. Today perhaps it is, as Richard Guy used to say, well known to those who well know it. Didier Pinchon nicely motivated his solution: Using $\dfrac{\pi}{12} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$, and $\tan\dfrac{\pi}{3} = \sqrt{3}$, $\tan\dfrac{\pi}{4} = 1$, it follows that

$$\tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{1+\sqrt{3}} = 2-\sqrt{3},$$

while $\dfrac{5\pi}{12} = \dfrac{2\pi}{3} - \dfrac{\pi}{4}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$ gives

$$\tan\frac{5\pi}{12} = \frac{-\sqrt{3}-1}{1-\sqrt{3}} = 2+\sqrt{3}.$$

4906. *Proposed by Cristinel Mortici.*

Find positive integers m and n such that m^3+n+12 is a perfect cube and n^2+m+13 is a perfect square.

We received 19 submissions, all of which were correct. We present the solution by Ioannis D. Sfikas.

We have

$$m^3+n+12 \geq (m+1)^3, \quad \text{or} \quad n \geq 3m^2+3m-11.$$

Also,

$$n^2+m+13 \geq (n+1)^2, \quad \text{or} \quad m \geq 2n-12.$$

Then

$$m \geq 6m^2+6m-34, \quad \text{or} \quad 6m^2+5m-34 \leq 0$$

so

$$(m-2)(6m+17) \leq 0, \quad \text{or} \quad m \leq 2.$$

Thus

$$n \leq \frac{m+12}{2} \leq 7.$$

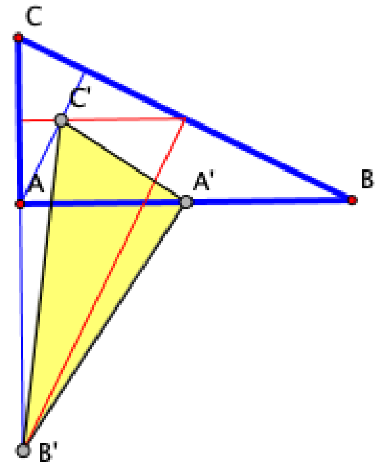
Checking gives $(m,n) = (2,7)$.

Editor's Comments. Almost all solvers argued as above. As noted by Á. Plaza, there was no need to show uniqueness (This reading is consistent with the French version "*Trouvez des entiers...*", p. 39.) Amusingly, $7^2+2+13 = 8^2$ is also a perfect cube. W. Janous asks: What happens if we replace 12 and 13 by positive integers a and b?

4907. *Proposed by J. Chris Fisher.*

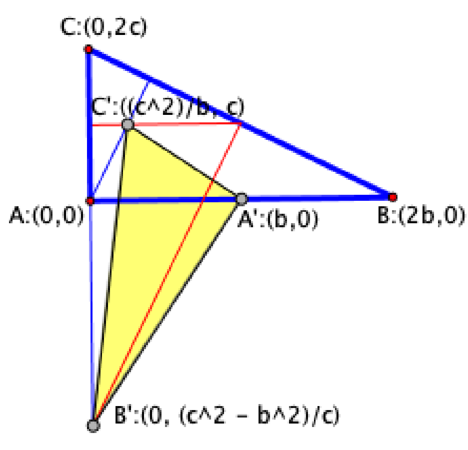
Given triangle ABC with a right angle at A, define A' to be the midpoint of the leg AB, B' to be the point where the perpendicular bisector of BC intersects the line AC, and C' to be the point where the perpendicular bisector of AC intersects the altitude from A to BC.

Prove that $\Delta A'B'C'$ is similar to ΔABC.



Twenty of the 22 submissions were correct and complete. Most made use of Cartesian coordinates, so our first featured solution will combine their best elements. We also provide one example of a proof that avoids coordinates.

Solution 1, a composite of most of the submissions.



