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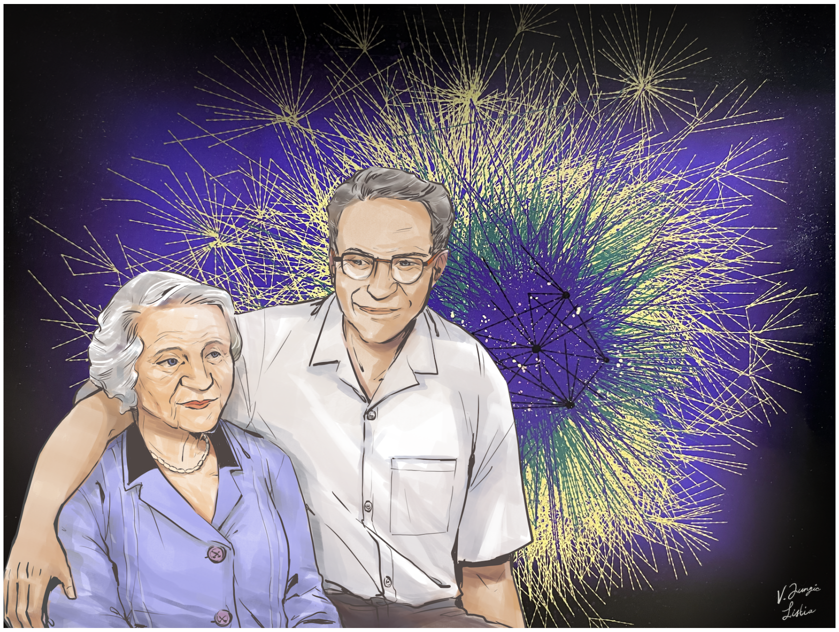
IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

I often tell my students that there are two steps towards becoming a math person: first, you should be a person and second, you should do some math. It sounds trivial, but the first step is important in many ways – we should always remember that we, our students and our teachers are humans, each with unique personalities, strengths and weaknesses, likes and dislikes. But in order to be a human, one needed to be born and raised.

May 8th marks Mother's Day in Canada this year. Without our mothers, we wouldn't be the persons that we are today, we wouldn't do math, we wouldn't even be! So let's use this opportunity to celebrate mothers in our lives and thank them for their love and support.

Kseniya Garaschuk



This drawing of Paul Erdős and his mother Anna Erdős is by Listiarini Listiarin and Veselin Jungić. The graph in the background is authors' take on Fan Chung-Graham's *Internet graph*.

MathemAttic

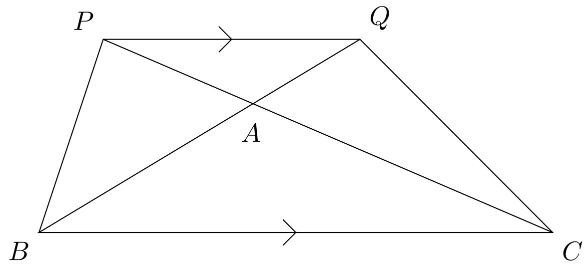
No. 35

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 15, 2022**.

MA171. $PBCQ$ is a trapezoid in which $PQ : BC = 2 : 3$. If the area of triangle ABC is 36, then determine the area of $PBCQ$.



MA172. Moe and Joe start together at point A and walk towards point B. Moe walks x times as fast as Joe. Moe reaches B, then travels back until he meets Joe. Determine the fraction of the distance AB that Joe has travelled at this point.

MA173. You are given an acute-angled triangle ABC in which J is the centre of the ascribed circle which touches BC (and touches AB and AC produced). Calculate the angle AJC in terms of the angles in the triangle.

MA174. If a, b, c are positive real numbers, find the least value of

$$\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right)$$

MA175. Given that p and q are two consecutive odd primes, show that their sum has three or more prime factors.

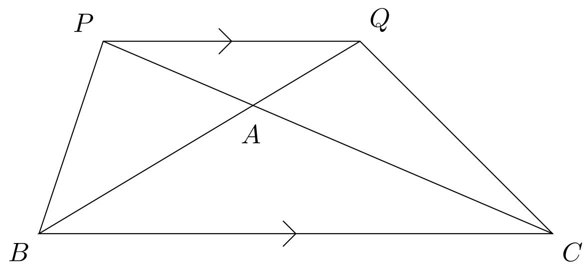
. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2022**.

MA171. $PBCQ$ est un trapèze tel que $PQ : BC = 2 : 3$. Si la surface du triangle ABC égale 36, déterminer la surface de $PBCQ$.



MA172. Marcelle et Jeannette quittent le point A au même moment, en destination du point B. Or, Marcelle marche à une vitesse x fois celle de Jeannette. Arrivant à B, Marcelle retrace ses pas jusqu'à ce qu'elle rencontre Jeannette. Déterminer la fraction de la distance AB parcourue par Jeannette en ce moment de rencontre.

MA173. Soit ABC un triangle acutangle. Soient aussi J le centre du cercle exinscrit touchant BC, les côtés AB et AC étant prolongés. Calculer l'angle AJC en termes des angles du triangle.

MA174. Si a, b et c sont des nombres réels positifs, déterminer la plus petite valeur de

$$\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right)$$

MA175. Soient p et q deux nombres premiers impairs consécutifs. Démontrer que leur somme possède au moins trois facteurs premiers.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(10), p. 475-477.

MA146. *Proposed by Alex Bloom.*

Solve the following equation for real numbers x and y:

$$(x^2+1)(y^2+1)-2(x+1)(y+1)+4=0.$$

We received 15 solutions, of which 11 were correct. There were several correct solutions that used the discriminant or derivative, but we have preferred a quick and elegant solution by a "sum of squares" technique. We present the solution by Bagheri Mehrsa.

Suppose x and y are real numbers that satisfy

$$(x^2+1)(y^2+1)-2(x+1)(y+1)+4=0.$$

Expanding and rearranging, we get

$$\begin{aligned} 0 &= (x^2y^2+x^2+y^2+1)-2(xy+x+y+1)+4 \\ &= x^2y^2+x^2+y^2+1-2xy-2x-2y-2+4 \\ &= (x^2y^2-2xy+1)+(x^2-2x+1)+(y^2-2y+1) \\ &= (xy-1)^2+(x-1)^2+(y-1)^2. \end{aligned}$$

Since a sum of squares of real numbers equals 0 if and only if each real number being squared is individually equal to 0, this yields that

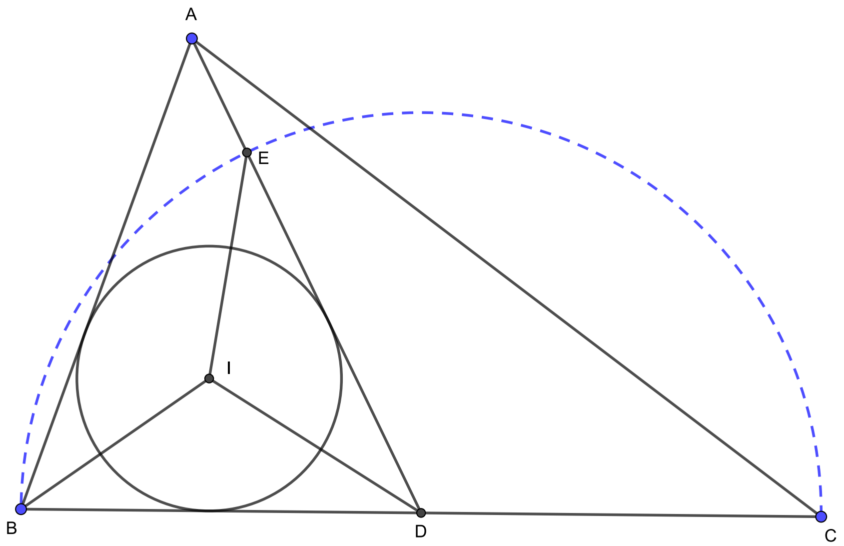
$$(xy-1)^2=(x-1)^2=(y-1)^2=0,$$

and so $x=y=1$. Substituting this into the original equation shows that $(x,y)=(1,1)$ is indeed a solution.

MA147. *Proposed by Didier Pinchon.*

Let ABC be an acute triangle, D the midpoint of BC, I the center of the incircle of triangle ABD, and E the intersection between the segment AD and the circle of diameter BC. Prove that the points A,B,E and I are concyclic.

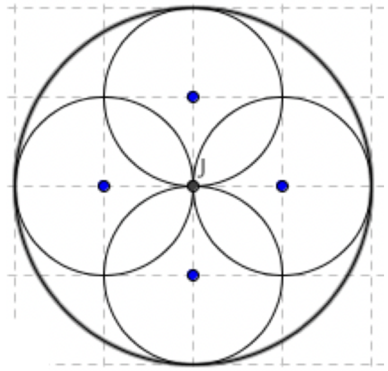
We received 7 submissions of which 4 were correct and complete. We present the solution by Aravind Mahadevan.



First we show that $\triangle IBD \cong \triangle IED$. Since I is the incentre of $\triangle ABD$, ID bisects $\angle BDE$; that is, $\angle IDB = \angle IDE$. Sides BD and DE are equal since they are radii in the circle drawn with BC as diameter. Finally, side ID is common to both triangles, so by side-angle-side $\triangle IBD \cong \triangle IED$. It follows that $\angle IBD = \angle IED$.

However, $\angle IBD = \angle IBA$ since BI is the angle bisector of $\angle ABD$. It follows that $\angle IED = \angle IBA$. Since $\angle IED$ is the opposite exterior angle to $\angle IBA$ in quadrilateral $AEIB$ we can conclude that $AEIB$ is cyclic. Therefore, A, B, E and I are concyclic points.

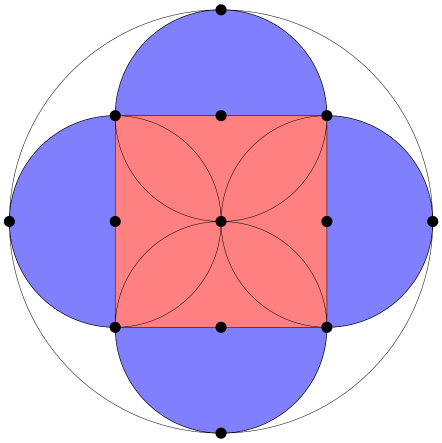
MA148. A large circle of radius 1 has centre at the point J and 4 small circles (with diameters equal to the radius of the larger circle) are drawn inside of it as shown below. Evaluate the area of the larger circle not inside any of the 4 small circles.



Originally problem 6b from the 2017 W. J. Blundon Mathematics Contest.

We received 7 solutions, out of which we present the one by Mehrsa Bagheri.

With the help of the following figure, it is easy to see that the area of the larger circle not inside any of the 4 small circles is equal to the area of the big circle (of radius 1) minus the sum of the area of two smaller circles (of radius $\frac{1}{2}$) and the area of a square with side length 1.



Hence the area of the larger circle not inside any of the 4 small circles is equal to

$$\pi - (1 + \frac{\pi}{2}) = \frac{\pi}{2} - 1.$$

MA149. Calculate

$$\frac{5^2+3}{5^2-1} + \frac{7^2+3}{7^2-1} + \frac{9^2+3}{9^2-1} + \cdots + \frac{2021^2+3}{2021^2-1}.$$

Originally problem 7b from the 2017 W. J. Blundon Mathematics Contest.

We received 16 submissions of which 11 were correct and complete. We present the solution by Alex Bloom.

In general, each term can be written as

$$\frac{n^2+3}{n^2-1} = 1 + 2\left(\frac{1}{n-1} - \frac{1}{n+1}\right),$$

so we can rewrite the sequence as

$$1 + 2\left(\frac{1}{4} - \frac{1}{6}\right) + 1 + 2\left(\frac{1}{6} - \frac{1}{8}\right) + 1 + 2\left(\frac{1}{8} - \frac{1}{10}\right) + \cdots + 1 + 2\left(\frac{1}{2020} - \frac{1}{2022}\right).$$

There are 1009 terms, so when we add the 1s together we get

$$\begin{aligned}
&1009 + 2\left(\frac{1}{4} - \frac{1}{6} + \frac{1}{6} - \frac{1}{8} + \frac{1}{8} - \frac{1}{10} + \cdots + \frac{1}{2020} - \frac{1}{2022}\right)\\
&= 1009 + 2\left(\frac{1}{4} - \frac{1}{2022}\right)\\
&= 1009 + \frac{1009}{2022} = \frac{2041207}{2022}.
\end{aligned}$$

MA150. Let us call a point an integer point if both its coordinates are integer numbers. For example, $(1, 2)$ and $(0, 5)$ are integer points, but $(1, 3/2)$ is not. What is the minimum number of integer points in the plane needed to guarantee that there is always a pair amongst them with an integer midpoint?

Originally problem 9 from the 2019 W. J. Blundon Mathematics Contest.

We received 11 *solutions, of which* 10 *were correct. We present the solution by Henry Ricardo.*

The answer is 5. To see this, we note that integer points (x_i, y_i) and (x_j, y_j) have an integer midpoint

$$\left(\frac{x_i + x_j}{2}, \frac{y_i + y_j}{2}\right)$$

if and only if $x_i + x_j$ and $y_i + y_j$ are even. This, in turn, is equivalent to saying that x_i and x_j have the same parity and y_i and y_j have the same parity.

If we define pigeonholes for integer points in the plane by the parity of the coordinates, we have the holes: (odd, odd), (odd, even), (even, odd), and (even, even). With 5 integer points, at least two must be in the same pigeonhole. That is, the two x-coordinates must have the same parity and the two y-coordinates must have the same parity and so the two points must have an integer midpoint. That this is not guaranteed for fewer than 5 integer points is shown by the counterexamples:

$$\begin{aligned}
&\{(1, 2), (2, 3)\},\\
&\{(1, 1), (2, 2), (2, 3)\},\\
&\{(1, 1), (1, 2), (2, 1), (2, 2)\}.
\end{aligned}$$

PROBLEM SOLVING VIGNETTES

No. 22

Shawn Godin

Revisiting MA108

Once a problem is solved, it is often fruitful to return to it to see if there is anything else to discover lurking beneath the surface. When the solution to MA108 appeared [2021: 47(7), p. 330], ***Crux*** editor Ed Barbeau suggested that the problem would be worth revisiting. As a reminder, here is the problem, as it appeared [2021: 47(2), p. 72]:

> **MA108.** Suppose that a, b, c and d are positive integers that satisfy the equations
>
> $$ab + cd = 38, \qquad ac + bd = 34, \qquad ad + bc = 43.$$
>
> What is the value of $a + b + c + d$?
>
> *Originally Question 29 of 2013 University of Cape Town Mathematics Competition (Grade 12).*

It was pointed out that the solution presented in ***Crux*** could be regarded as incomplete, so let's start with a complete solution. Adding the first two equations and factoring yields

$$\begin{aligned} ab + cd + ac + bd &= 72 \\ a(b + c) + d(c + b) &= 72 \\ (a + d)(b + c) &= 72 \end{aligned}$$

Similarly, adding and factoring other pairs of equations yields the following system

$$\begin{aligned} (a + d)(b + c) &= 72 \\ (a + c)(b + d) &= 81 \\ (a + b)(c + d) &= 77 \end{aligned}$$

The only ways to factor 77 into positive integers are 1×77 and 7×11. However, since our variables are positive integers, each of the factors in the system must be larger than 1. Hence we must use $77 = 7 \times 11$. Thus, if there is a solution to our original problem, we must have $a + b + c + d = 7 + 11 = 18$. Therefore, we can rewrite our system as

$$\begin{aligned} (a + d)(b + c) &= 72 = 6 \times 12 \\ (a + c)(b + d) &= 81 = 9 \times 9 \\ (a + b)(c + d) &= 77 = 7 \times 11 \end{aligned}$$

where each of the numbers on the right is factored as the product of two positive integers whose sum is 18.

It is important to notice that if any of the four variables are equal, it would produce equal numbers in the problem. However, since the values 38, 34, and 43 are unique, our four variables must be unique as well. I will leave it to interested readers to verify this.

Notice that the sum of every possible pair of variables appears as a factor in one of our equations. Suppose $A < B < C < D$ are four distinct numbers. Then we must have

$$A + B < A + C < A + D < B + D < C + D$$
$$A + C < B + C < B + D$$

This means, the smallest two sums are $A + B$ and $A + C$; and the largest two sums are $B + D$ and $C + D$. The middle two sums are $A + D$ and $B + C$, but *we do not know which is largest*! In fact, depending on the choice of numbers, $A + D < B + C$, $A + D > B + C$, and $A + D = B + C$ are all possible. I will leave it to the reader to construct sets of numbers for each of these cases.

As was noted in the solution, we can see from the original system that if (a, b, c, d) is a solution, then so are (b, a, d, c), (c, d, a, b), and (d, c, b, a). Hence, we will assume, without loss of generality, that a is the smallest of the four variables. How does this help? We know that the smallest sum is the sum of the two smallest numbers, hence since the smallest factor is 6, then it must be $a + d$, and d is the second smallest number. Similarly, the next largest factor is 7 which means $a + b = 7$ and we now know $a < d < b < c$ and we get the following system

$$\begin{aligned} a + d &= 6 \qquad & b + c &= 12 \\ a + c &= 9 \qquad & b + d &= 9 \\ a + b &= 7 \qquad & c + d &= 11 \end{aligned}$$

which yields as its only solution $(a, b, c, d) = (2, 5, 7, 4)$, which, with the permutations discussed earlier, are the solutions presented in [2021: 47(7), p. 330].

Hence we know that a solution to the original system exists with desired sum of 18, answering the question. As was pointed out in the solution, if the numbers 38, 34 and 43 from the original problem were replaced by 34, 46, and 31 we get two groups of four solutions for (a, b, c, d), namely $(1, 6, 4, 7)$ and $(2, 5, 3, 8)$, and their permutations. And for all these solutions, $a + b + c + d = 18$. What is going on?

If we go through a similar process as in the first case, we end up with the system

$$(a + d)(b + c) = 80 = 8 \times 10$$
$$(a + c)(b + d) = 65 = 5 \times 13$$
$$(a + b)(c + d) = 77 = 7 \times 11$$

where the presence of 77 forces $a + b + c + d = 7 + 11 = 18$ and the factorings shown. Again, without loss of generality we can assume that a is the smallest.

Then since 5 and 7 are the smallest factors we can determine $a < c < b < d$ and we know for sure that

$$a + c = 5 \qquad b + d = 13$$
$$a + b = 7 \qquad c + d = 11$$

However, we are not sure if $a + d > b + c$ or $a + d < b + c$. If $a + d > b + c$, then the other equations are

$$a + d = 10 \qquad b + c = 8$$

and the system has solution $(a, b, c, d) = (2, 5, 3, 8)$. Similarly, if $a+d < b+c$, then the other equations are

$$a + d = 8 \qquad b + c = 10$$

and the system has solution $(a, b, c, d) = (1, 6, 4, 7)$. Notice that in the original problem, the "middle two" sums corresponded to the factors of 81, which forces them both to be 9, yielding one solution. However, in this case there are two possibilities, each yielding a solution. This leads us to wonder how many different ways can we replace the values 38, 34, and 43 in the original problem with some other values p, q, and r and have a solution in positive integers such that $a + b + c + d = 18$?

