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Crux Mathematicorum

Crux Mathematicorum with Mathematical Mayhem

EDITORIAL

Have you come across an interesting problem lately? Do you have a comment to make about matters mathematical?

In March 1975, 6 mathematicians asked these questions in No. 1 issue of *Eureka*, a magazine that will eventually become known as ***Crux***. R. Duff Butterill, H. G. Dworschak, Viktors Linis, F. G. B. Maskell, Léo Sauvé and Richard J. Semple formed the original editorial board. As members of the Carleton-Ottawa Mathematics Association, they were looking for a forum where answers to the aforementioned questions will be shared amongst members of the mathematical community of Ottawa region. In the initial issue, editors asked the readership to show their support in 4 ways: join a mailing list (snail mailing list at that), propose problems for publication, send solutions to the proposed problems, send in any material with "some relevance, however far-fetched, to mathematics".

You are now looking at Volume 50. ***Crux*** is an international open-access journal, reaching audiences far beyond Ottawa region, but we remain true to our roots – our main goal is for our readership to get involved in the journal by sending in material with some relevance to mathematics. Without sounding too grandiose, the journal is a true testament to the never-ending joy of problem solving that defines the mathematical community.

Here's to the 50 years of ***Crux***. And many more to come!

Kseniya Garaschuk

MathemAttic

No. 51

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 15, 2024**.

MA251. Bob is practicing addition in base 2. Each time he adds two numbers in base 2, he counts the number of carries. For example, when summing the numbers 1001 and 1011 in base 2, there are three carries (shown on the top row).

$$\begin{array}{ccccc} {\scriptstyle 1} & & {\scriptstyle 1} & {\scriptstyle 1} & \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 \end{array}$$

Suppose that Bob starts with the number 0 and adds 111 (i.e. 7 in base 2) to it one hundred times to obtain the number 1010111100 (i.e. 700 in base 2). How many carries occur (in total) in these one hundred calculations?

MA252. An Indian raga has two kinds of notes: a short note, which lasts for 1 beat, and a long note, which lasts for 2 beats. For example, there are 3 ragas which are 3 beats long: 3 short notes, a short note followed by a long note, and a long note followed by a short note. How many Indian ragas are 11 beats long? Justify your answer.

MA253. Let $n \geq 2$ be an integer. There are n houses in a town. All distances between pairs of houses are different. Every house sends a visitor to the house closest to it. Find all possible values of n (with full justification) for which we can design a town with n houses where every house is visited.

MA254. A sequence $a_1, a_2, \ldots$ satisfies $a_1 = \frac{5}{2}$ and $a_{n+1} = a_n^2 - 2$ for all $n \geq 1$. Let M be the integer closest to a_{2023}. Find the last digit of M.

MA255. A $3 \times 3 \times 3$ cube of cheese is sliced into twenty-seven $1 \times 1 \times 1$ blocks. A mouse starts anywhere on the outside and eats one of the $1 \times 1 \times 1$ cubes. He then moves to an adjacent cube (in any direction), eats that cube, and continues until he has eaten all 27 cubes. (Two cubes are considered adjacent if they share a face.) Prove that no matter what strategy the mouse uses, he cannot eat the middle cube last. (Note: One should neglect gravity: intermediate configurations dont collapse.)

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mars 2024**.

MA251. Bob s'entraîne à faire des additions en base 2. Chaque fois qu'il additionne deux nombres en base 2, il compte le nombre de retenues. Par exemple, lorsqu'il additionne les nombres 1001 et 1011 en base 2, il y a trois retenues (illustrées sur la ligne supérieure).

1		1	1	
0	1	0	0	1
0	1	0	1	1
1	0	1	0	0

Supposons que Bob commence avec le nombre 0 auquel il ajoute cent fois 111 (c'est-à-dire 7 en base 2) pour obtenir le nombre 1010111100 (à savoir 700 en base 2). Combien de retenues y a-t-il eu (au total) dans ces cent calculs ?

MA252. Un raga indien comporte deux figures de notes : une figure de note courte, qui dure un temps, et une figure de note longue, qui dure deux temps. Par exemple, il y a 3 ragas qui durent 3 temps : 3 notes courtes, une note courte suivie d'une note longue, et une note longue suivie d'une note courte. Combien de ragas indiens ont une durée de 11 temps ? Justifiez votre réponse.

MA253. Soit $n \geq 2$ un entier. Il y a n maisons dans une ville. Toutes les distances entre les paires de maisons sont différentes. Chaque maison envoie un visiteur á la maison la plus proche. Trouvez toutes les valeurs possibles de n (avec justification complète) pour lesquelles on peut concevoir une ville avec n maisons où chaque maison est visitée.

MA254. Une suite $a_1, a_2, \ldots$ satisfait $a_1 = \frac{5}{2}$ et $a_{n+1} = a_n^2 - 2$ pour tout $n \geq 1$. Soit M l'entier le plus proche de a_{2023}. Trouvez le dernier chiffre de M.

MA255. Un cube de fromage de $3 \times 3 \times 3$ est découpé en vingt-sept blocs de $1 \times 1 \times 1$. Une souris commence n'importe où á l'extérieur et mange l'un des cubes de 1. Elle se déplace ensuite vers un cube adjacent (dans n'importe quelle direction), mange ce cube et continue jusqu'á ce qu'elle ait mangé les 27 cubes. (Deux cubes sont considérés comme adjacents s'ils partagent une face.) Montrez que, quelle que soit la stratégie utilisée par la souris, elle ne peut pas manger le cube du milieu en dernier. (Note : Il faut négliger la gravité: les configurations intermédiaires ne s'effondrent pas).

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(6), p. 287–289.

MA226. The numbers a, b and c are in arithmetic sequence. The numbers b, c and d are in geometric sequence. If $a = 1.27$, $d = 3.68$ and c is positive, determine c.

Originally from 1988 Illinois CTM, State Finals AA, Calculating Competition, Question 20.

We received 7 submissions, all correct and complete. We present the solution by the Missouri State University Problem Solving Group.

We have $b = a + x$, $c = a + 2x$, $c = rd$, and $b = r^2 d$. Eliminating x in the first pair of equations gives $a = 2b - c$. Using the second pair of equations gives

$$2dr^2 - dr - a = 0.$$

Solving for r, we have

$$r = \frac{d \pm \sqrt{8ad + d^2}}{4d}.$$

Letting $a = 127/100$ and $d = 368/100$, we find

$$r = \frac{46 + \sqrt{7958}}{184}$$

(r must be positive for c to be positive). Hence

$$c = rd = \frac{46 + \sqrt{7958}}{50} \approx 2.70415.$$

MA227. Find and prove the general formula for the square root of the product of four consecutive integers plus 1.

Originally from Mathematics Competitions *Vol. 34, #1 (2021),* A brief history of the South African Mathematics Olympiad*, Easy interesting problems with clever solutions, example 1.*

We received 7 submissions of which 5 were correct and complete. We present the solution by Mihika Bansal, slightly modified by the editor.

Let n be a positive integer.

We are interested in the formula for $\sqrt{n(n+1)(n+2)(n+3)+1}$. Rearranging the terms before multiplying the expression under the root, we get:

$$n(n+1)(n+2)(n+3)+1 = n(n+3)(n+1)(n+2)+1 = (n^2+3n)(n^2+3n+2)+1.$$

Let $x = n^2 + 3n$, then we have:

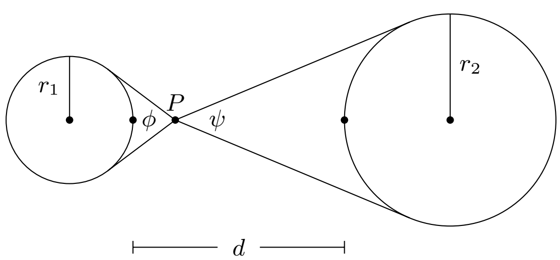
$$(n^2 + 3n)(n^2 + 3n + 2) + 1 = x(x + 2) + 1 = x^2 + 2x + 1 = (x + 1)^2.$$

Putting everything back together, we get:

$$\sqrt{n(n+1)(n+2)(n+3)+1} = \sqrt{(x+1)^2} = x + 1 = n^2 + 3n + 1.$$

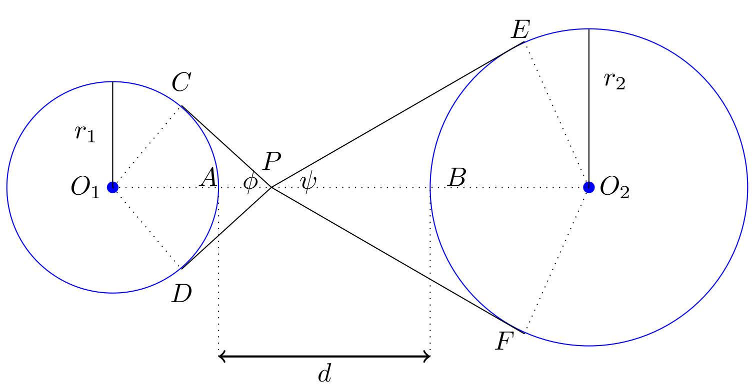
Note that this means that the square root of the product of four consecutive integers plus 1 is equal to the product of the smallest and the largest of these four consecutive integers plus 1.

MA228. Two circles with radii r_1 and of r_2 are a distance d apart from each other. A point P is to be placed on the line connecting the centers of the two circles so that the tangent lines to the circles go through the point forming angles ϕ and ψ as shown below. How far from the center of the left circle should one place the point P so that $\phi = \psi$? Write your answer in terms of r_1, r_2, and d.



Originally question 10 from the 35th University of Alabama High School Mathematics Tournament: Team Competition, 2016.

We received 3 solutions for this problem. The following is the solution by Luyu Han.



Connect O_1O_2. Denote the intersection of O_1O_2 with two circles by A and B, respectively. Denote C, D, E, and F the tangent points as shown in the figure.

Since PC and PD are tangent to circle O_1, $PC = PD$ and $\angle O_1CP = \angle O_1DP = 90^o$. On the other hand O_1P is the hypotenuse of $\triangle O_1CP$ and $\triangle O_1DP$. So, $\triangle O_1CP$ is congruent to $\triangle O_1DP$. Then we have $\angle CPO_1 = \angle DPO_1 = \frac{\phi}{2}$. Similarly, we can get $\angle EPO_2 = \angle FPO_2 = \frac{\psi}{2}$.

If $\phi = \psi$, then, in right-angled $\triangle O_1CP$ and $\triangle O_2EP$, we have $\angle CPO_1 = \angle EPO_2$. So, $\triangle O_1CP$ is similar to $\triangle O_2EP$. So,

$$\frac{O_1C}{O_2E} = \frac{O_1P}{O_2P}.$$

Let $PA = a$. Since A, P and B fall on O_1O_2, we have $PB = d - a$. Note that $O_1C = r_1$ and $O_2E = r_2$. So,

$$\frac{r_1}{r_2} = \frac{r_1 + a}{d - a + r_2}.$$

Solving this equation, we get $a = \frac{r_1 d}{r_1+r_2}$ and so

$$O_1P = r_1 + a = r_1 + \frac{r_1 d}{r_1 + r_2}.$$

MA229. Determine the largest real number t such that the two polynomials $x^4 + tx^2 + 1$ and $x^3 + tx + 1$ have a common root.

Originally from Mathematics Competitions *Vol. 25, #2 (2012),* Heaven and Earth*, heavenly problem 21.*

We received 8 submissions, 7 of which were correct. We present Amy Zhai's solution (slightly simplified).

Let $f(x) = x^4 + tx^2 + 1$ and $g(x) = x^3 + tx + 1$. A common root of $f(x)$ and $g(x)$ is a solution of $f(x) = 0$ and $g(x) = 0$. If $f(x) = g(x)$, then

$$x^4 + tx^2 + 1 = x^3 + tx + 1$$

which becomes

$$x(x-1)(x^2+t) = 0.$$

So $x = 0, 1, \pm\sqrt{-t}$.

1. When $x = 0$, $f(0) = 1$ and $g(0) = 1$. So, there is no t such that $f(x)$ and $g(x)$ have common root 0.
2. When $x = 1$, $f(1) = t + 2$ and $g(1) = 2 + t$. If $t = -2$, $f(1) = g(1) = 0$. So, when $t = -2$, $f(x)$ and $g(x)$ have common root 1.
3. When $x = \pm\sqrt{-t}$ then $f(\pm\sqrt{-t}) = g(\pm\sqrt{-t}) = 1$. So, there is no t such that $x = \pm\sqrt{-t}$ is a common root of $f(x)$ and $g(x)$.

Thus, just when $t = -2$ the two polynomials $f(x)$ and $g(x)$ have a common root (namely $x = 1$).

Editor's Comments. The reason f and g are both 1 in Case 3 is that

$$f(x) = x^2(x^2 + t) + 1 \quad \text{and} \quad g(x) = x(x^2 + t) + 1.$$

Amy's approach was most popular. A few other students instead considered

$$f(x) - xg(x) = 1 - x$$

this is 0 assuming $f(x) = g(x) = 0$, and instantly yields the candidate $x = 1$.

MA230. *Proposed by Titu Zvonaru, Comăneşti, Romania.*

Let ABC be an isosceles triangle with $AB = AC$ and AD, BE, and CF be its altitudes. A circle of diameter CE intersects the lines BC and CF at M and N, respectively. The line MN intersects the altitude AD at P. Prove that $DP = ME$.

There were 3 correct solutions, with one treating the acute triangle case only. We will present all submitted approaches.

Solution 1, by Ho Long Choi.

When $\angle BAC = 90°$, the orthocentre H, A, E, F, N, P all coincide as do M and D. The result is trivial since DP and ME coincide.

