Canadian Mathematical Olympiad Qualifying Repêchage 2022

A competition of the Canadian Mathematical Society.

Official Problem Set

1. Let $n \geq 2$ be a positive integer. On a spaceship, there are n crewmates. At most one accusation of being an imposter can occur from one crewmate to another crewmate. Multiple accusations are thrown, with the following properties:

   - Each crewmate made a different number of accusations.
   - Each crewmate received a different number of accusations.
   - A crewmate does not accuse themself.

   Prove that no two crewmates made accusations at each other.

2. Determine all pairs of integers (m, n) such that $m^2 + n$ and $n^2 + m$ are both perfect squares.

3. Consider n real numbers $x_0, x_1, \ldots, x_{n-1}$ for an integer $n \geq 2$. Moreover, suppose that for any integer i, $x_{i+n} = x_i$. Prove that

$$\sum_{i=0}^{n-1} x_i(3x_i - 4x_{i+1} + x_{i+2}) \geq 0.$$

4. For a non-negative integer n, call a one-variable polynomial F with integer coefficients n-good if:

   (a) $F(0) = 1$

   (b) For every positive integer $c, F(c) > 0$, and

   (c) There exist exactly n values of c such that $F(c)$ is prime.

   Show that there exist infinitely many non-constant polynomials that are not n-good for any n.

5. Alice has four boxes, 327 blue balls, and 2022 red balls. The blue balls are labeled 1 to 327. Alice first puts each of the balls into a box, possibly leaving some boxes empty. Then, a random label between 1 and 327 (inclusive) is selected, Alice finds the box the ball with the label is in, and selects a random ball from that box. What is the maximum probability that she selects a red ball?

6. Let a, b, c be real numbers, which are not all equal, such that

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3.$$

   Prove that at least one of a, b, c is negative.

7. Let ABC be a triangle with $|AB| < |AC|$, where $|\cdot|$ denotes length. Suppose D, E, F are points on side BC such that D is the foot of the perpendicular on BC from A, AE is the angle bisector of $\angle BAC$, and F is the midpoint of BC. Further suppose that $\angle BAD = \angle DAE = \angle EAF = \angle FAC$. Determine all possible values of $\angle ABC$.

8. Let m, n, k be positive integers. k coins are placed in the squares of an $m \times n$ grid. A square may contain any number of coins, including zero. Label the k coins $C_1, C_2, \cdots C_k$. Let r_i be the number of coins in the same row as C_i, including C_i itself. Let s_i be the number of coins in the same column as C_i, including C_i itself. Prove that

$$\sum_{i=1}^{k} \frac{1}{r_i + s_i} \leq \frac{m+n}{4}.$$