Note that if we just permute the values amongst the equations, we end up with the same set of values for our variables but also permuted. This makes sense as for any collection of 4 distinct numbers there are $4! = 24$ ways to arrange them. In our solutions, we see that solutions come in "packets" of 4. If we combine this with the $3! = 6$ ways we can permute 38, 34, and 43, we get our $4 \times 6 = 24$ permutations. Therefore, we can assume, without loss of generality, that $p < q < r$.

Trivially, if we just pick four unique positive integers that add to 18, we can run them through the expressions to come up with the p, q, and r. We could even use non-unique values if we don't care about p, q, and r being unique. Rather than do that, let's see if we can predict anything about p, q, and r.

Hence, our system is now

$$\begin{aligned} ab + cd &= p, \\ ac + bd &= q, \\ ad + bc &= r. \end{aligned}$$

Adding and factoring as before yields

$$\begin{aligned} (a + d)(b + c) &= p + q \\ (a + c)(b + d) &= p + r \\ (a + b)(c + d) &= q + r \end{aligned}$$

where $p + q < p + r < q + r$. Since we want $a + b + c + d = 18$, we must have

$$\begin{aligned} p + q &= f(x) \\ p + r &= f(y) \\ q + r &= f(z) \end{aligned}$$

where $f(x) = x(18 - x)$ and, without loss of generality, $2 \leq x < y < z \leq 9$ (we can even raise the lower limit to 3 if we want unique values). From this, we can solve and get

$$\begin{aligned}
p &= \frac{f(x) + f(y) - f(z)}{2}, \\
q &= \frac{f(x) - f(y) + f(z)}{2}, \\
r &= \frac{-f(x) + f(y) + f(z)}{2}.
\end{aligned}$$

The reader may enjoy verifying that $x = 6$, $y = 9$, $z = 7$ reproduces the original problem and $x = 8$, $y = 5$, $z = 7$ reproduces the other case we considered.

Note, however, that p, q, and r are positive integers so $f(x) + f(y) - f(z)$, $f(x) - f(y) + f(z)$, and $-f(x) + f(y) + f(z)$ should all be divisible by 2. It is easy to check that x and $f(x)$ have the same parity. Hence x, y, and z must either be all even, or two of them are odd and the other is even.

Does that mean that if we pick x, y, and z according to our conditions then we will have a solvable system? Furthermore, if $z = 9$, so that the "middle two" sums are both 9, will be have one set of four solutions and otherwise have two? Not quite. If we choose $x = 3$, $y = 6$ and $z = 9$ there are no solutions that satisfy the conditions of the problem. Similarly, $x = 3$, $y = 7$ and $z = 8$ only yields one set of four solutions while $x = 3$, $y = 5$ and $z = 8$ has none. However if we relax our condition in the original problem from positive integers to just integers, our predicted number of solutions holds. The reader may enjoy showing that any integers $x < y < z < 9$ will yield a system with two families of four solutions while $x < y < z = 9$ will yield one family of four.

This leaves us to explore other ideas with the original constraint of positive integers:

- What is the smallest possible sum $a + b + c + d$ that can be achieved by systems of this sort?
- Can we predict which combinations of x, y, and z will produce non-positive values in a solution?
- Are there values of $a + b + c + d$ where all solutions come in two groups of four?

We can think about generalizing the idea behind the problem but, unfortunately, the number of variables and equations increases, and the "fuzziness" about the ordering of the sums of collections of terms becomes more difficult to navigate. Braver and more patient readers may enjoy giving it a go. Have fun exploring this problem further.

MATHEMATICS FROM THE WEB

No. 1

This column features short commentaries or descriptions of mathematical items from the internet that may be of interest to high school students and teachers. Your contributions are welcomed and may be sent to `mathemattic@cms.math.ca`.

Snow art in New Brunswick
`https://www.cbc.ca/news/canada/new-brunswick/snow-art-nb-1.6358785`
While the cold winter months find many of us tucked away warm in our homes, John Panopoulos and Charmaigne Letourneau enjoy stamping out mathematical art in the snow. See some of their beautiful work and read about their process in this article from the CBC. The article includes an embedded time-lapse video of one of the pieces being created.

Ooodle
`https://mathszone.co.uk/resources/grid/ooodle`
Are you looking for a mathematical parallel to the Wordle craze? Give Ooodle a try to scratch that itch. The rules are very similar to Wordle but the letters have been replaced with digits and the challenge is to determine the mystery numerical expression. Ooodle is part of a larger site of fill in the grid challenges from Maths Zone (`https://mathszone.co.uk`). Check out the different challenges available on the site that are broadly separated into two categories: fluency and reasoning. (Submitted by Paul Alves, Resource Teacher - Mathematical Literacy, Peel District School Board, Mississauga Ontario.)

Can you solve this tricky sum from Indonesia?
`https://youtu.be/VO4QElbHv-0`
The YouTube channel *Mind Your Decisions* quite often contains problems that would be of interest to the readers of *MathemAttic*. The problem in the video is to solve the sum

$$\frac{1}{10^{-9}+1}+\frac{1}{10^{-8}+1}+\cdots+\frac{1}{10^{8}+1}+\frac{1}{10^{9}+1}$$

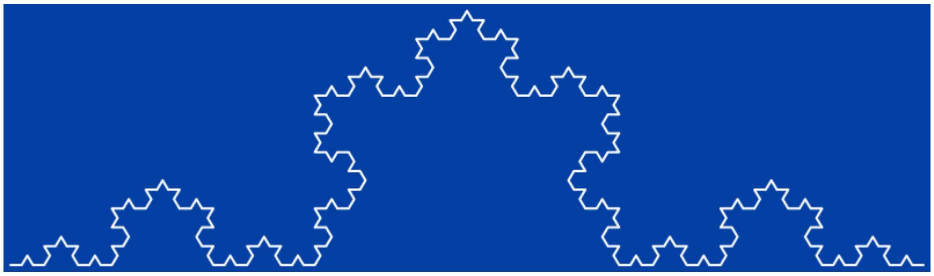
The video uses one method to solve the problem, different from the approach I would have used. It is nice to look in the discussions of the videos because you will usually see some other solutions (including the method that I would have used).

Google's 'DeepMind' does Mathematics
`https://www.numberphile.com/podcast/deep-mind`
For those interested in hearing about the impact AI is having on the work of mathematicians, check out this recent episode of the *Numberphile* podcast around AI and math. The episode explores the impact of Google's 'DeepMind' on the work of mathematicians - including the skepticism and embrace of this new tool

in mathematics. The AI teases out patterns that may have gone undetected in previous work that the mathematicians then use as the basis for their work.
(Submitted by Paul Alves, Resource Teacher - Mathematical Literacy, Peel District School Board, Mississauga Ontario.)

MacTutor History of Mathematics Archive
`https://mathshistory.st-andrews.ac.uk/`
Want to know more about your favourite mathematician? Curious about which mathematicians were born or died on today's date? Aching for some history of π? Then the MacTutor History of Mathematics Archive is for you. Containing over 3000 biographies of mathematicians, numerous essays on historical topics, and an index of famous mathematical curves, the archive contains a wealth of material for the teacher and interested student alike.

Online Math Tools – L-System Generator
`https://onlinemathtools.com/l-system-generator`
An L-system, also called the Lindenmayer system, is an alphabetic system for turtle geometry. It uses a system of symbols such as F (move forward one unit, tracing your path), f (move forward one unit), $+$ (rotate counter-clockwise by a fixed angle), and $-$ (rotate clockwise by a fixed angle). A replacement rule, or rules, are defined to replace one set of rules with another. For example, starting with F and using the replacement rule $F = F + F - -F + F$ where the angle of rotation is $60°$, the original line segment is replaced by the shape of a segment with an equilateral triangular "bump" in it. As this process is iterated fractal-like objects are created. Four iterations of the rule above give the Koch curve shown below (with width of 800 and height of 231). Other examples are given, have fun creating your own!



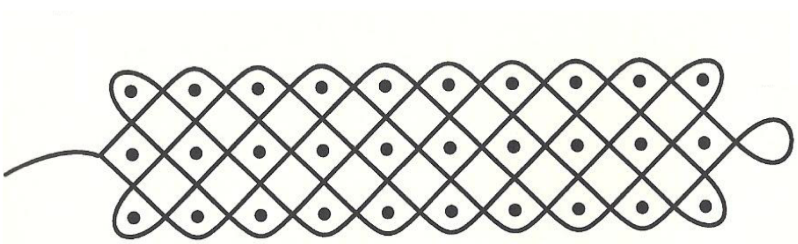
From the bookshelf of ...

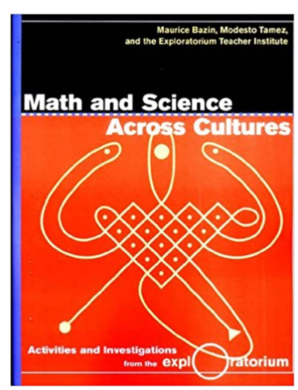
Kseniya Garaschuk

This new feature of MathemAttic *brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to `MathemAttic@cms.math.ca`. Publishers are also welcome to send along books for possible review.*

Math and Science Across Cultures: Activities and Investigations from the Exploratorium
by Maurice Bazin, Modesta Tamez and The Exploratorium Teacher Institute
ISBN 978-1565845411, 196 pages
Published by The New Press, New York, 2003.

What is the history of dye colours? How do you collect water in the desert? Why are there feathers on arrows? Can you make the following figure with just one continuous line?





I don't exactly remember who recommended this book to me or when I got it, but it's been on my shelf for a few years now: "Math and Science Across Cultures" by Maurice Bazin, Modesta Tamez and the Exploratorium Teacher Institute. Like I do with many of my math books, I occasionally grab it off the shelf, find something that catches my eye and dive in. I generally like math books arranged in self-contained chapters, so I can flip through and find something I am currently in the mood for – this book is no exception with its 14 distinct units. What is exceptional about this book though is the breadth and depth of material in terms of meaningful cultural presentations, diverse tasks and contexts, and, of course, mathematics.

The book was originally written for teachers, which is immediately clear: each unit has a list of materials needed, suggested time for activity and age of participants.

Furthermore, the activities are laid out with guiding questions to be posed in the classroom, with advice on how one might approach certain tasks. Each unit comes with recommended resources for further reading. At the end of the book, you will also find a detailed table connecting each unit to the National Council of Teachers of Mathematics math content standards.

While written for teachers, the book doesn't follow rigid lesson plan templates or overwhelm a non-teacher reader with targeted curriculum competencies. The units are written holistically as hands-on explorations, each in its own style, so that anyone is able to get engaged and follow. Some are written as narratives with proposed in-text questions to ponder, while others have step by step instructions for the exploration (very useful for things such as wool dyeing!). There really is no age restriction on the material due to the rich meaningful introduction of each topic and the open ended nature of suggested investigations. My 5-year-old daughter and I explored sona sand drawings and basket weaving, while students in my 4th year history of math course worked with quipus to learn about Inca counting.

What I really enjoy about this book is that it offers activities from a wide variety of cultures and, for each one, it provides a thoughtful context rooted in cultural traditions. The authors don't rush into exploring science connected to the curriculum; rather, they take the time to describe the cultural background, how it arose in the daily lives of the people and the world they live or lived in, and why it was important. Each unit provides a wholesome and engaging opening, leaving you wanting to learn more about Brazilian carnavals and Chinese tea traditions.

The book addresses one very important disconnect present in many math books when presenting real-life applications, namely the lack of connection to the physical materials related to the task. How often do we say 'consider the width of metal/paper/cardboard to be negligible' when solving a standard calculus problem of constructing a box or a can with some desired properties, such as maximum volume or minimal surface area? The application itself seems authentic enough: industry manufacturers truly have production constraints and wish to optimize material usage. But the habit of making simplifying assumptions without discussing them results in unexpected bafflement when we are faced with the actual task of building an object. This is what makes this book such a valuable resource as it makes you actually do things!

Take the Weaving Baskets unit. It has lots of pictures to guide you through creation of different kinds of weaves, so my daughter and I read through it, decided on a basket we wanted to build and set to work. First of all, preparation of the materials (strips of paper from recycled magazines in our case) took much longer than expected – you need lots of strips. Thinking of Scottish basket makers that use willow and Coast Salish people that use cedar bark, we were able to appreciate how much effort goes into collecting and preparing the materials before the weaving even begins. Then came the weaving. Our strips did not behave like the ones in the pictures: they slid around, didn't bend as expected and escaped from our novice hands. We had trouble sticking to the pattern as the simple 'over-under' didn't work out so simply with actual strips and we had to constantly re-check our work. No amount of reading about the craftsmanship will make you appreciate the difficulty of the work and the value of the artistry the way you do once you try it yourself. And then of course the time... our little basket took a couple of hours to make, so even accounting for an expert being a lot more efficient than we were, this is serious time commitment to produce something that can carry a non-trivial amount of things.

All in all, this book combines meaningful cultural context and inviting hands-on explorations of various math and science concepts. Work through it by yourself, with your students, friends or kids. Find a unit that speaks to you and get hooked on it.

This book is a recommendation from the bookshelf of Kseniya Garaschuk. Kseniya is an Assistant Professor at the University of the Fraser Valley and Editor-in-Chief of ***Crux***. She has been involved in outreach since grad school, having founded and organized several math camps and other outreach events. In her day job, Kseniya teaches math and pursues research in applied math education. Her current scholarly activities revolve around facilitating opportunities for student-created content and effective peer feedback exchange. Depending on when you ask, she lives in Abbotsford, BC or Ottawa, Ontario with her husband, their daughter and one menace of a dog.

From the lecture notes of ...

Andrijana Burazin, Veselin Jungić and Miroslav Lovrić

In the theory of sound waves, a sawtooth wave is described as a wave that ramps upwards and then sharply drops; see Figure 1 (left). The problem we propose relates to a modification of a sawtooth wave, in which the ramp (rise) is circular, and the amplitudes are limited by a specific line; see Figure 1 (right). This problem, suitable for a calculus class for science or mathematics majors, combines geometry, trigonometry, proof by induction and the sum of a geometric series.

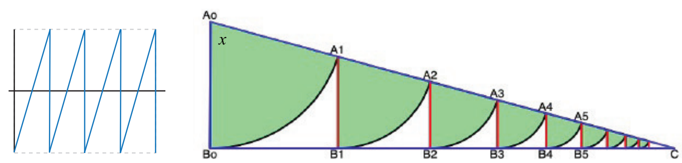


Figure 1. A sawtooth wave (left) and a modified sawtooth wave (right).

Start with a a right triangle ΔA_0B_0C whose hypotenuse $\overline{A_0C}$ is of length 1, and the angle $\angle CA_0B_0 = x$, where $0 < x < \pi/2$. Next, draw a circular arc centred at A_0 of radius $\overline{A_0B_0}$ and mark its intersection with the hypotenuse by A_1. Drop the perpendicular $\overline{A_1B_1}$ from the point A_1 onto the side $\overline{B_0C}$. Repeat: draw a circular arc centred at A_1 of radius $\overline{A_1B_1}$ and mark its intersection with the hypotenuse by A_2. Drop the perpendicular $\overline{A_2B_2}$ onto the side $\overline{B_0C}$.

By continuing this process, we will construct two sequences of points, $A_i \in \overline{A_0C}$ and $B_i \in \overline{B_0C}$ so that

$$|\overline{A_iA_{i+1}}| = |\overline{A_iB_i}| \quad \text{and} \quad \overline{A_iB_i} \perp \overline{B_0C}$$

where $i = 0, 1, 2, \ldots$. See Figure 1 (right).

Problem

Label the lengths $a_i = |\overline{A_iC}|$, $b_i = |\overline{A_iB_i}|$, and the arc length $c_i = |\widehat{B_iA_{i+1}}|$, where $i \geq 0$. Let d_i denote the area of the sector $B_iA_iA_{i+1}$. Find the following four quantities:

$$a = \sum_{i=0}^{\infty} a_i, \quad b = \sum_{i=0}^{\infty} b_i, \quad c = \sum_{i=0}^{\infty} c_i, \quad \text{and} \quad d = \sum_{i=0}^{\infty} d_i.$$

Solution. We observe that $a_0 = |\overline{A_0C}| = 1$ and, from $\cos x = |\overline{A_0B_0}|/|\overline{A_0C}| = b_0/a_0$, we obtain

$$b_0 = a_0 \cos x = \cos x.$$

Using the formula for the length of a circular arc (the radius times the angle in radians), we find that

$$c_0 = |\widehat{B_0A_1}| = |\overline{A_0B_0}| \cdot x = b_0 x = x \cos x.$$

From the formula for the area of a circle sector (1/2 times the angle in radians times the radius squared) we obtain

$$d_0 = \frac{1}{2}\, x\, |\overline{A_0B_0}|^2 = \frac{1}{2}\, x b_0^2 = \frac{1}{2}\, x \cos^2 x$$

Continue in the same way:

$$\begin{aligned}
a_1 &= |\overline{A_1C}| = |\overline{A_0C}| - |\overline{A_0A_1}| = |\overline{A_0C}| - |\overline{A_0B_0}| = a_0 - b_0 = 1 - \cos x \\
b_1 &= |\overline{A_1B_1}| = |\overline{A_1C}| \cos x = a_1 \cos x = \cos x(1 - \cos x) \\
c_1 &= |\widehat{B_1A_2}| = |\overline{A_1B_1}| \cdot x = b_1 x = x \cos x(1 - \cos x) \\
d_1 &= \frac{1}{2}\, x\, |\overline{A_1B_1}|^2 = \frac{1}{2}\, x b_1^2 = \frac{1}{2}\, x \cos^2 x(1 - \cos x)^2
\end{aligned}$$

Likewise,

$$\begin{aligned}
a_2 &= a_1 - b_1 = 1 - \cos x - \cos x(1 - \cos x) = (1 - \cos x)^2 \\
b_2 &= a_2 \cos x = \cos x(1 - \cos x)^2 \\
c_2 &= x b_2 = x \cos x(1 - \cos x)^2 \\
d_2 &= \frac{1}{2} x b_2^2 = \frac{1}{2} x \cos^2 x(1 - \cos x)^4
\end{aligned}$$

We now proceed by induction. Assume that $a_{i-1} = (1 - \cos x)^{i-1}$ and $b_{i-1} = \cos x(1 - \cos x)^{i-1}$ for $i \geq 1$. Then

$$a_i = a_{i-1} - b_{i-1} = (1 - \cos x)^{i-1} - \cos x(1 - \cos x)^{i-1} = (1 - \cos x)^i.$$

Now that we have a_i, we can calculate the remaining quantities:

$$\begin{aligned}
b_i &= a_i \cos x = \cos x(1 - \cos x)^i \\
c_i &= x b_i = x \cos x(1 - \cos x)^i \\
d_i &= \frac{1}{2} x b_i^2 = \frac{1}{2} x \cos^2 x(1 - \cos x)^{2i}
\end{aligned}$$

(for $i \geq 0$). We are now ready to calculate the sums.

(1) Because $0 < 1 - \cos x < 1$, the sum of the geometric series formula gives

$$a = \sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} (1 - \cos x)^i = \frac{1}{1 - (1 - \cos x)} = \frac{1}{\cos x}$$

Note that $a \to \infty$ as $x \to \pi/2$.