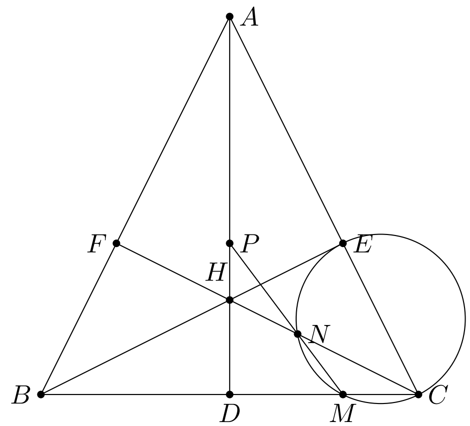
Suppose that $\angle BAC < 90°$. Observe that $HDCE$ is concyclic, having opposite right angles, and $ENMC$ is concyclic by hypothesis. Since

$$\angle PDE = \angle HDE = \angle HCE = \angle NCE = \angle NME = \angle PME,$$

the quadrilateral $PDME$ is concyclic. Since EC subtends a right angle at M,

$$90° = \angle PDM = \angle EMD = \angle DPE = \angle PEM.$$

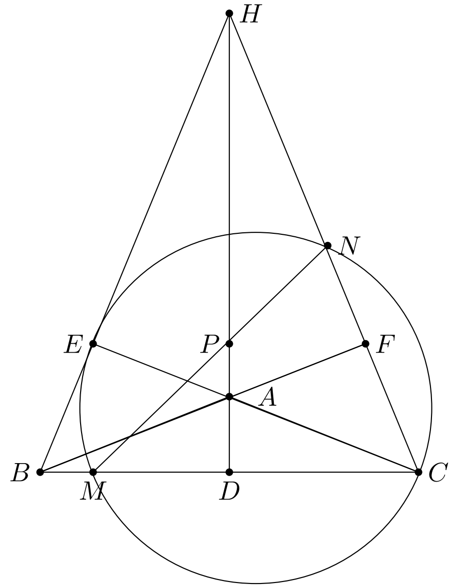
Therefore $PDME$ is a rectangle and $DP = ME$.



Suppose that $\angle BAC > 90°$. The rectangles $EBDA$, $EBCF$ and $EMCN$ are all concyclic, so that

$$\angle PDE = \angle ADE = \angle ABE = \angle FBE = \angle FCE = \angle NCE = \angle NME = \angle PME.$$

Therefore $PDME$ is concyclic with all angles right. Hence $DP = ME$.



Solution 2, by Ralf Roupec.

Using the fact that $DP \| ME$ (with transversals MP and DE), and that $ENMC$ and $HECD$ are both concyclic, we have that

$$\angle DPM = \angle PME = \angle NME = \angle NCE = \angle HCE = \angle HDE = \angle DEM,$$

so that $PEDM$ is concyclic. Therefore $PEDM$ is a rectangle and $DP = ME$.

Note that the case for $HECD$ being concyclic depends on opposite right angles when $\angle BAC < 90°$ and equal right angles subtended by HC when $\angle BAC > 90°$.

Solution 3 (for acute triangles), by Bing Jian.

Let $\angle BAC < 90°$. Since $HDCE$ and $ENMC$ are concyclic,

$$\angle PHE = \angle DCE = \angle PNE.$$

Therefore, $HNEP$ is concyclic and so $\angle HPE = \angle ENH = 90°$. Since DP is a common perpendicular to DM and PE, $PE \| DM$. Also $PD \| ME$, so that $PEDM$ is a rectangle and $DP = ME$.

PROBLEM SOLVING VIGNETTES

No. 30

Shawn Godin

Generalized Ordering

This past fall, the CMS introduced a new member to its growing family of mathematics competitions, the *Canada Lynx Mathematical Competition* (CLMC). The CLMC is a multiple choice competition, based off the Canadian grade 7 to 11 mathematics curriculum. In this issue we will look at one of the questions from the inaugural CLMC which was held on September 28, 2023.

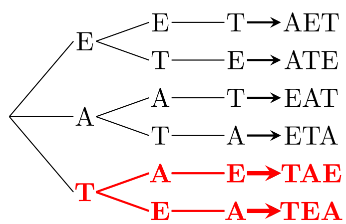
> **Question #5.** *There are 6 permutations (i.e. rearrangements) of the word EAT, namely*
>
> $$\{AET, ATE, EAT, ETA, TAE, TEA\}$$
>
> *There are 24 permutations of the word LYNX. Suppose we write these 24 permutations in alphabetical order, starting with the 1st word (LNXY) and ending with the 24th word (YXNL).*
>
> *What is the 11th word we will write down?*
>
> *(a) NXYL   (b) NYXL   (c) NYLX   (d) XLNY*

Following the method of the official solution, we note that there are $4! = 4 \times 3 \times 2 \times 1 = 24$ permutations of the letters in the word LYNX. As such, $\frac{1}{4}$ of all permutations begin with each letter. Hence, in alphabetical order, the first 6 permutations begin with L, the next 6 begin with N, the next 6 begin with X, and the last 6 begin with Y. Hence the 11th word we write down will be the 5th word in the list beginning with N. That is, the list: NLXY, NLYX, NXLY, NXYL, NYLX, NYXL. Consequently, our desired word is NYLX.

In this case, our insight saved us a little bit of work. If we had started listing things alphabetically, we would have had to list 11 words to get our answer. In our solution, we did a little calculation and only had to do 5. If the only insight we used in this problem was the one used in the solution, then the *most* we would have had to do is look through 6 words.

However, if the original word contained more letters, the savings are not so great. For example if we wanted the 65th word in the alphabetical list of permutations of the word ANGLE. In this case, there would be $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ words in our list. Of the permutations, $\frac{1}{5} \times 120 = 24$ begin with each letter. Accordingly, the first 24 words begin with A and the next 24 begin with E. Therefore, the 65th word will be the 17th word in the list of words that begin with G. At this point we could list the 17 words out and be done with it.

However, let's go back to the example in the problem and draw a tree diagram.



Notice that if we choose T to be the first letter, then the permutations that start with T are contained in a smaller subtree, shown in red in the diagram. We can use this idea to solve these types of problems recursively.

Returning to our problem, note that the 24 words that begin with G are just a G followed by a four letter word that is the permutation of the letters ANLE, which is equivalent to the original problem (i.e. we are focusing on a subtree). Of the 24 "subwords", 6 begin with each of the four letters. So the first 6 begin with A, the next 6 begin with E and hence the 17th word is the 5th that begins with L.

We could count off 5 at this point, but let's continue the pattern. There are 6 words that begin with GL and these are just the permutations of ANE, 2 starting with each of the three letters. Hence we get the next letter must be N and we are after the 1st of two words made up of A and E, which is AE. Therefore the 65th word in the list is GLNAE.

If we think of this in general we see that for a word made up of n unique letters, there are $n!$ permutations of the letters. Of the permutations, $\dfrac{n!}{n} = (n-1)!$ begin with each letter. Our job, at each step, is to decide which letter goes in the "next" position, starting with the first. In the original problem, we had 24 permutations, we wanted the 11th and 6 began with each letter. It might seem that some division by 6 is appropriate. However, let's take a closer look at all the words, their place on the list, and the ceiling function applied to what we get when we divide the place on the list by 6.

Recall that the ceiling function $\lceil x \rceil$ returns the smallest integer greater than or equal to x. Hence

$$\lceil 1.7 \rceil = 2, \qquad \lceil -4.373 \rceil = -4, \qquad \lceil 13 \rceil = 13.$$

We will see that $\lceil n \div 6 \rceil$ returns which letter, in alphabetical order, is the first letter in the nth word.

Word	Order (n)	$\lceil n \div 6 \rceil$	Word	Order (n)	$\lceil n \div 6 \rceil$
LNXY	1	1	XLNY	13	3
LNYX	2	1	XLYN	14	3
LXNY	3	1	XNLY	15	3
LXYN	4	1	XNYL	16	3
LYNX	5	1	XYLN	17	3
LYXN	6	1	XYNL	18	3
NLXY	7	2	YLNX	19	4
NLYX	8	2	YLXN	20	4
NXLY	9	2	YNLX	21	4
NXYL	10	2	YNXL	22	4
NYLX	11	2	YXLN	23	4
NYXL	12	2	YXNL	24	4

For the original problem there were 4 letters. So for each letter there is $3! = 6$ words starting with that letter. Since

$$\left\lceil \frac{11}{3!} \right\rceil = 2$$

then the 11th word starts with the 2nd letter, in alphabetical order, N. We remove the words that start with letters before this of which there are $3! = 6$. As

$$11 - 6 \times (2 - 1) = 5,$$

we are left to find the 5th word in the alphabetically arranged list of permutations of LXY. There are $2! = 2$ words starting with each of the three remaining letters. Since

$$\left\lceil \frac{5}{2!} \right\rceil = 3,$$

this tells us that the second letter is the 3rd letter that is left (alphabetically), that is, Y. Completing the process, $5 - 2 \times (3 - 1) = 1$, so the word we are after is the 1st word beginning with NY, which is NYLX, as we discovered before.

We can use the same process for ANGLE, shown below in a more compact form, Note that ℓ represents the number of letters "remaining".

Letters	Position (n)	$\left\lceil \frac{n}{(\ell-1)!} \right\rceil$	Letter
A, E, G, L, N	65	$\left\lceil \frac{65}{(5-1)!} \right\rceil = 3$	G
A, E, L, N	$65 - (3-1) \times (5-1)! = 17$	$\left\lceil \frac{17}{(4-1)!} \right\rceil = 3$	L
A, E, N	$17 - (3-1) \times (4-1)! = 5$	$\left\lceil \frac{5}{(3-1)!} \right\rceil = 3$	N
A, E	$5 - (3-1) \times (3-1)! = 1$	$\left\lceil \frac{1}{(2-1)!} \right\rceil = 1$	A
E	$1 - (1-1) \times (2-1)! = 1$	$\left\lceil \frac{1}{(1-1)!} \right\rceil = 1$	E

The reader may enjoy exploring this idea further with the following problems.

1. Determine the 357th word in the alphabetical list of permutations of the word RADIUS.
2. Reverse the process to determine the position in the list from the permutation. For example, at what position in the alphabetical list of permutations of the word ANGLE is the word GLEAN?
3. Things become more interesting with repeated letters. Determine the 100th word in the alphabetical list of permutations of the word CIRCLE.

Readers may also enjoy exploring other problems from the CLMC, or any of the other CMS competitions. These can be found on the CMS competitions web page at cms.math.ca/competitions.

OLYMPIAD CORNER

No. 419

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **March 15, 2024**.

OC661. Point N is the midpoint of side AD of a convex quadrilateral $ABCD$, and point M on side AB is such that $CM \perp BD$. Prove that if $BM > MA$, then $2BC + AD > 2CN$.

OC662. Let $a_1, \ldots, a_k$ be distinct positive integers such that the difference between the largest and smallest of them is less than 1000. What is the largest k for which it is possible that all quadratic equations $a_i x^2 + 2a_{i+1}x + a_{i+2} = 0$, where $1 \leq i \leq k-2$, have no real roots?

OC663. There are 100 cities in the Far Far Away Kingdom, and every two cities are connected by no more than one road. One day the king ordered the introduction of one-way traffic on every road, and at the same time every road was painted white or black. The Minister of Transport proudly announced that after carrying out the order, one can get from any city to any other along roads alternating their colors, and so that the first road along the way will be white. What is the smallest number of roads there could be in this country? When getting from city to city, you can pass through intermediate cities any number of times.

OC664. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$.

OC665. Let A, B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \qquad \text{and} \qquad B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mars 2024**.

OC661. Le point N est le milieu du côté AD d'un quadrilatère convexe $ABCD$, et le point M du côté AB est tel que $CM \perp BD$. Montrez que si $BM > MA$, alors $2BC + AD > 2CN$.

OC662. Soit $a_1, \ldots, a_k$ des entiers positifs distincts tels que la différence entre le plus grand et le plus petit d'entre eux est inférieure á 1000. Quel est le plus grand k pour lequel il est possible que toutes les équations quadratiques $a_i x^2 + 2a_{i+1}x + a_{i+2} = 0$, où $1 \leq i \leq k-2$, n'aient pas de racines reéles?

OC663. Il y a 100 villes dans le Royaume lointain, et toutes les paires de villes sont reliées entre elles par au plus une route. Un jour, le roi a ordonné la mise en place d'une circulation á sens unique sur toutes les routes et, du même coup, toutes les routes ont été peintes en blanc ou en noir. Le ministre des transports annonce fièrement qu'après l'exécution de l'ordre, on peut aller de n'importe quelle ville á n'importe quelle autre par des routes alternant leurs couleurs, et que la première route du chemin serait blanche. Quel est le plus petit nombre de routes qu'il puisse y avoir dans ce pays ?

Lorsque l'on se rend d'une ville á l'autre, on peut transiter par des villes intermédiaires autant de fois qu'on le souhaite.

OC664. Trouvez toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ qui ont une dérivée seconde continue et pour lesquelles l'égalité $f(7x+1) = 49f(x)$ est vérifiée pour tout $x \in \mathbb{R}$.

OC665. Soient A, B et C des matrices $n \times n$ á coefficients complexes satisfaisant aux conditions suivantes

$$A^2 = B^2 = C^2 \qquad \text{et} \qquad B^3 = ABC + 2I.$$

Montrez que $A^6 = I$.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2023: 49(6), p. 306–307.

OC636. Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

Originally Problem 3 from the 2012 Asian Pacific Mathematics Olympiad.

We received 9 submissions, 6 of which were correct and complete. We present the solution by Oliver Geupel, Brühl, NRW, Germany.

Such pairs are

$$(2,4) \quad \text{and} \quad (p,p) \text{ for every prime } p.$$

We show that there are no more solutions. In the following, we assume that (p, n) is a solution to the problem.

For $p = 2$, we have $n \notin \{1, 3\}$ by inspection, and a straightforward induction shows that $n^2 < 2^n$ when $n \geq 5$. Therefore we must have $n < 5$, which results in two valid values $n = 2$ and $n = 4$. This completes the case $p = 2$.

Next, we suppose that p is odd. Since $p^n + 1$ is even, so is $n^p + 1$, whence n is odd. We have

$$p^n + 1 = (p+1) \sum_{k=0}^{n-1} (-p)^k.$$

Hence $p + 1 \mid n^p + 1$, so that

$$n^p \equiv -1 \pmod{p+1}$$

and

$$n^{2p} \equiv 1 \pmod{p+1}. \tag{1}$$

As a consequence, the numbers n and $p + 1$ are coprime. It follows by Euler's theorem that

$$n^{\varphi(p+1)} \equiv 1 \pmod{p+1}. \tag{2}$$

Let d be the order of $n \pmod{p+1}$, that is, the smallest positive integer x such that $n^x \equiv 1 \pmod{p+1}$. Then d is a common divisor of $2p$ and $\varphi(p+1)$ by (1) and (2). Since $p + 1$ is nonprime, it holds $\varphi(p + 1) < p$; thus $d \leq 2$.

If $d = 1$, then we would have

$$-1 \equiv n^p \equiv 1 \pmod{p+1},$$

which is impossible.

