A **T**aste **O**f **M**athematics

Aime-**T**-**O**n Les **M**athématiques

Volume / Tome XVII

MATHEMATICAL LOGIC PUZZLES ON A GRID

Susan Milner
University of the Fraser Valley

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Booklets in the ATOM Series are designed to provide enrichment materials for pre-university students, and their teachers, who have an interest in mathematics. Some booklets in the series will cover materials useful for mathematical competitions at national and international levels. Other booklets may cover topics of broad interest to students and teachers such as puzzle collections, applications of mathematics, as well as treatises through a historical, social, or cultural lens. Booklets will be made available to the public, free of charge, on the CMS website.

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The Author

Susan Milner is professor emerita in the department of Mathematics & Statistics at the University of the Fraser Valley. She taught post-secondary mathematics in British Columbia for 29 years. For 11 of those years, she organised UFV's secondary math contest — her favourite part was coming up with post-contest activities for the participants. In 2009, her department started offering Math Mania events for local youngsters, parents, and teachers. This was so much fun that she devoted her sabbatical year to adapting math/logic puzzles and taking them into K-12 classrooms. In 2014, she was awarded the Pacific Institute for the Mathematical Sciences (PIMS) Education Prize.

Now retired and living in Nelson, BC, she greatly enjoys travelling the province, with support from Science World, to play games with K-12 students. She also enjoys giving professional development workshops for mathematics educators at all levels, often with the support of PIMS. Lately, she has been sharing games with the members of Nelson's Learning in Retirement group.

Her website contains printable math/logic puzzles and games, tested in many classrooms over the past decade.

Dedication

This work is dedicated to *Jim Totten*, who clued me in to the fact that teachers love having math people visit their classes to play games with their students. Jim's sheer delight in sharing mathematically-based games, tricks, and puzzles was highly infectious and a great inspiration to me, among many others.

Acknowledgements

I'd like to thank three people without whose support and hard work this booklet would not exist:

Malgorzata Dubiel encouraged me to get started on the project and saw it through its first few drafts. Over the years she has provided tremendous support for my efforts to share math/logic games with students and educators.

John Grant McLoughlin, proofreader extraordinaire, provided many thoughtful suggestions to improve the readability and consistency of this work. It is always a delight to talk shop with John.

Shawn Godin did a heroic job of completely reformatting the work for both on-line and off-line use and made many helpful suggestions to improve the flow and presentation. He went way beyond what I'd expect of an editor-in-chief.

And I would also like to express appreciation for the insights, feedback and suggestions made along the way by both *Ryan Jones*, a colleague of John's, and *Lyndia Belczewski*, a UNB student.

Thank you, all, for being so generous with your time and your excellent ideas.

Foreword

For students (and teachers)

These are some of the puzzles that have proven popular with junior and senior high school students all over British Columbia. I hope that you find one or two types that you really enjoy. The websites referred to at the ends of chapters contain many other types of logic puzzles which can be solved using similar general strategies.

General strategy #1:
In life there are things we know must be true, things that might be true, things we hope are true, and things we believe to be true. It's not always easy to tell the difference! When we are trying to solve logic puzzles such as the ones in this booklet, it's usually easier to distinguish among those possibilities. It is very helpful to keep checking - is this something I know **must** go there or just something that might work?

The goal is to place values or symbols only in places where we are absolutely certain they must go. If you make a mistake and end up with nonsense, it's easy to erase and start again. The next time through, you might even figure out where you made an assumption rather than a deduction.

General strategy #2:
Often we need to spend time looking for somewhere to begin, that special spot with very limited possibilities. This might take some patience but is well worth it in the end. Think about doing jigsaw puzzles: many people start with the edge pieces because there aren't as many places for them to go as there are for inside pieces. The next step is often to look for all pieces of a certain colour, if that colour turns up in easily-identifiable shapes in the puzzle. Each type of logic puzzle will train your brain to hunt for special conditions.

General strategy #3:
These puzzles are all about deduction. Guessing is usually more trouble than it is worth! Testing a hypothesis, on the other hand, can be a useful tool as puzzles get harder. This involves looking for a place where you have exactly two possible paths, one of which will cause a foreseeable problem for other clues. Then you'll know that the *other* path is the one required.

General strategy #4:
To begin with, most types of puzzles give more clues than absolutely necessary. As the puzzles get harder and provide fewer clues, you'll find that you need to pay close attention to every piece of information you are given and to everything you have deduced. It's a bit like hauling yourself up by your bootstraps, or jumaring on a rock face.

General strategy #5:
It may take time and a few tries to get all the rules of a particular puzzle into your head, to see the puzzle as a whole, rather than a collection of unrelated pieces. Many of the people I've shared these with report a similar experience: it all seems confusing or complicated at first, you struggle a bit, you re-read the examples, then something clicks in your brain and you can make progress. Then you try a harder puzzle, struggle a bit more, experience another aha! moment and feel you have control of that type of puzzle. If you aren't feeling the aha! moments, give the puzzle a rest and come back later. Our brains are remarkable - they can solve problems even when we aren't thinking about them.

Finally, you will notice that there are no answers in the back of the book! A gratifying part of solving puzzles like these is that you can check whether you have the right answer, all on your own, without referring to some "expert". This isn't the same as just checking every step in a word problem, where you can make the same mistake all over again. This is easier, as you can usually tell that you've got it if the last piece fits. If that's not enough, it's easy to check that your solution matches the given clues.

For teachers

Why puzzles?

I still remember the delight I felt when I discovered these kinds of logic puzzles — it felt like doing mathematical proofs, with the same rush of joy at a successful solution. There is a peculiar pleasure to be had in solving a problem by purely logical means which can be justified clearly to anyone who might be interested. I've watched people of all ages from about 5 to 90 display this pleasure in not only having solved a puzzle, but also in being able to articulate their reasoning.

Solving logic puzzles requires types of reasoning similar to proving mathematical theorems and is an excellent way to encourage the development of those skills with a minimum of anxiety. In particular, we can improve our spatial reasoning and sequential reasoning, without even noticing that we are doing it. And techniques with formal names such as modus ponens, modus tollens, and reductio ad absurdum (also known as proof by contradiction) arise naturally in the course of solving puzzles.

Beyond the techniques of reasoning are the desirable mathematical habits of mind which these puzzles encourage: being curious, paying attention to detail, being willing to erase and start again (no one gets too anxious when it's just a game), having fun with logical arguments, having the patience to tackle a big problem via small steps, and being tenacious but also able to leave a non-productive line of reasoning to hunt for a better one. And because it is not hard to confirm whether a solution is correct or not, students are encouraged to think over their results, which is a big part of making sense of a problem.

In the classroom

In this booklet, as far as is possible in written form, I have tried to illustrate the way I introduce puzzles in the classroom. This method has proven to be quite effective in getting students' attention and encouraging them to get started. You can read more about it at https://susansmathgames.ca/classroom-introductions/

If your students show real interest in a certain type of puzzle, you might increase the level of sophistication by asking them to create a puzzle of that sort. What makes a puzzle easy or hard? How many clues are needed to ensure only one correct solution? Can they make the puzzle harder by deleting some clues without adding any solutions?

Ultimately, while a strong case can be made that solving logic puzzles encourages many kinds of useful mathematical thought processes, I believe the most important feature is that people really enjoy the challenge.

I hope that you and your students have fun with these!

– Susan Milner

Note: Many of the *Rectangles*, *Three in a Row*, and *Towers* puzzles were originally generated by Simon Tatham's Portable Puzzle Collection website, and most of the *Kakurasu* puzzles came from puzzle-kakurasu.com . The articulation of ideas, accompanying discussions, development of strategies, along with commentary and the presentation/layout of content throughout the collection represent the work of the author.