(2) From (1), we obtain

$$b = \sum_{i=0}^{\infty} b_i = \sum_{i=0}^{\infty} \cos x (1-\cos x)^i = \cos x \sum_{i=0}^{\infty} (1-\cos x)^i = \cos x \cdot a = 1$$

Makes sense, as $b_i = |\overline{A_i B_i}| = |\overline{A_i A_{i+1}}|$, and thus all b_i, put together, form the hypotenuse $|\overline{A_0 C}|$.

(3) With the help of the sum in (2), we find

$$c = \sum_{i=0}^{\infty} c_i = \sum_{i=0}^{\infty} x \cos x (1-\cos x)^i = x \sum_{i=0}^{\infty} \cos x (1-\cos x)^i = xb = x$$

(4) For the sum of the areas of all circular sectors, we obtain

$$d = \sum_{i=0}^{\infty} d_i = \sum_{i=0}^{\infty} \frac{1}{2} x \cos^2 x (1-\cos x)^{2i} = \frac{1}{2} x \cos^2 x \sum_{i=0}^{\infty} (1-\cos x)^{2i}$$

Using the sum of the geometric series (note that $|(1-\cos x)^2| < 1$) we find

$$\sum_{i=0}^{\infty} (1-\cos x)^{2i} = \frac{1}{1-(1-\cos x)^2} = \frac{1}{1-(1-2\cos x+\cos^2 x)} = \frac{1}{\cos x(2-\cos x)}$$

and therefore

$$d = \frac{1}{2} x \cos^2 x \, \frac{1}{\cos x(2-\cos x)} = \frac{1}{2} x \, \frac{\cos x}{2-\cos x}.$$

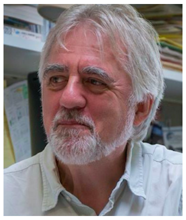
We are done.

For us, the beauty of the problem lies in the fact that it naturally combines several ideas and concepts that students do not often see in a single problem.

. . . .

Andie Burazin is a passionate mathematics practitioner at the University of Toronto Mississauga who enjoys supporting students in any capacity; you might catch her watching hockey or basketball. Veselin Jungic is a Teaching Professor at the Department of Mathematics, Simon Fraser University. Miroslav Lovric is a math-loving professor at McMaster University.





OLYMPIAD CORNER

No. 403

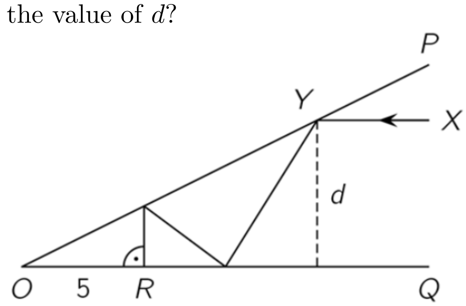
The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **July 15, 2022**.

OC581. Find the greatest positive integer n such that $n+3$ divides $1^3+2^3+\cdots+n^3$.

OC582. Two plane mirrors OP and OQ are inclined at an acute angle (diagram is not to scale). A ray of light XY parallel to QO strikes mirror OP at Y. The ray is refelcted and hits mirror OQ. Then it is reflected again and hits mirror OP. Finally, it is reflected for a third time and strikes mirror OQ at right angles at R, as shown. The distance OR is 5 cm. The ray XY is d cm from the mirror OQ. What is the value of d?



OC583. Reduce the following expression to a simplified rational

$$\cos^7\frac{\pi}{9}+\cos^7\frac{5\pi}{9}+\cos^7\frac{7\pi}{9}.$$

OC584. We say that two sequences $x,y:\ \mathbb{N}\to\mathbb{N}$ are completely different if $x(n)\neq y(n)$ holds for all $n\in\mathbb{N}$. Let F be a function assigning a natural number to every sequence of natural numbers such that $F(x)\neq F(y)$ for any pair of completely different sequences x, y, and for constant sequences we have $F((k,k,\dots))=k$. Prove that there exists $n\in\mathbb{N}$ such that $F(x)=x(n)$ for all sequences x.

OC585. Let $n \geq 3$ be a fixed integer. The number 1 is written n times on a blackboard. Below the blackboard, there are two buckets that are initially empty. A move consists of erasing two of the numbers a and b, replacing them with the numbers 1 and $a+b$, then adding one stone to the first bucket and $\gcd(a,b)$ stones to the second bucket. After some finite number of moves, there are s stones in the first bucket and t stones in the second bucket, where s and t are positive integers. Find all possible values of the ratio t/s.

. . . .

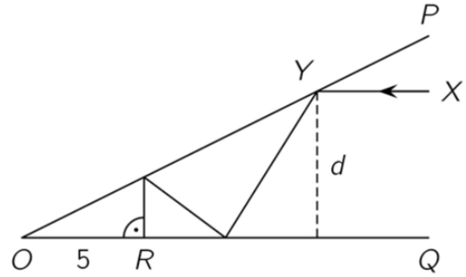
Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2022**.

OC581. Déterminer le plus grand entier positif n tel que $n+3$ divise $1^3 + 2^3 + \cdots + n^3$.

OC582. Le schéma qui se trouve ci-bas, pas nécessairement à l'échelle, illustre deux miroirs OP et OQ se rencontrant en un angle aigu. Un rayon de lumière XY, parallèle à OQ, frappe le miroir OP au point Y. Ce rayon est réfléchi et frappe le miroir OQ, puis est de nouveau réfléchi, frappant le miroir OP. Après une troisième réflexion, le rayon frappe le miroir OQ au point R, à un angle rectangle, tel qu'indiqué. Or, la distance OR est de 5 cm, et le rayon XY se trouve à une distance de d cm du miroir OQ. Déterminer la valeur de d.



OC583.

Déterminer le nombre rationnel simplifié égal à la valeur de l'expression

$$\cos^7 \frac{\pi}{9} + \cos^7 \frac{5\pi}{9} + \cos^7 \frac{7\pi}{9}.$$

OC584. Deux suites $x, y : \mathbb{N} \to \mathbb{N}$ sont dites complètement différentes si $x(n) \neq y(n)$ pour tout $n \in \mathbb{N}$. Une certaine fonction F assigne un nombre naturel à toute suite de nombres naturels de façon à ce que $F(x) \neq F(y)$ pour toutes suites complètement différentes x, y ; de plus, $F((k, k, \dots)) = k$ pour tout nombre naturel k. Démontrer qu'il existe $n \in \mathbb{N}$ tel que $F(x) = x(n)$ pour toute suite x.

OC585. Soit $n \geq 3$ un entier donné. Le nombre 1 est écrit n fois sur un tableau. Deux seaux initialement vides se trouvent tout près, et nous disposons d'un stock inépuisable de billes. Et puis, on joue. Un coup consiste à choisir et effacer deux des nombres au tableau, disons a et b, puis de les remplacer au tableau par 1 et $a + b$; de plus, on ajoute une bille au premier seau et $\mathrm{pgcd}(a, b)$ billes au deuxième seau. Après un certain nombre de coups, on observe qu'il se trouve s billes dans le premier seau et t billes dans le deuxième, où s et t sont des entiers positifs. Déterminer toutes les valeurs possibles du ratio t/s.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(10), p. 491–493.

OC556. Find all integer pairs (x, y) that satisfy the equation

$$7x^2 - 4xy + 7y^2 = (|x - y| + 2)^3.$$

Originally 2019 South Africa, Durban, Invitational World Youth Mathematics Intercity Competition, Individual Contest, Section B, Problem 2.

We received 3 submissions of which 1 was correct. We present the solution by UCLan Cyprus Problem Solving Group.

Writing $y = x + d$ we get

$$(d + 2)^3 = 7x^2 - 4x(x + d) + 7(x + d)^2 = 10x^2 + 10xd + 7d^2$$

so

$$4(d + 2)^3 - 28d^2 = 40x^2 + 40xd = 10(2x + d)^2 - 10d^2\,.$$

Thus

$$5(2x + d)^2 = 2(d + 2)^3 - 9d^2 = 2d^3 + 3d^2 + 24d + 16\,.$$

Multiplying both sides by 500 we get

$$(100x + 50d)^2 = (10d)^3 + 15(10d)^2 + 1200(10d) + 8000\,.$$

We are required to investigate the Diophantine equation

$$b^2 = a^3 + 15a^2 + 1200a + 8000\,.$$

It seems that this is quite hard. However, this is an Elliptic curve and there is a standard algorithm for computing integral points on it. Using the commands

```
E=EllipticCurve([0,15,0,1200,8000])
E.integral_points()
```

in Sage, we get that the only integral solutions (with positive y) are $a = 1, b = 96$ and $a = 10, b = 150$. Since $a = 10d$ is a multiple of 10 we must have $d = 1$ and $100x + 50d = 150$ giving $x = 1$. Thus $y = x + d = 2$. We also have the symmetric solution $x = 2, y = 1$.

OC557. A natural number k is given. For $n \in \mathbb{N}$ we define $f_k(n)$ as the smallest integer greater than kn such that $nf_k(n)$ is a perfect square. Prove that $f_k(m) = f_k(n)$ implies $m = n$.

Originally 9th Serbian Mathematical Olympiad for high school students, Belgrade, First Day, Problem 2 proposed by Nikola Petrovic.

We received 4 submissions of which 3 were correct and complete. We present the solution by Theo Koupelis, Broward College, Pembroke Pines, Florida, USA.

Let $f_k(n) = nk + a_n$ with $a_n \in \mathbb{N}$ being the smallest natural number so that $nf_k(n)$ is a perfect square. Then, if we write $n = n'b^2$, we have

$$f_k(n'b^2) = n'b^2k + a_n = n'c^2,$$

where $n', b, c \in \mathbb{N}$ and n' is square-free. Therefore $a_n = n'a'_n$, with $a'_n \in \mathbb{N}$ and $kb^2 + a'_n = c^2$. Similarly, if we write $m = m'd^2$, we have

$$f_k(m'd^2) = m'kd^2 + a_m = m'\ell^2,$$

with $m', d, \ell, a_m \in \mathbb{N}$ and m' is square-free. Therefore $a_m = m'a'_m$, with $a'_m \in \mathbb{N}$ and $kd^2 + a'_m = \ell^2$. When $f_k(m) = f_k(n)$ we have $n'c^2 = m'\ell^2$ and thus $n' = m' = p$ and $\ell = c$. Therefore $kb^2 + a'_n = kd^2 + a'_m = c^2$.

Then c is the smallest natural number greater than $b\sqrt{k}$ and also the smallest natural number greater than $d\sqrt{k}$. Therefore, we must have $|b\sqrt{k} - d\sqrt{k}| < 1$, or $|b - d| < \frac{1}{\sqrt{k}} < 1$. Thus $b = d$ and $m = n$.

OC558. Anne consecutively rolls a 2020-sided die with faces labeled from 1 to 2020 and keeps track of the running sum of all her previous dice rolls. She stops rolling the first time when her running sum is greater than 2019. Let X and Y be the running sums she is most and least likely to have stopped at with non-zero probability, respectively. What is the ratio between the probabilities of stopping at Y to stopping at X?

Originally April 2021 Stanford Math Tournament, Combinatorics test, Problem 7.

We received 3 correct submissions. We present the solution by UCLan Cyprus Problem Solving Group.

For $k = 0, 1, 2, \ldots, 2020$, we denote by p_k the probability that after some rolls we have a running sum equal to k. We set $p_0 = 1$ and notice that $p_1 = \frac{1}{2020}$.

For $k = 1, 2, \ldots, 2019$, there is a one to one correspondence between sequences of rolls having running sum k and sequences of rolls having running sum $k + 1$, with the last roll being greater than 1. The correspondence is given by $(r_1, \ldots, r_i) \mapsto (r_1, \ldots, r_{i-1}, r_i + 1)$. Both such sequences have equal probability of occurring.

The only other way to get a running sum of $k + 1$ is for the last roll to be equal to 1. From the above, it follows that

$$p_{k+1} = p_k + \frac{p_k}{2020} = \frac{2021}{2020}p_k \tag{1}$$

for each $k = 1, 2, \ldots, 2019$.

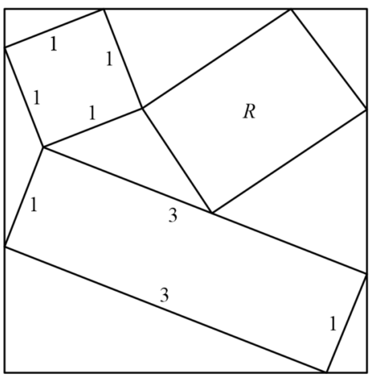
For $k = 2020, 2021, \ldots, 4039$, we denote by q_k the probability that we have a running sum equal to k, and all previous running sums are less than or equal to 2019. Observe now that we can stop at k, for $k = 2020, \ldots, 4039$, if we first reach a running sum of $l = k - 2020, \ldots, 2019$, and then roll $k - l$. The probability that this happens is

$$q_k = \frac{p_{(k-2020)}}{2020} + \cdots + \frac{p_{2019}}{2020}.$$

The formula above shows that the largest q is q_{2020} and smallest q is q_{4039}. Hence, the most likely running sum to have stopped at is $X = 2020$, and the least likely is $Y = 4039$. Also, we observe that the probabilities p_{2020} and q_{2020} are equal. Using (1) we get that the ratio

$$\frac{q_{4039}}{q_{2020}} = \frac{p_{2019}}{2020} \times \frac{1}{q_{2020}} = \frac{p_{2019}}{p_{2020}} \times \frac{1}{2020} = \frac{2020}{2021} \times \frac{1}{2020} = \frac{1}{2021}.$$

OC559. A rectangle with side lengths 1 and 3, a square with side length 1, and a rectangle R are inscribed inside a larger square as shown. The sum of all possible values for the area of R can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?



Originally Fall 2021 AMC 10B, Problem 25.

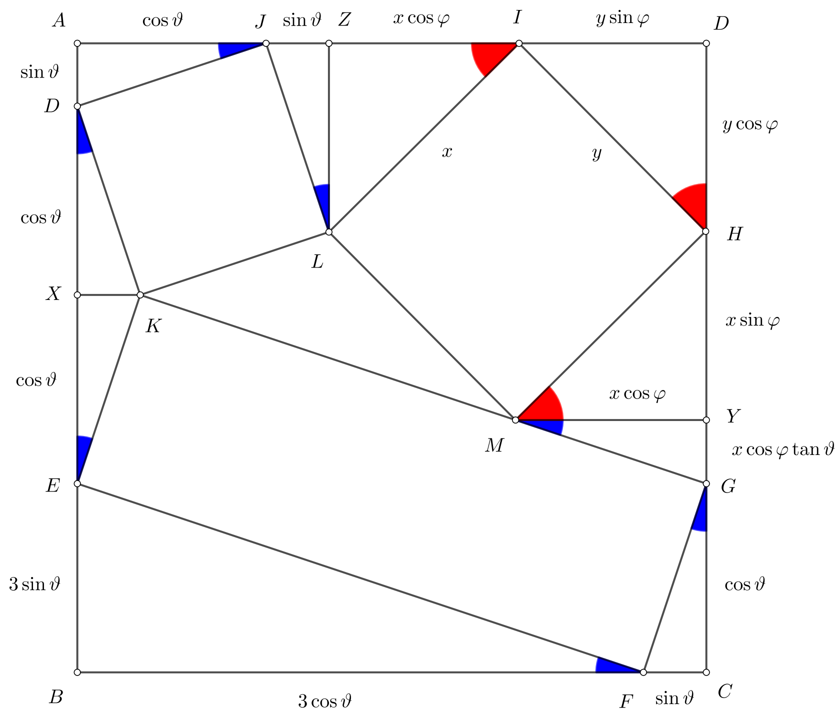
We received 6 correct submissions. We present two solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

We use the notation in the figure below. If $\angle BFE = \vartheta$ and $\angle DHI = \varphi$, then all blue angles are easily seen to be equal to ϑ, and all red angles are equal to φ. Many segment lengths are given in the figure below.

Since $ABCD$ is a square, then

$$3\cos\vartheta + \sin\vartheta = BC = AB = 4\sin\vartheta + 2\cos\vartheta$$



giving $\tan\vartheta = \frac{1}{3}$. This gives

$$\sin\vartheta = \sqrt{\frac{\tan^2\vartheta}{1+\tan^2\vartheta}} = \frac{1}{\sqrt{10}} \quad \text{and} \quad \cos\vartheta = \sqrt{\frac{1}{1+\tan^2\vartheta}} = \frac{3}{\sqrt{10}}\,.$$

So the square has side length

$$3\cos\vartheta + \sin\vartheta = \frac{10}{\sqrt{10}}\,.$$

Looking at the sides AD and CD, it follows that

$$x\cos\varphi + y\sin\varphi = \frac{6}{\sqrt{10}} \quad \text{and} \quad \left(\frac{x}{3}+y\right)\cos\varphi + x\sin\varphi = \frac{7}{\sqrt{10}}\,. \tag{1}$$

Note that $ZL = x\sin\varphi = \cos\vartheta = \frac{3}{\sqrt{10}}$ so (1) becomes

$$x\cos\varphi = \frac{6-3y/x}{\sqrt{10}} \quad \text{and} \quad \left(\frac{x}{3}+y\right)\cos\varphi = \frac{4}{\sqrt{10}}\,. \tag{2}$$

We get

$$4x = \left(6 - \frac{3y}{x}\right)\left(\frac{x}{3}+y\right) \implies 12x^2 = 6x^2 + 15xy - 9y^2 \implies 2x^2 - 5xy + 3y^2 = 0$$

This gives $(x-y)(2x-3y)=0$ and so $y=x$ or $y=2x/3$.

In the first case we have $x\cos\varphi=\frac{3}{\sqrt{10}}$ and since also $x\sin\varphi=\frac{3}{\sqrt{10}}$, then $\tan\varphi=1$. Thus $x=y=\frac{3}{\sqrt{5}}$ and the area of rectangle R is $xy=\frac{9}{5}$.

In the second case we have $x\cos\varphi=\frac{4}{\sqrt{10}}$ and since also $x\sin\varphi=\frac{3}{\sqrt{10}}$, then $\tan\varphi=\frac{3}{4}$. Then $\cos\varphi=\frac{4}{5}$ and $\sin\varphi=\frac{3}{5}$ giving $x=\frac{5}{\sqrt{10}}$ and $y=\frac{10}{3\sqrt{10}}$. So in this case the area of rectangle R is $\frac{5}{3}$.

Now the sum of these two values is $\frac{9}{5}+\frac{5}{3}=\frac{52}{15}$. Since 15,52 are relatively prime, then $m+n=52+15=67$.

Solution 2, by Oliver Geupel.

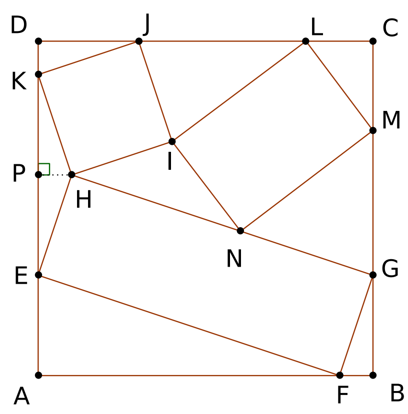
Let us use notations as shown in the diagram below, and let $a=PH$ and $b=PE$. We have $a+3b=AF+FB=AE+EK+KD=3a+2b+a$; whence $b=3a$ and $a=\frac{1}{\sqrt{10}}$.