Thus $d = 2$. With the notation $p = 2q + 1$, we obtain

$$n \equiv \left(n^2\right)^q \cdot n \equiv n^p \equiv -1 \pmod{p+1}. \tag{3}$$

Note that $n = 1$ is impossible. Hence $n \geq 3$. Since $p^n \leq n^p$, then $n \log p \leq p \log n$. Thus $(\log p)/p \leq (\log n)/n$. The function $f(x) = (\log x)/x$ is decreasing for $x > e$, because its derivative $f'(x) = (1 - \log x)/x^2$ is negative. We deduce that $n \leq p$. By (3), we finally conclude that $n = p$. The proof is complete.

Editor's Comments. We thank the readers for pointing out that this question has previously appeared as OC143 in ***Crux*** 39 (7). The current featured solution is an updated solution by Oliver Geupel for OC143.

OC637. For any positive integer x, we set

$$g(x) = \text{the largest odd divisor of } x,$$
$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{if } x \text{ is even;} \\ 2^{\frac{x+1}{2}} & \text{if } x \text{ is odd.} \end{cases}$$

Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_1 = 1$, $x_{n+1} = f(x_n)$. Show that the integer 2018 appears in this sequence, determine the last integer n such that $x_n = 2018$, and determine whether n with the property $x_n = 2018$ is unique.

Originally Problem 3 from the 2018 Pan African Mathematics Olympiad.

We received 5 submissions, all of which were correct and complete. We present first an intuitive solution followed by a technical solution.

Solution 1, by Theo Koupelis.

Let $x_i = 2k + 1$ be an odd, positive integer, where $i > 0$ and $k \geq 0$ are integers. Then $x_{i+1} = 2^{k+1}$ and thus $g(x_{i+1}) = 1$ and $x_{i+2} = 2^k + 2^{k+1} = 3 \cdot 2^k$.

If $k > 0$, repeating the process we get $g(x_{i+2}) = 3$ and $x_{i+3} = 3 \cdot 2^{k-1} + 2^k = 5 \cdot 2^{k-1}$.

If $k > 1$, then repeating the process we get $g(x_{i+3}) = 5$ and $x_{i+4} = 5 \cdot 2^{k-2} + 2^{k-1} = 7 \cdot 2^{k-2}$. Continuing the process until the exponent of the power of 2 in the expression for x is zero, we get $x_{i+k+2} = [2(k+1)+1] \cdot 2^0 = 2(k+1)+1$.

From the above, we see that every term in the sequence is unique; starting with an odd integer $(2k+1)$, the next term is a power of 2, namely $1 \cdot 2^{k+1}$, and then every other term in the sequence is a product of an increasing odd integer $(3, 5, 7, \ldots)$ with a power of 2 whose exponent decreases linearly, until we get a term that is the next odd integer in the sequence, $2(k+1)+1$. Therefore, there is no term that repeats, and there are $k+1$ terms between two consecutive odd integers in the sequence. Thus, $x_1 = 1$, $x_3 = 3$, $x_6 = 5$, $x_{10} = 7$, $x_{15} = 9$, $x_{21} = 11, \ldots$ with the general term given by $x_{n(n+1)/2} = 2n - 1$, with n a positive integer.

We note that $2018 = 2 \cdot 1009$, and thus, for $n = 505$ we get $x_{127765} = 1009$, and $x_{127765+505} = x_{128270} = 2 \cdot 1009 = 2018$.

Solution 2, by UCLan Problem Solving Group.

We claim that for any positive integer n and any $k \in \{1, 2, \ldots, n\}$ we have

$$x_{\binom{n}{2}+k} = (2k-1)2^{n-k}\,.$$

Above $\binom{n}{2} = n(n-1)/2$ and in the case $n = 1$, $\binom{n}{2} = 0$. We prove our claim by induction on n and for a fixed n by induction on k.

First, we establish the base case for induction on n. For $n = 1$ there is only one possible value for k, $k = 1$. In this case the claim is true: $x_{\binom{n}{2}+k} = x_1 = 1 = (2-1)2^{1-1}$.

Assume the claim is true for n and any $k = 1, 2, \ldots, n$. In particular,

$$x_{\frac{n(n+1)}{2}} = x_{\binom{n}{2}+n} = (2n-1)2^{n-n} = 2n-1\,.$$

Then we can prove our claim for $n+1$ and $k = 1$:

$$x_{\binom{n+1}{2}+1} = f(x_{\binom{n+1}{2}}) = f(x_{\frac{n(n+1)}{2}}) = f(2n-1) = 2^{\frac{(2n-1)+1}{2}} = 2^n\,.$$

The above establishes the base case for induction on k: n is fixed and $k = 1$. Assume n is fixed and that the claim is true for some $1 \le k < n$. Call $x = x_{\binom{n}{2}+k}$. Consequently,

$$x = (2k-1)2^{n-k}\,,$$

and x is even. The largest odd divisor of $(2k-1)2^{n-k}$ is $2k-1$ and $g(x) = 2k-1$. Therefore,

$$\begin{aligned} x_{\binom{n}{2}+k+1} = f(x) = \frac{x}{2} + \frac{x}{g(x)} &= (2k-1)2^{n-k-1} + 2^{n-k} \\ &= (2k+1)2^{n-k-1} = (2(k+1)-1)2^{n-(k+1)}\,. \end{aligned}$$

So the claim is true for $k+1$. Our claim is established for any positive integer n and any $k \in \{1, 2, \ldots, n\}$.

Now notice that there is a unique way to write a positive integer m in the form $\binom{n}{2} + k$ with $k \in \{1, 2, \ldots, n\}$. We have $x_m = x_{\binom{n}{2}+k} = 2018$ if and only if $(2k-1)2^{n-k} = 2 \cdot 1009$ if and only if $k = 505, n = 506$. So 2018 appears in the sequence (x_m) exactly once when $m = \binom{506}{2} + 505 = 254 \cdot 505 = 128270$.

OC638. Find all the real numbers x such that

$$\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3}$$

where $[x]$ denotes the integer part of x and $\{x\} = x - [x]$. For example $[2.5] = 2$, $\{2.5\} = 0.5$ and $[-1.7] = -2$, $\{-1.7\} = 0.3$.

Originally Problem 4 from the 2017 Pan African Mathematics Olympiad.

We received 14 submissions, 13 of which were correct and complete. We present the solution by Catherine Jian.

We notice that x must be positive since $\{x\}+\frac{1}{3}$ is always positive. Also, we claim that x cannot be an integer. Otherwise, $[x]=x$, $\{x\}=0$, and

$$\frac{1}{x}+\frac{1}{2x}=\frac{1}{3} \implies \frac{3}{2x}=\frac{1}{3} \implies x=4.5.$$

The last fact is a contradiction because 4.5 is not an integer.

Next, we know that $0<\{x\}<1$ and can do the following casework.

Case 1: $[2x]=2[x]$, $0<\{x\}<0.5$, hence

$$\frac{1}{[x]}+\frac{1}{2[x]}=\{x\}+\frac{1}{3} \implies \{x\}=\frac{3}{2[x]}-\frac{1}{3}=\frac{9-2[x]}{6[x]}.$$

Now, we have

$$0<\frac{9-2[x]}{6[x]}<0.5 \implies 0<9-2[x]<3[x] \implies \frac{9}{5}<[x]<\frac{9}{2}.$$

Since $[x]$ is an integer, $[x]$ can only be 2, 3, or 4.

Now, we can use these values of $[x]$ to find their corresponding values of $\{x\}$.

$$\begin{aligned}
[x]=2 &\implies \{x\}=\frac{5}{12} \implies x=\frac{29}{12},\\
[x]=3 &\implies \{x\}=\frac{1}{6} \implies x=\frac{19}{6},\\
[x]=4 &\implies \{x\}=\frac{1}{24} \implies x=\frac{97}{24}.
\end{aligned}$$

Case 2: $[2x]=2[x]+1$, $0.5\le\{x\}<1$, so

$$\begin{aligned}
&\frac{1}{[x]}+\frac{1}{2[x]+1}=\{x\}+\frac{1}{3}\\
\implies &\frac{3[x]+1}{2[x]^2+[x]}=\{x\}+\frac{1}{3}\\
\implies &\{x\}=\frac{3[x]+1}{2[x]^2+[x]}-\frac{1}{3}.
\end{aligned}$$

We get the inequality

$$\begin{aligned}
&\frac{1}{2}\le\frac{3[x]+1}{2[x]^2+[x]}-\frac{1}{3}<1\\
\implies &\frac{5}{6}\le\frac{3[x]+1}{2[x]^2+[x]}<\frac{4}{3}\\
\implies &\frac{5(2[x]^2+[x])}{6}\le 3[x]+1<\frac{4(2[x]^2+[x])}{3}\\
\implies &10[x]^2+5[x]\le 18[x]+6<16[x]^2+8[x].
\end{aligned}$$

We solve the first inequality $10[x]^2+5[x] \leq 18[x]+6$. Using the quadratic formula, we get

$$\frac{13-\sqrt{409}}{20} \leq [x] \leq \frac{13+\sqrt{409}}{20} < 2.$$

Since $[x]$ is an integer, $[x]$ can only be 1. However, before solving the other half of the inequality, let us plug in $[x]=1$ into the equation for $\{x\}$,

$$\{x\} = \frac{4}{3} - \frac{1}{3} = 1.$$

This is a contradiction as $\{x\}$ was assumed to be in $[0.5, 1)$. Therefore, this case leads to no solutions.

In summary, the only solutions for x are $\dfrac{29}{12}$, $\dfrac{19}{6}$, and $\dfrac{97}{24}$.

OC639. For the curve $\sin(x)+\sin(y)=1$ lying in the first quadrant, find the constant α such that

$$\lim_{x\to 0} x^\alpha \frac{d^2y}{dx^2}.$$

exists and is nonzero.

Originally Problem 7 from the 2011 Stanford Math Tournament.

We received 8 submissions, 7 of which were correct and complete. We present the solution by Oliver Geupel, Brühl, NRW, Germany.

We show that $\alpha = 3/2$.

For real functions $u(x)$ and $v(x)$ we write $u(x) \sim v(x)$ if there is a function $w(x)$ such that $u(x) = v(x)(1+w(x))$ and $\lim_{x\to 0} w(x) = 0$. It is well-known that $\sin x \sim x$.

The equation of the part of the curve that meets $x=0$ rewrites as

$$y = \begin{cases} 2k\pi + \arcsin(1-\sin x) \\ (2k+1)\pi - \arcsin(1-\sin x) \end{cases}$$

with $k = 0, 1, 2, \ldots$.

Let

$$f(x) = 2k\pi + \arcsin(1-\sin x)$$

for any $k \in \{0,1,2,\ldots\}$. Then

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -\frac{\cos x}{\sqrt{1-(1-\sin x)^2}} = -\frac{\cos x}{\sqrt{2\sin x - \sin^2 x}}$$

and

$$\begin{aligned}
\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} &= \frac{\sin x \cdot \sqrt{2\sin x - \sin^2 x} + \cos x \cdot \frac{2\cos x - 2\sin x \cos x}{2\sqrt{2\sin x - \sin^2 x}}}{2\sin x - \sin^2 x} \\
&= \frac{\sin x\left(2\sin x - \sin^2 x\right) + \left(1 - \sin^2 x\right)\left(1 - \sin x\right)}{\left(2\sin x - \sin^2 x\right)^{3/2}} \\
&= \frac{1 - \sin x + \sin^2 x}{\sin^{3/2} x(2 - \sin x)^{3/2}} \\
&\sim \frac{1}{(2\sin x)^{3/2}} \sim \frac{1}{(2x)^{3/2}}.
\end{aligned}$$

Therefore

$$\lim_{x\to 0} x^{3/2} \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} = \frac{1}{2^{3/2}}.$$

Similarly, if

$$g(x) = (2k+1)\pi - \arcsin(1 - \sin x),$$

with any $k \in \{0, 1, 2, \ldots\}$, then

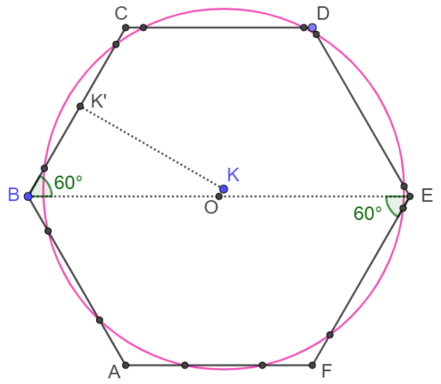
$$\lim_{x\to \infty} x^{3/2} \frac{\mathrm{d}^2 g(x)}{\mathrm{d}x^2} = -\frac{1}{2^{3/2}}.$$

Hence the result.

OC640. An equiangular hexagon has side lengths $1, 1, a, 1, 1, a$ in that order. Given that there exists a circle that intersects the hexagon at 12 distinct points, we have $M < a < N$ for some real numbers M and N. Determine the minimum possible value of the ratio $\frac{N}{M}$.

Originally Problem 4 (proposed by Yuan Yao) from the November 2017, HMMT Harvard MIT.

We received 3 submissions, 2 of which were correct and complete. We present the solution by Theo Koupelis.



All the angles of the hexagon are 60°, and $CD \parallel AF$, $BC \parallel FE$, and $AB \parallel DE$.

Let $O(0,0)$ be the origin and the vertices be $A = (-\frac{a}{2}, -\frac{\sqrt{3}}{2})$, $B = (-\frac{a+1}{2}, 0)$, $C = (-\frac{a}{2}, \frac{\sqrt{3}}{2})$, $D = (\frac{a}{2}, \frac{\sqrt{3}}{2})$, $E = (\frac{a+1}{2}, 0)$, and $F = (\frac{a}{2}, -\frac{\sqrt{3}}{2})$. Let $K(x_0, y_0)$ be the center of the circle that intersects the hexagon at 12 distinct points.

Without loss of generality, let $x_0, y_0 \geq 0$. By construction, the circle (K) must intersect each side of the hexagon at two distinct points. In order for (K) to intersect the sides CD and AF, its diameter must be greater than the distance between these two sides, and thus $2R > \sqrt{3}$. Similarly, in order for (K) to intersect the sides of length 1, its diameter must be less than the distance BE, and thus $2R < a + 1$. Therefore, $a > \sqrt{3} - 1$.