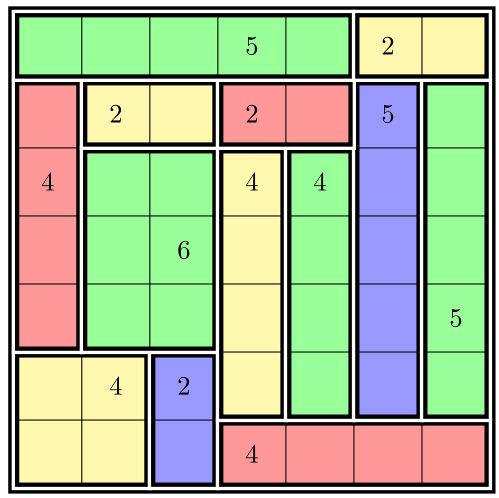
Chapter 1

Rectangles

See if you can figure out the rules, just by looking at a solved puzzle:

			5		2	
	2		2		5	
4			4	4		
		6				
						5
	4	2				
			4			

Puzzle 1.1



Puzzle 1.1 solved

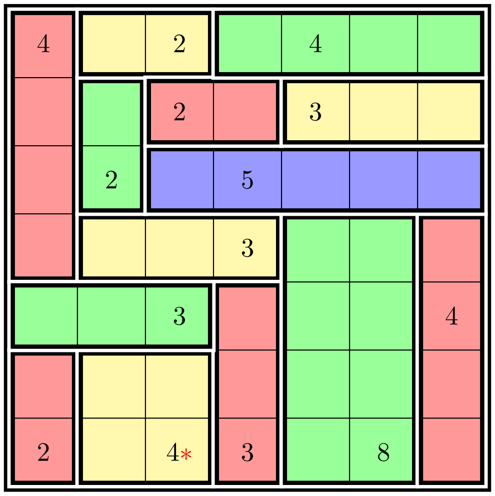
Now try out your understanding of the rules on this puzzle:

4		2		4		
		2		3		
	2		5			
			3			
		3				4
2		4	3		8	

Puzzle 1.2

If you think you've got it, turn the page and see! If you're not quite sure what's going on, you'll find the rules of the game on the next page.

This game is often called Rectangles, for obvious reasons. In Japanese it is called Shikaku, which means something like "divide by squares".



Puzzle 1.2 solved

The rules of Rectangles

- the background grid has to be covered in rectangles which do not overlap and which have no gaps between them
- each rectangle contains exactly one number, which tells us how many of the background grid-squares must be in the rectangle

A well-designed Rectangles puzzle will have only one possible solution. While guessing or relying on "intuition" might work on smaller puzzles, it doesn't work for bigger ones. It's a good idea to deliberately practice your logic on the easier puzzles before moving on to harder ones. At the heart of Rectangles is a relationship between number and shape, between arithmetic and geometry. Which numbers n can be enclosed only in a rectangle of the form $n \times 1$ or $1 \times n$? Which numbers give us more options? For fun, can you figure out how many rectangles could be used to picture the number 48?

* By the way, a square is a rectangle, so 4 can be pictured as 1×4, 4×1, or 2×2.

Some puzzles to try

As with all logic puzzles, it is worth staring at the puzzle for a few seconds, looking for something you know must be the case. If you'd like some specific hints about where to start either of these two puzzles, turn to page 4.

		4		2				
				2		2	4	2
			6	2				
			4		2			2
	2			9		3		2
6					8			
	2		2				2	
2			3	4		2	2	

Puzzle 1.3 (9×9)

	2	2						
	6						12	
					5		2	
			2		2			
3				3		2		3
		12		8	4			4
							3	
							2	
					4			

Puzzle 1.4 (9×9)

If you are comfortable with your answers to the puzzles on the previous page, try these harder puzzles. There is a list of general hints on the next page if you run into difficulties.

			4				3			
				2				2		
			8						2	9
		2								
				3			18			
6						12			6	
		10	8			8				
				6						
	2	2							2	
	2					2			2	

Puzzle 1.5 (11 × 11)

		2								6
7								4		
		4		3	3	2			3	
			3							2
					8				8	
	6								4	7
		2		15				3		
						2				
		12				3				
							4	8		

Puzzle 1.6 (11 × 11)

			10		4		2					
												4
					24	2				15		
						3						
							2					18
								2	2			
					11							
				21			2		2			
							2		2			
									2			
4							18			5		
	4										2	2
	2	2										

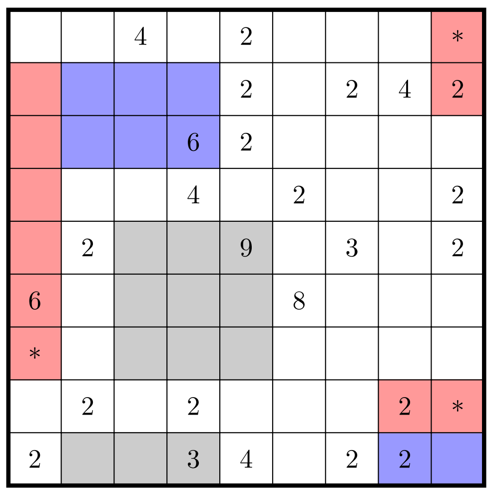
Puzzle 1.7 (13 × 13)

If you'd like some specific hints for Puzzle 1.7 turn to page 5.

General hints

- A good starting place is a number for which there are no options.
- Another good starting point is a grid-square that can be reached by only one rectangle.
- It can help to look for possibilities that would block another number's rectangle, leaving no further choice for that second rectangle. Then you can reject those possibilities.
- Look also for possible rectangles that would leave a grid-square unreachable by any other rectangle. Then those possible rectangles can be rejected.
- It can help to mark in grid-squares that you know must form part of a given rectangle.
- Guessing or making assumptions is likely to cause frustration! It can be difficult to catch yourself making assumptions, so sometimes all you can do is erase the entire puzzle and start again, being careful to identify your reasoning for each step you make.
- If you think you've been absolutely logical but still can't solve the puzzle, double-check your counting in the larger rectangles.

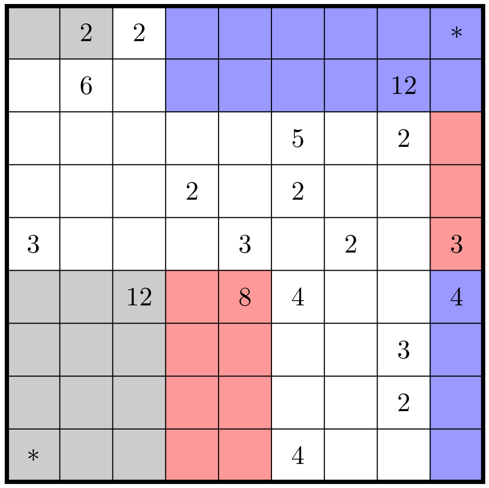
Specific suggestions for starting some puzzles



Puzzle 1.3 hint

Some of these suggestions are independent of each other:

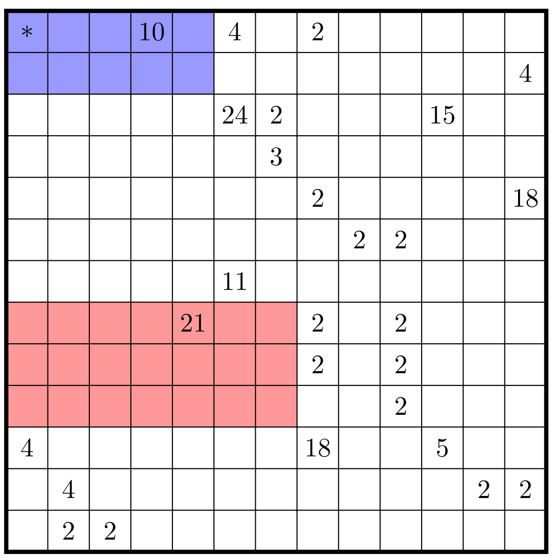
- That 9 in the middle jumped out at me. The only one way to fit a rectangle around it is to use a 3×3 square, as shown.
- Similarly, there is only one rectangle possible for the 6 in row 3.
- The 3 in the bottom row can go in only one way.
- There is only one direction possible for the right-most 2 in the bottom row.
- The squares labelled with $*$ can be reached only by the values shown.



Puzzle 1.4 hint

These are independent observations, so you can start with any of them.

- Only one number can reach the bottom left corner.
- Only one number can reach the top right corner.
- Only one rectangle is possible for the 8 near the middle of the puzzle.
- There is only one direction possible for the left-most 2 in the top row.
- Only one direction is possible for each of the 3 and the 4 in the rightmost column.



Puzzle 1.7 hint

- Only 10 can reach the square with the *, which fixes the rectangle.
- There is only one way to make a rectangle around 21. That's not obvious, but if you try all the possibilities, you'll find that any other rectangle than the one shown will either block rectangles for the 24 and/or the 10, or it will create a square that cannot be reached by any other number.

If you left any earlier puzzles unfinished, these hints might help you complete them. Then you're set to try some more challenging puzzles on the next page.

