2021 Canadian Mathematical Gray Jay Competition

Official Solutions

A competition of the Canadian Mathematical Society.

Part A: 4 marks each

1. Calculate the following (hint: remember order of operations, BEDMAS).

$$1 - 2 \times 2 + 3 \times 3 \times 3$$

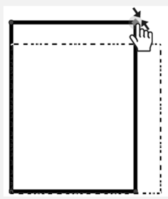
(A) 9 **(B)** 10 **(C)** 24 **(D)** 25 **(E)** 45

Solution: Using BEDMAS we know we must perform multiplications and divisions, from left to right, before addition and subtraction. Since $2 \times 2 = 4$ and $3 \times 3 \times 3 = 27$, we have

$$\begin{aligned} 1 - 2 \times 2 + 3 \times 3 \times 3 &= 1 - 4 + 27 \\ &= 24 \end{aligned}$$

Answer: **(C)**

2. Romina is working with a rectangle on her computer. It started as a 45×45 square.
She resized it, keeping its perimeter the same. Its area is now 2021. What are the new length and width?



(A) 33×39 **(B)** 33×37
(C) 39×49 **(D)** 43×47
(E) 41×49

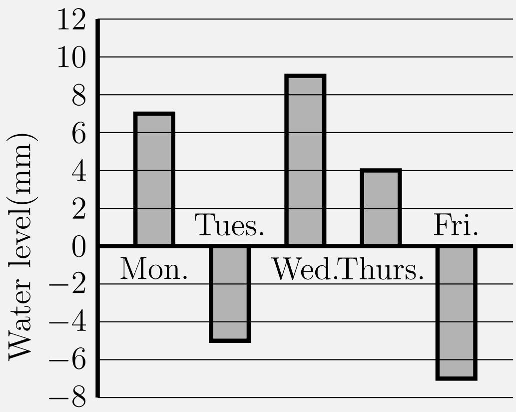
Solution: Note that the perimeter of the square is $4 \times 45 = 180$. Therefore, if our new rectangle has length ℓ and width w, then $2\ell + 2w = 180$. Dividing this by 2 we see that we must have $\ell + w = 90$. We can immediately dismiss all answers where the length and width do not have a sum of 90. (D) and (E) are the only viable options. Since 2021 is not divisible by 7, it's not divisible by $49 = 7 \times 7$ which eliminates (E). Note that $2021 = 43 \times 47$ and $2 \times 43 + 2 \times 47 = 180$.

Answer: **(D)**

Thoughts for further investigation:
Two rectangles who have the same perimeter have the same *average dimension*, that is $\frac{\ell+w}{2}$ is the same. How are the areas of two rectangles who have the same average dimension related?

Note the average, $\frac{\ell+w}{2}$, is also called the *arithmetic mean.* There are other types of "averages", for example the *geometric mean* of the length and the width is $\sqrt{\ell \times w}$. How do two rectangles with the same average dimensions compare if we are using the geometric mean as our average?

3. Sophia does an experiment at school. On Monday morning she puts a beaker of water outside and marks the water level. Each day the water level changes, due to rainfall and evaporation. At the end of each school day Sophia measures the water level with respect to the starting level on Monday. She draws a bar chart of the water level at the end of each day from Monday to Friday, as illustrated on the right. By how much did the water level increase from the end of the school day on Tuesday to the end of the school day on Wednesday?



(A) 7 mm **(B)** 9 mm **(C)** 11 mm **(D)** 12 mm **(E)** 14 mm

Solution: At the end of the day on Tuesday the water level was 5 mm below the starting level. At the end of the day on Wednesday the water level was 9 cm above the starting level. So the water level increased by $5 + 9 = 14$ mm. Note you could also have calculated $9 - (-5) = 14$ mm.

Answer: **(E)**

4. Tarik rolls a standard die numbered 1, 2, 3, 4, 5, 6 a number of times and writes down the results. When he is done, he adds all the numbers and finds the sum is 10. He is shocked to find out when he multiplies all the numbers together that the product is also 10. How many times did he roll the die?