Finally, in order for (K) to intersect the segment BC, the projection K' of K on BC must be between points B and C. But the equations of the lines BC and KK' are $y_{_{BC}} = \sqrt{3}x + \frac{\sqrt{3}}{2}(a+1)$, and $y_{_{KK'}} = -\frac{1}{\sqrt{3}}x + y_0 + \frac{x_0}{\sqrt{3}}$, respectively. Therefore,

$$x_{_{K'}} = \frac{x_0}{4} + \frac{\sqrt{3}y_0}{4} - \frac{3}{8}(a+1);$$

but $-\frac{a+1}{2} < x_{_{K'}} < -\frac{a}{2}$, and thus

$$-\frac{a+1}{2} < x_0 + \sqrt{3}y_0 < \frac{3-a}{2}.$$

However, $x_0, y_0 > 0$, and thus $a < 3$. For every value $a \in (\sqrt{3} - 1, 3)$ there is a circle that intersects the hexagon at 12 distinct points.

When $a = \sqrt{3} - 1 + \epsilon$, where ϵ is a small, positive real number, the circle with center $K \equiv O$ and radius $R = \frac{\sqrt{3}}{2} + \epsilon'$ satisfies the requirement. Indeed, this circle clearly intersects the sides CD and AF at two points. Also, in this case $K' = (-\frac{3}{8}(a+1), \frac{\sqrt{3}}{8}(a+1))$ and thus $KK' = \frac{\sqrt{3}}{4}(a+1) = \frac{3}{4} + \frac{\sqrt{3}}{4}\epsilon < R$. Therefore, this circle also intersects the sides of length 1 at two points.

On the other hand, when $a = 3 - \epsilon$, a circle with center $K \equiv O$ and radius $R = \frac{\sqrt{3}}{4}(a+1) + \epsilon'$ satisfies the requirement. Indeed, as shown above, we have $R > KK'$, and thus the circle intersects the sides BC, DE, EF, and AB at two points. Also, $R > \frac{\sqrt{3}}{2}$, and thus the circle also intersects the sides CD and AF at two points.

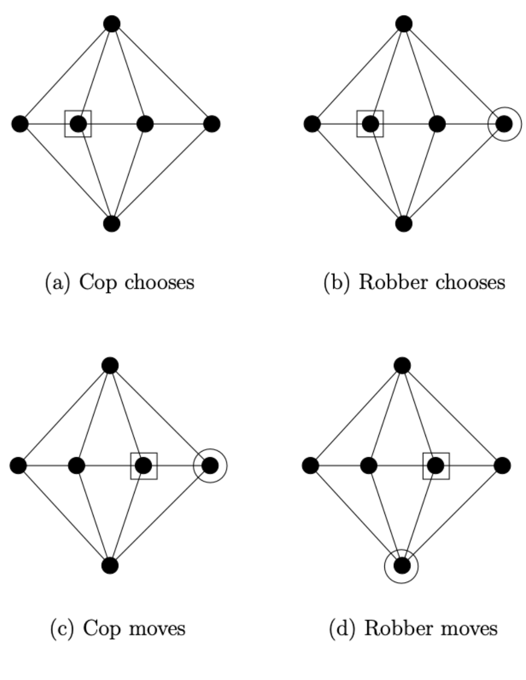
Therefore, we have $M \leq \sqrt{3} - 1$, and $N \geq 3$, and thus $\frac{N}{M} \geq \frac{3}{2} \cdot (\sqrt{3} + 1)$.

Exploring Mathematics Through the Game of Cop and Robber

Kseniya Garaschuk and Gary MacGillivray[+]

1 Introduction

Cop and Robber is a pursuit-evasion game played on a graph. There are two players: the Cop and the Robber. To start play, the Cop chooses a vertex of the graph to occupy (Figure 1(a)), and then the Robber chooses a vertex to occupy (Figure 1(b)). The two sides then alternate making moves (Cop, then Robber, then Cop, and so on). A *move* consists of either remaining at the currently occupied vertex, or travelling along an edge to an adjacent vertex (Figure 1(c) and (d)). The Cop's goal is to *catch* the Robber, that is, to occupy the same vertex as the Robber. If this ever happens, the Cop wins the game (no matter whose turn it is when it happens). The Robber wins the game if she can avoid ever being caught. The game is played with *perfect information*, which means that each player knows the graph, position of their opposite, and all options available.



(a) Cop chooses

(b) Robber chooses

(c) Cop moves

(d) Robber moves

Figure 1: Illustrating the beginning of a game

[+]supported by NSERC

Looking at the game shown in Figure 1, after the Cop has chosen her starting vertex, there is only one vertex that the Robber can choose to avoid being caught on the Cop's next move. She chooses that vertex in Figure 1(b). After the Cop travels along an edge to the position shown in Figure 1(c), the Robber has no option that will allow her to avoid capture on the Cop's next move. Since the Cop moves first, the strategy described in this paragraph guarantees that she can catch the Robber in at most three Cop's moves.

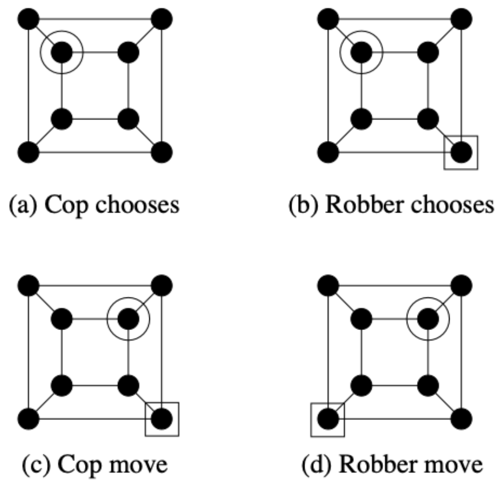


Figure 2: A game that the Robber wins

The start of another game is illustrated in Figure 2. In this one, the Robber seems to always have a move that allows her to avoid being adjacent to (on a vertex joined to the one occupied by) the Cop. Hence, it appears that she can avoid ever being caught. But we can't play the game forever, and just because the Cop has not caught the Robber by the time the players are tired does not mean that the Cop can never catch the Robber.

It is clear when the Cop has won the game because she is on the same vertex as the Robber. But how can situations in which the Robber wins be identified?

Cop and Robber was introduced, independently, by Quilliot in 1978 [14], and in a paper of Nowakowski and Winkler that was published in 1983 [12]. Since that time, there have been many papers and theses written on this game, and its variants (see [3] for a list that is current as of 2011).

Very little graph theory is needed to describe the game and do an informal analysis. It suffices to know that a *graph* consists of a finite, non-empty, collection of objects called *vertices*, which are usually represented by dots in the plane, and a collection of pairs of vertices called *edges*, which are usually represented by line segments or curves joining the vertices in each pair.

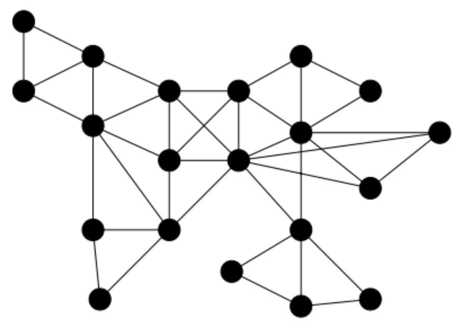


Figure 3: Do you want to be the Cop or the Robber?

Each different graph gives rise to a different version of the game. The main question to be answered is whether the Cop can catch the Robber when the game is played on the given graph. That is, *is the graph cop-win or is it robber-win?* One or the other is always true because either the Cop has a *winning strategy* or she doesn't. There turns out to be a straightforward way to decide which player has a winning strategy [12, 14], but if someone does not know a theorem that characterizes cop-win graphs, the question can be framed differently by displaying a moderately complicated graph like the one in Figure 3 and asking one of the players which role she prefers: *"Do you want to be the Cop or the Robber?"* Other questions can be asked that bring to light some more subtle aspects of the analysis of this game. Three such questions are listed below:

- Is it possible to describe the graphs on which the Cop has a winning strategy? (From above, the answer to this question is yes.)

- Suppose we can determine that the Cop can always catch the Robber. Can the *winning strategy*, ie the strategy that guarantees the Cop will win, be described?

- Given that we can't play forever, how long must we play the game before it is legitimate to conclude that the Robber will never be caught? A different way to phrase this question is to ask if there is an upper bound on the number of moves in a winning strategy.

These three questions are interrelated. Ways of approaching the first question will lead to ways of approaching the second and third questions.

2 A first analysis of the game

One analysis of the game is recursive. Think about the position of the Cop and Robber, on the Robber's move, just before the Cop catches her. In order for the Robber to be caught on the next Cop's move, every possible vertex that the Robber can occupy must be adjacent to the vertex currently occupied by the Cop. A vertex r, such as the one occupied by the Robber in the situation just described, is called a *corner*. More precisely, a vertex r is a corner if there is another vertex c adjacent to r and all the other vertices to which r is adjacent (and maybe some more vertices too). The name is intended to be descriptive of the situation: if the Robber is at r and the Cop is at c then the Robber is *corner*ed in the sense that any available move leads to the Cop being able to win on her next move. The vertex that the Robber occupies in Figure 1(c) is a corner. The extreme left-hand vertex is also a corner, but no other vertex of this graph is a corner. No vertex of the graph in Figure 2 is a corner.

If a graph G has at least two vertices and is cop-win, then it must have a corner, otherwise from any position the Robber has a move that makes it possible to avoid being caught on the next Cop's move.

Exercise 1. Suppose that x is a corner of G. Prove that G is cop-win if and only if $G - x$ (the graph obtained by deleting x and every edge that involves x) is cop-win. Hint: If $G - x$ isn't cop-win, then when playing the game on G why would the Robber ever move to x? Also, if $G - x$ is cop-win, then what describes the situation when the Cop is about to catch the Robber on that graph?

The if and only if condition gives a straightforward way to determine if a given graph G is cop-win. If G has at least two vertices and has a corner x, delete x and apply the same reasoning to $G - x$. If this procedure eventually reduces G to a single vertex, then G is cop-win. If it doesn't, then G is robber-win.

Exercise 2. Prove the assertion in the above paragraph. That is, show that G is cop-win if and only if it can be reduced to a single vertex by deleting corners.

We now have a way of *proving* that a graph is cop-win: reduce it to a single vertex by iteratively deleting corners. We also have a way of proving a graph is not cop-win: show that after some number (which may be zero) of corners are deleted, a graph with at least two vertices and no corner remains. Since a cop-win graph must have a corner, a graph with no corner must be robber-win. By the if and only if condition, a cop-win graph can not be reduced to a robber-win graph by repeatedly deleting corners, so the original graph G must be robber-win.

The graphs in Figure 1 and Figure 3 can be reduced to a single vertex by iteratively deleting corners, hence they are both cop-win. The graph in Figure 2 has no corner and therefore can not be so reduced. It is therefore robber-win.

There is a winning strategy for the Cop subtly buried in the above analysis. Finding this strategy is left as an exercise in thinking recursively.

3 A second analysis of the game

A different analysis of the game reveals a winning strategy for the Cop (if one exists) and an upper bound on the number of moves needed for the Cop to catch the Robber. The analysis arises from looking at the game beginning with what appears to be a truism: *the Cop can catch the Robber if, no matter to which vertex the Robber moves, the Cop can relocate so that fewer moves than before are needed to catch her.*

This obvious fact leads to the recognition that, after the Cop has chosen her starting vertex, the game is a sequence of *positions* (c, r), where c is the vertex occupied by the Cop and r is the vertex occupied by the Robber, *when it is the Robber's turn to move.* The Robber's first move involves choosing among all such positions given the Cop's initial location. It also leads to the recognition that the number of further Cop's moves needed to win, if possible, depends only on the current position (this assumes optimal play, that is each player always makes her best possible move).

If the Cop and Robber are located at the same vertex, then the game is over, so no more Cop's moves are needed for the Cop to catch the Robber. If the Robber's vertex r is a corner and the Cop's vertex c is adjacent to r and every other vertex besides itself adjacent to r, then one more Cop's move is needed for the Cop to catch the Robber (assuming the robber plays to make the game go on as long as possible). This suggests the idea of assigning a pair (c, r) the integer k if the Cop, located at c, can catch the Robber, located at r, in at most k more Cop's moves. We will analyse the following way of assigning non-negative integers to ordered pairs of vertices.

- For each vertex w, assign the pair (w, w) the integer 0.
- If possible, assign the unnumbered pair (c, r) the smallest integer k such that, for every vertex r_1 which either equals r or is adjacent to r, there is a vertex c_1 which either equals c or is adjacent to c, such that the pair (c_1, r_1) is numbered less than k.

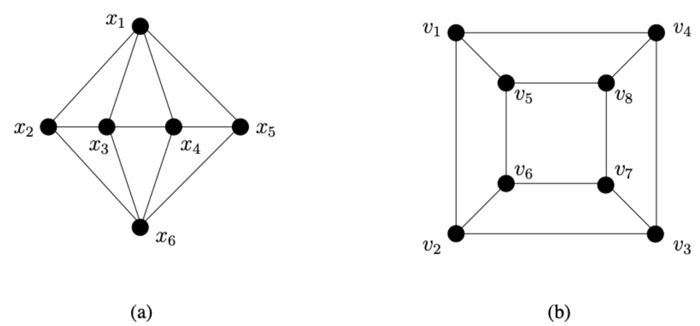


Figure 4: Graphs used to illustrate numbering positions

The numbering of positions (pairs) becomes less cumbersome to understand once some notation is introduced. We can use an $n \times n$ array whose columns and rows are indexed by the vertices of the graph, and in which the entry in column c and row r is either the number assigned to the pair (c, r) or "$-$" if the pair is unnumbered. (While it is tempting to read the first coordinate as the row index, we have deliberately chosen the same order of coordinates as is used for the Cartesian plane.) The condition "*For each vertex w, assign the pair (w, w) the integer zero*" says that the entries on the main diagonal of the array (positions $(x_1, x_1), (x_2, x_2), \ldots, (n, n)$) are all 0. For the graph in Figure 4(a), we have $n = 6$, and when this is done the array is:

$$\begin{array}{ccccccc}
x_1 & 0 & - & - & - & - & - \\
x_2 & - & 0 & - & - & - & - \\
x_3 & - & - & 0 & - & - & - \\
x_4 & - & - & - & 0 & - & - \\
x_5 & - & - & - & - & 0 & - \\
x_6 & - & - & - & - & - & 0 \\
r/c & x_1 & x_2 & x_3 & x_4 & x_5 & x_6
\end{array}$$

We will do examples before moving on to the general case.