Let

$$A = (0, 0), B = (2b, 0), C = (0, 2c);$$

Then $A' = (b, 0)$, while the perpendicular bisector of AC, namely $y = c$, meets the altitude from A, namely $y = \frac{b}{c}x$ at $C' = \left(\frac{c^2}{b}, c\right)$. Finally, for B' to lie on the y-axis and be equidistant from B and C, its coordinates $(0, y)$ must satisfy $4b^2 + y^2 = (2c - y)^2$, so that $y = \frac{c^2-b^2}{c}$, and

$$B' = \left(0, \frac{c^2 - b^2}{c}\right).$$

There are now three ways to finish the argument that $\Delta ABC \sim \Delta A'B'C'$. One can show that corresponding angles are equal: The relevant slopes are

$$\frac{b^2 - c^2}{bc} \text{ of } B'A', \quad \frac{bc}{c^2 - b^2} \text{ of } A'C', \text{ and } \frac{b^3}{c^3} \text{ of } C'B'.$$

It immediately follows that there is a right angle at A' (because the slope of $B'A'$ is the negative reciprocal of the slope of $A'C'$); moreover, the tangent of the angle at B' equals

$$\frac{\dfrac{b^3}{c^3} - \dfrac{b^2 - c^2}{bc}}{1 + \dfrac{b^3}{c^3} \cdot \dfrac{b^2 - c^2}{bc}} = \frac{c}{b},$$

which equals the negative of the slope of BC. That is enough to prove that the triangles ABC and $A'B'C'$ are oppositely similar.

Alternatively, one can show that the ratios of corresponding sides are equal. The relevant lengths are

$$\begin{aligned}(A'B')^2 = b^2 + \left(c - \frac{b^2}{c}\right)^2 = b^2 + c^2 - 2b^2 + \frac{b^4}{c^2} &= b^2\left(\frac{c^2}{b^2} + \frac{b^2}{c^2} - 1\right) \\ &= \frac{(AB)^2}{4}\left(\frac{c^2}{b^2} + \frac{b^2}{c^2} - 1\right).\end{aligned}$$

By interchanging the roles of b and c we find

$$(A'C')^2 = \left(b - \frac{c^2}{b}\right)^2 + c^2 = \frac{(AC)^2}{4}\left(\frac{c^2}{b^2} + \frac{b^2}{c^2} - 1\right).$$

And finally,

$$(B'C')^2 = \frac{c^4}{b^2} + \frac{b^4}{c^2} = \frac{b^6 + c^6}{b^2c^2} = (b^2 + c^2) \cdot \frac{b^4 - b^2c^2 + c^4}{b^2c^2} = \frac{(BC)^2}{4}\left(\frac{c^2}{b^2} + \frac{b^2}{c^2} - 1\right).$$

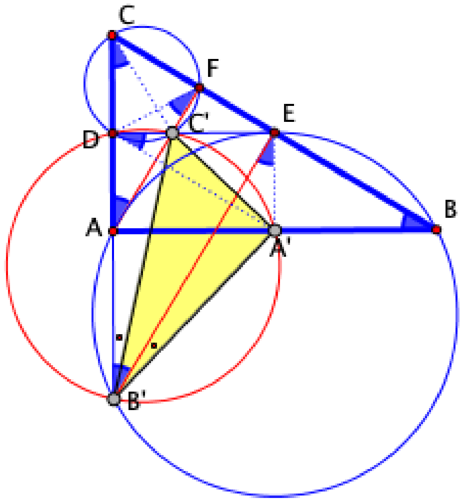
It follows that

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}$$

whence the two triangles are similar.

The third (and easiest) approach is to use side-angle-side: Use the slopes of $B'A'$ and $A'C'$ from the first method to get the right angle at A' and, from the second method, the lengths of those two legs to get two proportional corresponding sides.

Solution 2, by Theo Koupelis.



Let D, E be the midpoints of AC, BC, respectively, and let F be the foot of the altitude from A to BC. Note that

$$C, D, C', F \text{ lie on the circle with diameter } CC', \tag{1}$$

and

$$B, E, A, B' \text{ lie on the circle with diameter } BB'. \tag{2}$$

Moreover, in the right triangle AFC, point D is the midpoint of AC; consequently,

$$CD = DA = DF = EA'. \tag{3}$$

Thus,

$$\begin{aligned} \angle ABC \quad &= \quad \angle ABE =_{(2)} \angle AB'E = \angle DAF \quad (\text{because } \ AF \parallel B'E) \\ &=_{(3)} \quad \angle DFA = \angle DFC' =_{(1)} \angle DCC'. \end{aligned}$$