Some More Challenging Puzzles

				24								
5	5	3									2	
								16				
										2	3	6
2							11					
	2										10	
												3
									20			
							7					
		4		3	3	4						
	8						2		2			
			3			2	2				15	

Puzzle 1.8 (13 × 13)

			14											3
4											2		2	
				4				5	13			9		
							27			3				
												2		5
				3						10				
6				2									9	
			30					4						
					3						18			3
		18					6							3
						3								
							2							
									10					2

Puzzle 1.9 (15 × 15)

2			6						8	14						
		4			6			6				2		2	2	
						12								2		
	2	3													4	
4																2
												36				6
9	4					10			12							
							2							2		
							7					18				
				2				3								
			14						3							8
	2														15	
	3															
										8						
				6		20								8	2	
2						3				3						

Puzzle 1.10 (17×17)

Find many more

You'll find many more Rectangles puzzles to download or play on-line at Simon Tatham's Portable Puzzles.

Chapter 2

Three in a Row

See if you can figure out the rules by looking at a solved puzzle.

			O		X	O	
	X	X				O	
	X						
					O		
		O			O		O
				X			
							X
O							

Puzzle 2.1

X	O	X	O	X	X	O	O
O	X	X	O	X	O	O	X
O	X	O	X	O	X	X	O
X	O	X	O	X	O	O	X
X	X	O	X	O	O	X	O
O	O	X	O	X	X	O	X
X	O	O	X	O	O	X	X
O	X	O	X	O	X	X	O

Puzzle 2.1 solved

Try out your ideas by solving these two puzzles; that is, fill in the Xs and Os according to what you think the rules of the game are. Then turn the page to see if you are correct. (You don't need to use shading – that is just to help you see the patterns.)

If you are not sure how to begin, turn the page to see the answers, read the rules, and find a step-by-step description of a solution to a new puzzle.

X			X				
		O					
		O			X		
			O		X		
		X				O	
					X		
			O	O			X
X							

Puzzle 2.2

X			O			X	
	X		O	O			
				X		O	
X			X				
X		X		X			
				X			X
X					O		

Puzzle 2.3

X	O	X	X	O	O	X	O
X	X	O	O	X	O	O	X
O	X	O	X	O	X	X	O
O	O	X	O	X	X	O	X
X	O	X	O	X	O	O	X
O	X	O	X	O	X	X	O
O	X	X	O	O	X	O	X
X	O	O	X	X	O	X	O

Puzzle 2.2 solved

X	O	X	O	O	X	X	O
O	X	X	O	O	X	X	O
O	X	O	X	X	O	O	X
X	O	O	X	X	O	O	X
O	X	X	O	O	X	X	O
X	O	X	O	X	X	O	O
O	X	O	X	X	O	O	X
X	O	O	X	O	O	X	X

Puzzle 2.3 solved

The rules of Three in a Row

- There must be the same number of Xs and Os in each row and in each column (4 of each in this case).
- We can never have 3 or more of the same letter adjacent in a horizontal or vertical line.

Tempting as it might be, guessing is not a useful technique. These puzzles can always be solved by a completely logical process.

Techniques

There are two basic types of configurations to look for. We have illustrated them horizontally in order to save space, but of course they could appear vertically.

	X	X	

	O	O	

or

X		X

O		O

And remember to count!

What's missing from this row from a 6×6 puzzle?

O	O	X	O		

What's missing from this row from an 8×8 puzzle?

O	X	O	X		O	X	X

Finally, here is an example of a more sophisticated technique that will be needed for harder puzzles. Consider the puzzle below:

O	X	O	O	X	X	O	X
			X				
			X				
	X	O	O	X	O		
					X		
O			X		O	X	O
			O	X	O		X
	X				X	O	X

Puzzle 2.4

1. What is special about the shaded row?
2. Can the last O go in the fifth position, between the X and the O? Why? What can you conclude?
3. Complete the rest of the row.
4. Complete the puzzle using all the techniques you know.

A step-by-step solution

Try this puzzle yourself. If you're comfortable with it, go to page 15 for some harder puzzles. If you'd like to walk through the solution, turn to the next page.

X	X				O		O
X				O		X	
				O			
	O						
		X			O		
				X			
		O				X	
X				X			O

Puzzle 2.5

We will use coloured shading to indicate the parts of the puzzle we are focussing on. As you are walking through the solution, start by covering up the writing beneath each grid and focussing on the shaded parts. Then check to see if your reasoning agrees with that given, and carry on from there.

X	X				O		O
X				O		X	
				O			
	O						
		X			O		
				X			
		O				X	
X				X			O

Technique (1): Look for 2 Xs or 2 Os appearing side by side.

Technique (2): Look for anywhere that 2 Xs or 2 Os in a vertical or horizontal line are separated by exactly 1 square.

X	X	O		X	O	X	O
X				O		X	
O				O			
	O			X			
		X			O		
				X			
		O		O		X	
X				X			O

Check out the new Xs and Os for the same two situations described above.

X	X	O		X	O	X	O
X				O		X	
O				O		O	
	O			X			
		X		O	O		
				X			
		O	X	O		X	
X				X			O

Do it again!

X	X	O		X	O	X	O
X				O		X	
O				O	X	O	
	O			X			
		X	X	O	O	X	
				X			
		O	X	O		X	
X				X			O

And again . . .

Technique (3): If there is a row or column with 4 of one symbol, we know what must go in the remaining squares, as there must be 4 Xs and 4 Os in each line.

X	X	O	O	X	O	X	O
X				O		X	
O				O	X	O	
	O			X			
	O	X	X	O	O	X	
			O	X		O	
		O	X	O		X	
X				X			O

Keep checking for those two situations after each move you make.

Once the puzzle starts to fill in, it's time to look for rows or columns with 4 Xs or 4 Os. Then you can fill in the rest of the line.

X	X	O	O	X	O	X	O
X				O		X	
O	X			O	X	O	
	O			X		O	
	O	X	X	O	O	X	
	X		O	X		O	
		O	X	O		X	
X				X		O	O

X	X	O	O	X	O	X	O
X	O			O		X	
O	X			O	X	O	
	O			X		O	
	O	X	X	O	O	X	
	X		O	X		O	
		O	X	O		X	
X				X	X	O	O

Now we've exhausted the simplest techniques, it's time for something a bit more sophisticated.

X	X	O	O	X	O	X	O
X	O			O		X	
O	X			O	X	O	⋆
	O			X		O	
	O	X	X	O	O	X	
	X		O	X		O	
		O	X	O		X	
X			O	X	X	O	O

Technique (4): In this puzzle we need 4Xs and 4Os. Row 3 has 3 Os and fewer than 3 Xs, so we check each blank square to see if putting the 4th O there will cause three-in-a-row Xs.

The ⋆ marks a square that would create three-in-a-row Xs if we placed the O in it, so we cannot put the O in the ⋆ square; we must put an X there.

X	X	O	O	X	O	X	O
X	O		⋆	O		X	
O	X			O	X	O	X
	O			X		O	
	O	X	X	O	O	X	
⋆	X		O	X		O	
⋆		O	X	O		X	
X			O	X	X	O	O

Repeat that reasoning on the 4th column.

If we were to put an O in the ⋆ square, we'd get three-in-a-row Xs, so we can't put an O there; it must be an X.

In the first column, we see 3 Xs. There are two ⋆ squares, where putting the 4th X would create three-in-a-row Os.

From here you can use techniques (1), (2), and (3) to finish the puzzle, arriving at:

X	X	O	O	X	O	X	O
X	O	X	X	O	O	X	O
O	X	O	X	O	X	O	X
O	O	X	O	X	X	O	X
X	O	X	X	O	O	X	O
O	X	O	O	X	X	O	X
O	X	O	X	O	O	X	X
X	O	X	O	X	X	O	O

Puzzle 2.5 solved

Note: Technique (4) is a possibility for a $2n \times 2n$ puzzle only when there are $n-1$ of one symbol and fewer than $n-1$ of the other. For example, in a 14×14 puzzle, we need 7 Xs and 7 Os in each line. Look for a line with 6 of one letter and fewer than 6 of the other. Think about it – why this is the only time that technique (4) might be helpful!

Two 8×8 puzzles

		O				O	
X		X		X			
	O				O		O
				O		X	
				O			
							X
		O	O			O	O

Puzzle 2.6

	O	O					X
		O		O			
				X			
X							X
X							
				O		X	
		X		O	O		
X							

Puzzle 2.7

Now for some bigger puzzles

Once you've put in all the obvious letters in a bigger puzzle, it is helpful to keep a running tally of each type of letter for each row and column, rather than having to re-count over and over. For example, 3/4 could mean 3Xs and 4Os in a row or column.