Applying coordinates with origin A and base vectors $\overrightarrow{FB}$ and $\overrightarrow{KD}$ of length a, we obtain $G=(10,3)$, $H=(1,6)$, and $I=(4,7)$. Let $L=(x,10)$ and $M=(10,y)$. Since N lies on GH, there exists a $t\in\mathbb{R}$ such that

$$(14-x,y-3)=\vec{I}+\vec{M}-\vec{L}=\vec{N}=t\vec{G}+(1-t)\vec{H}=(1+9t,6-3t).$$

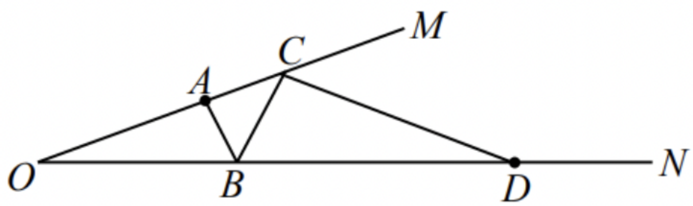
This implies that $x=13-9t$ and $y=9-3t$. Also

$$0=\overrightarrow{LI}\cdot\overrightarrow{LM}=(4-x,-3)\cdot(10-x,y-10)=81\left(t-\frac{5}{9}\right)\left(t-\frac{2}{3}\right).$$



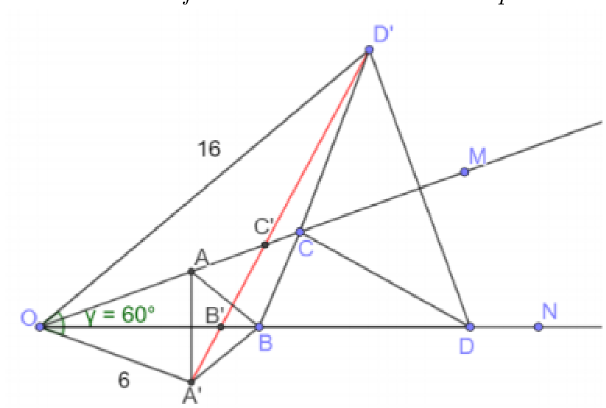
It is straightforward to verify that for $t\in\left\{\frac{5}{9},\frac{2}{3}\right\}$ the quadrilaterals $ILMN$ are rectangles with $N\in GH$. For $t=\frac{5}{9}$, we obtain $L=(8,10)$ and $M=\left(10,\frac{22}{3}\right)$. Since the base vectors have length a, the area of rectangle R is $IL\cdot LM=\frac{5}{\sqrt{10}}\cdot\frac{\sqrt{10}}{3}=\frac{5}{3}$. For $t=\frac{2}{3}$, we find $L=(7,10)$, $M=(10,7)$, and $IL\cdot LM=\frac{3}{\sqrt{5}}\cdot\frac{3}{\sqrt{5}}=\frac{9}{5}$. Since the sum of the areas is $\frac{5}{3}+\frac{9}{5}=\frac{52}{15}$, we finally obtain $m=52$, $n=15$, and $m+n=67$.

OC560. In the figure below, points A, C are on ray OM and B, D are on ray ON. It is given that $OA = 6$cm, $OD = 16$cm and $\angle NOM = 20°$. What is the minimum length, in cm, of $AB + BC + CD$?



Originally 2019 South Africa, Durban, Invitational World Youth Mathematics Intercity Competition, Team Contest, Problem 3.

We received 10 submissions of which 9 were correct. We present a typical solution.



It is helpful to think of OM and ON as two mirrors. Let A' be the reflection point of A about ON and D' the reflection point of D about OM. $A'D'$ intersects OM at C' and ON at B'. We have $AB + BC + CD = A'B + BC + CD'$ because $AB = A'B$ and $CD = CD'$. Since $A'D'$ is a straight segment we have that $A'B + BC + CD' \geq A'D'$. Therefore $A'D'$ is the minimum of $AB + BC + CD$ as B and C vary. Using the Law of Cosines in $\triangle A'OD'$ we calculate $A'D'^2 = 16^2 + 6^2 - 2 \times 16 \times 6 \times \cos 60° = 196$. The minimum of $AB + BC + CD$ is 14 cm.

FOCUS ON...

No. 51

Michel Bataille

Linear algebra (II)

Introduction

In this number, we continue to present examples of problems involving linear algebra. All of them will be connected with eigenvalues. After the usual reduced forms of a square matrix, we will consider some interventions of the characteristic and minimum polynomials in the solutions of problems.

Reduction to a diagonal or triangular matrix

The conditions insuring that a given $n \times n$ matrix is similar to a diagonal one are well-known. The following exercise gives the opportunity to review them.

Let u, v, w be nonzero real numbers and

$$A = -\frac{2}{3}\begin{pmatrix} -1/2 & v/u & w/u \\ u/v & -1/2 & w/v \\ u/w & v/w & -1/2 \end{pmatrix}.$$

Show that A is diagonalizable.

A direct calculation (left to the reader) shows that the characteristic polynomial $\det(\lambda I_3 - A)$ is independent of u, v, w and equal to $(\lambda+1)(\lambda-1)^2$, hence the eigenvalues are -1 and 1 with respective algebraic multiplicity 1 and 2. Thus, A will be diagonalizable if (and only if) the eigenspace E_1 associated with the eigenvalue 1 is of dimension 2. Now, if $X = (x\ y\ z)$, the equation $(A - I_3)X^T = 0$ is equivalent to $ux + vy + wz = 0$, hence $\dim E_1 = 2$ and A is diagonalizable.

Note that since $A = P\operatorname{diag}(1, 1, -1)P^{-1}$ for some invertible matrix P we have $A^2 = I_3$, a result which was not obvious *a priori*. It is also worth remarking that if $u^2 = v^2 = w^2$, then in the Euclidean space $\mathbb{R}^3$, the vector $(1/u\ 1/v\ 1/w)^T$ (which spans the eigenspace associated with -1) and the vector $(u\ v\ w)^T$ (orthogonal to E_1) are collinear, so that A represents the reflection in the plane E_1.

The reduction of a matrix to a diagonal form has numerous applications. Perhaps the more classical one is the calculation of the powers of the matrix. For a simple example, we refer the reader to the published solution to problem **3390** [2008 : 486 ; 2009 : 521].

The following problem set in the December 2013 issue of *The American Mathematical Monthly* offers an application to the computation of an inverse:

Let n be a positive integer, let x be a real number, and let B be the n-by-n matrix with $2x$ in all diagonal entries, 1 in all sub- and superdiagonal entries, and 0 in all other entries. Compute the inverse, when it exists, of B as a function of x.

We propose a solution different from the featured one. Some trigonometric identities will be required and we leave their proofs to the reader.

Let $A = (2x+2)I_n - B$. From the identity

$$-\sin(n-1)\theta + 2\sin(n\theta) - \sin(n+1)\theta = 2(1-\cos\theta)\sin n\theta \qquad (\theta \in \mathbb{R},\ n \in \mathbb{N}),$$

we readily deduce that the column vector $\Omega = (\sin\theta \quad \sin 2\theta \quad \cdots \quad \sin n\theta)^T$ satisfies $A\Omega = 2(1-\cos\theta)\Omega$ provided that $\sin(n+1)\theta = 0$.

It follows that for $k \in \{1, 2, \ldots, n\}$,

$$B\Omega_k = 2(x + \cos\theta_k)\Omega_k,$$

where $\theta_k = \frac{k\pi}{n+1}$ and $\Omega_k = (\sin\theta_k \quad \sin 2\theta_k \quad \cdots \quad \sin n\theta_k)^T$. Thus, the real numbers $2\rho_k = 2(x + \cos\theta_k), \quad k = 1, 2, \ldots, n$ are n distinct eigenvalues of B and B is diagonalizable: $B = PDP^{-1}$ where P is the matrix whose column vectors are $\Omega_1,\ \Omega_2,\ \cdots\ \Omega_n$ and $D = \operatorname{diag}(2\rho_1, 2\rho_2, \cdots, 2\rho_n)$. In consequence, B is invertible if and only if $x \neq -\cos\theta_k$ for $k \in \{1, 2, \ldots, n\}$.

Using the formulas

$$\sum_{j=1}^{n} \sin^2(j\theta_k) = \frac{n+1}{2} \quad \text{and} \quad \sum_{j=1}^{n} \sin(j\theta_r)\sin(j\theta_s) = 0$$

for $k, r, s \in \{1, 2, \ldots, n\}$, $r \neq s$, we obtain $P^TP = \frac{n+1}{2} I_n$, that is, $P^{-1} = \frac{2}{n+1} P^T$.

Now, $B^{-1} = \frac{1}{n+1}\, P \cdot \operatorname{diag}(\rho_1^{-1}, \rho_2^{-1}, \cdots, \rho_n^{-1}) \cdot P^T = (b_{r\,s})$ where for $r, s \in \{1, 2, \ldots, n\}$,

$$b_{r\,s} = \frac{1}{n+1} \sum_{j=1}^{n} \frac{\sin\left(\frac{jr\pi}{n+1}\right)\sin\left(\frac{js\pi}{n+1}\right)}{x + \cos\left(\frac{j\pi}{n+1}\right)}.$$

A simple condition allows a reduction to a triangular form: if $\mathbb{F}$ is a field and $A \in \mathcal{M}_n(\mathbb{F})$, then A is similar (over $\mathbb{F}$) to a triangular matrix T if and only if all the roots of the characteristic polynomial are in $\mathbb{F}$. If this is the case, the principal diagonal of T is formed by a list of all these roots.

Here is an easy application:

> Let $A \in \mathcal{M}_n(\mathbb{C})$ be such that $A^p = O_n$ for some positive integer p. Evaluate $\det(A + I_n)$.

Let λ be an eigenvalue of A and X be an associated eigenvector. From $AX = \lambda X$, we readily deduce that $A^pX = \lambda^pX$ so that $\lambda^pX = 0$ and therefore $\lambda = 0$ (since $X \neq 0$). It follows that $A = PTP^{-1}$ where T is triangular with 0 as each diagonal entry.

Then, from $A+I_n = P(T+I_n)P^{-1}$ we deduce that $\det(A+I_n) = \det(T+I_n) = 1$.

We conclude this section with a problem (proposed in *The American Mathematical Monthly* in 2002, here slightly reworded) in which both the diagonal and triangular reduced forms are used:

> Let $A, B \in \mathcal{M}_2(\mathbb{C})$ and let $C = AB - BA$. Prove that if $CA = AC$ and $CB = BC$, then $C = O_2$.

We shall use the following remark: if M, N, P are in $\mathcal{M}_2(\mathbb{C})$ and P is invertible, then $MN = NM$ if and only if $(P^{-1}MP)(P^{-1}NP) = (P^{-1}NP)(P^{-1}MP)$.

Assume that C has two distinct eigenvalues λ_1, λ_2. Then $C = PDP^{-1}$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and P is invertible. Since $CA = AC$ and $CB = BC$, the above remark shows that $A' = P^{-1}AP$ and $B' = P^{-1}BP$ satisfy $DA' = A'D$ and $DB' = B'D$. But then A' and B' must be diagonal matrices (easily checked), hence $A'B' = B'A'$ and therefore $AB = BA$, that is, $C = O_2$, a contradiction with the assumption about C.

In consequence, C must have a unique eigenvalue λ_0 (with algebraic multiplicity 2). Then we have $\lambda_0 = 0$ (since $\operatorname{tr}(C) = 0$) and $C = QTQ^{-1}$ where T is a triangular matrix, say, $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$. If $t \neq 0$, then the matrices $A'' = Q^{-1}AQ$ and $B'' = Q^{-1}BQ$ satisfy $TA'' = A''T$ and $TB'' = B''T$, hence $A'' = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ and $B'' = \begin{pmatrix} \gamma & \delta \\ 0 & \gamma \end{pmatrix}$ for some complex numbers $\alpha, \beta, \gamma, \delta$. But then, $A''B'' = B''A''$ and $AB = BA$ again, contradicting $t \neq 0$. Thus, we must have $t = 0$ and therefore $C = O_2$.

The Cayley-Hamilton theorem

We offer two interventions of this important, easily stated theorem: If $A \in \mathcal{M}_n(\mathbb{F})$ and $\det(\lambda I_n - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$, then we have

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = O_n.$$

First, we consider problem **4145** [2016 : 221 ; 2017 : 224]:

> Prove that the system
>
> $$\begin{cases} A^3 + A^2B + AB^2 + ABA & = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ \\ B^3 + B^2A + BA^2 + BAB & = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{cases}$$
>
> has no solutions in the set of 3×3 matrices over complex numbers.

We propose a variant of solution: Assume that such matrices A, B exist. Then, since

$$\begin{aligned}(A+B)^3 &= (A+B)(A^2+AB+BA+B^2)\\ &= (A^3+A^2B+ABA+AB^2)+(BA^2+BAB+B^2A+B^3),\end{aligned}$$

we obtain

$$(A+B)^3 = \begin{bmatrix} 0 & 2 & 6\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix} \qquad (1).$$

Now, let λ be any complex eigenvalue of $A+B$. Then λ^3 is an eigenvalue of $(A+B)^3$, hence $\lambda^3 = 0$ (by (1)) that is, $\lambda = 0$. Thus, $A+B$ has no other eigenvalue than 0 and its characteristic polynomial must be λ^3. From the Cayley-Hamilton theorem, we deduce $(A+B)^3 = O_3$, in contradiction with (1). The required result follows.

Our second example, proposed in 2008 in *The Mathematical Gazette*, has a geometric flavor:

> Let $\mathbf{a}$ and $\mathbf{b}$ be linearly independent column vectors in $\mathbb{R}^3$. Determine a real square root of $\mathbf{b}\mathbf{a}^t - \mathbf{a}\mathbf{b}^t$; that is, find a 3×3 matrix A with real entries so that $A^2 = \mathbf{b}\mathbf{a}^t - \mathbf{a}\mathbf{b}^t$. (Here, $\mathbf{x}^t$ denotes the transpose of $\mathbf{x}$.)

There are two solutions, namely

$$A = \frac{\pm 1}{m\sqrt{2m}}(\mathbf{b}\mathbf{a}^t - \mathbf{a}\mathbf{b}^t)(mI_3 - (\mathbf{b}\mathbf{a}^t - \mathbf{a}\mathbf{b}^t))$$

where $m = \left((\mathbf{a}^t\mathbf{a})(\mathbf{b}^t\mathbf{b}) - (\mathbf{a}^t\mathbf{b})^2\right)^{1/2}$.

(Note that $(\mathbf{a}^t\mathbf{a})(\mathbf{b}^t\mathbf{b}) - (\mathbf{a}^t\mathbf{b})^2 > 0$ follows from Cauchy-Schwarz inequality and the independence of $\mathbf{a}$ and $\mathbf{b}$.)

We denote by C the 3×3 matrix $\mathbf{b}\mathbf{a}^t - \mathbf{a}\mathbf{b}^t$ and by $\mathbf{c} = (c_1\ c_2\ c_3)^t$ the cross product $\mathbf{a}\times\mathbf{b}$. Note that $c_1^2 + c_2^2 + c_3^2 = \mathbf{c}^t\mathbf{c} = m^2$. The expression of C as a function of c_1, c_2, c_3 is readily obtained as well as its characteristic polynomial $\det(xI_3 - C) = x^3 + m^2x$. Thus, $C^3 + m^2C = 0$ (by the Cayley-Hamilton theorem) and we have

$$(C^2 - mC + 2m^2I_3)(C + mI_3) = 2m^3I_3 \qquad (2).$$

Now, let $A \in \mathcal{M}_3(\mathbb{R})$ be such that $A^2 = C$ and let α, β, γ be the (complex) eigenvalues of A. Then $\alpha^2, \beta^2, \gamma^2$ are the complex eigenvalues $0, im, -im$ of C. Thus, the characteristic polynomial of A must be of the form $\chi(x) = x(x-\omega)(x-\overline{\omega})$ with $\omega^2\overline{\omega}^2 = m^2$ and $\omega^2 + \overline{\omega}^2 = 0$, that is, $\omega\overline{\omega} = m$ and $\omega + \overline{\omega} = \pm\sqrt{2m}$. It follows that $\chi(x) = x(x^2 \pm \sqrt{2m}x + m)$.

Thus, $A^3 \pm \sqrt{2m}A^2 + mA = 0$ or (since $A^2 = C$), $A(C + mI_3) = \pm\sqrt{2m}C$. Using (2), this yields

$$A = \pm\sqrt{2m}C \cdot \frac{1}{2m^3}(C^2 - mC + 2m^2 I_3) = \frac{\pm 1}{m^2\sqrt{2m}}(-m^2C - mC^2 + 2m^2C)$$
$$= \frac{\pm 1}{m\sqrt{2m}}C(mI_3 - C).$$

Conversely, $\left(\pm\frac{1}{m\sqrt{2m}}C(mI_3 - C)\right)^2 = C$ is easily checked.

The minimum polynomial

Recall that the minimal polynomial $\mu(x)$ of A ($A \in \mathcal{M}_n(\mathbb{F})$), is the monic polynomial of least degree dividing all the polynomials $p(x)$ such that $p(A) = O_n$ ($\mu(x), p(x) \in \mathbb{F}[x]$).

In particular, $\mu(x)$ divides the characteristic polynomial $\chi(x)$. A nice application is problem **4229** [2017 : 102 ; 2018 : 128]:

> Let n be an integer with $n \geq 2$ and let p be a prime number. Consider an $n \times n$ matrix X over $\mathbb{Z}_p$ with $X^p = I_n$. Prove that $(X - I_n)^n = O_n$.

Since $X^p - I_n = O_n$, the minimum polynomial of X is a divisor of $x^p - 1 = (x-1)^p$, hence is of the form $(x-1)^k$ for some integer k with $1 \leq k \leq p$. Now, the characteristic polynomial of X, which is of degree n, is divisible by the minimum polynomial, which is of degree k. Thus, we must have $k \leq n$ and since $(X - I_n)^k = O_n$, we deduce that $(X - I_n)^n = O_n$.

Another recent example is problem **4446** [2019 : 266 ; 2019 : 578]:

> Let n be a prime number greater than 4 and let $A \in M_{n-1}(\mathbb{Q})$ be such that $A^n = I_{n-1}$. Evaluate
>
> $$\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1})$$
>
> in terms of n.

Let
$$B = A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1}$$
and let
$$p(x) = x^{n-1} + x^{n-2} + \cdots + x + 1.$$

Since $O_{n-1} = A^n - I_{n-1} = (A - I_{n-1})p(A)$, the minimum polynomial $\mu(x)$ of A divides $(x-1)p(x)$. Recalling the well-known fact that $p(x)$ is an irreducible polynomial of $\mathbb{Q}[x]$ (n being a prime) and because $1 \leq \text{degree}(\mu(x)) \leq n-1$, we must have either $\mu(x) = x - 1$ or $\mu(x) = p(x)$.

In the former case, $A = I_{n-1}$ and $B = (1 + 2 + \cdots + (n-1))I_{n-1} = \frac{n(n-1)}{2}I_{n-1}$. Thus, $\det(B) = \left(\frac{n(n-1)}{2}\right)^{n-1}$.