Therefore, the right triangles CDC' and $B'EC$ are similar, and thus

$$CD/B'E = CC'/B'C.$$

But by (3), $CD = EA'$, and (because $CB' \parallel EA'$), we have

$$\angle B'EA' = \angle EB'C = \angle B'CC',$$

whence triangles $B'CC'$, $B'EA'$ are similar. Therefore, $\angle A'B'E = \angle C'B'C$, and thus

$$\angle A'B'C' = \angle EB'A = \angle ABE = \angle A'DC'$$

(the last equality because $DA'BE$ is a parallelogram). Therefore, points A', B', D, C' are concyclic and

$$\angle B'A'C' = B'DC' = 90^\circ.$$

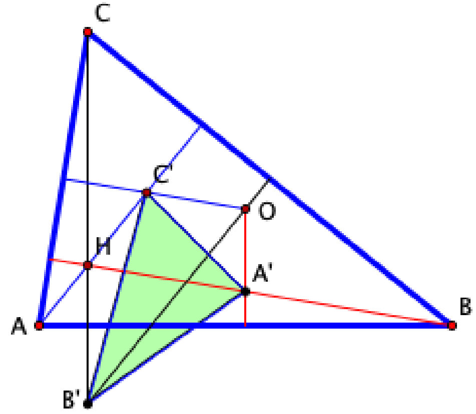
Therefore, finally,

$$\angle B'A'C' = \angle BAC = 90^\circ,$$

while $\angle ABC = \angle ABE = \angle A'B'C'$ which implies that $\Delta A'B'C' \sim \Delta ABC$, as desired.

Editor's comments. Our result is a special case of a theorem of Dixon J. Jones:

> *In any triangle that is not equilateral, the six intersection points of its altitudes with the perpendicular bisectors of the sides are the vertices of two congruent triangles that are oppositely similar to the original triangle.* [*Forum Geometricorum* Vol. **17** (2017) 383-399].



Denoting the altitudes with h's and the perpendicular bisectors with p's, all three figures above display one triangle of the congruent pair, namely

$$A' = h_b \cap p_c, \quad B' = h_c \cap p_a, \quad \text{and} \quad C' = h_a \cap p_b.$$

Dixon's proof used coordinates in an argument that was lengthy and unappealing, but then he challenged his readers to devise a more informative proof. The proposer of our problem hoped that a special case might suggest such a proof for the general case. Dixon's sequel article, "What's New with Altitudes and Perpendicular Bisectors?", *Mathematics Magazine* 93:5 (December, 2020)347-351), indicates that a better proof of his theorem remains an open challenge.

4908. *Proposed by Mihaela Berindeanu.*

In the square $ABCD$, the points X, Y and Z are respectively on the segments AB, AD and AX so that $XC = XY$ and $\angle ZYX = \angle XCB$. Show that

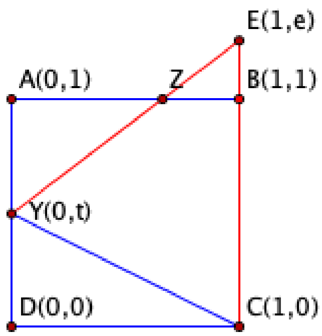
$$AY \cdot ZC^2 = 2 \cdot ZB \cdot BC \cdot YZ.$$

The submissions received from all 17 readers were correct and complete, but only Kee-Wai Lau stated explicitly that the point X in the statement of the problem *was redundant. We begin with his solution, which uses coordinates to verify the problem's assertion; we follow with a composite of the similar solutions whose first step tacitly removed X.*

Solution 1, by Kee-Wai Lau.

Without loss of generality, let the lengths of the sides of the square $ABCD$ be 1. As in the accompanying diagram, let

$$D = (0, 0),\ C = (1, 0),\ B = (1, 1),\ A = (0, 1), \quad \text{and} \quad Y = (0, t).$$