				O				X	
						X			
X				O		X			
		O					X		
		O	O						
					X		X		
									O
O		O				O			O
O							O		
	X						O		

Puzzle 2.8 (10×10)

	X							X	
X	X						X	X	
	O	O		O	O				
		X							
									O
		X			X		X		
			O						O
O			O				X	X	
				X					

Puzzle 2.9 (10 × 10)

These require repeated use of Technique (4), but be sure to keep checking for places where you can use the earlier techniques.

	O	O			O				O		
X				X	X						X
	O							O			
X											
		O		X				O			
		O						X	X		
				X			O				
				X				X		O	
							X		O		
		O	O								
				X	X					O	
X					X				X		

Puzzle 2.10 (12 × 12)

											O
	O	O				X	X				
							X			X	
						O				X	
O						O	X		X		
									X		
	O	O						O			
									X		
O	O					X		O			
O					O	O					O
		X	X						X		
X		X			O		O				

Puzzle 2.11 (12×12)

Find many more

At Brainbashers this is called 3 in a Row.You'll find puzzles from 8×8 to 18×18, in easy and medium difficulty. Brainbashers provides an automatic counter to keep track of the number of Xs and Os in each row and column, which is very helpful with big puzzles.

At Simon Tatham's Portable Puzzles the puzzle is called Unruly.You can customise and get even larger puzzles, in two levels of difficulty.

Chapter 3

Kakurasu

Try to figure out the rules for this game by looking at these two solved puzzles. A row and a column have been shaded to give you a hint.

	1	2	3	4	
1	✓	✓	✓	✓	**10**
2	✗	✓	✗	✗	**2**
3	✗	✗	✓	✓	**7**
4	✓	✗	✓	✗	**4**
	5	**3**	**8**	**4**	

Puzzle 3.1 solved

	1	2	3	4	
1	✗	✓	✓	✗	**5**
2	✓	✓	✓	✗	**6**
3	✗	✗	✓	✓	**7**
4	✓	✓	✗	✗	**3**
	6	**7**	**6**	**3**	

Puzzle 3.2 solved

If you think you know what's going on, try to fill in the **blanks** on the right and bottom of this puzzle. The answer is on the next page. If you are not sure what's going on, turn the page to see the rules of Kakurasu.

	1	2	3	4	
1	✗	✗	✗	✓	
2	✓	✓	✗	✗	
3	✗	✓	✓	✗	
4	✗	✓	✓	✓	

Puzzle 3.3

	1	2	3	4	
1	✗	✗	✗	✓	**4**
2	✓	✓	✗	✗	**3**
3	✗	✓	✓	✗	**5**
4	✗	✓	✓	✓	**9**
	2	**9**	**7**	**5**	

Puzzle 3.3 solved

Did you come up with this? If you did, try to solve the next two puzzles.

Your goal is to place the ✓s and ✗s appropriately. If you get stuck, see below for the rules of Kakurasu.

	1	2	3	4	
1					**7**
2					**2**
3					**3**
4					**9**
	4	**10**	**4**	**5**	

Puzzle 3.4

	1	2	3	4	
1					**1**
2					**9**
3					**6**
4					**7**
	1	**5**	**6**	**9**	

Puzzle 3.5

The rules of Kakurasu

Our goal is to place the ✓ marks where they will produce the given totals on the right and the bottom of the grid.

- The clues on the *right* give the totals for the rows.
- The clues across the *bottom* give the totals for the columns.
- The numbers across the top and on the left give the values that contribute to the row and column totals, respectively.
- Marking a square with a ✓ means that square's row value gets added to the row's total and the square's column value gets added to the column's total.
- ✗ represents a box that is not used in the row sum or column sum.
- There is only one possible solution.

Guessing or relying on "intuition" might work on smaller puzzles, but it doesn't work for bigger ones. It's a good idea to deliberately practice your logic on the easier puzzles before moving on to harder ones. You may notice that solving these puzzles is at least as much about subtraction as it is about addition.

Here are two more puzzles for you to try. They are followed by step-by-step discussions of the solutions, which we hope will clarify the rules and some puzzle-solving strategies.

	1	2	3	4	
1					**7**
2					**5**
3					**7**
4					**4**
	4	**2**	**10**	**4**	

Puzzle 3.6

	1	2	3	4	
1					**7**
2					**5**
3					**4**
4					**8**
	4	**2**	**7**	**8**	

Puzzle 3.7

Answers for Puzzle 3.4 and Puzzle 3.5

	1	2	3	4	
1	✓	✓	✗	✓	**7**
2	✗	✓	✗	✗	**2**
3	✓	✓	✗	✗	**3**
4	✗	✓	✓	✓	**9**
	4	**10**	**4**	**5**	

Puzzle 3.4 solved

	1	2	3	4	
1	✓	✗	✗	✗	**1**
2	✗	✓	✓	✓	**9**
3	✗	✓	✗	✓	**6**
4	✗	✗	✓	✓	**7**
	1	**5**	**6**	**9**	

Puzzle 3.5 solved

A step-by-step solution of Puzzle 3.6

It's useful to start by looking for "big" numbers and "small" numbers.

	1	2	3	4	
1			✓		**7**
2			✓		**5**
3			✓		**7**
4			✓		**4**
	4	**2**	**10**	**4**	

The biggest number in a 4×4 puzzle is 10, as $1 + 2 + 3 + 4 = 10$.

	1	2	3	4	
1		✗	✓		**7**
2		✓	✓		**5**
3		✗	✓		**7**
4		✗	✓		**4**
	4	**2**	**10**	**4**	

The smallest number here is 2, which can be achieved in only one way, so that is a good total to tackle next.

	1	2	3	4	
1		✗	✓		**7**
2	✗	✓	✓	✗	**5**
3		✗	✓		**7**
4	✓	✗	✓	✗	**4**
	4	**2**	**10**	**4**	

There are two ways to get those 4s along the bottom ($1 + 3$ or just 4) and only one is correct, so instead of guessing, let's consider the rows instead.
5 is already complete: $2 + 3 = 5$, so finish the second row by placing the ✗s.
In the bottom row we already have 3, so we just need 1 to get $1 + 3 = 4$.

	1	2	3	4	
1	✗	✗	✓	✓	**7**
2	✗	✓	✓	✗	**5**
3	✗	✗	✓	✓	**7**
4	✓	✗	✓	✗	**4**
	4	**2**	**10**	**4**	

It's easy to finish either the remaining columns or the remaining rows. Let's do the columns.
Column 1: $4 = 4$
Column 4: $4 = 1 + 3$

If you now check the remaining rows, you'll see that everything works out correctly.

A step-by-step solution of Puzzle 3.7

As with the last puzzle, we will start by looking for "big" numbers and "small" numbers.

	1	2	3	4	
1		✗			**7**
2		✓			**5**
3		✗			**4**
4		✗			**8**
	4	**2**	**7**	**8**	

The only way to get 2 is to use 2 itself.

	1	2	3	4	
1		✗		✓	**7**
2		✓		✗	**5**
3		✗		✓	**4**
4	✓	✗	✓	✓	**8**
	4	**2**	**7**	**8**	

Similarly, the only way to reach 8 is to subtract 2 from 10.

	1	2	3	4	
1	✗	✗		✓	**7**
2	✗	✓		✗	**5**
3	✗	✗	✗	✓	**4**
4	✓	✗	✓	✓	**8**
	4	**2**	**7**	**8**	

Now row 3 and column 1 are done, as $4 = 4$, and the rest can be filled with ✗s.

	1	2	3	4	
1	✗	✗	✓	✓	**7**
2	✗	✓	✓	✗	**5**
3	✗	✗	✗	✓	**4**
4	✓	✗	✓	✓	**8**
	4	**2**	**7**	**8**	

Each of the other totals now has only one possible solution.
Row 1 : $7 = 3 + 4$
Row 2: $5 = 2 + 3$
Column 3: $7 = 1 + 2 + 4$

Two for you to try

Remember: if the last couple of totals in any puzzle work out, then you'll know that you have solved the puzzle correctly.

	1	2	3	4	
1					**8**
2					**7**
3					**6**
4					**2**
	1	**7**	**3**	**6**	

Puzzle 3.8

	1	2	3	4	
1					**10**
2					**6**
3					**6**
4					**1**
	7	**6**	**3**	**4**	

Puzzle 3.9

Some useful strategies

Feel free to skip these if you want to figure everything out yourself!

Look for small numbers

- Anywhere there is a 1 or a 2 is a good place to start, as there is only one way to make either of those work. This true, no matter how large the puzzle.
- While there are two ways to reach 3, we know for sure that no number larger than 3 can be used.