Suppose we want to see if the pair (x_3, x_2) can be numbered 1. By the condition, this happens if the "*smallest integer such that, for every vertex r_1 which either equals x_2 or is adjacent to x_2, there is a vertex c_1 which either equals x_3 or is adjacent to x_3, such that the pair (c_1, r_1) is numbered less than* 1". Referring to Figure 4(a), the possibilities for r_1 are x_1, x_2, x_3 and x_6. We need to consider each of these in turn. Let's look at x_1 first. The first question to be answered is whether, in the row x_1 there is a pair numbered less than 1 in the column of some vertex equal or adjacent to x_3. And there is: (x_1, x_1) is numbered zero. Similarly, in each of rows x_2, x_3 and x_6 there is a pair numbered 0 in the column of a vertex adjacent to x_3. The pairs are $(x_2, x_2), (x_3, x_3)$ and (x_6, x_6), respectively. Before assigning (x_3, x_2) the number 1 we need to check that 1 is the smallest number that it can be assigned. This is true because, by definition of the numbering procedure, the pair can not be assigned 0. We can therefore put a 1 in *column* x_3 and *row* x_2, and the array becomes:

$$\begin{array}{ccccccc}
x_1 & 0 & - & - & - & - & - \\
x_2 & - & 0 & 1 & - & - & - \\
x_3 & - & - & 0 & - & - & - \\
x_4 & - & - & - & 0 & - & - \\
x_5 & - & - & - & - & 0 & - \\
x_6 & - & - & - & - & - & 0 \\
r/c & x_1 & x_2 & x_3 & x_4 & x_5 & x_6
\end{array}$$

Exercise 3. Show that pair (x_3, x_1) cannot be numbered 1.

Exercise 4. Show that the only other pair that can be numbered 1 is (x_4, x_5).

After as many vertices as possible can be numbered 0 or 1, the array is:

x_1	0	$-$	$-$	$-$	$-$	$-$
x_2	$-$	0	1	$-$	$-$	$-$
x_3	$-$	0	0	$-$	$-$	$-$
x_4	$-$	$-$	$-$	0	$-$	$-$
x_5	$-$	$-$	$-$	1	0	$-$
x_6	$-$	$-$	$-$	$-$	$-$	0
r/c	x_1	x_2	x_3	x_4	x_5	x_6

Next we consider all pairs one at a time to see if any of them can be numbered 2. The order in which they are considered does not matter because whether a pair can be numbered 2 depends only on pairs that have been numbered 0 or 1. Once all unnumbered pairs have been considered for numbering with 2 we move on to trying to number pairs 3, then 4, and so on until, finally, no more pairs can be numbered. By proceeding in this way, the question of using *the smallest* possible number for each pair is handled automatically. If we have proceeded systematically, then by the time we try to number a pair with k, we know it can not be numbered less than k and therefore k *is* the smallest number that can be assigned.

Exercise 5. Assign the number 2 to as many pairs as possible. (Hint: there are 8 such pairs.)

The array has 6^2 total entries. In general, if G has n vertices the array will have n^2 entries. Proceeding systematically, for each integer k we either fail to number any pairs or else number a pair with k. Thus the numbering process eventually stops. This can be because the array is filled in (all pairs are numbered) or because no more unnumbered pairs can be assigned a number. However, it is not necessary to number as many vertices as possible in order to determine that the Cop can win the game. Suppose there is a vertex s such that every entry in column s is a number. If the Cop is at the vertex s then, according to the meaning of the numbers (and as we will argue below) no matter what vertex the Robber is at, she will get caught. That is, we will show that *a graph is cop-win if and only if the numbering procedure eventually results in a column with no "$-$"*. In order to determine that a graph is robber-win, it is necessary to verify that this condition never holds. The only way to do that is to continue numbering vertices until no more can be numbered. The completed array with a "$-$" in each column is then a certificate that the graph is robber-win.

Suppose the Cop has started the game by choosing the vertex s, and the Robber has responded by choosing vertex r_0. If the pair (s, r_0) is numbered, then the Cop can win. The strategy is: *Move to a vertex c_0 for which is (c_0, r_0) numbered less than (s, r_0). Then after each subsequent Robber's move, move to a vertex so that the new position is numbered less than the position the game was in before the Robber moved.* The first part is possible by definition of the numbering: take $y_1 = y_0$ in the second bullet point. The second part is possible by the definition of the numbering. For any possible Robber's move there is a Cop's move to a position which is numbered lower still. Suppose the position (c_0, r_0) is numbered k and it is the Robber's turn to move. By definition of the numbering, for any vertex r_1 to which the Robber can move, there is a vertex c_1 to which the Cop can move so that the position (c_1, r_1) is numbered less than k (therefore, at most $k-1$). In turn, if (c_1, r_1) is not numbered zero then, by definition of the numbering, for every possible Robber's move from r_1 to a vertex r_2 there is Cop's move from c_1 to a vertex c_2 so that the position (c_2, r_2) is numbered less than (c_1, r_1). Continuing in this way, the sequence of positions that arise as the game is played optimally gives rise to a corresponding sequence of positive integers in which the largest term is k, and each subsequent term is smaller than its immediate predecessor. The sequence of positions therefore has as most $k+1$ terms, and must eventually terminate in a position numbered 0. That is, the Cop catches the Robber in at most k moves.

In order for the Cop to win the game, she must be able to choose an initial vertex from which she can catch the Robber (in some number of moves) no matter which initial vertex the robber chooses.

Exercise 6. Argue that the Cop can win if and only if she is able to choose a start vertex s so that the pair (s, r_0) is numbered for all possible choices of r_0. Hint: Suppose the current position is indicated in the array with a $-$. Use the fact that this pair could not be numbered to argue that the Robber has a move to a vertex so that, no matter to which vertex the Cop moves, the resulting position is also unnumbered. Why does this mean the Robber can play so as to never be caught?

In the array, a vertex as in Exercise 6 is one for which every entry in its column is a number (none are "$-$"), so it is easy to see if such a vertex exists.

Suppose the game is played on the graph in Figure 4(a), and that the Cop chooses x_3 on her first move. Since each entry in column x_3 equals 0, 1 or 2, the Cop can catch the Robber in at most 2 more moves. This agrees with our discussion of the play in Figure 1. (The vertex x_4 would also work as a start vertex.)

For the graph in Figure 4(b) the only pairs that can be numbered are (v_1, v_1), (v_2, v_2), $\ldots$, (v_8, v_8). By the discussion above, this graph is robber-win. This agrees with our analysis of the game using the method of iteratively deleting corners discussed in the previous section.

The analysis in this section can be used to calculate the maximum length of the game on any cop-win graph assuming optimal play: it is 1 (for the initial move to start the game) plus the smallest integer k such that there is a vertex s for which all entries in column s are at most k. Stated differently, if the numbering has been done systematically by assigning all possible zeros, then all possible ones, and so on, then the integer k is the number that is being assigned when the first column without a "$-$" appears. An upper bound on this maximum in a cop-win graph with n vertices follows from the numbering procedure. There are n pairs which are assigned the number 0. If all pairs are eventually numbered, then each of the integers $1, 2, \ldots, t$, where t is the maximum number assigned to a pair, is assigned to at least one pair. Since each of $n^2 - n$ pairs which are numbered greater than zero could be assigned a different number, it follows that no pair can be numbered greater than $n^2 - n$. Hence, *if the Cop can catch the Robber, then she can do so in at most $n^2 - n$ Cop's moves*, and *assuming optimal play, if the Robber has avoided capture for $n^2 - n + 1$ Cop's moves, then she will never be caught.*

The bound $n^2 - n$ is not the best possible. That is, there is no graph with n vertices in which $n^2 - n$ Cop's moves are required for the Cop to catch the Robber. It can be improved to $n^2 - n - (n-1) = n^2 - 2n + 1$ by refining the above argument slightly. By contrast, using the first analysis of the game, Nancy Clarke has proved that $n - 1$ Cop's moves suffice [5]. Is this bound is best possible? (It turns out not to be except for small graphs – see Section 8.6 of [3].)

4 Concluding remarks

An informal analysis of the game Cop and Robber has been used to talk about proofs, characterizations, recursive thinking (induction), quantifiers, relations, inequalities (bounds) and algorithms. Let us now mention some generalizations of the game and several more unsolved problems.

The game can be extended to the situation where there is more than one Cop. In this situation the proper name for the game is Cop**s** and Robber. It is described similarly. There are two sides: a group of $k > 0$ Cops and a single Robber. To start the game, a vertex is chosen for each Cop, and then the Robber chooses a vertex. The two sides move alternately as before. A move for the Cops consists of each Cop either staying at her current vertex or sliding along an edge to an adjacent vertex. The Cops win the game if any one of them ever occupies the same vertex as the Robber. There are potentially two different versions of the game: it is unknown whether there is an advantage to the Cops if more than one Cop is allowed to be on the same vertex at the same time. Clarke and MacGillivray have extended the method of numbering positions to Cops and Robber games with any fixed number of Cops [6].

Putting a Cop on each vertex of a graph guarantees that the Robber will be caught. Therefore, for each graph G there is a least number of Cops that suffice to catch the Robber. This is called the *cop number* of G. It was first introduced by Aigner and Fromme in 1984 [1]. Meniel has conjectured that essentially $\sqrt{n}$ Cops always suffice to catch the Robber if the graph is *connected* (informally, if it is possible to get between any two vertices by travelling along edges of the graph). The best known upper bound on the cop number is due to Chiniforooshan: there is a number c such that the cop number of a connected graph is always at most $c \cdot \frac{n}{\log n}$ [4].

For any fixed number of Cops it is feasible to determine whether the Cops can catch the Robber [2, 6, 8, 9, 13]. By *feasible* we mean that it is possible to write a computer program that will answer the question in a "reasonable" amount of time. On the other hand, it is not feasible to determine the cop number of a graph [7, 10]. That is, even for relatively small graphs it could take decades or worse for a computer program to find the answer.

Finally, we mention that graphs do not actually need to be finite. It is possible to consider Cops and Robber games on infinite graphs (for references, see [6]). The situation is much more complex. Many of the counter-intuitive subtleties of dealing with infinite objects quickly come into play.

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PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 15, 2024**.

4901. *Proposed by Michel Bataille.*

Let ABC be a triangle and I its incenter. Let M, N on the line BI and P, Q on the line CI be such that AM, CN (resp. AP, BQ) are perpendicular to BI (resp. CI). Prove that M, N, P, Q are concyclic and that MP is parallel to BC.

4902. *Proposed by Titu Zvonaru.*

Let O be the circumcenter of triangle ABC. Let AN be the altitude from A. Lines BO and CO intersect the lines AC and AB at E and F, respectively. Prove that if $\angle BAC = 45°$, then the lines NO and EF are perpendicular.

4903. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{2n-1}-\frac{1}{2n+1}+\frac{1}{2n+3}-\cdots\right)-\frac{1}{4n}\right].$$

4904. *Proposed by Ivan Hadinata.*

Find all pairs (x, y) of prime numbers x and y such that $x \geq y$, $x+y$ is prime and $x^x + y^y$ is divisible by $x + y$.

4905. *Proposed by Aravind Mahadevan.*

In a right-angled triangle, the acute angles x and y satisfy the following equation:

$$\tan x + \tan y + \tan^2 x + \tan^2 y + \tan^3 x + \tan^3 y = 70.$$

Find x and y.

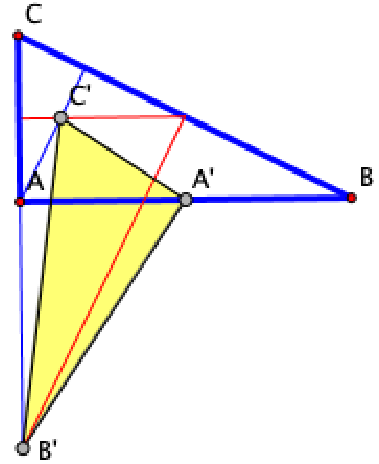
4906. *Proposed by Cristinel Mortici.*

Find positive integers m and n such that $m^3 + n + 12$ is a perfect cube and $n^2 + m + 13$ is a perfect square.

4907. *Proposed by J. Chris Fisher.*

Given triangle ABC with a right angle at A, define A' to be the midpoint of the leg AB, B' to be the point where the perpendicular bisector of BC intersects the line AC, and C' to be the point where the perpendicular bisector of AC intersects the altitude from A to BC.

Prove that $\Delta A'B'C'$ is similar to ΔABC.



4908. *Proposed by Mihaela Berindeanu.*

In the square $ABCD$, the points X, Y and Z are respectively on the segments AB, AD and AX so that $XC = XY$ and $\angle ZYX = \angle XCB$. Show that $AY \cdot ZC^2 = 2 \cdot ZB \cdot BC \cdot YZ$.

4909. *Proposed by Michel Bataille.*

For each positive integer n, let $P_n(x) = (x-1)^{2n+1}(x^2-(2n+1)x-1)$. Show that the equation $P_n(x) = 1$ has a unique solution x_n in the interval $(0, \infty)$. Prove that $\lim\limits_{n\to\infty}(x_n - 2n) = 1$ and find $\lim\limits_{n\to\infty} n(x_n - 2n - 1)$.

4910. *Proposed by Paul Bracken.* Let m and n be non-negative integers and let

$$J_{m,n} = \int_0^\infty \left(\left(\frac{\sin t}{t}\right)^m - \left(\frac{\sin t}{t}\right)^n \right) \frac{dt}{t^2}.$$

Prove that the $J_{m,n}$ are rational multiples of π.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mars 2024**.

4901. *Soumis par Michel Bataille.*

Soit ABC un triangle et I le centre de son cercle inscrit. Soit M et N des points de la droite BI, et P et Q des points de la droite CI tels que AM et CN (respectivement AP et BQ) sont perpendiculaires á BI (respectivement CI). Montrez que M, N, P et Q sont concycliques et que MP est parallèle à BC.

4902. *Soumis par Titu Zvonaru.*

Soit O le centre du centre du cercle circonscrit au triangle ABC. Soit AN la hauteur issue de A. Les droites BO et CO coupent les droites AC et AB respectivement en E et F. Montrez que si $\angle BAC = 45°$, alors les droites NO et EF sont perpendiculaires.

4903. *Soumis par Ovidiu Furdui et Alina Sîntămărian.*

Calculez

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{2n-1}-\frac{1}{2n+1}+\frac{1}{2n+3}-\cdots\right)-\frac{1}{4n}\right].$$

4904. *Soumis par Ivan Hadinata.*

Trouvez toutes les paires (x, y) de nombres premiers x et y telles que $x \geq y$, $x+y$ est premier et $x^x + y^y$ est divisible par $x + y$.

4905. *Soumis par Aravind Mahadevan.*

Dans un triangle rectangle, les angles aigus x et y vérifient l'équation suivante :

$$\tan x + \tan y + \tan^2 x + \tan^2 y + \tan^3 x + \tan^3 y = 70.$$

Trouvez x et y.