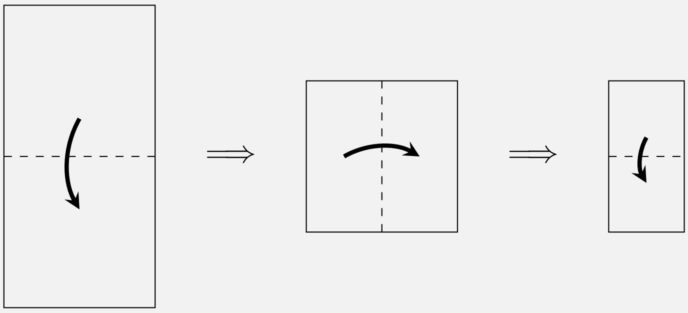
(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Solution: Since $10 = 2 \times 5$, Tarik must have rolled a 2 and a 5 at some point. If he rolls anything other than a 1, the product of his rolls will be bigger than 10. So he must have rolled a 2, a 5 and a number of 1s. Since $2+5+1+1+1 = 10$, Tarik rolled the die 5 times.

Answer: **(D)**

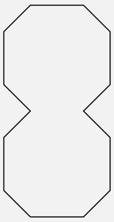
Thoughts for further investigation:
If the sum and product of the rolls was the same, but some number other than 10, would you be able to solve the problem?

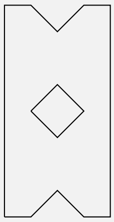
5. Chen folded a rectangular-shaped piece of paper in the directions shown by the arrows below

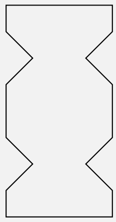


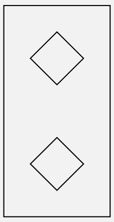
and then he cuts off the shaded part as in the figure below.

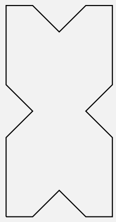
What shape is Chen going to get when he opens the folded piece of paper?

(A) 

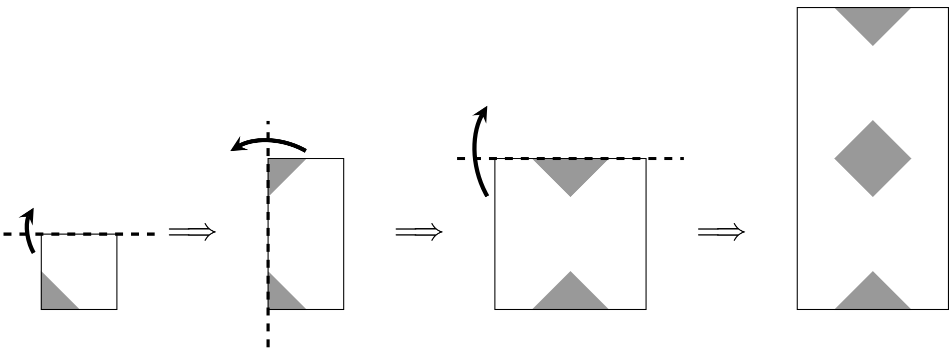
(B) 

(C) 

(D) 

(E) 

Solution: Unfolding the paper we get



Answer: (B)

Thoughts for further investigation:
If we cut out a different corner would it have changed the final result?

Part B: 5 marks each

6. Waneek picks a two-digit number, subtracts the tens digit and then subtracts the ones digit to get a new number. For example if she had picked 37, she would get

$$37 - 3 - 7 = 27$$

so her new number would be 27. How many different numbers can be formed using Waneek's process?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 90 **(E)** 99

Solution: If we let the tens and ones digits be T and O, respectively then the original number was $10T + O$ and after the process we get

$$10T + O - T - O = 9T$$

So the end result is 9 times the tens digit (notice the example, we started with 37 and ended with $27 = 9 \times 3$). Since we are dealing with two-digit numbers, the tens digits range from 1 to 9, hence there are 9 different numbers formed this way.

Answer: **(A)**

Thoughts for further investigation:
Can the question be answered for three-digit numbers?

7. The digits 1, 2, 3, 4, 5, 6 are used once each to make two three-digit numbers. What is the smallest possible difference between these two numbers?

(A) 31 **(B)** 39 **(C)** 47 **(D)** 60 **(E)** 76

Solution: To make the difference as small as possible we must:

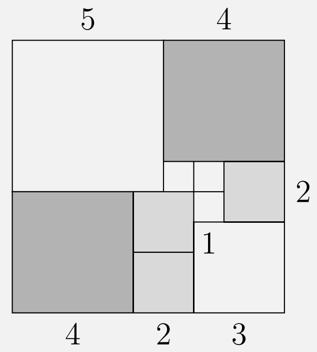
- make the hundreds digits as close as possible,
- make the smaller number be as close to the next multiple of 100 as possible,
- make the larger number be as close to the previous multiple of 100 as possible.

We can satisfy the second point by choosing the largest possible tens digit and the next possible largest ones digit for the smaller number. So the smaller number is of the form $X65$ for some digit X. Similarly we can satisfy the third point by making the larger number of the form $Y12$. Our two numbers are then 365 and 412 and their difference is $412 - 365 = 47$.