1	2	3	4	
			✗	**3**

- What can we conclude about 4? There are only two ways to reach 4, either $4 = 4$ or $4 = 1 + 3$. There is no way to use 2, so we can get rid of that possibility.

1	2	3	4	
	✗			**4**

Look for large numbers

- Other good places to start a puzzle involve the appropriate triangular number (see Appendix B).
- $1 + 2 + 3 + 4 = 10$, so in a 4×4 puzzle 10 is great place to start! So are 9 and 8, as there is only one way to get each of them: $9 = 10 - 1$ and $8 = 10 - 2$.

1	2	3	4	
✗	✓	✓	✓	**9**

1	2	3	4	
✓	✗	✓	✓	**8**

- On the other hand, 7 does not have a unique decomposition, because $10 - 3$ can be found by deleting the 3 or by deleting 1 and 2. We can, however, conclude something about 7 in a 4×4 puzzle:

1	2	3	4	
			✓	**7**

- Larger puzzles give us more to think about. In a 5×5 puzzle, 10 is no longer a good place to start. The most useful large numbers here are 15, 14, and 13. What are the most useful large numbers for a 6×6 puzzle?

Some 5×5 puzzles for you to try

	1	2	3	4	5	
1						**3**
2						**15**
3						**8**
4						**7**
5						**4**
	6	**10**	**2**	**7**	**9**	

Puzzle 3.10

	1	2	3	4	5	
1						**9**
2						**5**
3						**3**
4						**12**
5						**8**
	12	**9**	**7**	**10**	**5**	

Puzzle 3.11

	1	2	3	4	5	
1						**7**
2						**9**
3						**9**
4						**7**
5						**9**
	10	**4**	**11**	**1**	**14**	

Puzzle 3.12

	1	2	3	4	5	
1						**10**
2						**8**
3						**2**
4						**13**
5						**9**
	12	**5**	**9**	**5**	**12**	

Puzzle 3.13

Puzzles with missing clues

Here are some puzzles that are a bit trickier because they are missing some clues. Solutions are still unique. To avoid serious frustration, don't ever put in a number unless you are absolutely sure it must go there!

	1	2	3	4	
1					**4**
2					**7**
3					**2**
4					**6**
	5		**7**		

Puzzle 3.14

	1	2	3	4	
1					
2					**5**
3					**6**
4					
	4	**6**	**5**	**4**	

Puzzle 3.15

	1	2	3	4	5	
1						**11**
2						**7**
3						**5**
4						**8**
5						**12**
	1	**4**	**15**			

Puzzle 3.16

	1	2	3	4	5	
1						
2						**3**
3						**11**
4						
5						**8**
	14	**5**	**13**	**9**	**3**	

Puzzle 3.17

Larger puzzles

	1	2	3	4	5	6	
1							**16**
2							**6**
3							**11**
4							**20**
5							**14**
6							**12**
	8	**13**	**13**	**9**	**18**	**14**	

Puzzle 3.18

As the puzzles get larger, you may find it useful to note down subtotals as you reach them. For example, suppose that as part of the puzzle we have already been able to tick the 3 and the 6. That adds to 9, leaving us with 6 more to go.

1	2	3	4	5	6	7	
		✓			✓		**15**(9)

Then it is clear that we cannot use the 7, so we can ✗ it out. This may provide a key clue for a column total. Solving a larger puzzle often requires us to use every possible deduction we can make from any given row or column. Remember to put in those ✗s and ✓s whenever you can!

Here is a partial solution of the 6×6 Puzzle 3.18. Note that there is often more than one sequence of logical steps that will produce the same result.

	1	2	3	4	5	6	
1							**16**
2							**6**
3							**11**
4	✗	✓	✓	✓	✓	✓	**20**
5							**14**
6							**12**
	8	**13**	**13**	**9**	**18**	**14**	

	1	2	3	4	5	6	
1							**16**
2							**6**
3							**11**
4	✗	✓	✓	✓	✓	✓	**20**
5					✓		**14**
6					✓		**12**
	8	**13**	**13**	**9**	**18**	**14**	

1) Row 4: Since the biggest possible sum is $1+2+3+4+5+6=21$, the target of 20 is a good place to start.
2) Column 5: The target of 18 is also a large number. Since $21-18=3$, and we can leave out 3 either by leaving out $1+2$ or 3 itself, we need all the values greater than 3. We don't know which way to leave out the total of 3, so leave it at that for now.

	1	2	3	4	5	6	
1							**16**
2							**6**
3							**11**
4	✗	✓	✓	✓	✓	✓	**20**
5					✓		**14 (5)**
6					✓		**12 (5)**
	8	**13 (4)**	**13 (4)**	**9 (4)**	**18 (15)**	**14 (4)**	

Hint: Once we get to bigger puzzles such as this one, it can be helpful to keep a running tab of the sums reached for each uncompleted row and column, which we have done using parentheses. It is also possible to indicate the new targets rather than the current sums, but let's use the format as shown in puzzle-kakurasu.com.

	1	2	3	4	5	6	
1				✗			**16**
2							**6**
3							**11**
4	✗	✓	✓	✓	✓	✓	**20**
5					✓		**14 (5)**
6				✗	✓		**12 (5)**
	8	**13 (4)**	**13 (4)**	**9 (4)**	**18 (15)**	**14 (4)**	

3) Column 4: We can't use either the values 6 or 1 to reach the new target of $9-4=5$, so we ✗ them out.

	1	2	3	4	5	6	
1	✗	✓	✓	✗	✓	✓	16
2							6
3							11
4	✗	✓	✓	✓	✓	✓	20
5					✓		14 (5)
6	✓	✗	✗	✗	✓	✓	12
	8	13	13	9	18	14	
	(6)	(5)	(5)	(4)	(16)	(11)	

4) Row 1 We have left in the row a total of $21 - 4 = 17$. This is only 1 more than the target of 16, so we need everything but the value 1.
5) Row 6 We now have a target of $12 - 5 = 7$. The only way to reach that is $6 + 1$. Remember to update the ✗s and the current sums.

	1	2	3	4	5	6	
1	✗	✓	✓	✗	✓	✓	16
2	✓				✓		6
3	✗				✗		11
4	✗	✓	✓	✓	✓	✓	20
5	✗				✓		14 (5)
6	✓	✗	✗	✗	✓	✓	12
	8	13	13	9	18	14	
		(5)	(5)	(4)		(11)	

6) Columns 1 and 5: After updating the targets along the bottom, we see that there is only one way to reach the target 8 in column 1 and the target 18 in column 5. Update the ✗s and the running sums.

	1	2	3	4	5	6	
1	✗	✓	✓	✗	✓	✓	16
2	✓	✗	✗		✓		6
3	✗	✓	✓		✗		11 (5)
4	✗	✓	✓	✓	✓	✓	20
5	✗	✓	✓		✓		14 (10)
6	✓	✗	✗	✗	✓	✓	12
	8	13	13	9	18	14	
				(4)		(11)	

7) Columns 2 and 3: In each column we have only the values 3 and 5 left to reach the target of $13 - 5 = 8$, which works out nicely. Update any running sums on the right.
8) See if you can finish it from here!

Finally, here are two 6×6 puzzles for you to try:

	1	2	3	4	5	6	
1							**10**
2							**7**
3							**5**
4							**8**
5							**17**
6							**9**
	16	**12**	**12**	**5**	**12**	**9**	

Puzzle 3.19

	1	2	3	4	5	6	
1							**8**
2							**15**
3							**7**
4							**5**
5							**14**
6							**9**
	15	**19**					

Puzzle 3.20

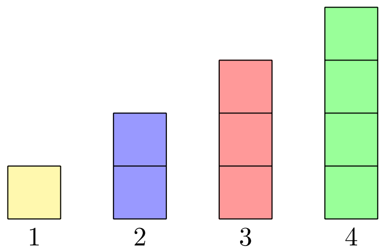
Find many more

A source of Kakurasu puzzles to play on-line is Puzzle-Kakurasu.