We show that in the latter case, $\det(B) = n^{n-2}$. First, we observe that since $\mu(x)$ divides the characteristic polynomial $\chi(x) = \det(xI_{n-1} - A)$ of A we must have $\chi(x) = \mu(x) = p(x)$. Therefore, if $w = \exp(2\pi i/n)$, the complex eigenvalues of A are the distinct roots $w, w^2, \dots, w^{n-1}$ of $p(x)$ and A is diagonalizable (over $\mathbb{C}$): $A = PDP^{-1}$ for some $P \in GL_{n-1}(\mathbb{C})$ and where $D = diag(w, w^2, \dots, w^{n-1})$. It follows that

$$\begin{aligned} B &= P(D^{n-2} + 2D^{n-3} + \cdots + (n-2)D + (n-1)I_{n-1})P^{-1} \\ &= Pdiag(q(w), q(w^2), \dots, q(w^{n-1}))P^{-1}, \end{aligned}$$

where $q(x) = x^{n-2} + 2x^{n-3} + \cdots + (n-2)x + n - 1$. As a result, we have $\det(B) = \prod_{k=1}^{n-1} q(w^k)$.

Now, we have $q(x) = x^{n-2}p'(1/x)$ where p' is the derivative of p. With the help of $p(x) = \dfrac{x^n - 1}{x - 1}$, we obtain

$$p'(x) = \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}.$$

Using $w^n = 1$, we easily deduce that $q(w^k) = \frac{n}{1-w^k}$ and therefore

$$\det(B) = \frac{n^{n-1}}{p(1)} = \frac{n^{n-1}}{n} = n^{n-2}.$$

To conclude, we recall an important result about the minimum polynomial: if $A \in \mathcal{M}_n(\mathbb{F})$, then A is diagonalizable over $\mathbb{F}$ if and only if the roots of its minimum polynomial are all in $\mathbb{F}$ and are simple roots. These roots are the distinct eigenvalues of A.

Here is an example of application:

> Let $A \in \mathcal{M}_n(\mathbb{R})$ satisfy the property $A^T = -A$. Prove that A is diagonalizable over $\mathbb{C}$.

The matrix $A^2 = -AA^T$ is symmetric with real entries, hence is diagonalizable over $\mathbb{R}$. Moreover its eigenvalues are nonpositive: indeed, if $-AA^TX = \lambda X$ where $\lambda \in \mathbb{R}$ and $X \in \mathcal{M}_{n,1}(\mathbb{R})$, with $X \neq 0$, then

$$\lambda X^TX = -(A^TX)^TA^TX,$$

hence $\lambda \leq 0$ since $X^TX > 0$ and $(A^TX)^TA^TX \geq 0$. Thus, the minimum polynomial $\mu_2(x)$ of A^2 writes as $\mu_2(x) = \prod_{j=1}^k (x + a_j^2)$ for some real numbers $a_1, \dots, a_k$ satisfying $0 \leq a_1 < a_2 < \cdots < a_k$ and $\prod_{j=1}^k (A^2 + a_j^2 I_n) = O_n$.

If $a_1 > 0$, then

$$\prod_{j=1}^k (A - ia_jI_n)(A + ia_jI_n) = O_n,$$

hence the minimum polynomial $\mu_1(x)$ of A divides $\prod_{j=1}^{k}(x-ia_j)(x+ia_j)$ and therefore all its complex roots are simple. Otherwise, we have $A^2q(A) = O_n$ where $q(A) = \prod_{j=2}^{k}(A^2 + a_j^2 I_n)$. Let $Z \in \mathcal{M}_{n,1}(\mathbb{C})$ be such that $A^2Z = 0$. By conjugation, we have $\overline{AZ} = A\overline{Z}$, hence

$$(\overline{AZ})^T AZ = \overline{Z}^T A^T AZ = -\overline{Z}^T A^2 Z = 0.$$

It follows that $AZ = 0$. In particular, for any $X \in \mathcal{M}_{n,1}(\mathbb{C})$ we obtain $Aq(A)X = 0$ (since $A^2q(A)X = 0$), hence $Aq(A) = O_n$. As a result, $\mu_1(x)$ divides

$$x\prod_{j=2}^{k}(x-ia_j)(x+ia_j).$$

Again, the complex roots of $\mu_1(x)$ are simple. The result follows.

Exercises

1. Let $A \in \mathcal{M}_n(\mathbb{F})$ have n distinct eigenvalues in $\mathbb{F}$ and let $M \in \mathcal{M}_n(\mathbb{F})$ be such that $MA = AM$. Prove that $M = p(A)$ for some polynomial $p(x) \in \mathbb{F}[x]$.

2. Let $A \in \mathcal{M}_3(\mathbb{R})$ satisfying $A \neq O_3$ and $A^3 = -A$. Find the minimum polynomial of A.

3. Let m, n, p be positive integers with $p \geq 2$ and let $A, B \in \mathcal{M}_n(\mathbb{C})$ be such that $A^m = I_n$ and $BA = AB^p$. Show that B is diagonalizable.

An Amazing Pattern involving Lucas Sequences

Hajime Katsumoto

The well-known **Fibonacci** sequence 1, 1, 2, 3, 5, 8, ... is defined by the initial conditions that the first two terms are 1s, and the recurrence relation that from the third term on, each is the sum of the preceding two terms. Thus 1+1=2, 1+2=3, 2+3=5, 3+5=8, and so on. If we keep the recurrence relation, but allow the first two terms to be arbitrary positive integers, then we have a **Lucas** sequence.

For any positive integer n, define $m(n)$ to be the smallest positive integer which does not occur in any Lucas sequence in which the first term is at most n and the second term is at most the first term.

For $n = 1$, the first two terms must be (1,1). We have the Fibonacci sequence, and $m(1) = 4$.

For $n = 2$, the first two terms may be (1,1), (2,1) or (2,2). In addition to the Fibonacci sequence, we also have the Lucas sequences 2, 1, 3, 4, 7, 11, ... and 2, 2, 4, 6, 10, ... Hence $m(2) = 9$.

The following table gives some empirical data.

n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$	n	$m(n)$
1	4	11	148	21	496	31	1046	41	1868	51	2758
2	9	12	169	22	551	32	1101	42	1957	52	2847
3	17	13	203	23	606	33	1156	43	2046	53	2936
4	25	14	237	24	661	34	1245	44	2135	54	3025
5	38	15	271	25	716	35	1334	45	2224	55	3169
6	51	16	305	26	771	36	1423	46	2313	56	3313
7	64	17	339	27	826	37	1512	47	2402	57	3457
8	85	18	373	28	881	38	1601	48	2491	58	3601
9	106	19	407	29	936	39	1690	49	2580	59	3745
10	127	20	441	30	991	40	1779	50	2669	60	3889

This pattern does not seem remarkable, but wait! Define $\Delta(n) = m(n) - m(n-1)$. We take $m(0) = 1$ because for $n = 0$, the first two terms are 0s, and the sequence generated is the all-0 sequence, even though it is not a Lucas sequence by our definition. We have the following table.

n	$\Delta(n)$	n	$\Delta(n)$	n	$\Delta(n)$	n	$\Delta(n)$	n	$\Delta(n)$	n	$\Delta(n)$
1	3	11	21	21	55	31	55	41	89	51	89
2	5	12	21	22	55	32	55	42	89	52	89
3	8	13	34	23	55	33	55	43	89	53	89
4	8	14	34	24	55	34	89	44	89	54	144
5	13	15	34	25	55	35	89	45	89	55	144
6	13	16	34	26	55	36	89	46	89	56	144
7	13	17	34	27	55	37	89	47	89	57	144
8	21	18	34	28	55	38	89	48	89	58	144
9	21	19	34	29	55	39	89	49	89	59	144
10	21	20	34	30	55	40	89	50	89	60	144

These values are the Fibonacci numbers starting with 3 and 5, each of which appears once. They are followed by two appearances of 8, three appearances of 13, five appearances of 21, eight appearances of 34, thirteen appearances of 55, twenty-one appearances of 89, and so on.

The pattern is amazing! Can anyone prove that it will continue?

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 15, 2022**.

4741. *Proposed by Alexander Bloom.*

A large pond has 100 lily pads lying in a straight line, numbered consecutively 1 to 100, with a frog making jumps between adjacent lily pads. At any given lily pad x, $1 \leq x \leq 99$, the probability of the frog moving forwards to the next lily pad is $\dfrac{1}{x}$ and the probability of retreating to the previous lily pad is $\dfrac{x-1}{x}$. For some y, if the frog starts on lily pad y, then the probability that the frog reaches lily pad 100 without ever touching lily pad $y-1$ can be written as $\dfrac{9!}{10 \cdot 98!}$. Find the value of y.

4742. *Proposed by Michel Bataille.*

Let m, n be non-negative integers. Prove that

$$\sum_{k=0}^{n} \binom{2n+2m}{k+2m} \binom{2n-k}{k} 4^{-k} = \binom{4n+4m}{2n} 4^{-n}.$$

4743. *Proposed by Cristian Chiser.*

Let ABC be an acute triangle. Suppose that D, E, F are points on sides BC, CA and AB, respectively, such that FD is perpendicular to BC, DE is perpendicular to CA, and EF is perpendicular to AB. Let a, b, c be the side lengths of the triangle ABC and let R_{DEF} be the circumradius of triangle DEF. Show that

$$R_{DEF} = \frac{a+b+c}{9}$$

if and only if triangle ABC is equilateral.

4744. *Proposed by Olimjon Jalilov.*

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that the equation

$$\tan^2 x \cdot f''(x) + (\tan^3 x + 4\tan x) f'(x) + 2f(x) = 0, \quad x \neq \pm\frac{\pi}{2}$$

has at least one root in the open interval $(-\pi, \pi)$.

4745. *Proposed by Marius Stănean.*

Let a, b, c be nonnegative real numbers such that $ab+bc+ca=4$. Prove that

$$(a^2+b^2+c^2+1)\left(\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\right)\geq\frac{45}{8}.$$

When does equality hold?

4746. *Proposed by George Stoica.*

Let A_n be the $n\times n$ matrix with elements $a_{ij}=\binom{n+1}{2i-j}$, $i,j=1,\ldots,n$. Prove that $\det A_n=2^{n(n+1)/2}$.

4747. *Proposed by Stanescu Florin.*

Determine all the functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^2f(x)+f(y))=f(f(x^3))+y$$

for all $x,y\in\mathbb{R}$.

4748. *Proposed by Mihaela Berindeanu.*

Let Γ be the circumcircle of a given triangle ABC, and define T to be the intersection of the tangents to Γ at B and at C, S to be the second point where Γ intersects AT, and $\widehat{A}$ to be the reflection of the vertex A in the line BC. Prove that the line $\widehat{A}S$ intersects the side BC at its midpoint.

4749. *Proposed by Julie Z.Y. Dong and Clarence C.Y. Kwan.*

For an arbitrary, monotonically increasing, strictly concave, and twice differentiable function $f(x)$ of a continuous variable x, defined for $h\leq x\leq k$, prove that

$$\frac{1}{b-a}\int_{x=a}^{b}f(x)dx>\frac{1}{k-h}\int_{x=h}^{k}f(x)dx,\ \text{if } a+b>h+k,$$

where $h<a<b<k$ are arbitrary constants.

4750. *Proposed by Nguyen Viet Hung.*

Given a triangle ABC with orthocentre H, let M be any point inside the triangle and let D,E,F be respectively projections of M onto the sides BC,CA,AB. Denote by S,S_a,S_b,S_c the areas of triangles ABC,HBC,HCA,HAB, respectively. Prove that

$$(S_b+S_c)\overrightarrow{MD}+(S_c+S_a)\overrightarrow{ME}+(S_a+S_b)\overrightarrow{MF}=S\cdot\overrightarrow{MH}.$$

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2022**.

4741. *Soumis par Alexander Bloom.*

Un grand étang contient 100 nénuphars positionnés de sorte à former une ligne droite, ce qui permet de leur assigner un numéro de 1 à 100. Une grenouille bondit entre des nénuphars adjacents. Étant donné un nénuphar x, $1 \leq x \leq 99$, la probabilité que la grenouille bondisse sur le nénuphar suivant est $\frac{1}{x}$ et la probabilité qu'elle bondisse plutôt sur le nénuphar précédant est $\frac{x-1}{x}$. Pour un y donné, si la grenouille se trouve initialement sur le nénuphar y, alors la probabilité qu'elle atteigne le nénuphar 100 sans jamais s'être rendu sur le nénuphar $y-1$ peut être écrite comme suit : $\dfrac{9!}{10 \cdot 98!}$. Trouvez la valeur de y.

4742. *Soumis par Michel Bataille.*

Soit m et n des entiers non négatifs. Montrez que

$$\sum_{k=0}^{n} \binom{2n+2m}{k+2m} \binom{2n-k}{k} 4^{-k} = \binom{4n+4m}{2n} 4^{-n}.$$

4743. *Soumis par Cristian Chiser.*

Soit ABC un triangle aigu. Supposons que D, E et F soit les points respectivement inscrits sur les côtés BC, CA et AB tel que FD est perpendiculaire à BC, DE est perpendiculaire à CA et EF est perpendiculaire à AB. Soit a, b et c la longueur des côtés du triangle ABC et soit R_{DEF} le rayon du cercle circonscrit au triangle DEF. Montrez que

$$R_{DEF} = \frac{a+b+c}{9}$$

si et seulement si le triangle ABC est équilatéral.

4744. *Soumis par Olimjon Jalilov.*

Soit $f : \mathbb{R} \mapsto \mathbb{R}$ une fonction deux fois dérivable. Supposons que $f(0) = 0$. Montrez que l'équation

$$\tan^2 x \cdot f''(x) + (\tan^3 x + 4 \tan x) f'(x) + 2f(x) = 0, \quad x \neq \pm\frac{\pi}{2}$$

possède au moins une racine dans l'intervalle ouvert $(-\pi, \pi)$.

4745. *Soumis par Marius Stănean.*

Soit a, b et c des nombres réels non négatifs vérifiant $ab + bc + ca = 4$. Montrez que

$$\left(a^2+b^2+c^2+1\right)\left(\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\right) \geq \frac{45}{8}.$$

Sous quelles conditions a-t-on l'égalité ?

4746. *Soumis par George Stoica.*

Soit A_n une matrice $n \times n$ dont les éléments sont $a_{ij} = \binom{n+1}{2i-j}$, $i, j = 1, \ldots, n$. Montrez que $\det A_n = 2^{n(n+1)/2}$.

4747. *Soumis par Stanescu Florin.*

Déterminez toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ vérifiant

$$f(x^2 f(x) + f(y)) = f(f(x^3)) + y$$

pour tout $x, y \in \mathbb{R}$.

4748. *Soumis par Mihaela Berindeanu.*

Soit Γ le cercle circonscrit du triangle ABC. Définissons T comme étant l'intersection des tangentes à Γ en B et en C. De plus, définissons S comme étant le second point où Γ rencontre AT. Soit $\widehat{A}$ la réflexion du sommet A par rapport à la droite BC. Montrez que la droite $\widehat{A}S$ rencontre le côté BC en son point milieu.

4749. *Soumis par Julie Z.Y. Dong et Clarence C.Y. Kwan.*

Étant donné une fonction quelconque $f(x)$ d'une variable continue x qui est monotone croissante, strictement concave, deux fois dérivable et qui est définie pour $h \leq x \leq k$, montrez que

$$\frac{1}{b-a}\int_{x=a}^{b} f(x)\,dx > \frac{1}{k-h}\int_{x=h}^{k} f(x)\,dx, \text{ si } a+b > h+k,$$

où $h < a < b < k$ sont des constantes arbitraires.

4750. *Soumis par Nguyen Viet Hung.*

Soit un triangle ABC d'orthocentre H. Soit M un point quelconque de l'intérieur de ce triangle. Soit encore D, E, F les projections de M sur les côtés BC, CA et AB respectivement. Notons par S, S_a, S_b et S_c l'aire des triangles ABC, HBC, HCA, HAB, respectivement. Montrez que

$$(S_b+S_c)\overrightarrow{MD} + (S_c+S_a)\overrightarrow{ME} + (S_a+S_b)\overrightarrow{MF} = S \cdot \overrightarrow{MH}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2022: 47(10), p. 508–511.

4691. *Proposed by Michel Bataille.*

Let ABC be a triangle inscribed in a circle Γ and let U_1, U_2, U_3 be distinct points of Γ. Let σ_i be the Simson line of U_i $(i = 1, 2, 3)$ and let V_k be the point of intersection of σ_i and σ_j $(\{i, j, k\} = \{1, 2, 3\})$. Given that $\Delta V_1V_2V_3$ is congruent to $\Delta U_1U_2U_3$, prove that $\Delta V_1V_2V_3$ and $\Delta U_1U_2U_3$ are symmetrical about a point and identify this point.

We received 5 solutions, all of which were correct. Four of them took Γ to be the unit circle and used complex numbers to represent the points, but we feature the solution that avoided the use of coordinates by Theo Koupelis with details added by the editor.

We base the proof on three nineteenth century theorems, all of which can be found in geometry texts such as *Advanced Euclidean Geometry* by Roger A. Johnson, a Dover Reprint (1960).