Answer: **(C)**

Thoughts for further investigation:
Have fun exploring variations of this problem. For example, what is the largest possible sum of the two numbers?

8. A square is broken up into smaller squares of side length 1, 2, 3, 4, or 5, as shown in the diagram below.



If a dart is thrown randomly at the square, what is the probability that it strikes one of the squares with side length 2?

(A) $\frac{4}{81}$ **(B)** $\frac{3}{10}$ **(C)** $\frac{6}{25}$ **(D)** $\frac{4}{27}$ **(E)** $\frac{2}{3}$

Solution: The probability will depend on the areas involved. The total area of the three squares of side length 2 is $3 \times 2^2 = 12$, whereas the total area of the large square is $9^2 = 81$. The desired probability is then

$$\frac{12}{81} = \frac{4}{27}$$

Answer: **(D)**

Thoughts for further investigation:
The diagram above is an example of what mathematicians call a *Mrs. Perkins' Quilt* (who said mathematicians don't have a sense of humour!). A Mrs. Perkins' Quilt is a square broken up exactly into smaller squares. The term originated in a puzzle by Henry Dudeney from the early 1900s. In the puzzle, he asked for the smallest number of squares needed to completely cover a 13×13 square.

In the given problem, a 9×9 is broken up into the smallest number of squares possible, 10, and it turns out the side lengths are consecutive numbers $1, 2, 3, 4, 5$. What other squares of side length 10 or smaller have this property?

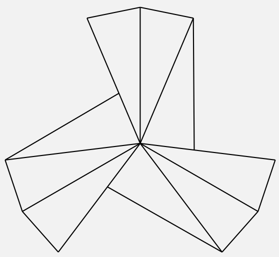
9. The 2021^{st} number which is not divisible by 3 is

(A) 2021 **(B)** 3019 **(C)** 3031 **(D)** 3033 **(E)** 6064

Solution: Each 2 numbers not divisible by 3 are followed by a multiple of 3. This means that the 2020^{th} number not divisible by 3 is followed by the $2020 \div 2 = 1010^{th}$ multiple of 3 which is $3 \times 1010 = 3030$. Therefore the 2021^{st} number is the next number, which is 3031.

Answer: **(C)**

10. In the diagram below, 9 identical copies of an isosceles triangle share a common vertex and fit together as shown.



What is the difference between the largest and smallest angles in the identical triangles?

(A) 60° **(B)** 65° **(C)** 70° **(D)** 75° **(E)** 80°

Solution: If we let L and S represent the size of the largest and smallest angles in the triangle, respectively, then since the triangle is isosceles we get $2L + S = 180°$. In the diagram, we see that there are 3 large and 6 small angles surrounding the common vertex, hence $3L + 6S = 360°$. If we multiply the first equation by 6 and then subtract the second equation we get

$$\begin{array}{rr} & 12L + 6S = 1080^\circ \\ - & 3L + 6S = 360^\circ \\ \hline & 9L = 720^\circ \end{array}$$

which gives $L = 80°$. Substituting into the first equation gives

$$\begin{aligned} 2 \times 80^\circ + S &= 180^\circ \\ 160^\circ + S &= 180^\circ \\ S &= 20^\circ \end{aligned}$$

So the difference between the largest and smallest angle is $80° - 20° = 60°$.

Similarly, if we divide the second equation by three and subtract it from the first we would get

$$\begin{array}{rr} & 2L + S = 180° \\ - & L + 2S = 120° \\ \hline & L - S = 60° \end{array}$$

Answer: **(A)**

Part C: 7 marks each

11. At the start of the day a vending machine had 42 sandwiches in it:

- 13 chicken sandwiches at \$3 each,
- 14 cheese sandwiches at \$3 each,
- 15 egg sandwiches at \$2.50 each.

At the end of the day the total money made was \$61, of which \$21 came from cheese sandwich sales.

What additional piece of information would allow us to solve the question "How many sandwiches of each type were left at the end of the day?"

(A) Total price of remaining sandwiches.
(B) Number of remaining sandwiches.
(C) Total price of sandwiches at the beginning of the day.
(D) Number of remaining cheese sandwiches.
(E) No additional information is needed.

Solution: Note that since we know the price of the sandwiches and the number of sandwiches of each type we started with, we can calculate the total price of the sandwiches at the beginning of the day, so the answer is not (C). Also, since we know the total money made, we can calculate the total price of the remaining sandwiches we can eliminate (A). Since \$21 came from cheese sandwich sales, we know that we must have sold $\$21 \div \$3 = 7$, cheese sandwiches, leaving $14 - 7 = 7$ cheese sandwiches which eliminates (D).