[We shall see that for the conclusion of the result to hold, t can take on any real value except -1, but the proposer requires the restriction $0 \leq t \leq 1$.] Note that because X is defined to be equidistant from C and Y, the angles at Y and C of the isosceles triangle XYC are equal. Because Z is defined such that the angle $\angle XYZ$ on side XY equals the angle BCX on side XC, Z could be defined as the unique point of the line AB for which $\angle CYZ = \angle YCB$. Consequently, the lines CB and YZ meet at the point of the line CB, call it $E = (1, e)$, that is equidistant from Y and C. That is,

$$1 + (e - t)^2 = e^2 \quad \text{and} \quad e = \frac{1 + t^2}{2t}.$$

Thus

$$E = \left(1, \frac{1 + t^2}{2t}\right).$$

The equation of the line YE is, therefore,

$$(1 - t^2)x - 2ty + 2t^2 = 0,$$

and $Z = YE \cap (y = 1)$ has coordinates

$$Z = \left(\frac{2t}{1 + t}, 1\right).$$

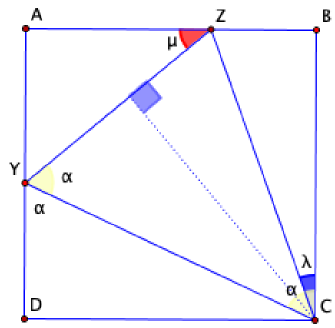
It follows that

$$AY = 1 - t, \quad ZC^2 = \frac{2(1 + t^2)}{(1 + t)^2}, \quad ZB = \frac{1 - t}{1 + t}, \quad \text{and} \quad YZ = \frac{1 + t^2}{1 + t}.$$

Hence,

$$AY \cdot ZC^2 = 2 \cdot ZB \cdot BC \cdot YZ = \frac{2(1-t)(1+t^2)}{(1+t)^2}.$$

Solution 2 is a composite of similar solutions by Michel Bataille, Ilkin Hasanov, and the proposer.



As in the first solution, we have $\angle CYZ = \angle BCY$, denoted by α in the accompanying figure. Let $\lambda = \angle BCZ$ and $\mu = \angle AZY$. Since

$$\frac{YA}{YZ} = \sin\mu \quad \text{and} \quad 2 \cdot \frac{CB}{CZ} \cdot \frac{ZB}{ZC} = 2\cos\lambda\sin\lambda = \sin 2\lambda,$$

the required result will be proved if we show that $\mu = 2\lambda$.

Now, since $BC \parallel AD$, we have $\angle DYC = \angle BCY = \alpha$, hence $\angle DYC = \angle CYZ$. Consequently, CY bisects $\angle DYZ$, and $d(C, YZ) = CD$ (the perpendicular distance to YZ from C). Since

$$d(C, BZ) = CB = CD,$$

we see that $d(C, YZ) = d(C, BZ)$ and deduce that CZ bisects $\angle YZB$. Thus,

$$\mu = 180^\circ - 2\angle CZB = 180^\circ - 2(90^\circ - \angle BCZ) = 2\lambda,$$

and we are done.

4909. *Proposed by Michel Bataille.*

For each positive integer n, let $P_n(x) = (x-1)^{2n+1}(x^2 - (2n+1)x - 1)$. Show that the equation $P_n(x) = 1$ has a unique solution x_n in the interval $(0, \infty)$. Prove that $\lim\limits_{n\to\infty}(x_n - 2n) = 1$ and find $\lim\limits_{n\to\infty} n(x_n - 2n - 1)$.

A total of 11 correct solutions were received. There were also 1 incomplete and 2 incorrect solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

The derivative

$$P_n'(x) = x(x-1)^{2n}[(2n+3)x - (4n^2+6n+4)]$$

is nonpositive on the interval $[0, 2n+c_n)$ and positive on $(2n+c_n, \infty)$, where $c_n = 4(2n+3)^{-1} < 1$. Since $P_n(0) = 1$ and

$$P_n(2n+1) = (2n)^{2n+1}(-1) < 0,$$

it follows that $P_n(x)$ is less than 1 on for $0 < x \leq 2n+1$ and strictly increasing without bound for $x \geq 2n+c_n$. Therefore there is a unique number $x_n > 2n+1$ for which $P_n(x_n) = 1$.

Observe that, for $n \geq 2$,

$$P_n\left(2n+1+\frac{1}{2n+1}\right) = \left(2n+\frac{1}{2n+1}\right)^{2n+1} \cdot \frac{1}{(2n+1)^2} > \frac{(2n)^{2n+1}}{(2n+1)^2} > 1,$$

from which $x_n < 2n+1+(2n+1)^{-1}$. Therefore

$$1 < x_n - 2n < 1 + \frac{1}{2n+1}$$

for $n \geq 2$. Hence $\lim_{n\to\infty}(x_n - 2n) = 1$.