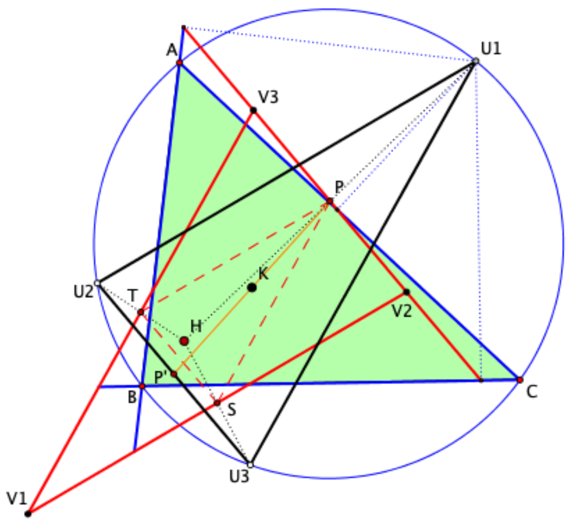
Theorem 1: Given points Q_1 and Q_2 on the circumcircle (O) of $\triangle ABC$, the angle between their corresponding Simson lines is equal to half of the central angle $\angle Q_1OQ_2$. [Johnson, paragraph 326, Cor. 1, p. 207]

Theorem 2: If H is the orthocenter, and Q a point on the circumcircle of $\triangle ABC$, then the Simson line of Q bisects the segment HQ. [Johnson, paragraph 327, p. 207]

Theorem 3: Given a triangle XYZ with circumcenter O' and a point S on its side YZ there is a unique triangle SPT directly similar to XYZ and inscribed in it with P the point where the circle SZO' again meets XZ, and T the point where the circle YSO' again meets XY; moreover, the unique smallest inscribed triangle that is directly similar to XYZ occurs when S is the midpoint of YZ (and thus P and T the midpoints of their respective sides). [A special case of Johnson, paragraphs 188b and 190b, pp. 134-136]

From Theorem 1 we deduce that for any triple of points U_1, U_2, U_3 on Γ, the triangle $V_1V_2V_3$ formed by the corresponding Simson lines is directly similar to $\Delta U_1U_2U_3$ (because $\angle U_1U_3U_2 = \frac{1}{2}\angle U_1OU_2 = \angle V_1V_3V_2$, and so on). We are given, however, that the two triangles are congruent, and thus we know that they are *directly* congruent, with

$$U_1U_2 = V_1V_2, \quad U_1U_3 = V_1V_3, \quad \text{and} \quad U_2U_3 = V_2V_3.$$



Let P, T, S be the midpoints of HU_1, HU_2, HU_3, respectively. Then in triangle HU_1U_2 we have PT parallel to the base U_1U_2 and $PT = \frac{1}{2}U_1U_2$, and similarly for TS and SP. That is, $\triangle PTS$ is similar to $\triangle U_1U_2U_3$ with sides parallel to the corresponding sides of $\triangle U_1U_2U_3$ and half as long. But from Theorem 2 we have

$$P \in V_2V_3, \quad T \in V_3V_1, \quad S \in V_1V_2.$$

Furthermore, from the congruency between triangles $U_1U_2U_3$ and $V_1V_2V_3$, we get that $\triangle PTS$ is not only inscribed in $\triangle V_1V_2V_3$, but it is directly similar to it, and the lengths of its sides are equal to half of the lengths of the corresponding sides of $\triangle V_1V_2V_3$. From Theorem 3 it follows that the vertices of $\triangle PTS$ are the midpoints of the sides of $\triangle V_1V_2V_3$. Therefore, $PT \parallel V_1V_2 \parallel U_1U_2$, and similarly $PS \parallel V_1V_3 \parallel U_1U_3$, and $TS \parallel V_2V_3 \parallel U_2U_3$. Thus the quadrilaterals $V_1V_2U_1U_2$, $V_2V_3U_2U_3$, and $V_3V_1U_3U_1$ are parallelograms, and their diagonals are concurrent at a point K. Consequently, the two triangles $U_1U_2U_3$ and $V_1V_2V_3$ are related by a reflection in K. Because P is the midpoint of V_2V_3, this reflection must interchange P with the midpoint P' of U_2U_3. Thus, given ΔABC with points U_1, U_2, U_3 on its circumcircle, we define P to be the midpoint of the segment joining H to U_1, in which case the fixed point K is the midpoint of the segment joining P to the midpoint of U_1U_2.

Editor's comments. Given ΔABC with points U_1 and U_2 on its circumcircle, there is a unique point U_3 on the circumcircle for which the triangle formed by the corresponding Simson lines is directly congruent to $\Delta U_1U_2U_3$. According to [Johnson, paragraph 326, Cor. 3, p. 207], the construction of U_3 consists of dropping a perpendicular from a vertex, say B, to the line U_1U_2; U_3 is then the point where its isogonal (that is, its reflection in the bisector of $\angle ABC$) again meets the circumcircle.

4692. *Proposed by Todor Zaharinov.*

Let a, b and c be nonzero real numbers such that $a^3 + b^3 + c^3 = 0$. Find the minimum possible value of

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

and determine where the minimum holds.

We received 12 submissions, all correct. We present the solution by Theo Koupelis.

Clearly, the numbers a, b, c cannot be all positive or all negative. Without loss of generality, let $a, b > 0$ and $c = -(a^3+b^3)^{1/3} < 0$. We set $x = a+b, y = ab$ and

$$f := (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \left[x - (x^3-3xy)^{1/3}\right]\left[\frac{x}{y} - \frac{1}{(x^3-3xy)^{1/3}}\right].$$

Note that

$$x^3 - 3xy = (a+b)^3 - 3ab(a+b) = a^3 + b^3$$

and $x^2 > 4y$. Setting $x^2 = k^3 y$ with $k \geq 4^{1/3}$, we have

$$f(k) = \left[k - (k^3-3)^{1/3}\right]\left[k^2 - \frac{1}{(k^3-3)^{1/3}}\right]$$

We show that $f(k)$ is an increasing function. Indeed,

$$f'(k) = \frac{3}{(k^3-3)^{4/3}}\left[k^2(k^3-3)^{4/3} - (k^3-2)\cdot k(k^3-3)^{2/3} + 1\right].$$

The expression in brackets above can be considered as a quadratic in $k(k^3-3)^{2/3}$. Since $k^3 \geq 4$, we have

$$f'(k) > 0 \Longleftrightarrow 2k(k^3-3)^{2/3} > (k^3-2) + k\sqrt{k(k^3-4)}.$$

Raising to the third power and simplifying, we get the equivalent expression

$$4k^3(k^3-3)^2 + 8 > 4k\sqrt{k(k^3-4)}(k^3-3)(k^3-1),$$

which after squaring becomes

$$2k^3(k^3-4)(k^3-2) + 2k^3 + 1 > 0,$$

which is obvious. Therefore,

$$f_{\min}(k) = f(4^{1/3}) = (4^{1/3}-1)(4^{2/3}-1) = 5 - 2^{4/3} - 2^{2/3}.$$

The minimum occurs when $(a, b, c) = (z, z, -2^{1/3}z)$ (where z is a nonzero real number) and all of its cyclic permutations.

4693. *Proposed by Michel Bataille.*

Prove that
$$\sum_{k=1}^{n} \sec^4 \frac{k\pi}{2n+1} = \frac{8n(n+1)(n^2+n+1)}{3}$$
for any positive integer n.

We received 20 submissions, of which 17 were correct and complete. We present a solution submitted by Sean M. Stewart.

Define the polynomial
$$P_n(z) = (z-i)^{2n+1} + (z+i)^{2n+1},$$
where $z \in \mathbb{C}$, n is a non-negative integer, and i is the imaginary unit. To find the roots of this polynomial, one has
$$\begin{aligned}(z-i)^{2n+1} + (z+i)^{2n+1} &= 0\\ \Rightarrow (z-i)^{2n+1} &= (-z-i)^{2n+1}\\ \Rightarrow z-i &= e^{\frac{2k\pi i}{2n+1}}(-z-i),\end{aligned}$$
where k is an integer such that $0 \leqslant k \leqslant 2n$. On solving for z we find
$$z = \frac{i\left(1-e^{\frac{2k\pi i}{2n+1}}\right)}{1+e^{\frac{2k\pi i}{2n+1}}} = \frac{\left(\frac{e^{\frac{k\pi i}{2n+1}} - e^{-\frac{k\pi i}{2n+1}}}{2i}\right)}{\left(\frac{e^{\frac{k\pi i}{2n+1}} + e^{-\frac{k\pi i}{2n+1}}}{2}\right)} = \frac{\sin\left(\frac{k\pi}{2n+1}\right)}{\cos\left(\frac{k\pi}{2n+1}\right)} = \tan\left(\frac{k\pi}{2n+1}\right).$$

So $z_k = \tan\left(\dfrac{k\pi}{2n+1}\right)$, $0 \leqslant k \leqslant 2n$ are the roots of $P_n(z)$.

From the binomial theorem $P_n(z)$ can be written as
$$\begin{aligned}P_n(z) &= (z-i)^{2n+1} + (z+i)^{2n+1}\\ &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} z^{2n-k+1}(-i)^k + \sum_{k=0}^{2n+1} \binom{2n+1}{k} z^{2n-k+1} i^k\\ &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} z^{2n-k+1}\left(i^k + (-i)^k\right)\\ &= 2z\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} (z^2)^{n-k} = 2zQ_n(z^2),\end{aligned}$$
where
$$Q_n(x) = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} x^{n-k}.$$
So,
$$z_k^2 = \tan^2\left(\frac{k\pi}{2n+1}\right), \quad 1 \leqslant k \leqslant 2n,$$

are roots of the polynomial $Q_n(z^2)$. Since $Q_n(x)$ can be expressed

$$Q_n(x) = \binom{2n+1}{0}x^n - \binom{2n+1}{2}x^{n-1} + \binom{2n+1}{4}x^{n-2} - \binom{2n+1}{6}x^{n-3} + \cdots,$$

by Vièta's formula for the sum of the roots of the polynomial $Q_n(x)$, we have

$$\sum_{cyc} \alpha = -\frac{-\binom{2n+1}{2}}{\binom{2n+1}{0}} = \binom{2n+1}{2} = n(2n+1).$$

Likewise, the sum of the products of pairs of roots, denoted by $\sum_{cyc} \alpha\beta$, is

$$\sum_{cyc} \alpha\beta = \frac{\binom{2n+1}{4}}{\binom{2n+1}{0}} = \binom{2n+1}{4} = \frac{n(n-1)(4n^2-1)}{6}.$$

Thus,

$$\sum_{k=1}^{n} \tan^2\left(\frac{k\pi}{2n+1}\right) = \sum_{cyc} \alpha = n(2n+1),$$

while

$$\begin{aligned}\sum_{k=1}^{n} \tan^4\left(\frac{k\pi}{2n+1}\right) &= \sum_{cyc} \alpha^2 = \left(\sum_{cyc} \alpha\right)^2 - 2\sum_{cyc} \alpha\beta \\ &= [n(2n+1)]^2 - \frac{n(n-1)(4n^2-1)}{3} \\ &= \frac{n(2n+1)(4n^2+6n-1)}{3}.\end{aligned}$$

Since

$$\sec^4\theta = (1+\tan^2\theta)^2 = 1 + 2\tan^2\theta + \tan^4\theta,$$

we have

$$\begin{aligned}\sum_{k=1}^{n} \sec^4\left(\frac{k\pi}{2n+1}\right) &= \sum_{k=1}^{n} \left\{1 + 2\tan^2\left(\frac{k\pi}{2n+1}\right) + \tan^4\left(\frac{k\pi}{2n+1}\right)\right\} \\ &= \sum_{k=1}^{n} 1 + 2\sum_{k=1}^{n} \tan^2\left(\frac{k\pi}{2n+1}\right) + \sum_{k=1}^{n} \tan^4\left(\frac{k\pi}{2n+1}\right) \\ &= n + 2n(2n+1) + \frac{1}{3}n(2n+1)(4n^2+6n-1) \\ &= \frac{8n(n+1)(n^2+n+1)}{3}.\end{aligned}$$

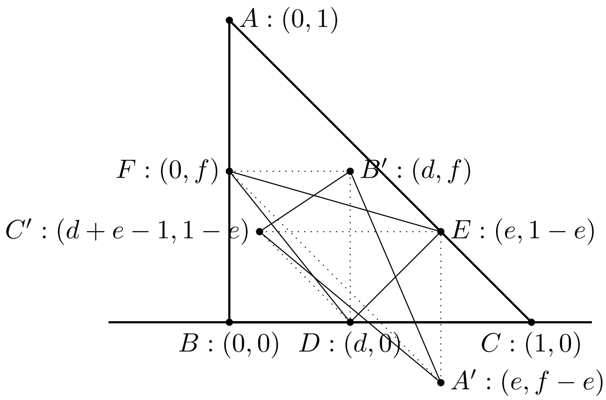
4694. *Proposed by Chen Jiahao.*

In triangle ABC, the inscribed circle touches side BC, CA and AB at D, E and F, respectively. Let A', B' and C' be the reflection of A, B and C in line EF, DF and DE, respectively. Show that the area of triangle DEF equals the area of triangle $A'B'C'$.

Solution 1, by Chris Fisher.

Because the lines AE and AF are tangents to the incircle, the line segments AE and AF are congruent. Since A' is the reflection of A in the line EF, $AFA'E$ is a rhombus with $AE \| FA'$. This suggests the following generalization:

> Given triangle ABC, let D, E, F be arbitrary points on the sides BC, CA, AB respectively. Define
>
> A' to be the point where the line parallel to AB through E intersects the line parallel to AC through F,
> B' to be the point where the line parallel to BC through F intersects the line parallel to BA through D,
> C' to be the point where the line parallel to CA through D intersects the line parallel to CB through E.
>
> Then the triangles DEF and $A'B'C'$ have equal areas.



Because the ratio of areas is invariant under an affine transformation, we may wolog take the origin at B, and let $C = (1, 0)$ and $A = (0, 1)$. We let $D = (d, 0)$, $E = (e, 1-e)$ and $F = (0, f)$, whereupon $A' = (e, f-e)$, $B' = (d, f)$ and $C' = (d+e-1, 1-e)$.

The respective areas of triangle DEF and $A'B'C'$ are

$$\begin{vmatrix} d & 0 & 1 \\ e & 1-e & 1 \\ 0 & f & 1 \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} e & f-e & 1 \\ d & f & 1 \\ d+e-1 & 1-e & 1 \end{vmatrix}$$

which both equal $d - de - df + ef$.

Solution 2, by Oliver Geupel, Mehra Vivek and Somasundaram Muralidharan (done independently).

Let each point X in the problem be represented in the complex plane by its corresponding lower case letter x. Wolog, we assume that d, e, f lie on the unit circle. Thus $\bar{d} = 1/d$, $\bar{e} = 1/e$ and $\bar{f} = 1/f$.

Recall that the area of a triangle (up to sign) with vertices z_1, z_2, z_3 in the complex plane is given by

$$\frac{i}{4}\begin{vmatrix} 1 & z_1 & \bar{z_1} \\ 1 & z_2 & \bar{z_2} \\ 1 & z_3 & \bar{z_3} \end{vmatrix}.$$

Therefore

$$[DEF] = \frac{i}{4def}(d-e)(e-f)(f-d).$$

Since $AE = AF$, $IE \perp EA$, $IF \perp FA$ (I the centre of the incircle), $a - e = \lambda ie$ and $a - f = -\lambda if$ for some nonzero real λ. Hence $f(a-e) = -e(a-f)$, so that $a = 2ef/(e+f)$. Similarly, $b = 2fd/(f+d)$ and $c = 2de/(d+e)$. Since AA' and DE bisect each other, we have $a + a' = d + e$, so that

$$a' = e + f - a = \frac{e^2 + f^2}{e + f}$$

and $\bar{a'} = a'/ef$. Similarly

$$b' = \frac{f^2 + d^2}{f + d}, \quad \bar{b'} = \frac{b'}{fd}, \quad c' = \frac{d^2 + e^2}{d + e}, \quad \bar{c'} = \frac{c'}{de}.$$

$$\begin{aligned}
[A'B'C'] &= \frac{i}{4}\begin{vmatrix} 1 & a' & a'/ef \\ 1 & b' & b'/fd \\ 1 & c' & c'/de \end{vmatrix} \\
&= \frac{i}{4}\left[b'c'\left(\frac{1}{de} - \frac{1}{fd}\right) + c'a'\left(\frac{1}{ef} - \frac{1}{de}\right) + a'b'\left(\frac{1}{fd} - \frac{1}{ef}\right)\right] \\
&= \frac{i}{4def}[b'c'(f-e) + c'a'(d-f) + a'b'(e-d)] \\
&= \frac{i[(f^2+d^2)(d^2+e^2)(f^2-e^2) + (d^2+e^2)(e^2+f^2)(d^2-f^2) + (e^2+f^2)(f^2+d^2)(e^2-d^2)]}{4def(d+e)(e+f)(f+d)} \\
&= \frac{i(d^2-e^2)(e^2-f^2)(f^2-d^2)}{4def(d+e)(e+f)(f+d)} = [DEF].
\end{aligned}$$

Solution 3, by Michel Bataille.

Consider the points as vectors in the plane. Let $a = BC$, $b = CA$, $c = AB$,

$2s = a+b+c$, $u = s-a = AF = AE$, $v = s-b = BF = BD$, $w = s-c = CD = CF$.

We have

$$aD = (v+w)D = wB+vC, \quad bE = (w+u)E = uC+wA, \quad cF = (u+v)F = vA+uB.$$

Multiplying these equations by $-u$, v, w respectively and adding yields

$$2vwA = -u(v+w)D + v(w+u)E + w(u+v)F.$$

Since AA' and EF right bisect each other, we have $A' + A = E + F$, whence

$$\begin{aligned} 2vwA' &= 2vw[-A + E + F] = -2vwA + 2vwE + 2vwF \\ &= u(v+w)D + v(w-u)E + w(v-u)F. \end{aligned}$$

Similarly,

$$2wuB' = u(w-v)D + v(w+u)E + w(u-v)F$$

and

$$2uvC' = u(v-w)D + v(u-w)E + w(u+v)F.$$

$$\begin{aligned} \frac{[A'B'C']}{[DEF]} &= \frac{1}{8u^2v^2w^2}\begin{vmatrix} u(v+w) & v(w-u) & w(v-u) \\ u(w-v) & v(w+u) & w(u-v) \\ u(v-w) & v(u-w) & w(u+v) \end{vmatrix} \\ &= \frac{1}{8uvw}\begin{vmatrix} v+w & w-u & v-u \\ 2w & 2w & 0 \\ 2v & 0 & 2v \end{vmatrix} = \frac{1}{2u}\begin{vmatrix} v+w & w-u & v-u \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1. \end{aligned}$$

Solution 4, by Prithwijit De.

Let r, R, a, b, c, I, α, β, γ be the inradius, circumradius, sides, incentre and angles of triangle ABC. Observe that AA', BB' and CC' intersect at I, and that AA' and EF right bisect each other. We have

$$\begin{aligned} &AI = r\csc\alpha/2, \quad AF = r\cot(\alpha/2) \\ &AA' = 2AF\cos(\alpha/2) = 2r\cot(\alpha/2)\cos(\alpha/2) = 2r\csc(\alpha/2)\cos^2(\alpha/2) \\ &A'I = AA' - AI = r\csc(\alpha/2)\cos\alpha. \end{aligned}$$

We note that, if A is obtuse, then $\cos\alpha < 0$, $\cos^2(\alpha/2) < 1/2$ and $AA' < AI$, I lies outside triangle $A'B'C'$ and angles $A'IB'$ and AIB are supplementary. If one angle of ABC is obtuse, we can regard expressions for length and area as signed.

Since

$$r = 4R\sin(\alpha/2)\sin(\beta/2)\sin(\gamma/2), \; B'I = r\csc(\beta/2)\cos\beta, \; \angle AIB = 90^\circ + (\gamma/2),$$

we get

$$\begin{aligned} [A'IB'] &= \tfrac{1}{2}(A'I)(B'I)\sin(\angle A'IB') = \tfrac{1}{2}(A'I)(B'I)\sin(\angle AIB) \\ &= \tfrac{1}{2}r^2\cos\alpha\cos\beta\csc(\alpha/2)\csc(\beta/2)\cos(\gamma/2) \\ &= 2rR\cos\alpha\cos\beta\sin(\gamma/2)\cos(\gamma/2) = rR\cos\alpha\cos\beta\sin\gamma. \end{aligned}$$

Using analogous expressions for $[B'IC']$ and $[C'IA']$ and additionally the fact that $\tan(\alpha+\beta+\gamma)=0$. we find that

$$\begin{aligned}[A'B'C'] &= [A'IB'] + [B'IC'] + [C'IA'] \\ &= rR(\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha)\sin\alpha\sin\beta\sin\gamma \\ &= rR\sin\alpha\sin\beta\sin\gamma.\end{aligned}$$

On the other hand,

$$\begin{aligned}[DEF] &= \tfrac{1}{2}(DE)(DF)\sin(\angle FDE) \\ &= \tfrac{1}{2}(2r\cos\gamma/2)(2r\cos\beta/2)\sin(90^\circ - \alpha/2) \\ &= 2r^2\cos(\alpha/2)\cos(\beta/2)\cos(\gamma/2) \\ &= 2r(4R\sin(\alpha/2)\cos(\alpha/2)\sin(\beta/2)\cos(\beta/2)\sin(\gamma/2)\cos(\gamma/2) \\ &= rR\sin\alpha\sin\beta\sin\gamma \\ &= [A'B'C'].\end{aligned}$$

Solution 5, by the proposer.