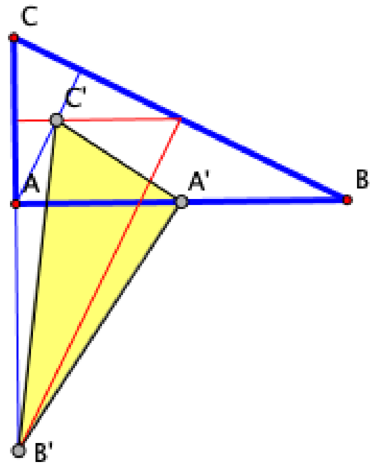
4906. *Soumis par Cristinel Mortici.*

Trouvez des entiers positifs m et n tels que $m^3 + n + 12$ est un cube parfait et $n^2 + m + 13$ est un carré parfait.

4907. *Soumis par J. Chris Fisher.*

Étant donné un triangle ABC avec un angle droit en A, soient A' le milieu du côté AB, B' le point où la médiatrice de BC coupe la droite AC, et C' le point où la médiatrice de AC coupe la hauteur du triangle ABC issue de A.

Montrez que $\Delta A'B'C'$ est semblable à ΔABC.



4908. *Soumis par Mihaela Berindeanu.*

Dans le carré $ABCD$, les points X, Y et Z sont respectivement sur les segments AB, AD et AX de sorte que $XC = XY$ et $\angle ZYX = \angle XCB$. Montrez que $AY \cdot ZC^2 = 2 \cdot ZB \cdot BC \cdot YZ$.

4909. *Soumis par Michel Bataille.*

Pour chaque entier positif n, soit $P_n(x) = (x-1)^{2n+1}(x^2-(2n+1)x-1)$. Montrez que l'équation $P_n(x) = 1$ a une solution unique x_n dans l'intervalle $(0, \infty)$. Montrez que $\lim\limits_{n\to\infty} (x_n - 2n) = 1$ et trouvez $\lim\limits_{n\to\infty} n(x_n - 2n - 1)$.

4910. *Soumis par Paul Bracken.*

Soient m et n des entiers non négatifs et soit

$$J_{m,n} = \int_0^\infty \left(\left(\frac{\sin t}{t} \right)^m - \left(\frac{\sin t}{t} \right)^n \right) \frac{dt}{t^2}.$$

Montrez que les $J_{m,n}$ sont des multiples rationnels de π.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

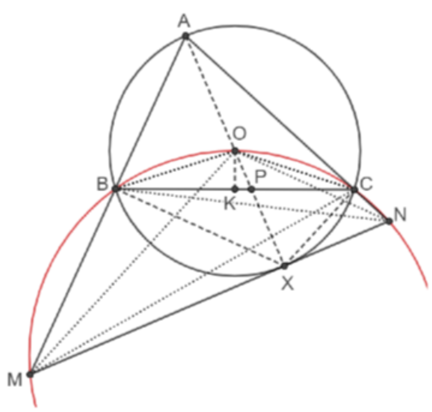
Statements of the problems in this section originally appear in 2023: 49(6), p. 322–325.

4851. *Proposed by Mihaela Berindeanu.*

Let Γ be the circumcircle of $\triangle ABC$, with circumcenter O and radius R. Point X is diametrically opposed to A, $AX \cap BC = \{P\}$ and$AP = 2PX$. The tangent to the circle Γ through X cuts AB in M and AC in N. Show that

$$2R \cdot MN + OM \cdot CN + ON \cdot BM = MC \cdot ON + BN \cdot OM.$$

All seven of the submissions we received are correct, and we feature the solution by Theo Koupelis, supplemented by the editor.



Points A, O, P, X are collinear and $2R = AX = 3PX$; thus, the circumcenter O is inside the triangle ABC. Let K be the projection of O onto BC, and assume without loss of generality that $\angle B \geq \angle C$ as in the accompanying diagram. Our initial goal is to prove that the points M, B, O, C, N are concyclic.

Because AX is a diameter of Γ, in the right triangles AXM and AXN we have $XB \perp AM$ and $XC \perp AN$. Thus,

$$AX^2 = AB \cdot AM = AC \cdot AN,$$

and so $MBCN$ is a cyclic quadrilateral; furthermore (as $AB = 2R \sin C$ by the sine law, while $AX = 2R$),

$$AM = AX^2/AB = 2R/\sin C. \tag{1}$$

Moreover, $OP = R - PX = R/3$.

Consider $\angle OPK$ in the right triangle KOP as an exterior angle of ΔAPC; we have

$$\angle KOP = 90^\circ - \angle OPK = 90^\circ - \angle C - \angle PAC = 90^\circ - \angle C - \angle XBC = \angle B - \angle C.$$

Thus, from the right triangles BKO and OKP, we get

$$OK = R \cdot \cos A = \frac{1}{3} R \cos(B - C).$$

Therefore,

$$3 \cos A = \cos(B - C) = \cos B \cos C + \sin B \sin C. \tag{2}$$

Recalling that in any triangle ABC, we have

$$\cos A = -\cos(B + C) = \sin B \sin C - \cos B \cos C,$$

add this equation to (2) to obtain

$$\sin B \cdot \sin C = 2 \cos A,$$

or

$$2R \sin B = 2 \cdot \frac{2R}{\sin C} \cdot \cos A,$$

or by (1),

$$AC = 2 \cdot AM \cdot \cos A.$$

Thus, $AM = MC$ (because $\frac{AC/2}{AM} = \cos A$ implies that the midpoint of AC is the third vertex of the right triangle whose hypotenuse is AM, so that the line MO is both the altitude and median of ΔAMC); consequently, ΔAMC is isosceles, whence $\angle AMC = 180^\circ - 2\angle A = 180^\circ - \angle BOC$. Therefore, $MBOC$ is a cyclic quadrilateral, as claimed; furthermore, the points lie on the circle in the order M, B, O, C, N, as implied by the opening paragraph.

Finally, applying Ptolemy's theorem to the cyclic quadrilaterals $BONM$ and $COMN$ we get

$$OC \cdot MN + OM \cdot CN = MC \cdot ON,$$

and

$$OB \cdot MN + ON \cdot BM = BN \cdot OM.$$

The desired result follows by addition since $OB = OC = R$.

Editor's comments. Bataille was the only correspondent to investigate the case where $\overrightarrow{AP} = -2\overrightarrow{PX}$. He observed that the required equality is not valid with that arrangement — M, B, C, N would still lie on a circle, but that circle would never contain O, in which case Ptolemy's theorem implies that the left-hand-side would always exceed the right-hand-side. He concluded (as did all solvers) that the proposer had intended the stated requirement (namely, $AP = 2PX$) to mean $\overrightarrow{AP} = 2\overrightarrow{PX}$.

4852. *Proposed by Aravind Mahadevan.*

In triangle ABC, the bisectors of angles A, B and C meet the sides BC, CA and AB at D, E and F respectively. If $\angle ADC = x$, $\angle AEB = y$ and $\angle BFC = z$, prove that $a\sin 2x + b\sin 2y + c\sin 2z = 0$ where a, b and c are the lengths of BC, CA and AB respectively.

All 18 submissions were correct, and almost all were essentially the same. We feature a typical solution by C. R. Pranesachar.

Since x measures the exterior angle at D of ΔABD, we have $x = \frac{A}{2} + B$. Therefore,

$$2x = A + 2B = (A + B + C) + (B - C) = 180^\circ + (B - C).$$

Hence,

$$\begin{aligned} a\sin 2x &= a \cdot (-\sin(B - C)) \\ &= -2R \cdot \sin A \cdot \sin(B - C) \\ &= -2R\sin(B + C) \cdot \sin(B - C) \\ &= R(\cos 2B - \cos 2C). \end{aligned}$$

Similarly,

$$b\sin 2y = R(\cos 2C - \cos 2A) \quad \text{and} \quad c\sin 2z = R(\cos 2A - \cos 2B).$$

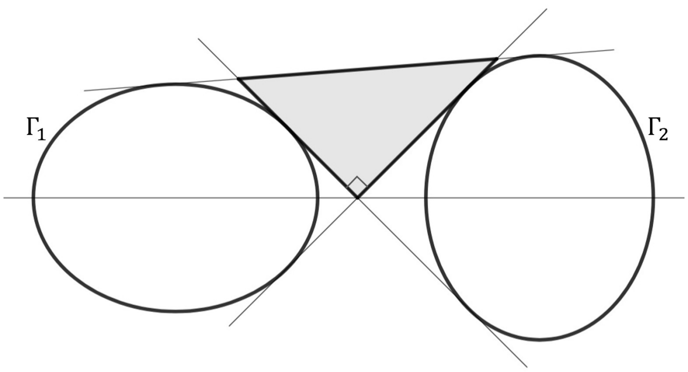
Adding, we get

$$a\sin 2x + b\sin 2y + c\sin 2z = 0.$$

This completes the proof.

4853. *Proposed by Byungjun Lee.*

Two congruent ellipses Γ_1 and Γ_2 with semi-major axis a and semi-minor axis b are given. The major axis of Γ_1 and the minor axis of Γ_2 lie on the same line, and two common internal tangents of Γ_1 and Γ_2 are perpendicular. Find the area of the triangle formed by two common internal tangents and one common external tangent.



There were 13 correct solutions to the problem from 12 solvers. The proposer provided a second solution that did not involve analytic geometry; a version of this is given as Solution 2.

Solution 1.

The area is equal to $\frac{1}{2}(a^2+b^2)$.

The common internal tangents t_1 and t_2 of Γ_1 and Γ_2 meet at a point O on the common axis of the two ellipses and each is the reflected image of the other in that axis. Since O is a point from which perpendicular tangents can be drawn to each ellipse, it is on the director (orthoptic) circle of each. The centres of the director circles coincide with the centres of the ellipses and their radii are $c=\sqrt{a^2+b^2}$.

We place O at the origin and let the equations of Γ_1 and Γ_2 be respectively

$$\frac{(x+c)^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{and} \qquad \frac{(x-c)^2}{b^2}+\frac{y^2}{a^2}=1,$$

with $c>0$. Then the equations of t_1 and t_2 are $y=sx$ and $y=-sx$ for some real number s. Since t_1 and t_2 are perpendicular, $-s^2=-1$, so we may assume that t_1 has equation $y=-x$ and t_2 equation $y=x$.

Suppose that the common external tangent t_0 has equation $y=mx+k$ where $0<m<1$. Then t_0 intersects t_1 and t_2 at the points

$$\left(\frac{-k}{1+m},\frac{k}{1+m}\right) \qquad \text{and} \qquad \left(\frac{k}{1-m},\frac{k}{1-m}\right).$$

The two legs of the right triangle formed by the three tangents have lengths $k\sqrt{2}/(1+m)$ and $k\sqrt{2}/(1-m)$ and the area of this triangle is equal to

$$\frac{1}{2}\left(\frac{2k^2}{1-m^2}\right)=\frac{k^2}{1-m^2}.$$

It remains to express this area in terms of a and b. The tangent with slope m to Γ_1 has equation $y=m(x+c)+u$ and to Γ_2 has equation $y=m(x-c)+v$, where $u^2=a^2m^2+b^2$ and $v^2=b^2m^2+a^2$. Observe that

$$u^2+v^2=(a^2+b^2)(m^2+1)=c^2(m^2+1)$$

and

$$\begin{aligned} u^2v^2 &= a^2b^2(m^4+1)+(a^4+b^4)m^2=a^2b^2(m^4+1)+(c^4-2a^2b^2)m^2\\ &= a^2b^2(1-m^2)^2+c^4m^2. \end{aligned}$$

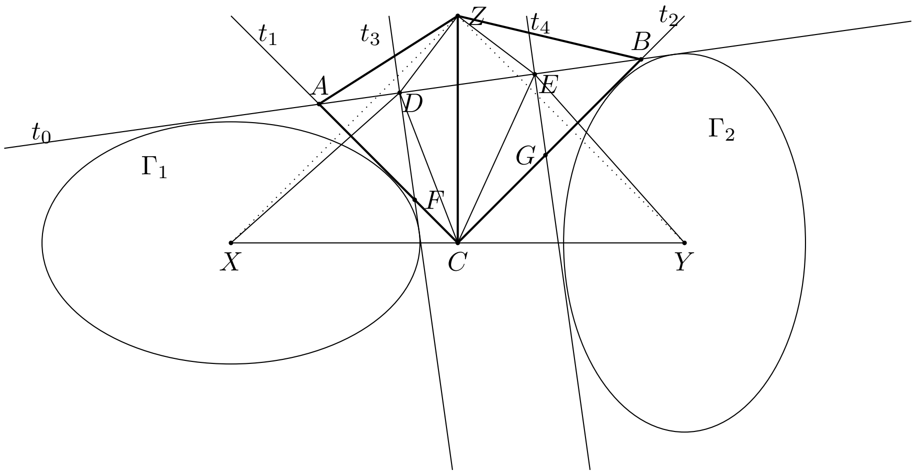
When the tangents to the two ellipses are the same line, then $mc+u=-mc+v$, so that $2mc=v-u$ and $4m^2c^2=v^2+u^2-2uv=c^2(m^2+1)-2uv$. Therefore $2uv=-(3m^2-1)c^2$. Since $k=mc+u=-mc+v$,

$$k^2=-m^2c^2+mc(v-u)+uv=-m^2c^2+2m^2c^2+\tfrac{1}{2}(-3m^2+1)c^2=\frac{1}{2}c^2(1-m^2).$$

Hence the area of the triangle is $\frac{1}{2}c^2=\frac{1}{2}(a^2+b^2)$.

Solution 2.

Let the centres of Γ_1 and Γ_2 respectively be X and Y. Let t_0 be the common external tangent above XY, t_1 and t_2 be the common internal tangents, with t_1 touching Γ_1 and t_2 touching Γ_2 above XY. Since the point C of intersection of the perpendicular tangents t_1 and t_2 lies on XY as well as the congruent director circles of Γ_1 and Γ_2, $XC = YC = \sqrt{a^2+b^2}$. Let CZ right bisect XY, where Z lies above the common axis and $CZ = \sqrt{a^2+b^2}$.