Since

$$P_n\left(2n+1+\frac{1}{2n+2}\right) = \left(2n+\frac{1}{2n+2}\right)^{2n+1}\left(\frac{2n+1}{2n+2}+\frac{1}{(2n+2)^2}-1\right) < 0,$$

we find that

$$2n+1+\frac{1}{2n+2} < x_n < 2n+1+\frac{1}{2n+1}.$$

Hence

$$\frac{n}{2n+2} < n(x_n-2n-1) < \frac{n}{2n+1},$$

so that $\lim_{n\to\infty} n(x_n-2n-1) = \frac{1}{2}$.

Solution 2, by Raymond Mortini.

As in solution 1, we show that $P(x) = 1$ has a unique solution x_n that exceeds $2n+4(2n+3)^{-1}$, the unique positive root of $P_n(x)$. Since

$$1 = P_n(x_n) = x_n(x_n-1)^{2n+1}(x_n-2n-1-x_n^{-1}),$$

then

$$x_n - 2n - 1 = \frac{1}{x_n(x_n-1)^{2n+1}} + \frac{1}{x_n}.$$

Therefore $\lim_{n\to\infty}(x_n-2n-1) = 0$ and $\lim_{n\to\infty}(\frac{x_n}{n} - 2 - \frac{1}{n}) = 0$.

It follows that $\lim_{n\to\infty}(x_n - 2n) = 1$ and $\lim_{n\to\infty}(x_n/n) = 2$. Accordingly, since

$$n(x_n - 2n - 1) = \frac{n}{x_n} + \frac{n}{x_n(x_n-1)^{2n+1}} = \frac{n}{x_n}\left(1 + \frac{1}{(x_n-1)^{2n+1}}\right),$$

$\lim_{n\to\infty} n(x_n - 2n - 1) = 1/2$.

Solution 3, adapted from that of Aryan Desai.

The polynomial $P_n(x)$ has a unique positive root

$$r_n = \frac{1}{2}[(2n+1) + \sqrt{4n^2+4n+5}]$$

that satisfies $2n+1 < r_n < x_n$; its derivative

$$P_n'(x) = x(x-1)^{2n}[(2n+3)x - (4n^2+6n+4)]$$

is positive and increasing when $x \geq 2n+1$. By the Mean Value Theorem,

$$1 = P_n(x_n) - P_n(r_n) = (x_n - r_n)P_n'(s_n),$$

where $r_n < s_n < x_n$.

Since $P_n'(s_n) \geq P_n'(2n+1)$,

$$x_n - r_n \leq \frac{1}{P_n'(2n+1)} = \frac{1}{(2n+1)(2n)^{2n}(2n-1)} \leq \frac{1}{4(2n+1)^3}$$

for $n \geq 3$ (using the fact that $(2n)^2 > 2n+2$).

Now

$$0 < r_n - (2n+1) = \frac{\sqrt{4n^2+4n+4} - (2n+1)}{2} = \frac{2}{\sqrt{4n^2+4n+5}+2n+1}.$$

Since $(1+t)^{1/2} \geq 1 + \frac{t}{4}$ for $0 < t < 8$,

$$\begin{aligned}\sqrt{4n^2+4n+5} + (2n+1) &= (2n+1)\left[\left(1+\frac{4}{(2n+1)^2}\right)^{1/2} + 1\right]\\ &\geq (2n+1)\left[2 + \frac{1}{(2n+1)^2}\right]\\ &= 2(2n+1)\left[1 + \frac{1}{2(2n+1)^2}\right].\end{aligned}$$

Therefore, since $(1+t)^{-1} \leq 1 - \frac{t}{2}$ for $0 < t < 1$,

$$0 < r_n - (2n+1) \leq \frac{1}{2n+1}\left[1 + \frac{1}{2(2n+1)^2}\right]^{-1} < \frac{1}{2n+1} - \frac{1}{4(2n+1)^3}.$$

Hence

$$0 < x_n - (2n+1) = (x_n - r_n) + (r_n - (2n+1)) < \frac{1}{2n+1},$$

from which it follows that

$$\lim_{n\to\infty}(x_n - 2n) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} n(x_n - 2n - 1) = \frac{1}{2}.$$