Chapter 4

Towers

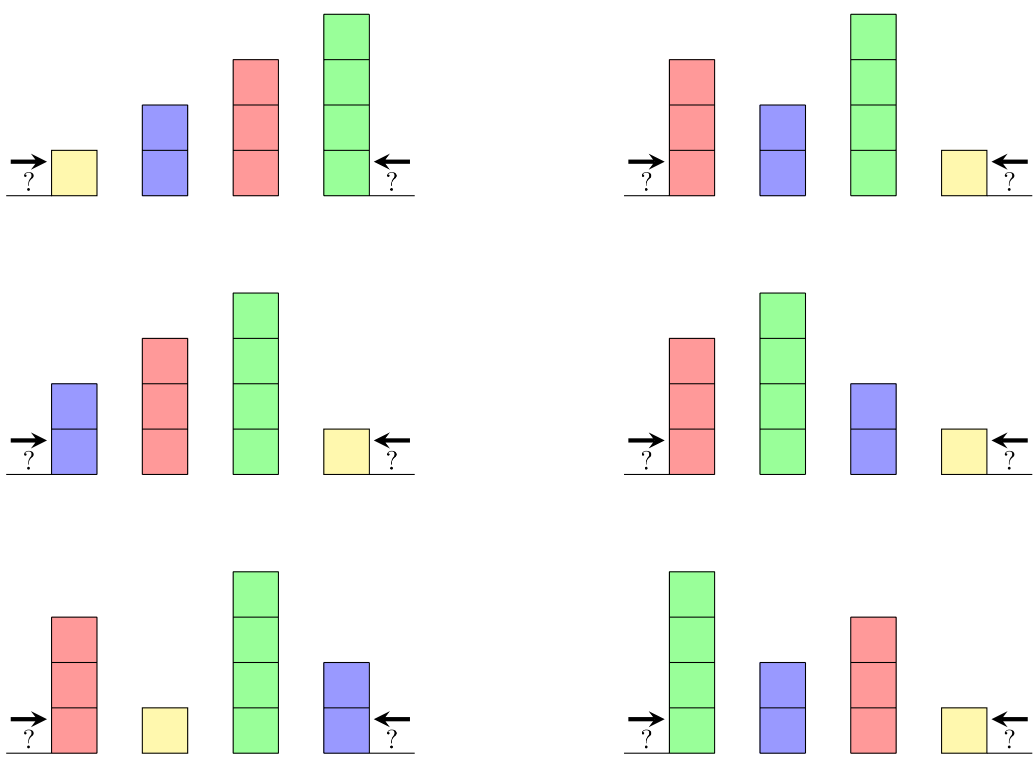
This puzzle requires us to think in three dimensions using two-dimensional clues. Many people find it easier to get started if they use physical towers, a different colour for every height. Snap-together cubes work well for this.



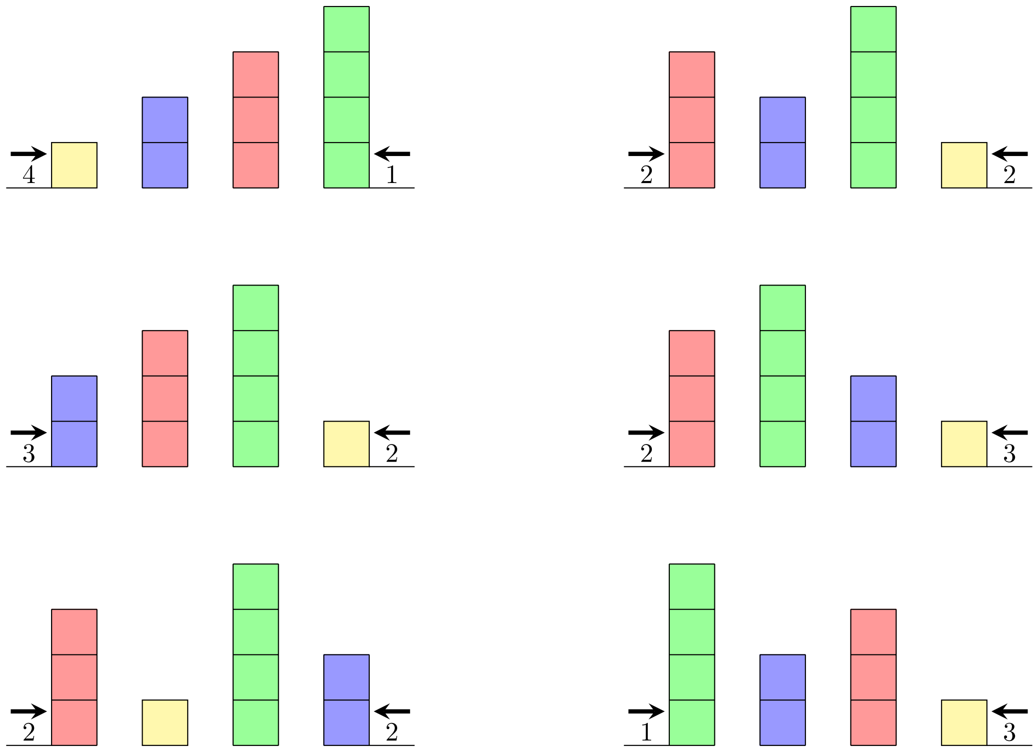
We'll start by using single yellow blocks, blue blocks of twice the height, red blocks of height three, and green blocks of height four.

Warm-up

Imagine you're looking at each row of four towers from the sides, where the ? marks are. How many would you be able to see from each side? You can confirm that you have the idea by checking the answers on the following page.



Answers to warm-up



Lining up four rows of towers, we get a grid

Now we can look at each column from above and below in order to count the visible towers from those points of view. You can tell from the Towers Puzzle 4.1 4×4 solved below that, inside the heavy border, a Latin square is involved (see Latin Squares in the Appendix A).

	1	**2**	**3**	**2**	
1	**4**	**3**	**2**	**1**	**4**
2	**3**	**2**	**1**	**4**	**1**
2	**1**	**4**	**3**	**2**	**3**
2	**2**	**1**	**4**	**3**	**2**
	3	**2**	**1**	**2**	

Puzzle 4.1 solved

	3	**4**	**1**	**2**	
	2	**1**	**3**	**4**	
	1	**2**	**4**	**3**	
	4	**3**	**2**	**1**	

Puzzle 4.2 fill in the outside numbers

The actual puzzle will be to find the values in the interior of the heavy border, given some clues around the outside. To complete our warm-up, though, let's do it backwards and fill in the outside numbers in Puzzle 4.2, shown above, instead. The answer is on the following page.

	2	**1**	**3**	**2**	
2	**3**	**4**	**1**	**2**	**2**
3	**2**	**1**	**3**	**4**	**1**
3	**1**	**2**	**4**	**3**	**2**
1	**4**	**3**	**2**	**1**	**4**
	1	**2**	**2**	**3**	

Puzzle 4.2 solved

The rules of Towers

- The numbers inside the heavy border tell us how many floors each tower has.
- The numbers outside the heavy border tell us how many towers we could see if we stood in that position and looked along the appropriate row or column.
- We can't see a shorter tower behind a taller one.
- Each row and each column must contain a tower of each height exactly once.
- A well-designed Towers puzzle will have only one possible solution.
- As usual, although guessing might work on easy puzzles, you'll get tangled up very quickly if you try to guess on the harder ones.

In this chapter all puzzles, with the exception on one challenge at the end of the chapter, are on 4×4 grids. We will therefore indicate the difficulty of the puzzle using levels, with higher levels indicating more challenging puzzles.

Two level I puzzles

If you get stuck, turn the page for a step-by step solution of a puzzle of similar difficulty, then come back to these.

If you are having a hard time picturing the towers, make yourself a set, 4 of each height, out of snap-together blocks or something similar.

	1	**2**	**4**	**2**	
1					**2**
2					**3**
3					**1**
3					**2**
	4	**2**	**1**	**2**	

Puzzle 4.3

	2	**3**	**1**	**2**	
2					**2**
1					**3**
4					**1**
2					**3**
	2	**1**	**3**	**2**	

Puzzle 4.4

A step-by-step solution of a level I puzzle

	2	1	2	3	
2					3
2					2
1					3
4					1
	2	3	2	1	

Puzzle 4.5

If you can solve Puzzle 4.5, shown above, on your own right now, turn to page 36 to try a harder one.

	2	1	2	3	
2					3
2					2
1					3
4	1	2	3	4	1
	2	3	2	1	

First, there is only one way to see all 4 towers from the left of the bottom row, so we can place them right away.

	2	1	2	3	
2		4			3
2					2
1	4				3
4	1	2	3	4	1
	2	3	2	1	

The only way to see 1 tower is for it to be of height 4, so look for all 1-clues.

	2	1	2	3	
2		4			3
2			4		2
1	4				3
4	1	2	3	4	1
	2	3	2	1	

Now that three 4-towers have been put in place, there is only one spot for the fourth, avoiding any row or column which already contains a 4-tower.