That means the total amount collected at the end of the day from selling egg and chicken sandwiches is $\$61 - \$21 = \$40$. Thus if we let C and E represent the number of chicken and egg sandwiches sold, respectively, we have $\$3C + \$2.5E = \$40$. If we multiply the equation by 2 and rearrange we get

$$6C = 80 - 5E$$

Since everything on the right side of the equation is divisible by 5, then clearly C is a multiple of 5. Since $6 \times 15 = 90 > 80$, the only possible values for C are 0, 5, and 10. A quick check shows that these numbers correspond to E being 16, 10, and 4. Since we started with 15 egg sandwiches, we couldn't have sold 16, so we could have sold 5 chicken sandwiches and 10 egg sandwiches or 10 chicken sandwiches and 4 egg sandwiches. We need more information, so we can eliminate (E).

If we sold 5 chicken sandwiches and 10 egg sandwiches we would have 8 chicken, 7 cheese, and 5 egg sandwiches left, or $8 + 7 + 5 = 20$ sandwiches left. On the other hand, if we sold 10 chicken sandwiches and 4 egg sandwiches we would have 3 chicken, 7 cheese, and 11 egg sandwiches left, or $3 + 7 + 11 = 21$ sandwiches left. Therefore if we knew how many sandwiches were left we could answer the question.

Answer: **(B)**

12. Which of the numbers $3, 5, 11$ and 37 is a divisor of

$$\underbrace{111\ 111\ 111\ 111\ 111\ 111\ 111}_{21\ \text{ones}}$$

(A) $3, 11, 37$ **(B)** $5, 11$ **(C)** $3, 11$ **(D)** $3, 37$ **(E)** $3, 5, 11, 37$

Solution: If we know divisibility tests for 3, 11 and 37 we can use them. If not, if we start doing long division, we notice that

$$3 \times 37 = 111$$

Our number is made up of 21 ones, or seven groups of three ones (111). Hence the number is divisible by both 3 and 37. Since the number ends with 1 it is not divisible by 5.

Using similar logic, our number is made up of ten groups of two ones (11) with one extra. That means, if we divide the number by 11 we will have a remainder of 1 at the end. Therefore the number is only divisible by 3 and 37.

Answer: **(D)**

Thoughts for further investigation:
Numbers made up only of the digit 1 are called *repunit* numbers (for repeated unit, similarly a *repdigit* number is a repeating digit like 55, or 7 777 777). Can you classify

which repunit numbers are divisible by 3? by 11? by 37? by any combination of the three?

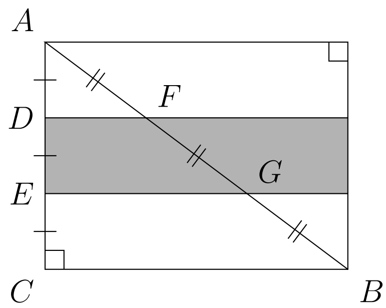
13. Right angled triangle ABC has its right angle at C, side AC has length 9 cm and side BC has length 12 cm. Points D and E are on AC so that $AD = DE = EC$. Similarly, points F and G are on AB so that $AF = FG = GB$. What is the area of trapezoid $DFGE$? **Hint:** the area of a triangle is given by $A = b \times h \div 2$.

(A) 9 cm^2 **(B)** 12 cm^2 **(C)** 15 cm^2 **(D)** 18 cm^2 **(E)** 27 cm^2

Solution 1: Since D and E divide AB into thirds as do F and G to AB, triangles ADF, AEG and ACB are similar. Hence DF is one third the length of CB, that is, 4 units and similarly EG is 8 units long. Therefore

$$\begin{aligned}\text{Area of } DFGE &= \text{Area of } AEG - \text{Area of } ADF \\ &= \frac{1}{2}(8)(6) - \frac{1}{2}(4)(3) \\ &= 24 - 6 \\ &= 18\end{aligned}$$

Solution 2: If we draw a second copy of ABC and rearrange it to form a rectangle we see that the shaded strip has area one-third of the rectangle, so our trapezoid has area one-third of the triangle.