Comments from the editor. Walther Janous noted that, for large n, x_n is approximately $2n + 1 + \frac{1}{2n}$; with the aid of a computer algebra system, he investigated the extension into a Laurent expansion and conjectures that

$$x_n = 2n + 1 + \sum_{i=1}^{m} \frac{p_i}{(2n)^i} + O\left(\frac{1}{n^{m+1}}\right)$$

as $m \to \infty$. The sequence $\{p_i\}$ starts out

$$\{1, -1, 0, 1, -3, -1, 11, -15, -13, 77, -86, -144, 595, -495, -1520, 4810, -2485, \ldots\}.$$

He made the surprising discovery that this sequence agrees with sequence A007440 in the Online Encyclopedia of Integer Sequences (OEIS) *with at least one difference*: the fourth term in A007440 is equal to 2 and not 1.

The sequence A007440 is the *reversion* of the Fibonacci sequence $\{f_n\}$ with initial values $f_1 = f_2 = 1$. This means that if

$$y = f_1 x + f_2 x^2 + f_3 x^3 + \cdots = x + x^2 + 2x^3 + 5x^5 + 8x^6 + \cdots ,$$

then

$$x = p_1 y - p_2 y^2 + 2y^4 + p_5 y^5 + \cdots = y - y^2 + 2y^4 - 3y^5 - y^6 + 11y^7 - \cdots .$$

This can be verified by substitution of the series for one variable into the equation for the other.

4910. *Proposed by Paul Bracken.* Let m and n be non-negative integers and let

$$J_{m,n} = \int_0^\infty \left(\left(\frac{\sin t}{t}\right)^m - \left(\frac{\sin t}{t}\right)^n\right)\frac{dt}{t^2}.$$

Prove that the $J_{m,n}$ are rational multiples of π.

We received 10 submissions including the solution from the proposer. Among these 8 are correct and complete. We present 2 solutions; one is being purely real, while another follows a complex-analytic approach.

Solution 1, by Raymond Mortini and Rudolf Rupp.

It is sufficient to consider the case $n = 0$, otherwise write

$$\int \frac{S^m - S^n}{x^2} dx = \int \frac{S^m - 1}{x^2} dx + \int \frac{1 - S^n}{x^2} dx,$$

where $S = \dfrac{\sin x}{x}$. Since $|S| \leq 1$, we see that $\int_1^\infty \frac{S^m-1}{x^2}dx$ converges. Now use that

$$1 - S = \frac{x^2}{3!} - \frac{x^4}{5!} \pm \cdots = x^2\left(\frac{1}{6} + \mathcal{O}(x)\right) \quad \text{as} \quad x \to 0,$$

and

$$|S^m - 1| = |S - 1| \left| \sum_{j=0}^{m-1} S^j \right| \leq m|S - 1|,$$

to conclude that $\displaystyle\int_0^1 \frac{S^m - 1}{x^2}dx$ converges, too. Hence

$$I(m) := \int_0^\infty \frac{S^m - 1}{x^2}dx$$

converges. Next we write

$$J := \frac{S^m - 1}{x^2} = \frac{(\sin x)^m - x^m}{x^{m+2}} =: \frac{f(x)}{x^{m+2}}.$$

Now we apply Apostol's method integrating by parts $\int uv' = uv - \int u'v$ with $u = f$ and $v' = x^{-m-2}$. This yields

$$I(m) = \frac{1}{m+1} \int_0^\infty \frac{f'(x)}{x^{m+1}}dx,$$

since $\displaystyle\lim_{x\to 0} \frac{f(x)}{x^{m+1}} = 0$ (and $\displaystyle\lim_{x\to\infty} \frac{f(x)}{x^{m+1}} = 0$) because

$$\left|\frac{f(x)}{x^{m+1}}\right| \quad \leq \quad m\frac{|S - 1|}{x} = mx\left(\frac{1}{6} + \mathcal{O}(x)\right) \quad \text{as} \quad x \to 0.$$

Similarly, since 0 is a zero of order 1 of the analytic function $J(z) := \dfrac{(\sin z)^m - z^m}{z^{m+1}}$, we have that for all $j = 0, 1, \ldots, m$

$$\lim_{x\to 0} \frac{f^{(j)}(x)}{x^{m+1-j}} = 0.$$

[Note that

$$J(z) = z\frac{\left(\frac{\sin z}{z}\right)^m - 1}{z^2} = z\left(-\frac{1}{3!} + \frac{z^2}{5!} + \cdots\right)R(z),$$

where $\displaystyle\lim_{z\to 0} R(z) = \lim_{z\to 0} \sum_{j=0}^{m-1} \left(\frac{\sin z}{z}\right)^j = m.$]