	2	1	2	3	
2	3	4			3
2	2		4		2
1	4				3
4	1	2	3	4	1
	2	3	2	1	

Look at the first column: we need to place a 2-tower and a 3-tower. In order to see only two towers from the *top* of that column, the order has to be 3, 2.

	2	1	2	3	
2	3	4	2	1	3
2	2		4		2
1	4				3
4	1	2	3	4	1
	2	3	2	1	

In order to see three towers looking along the top row from the *right*, we need to place a 1-tower and a 2-tower as shown.

	2	1	2	3	
2	3	4	2	1	3
2	2	1	4		2
1	4				3
4	1	2	3	4	1
	2	3	2	1	

Looking along the second row from the *left*, we can see only two towers if we place a 1-tower between the 2-tower and the 4-tower.

	2	1	2	3	
2	3	4	2	1	3
2	2	1	4	3	2
1	4	3	1	2	3
4	1	2	3	4	1
	2	3	2	1	

Finally, we can place the missing values in each row or column that already has three values.

Some level I puzzles to try

	2	1	2	2	
2					3
3					1
1					2
2					2
	2	4	1	3	

Puzzle 4.6

	2	3	2	1	
4					1
1					3
2					2
2					3
	2	1	2	3	

Puzzle 4.7

	1	2	3	3	
1					3
2					2
2					2
3					1
	3	3	2	1	

Puzzle 4.8

	1	2	4	3	
1					3
2					2
3					1
2					2
	2	2	1	2	

Puzzle 4.9

Some level II puzzles to try

You may have noticed that sometimes you don't need to use all the given clues. Here are three puzzles with some redundant clues removed.

	2	3	2	1	
3					
1					
2					2
3					2
	3	2	1	2	

Puzzle 4.10

	2	3	3		
2					1
					4
2					
					3
	3	1	2	3	

Puzzle 4.11

		3	4		
1					
		3			
				1	
				2	

Puzzle 4.12

	2		2	1	
3					
1					3
2					
2					2
	3	1	2	2	

Puzzle 4.13

Some observations

- There is only one way to be able to see all four towers in a row or in a column (heights 1,2,3,4).
- If we can see only a single tower, we know it has to be the tallest (height 4).
- If a clue says we can see three towers, we know that the 4 can't appear in the first or second place.
- Don't forget that each number must appear exactly once in each row and each column. Look for any rows or columns that have only one possible place for a given number.

Warming up for level III puzzles

You may have noticed that some of the clues in the level II puzzles are unnecessary. Below, we have eliminated four clues from puzzle 4.14. Try to solve it from these clues and see if you get the same answer as you did before.

			2	1	
3					
1					3
2					
		1		2	

Variation of puzzle 4.13

For fun, let's eliminate one more clue, as in the version below. We ended up with two possible solutions. Do you?

			2	**1**	
1					**3**
2					
		1		**2**	

New variation of puzzle 4.13

Clearly, making a puzzle more difficult by deleting clues has to be done carefully!

A step-by-step solution of a level III puzzle

If you find you can solve puzzle 4.14, shown below, on your own, turn to page 39 for some harder ones.

	3		**3**		
					3
					3

Puzzle 4.14

Note: Here we'll use shading just for the row, column or squares on which we are focussing.

	3		3		
	12		12		
	123		123	12	3
			123	12	3

In order to see three towers from any position, we can only place either a 1-tower or a 2-tower right next to the viewing position.
Next to that we can only place a 1-tower, a 2-tower, or a 3-tower.

	3		3		
	12		12		
	123	4	123	12	3
			4		
			123	12	3

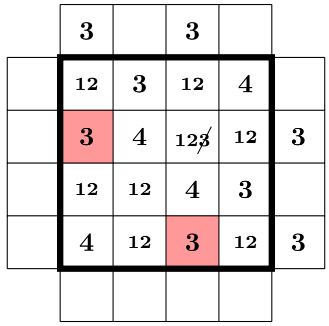
In each of row 2 and column 3 there is now only one place for a 4-tower.

	3		3		
	12		12	4	
	123	4	123	12	3
			4		
	4		123	12	3

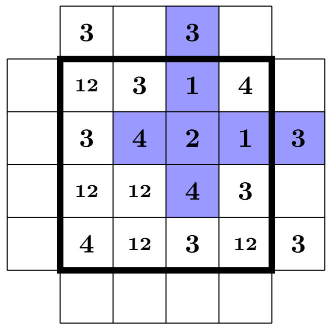
Look at column 1: there is only one place for a 4-tower. This is also true of column 4.

	3		3		
	12	3	12	4	
	123	4	123	12	3
			4	3	
	4		123	12	3

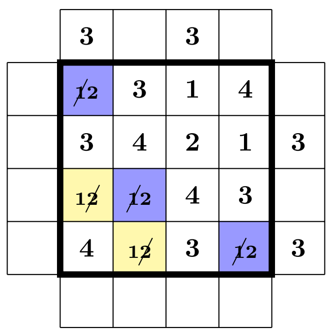
In each of row 1 and column 4 there is now only one place for a 3-tower.



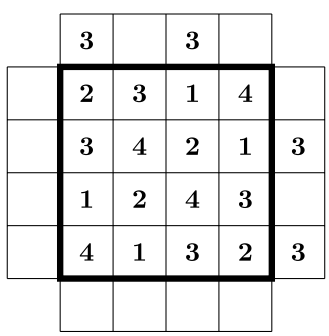
In each of column 1 and row 4 there is only one place left for a 3-tower. Now we'll fill in the remaining squares with the possibilities.



Looking at column 3 from the top and at row 2 from the right, we find there is only one way to place the towers in order to see three of them: from the viewpoint they must read 1, 2, 4.



And finally we eliminate the towers we've used in the remaining rows and columns.



Puzzle 4.14 solved

Done!

Now for some level III puzzles

		2	2	2	
					3
					1
2					
3					2
	4		1	3	

Puzzle 4.15

				4	
					2
		3			

Puzzle 4.16

	1	2	2	3	
					3
2					
3					2
	3	2		1	

Puzzle 4.17

		2		3	
				1	
		2			
					1
			2		

Puzzle 4.18

				1	
3					
					4
2					

Puzzle 4.19

		1	3		
					3
					1

Puzzle 4.20

		2			
3					
			2		**1**
					2
1					

Puzzle 4.21

			3	**2**	
					2

Puzzle 4.22

A challenge

An interesting problem is to try to create a puzzle with the fewest possible clues, remembering of course that there must be only one possible solution. Here is one more version of puzzle 4.13 from page 35 with what we believe is a minimum of clues. Now that you've had some practice with level III puzzles, see if you get the same solution you did before.

			2	**1**	
3					
1					**3**
2					
		1			

Third variation of puzzle 4.13 revisited

You might like to see how many clues you can eliminate from the following puzzle and still get the same solution as for the original. Some redundant clues can be deleted quickly, but after that things get harder! And there may be more than one possible set of minimal clues, which makes it even more interesting.

	2	**1**	**2**	**3**	
2					**2**
2					**2**
4					**1**
1					**4**
	1	**2**	**3**	**2**	

Puzzle 4.23

A 5×5 puzzle

Bigger puzzles can get harder very quickly. Here is a 5×5 puzzle to whet your appetite.

	4	**2**	**1**	**3**	**3**	
3						**2**
2						**2**
4						**1**
2						**2**
1						**4**
	1	**4**	**2**	**2**	**2**	

Puzzle 4.24

Find many more

At Brainbashers they are called Skyscrapers. You'll find puzzles in sizes from 4×4 to 9×9, in Easy, Medium, and Hard.

At Simon Tatham's Portable Puzzles they are called Towers. Here the diagrams use perspective to try to show the three-dimensional nature of the puzzle. They come in the same sizes, in Easy, Hard, Extreme, and Unreasonable. (You can customise beyond the options in the drop-down menu.)

Popular Playthings used to make a physical game called Utopia, which consists of two types of 4×4 puzzles, each with several levels of difficulty.