So the area of the trapezoid is

$$\begin{aligned}A &= \frac{1}{2}(12)(9) \div 3 \\ &= 18\end{aligned}$$

Solution 3: If we knew the formula for the area of a trapezoid we could have calculated it directly

$$\begin{aligned} A &= \frac{1}{2}(b_1 + b_2)h \\ &= \frac{1}{2}(4+8)(3) \\ &= 18 \end{aligned}$$

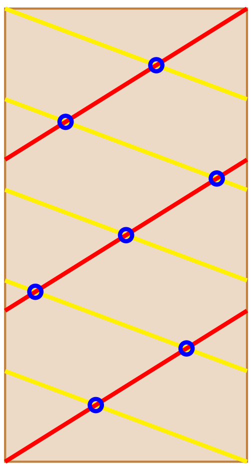
Answer: **(D)**

14. Two creeper plants, one golden hop and the other tea rose, are both climbing up and around a cylindrical tree trunk. The golden hop twists clockwise and the tea rose counterclockwise, and they both start at the same point on the ground. Before they reach the first tree branch, the golden hop has made 5 complete turns around the tree trunk and the tea rose has made 3 complete turns. The turns are equidistant for each creeper.

Without counting the ground and the first tree branch, how many times do the two creepers cross before reaching the first branch?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

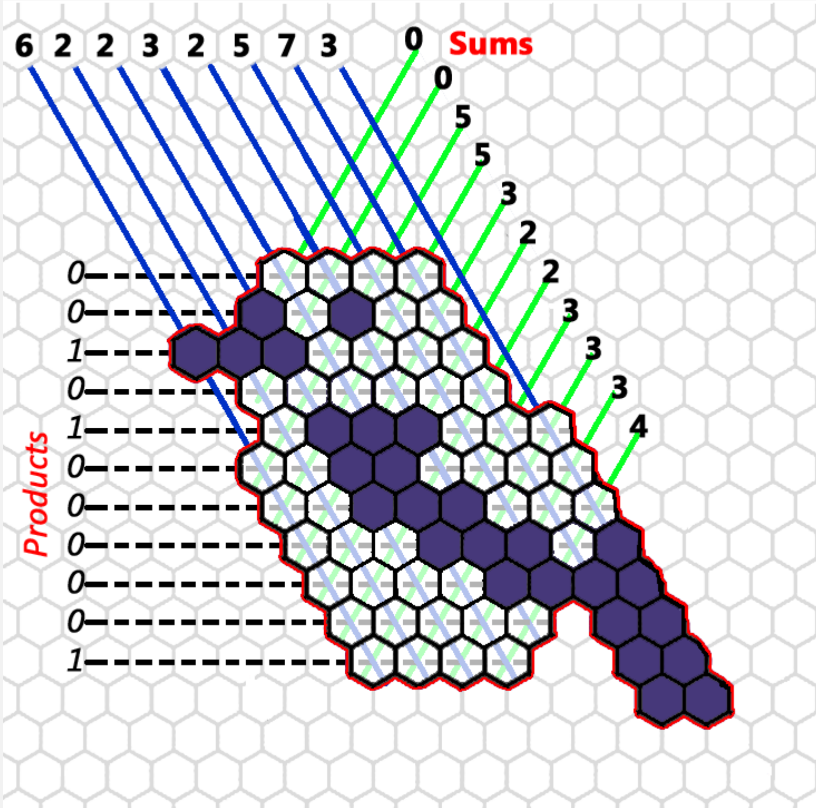
Solution: If we "unroll" the cylindrical tree trunk we get a rectangle. The golden hop is shown in gold and the tea rose in red.



The seven desired crossings are circled in blue.

Answer: **(B)**

15. Nick needs to fill in the picture below as follows: in every hexagon within the bird outline, there is a zero or a one. The **sum** of the numbers on each diagonal is given in **bold** above the picture and the *product* of the numbers in each row is given in *italics* along the side.



Once the picture is completed, how many zeros will there be? **Note:** the shaded hexagons help to form the bird and are **NOT** part of the sum/product counts.

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

Solution 1: Note if we total the sums along the top or right we get

$$6 + 2 + 2 + 3 + 2 + 5 + 7 + 3 = 30 = 0 + 0 + 5 + 5 + 3 + 2 + 2 + 3 + 3 + 3 + 4$$

which means there must be 30 ones in the cells. There are 49 cells in total, so there must be $49 - 30 = 19$ zeroes.

Solution 2: We can fill in the cells using the information in the problem. First, we note that the only way a row can have a product of 1 is if every entry is a 1.

Next, the only way the sum of a column can be 0 is if every entry is a 0.

Finally, going through the columns some of them need all the entries to be 1 or 0 to work out, for example the 6. Fill these in, then go over the missed ones, you can fill some of them now because of earlier entries. The figure can be completely filled this way.

Hence, there are 19 zeros.

Answer: **(B)**