Hence, by repeating this procedure another m-times, we obtain

$$I(m) = \frac{1}{(m+1)!}\int_0^\infty \frac{f^{(m+1)}(x)}{x}dx.$$

Now $f^{(m+1)}(x) = \frac{d^{m+1}}{dx}(\sin x)^m - 0$. Next we "linearize" the sinus-power:

$$\begin{aligned}(\sin x)^m &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^m = \left(\frac{1}{2i}\right)^m \sum_{j=0}^m (-1)^{m-j}\binom{m}{j} e^{ijx}e^{-i(m-j)x} \\ &= (-1)^m \frac{1}{(2i)^m}\sum_{j=0}^m (-1)^{-j}\binom{m}{j} e^{ix(2j-m)}.\end{aligned}$$

Since the "constant" term (appearing for $j = m/2$ when m is even) is annihilated by the derivative, we find

$$\begin{aligned}\frac{d^{m+1}}{dx}(\sin x)^m &= (-1)^m(-i)^{m+1}\frac{1}{(2i)^m}\sum_{0\le j<\frac{m}{2}}(m-2j)^{m+1}(-1)^j\binom{m}{j}e^{-ix(m-2j)} \\ &\quad +(-1)^m i^{m+1}\frac{1}{(2i)^m}\sum_{\frac{m}{2}<j\le m}(2j-m)^{m+1}(-1)^j\binom{m}{j}e^{ix(2j-m)}.\end{aligned}$$

As the left hand side is real, we may take the real part on the right hand side and get (by observing $\operatorname{Re} iz = -\operatorname{Im} z$)

$$\begin{aligned}\frac{d^{m+1}}{dx}(\sin x)^m &= -\frac{1}{2^m}\sum_{0\le j<\frac{m}{2}}(m-2j)^{m+1}(-1)^j\binom{m}{j}\sin((m-2j)x) \\ &\quad +(-1)^{m+1}\frac{1}{2^m}\sum_{\frac{m}{2}<j\le m}(2j-m)^{m+1}(-1)^j\binom{m}{j}\sin((2j-m)x).\end{aligned}$$

Finally, as

$$\int_0^\infty \frac{\sin(px)}{x}dx = \int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$$

whenever $p > 0$, we deduce that

$$\begin{aligned}I(m) = \frac{1}{(m+1)!}\int_0^\infty \frac{f^{(m+1)}(x)}{x}dx &= \frac{\pi}{2}\frac{1}{(m+1)!}\Bigg[-\frac{1}{2^m}\sum_{0\le j<\frac{m}{2}}(m-2j)^{m+1}(-1)^j\binom{m}{j} \\ &\quad +(-1)^{m+1}\frac{1}{2^m}\sum_{\frac{m}{2}<j\le m}(2j-m)^{m+1}(-1)^j\binom{m}{j}\Bigg]\end{aligned}$$

which surely is a rational multiple of π. Making in the second summand the substitution $k = m - j$, then we obtain

$$I(m) = -\frac{\pi}{2^m (m+1)!} \sum_{0 \le j < \frac{m}{2}} (m-2j)^{m+1} (-1)^j \binom{m}{j}.$$

For instance $I(1) = -\frac{\pi}{4}, I(2) = -\frac{\pi}{3}, I(3) = -\frac{13\pi}{32}$.

Solution 2, by Didier Pinchon.

For any non-negative integers m and n, $J_{m,n} = J_{0,n} - J_{0,m} = I_n - I_m$ with

$$I_n = \int_0^{+\infty} \left(\frac{1}{t^2} - \frac{\sin^n t}{t^{n+2}} \right) dt,$$

and it is therefore sufficient to prove that I_n, $n > 1$, are rational multiples of π.