Let ρ be the counterclockwise rotation with centre Z through $90°$. Then we have

$$\rho(X) = Y,\ \rho(t_1) = t_2,\ \rho(t_2) = t_1.$$

Suppose that $t_4 = \rho(t_0)$ and $t_3 = \rho^{-1}(t_0)$ so that $\rho(t_3) = t_0$. The intersection $D = t_0 \cap t_3$, with $t_0 \perp t_3$, lies on the director circle of Γ_1. Thus $XD = XC$. Also $\rho(D) = E = t_4 \cap t_0$, the intersection of perpendicular tangents to Γ_2, so that

$$YE = YC = XC = XD.$$

Observe that $\rho(XD) = \rho(YE)$ so that $YE \perp XE$. Therefore

$$\angle DXC + \angle EYC = 90°.$$

Observe also that $\angle XDC = \angle XCD$ and $\angle YEC = \angle YCE$.

$$\begin{aligned}\angle DCE &= 180° - (\angle DCX + \angle ECY)\\ &= 180° - (90° - \tfrac{1}{2}\angle DXC + 90° - \tfrac{1}{2}\angle EYC)\\ &= \tfrac{1}{2}(\angle DXC + \angle EYC) = 45° = \angle ACZ = \angle BCZ.\end{aligned}$$

Consider the five points A, C, G, E, Z. Since $\angle AEG = \angle ACG = 90°$, $AEGC$ is concyclic. Since

$$\rho(A) = \rho(t_0 \cap t_1) = t_4 \cap t_2 = G,$$

we have $\angle AZG = 90^\circ = \angle ACG$, so that $AZGC$ is concyclic. Therefore $AZEC$ is concyclic and $\angle AZC = \angle AEC = \angle DEC$. Since $\angle ACZ = \angle DCE = 45^\circ$, triangles CAZ and CDE are similar.

Similarly, the points B, C, F, D, Z are concyclic and $\angle BZC = \angle EDC$. Since $\angle BCZ = \angle DCE = 45^\circ$, triangles CDE and CZB are similar. Therefore CAZ and CZB are similar triangles, and $CA : CZ = CZ : CB$. Hence $CA \cdot CB = CZ^2$. The area of the right triangle ABC is

$$\frac{1}{2}CA \cdot CB = \frac{1}{2}CZ^2 = \frac{1}{2}(a^2 + b^2).$$

Editor's Comments. In solution 1, we can determine m in terms of a and b. Since

$$(9m^4 - 6m^2 + 1)c^4 = 4u^2v^2 = 4a^2b^2(1 - m^2)^2 + 4c^4m^2,$$

then

$$(9m^4 - 10m^2 + 1)c^4 = 4a^2b^2(m^2 - 1)^2.$$

We know that $m = \pm 1$ satisfy this equation, so we divide by $m^2 - 1$ to get

$$(9m^2 - 1)c^4 = 4a^2b^2(m^2 - 1).$$

Solving for m^2, we get

$$m^2 = \frac{c^4 - 4a^2b^2}{9c^4 - 4a^2b^2} = \frac{(a^2 - b^2)^2}{8(a^2 + b^2)^2 + (a^2 - b^2)^2}.$$

The equations of the common external tangents are

$$(a^2 - b^2)x \pm (\sqrt{9c^4 - 4a^2b^2})y + 2c^3 = 0.$$

4854. *Proposed by Michel Bataille.*

Let n be a positive integer and let $\theta_k = \frac{k\pi}{n+1}$. For $r, s \in \{1, 2, \ldots, n\}$, evaluate

$$\sum_{j=1}^{n} (\sin\theta_{jr} + \sin\theta_{js})^2.$$

We received 7 submissions, 4 of which were correct. We present a blend of the solutions by Giuseppe Fera and the UCLan Cyprus Problem Solving Group.

First, we evaluate

$$C_k := \sum_{j=1}^{n} \cos\theta_{jk} = -\frac{(-1)^k + 1}{2} \tag{1}$$

assuming $2(n+1) \nmid k$. To prove this, let $\omega = e^{\pi i/(n+1)}$ and note that

$$\cos\theta_{jk} = \frac{\omega^{jk} + \omega^{-jk}}{2}.$$

Evaluating the finite geometric series, we have

$$C_k = \frac{1}{2}\sum_{j=1}^{n}\left(\omega^{jk}+\omega^{-jk}\right) = \frac{1}{2}\left(\frac{\omega^{(n+1)k}-\omega^k}{\omega^k-1}+\frac{\omega^{-(n+1)k}-\omega^{-k}}{\omega^{-k}-1}\right)$$

as $\omega^k \neq 1$ iff $2(n+1)\nmid k$. Since $\omega^{n+1}=-1$ and $(-1)^{-k}=(-1)^k$ we get

$$\begin{aligned} C_k &= \frac{1}{2}\left(\frac{(-1)^k-\omega^k}{\omega^k-1}+\frac{(-1)^k-\omega^{-k}}{\omega^{-k}-1}\right) \\ &= \frac{1}{2}\left(\frac{(-1)^k-\omega^k}{\omega^k-1}+\frac{(-1)^k\omega^k-1}{1-\omega^k}\right) \\ &= \frac{1}{2}\left(\frac{(1-\omega^k)\big((-1)^k+1\big)}{\omega^k-1}\right) \end{aligned}$$

proving (1). It follows that $C_k=0$ if k is odd and $C_k=-1$ if k is even (at least when $2(n+1)\nmid k$).

Now by the well-known trigonometric identity

$$\sin x\sin y = \frac{1}{2}\big(\cos(x-y)-\cos(x+y)\big)$$

and its corollary

$$\sin^2 x = \frac{1}{2}\big(1-\cos 2x\big)$$

we have

$$\begin{aligned} S &:= \sum_{j=1}^{n}\left(\sin\theta_{jr}+\sin\theta_{js}\right)^2 \\ &= \sum_{j=1}^{n}\left(\sin^2\theta_{jr}+2\sin\theta_{jr}\sin\theta_{js}+\sin^2\theta_{js}\right) \\ &= \sum_{j=1}^{n}\left(\frac{1-\cos 2\theta_{jr}}{2}+\cos(\theta_{jr}-\theta_{js})-\cos(\theta_{jr}+\theta_{js})+\frac{1-\cos 2\theta_{js}}{2}\right) \\ &= \sum_{j=1}^{n}\left(1-\frac{\cos\theta_{j2r}}{2}-\frac{\cos\theta_{j2s}}{2}+\cos\theta_{j(r-s)}-\cos\theta_{j(r+s)}\right) \\ &= n-\frac{1}{2}C_{2r}-\frac{1}{2}C_{2s}+C_{r-s}-C_{r+s}. \end{aligned}$$

Note that $2r$ and $2s$ are always even, and that $r-s$ and $r+s$ have the same parity. Furthermore, since $r,s\in\{1,2,\ldots,n\}$, none of $2r,2s,r-s,r+s$ is divisible by $2(n+1)$ except in the case $r=s$ when $r-s=0$.

Substituting (1) we get $C_{2r}=C_{2s}=-1$ and $C_{r-s}-C_{r+s}=0$ unless $r=s$, in which case $C_{r-s}-C_{r+s}=C_0-C_{2r}=n+1$ since $C_0=n$ trivially. It follows that $S=n+1$ if $r\neq s$ and $S=2(n+1)$ if $r=s$.

Editor's Comments. In terms of the Iverson bracket $[P]$, defined as 1 when P is true and 0 when P is false, we have

$$C_k = -[2 \mid k] + (n+1)[2(n+1) \mid k] \qquad (1^+)$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Using Equation (1^+), one can show that $S/(n+1) \in \{0, \frac{1}{2}, 1, 2\}$ for all $n \in \mathbb{N}$ and all $r, s \in \mathbb{Z}$ according to the pattern in Figure 5.

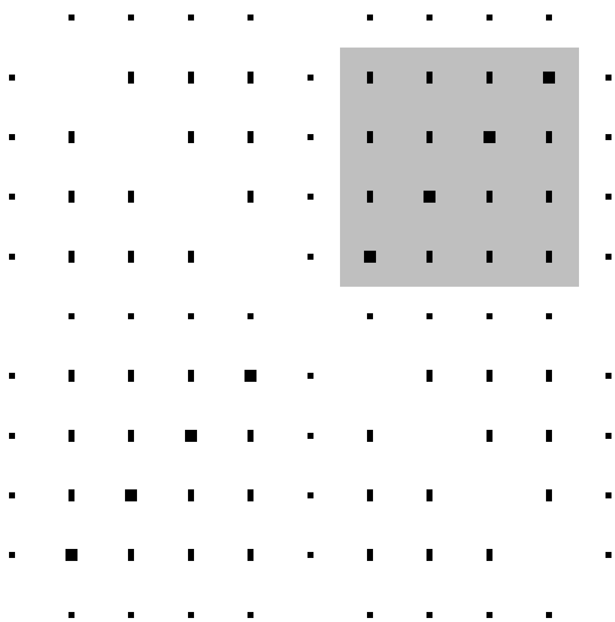


Figure 5: Plot of $S/(n+1)$ for $|r|, |s| \le n+1$. Bullet size is proportional to value $(0, \frac{1}{2}, 1, \text{ or } 2)$. Problem **4854** covered the shaded region. Here, $n = 4$.

4855. *Proposed by Ivan Hadinata.*

Find all pairs of positive integers (a, b) such that $a^b - b^a = a - b$.

We received 15 correct and 1 incomplete solutions. Almost all solutions were based on some inequalities reducing the number of considered cases to a few obvious ones. The following is the solution by UCLan Cyprus Problem Solving Group.

All pairs with $a = b$ or $a = 1$ or $b = 1$ are solutions. We will show that the only other solutions are $(3, 2)$ and $(2, 3)$.

By symmetry we may assume that $a > b \geqslant 2$. Then

$$a^b > b^a \implies b \log a > a \log b \implies \frac{b}{\log b} > \frac{a}{\log a}.$$

The function defined by $f(x) = \dfrac{x}{\log x}$ satisfies

$$f'(x) = \frac{\log x - 1}{\log^2 x}$$

so it is strictly increasing for $x > e$. So if $b \geqslant 3$ then we must have $f(a) > f(b)$, a contradiction.

So we must have $b = 2$. We observe that $a = 3$ gives a solution while $a = 4$ does not. Also, for $a > 5$ we have

$$f(a) > f(4) = \frac{4}{\log 4} = \frac{4}{\log 2^2} = \frac{2}{\log 2} = f(2) = f(b)$$

giving a contradiction.

Thus the only additional solution under these assumptions is $(3, 2)$ as claimed.

4856. *Proposed by Titu Zvonaru.*

Let ABC be a triangle with $\angle A = 30^\circ$ and $\angle B = 100^\circ$. Consider the points D and E on the sides AC and BC, respectively, such that $\angle ABD = \angle DBC$ and $DE || AB$. Find $\angle EAC$.

We received 14 submissions, all correct. We present the solution provided by Theo Koupelis.

Let $\theta = \angle EAC$. Using the law of sines in triangles AEC and ABE we get

$$\frac{\sin\theta}{EC} = \frac{\sin 50^\circ}{AE}, \qquad \text{and} \qquad \frac{\sin(30^\circ - \theta)}{BE} = \frac{\sin 100^\circ}{AE} = \frac{\cos 10^\circ}{AE}.$$

But BD is the angle bisector of $\angle ABC$ and $DE \parallel AB$, and thus

$$\frac{BE}{EC} = \frac{AD}{DC} = \frac{AB}{BC} = \frac{\sin 50^\circ}{\sin 30^\circ}.$$

From the above we get

$$\frac{\sin\theta}{\sin(30^\circ - \theta)} = \frac{\sin 30^\circ}{\cos 10^\circ} = \frac{1}{2\cos 10^\circ} = \frac{\sin 10^\circ}{\sin 20^\circ}.$$

Therefore,

$$\sin\theta \sin 20^\circ = \sin 10^\circ \sin(30^\circ - \theta)$$

or

$$\cos(20^\circ - \theta) - \cos(20^\circ + \theta) = \cos(20^\circ - \theta) - \cos(40^\circ - \theta),$$

or

$$2\sin 30^\circ \sin(\theta - 10^\circ) = 0.$$

But $0^\circ < \theta < 30^\circ$, and thus $\theta = 10^\circ$.

4857. *Proposed by Toyesh Prakash Sharma.*

Let a, b, c be positive real numbers such that $a+b+c=\dfrac{3}{2}$. Show that

$$a^a b^b + b^b c^c + c^c a^a \geq \frac{3}{2}.$$

We received 18 submissions, all correct and complete. We present two solutions, slightly altered by the editor.

Solution 1, by Arkady Alt.

By the AM-GM inequality we have

$$a^a b^b + b^b c^c + c^c a^a \geq 3\sqrt[3]{a^a b^b \cdot b^b c^c \cdot c^c a^a} = 3\left(a^a b^b c^c\right)^{2/3}.$$

Also, by the weighted AM-GM inequality

$$\frac{1}{a^a b^b c^c} = \left(\frac{1}{a}\right)^a \left(\frac{1}{b}\right)^b \left(\frac{1}{c}\right)^c \leq \left(\frac{a \cdot \dfrac{1}{a} + b \cdot \dfrac{1}{b} + c \cdot \dfrac{1}{c}}{a+b+c}\right)^{a+b+c} = \left(\frac{3}{a+b+c}\right)^{a+b+c},$$

which is equivalent to each of:

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c} = \left(\frac{1}{3} \cdot \frac{3}{2}\right)^{3/2} = \left(\frac{1}{2}\right)^{3/2} \quad \text{and} \quad \left(a^a b^b c^c\right)^{2/3} \geq \frac{1}{2}.$$

Hence,

$$a^a b^b + b^b c^c + c^c a^a \geq 3\left(a^a b^b c^c\right)^{2/3} \geq \frac{3}{2}.$$

Solution 2, by Michel Bataille.

The functions $f(x) = x \ln x$ and $g(x) = x^x$ are convex on $(0, \infty)$ (since $f''(x) = \frac{1}{x}$ and $g''(x) = \left(\frac{1}{x} + (1 + \ln x)^2\right) x^x$ are positive on $(0, \infty)$). We deduce that

$$a^a b^b = e^{f(a)+f(b)} \geq e^{2f((a+b)/2)} = (g((a+b)/2))^2$$

and therefore

$$a^a b^b + b^b c^c + c^c a^a \geq (g((a+b)/2))^2 + (g((b+c)/2))^2 + (g((c+a)/2))^2.$$

Now, using the fact that the function $x \mapsto x^2$ is increasing and convex on $(0, \infty)$ and the convexity of g, we obtain

$$\begin{aligned} a^a b^b + b^b c^c + c^c a^a &\geq 3\left(\frac{g((a+b)/2) + g((b+c)/2) + g((c+a)/2)}{3}\right)^2 \\ &\geq \frac{1}{3}\left(3g\left(\frac{a+b+c}{3}\right)\right)^2 = 3\left(g(1/2)\right)^2 = \frac{3}{2}. \end{aligned}$$

Editor's Comment. Most solutions did not mention that equality occurs only when $a = b = c$.