This solution is based on the diagram supplied with the problem, but can be adapted for different positioning of the points. Let α, β, γ be the angles at the respective vertices A, B, C of the triangle. Then the angles of the triangle DEF are $90^\circ-(\alpha/2)$, $90^\circ-(\beta/2)$, $90^\circ-(\gamma/2)$ at the respective vertices D, E, F, and the angle at BFB' and CEC' are respectively $180^\circ-\beta$ and $180^\circ-\gamma$. Therefore

$$\begin{aligned}\angle B'FE &= \angle BFD + \angle DFE - \angle BFB' \\ &= (90^\circ - \tfrac{\beta}{2}) + (90^\circ - \tfrac{\gamma}{2}) - (180^\circ - \beta) = \tfrac{1}{2}(\beta-\gamma),\end{aligned}$$

and

$$\begin{aligned}\angle C'EF &= \angle C'EC - \angle FED - \angle CED \\ &= (180^\circ - \gamma) - (90^\circ - \tfrac{\beta}{2}) - (90^\circ - \tfrac{\gamma}{2}) = \tfrac{1}{2}(\beta-\gamma).\end{aligned}$$

It follows that $B'F\|C'E$. Similarly $A'F\|C'D$ and $A'E\|B'D$.

Since triangles on the same base between the same parallels have equal area, $[B'C'F]=[B'EF]$, $[A'C'F]=[A'DF]$, $[B'DE]=[A'DB']$. Therefore

$$\begin{aligned}[A'B'C'] &= [B'C'F] + [A'C'F] + [FB'D] - [A'DB'] - [A'DF] \\ &= [B'EF] + [A'DF] + [F'BD] - [B'DE] - [A'DF] \\ &= [B'EF] + [FB'D] - [B'DE] = [DEF].\end{aligned}$$

Comments from the editor. J.-C. Andrieux used barycentric coordinates with $A\sim(1,0,0)$, $B\sim(0,1,0)$, $C\sim(0,0,1)$, $D\sim(0,w/a,v/a)$, $E\sim(w/b,0,u/b)$,

$F \sim (v/c, u/c, 0)$. Using a determination for the ratio of areas related to those in Solutions 2 and 3, he found that

$$\frac{[DEF]}{[ABC]} = \frac{1}{abc}\begin{vmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{vmatrix} = \frac{2uvw}{abc}.$$

The barycentric coordinates for the vertices of triangle $A'B'C'$ are

$$\left(\frac{w}{b} + \frac{v}{c} - 1, \frac{u}{c}, \frac{u}{b}\right); \quad B' \sim \left(\frac{v}{c}, \frac{u}{c} + \frac{w}{a} - 1, \frac{v}{a}\right); \quad C' \sim (\frac{w}{b}, \frac{w}{a}, \frac{v}{a} + \frac{u}{b} - 1).$$

The determinant that represents $[A'B'C']/[ABC]$ is pretty indigestible and seems to require the muscle of a computer to sort out. When the dust settles, this ratio equals that of $[DEF]/[ABC]$. There were two other solvers who had to resort to a computer programme to simplify a determinant.

The solver who submitted an incorrect solution used the barycentric coordinates for the situation that D, E and F were the intersection points of the angle bisectors with the respective sides BC, CA and AB. It turns out that, if A' is the result of reflecting A through the midpoint of EF (and similarly for B' and C'), then the conditions of the result in Solution 1 are satisfied. The determinants in this situation turn out to be easy to determine. In the ratio $[A'B'C']/[ABC]$, the determinant $f(a, b, c)$ to be evaluated is a polynomial of degree 6 that is easily seen to vanish when any of a, b, c, $a + b$, $b + c$, $c + a$ vanish, so has the form $kabc(a + b)(b + c)(c + a)$, where $k = f(1, 1, 1)/8$.

4695. *Proposed by George Apostolopoulos.*

Let triangle ABC have sides $BC = a, CA = b$ and $AB = c$ and circumradius R. Equilateral triangles A_1BC, B_1CA and C_1AB are drawn externally to triangle ABC. Let K, L and M be the centroids of the equilateral triangles, respectively. Prove that

$$[ALM] + [BMK] + [CKL] \leq \frac{3\sqrt{3}}{4}R^2,$$

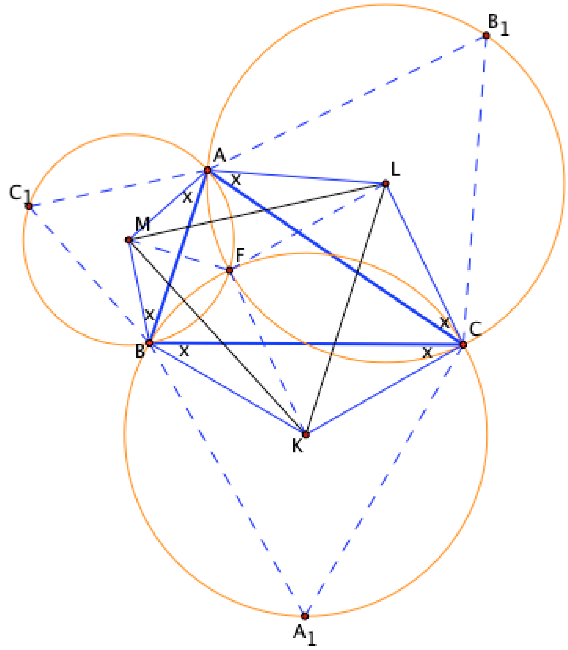
where $[\cdot]$ denotes the area of the corresponding triangle.

We received 14 submissions, all but one of which were correct. We feature a composite of the similar solutions by Subhankar Gayen and by Marie-Nicole Gras.

Let r be the inradius and s the semi-perimeter of $\triangle ABC$; we denote by A, B and C the angles $\angle BAC$, $\angle CBA$ and $\angle ACB$, respectively.

Since L is the centroid of equilateral $\triangle B_1AC$, and M is the centroid of equilateral $\triangle C_1AB$, we deduce the relations

$$AL = \frac{b\sqrt{3}}{3}, \quad AM = \frac{c\sqrt{3}}{3} \quad \text{and} \quad \angle LAC = \angle MAB = 30^{\circ}.$$



It follows $\angle LAM = A + 60^{\mathrm{o}}$, and

$$\begin{aligned}
[ALM] &= \frac{1}{2} AL \cdot AM \sin(\angle LAM) \\
&= \frac{1}{2} \frac{b\sqrt{3}}{3} \frac{c\sqrt{3}}{3} \sin(A + 60^{\mathrm{o}}) = \frac{bc}{6} \sin(A + 60^{\mathrm{o}}) \\
&= \frac{bc}{6} \left[\frac{1}{2} \sin A + \frac{\sqrt{3}}{2} \cos A \right] = \frac{bc}{12} \sin A + \frac{\sqrt{3}}{24} \left(2bc \cos A \right) \\
&= \frac{bc}{12} \cdot \frac{a}{2R} + \frac{\sqrt{3}}{24} \left(b^2 + c^2 - a^2 \right) \quad \text{[Sine Law and Cosine Law].}
\end{aligned}$$

By permuting a, b, c, we deduce $[BMK]$ and $[CKL]$, and obtain

$$F := [ALM] + [BMK] + [CKL] = \frac{abc}{8R} + \frac{\sqrt{3}}{24} \left(a^2 + b^2 + c^2 \right). \tag{1}$$

Since $abc = 4srR$, we obtain

$$F = \frac{sr}{2} + \frac{\sqrt{3}}{24} \left(a^2 + b^2 + c^2 \right).$$

Now, we use the following relations (see, for example, O. Bottema et al, *Geometric Inequalities* (1968)),

$$\begin{aligned}
& r \leq \tfrac{R}{2} && (5.1) \\
& s \leq \tfrac{3\sqrt{3}R}{2} && (5.3) \\
& a^2 + b^2 + c^2 \leq 9R^2 && (5.13)
\end{aligned}$$

with equality in all when ΔABC is equilateral. We deduce

$$F \leq \frac{1}{2} \cdot \frac{3R\sqrt{3}}{2} \cdot \frac{R}{2} + \frac{\sqrt{3}}{24}(9R^2) = \frac{3\sqrt{3}}{4} R^2,$$

with equality if and only if the given triangle is equilateral. Alternatively, we can combine the Weitzenböck inequality, namely

$$sr \leq \frac{a^2 + b^2 + c^2}{4\sqrt{3}},$$

with Bottema's equation (5.13) to get the same result.

Editor's comments. Readers will certainly recognize ΔKLM as the outer Napoleon triangle of the given triangle ABC. (See, for example, *Geometry Revisited* by H.S.M. Coxeter and S.L. Greitzer (1967), Section 3.3, pages 60-65.) Among other properties of the configuration, the circumcircles of the triangles such as A_1BC (erected outwardly on the sides of ΔABC) meet in the Fermat point F of ΔABC. The three triangles with common vertex F and opposite sides LM, MK, KL are congruent, respectively, to the three triangles ALM, BMK, CKL of problem 4695. Consequently the sum of interest, namely equation (1), equals the area of the outer Napoleon triangle. In other words, we have shown that the area of the outer Napoleon triangle is bounded above by $\frac{3\sqrt{3}}{4}R^2$. By anaolgy, if equilateral triangles are erected internally on the sides of ΔABC, their centers are the vertices of the inner Napoleon triangle, whose area is

$$\frac{\sqrt{3}}{24}(a^2 + b^2 + c^2) - \frac{sr}{2}.$$

Note that the outer and inner Napoleon triangles of any triangle ABC differ in area by $[ABC]$.

4696. *Proposed by Elena Corobea.*

Find the following limit:

$$\lim_{n\to\infty} \int_0^1 \frac{\left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-1}}{n-1}\right)^{n+1}}{\left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}\right)^n} dx.$$

There were 16 solutions submitted, of which 14 were correct. We present a solution followed by many solvers.

For $n \geq 1$, let

$$f_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}.$$

For $n \geq 2$ and $x > 0$, $f_{n-1}(x) < f_n(x)$. Denote the integral in the problem by I_n.

$$\begin{aligned} I_n &= \int_0^1 \left[\frac{f_{n-1}(x)}{f_n(x)}\right]^n f_{n-1}(x)\,dx < \int_0^1 f_{n-1}(x)\,dx \\ &= 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} = 2 - \frac{1}{n}. \end{aligned}$$

For the lower bound of I_n, we first note that, when $b > a \geq 0$,

$$\begin{aligned}\frac{a^{n+1}}{b^n} &= b - \frac{b^{n+1} - a^{n+1}}{b^n} \\ &= b - (b-a)\left(1 + \frac{a}{b} + \frac{a^2}{b^2} + \cdots + \frac{a^n}{b^n}\right) \\ &\geq b - (b-a)(n+1).\end{aligned}$$

Hence

$$\frac{f_{n-1}(x)^{n+1}}{f_n(x)^n} \geq f_n(x) - \frac{(n+1)x^n}{n},$$

from which

$$\begin{aligned}I_n &\geq \int_0^1 f_n(x)\,dx - \int_0^1 \frac{(n+1)x^n}{n}\,dx \\ &= 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} - \frac{1}{n} \\ &= 2 - \frac{1}{n+1} - \frac{1}{n}.\end{aligned}$$

Therefore

$$2 - \frac{1}{n} - \frac{1}{n+1} < I_n < 2 - \frac{1}{n}.$$

from which $\lim_{n\to\infty} I_n = 2$.

Comments from the editor. About half the solvers essentially used the foregoing argument. Another approach followed by others relied on the fact that the limit of the integrand was $1 - \ln(1-x)$ and used either the Dominated or Monotone Convergence Theorem to show that the limit is

$$\int_0^1 (1 - \ln(1-x))\,dx = [2x - (x-1)\ln(1-x)]_0^1 = 2.$$

Showing that the integrand increases with respect to n involves showing that $f_{n-1}(x)f_{n+1}(x) \leq f_n(x)^2$.

It was pointed out to us by a reader that a similar problem by the same proposer appeared in *American Mathematical Monthly* 128:3 (March, 2021), 276-277, to wit

> **12242.** For $n \geq 1$, let
>
> $$I_n = \int_0^1 \frac{\left(\sum_{k=0}^n x_k/(2k+1)\right)^{2022}}{\sum_{k=0}^{n+1}(x^k/(2k+1))^{2021}}\,dx$$
>
> and $L = \lim_{n\to\infty} I_n$. Compute L and $\lim_{N\to\infty} n(I_n - L)$.

We will take this opportunity to point out that proposers should not submit identical or equivalent problems to multiple sources at the same time. This is an unethical and disrespectful practice.

4697. *Proposed by Amit Kumar Basistha.*

Let $f : \mathbb{N} \to \mathbb{N}$ with $f(1) = 1, f(2) = a$ for some $a \in \mathbb{N}$ and, for each positive integer $n \geq 3$, $f(n)$ is the smallest value not assumed at lower integers that is coprime with $f(n-1)$. Prove that f is onto.

We received 7 submissions and they were all correct. We present the solution by the majority of solvers, slightly modified by the editor.

We first show that the image of the function f contains the set of primes. Assume for contradiction that there is a prime p which is not in the image of f. Since f is one-to-one, there is an $N \in \mathbb{N}$ such that $f(m) > p$ for each $m > N$. Given any $m > N$, since $f(m) > p$, it follows that $f(m-1)$ has a common factor with p, that is, $p \mid f(m-1)$. In particular, $p \mid f(N)$ and $p \mid f(N+1)$, contradicting the assumption that $f(N)$ and $f(N+1)$ are coprime.

Next we show that f is onto. Assume for contradiction that there is a natural number k which is not in the image of f. From the definition of the function f, it is clear that f is one-to-one. Thus, there is an $N \in \mathbb{N}$ such that $f(m) > k$ for each $m > N$. Given any $m > N$, since $f(m) > k$, it follows that $f(m-1)$ has a common factor with k. In particular, for every prime p with $p \nmid k$, we have that $f(m-1) \neq p$. Therefore, if p is a prime with $p \nmid k$, then $p \in \{f(1), f(2), \ldots, f(N-1)\}$. This is impossible as the set of primes not dividing k is infinite.

Editor's Comment. Barbara Roy pointed out that the above proof in fact shows the following more general statement:

> Let $f : \mathbb{N} \to \mathbb{N}$ be so that for all sufficiently large n, $f(n)$ is the smallest value not assumed at lower integers that is coprime with $f(n-1)$. Then f is onto.

4698. *Proposed by Goran Conar.*

Let $x_1, \ldots, x_n > 0$ be real numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$\sum_{i=1}^{n} x_i \ln(1 + x_i) < \ln 2\,.$$

We received 30 solutions, of which two were incorrect. We present the solution by Amit Kumar Basistha, modified slightly by the editor.

First (as several solvers pointed out), equality holds when $n = 1$. We thus assume $n \geq 2$. We then have, since $x_i < 1$ for all i,

$$\begin{aligned}\sum_{i=1}^{n} x_i \ln(1+x_i) &= \sum_{i=1}^{n} \ln\left[(1+x_i)^{x_i}\right] = \ln \prod_{i=1}^{n} \left[(1+x_i)^{x_i}\right] \\ &< \ln \prod_{i=1}^{n} 2^{x_i} = \ln\left(2^{\sum_{i=1}^{n} x_i}\right) = \ln 2.\end{aligned}$$

4699. *Proposed by Mihaela Berindeanu.*

Let ABC be a non-isosceles triangle with acute angles B and C. Let D and E be two points outside the triangle and F be the foot of the altitude from A. Show that if

$$\angle EAC = \angle ECA = \angle ABC, \quad \angle DAB = \angle ABD = \angle BCA$$

and $BE \cap CD \cap AF = \{X\}$, then $AX = XF$.

All 11 submissions that we received were correct; we shall feature two of them.

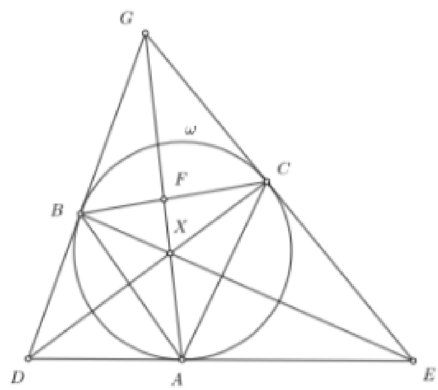
Solution 1 by the UCLan Cyprus Problem Solving Group.

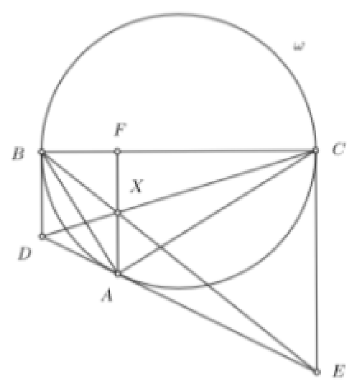
Let ω be the circumcircle of triangle ABC. Since $\angle DAB = \angle BCA$, then DA is tangent to ω at A. Similarly, DB, EA, EC are also tangent to ω. In particular D, A, E are collinear.

We claim that BD and CE are parallel. If this is not the case, then let G be their point of intersection. Since

$$\frac{DA}{AE} \cdot \frac{EC}{CG} \cdot \frac{GB}{BD} = \frac{DA}{DB} \cdot \frac{EC}{EA} \cdot \frac{GB}{GC} = 1\,,$$

while BE and CD are assumed to intersect at X, Ceva's theorem implies that AG must also pass through X. It follows that GF is perpendicular to BC and, since $GB = GC$, then $BF = CF$. It now follows that $AB = AC$, a contradiction as the triangle ABC is nonisosceles.





Since the parallel lines BD and CE are tangent to ω, BC must be a diameter. Thus, $\angle DBC = \angle ECB = 90°$. In particular, AF is also parallel to BD and CE. So we have the similarities of triangles $EAX \sim EDB, CXF \sim CDB$ and $XBD \sim XEC$ from which we get

$$\frac{AX}{BD} = \frac{EX}{EB} = \frac{CX}{CD} = \frac{FX}{BD}\,.$$

Therefore $AX = FX$ as required.

Solution 2 by Titu Zvonaru.