The complex representation of $\sin t$ for $t \in \mathbb{C}$ $(i^2 = -1)$:

$$\sin t = \frac{e^{it} - e^{-it}}{2i},$$

is used, and also the binomial expansion

$$(e^{it} - e^{-it})^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{i(n-2k)t}.$$

Let $f_n(t)$ be the function defined for $t \in \mathbb{C} - \{0\}$ by

$$f_n(t) = \frac{1}{t^2} - \frac{\sin^n t}{t^{n+2}},$$

and the $g_n(t)$ be defined for $t \in \mathbb{C} - \{0\}$ by

$$g_n(t) = \begin{cases} \dfrac{1}{2t^2} - \dfrac{1}{(2i)^n\, t^{n+2}} \displaystyle\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} e^{i(n-2k)t} & \text{if } n \text{ is odd,} \\ \dfrac{1}{2t^2} - \dfrac{1}{(2i)^n\, t^{n+2}} \displaystyle\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{k} e^{i(n-2k)t} - \dfrac{1}{2(2i)^n t^{n+2}} (-1)^{\frac{n}{2}} \binom{n}{\frac{n}{2}} & \text{if } n \text{ is even,} \end{cases}$$

in such a way that $f_n(t) = g_n(t) + g_n(-t)$, $t \in \mathbb{C} - \{0\}$.

Let $D_{\varepsilon,R}$ be the domain in the complex plane whose boundary consists of the real segment $[-R, -\varepsilon]$, a half circle C_ε, the real segment $[\varepsilon, R]$ and a half circle C_R, as shown in Figure 1. As $g_n(t)$ is an analytic function in $D_{\varepsilon,R}$, $\int_{\partial D_{\varepsilon,R}} g_n(t)\, dt = 0$, and

$$\int_{\partial D_{\varepsilon,R}} g_n(t)\, dt = \int_{C_R} g_n(t)\, dt + \int_{-R}^{-\varepsilon} g_n(t)\, dt + \int_{C_\varepsilon} g_n(t)\, dt + \int_{\varepsilon}^{R} g_n(t)\, dt = 0.$$

As $R^2|g_n(t)|$ is bounded for $R > 0$ and $t \in C_R$, $\lim_{R\to+\infty} \int_{C_R} g_n(t)\ dt = 0$. On the other hand,

$$\lim_{\varepsilon\to 0,\ R\to+\infty} \left[\int_{-R}^{-\varepsilon} g_n(t)\ dt + \int_{\varepsilon}^{R} g_n(t)\ dt\right] = \lim_{\varepsilon\to 0,\ R\to+\infty} \int_{\varepsilon}^{R} f_n(t)\ dt = \int_0^{+\infty} f_n(t) = I_n.$$

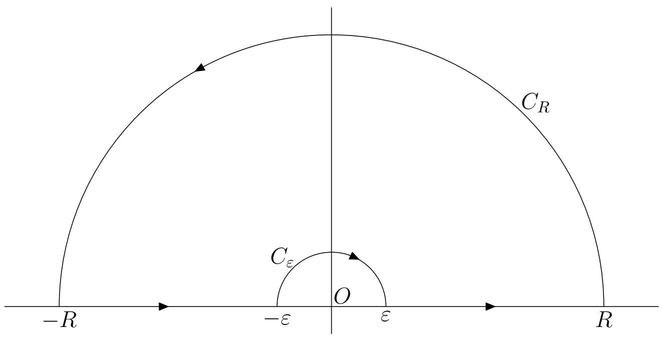


Figure 1: The domain $D_{\varepsilon,R}$.

Using Cauchy's residue theorem, it follows, because C_ε is oriented in the opposite trigonometric sense,

$$\int_{\varepsilon}^{R} g_n(t)\ dt = -i\pi \operatorname{Res}_0 g_n(t),$$

where $\operatorname{Res}_0 g_n(t)$ is the residue at 0 of $g_n(t)$.

Therefore $I_n = i\pi \operatorname{Res}_0 g_n(t)$.

$\operatorname{Res}_0\ g_n(t)$ is computed using Laurent's expansion at $t = 0$ of the functions $\frac{1}{t^{n+2}} e^{i(n-2k)t}$, that gives, for every $n > 0$,

$$\begin{aligned}
\operatorname{Res}_0 g_n(t) &= -\frac{1}{(2i)^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \frac{(i(n-2k))^{n+1}}{(n+1)!} \\
&= -\frac{i}{(n+1)2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k\,(n-2k)^{n+1}}{k!(n-k)!}.
\end{aligned}$$

It follows that

$$I_n = \frac{\pi}{(n+1)2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k\,(n-2k)^{n+1}}{k!(n-k)!},$$

which proves the result.