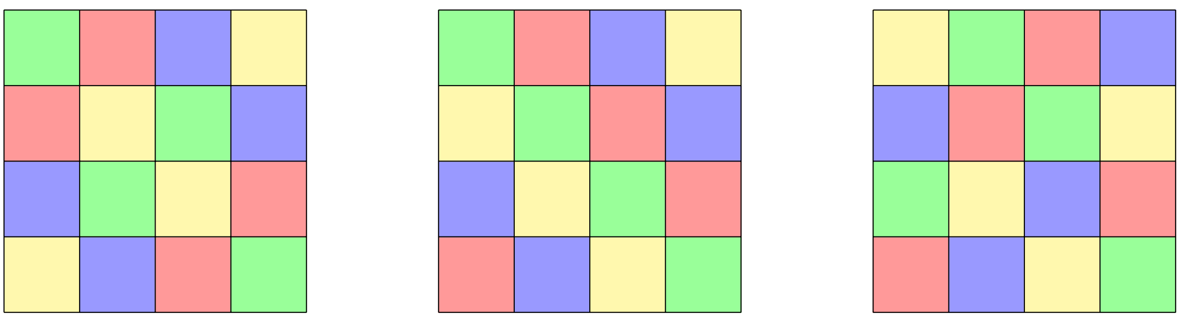
Appendix A

Latin Squares

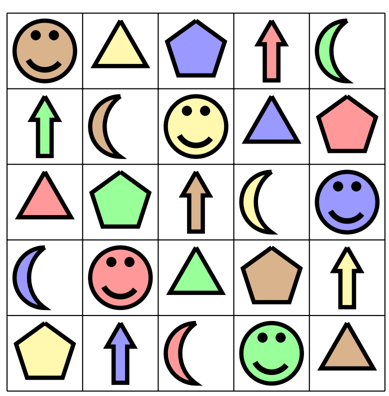
Latin squares form the basis of many puzzles. They may already be familiar to you, although you might not have known that they have a name or that a great deal of mathematics is associated with them. Look at the following examples – see what you can deduce!

C	A	T
T	C	A
A	T	C

C	A	T
A	T	C
T	C	A



Clearly, Latin squares involve shapes or colours arranged in a square grid. The goal is to arrange the shapes or colours so that there is no repetition in any row or column. This can be done in many ways.



Our example of a 5×5 Latin square is special because two attributes are involved: shape and colour. Neither shape nor colour is repeated in any row or column. This is called a Graeco-Latin square, an Euler square, or an orthogonal Latin square. For fun, you can try making a 4×4 Euler square using the Jack, Queen, King and Ace of each of the four suits from an ordinary deck of cards.

Why Latin and Graeco-Latin?

As far as we know, the mathematical properties of these squares were first studied in the 1700s by the Swiss genius Leonhard Euler (pronounced "*oiler*"). He used two sets of letters: upper-case letters from the Latin alphabet and lower-case letters from the Greek alphabet. This would have been easy to type-set but can be quite hard on the eyes!

$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
$B\delta$	$C\alpha$	$D\beta$	$A\gamma$
$C\gamma$	$D\delta$	$A\alpha$	$B\beta$
$D\beta$	$A\gamma$	$B\delta$	$C\alpha$

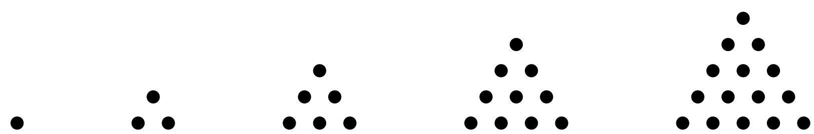
Uses

Latin squares can be very decorative and are often used in quilting and folk art patterns. They form the basis of a large number of puzzles and games, including Kakurasu and Towers in this booklet. Latin squares also have many practical uses, for example in the design of experiments, in error-correcting computer codes, and in creating tournament schedules. Their properties are thoroughly studied in a branch of mathematics called *discrete mathematics*.

Appendix B

Triangular Numbers

Think about arranging rows of dots, one below the other, where each row has one more dot than the previous row. Then count the dots. The resulting numbers are called *triangular*.



$T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$ $T_5 = 15$

The first five triangular numbers

From the diagram we can see that the **4**th triangular number, $T_{\mathbf{4}}$ is 10, because $1 + 2 + 3 + \mathbf{4} = 10$. The **5**th triangular number, $T_{\mathbf{5}}$, is $1 + 2 + 3 + 4 + \mathbf{5} = 15$. So the **6**th triangular number must be what?

In general, the $\boldsymbol{n}$th triangular number is the sum of the first n rows of a triangle built of dots, that is,

$$T_n = 1 + 2 + 3 + 4 + \cdots + n.$$

Triangular numbers turn up in many different places and have many interesting properties. They have a long history: as far as we know, the first people to study the properties of triangular numbers were the members of the Pythagorean Society in ancient Greece, around the 6th century BCE.

Here we are interested in the relationship between triangular numbers and Kakurasu puzzles. The largest sum possible in a 4×4 Kakurasu puzzle is 10, the 4th triangular number T_4. The largest sum possible in a 5×5 Kakurasu puzzles is 15, the 5th triangular number T_5. Thinking about how Kakurasu puzzles are structured, we can see that the largest sum possible in an $n \times n$ puzzle is the nth triangular number T_n.

Kakurasu size	**largest sum**
4×4	$T_4 = 10$
5×5	$T_5 = 15$
6×6	$T_6 = 21$
7×7	$T_7 = 28$
8×8	$T_8 = 36$
9×9	$T_9 = 45$
10×10	$T_{10} = 55$

These values also play a large part in solving Kakuro or cross-sum puzzles.

Appendix C

Online Resources

Below are five recommended websites for math/logic puzzles. The puzzles are designed to be played online but many can be printed off as well. The links were accessible in August 2022, shortly before the publication of this booklet.

Most sites have a large supply of puzzles of a given size and degree of difficulty. Puzzles sometimes appear under different names.

Some sites provide ways of keeping track of the counts for rows and columns, where that is appropriate. For example, this feature is helpful for the larger grids in *Kakurasu* and *Three in a Row*.

All of these sites have many other types of puzzles for you to explore.

- **Brainbashers** (https://www.brainbashers.com/) provides daily logic puzzles of about two dozen types. You can easily access puzzles from the rest of the year.

- **Simon Tatham's Portable Puzzles** (https://www.chiark.greenend.org.uk/ sgtatham/puzzles/) These puzzles are designed to run on a variety of platforms. This site makes it easy to print appropriate puzzles in large quantities, which is great for the classroom.

- **The site with no name** can be accessed via any one of the many puzzles it contains. We mentioned it in the chapter on *Kakurasu*. (https://www.puzzle-kakurasu.com/) Once you are on the page for a particular puzzle, the full collection of puzzles appears at the bottom of the page.

- **Otto and Angela Janko's site** (https://www.janko.at/Raetsel/)has a truly remarkable collection of puzzles from all over the world. The easiest way to find the site is to Google "Janko puzzle". The site is written in German but you can ask for the English translation.
  Note: The phrase "Most of the puzzles here have a solution that can be derived purely logically without trying", should be translated as "without using trial and error". You might have to try quite hard to solve some of them!

Finally, my own site is **Susan's Math Games** (https://susansmathgames.ca/). The site is designed as an online resource for teachers of kindergarten through secondary school. The site features a wide range of classroom-tested puzzles and games along with links to even more online resources. Materials on this site are meant to be printed for playing purposes, rather than being used online.

ATOM

A Taste Of Mathematics / Aime-T-On les Mathématiques

1. Edward J. Barbeau *Mathematical Olympiads' Correspondence Program (1995–1996)*
2. Bruce L.R. Shawyer *Algebra — Intermediate Methods*
3. Peter I. Booth, John Grant McLoughlin and Bruce L.R. Shawyer *Problems for Mathematics Leagues*
4. Edward J. Barbeau and Bruce L.R. Shawyer *Inequalities*
5. Richard Hoshino and John Grant McLoughlin *Combinatorial Explorations*
6. Peter I. Booth, John Grant McLoughlin and Bruce L.R. Shawyer *Problems for Mathematics Leagues — II*
7. Jim Totten *Problems of the Week*
8. Peter I. Booth, John Grant McLoughlin and Bruce L.R. Shawyer *Problems for Mathematics Leagues — III*
9. Edward J. Barbeau *The CAUT Problems*
10. Naoki Sato *Modular Arithmetic*
11. Bruce L.R. Shawyer and Bruce B. Watson *Problems for Junior Mathematics Leagues*
12. Edward J. Barbeau *Transformational Geometry*
13. Edward J. Barbeau *Quadratics and Complex Numbers*
14. Margo Kondratieva with Justin Rowsell *Sequences and Series*
15. Michel Bataille *Géométrie Plane, avec des Nombres*
16. Iliya Bluskov *Recurrence Relations*
17. **Susan Milner** ***Mathematical Logic Puzzles on a Grid***

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