Since $\angle EAC = \angle ECA = \angle ABC$, the lines EA and EC are tangent to the circumcircle of ΔABC; analogously, DA and DB are also tangents to the circumcircle. It follows that BE and CD are symmedians in ΔABC and, consequently, so is AF. It is known that the altitude AF is a symmedian if and only if $AB = AC$ or $\angle BAC = 90°$; here is an easy proof:

AF is a symmedian if and only if $\frac{FB}{FC} = \frac{c^2}{b^2}$; AF is an altitude if and only if $\frac{FB}{FC} = \frac{c\cos B}{b\cos C}$; consequently,

$$\begin{aligned} & \frac{c\cos B}{b\cos C} = \frac{c^2}{b^2} \\ \Leftrightarrow\ & b\cos B = c\cos C \\ \Leftrightarrow\ & \sin 2B = \sin 2C \\ \Leftrightarrow\ & \angle B = \angle C \ \textit{or}\ \ 2\angle B + 2\angle C = 180°. \end{aligned}$$

Because the triangle is assumed to be nonisosceles, we conclude that $\angle A = 90°$. We denote $\{M\} = BE \cap AC$ and $\{N\} = CD \cap AB$. Van Aubel's theorem implies that

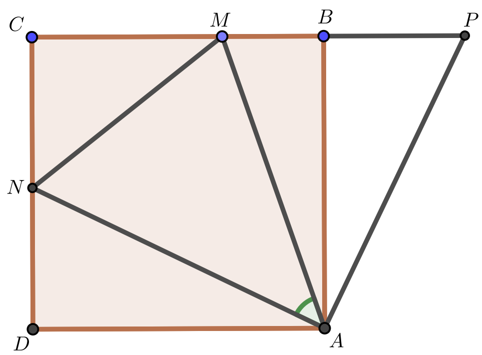
$$\frac{AX}{XF} = \frac{AM}{MC} + \frac{AN}{NB} = \frac{AB^2}{BC^2} + \frac{AC^2}{BC^2} = 1;$$

hence $AX = XF$, as claimed.

4700. *Proposed by Hung Nguyen Viet.*

Let $ABCD$ be a unit square. The points M and N lie on the sides BC and CD respectively such that $\angle MAN = 45°$. Prove that

$$MN + BM \cdot DN = 1.$$



We received more than 44 solutions. We selected the following 11 solutions to show the variety of methods that the solvers used.

Solution 1, by the proposer.

We take point P on the ray CB such that $BP = DN$. This leads us to $AN = AP$ and consequently $\Delta AMN = \Delta AMP$. Hence $MN = MP = MB + ND$. Now we have

$$\angle BAM + \angle DAN = 45^\circ.$$

Therefore

$$1 = \tan(\angle BAM + \angle DAN) = \frac{\tan\angle BAM + \tan\angle DAN}{1 - \tan\angle BAM \tan\angle DAN}$$

or

$$1 - \tan\angle BAM \tan\angle DAN = \tan\angle BAM + \tan\angle DAN$$

or

$$1 - \frac{BM}{AB} \cdot \frac{DN}{AD} = \frac{BM}{AB} + \frac{DN}{AD}.$$

From here and noting that $AB = AD = 1$, we get

$$1 - BM \cdot DN = BM + DN = MN.$$

The conclusion follows.

Solution 2, by Brian D. Beasley.

Let $\alpha = \angle DAN$ (with $0^\circ \leq \alpha \leq 45^\circ$) and $\beta = \angle BAM$, so that $\alpha + \beta = 45^\circ$. Then $DN = \tan\alpha$, $BM = \tan\beta$, and

$$MN = \sqrt{(1-\tan\alpha)^2 + (1-\tan\beta)^2}.$$

Applying the tangent subtraction formula yields $\tan\beta = (1-\tan\alpha)/(1+\tan\alpha)$ and hence

$$\begin{aligned} MN + BM \cdot DN &= \sqrt{(1-\tan\alpha)^2 + \left(\frac{2\tan\alpha}{1+\tan\alpha}\right)^2} + \frac{\tan\alpha(1-\tan\alpha)}{1+\tan\alpha} \\ &= \frac{1+\tan^2\alpha}{1+\tan\alpha} + \frac{\tan\alpha - \tan^2\alpha}{1+\tan\alpha} \\ &= 1. \end{aligned}$$

Solution 3, by Brian Bradie.

Let $\theta = \angle BAM$. Then $\angle DAN = 45^\circ - \theta$, $BM = \tan\theta$, and

$$DN = \tan(45^\circ - \theta) = \frac{1-\tan\theta}{1+\tan\theta}.$$

Next, $CM = 1 - \tan\theta$,

$$CN = 1 - \frac{1-\tan\theta}{1+\tan\theta} = \frac{2\tan\theta}{1+\tan\theta},$$

and

$$MN^2 = (1 - \tan\theta)^2 + \frac{4\tan^2\theta}{(1+\tan\theta)^2} = \frac{(1+\tan^2\theta)^2}{(1+\tan\theta)^2},$$

so

$$MN = \frac{1+\tan^2\theta}{1+\tan\theta}.$$

Finally,

$$MN + BM \cdot DN = \frac{1+\tan^2\theta}{1+\tan\theta} + \frac{\tan\theta - \tan^2\theta}{1+\tan\theta} = \frac{1+\tan\theta}{1+\tan\theta} = 1.$$

Solution 4, by the Eagle Problem Solvers.

Let $b = BM$ and $d = DN$. Then $AM = \sqrt{b^2+1}$, $AN = \sqrt{d^2+1}$, $\tan\angle BAM = b$, and $\tan\angle NAD = d$. Since

$$\angle BAM + \angle NAD = \frac{\pi}{2} - \angle MAN = \frac{\pi}{4},$$

then

$$d = \tan\angle NAD = \tan\left(\frac{\pi}{4} - \angle BAM\right) = \frac{1-b}{1+b}.$$

Notice that

$$d^2 + 1 = \left(\frac{1-b}{1+b}\right)^2 + 1 = \frac{2(b^2+1)}{(b+1)^2}$$

and

$$(b^2+1)(d^2+1) = 2\left(\frac{b^2+1}{b+1}\right)^2.$$

From the Law of Cosines,

$$\begin{aligned}
MN^2 &= AM^2 + AN^2 - 2(AM)(AN)\cos\frac{\pi}{4} \\
&= (b^2+1) + (d^2+1) - \sqrt{2(b^2+1)(d^2+1)} \\
&= (b^2+1) + \frac{2(b^2+1)}{(b+1)^2} - 2\left(\frac{b^2+1}{b+1}\right) \\
&= \frac{b^2+1}{(b+1)^2}\left[b^2+2b+1+2-2(b+1)\right] \\
&= \left(\frac{b^2+1}{b+1}\right)^2.
\end{aligned}$$

Thus, $MN = \frac{b^2+1}{b+1}$ and

$$MN + BM \cdot DN = \frac{b^2+1}{b+1} + b\left(\frac{1-b}{1+b}\right) = \frac{b^2+1+b-b^2}{1+b} = 1.$$

Solution 5, by Prithwijit De.

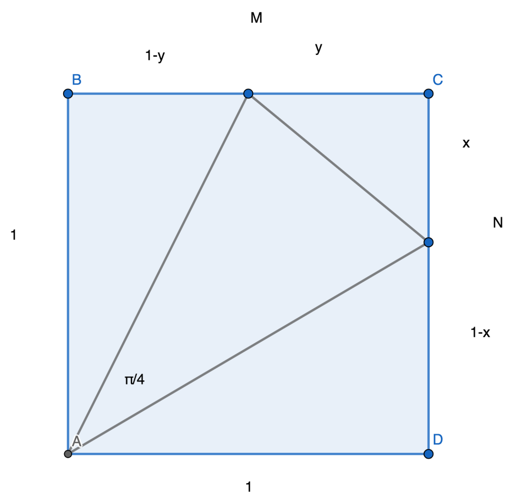
Let $BM = x$, $DN = y$ and $\angle BAM = \alpha$. Then $x = \tan\alpha$ and $y = \tan(45^\circ - \alpha)$, whence $y = \dfrac{1-x}{1+x}$. Now,

$$\begin{aligned} MN^2 &= (1-x)^2 + (1-y)^2 = (y+xy)^2 + (1-y)^2 = 1 + (xy)^2 + 2y(y+xy-1) \\ &= (1 - 2xy + x^2y^2) = (1-xy)^2. \end{aligned}$$

Therefore

$$MN + BM.DN = (1-xy) + xy = 1.$$

Solution 6, by Noah Garson.



Note that in the above diagram we have defined the variables we will be using in the proof; y is the distance from C to M, thus $1-y$ is the distance from B to M. Similarily, we define x as the distance between C and N and $1-x$ as the distance between N and D.

If we refer to the above diagram, it is clear that $\arctan(1-y)+\arctan(1-x) = \pi/4$. By the formula for addition in arctangent, it follows that

$$\arctan\left(\frac{1-x+1-y}{1-(1-x)(1-y)}\right) = \pi/4, \text{ and so } \frac{2-x-y}{1-1+x+y-xy} = \tan(\pi/4) = 1.$$

Thus we have:

$$\begin{aligned} 2-x-y &= x+y-xy, \\ 2-2x-2y+xy &= 0. \end{aligned}$$

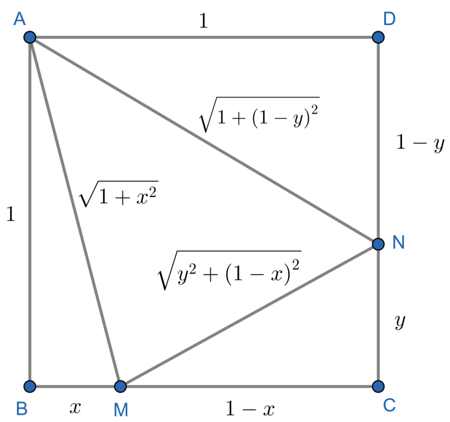
Now we just have to prove that $MN + BM \cdot DN = \sqrt{x^2+y^2} + (1-y)(1-x) = 1$:

$$\begin{aligned}
&(2-2x-2y+xy)(xy) = 0,\\
&2xy - 2x^2y - 2xy^2 + x^2y^2 + x^2 + y^2 = x^2 + y^2,\\
&(y+x-xy)^2 = x^2+y^2,\\
&y + x - xy = \sqrt{x^2+y^2},\\
&1-(1-y-x+xy) = \sqrt{x^2+y^2},\\
&1-(1-x)(1-y) = \sqrt{x^2+y^2},\\
&1 = \sqrt{x^2+y^2} + (1-x)(1-y).
\end{aligned}$$

As required, hence the claim that $MN + BM \cdot DN = 1$ has been shown.

Solution 7, by David A. Huckaby.

Let $BM = x$ and $CN = y$ so that $AM = \sqrt{1+x^2}$, $AN = \sqrt{1+(1-y)^2}$, and $MN = \sqrt{y^2+(1-x)^2}$. See the figure below.



The Law of Cosines applied to triangle AMN gives

$$\begin{aligned}
MN^2 &= AM^2 + AN^2 - 2 \cdot AM \cdot AN \cdot \cos 45^\circ\\
y^2 + (1-x)^2 &= 1 + x^2 + 1 + (1-y)^2 - 2\sqrt{1+x^2}\sqrt{1+(1-y)^2} \cdot \frac{\sqrt{2}}{2}\\
y^2 + 1 - 2x + x^2 &= 3 + x^2 - 2y + y^2 - \sqrt{2}\sqrt{1+x^2}\sqrt{1+(1-y)^2}\\
2 + 2x - 2y &= \sqrt{2}\sqrt{1+x^2}\sqrt{1+(1-y)^2}\\
\sqrt{2}(1+x-y) &= \sqrt{1+x^2}\sqrt{1+(1-y)^2} \qquad (1)
\end{aligned}$$

The area Δ of triangle AMN is the area of square $ABCD$ minus the sum of the areas of triangles ABM, CMN and ADN, so that

$$\Delta = 1 - \frac{1}{2}(1)(x) - \frac{1}{2}(y)(1-x) - \frac{1}{2}(1)(1-y) = \frac{1}{2}(1-x+xy).$$

Now from the Law of Sines, $\frac{MN}{\sin\angle MAN} = 2R$, where R is the circumradius of triangle AMN. Combining this with the well-known formula $R = \frac{abc}{4\Delta}$, where a, b, and c are the sides of the triangle, gives

$$\frac{MN}{\sin\angle MAN} = 2R = 2\cdot\frac{AM\cdot AN\cdot MN}{4\Delta},$$

that is,

$$\begin{aligned}
\frac{\sqrt{y^2+(1-x)^2}}{\frac{\sqrt{2}}{2}} &= 2\cdot\frac{\sqrt{1+x^2}\sqrt{1+(1-y)^2}\sqrt{y^2+(1-x)^2}}{4\cdot\frac{1}{2}(1-x+xy)}\\
\frac{1}{\frac{\sqrt{2}}{2}} &= \frac{\sqrt{1+x^2}\sqrt{1+(1-y)^2}}{1-x+xy}\\
\sqrt{2}(1-x+xy) &= \sqrt{1+x^2}\sqrt{1+(1-y)^2} \qquad (2)
\end{aligned}$$

Equations (1) and (2) give $1+x-y = 1-x+xy$, whence $y = \frac{2x}{x+1}$.

So,

$$\begin{aligned}
MN + BM\cdot DN &= \sqrt{y^2+(1-x)^2} + x(1-y)\\
&= \sqrt{\left(\frac{2x}{x+1}\right)^2 + (1-x)^2} + x\left(1-\frac{2x}{x+1}\right)\\
&= \sqrt{\frac{(2x)^2}{(x+1)^2} + \frac{(x^2-1)^2}{(x+1)^2}} + x - \frac{2x^2}{x+1}\\
&= \sqrt{\frac{(x^2+1)^2}{(x+1)^2}} + x - \frac{2x^2}{x+1}\\
&= \frac{x^2+1+x(x+1)-2x^2}{x+1}\\
&= \frac{x+1}{x+1}\\
&= 1.
\end{aligned}$$

Solution 8, by Parviz Khalili.

Let $\angle AND = \alpha$ and $\angle AMB = \beta$ then $\alpha+\beta = 135^\circ$. From triangles ABM and ADN we have

$$DN = \cot\alpha,\ \ BM = \cot\beta,\ \ AN = \csc\alpha,\ \ AM = \csc\beta$$

Employ the Law of Cotangent

$$\cot(\alpha+\beta) = \cot(135^\circ) = -1 = \frac{\cot\alpha+\cot\beta}{\cot\alpha\cot\beta-1} = \frac{DN+BM}{DN\cdot BM-1}$$

Thus, we have

$$BM + DN + BM \cdot DN = 1$$

Therefore, we need to show

$$MN = BM + DN = \cot\alpha + \cot\beta.$$

Next, apply the Law of Cosine to triangle AMN. (It is useful to note that $\cos(\alpha+\beta) = \cos(135^\circ) = -\cos(45^\circ)$.)

$$\begin{aligned}
MN^2 &= AM^2 + AN^2 - 2\,AM\cdot AN\cos 45^\circ \\
&= \csc^2\alpha + \csc^2\beta + 2\csc\alpha\csc\beta\cos(\alpha+\beta) \\
&= 1 + \cot^2\alpha + 1 + \cot^2\beta + 2\csc\alpha\csc\beta\cos(\alpha+\beta) \\
&= \cot^2\alpha + \cot^2\beta + 2\Big(1 + \frac{\cos(\alpha+\beta)}{\sin\alpha\sin\beta}\Big).
\end{aligned}$$

Simplifying we obtain

$$MN^2 = \cot^2\alpha + \cot^2\beta + 2\cot\alpha\cot\beta = (\cot\alpha + \cot\beta)^2$$

Thus the equality is verified and the proof is complete.

Solution 9, by Theo Koupelis.

Let $\angle BAM = \theta$. Then $\angle DAN = 45^\circ - \theta$, and therefore

$$BM = \tan\theta,\ MC = 1 - \tan\theta,$$

$$DN = \tan(45^\circ - \theta) = \frac{1-\tan\theta}{1+\tan\theta},\ CN = 1 - DN = \frac{2\tan\theta}{1+\tan\theta}.$$

From the right triangle MCN, we have

$$MN^2 = (1-\tan\theta)^2 + \left(\frac{2\tan\theta}{1+\tan\theta}\right)^2 = \left(\frac{1+\tan^2\theta}{1+\tan\theta}\right)^2,$$

and therefore

$$MN = \frac{1+\tan^2\theta}{1+\tan\theta} = 1 - \tan\theta\cdot\frac{1-\tan\theta}{1+\tan\theta} = 1 - BM\cdot DN.$$

Solution 10, by Marie-Nicole Gras.

Put $u = BM$, $v = DN$, $\theta = \angle BAM$ and $\omega = \angle DAN$. In the right angled $\triangle NCM$, we have

$$MN^2 = (1-u)^2 + (1-v)^2,$$

and then, it is equivalent to prove that $MN^2 = (1 - BM\cdot DN)^2$, whence

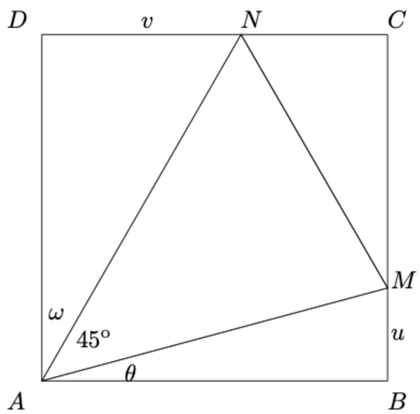
$$(1-u)^2 + (1-v)^2 = (1-uv)^2.$$

By assumption, $\theta + \omega = 45^{\mathrm{o}}$; it follows

$$\tan(\theta + \omega) = \frac{\tan(\theta) + \tan(\omega)}{1 - \tan(\theta)\tan(\omega)} = 1.$$

However, $\tan(\theta) = u$ and $\tan(\omega) = v$; we deduce

$$u + v = 1 - uv.$$



It results

$$\begin{aligned}(1-u)^2 + (1-v)^2 - (1-uv)^2 &= (1-u)^2 + (1-v)^2 - (u+v)^2\\ &= -2u - 2v + 2 - 2uv\\ &= -2(u + v - 1 + uv)\\ &= 0.\end{aligned}$$

Then, we have shown the equality.

Solution 11, by Sorin Rubinescu.

We denote $BM = x, DN = y, x, y < 1$. In $\triangle MAN$ we have

$$MN^2 = AN^2 + AM^2 - 2\cdot AM\cdot AN\cdot\cos 45^\circ = 1 + y^2 + 1 + x^2 - \sqrt{2\left(1+x^2\right)\left(1+y^2\right)}.$$

In $\triangle NCM$, we have

$$MN^2 = (1-x)^2 + (1-y)^2 = x^2 + y^2 - 2x - 2y + 2. \qquad \text{(i)}$$

Equating the two relations from above, we get that

$$\begin{aligned}&x^2 + y^2 - x^2y^2 + 4xy - 1 = 0\\ \Leftrightarrow\ &\left(x^2 + y^2 + 2xy\right) - \left(x^2y^2 - 2xy + 1\right) = 0\\ \Leftrightarrow\ &(x+y)^2 = (1-xy)^2.\end{aligned}$$

Thus,

$$\begin{aligned} & x+y=\mid 1-xy \mid \\ \Rightarrow\ & x+y=1-xy,\ x\cdot y<1\,(x<1,y<1) \\ \Leftrightarrow\ & x+y+xy=1 \\ \Leftrightarrow\ & y=\frac{1-x}{1+x}. \end{aligned}$$

Substituting this into (i), we get that

$$MN^2=\frac{x^4+2x^2+1}{(1+x)^2}\Leftrightarrow MN=\frac{x^2+1}{x+1}. \qquad \text{(ii)}$$

Then $BM\cdot DN=xy=\dfrac{x-x^2}{1+x}$, so by (ii) the conclusion now follows.