4858. *Proposed by Anton Mosunov.*

Prove that for every positive integer n and for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, we have

$$\int_0^{\pi} \sqrt[n]{\prod_{k=1}^{n} \csc^2(x-\alpha_k)}dx \geq 4\pi.$$

We receive two submissions and only one is correct and complete. This solution, presented here, is the proposer's one. In a second part, we reproduce the proof by the UCLan Cyprus Problem Solving group that 4π is the best lower bound for the integral.

Solution 1, by the proposer.

By Jensen's inequality,

$$\phi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \geq \frac{1}{b-a}\int_a^b \phi\left(f(x)\right)\,dx,$$

for any concave function $\phi(x)$ and any nonnegative real integrable function $f(x)$ on $[a,b]$. We take $\phi(x) = \log(x)$, so that

$$\log\left(\frac{1}{\pi}\int_0^{\pi} \sqrt[n]{\prod_{k=1}^{n}\csc^2(x-\alpha_k)}\,dx\right) \geq \frac{1}{\pi}\int_0^{\pi} \log \sqrt[n]{\prod_{k=1}^{n}\csc^2(x-\alpha_k)}\,dx.$$

Observing that

$$\int_0^{\pi} \log \sqrt[n]{\prod_{k=1}^{n}\csc^2(x-\alpha_k)}\,dx = \frac{1}{n}\sum_{k=1}^{n}\int_0^{\pi} \log\left(\csc^2(x-\alpha_k)\right)\,dx,$$

and because the functions $\log\left(\csc^2(x-\alpha_k)\right)$ has period π, the right member of the last equality is

$$\int_0^{\pi} \log\left(\csc^2(x)\right)\,dx = -2\int_0^{\pi} \log\left(\sin(x)\right)\,dx = -4I,$$

where

$$I = \int_0^{\frac{\pi}{2}} \log\left(\sin(x)\right)\,dx.$$

We claim that $I = -\frac{\pi}{2}\log 2$. To show that it is the case, observe that

$$I = \int_0^{\frac{\pi}{2}} \log\left(\cos\left(\frac{\pi}{2}-x\right)\right)\,dx = \int_0^{\frac{\pi}{2}} \log\left(\cos(x)\right)\,dx.$$

So

$$\begin{aligned}
2I &= \int_0^{\frac{\pi}{2}} \log\left(\sin(x)\right)\,dx + \int_0^{\frac{\pi}{2}} \log\left(\cos(x)\right)\,dx \\
&= \int_0^{\frac{\pi}{2}} \log\left(\sin(x)\cos(x)\right)\,dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin(2x)}{2}\right)\,dx \\
&= \int_0^{\frac{\pi}{2}} \log\left(\sin(2x)\right)\,dx - \frac{\pi}{2}\log 2 = \frac{1}{2}\int_0^{\pi} \log\left(\sin x\right)\,dx - \frac{\pi}{2}\log 2 \\
&= I - \frac{\pi}{2}\log 2.
\end{aligned}$$

Hence $I = -\frac{\pi}{2}\log 2$. We conclude that

$$\log\left(\frac{1}{\pi}\int_0^{\pi} \sqrt[n]{\prod_{k=1}^{n} \csc^2(x-\alpha_k)}\,dx\right) \geq \frac{1}{\pi}(-4I) = \log 4.$$

Exponentiating both sides of the above inequality yields

$$\int_0^{\pi} \sqrt[n]{\prod_{k=1}^{n} \csc^2(x-\alpha_k)}\,dx \geq 4\pi.$$

Proof of the lower bound condition by the UCLan Cyprus Problem Solving group, modified by the editor.

We show here that this lower bound cannot be increased.

Lemma. For any integer $m \geq 0$, $n = 2^m$, and for $x \in \mathbb{R}$, we can pick $\alpha_1, \ldots, \alpha_n$ such that,

$$\prod_{k=1}^{n} \sin^2(x-\alpha_k) = \frac{\sin^2(2^m x)}{2^{2^{m+1}-2}}.$$

Proof of the lemma. For $m = 0$ this is immediate by taking $\alpha_1 = 0$. Assume it is true for $m = r$ and let $\alpha_1, \ldots, \alpha_{2^r}$ be the choices giving

$$\prod_{k=1}^{2^r} \sin^2(x-\alpha_k) = \frac{\sin^2(2^r x)}{2^{2^{r+1}-2}}.$$

Defining $\alpha_{2^r+i} = \alpha_i + \frac{\pi}{2^{r+1}}$ for each $i = 1, \ldots, 2^r$ we get

$$\prod_{k=2^r+1}^{2^{r+1}} \sin^2(x-\alpha_k) = \frac{\sin^2\left(2^r\left(x+\frac{\pi}{2^{r+1}}\right)\right)}{2^{2^{r+1}-2}} = \frac{\sin^2\left(2^r x + \frac{\pi}{2}\right)}{2^{2^{r+1}-2}} = \frac{\cos^2(2^r x)}{2^{2^{r+1}-2}}.$$

Thus

$$\prod_{k=1}^{2^{r+1}} \sin^2(x-\alpha_k) = \frac{\sin^2(2^r x)}{2^{2^{r+1}-2}} \cdot \frac{\cos^2(2^r x)}{2^{2^{r+1}-2}} = \frac{\left(\frac{\sin\left(2^{r+1}x\right)}{2}\right)^2}{2^{2^{r+2}-4}} = \frac{\sin^2\left(2^{r+1}x\right)}{2^{2^{r+2}-2}}.$$

So the lemma follows by induction. □

Picking $\alpha_1, \ldots, \alpha_{2^m}$ as in the lemma, we get

$$I = \int_0^\pi \sqrt[2^m]{\frac{2^{2^{m+1}-2}}{\sin^2(2^m x)}}\,\mathrm{d}x = 2^{2-1/2^{m-1}} \int_0^\pi \sqrt[2^m]{\frac{1}{\sin^2(2^m x)}}\,\mathrm{d}x$$
$$= 2^{2-1/2^{m-1}} \cdot \frac{1}{2^m} \int_0^{2^m\pi} \sqrt[2^m]{\frac{1}{\sin^2 t}}\,\mathrm{d}t$$
$$= 2^{3-1/2^{m-1}} \int_0^{\pi/2} \sqrt[2^m]{\frac{1}{\sin^2 t}}\,\mathrm{d}t\,.$$

By Jordan's concave function inequality, we have $\sin(t) \geqslant \frac{2t}{\pi}$ on $[0, \pi/2]$, thus

$$I \leqslant 2^{3-1/2^{m-1}} \int_0^{\pi/2} \left(\frac{\pi^2}{4t^2}\right)^{2^{-m}} \mathrm{d}t \quad = \quad 2^{3-1/2^{m-1}} \cdot \left(\frac{\pi^2}{4}\right)^{2^{-m}} \cdot \frac{(\pi/2)^{1-2^{1-m}}}{1-2^{1-m}}$$
$$= \quad \frac{2^{2-2^{1-m}}}{1-2^{1-m}} \cdot \pi\,.$$

As m tends to infinity the right hand side tends to 4π, so the lower bound cannot be improved.

Editor's comment. For $n > 0$, let us consider the n points regular subdivision of $(0, \pi)$ defined by $\alpha_k = \frac{k\pi}{n+1}$, $1 \le k \le n$, and $g_n(x)$ the real function defined on $\mathbb{R} \setminus \pi\mathbb{Q}$ (the real numbers minus the products of rational numbers by π) by

$$g_n(x) = \sqrt[n]{\prod_{k=1}^n \csc^2(x - \alpha_k)}.$$

From

$$\log g_n(x) = -\frac{1}{n}\sum_{k=1}^n \log\sin^2(x-\alpha_k) = -\frac{1}{\pi}\cdot\frac{\pi}{n}\sum_{k=1}^n \log\sin^2(x-\alpha_k),$$

and considering the last sum as a Riemann sum, we get

$$\lim_{n\to\infty} \log g_n(x) \quad = \quad -\frac{1}{\pi}\int_0^\pi \log\sin^2(x-y)\,dy = -\frac{1}{\pi}\int_0^\pi \log\sin^2 y\,dy$$
$$= \quad -\frac{4}{\pi}\int_0^{\frac{\pi}{2}} \log\sin y\,dy = \log 4.$$

Therefore $\lim_{n\to\infty} g_n(x) = 4$,

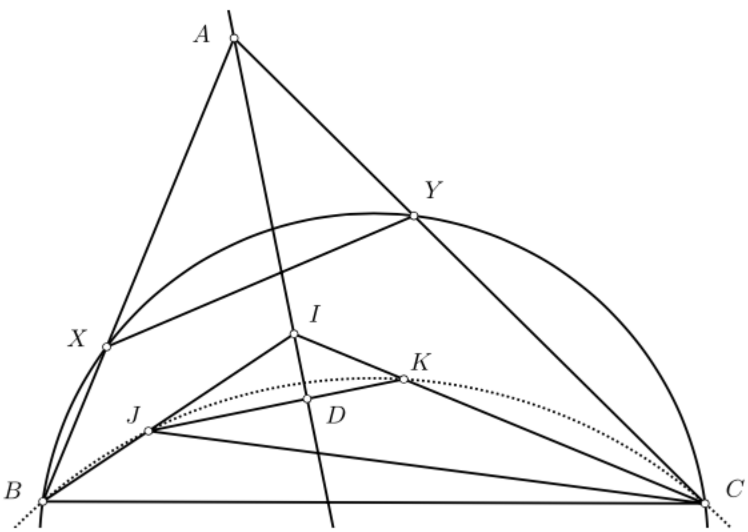
$$\lim_{n\to\infty}\int_0^\pi g_n(x)\,dx = 4\pi,$$

and 4π is the best lower bound.

4859. *Proposed by Trinh Quoc Khanh, modified by the Editorial Board.*

Given a triangle ABC, a point X on segment AB and a point Y on segment AC, such that B, X, Y, C are concyclic, let I, J, K be the incenters of triangles ABC, XBC, and YBC, respectively. Prove that AI is orthogonal to JK.

All 9 submissions were correct; we feature a composite of the similar solutions by the UCLan Cyprus Problem-Solving Group and by C.R. Pranesachar.



Since B, X, Y, C are concyclic while J and K are incenters of their respective triangles, we have

$$\angle BJC = \frac{180^\circ + \angle BXC}{2} = \frac{180^\circ + \angle BYC}{2} = \angle BKC\,.$$

It follows that B, J, K, C are concyclic. Note, moreover, that J is between B and I while K is between I and C. Thus

$$\angle IJK = \angle KCB = \angle ICB = \frac{1}{2}\widehat{C}.$$

Let D be the point of intersection of AI with JK. Note that I is between A and D. The external angle at I of ΔABI satisfies

$$\angle JID = \angle BAI + \angle IBA = \frac{\widehat{A} + \widehat{B}}{2}\,.$$

Thus, in ΔIDJ we have

$$\angle IDJ = 180^\circ - \angle IJK - \angle JID = 180^\circ - \frac{\widehat{A} + \widehat{B} + \widehat{C}}{2} = 90^\circ\,.$$

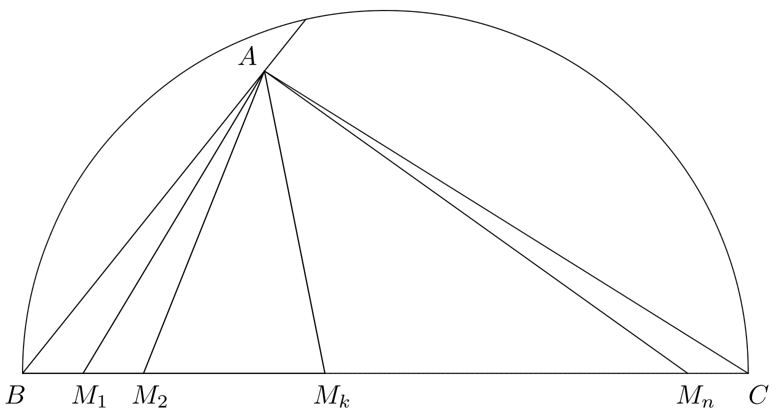
So $AI \perp JK$, as desired.

4860. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with $\angle A > 90^\circ$. Let $M_1, M_2, \ldots, M_n$ $(n \geq 1)$ be internal points on the side BC such that $BM_1 = M_1M_2 = \cdots = M_{n-1}M_n = M_nC$. Prove that

$$AM_1 + AM_2 + \cdots + AM_n < n\sqrt{\frac{2n+1}{6(n+1)}}BC.$$

We received 9 solutions, all correct and complete. We present the solution by Marie-Nicole Gras.



We put $a = BC$, $b = CA$ and $c = AB$. For all $k = 1, \ldots, n$, we have $BM_k = \dfrac{ak}{n+1}$, and using the cosine Law in $\triangle ABM_k$, we obtain

$$\begin{aligned} AM_k^2 &= AB^2 + BM_k^2 - 2AB{\cdot}BM_k \cos(\angle B) \\ &= c^2 + \frac{a^2k^2}{(n+1)^2} - 2\frac{cak}{n+1}\cos(\angle B). \end{aligned}$$

We deduce

$$\begin{aligned} \sum_{k=1}^{n} AM_k^2 &= nc^2 + \frac{a^2}{(n+1)^2}\sum_{k=1}^{n} k^2 - \frac{2ac}{(n+1)}\Big(\sum_{k=1}^{n} k\Big)\cos(\angle B) \\ &= nc^2 + \frac{a^2}{(n+1)^2}\frac{n(n+1)(2n+1)}{6} - \frac{2ac}{n+1}\frac{n(n+1)}{2}\cos(\angle B) \\ &= \frac{a^2n(2n+1)}{6(n+1)} + nc^2 - acn\cos(\angle B); \end{aligned}$$

we have $c - a\cos(\angle B) = b\cos(\angle A) < 0$, since $\angle A > 90^{\text{o}}$, which implies

$$\sum_{k=1}^{n} AM_k^2 < \frac{a^2n(2n+1)}{6(n+1)}. \tag{1}$$

The AM-QM inequality implies $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} AM_k \leq \sqrt{\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} AM_k^2}$, whence:

$$AM_1 + AM_2 + \cdots + AM_n < n\sqrt{\frac{2n+1}{6(n+1)}}\,BC.$$